F-Square Geometries for $s = p^m$ Illustrated With $s = 4, 8$, and 9.

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ABSTRACT

Complete sets of pairwise orthogonal F-squares which are orthogonal to a cyclic latin square of order $s = 2^m$ have been obtained by Schwager, Federer, and Mandeli (1984). The feasibility of doing the same thing for $s = p^m$, p a prime number and m an integer greater than zero, is investigated here using factorial and confounding theory. Complete sets of pairwise orthogonal F-squares which are orthogonal to a cyclic latin square of orders 4, 8, and 9 and for a pair of cyclic latin squares of order 9 were constructed using factorial and confounding concepts. The general case is discussed.

1. Introduction

Complete sets of F-squares orthogonal to a cyclic latin square of order $s = 2^m$ were given by Schwager, Federer, and Mandeli (1984). Note that a cyclic latin square of even order has no orthogonal mate but that it is possible to form a complete set of F(s; 2^{m-1}, 2^{m-1})-squares to construct a new F-square geometry. This set is nonisomorphic to the geometry formed from a complete set of orthogonal latin squares of order 2^m.

Herein we show how to construct complete sets of mutually orthogonal F-squares which are orthogonal to a cyclic latin square of orders 4, 8, and 9 and for a pair of orthogonal cyclic latin squares of order 9. We use confounding of factorial effects to construct these sets.

Federer (1992) has given a method for obtaining the aliasing (confounding) structure for any specific fractional replicate from a symmetrical prime power factorial. This method will be used to determine the confounding scheme between factorial effects and treatments in a latin square.

A useful result from linear model theory is that for any set of N orthogonal effects and for any subset of K orthogonal effects, any linear combination of the K orthogonal effects will remain orthogonal to the $N-K$ effects not in the subset. This result will be useful in constructing the complete set of mutually orthogonal F-squares orthogonal to the cyclic latin square or squares. A subset of the factorial effects will be unconfounded with the treatments in the latin square. These may be used to form a set of mutually orthogonal F-squares. Another subset of factorial effects will be partially confounded with treatments in the latin squares. Linear combinations of these effects where they are unconfounded are used to form F-squares and complete the set of mutually orthogonal F-squares. The above linear model result negates the need to check for orthogonality of the F-squares constructed from the effects which are partially confounded with treatments in the cyclic latin squares.

2. Complete Set of F-squares for a Cyclic Latin Square of Order 4

There are $4^2 = 16$ row-column intersections (combinations) in a 4×4 square. Denote these 16 observations as combinations from a 2^4 factorial. Select two effects and their interaction, say A, B, and AB, to represent row effects, and two other effects and their interactions, say C, D, and CD, to represent column effects. Then, write out the 16 resulting combinations and super-impose a cyclic latin square of order four in standard order upon the 4×4 square as follows:

	Columns							
Rows	00		01		10		11	
00	0000	E	0001	F	0010	G	0011	H
01	0100	F	0101	G	0110	H	0111	E
10	1000	G	1001	H	1010	E	1011	F
11	1100	H	1101	E	1110	F	1111	G

where the letters E, F, G, and H represent the treatments in a latin square of order four. The relationship of treatments to factorial effects is as follows:

	Effect								
Treatment	AC	AD	BC	BD	ABC	ABD	ACD	BCD	ABCD
E	u	u	u	0	0	u	0	u	u
F	0	u	u	1	u	u	u	u	1
G	u	u	u	0	1	u	1	u	u
H	1	u	u	1	u	u	u	u	0

From the above, we note that the four F(4; 2, 2) squares formed by effects AD, BC, ABD, and BCD are orthogonal to the original latin square and to each other. The remaining problem is to make two F(4; 2, 2) squares from the remaining partially confounded effects, i.e., AC, BD, ABC, and ABCD.

AC and ABCD are unconfounded with treatments E and G; ABC and ACD are unconfounded with treatments F and H. For the first F(4; 2, 2) square where E and G occur in the latin square, insert 0 and 1 to correspond to $(AC)_0$ and $(AC)_1$ as follows:

E 0		G 1	
	G 0		E 1
G 1		E 0	
	E 1		G 0

Doing the same thing for treatments F and H and ABC (or ACD), we obtain the F(4; 2, 2)-square:

E 0	F 0	G 1	H 1
F 1	G 0	H 0	E 1
G 1	H 1	E 0	F 0
H 0	E 1	F 1	G 0

To form the sixth F(4; 2, 2)-square and complete the orthogonal set, use levels of ABCD for determining the 0 or 1 for the cell, where treatments E and G occur and ACD where treatments F and H occur to obtain the F(4; 2, 2)-square:

E 0	F 1	G 1	H 0
F 0	G 0	H 1	E 1
G 1	H 0	E 0	F 1
H 1	E 1	F 0	G 0

The above six F(4; 2, 2)-squares and the latin square form a complete F-square set or geometry.

The four F(4; 2, 2)-squares formed by AD, BC, ABD, and BCD are:

AD

0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

BC

0	0	1	1
1	1	0	0
0	0	1	1
1	1	0	0

ABD

0	1	0	1
1	0	1	0
1	0	1	0
0	1	0	1

BCD

0	1	1	0
1	0	0	1
0	1	1	0
1	0	0	1

To obtain confounding relationships between levels of effects and treatments, such a program as GAUSS does much to alleviate the tedium of this operation (see Federer, 1992). For the example in this section, there is little need for a computer program but for the larger examples in the following sections, GAUSS has been invaluable. The procedure follows. Construct a matrix of superscripts (exponents) of effects not confounded with rows and columns which for this example is:

$$
S4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\qquad
\begin{array}{c} \underline{\text{Exponent of factor}} \\ \text{AC} \\ \text{AD} \\ \text{BC} \\ \text{BD} \\ \text{ABC} \\ \text{ABD} \\ \text{ACD} \\ \text{BCD} \\ \text{ABCD} \end{array}
$$

The matrix corresponding to the combinations where treatment E occurs is

$$
YE = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},
$$

where the columns of the above matrix are the factorial combinations where treatment E occurs. Then form the product, modulo 2, of S4 and YE with the GAUSS command "S4*YE % 2;", which results in

S4*YE				Effect	Confounding	
0	1	0	1	AC	2:2 = u	unconfounded
0	1	1	0	AD	2:2 = u	unconfounded
0	0	1	1	BC	2:2 = u	unconfounded
0	0	0	0	BD	0 = 4:0	confounded
0	0	0	0	ABC	0 = 4:0	confounded
0	0	1	1	ABD	2:2 = u	unconfounded
0	0	0	0	ACD	0 = 4:0	confounded
0	1	1	0	BCD	2:2 = u	unconfounded
0	1	0	1	ABCD	2:2 = u	unconfounded

3. Complete Set of F-squares for a Cyclic Latin Square of Order 8

There are $8^2 = (2^3)^2 = 64$ row-column combinations in an 8×8 square. Denote these 64 combinations as coming from a 2^6 factorial with 64 effects. Select three effects, say A, B, