Derivations of Conditional Distributions in Hierarchical Normal Linear Models

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ABSTRACT

These are notes. They give details of deriving conditional distributions occurring in hierarchical modeling of familiar analysis of variance mixed models. All the results, and most of the derivations, are to be found in Section 4.3b of *Variance Components*, Searle, Casella and McCulloch (Wiley, 1992). The main distinctions between that section and these notes are the sequencing of the derivations, the details displayed, and the tabular summaries of general results and special cases.

MODELS

The usual mixed model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$$

$\boldsymbol{\beta}$ is fixed; $\mathbf{X}$ and $\mathbf{Z}$ are known

$\mathbf{u}$ is random: $\mathrm{E}(\mathbf{u}) = \mathbf{0}$ $\mathrm{var}(\mathbf{u}) = \mathbf{D}$

$\mathbf{e}$ is residual: $\mathrm{E}(\mathbf{e}) = \mathbf{0}$ $\mathrm{var}(\mathbf{e}) = \mathbf{R}$

$\mathrm{cov}(\mathbf{u}, \mathbf{e}') = \mathbf{0}$.

$$\mathrm{var}(\mathbf{y}) = \mathbf{V} = \mathbf{ZDZ}' + \mathbf{R} . \qquad (1)$$

Hierarchical modeling (Bayes, e.g., p. 328)

In addition to the above:

$$\mathrm{E}(\boldsymbol{\beta}) = \boldsymbol{\beta}_0 \qquad \mathrm{var}(\boldsymbol{\beta}) = \mathbf{B} \qquad \mathrm{cov}(\boldsymbol{\beta}, \mathbf{y}') = \mathbf{BX}' . \qquad (2)$$

$$\mathrm{var}(\mathbf{y}) = \mathbf{W} = \mathbf{XBX}' + \mathbf{V} = \mathbf{XBX}' + \mathbf{ZDZ}' + \mathbf{R} . \qquad (3)$$

$[57]^*_{332}$ Define $$\mathcal{L}^{-1} = \mathbf{XBX}' + \mathbf{R} \qquad \text{and} \qquad \mathcal{C} = \mathbf{D}^{-1} + \mathbf{X}'\mathcal{L}\mathbf{Z} . \qquad (4)$$

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* Equation numbers from Section 9.3b of Searle *et al.* (1982) are shown here on the left-hand margin, in square brackets, e.g., $[57]_{332}$ is equation (57) on page 332.

Normality

[49]$_{331}$

$$\begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}\left\{ \begin{bmatrix} \boldsymbol{\beta}_0 \\ \mathbf{u}_0 \\ \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{Z}\mathbf{u}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{BX'} \\ \mathbf{0} & \mathbf{D} & \mathbf{DZ'} \\ \mathbf{XB} & \mathbf{ZD} & \mathbf{W} \end{bmatrix} \right\}. \tag{5}$$

Conditional variables, under normality

A general result

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} \right]. \tag{6}$$

[50]$_{332}$

$$\mathbf{x}_1 \,|\, \mathbf{x}_2 \sim \mathcal{N}\left[\boldsymbol{\mu}_1 + \mathbf{V}_{12}\mathbf{V}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2), \quad \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\right]. \tag{7}$$

By applying (7) to (5) we get distributional results for four conditional variables that arise in the hierarchical model.

FOUR CONDITIONAL VARIABLES

The four conditional variables to be considered come in pairs: $\mathbf{u}\,|\,\boldsymbol{\beta}, \mathbf{y}$ and $\mathbf{u}\,|\,\mathbf{y}$, and $\boldsymbol{\beta}\,|\,\mathbf{u}, \mathbf{y}$ and $\boldsymbol{\beta}\,|\,\mathbf{y}$. General results for the four are derived first, and special cases are then summarized in tables. Initially, the presentation is directed towards establishing results [51]$_{332}$ through [62]$_{333}$ of Section 4.3b.

First: $\mathbf{u}\,|\,\boldsymbol{\beta}, \mathbf{y}$

The appropriate application of (7) to (5) is

$$\mathbf{x}_1 = \mathbf{u} \qquad \mathbf{x}_2' = [\boldsymbol{\beta}' \;\; \mathbf{y}']$$
$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 \qquad \boldsymbol{\mu}_2' = [\boldsymbol{\beta}_0' \;\; (\mathbf{X}\boldsymbol{\beta}_0 + \mathbf{Z}\mathbf{u}_0)'] \tag{8}$$

$$\mathbf{V}_{11} = \mathbf{D} \qquad \mathbf{V}_{12} = [\mathbf{0} \;\; \mathbf{DZ'}] \qquad \mathbf{V}_{22} = \begin{bmatrix} \mathbf{B} & \mathbf{BX'} \\ \mathbf{XB} & \mathbf{W} \end{bmatrix}. \tag{9}$$

Matrix Result (M1)

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{I} \end{bmatrix} (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} [-\mathbf{C}\mathbf{A}^{-1} \quad \mathbf{I}]$$

[See (27), p. 453, plus errata.]

Hence from (9)

$$\mathbf{V}_{22}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{B}\mathbf{X}' \\ \mathbf{I} \end{bmatrix} (\mathbf{W} - \mathbf{X}\mathbf{B}\mathbf{B}^{-1}\mathbf{B}\mathbf{X}')^{-1} [-\mathbf{X}\mathbf{B}\mathbf{B}^{-1} \quad \mathbf{I}]$$

$$= \begin{bmatrix} \mathbf{B}^{-1} + \mathbf{X}'\mathbf{V}^{-1}\mathbf{X} & -\mathbf{X}'\mathbf{V}^{-1} \\ -\mathbf{V}^{-1}\mathbf{X} & \mathbf{V}^{-1} \end{bmatrix} . \tag{10}$$

Hence from using (9) and (10) in (7)

$$\begin{aligned} E(\mathbf{u} \,|\, \boldsymbol{\beta}, \mathbf{y}) &= \mathbf{u}_0 + [-\mathbf{D}\mathbf{Z}'\mathbf{V}^{-1}\mathbf{X} \quad \mathbf{D}\mathbf{Z}'\mathbf{V}^{-1}] \begin{bmatrix} \boldsymbol{\beta} - \boldsymbol{\beta}_0 \\ \mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Z}\mathbf{u}_0 \end{bmatrix} \\ &= \mathbf{u}_0 + \mathbf{D}\mathbf{Z}'\mathbf{V}^{-1}[-\mathbf{X}(\boldsymbol{\beta} - \boldsymbol{\beta}_0) + \mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Z}\mathbf{u}_0] \\ &= \mathbf{u}_0 + \mathbf{D}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}_0) \ . \end{aligned} \tag{11}$$

Matrix Result (M2)

$$\begin{aligned} (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})\,\mathbf{D}\mathbf{Z}' &= \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{Z}' \\ &= \mathbf{Z}'\mathbf{R}^{-1}(\mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{R}) \\ &= \mathbf{Z}'\mathbf{R}^{-1}\mathbf{V} \end{aligned}$$

$$\mathbf{D}\mathbf{Z}'\mathbf{V}^{-1} \equiv \mathbf{D}\mathbf{Z}'(\mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{R})^{-1} = (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1} \tag{12}$$

$$= \mathbf{A}^{-1}\mathbf{Z}'\mathbf{R}^{-1} \quad \text{for} \quad \mathbf{A} = (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} \ . \tag{13}$$

$[39]_{329}$

$$\begin{aligned} E(\mathbf{u} \,|\, \boldsymbol{\beta}, \mathbf{y}) &= \mathbf{u}_0 + \mathbf{D}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}_0) \\ &= \mathbf{u}_0 + (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}_0) \\ &= (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}[\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} - \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z})\mathbf{u}_0] \\ &= (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}[\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \mathbf{D}^{-1}\mathbf{u}_0] \end{aligned} \tag{14}$$

$[55]_{332}$

$$= \mathbf{u}_0 + \mathbf{A}^{-1}\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Z}\mathbf{u}_0) \quad \text{for} \quad \mathbf{A} = \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} \tag{15}$$

$$= \mathbf{A}^{-1}[\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \mathbf{D}^{-1}\mathbf{u}_0] \ .$$

Also, from (7)

$$\mathrm{var}(\mathbf{u} \mid \boldsymbol{\beta}, \mathbf{y}) = \mathbf{D} - \mathbf{DZ'V^{-1}ZD'} \ . \tag{16}$$

Matrix Result (M3)

$$\begin{aligned}\mathbf{D} - \mathbf{DZ'V^{-1}ZD} &= \mathbf{D} - (\mathbf{Z'R^{-1}Z} + \mathbf{D^{-1}})^{-1}\mathbf{Z'R^{-1}ZD} \\ &= [\mathbf{I} - (\mathbf{Z'R^{-1}Z} + \mathbf{D^{-1}})^{-1}(\mathbf{Z'R^{-1}Z} + \mathbf{D^{-1}} - \mathbf{D^{-1}})]\,\mathbf{D} \\ &= (\mathbf{Z'R^{-1}Z} + \mathbf{D^{-1}})^{-1} = \mathbf{A^{-1}} \text{ from (13)} \ .\end{aligned}$$

Hence from (16)

[55] $$\mathrm{var}(\mathbf{u} \mid \boldsymbol{\beta}, \mathbf{y}) = (\mathbf{Z'R^{-1}Z} + \mathbf{D^{-1}} = \mathbf{A^{-1}} \tag{17}$$

Hence $$\mathbf{u} \mid \boldsymbol{\beta}, \mathbf{y} \sim \mathcal{N}[\mathbf{u}_0 + \mathbf{A^{-1}Z'R^{-1}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu}_0), \quad \mathbf{A^{-1}}] \ . \tag{18}$$

<u>**Second: u | y**</u>

For (7): $$\boldsymbol{\mu}_1 = \mathbf{u}_0 \ , \qquad \boldsymbol{\mu}_2 = \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{Zu}_0 \ , \qquad \mathbf{V}_{11} = \mathbf{D} \ , \qquad \mathbf{V}_{12} = \mathbf{DZ'}$$

$$\mathbf{V}_{22} = \mathbf{W} = \mathbf{XBX'} + \mathbf{ZDZ'} + \mathbf{R} \ , \qquad \mathbf{V}_{12}\mathbf{V}_{22}^{-1} = \mathbf{DZ'W^{-1}} \ .$$

$$\mathbf{u} \mid \mathbf{y} \sim \mathcal{N}[\mathbf{u}_0 + \mathbf{DZ'W^{-1}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{Zu}_0), \qquad \mathbf{D} - \mathbf{DZ'W^{-1}ZD}] \ . \tag{19}$$

Matrix Result (M3)

In general, as in [28b], p. 453

$$(\mathbf{D} + \mathbf{CA^{-1}B})^{-1} = \mathbf{D^{-1}} - \mathbf{D^{-1}C}(\mathbf{A} + \mathbf{BD^{-1}C})^{-1}\mathbf{BD^{-1}} \tag{20}$$

Hence

$$\mathbf{V}_{12}\mathbf{V}_{22}^{-1} = \mathbf{DZ'W^{-1}} = \mathbf{DZ'}(\mathbf{XBX'} + \mathbf{ZDZ'} + \mathbf{R})^{-1} \ . \tag{21}$$

$$= \mathbf{DZ'}(\boldsymbol{\mathcal{L}}^{-1} + \mathbf{ZDZ'})^{-1}, \qquad \text{from (4)}$$

$$= (\mathbf{Z'}\boldsymbol{\mathcal{L}}\mathbf{Z} + \mathbf{D^{-1}})^{-1}\mathbf{Z'}\boldsymbol{\mathcal{L}}, \quad \text{replacing } \mathbf{R} \text{ in (12) by } \boldsymbol{\mathcal{L}}^{-1} \ . \tag{22}$$

$$= \boldsymbol{\mathcal{C}}^{-1}\mathbf{Z'}\boldsymbol{\mathcal{L}}, \text{ using } \boldsymbol{\mathcal{C}} \text{ from (4)} \ . \tag{23}$$

Also in (19)

$$\mathbf{W^{-1}} = (\mathbf{V} + \mathbf{XBX'})^{-1} = \mathbf{V^{-1}} - \mathbf{V^{-1}X}(\mathbf{X'V^{-1}X} + \mathbf{B^{-1}})^{-1}\mathbf{X'V^{-1}}$$

so that

$$E(\mathbf{u} \mid \mathbf{y}) = \mathbf{u}_0 + \mathbf{DZ'}[\mathbf{V^{-1}} - \mathbf{V^{-1}X}(\mathbf{X'V^{-1}X} + \mathbf{B^{-1}})^{-1}\mathbf{X'V^{-1}}](\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}_0) \ . \tag{24}$$

$$\begin{aligned}\mathbf{D} - \mathbf{DZ'W^{-1}ZD} &= \mathbf{D} - (\mathbf{Z'}\mathcal{L}\mathbf{Z} + \mathbf{D}^{-1})^{-1}\mathbf{Z'}\mathcal{L}\mathbf{ZD}, \quad \text{from 22} \\ &= \mathbf{D} - (\mathbf{Z'}\mathcal{L}\mathbf{Z} + \mathbf{D}^{-1})^{-1}(\mathbf{Z'}\mathcal{L}\mathbf{Z} + \mathbf{D}^{-1} - \mathbf{D}^{-1})\,\mathbf{D} \\ &= (\mathbf{Z'}\mathcal{L}\mathbf{Z} + \mathbf{D}^{-1})^{-1} = \mathcal{C}^{-1}\,. \end{aligned} \tag{25}$$

Hence in (19), using (23) and (25),

$$\mathbf{u}\,|\,\mathbf{y} \sim \mathcal{N}[\mathbf{u}_0 + \mathcal{C}^{-1}\mathbf{Z'}\mathcal{L}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}_0), \quad \mathcal{C}^{-1}]\,. \tag{26}$$

But

$$\mathbf{u}_0 - \mathcal{C}^{-1}\mathbf{Z'}\mathcal{L}\mathbf{Zu}_0 = \mathcal{C}^{-1}(\mathcal{C} - \mathbf{Z'}\mathcal{L}\mathbf{Z})\mathbf{u}_0 = \mathcal{C}^{-1}\mathbf{D}^{-1}\mathbf{u}_0\,,$$

so that

[below 57]$_{332}$

$$\mathbf{u}\,|\,\mathbf{y} \sim \mathcal{N}\left\{\mathcal{C}^{-1}[\mathbf{Z'}\mathcal{L}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0) + \mathbf{D}^{-1}\mathbf{u}_0], \quad \mathcal{C}^{-1}\right\}. \tag{27}$$

Third: u | β, y

Take the results for $\mathbf{u}\,|\,\boldsymbol{\beta}$, $\mathbf{y}$ and have

$$\left.\begin{matrix}\mathbf{u}\\ \mathbf{u}_0\\ \mathbf{Z}\\ \mathbf{D}\\ \mathbf{V} = \mathbf{ZDZ'} + \mathbf{R}\end{matrix}\right\} \quad \text{interchanged with} \quad \left\{\begin{matrix}\boldsymbol{\beta}\\ \boldsymbol{\beta}_0\\ \mathbf{X}\\ \mathbf{B}\\ \mathcal{L}^{-1} = \mathbf{XBX'} + \mathbf{R}\end{matrix}\right. .$$

Making these interchanges in (11), (14) and (15) gives

$$\mathrm{E}(\boldsymbol{\beta}\,|\,\mathbf{u}, \mathbf{y}) = \boldsymbol{\beta}_0 + \mathbf{BX'}(\mathbf{XBX'} + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}) \tag{28}$$

$$= \boldsymbol{\beta}_0 + (\mathbf{X'R^{-1}X} + \mathbf{B}^{-1})^{-1}\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}) \tag{29}$$

[59]$_{333}$

$$= \boldsymbol{\beta}_0 + \mathcal{A}^{-1}\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}), \quad \text{for} \quad \mathcal{A} = \mathbf{X'R^{-1}X} + \mathbf{B}^{-1} \tag{30}$$

$$= \boldsymbol{\beta}_0 + \mathcal{A}^{-1}\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{Zu}) - \mathcal{A}^{-1}\mathbf{X'R^{-1}X} + \boldsymbol{\beta}_0$$

$$= \boldsymbol{\beta}_0 + \mathcal{A}^{-1}\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{Zu}) - \mathcal{A}^{-1}(\mathcal{A} - \mathbf{B}^{-1})\,\boldsymbol{\beta}_0$$

[below 59]$_{333}$

$$= \mathcal{A}^{-1}[\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{Zu}) + \mathbf{B}^{-1}\,\boldsymbol{\beta}_0]\,. \tag{31}$$

And also from (17)

$$\mathrm{var}(\boldsymbol{\beta}\,|\,\mathbf{u}, \mathbf{y}) = (\mathbf{X'R^{-1}X} + \mathbf{B}^{-1})^{-1}\mathcal{A}^{-1}\,. \tag{32}$$

Hence

[60]$_{333}$

$$\boldsymbol{\beta}\,|\,\mathbf{u}, \mathbf{y} \sim \mathcal{N}\{\mathcal{A}^{-1}[\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{Zu}) + \mathbf{B}^{-1}\,\boldsymbol{\beta}_0], \quad \mathcal{A}^{-1}\}\,. \tag{33}$$

Fourth: $\boldsymbol{\beta}|\mathbf{y}$

Make the same interchanges in results for $\mathbf{u}|\mathbf{y}$ as were used to derive results for $\boldsymbol{\beta}|\mathbf{u}, \mathbf{y}$ from those for $\mathbf{u}|\boldsymbol{\beta}, \mathbf{y}$.

(19) becomes

$$[52]_{332} \qquad \boldsymbol{\beta}|\mathbf{y} \sim \mathcal{N}[\boldsymbol{\beta}_0 + \mathbf{BX'W}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}_0), \quad \mathbf{B} - \mathbf{BX'W}^{-1}\mathbf{XB}] \; . \tag{34}$$

(26) gives this as

$$[41]_{329} \qquad \boldsymbol{\beta}|\mathbf{y} \sim \mathcal{N}[\boldsymbol{\beta}_0 + \mathbf{C}^{-1}\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{Zu}_0), \quad \mathbf{C}^{-1}] \tag{35}$$

and so (27) is

$$\begin{aligned} \boldsymbol{\beta}|\mathbf{y} &\sim \mathcal{N}\left\{\mathbf{C}^{-1}[\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{Zu}_0) + \mathbf{B}^{-1}\boldsymbol{\beta}_0], \quad \mathbf{C}^{-1}\right\} \\ &\sim \mathcal{N}\left\{(\mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}[\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{Zu}_0) + \mathbf{B}^{-1}\boldsymbol{\beta}_0], \quad (\mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}\right\} . \end{aligned} \tag{36}$$

TABLE 1.

Normal distributions of conditional variables in the linear model $\mathbf{y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{u} + \mathbf{e}$.

$$\mathbf{W} = \mathbf{XBX'} + \mathbf{ZDZ'} + \mathbf{R} \qquad \mathbf{V} = \mathbf{ZDZ'} + \mathbf{R} \qquad \mathcal{L}^{-1} = \mathbf{XBX'} + \mathbf{R}$$

$$\mathbf{A} = \mathbf{Z'R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} \qquad \mathbf{C} = \mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1}$$

$$\mathcal{A} = \mathbf{X'R}^{-1}\mathbf{X} + \mathbf{B}^{-1} \qquad \mathcal{C} = \mathbf{Z'}\mathcal{L}\mathbf{Z} + \mathbf{D}^{-1}$$

$$\begin{bmatrix} \beta \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}\left[\begin{bmatrix} \beta_0 \\ \mathbf{u}_0 \\ \mathbf{X}\beta_0 + \mathbf{Zu}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{BX'} \\ \mathbf{0} & \mathbf{D} & \mathbf{DZ'} \\ \mathbf{XB} & \mathbf{ZD} & (\mathbf{XBX'} + \mathbf{ZDZ'} + \mathbf{R}) \end{bmatrix} \right]$$

Conditional Variable	Normal Distribution			
		Mean	Variance	
	Equ.			Equ.
$\mathbf{u} \mid \beta, \mathbf{y}$	(11)	$\mathbf{u}_0 + \mathbf{DZ'V}^{-1}(\mathbf{y} - \mathbf{X}\beta - \mathbf{Zu}_0)$	$\mathbf{D} - \mathbf{DZ'V}^{-1}\mathbf{ZD}$	(10)
	(15), [55]	$= \mathbf{u}_0 + \mathbf{A}^{-1}\mathbf{Z'R}^{-1}(\mathbf{y} - \mathbf{X}\beta - \mathbf{Zu}_0)$	$= (\mathbf{Z'R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}$	(17)
	(14)	$= (\mathbf{Z'R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}[\mathbf{Z'R}^{-1}(\mathbf{y} - \mathbf{X}\beta) + \mathbf{D}^{-1}\mathbf{u}_0]$	$= \mathbf{A}^{-1}$	(25), [55]
$\mathbf{u} \mid \mathbf{y}$	(19)	$\mathbf{u}_0 + \mathbf{DZ'W}^{-1}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu}_0)$	$\mathbf{D} - \mathbf{DZ'W}^{-1}\mathbf{ZD}$	(19)
	(24)	$= \mathbf{u}_0 + \mathbf{DZ'}[\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}\mathbf{X'V}^{-1}] \times (\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu}_0)$		
	(26)	$= \mathbf{u}_0 + \mathcal{C}^{-1}\mathbf{Z'}\mathcal{L}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu}_0)$	$= \mathcal{C}^{-1}$	(26), [47]
	(27), [below 57]	$= \mathcal{C}^{-1}[\mathbf{Z'}\mathcal{L}(\mathbf{y} - \mathbf{X}\beta_0) + \mathbf{D}^{-1}\mathbf{u}_0]$	$= [\mathbf{Z'}(\mathbf{XBX'} + \mathbf{R})^{-1}\mathbf{Z} + \mathbf{D}^{-1}$	
$\beta \mid \mathbf{u}, \mathbf{y}$	(28)	$\beta_0 + \mathbf{BX'}(\mathbf{XBX'} + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu})$	$\mathbf{B} - \mathbf{BX'}\mathcal{L}\mathbf{XB}$	
	(29)	$= \beta_0 + (\mathbf{X'R}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}\mathbf{X'R}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu})$	$= \mathbf{B} - \mathbf{BX'}(\mathbf{XBX'} + \mathbf{R})^{-1}\mathbf{XB}$	
	(30), [60]	$= (\mathbf{X'R}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}[\mathbf{X'R}^{-1}(\mathbf{y} - \mathbf{Zu}) + \mathbf{B}^{-1}\beta_0]$	$= (\mathbf{X'R}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}$	[60]
$\beta \mid \mathbf{y}$	(34), [52]	$\beta_0 + \mathbf{BX'W}^{-1}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu}_0)$	$\mathbf{B} - \mathbf{BX'W}^{-1}\mathbf{XB}$	(34), [52]
	(35)	$= \beta_0 + \mathbf{C}^{-1}\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{X}\beta_0 - \mathbf{Zu}_0)$	$= \mathbf{C}^{-1}$	(35), [41]
	(36), [53]	$= \mathbf{C}^{-1}[\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{Zu}_0) + \mathbf{B}^{-1}\beta_0]$	$= (\mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}$	(36), [53]
		$= (\mathbf{X'V}^{-1}\mathbf{X} + \mathbf{B}^{-1})^{-1}[\mathbf{X'V}^{-1}(\mathbf{y} - \mathbf{Zu}_0) + \mathbf{B}^{-1}\beta_0]$		

TABLE 2a

Special Case: Table 1 with $\mathbf{u}_0 = \mathbf{0}$.

Variable	Equation	Mean	Variance
$\mathbf{u} \mid \beta, \mathbf{y}$	[40]	$\mathbf{DZ'V^{-1}(y - X\beta)}$	$\mathbf{D - DZ'V^{-1}ZD}$
	[39], [56]	$= \mathbf{A^{-1}Z'R^{-1}(y - X\beta)}$	$= \mathbf{A^{-1}}$
		$= \mathbf{(Z'R^{-1}Z + D^{-1})^{-1}Z'R^{-1}(y - X\beta)}$	$= \mathbf{(Z'R^{-1}Z + D^{-1})^{-1}}$
$\mathbf{u} \mid \mathbf{y}$		$\mathbf{DZ'W^{-1}(y - X\beta_0)}$	$\mathbf{D - DZ'W^{-1}ZD}$
		$= \mathbf{DZ'\left[V^{-1} - V^{-1}X(X'V^{-1}X + B^{-1})^{-1}X'V^{-1}(y - X\beta_0)\right]}$	
	[46], [58]	$= \mathbf{\mathcal{C}^{-1}Z'\mathcal{L}(y - X\beta_0)}$	$= \mathbf{\left[Z'(XBX' + R)^{-1}Z + D^{-1}\right]^{-1}}$
$\beta \mid \mathbf{u}, \mathbf{y}$		$\mathbf{\beta_0 + BX'(XBX + R)^{-1}(y - X\beta_0 - Zu)}$	$\mathbf{B - BX'(XBX + R)^{-1}XB}$
	[46], [59] [60]	$= \mathbf{(X'R^{-1}X + B^{-1})^{-1}\left[X'R^{-1}(y - Zu) + B^{-1}\beta_0\right]}$	$= \mathbf{(X'R^{-1}X + B^{-1})^{-1}}$
		$= \mathbf{\mathcal{A}^{-1}\left[X'R^{-1}(y - Zu) + B^{-1}\beta_0\right]}$	$= \mathbf{\mathcal{A}^{-1}}$
$\beta \mid \mathbf{y}$		$\mathbf{\beta_0 + BX'W^{-1}(y - X\beta_0)}$	$\mathbf{B - BX'W^{-1}XB}$
	[39], [54]	$= \mathbf{C^{-1}(X'V^{-1}y + B^{-1}\beta_0)}$	
	[40]	$= \mathbf{(X'V^{-1}X + B^{-1})^{-1}(X'V^{-1}y + B^{-1}\beta_0)}$	$= \mathbf{(X'V^{-1}X + B^{-1})^{-1}}$

TABLE 2b

Special case: Table 1 with $\mathbf{u}_0 = \mathbf{0}, \mathbf{B}^{-1} \to \mathbf{0}$

Table 2a with $\mathbf{B}^{-1} \to \mathbf{0}$

Variable	Equation	Mean	Variance
$\mathbf{u} \mid \beta, \mathbf{y}$		Same as Table 2a	
$\mathbf{u} \mid \mathbf{y}$	[62]	$\mathbf{DZ'V^{-1}(y - X\hat{\beta})} = \text{BLUP}(\mathbf{u})$	$\mathbf{D - DZ'V^{-1}ZD}$
$\beta \mid \mathbf{u}, \mathbf{y}$		Same as Table 2a	
$\beta \mid \mathbf{y}$	[43]	$\mathbf{(X'V^{-1}X)^{-}X'V^{-1}y} = \text{BLUE}(\beta)$	$\mathbf{D - DZ'V^{-1}ZD}$

TABLE 2c

Special case: Table 1 with $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{B}^{-1} \to \mathbf{0}$

Table 2a with $\mathbf{R} = \sigma_e^2 \mathbf{I}$*

Variable	Mean	Variance
$\mathbf{u} \mid \beta, \mathbf{y}$	$\left(\mathbf{Z}'\mathbf{Z} + \sigma_e^2 \mathbf{D}^{-1}\right)^{-1} \mathbf{Z}'(\mathbf{y} - \mathbf{X}\beta)$	$\left(\mathbf{Z}'\mathbf{Z} / \sigma_e^2 + \mathbf{D}^{-1}\right)^{-1}$
$\mathbf{u} \mid \mathbf{y}$	$\left[\mathbf{Z}'\left(\mathbf{XBX}' + \sigma_e^2 \mathbf{I}\right)^{-1} \mathbf{Z} + \mathbf{D}^{-1}\right]^{-1} \left(\mathbf{XBX}' + \sigma_e^2 \mathbf{I}\right)^{-1} (\mathbf{y} - \mathbf{X}\beta_0)$	$\left[\mathbf{Z}'\left(\mathbf{XBX}' + \sigma_e^2 \mathbf{I}\right)^{-1} \mathbf{Z} + \mathbf{D}^{-1}\right]^{-1}$
$\beta \mid \mathbf{u}, \mathbf{y}$	$\left(\mathbf{X}'\mathbf{X} + \sigma_e^2 \mathbf{B}^{-1}\right)^{-1} \left[\mathbf{X}'(\mathbf{y} - \mathbf{X}\beta) + \sigma_e^2 \mathbf{B}^{-1} \beta_0\right]$	$\left(\mathbf{X}'\mathbf{X} / \sigma_e^2 + \mathbf{B}^{-1}\right)^{-1}$
$\beta \mid \mathbf{y}$	$\left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} + \mathbf{B}^{-1}\right)^{-1} \left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} + \mathbf{B}^{-1}\beta_0\right)$	$\left(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} + \mathbf{B}^{-1}\right)^{-1}$

TABLE 2d

Special case: Table 1 with $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{R} = \sigma_e^2 \mathbf{I}$, and $\mathbf{B}^{-1} \to \mathbf{0}$

Table 2a with $\mathbf{R} = \sigma_e^2 \mathbf{I}$ and $\mathbf{B}^{-1} \to \mathbf{0}$

Table 2b with $\mathbf{R} = \sigma_e^2 \mathbf{I}$*

Table 2c with $\mathbf{B}^{-1} \to \mathbf{0}$

Variable	Mean	Variance
$\mathbf{u} \mid \beta, \mathbf{y}$	Same as Table 2c	$\left(\mathbf{Z}'\mathbf{Z} / \sigma_e^2 + \mathbf{D}^{-1}\right)^{-1}$
$\mathbf{u} \mid \mathbf{y}$	Same as Table 2c	
$\beta \mid \mathbf{u}, \mathbf{y}$	$(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'(\mathbf{y} - \mathbf{Zu})$	$(\mathbf{X}'\mathbf{X})^{-}\sigma_e^2$
$\beta \mid \mathbf{y}$ [43]	Same as Table 2b	$\mathbf{C}^{-1}$

* Searle *et al.* (1992) does not show details for $\mathbf{R} = \sigma_e^2 \mathbf{I}$.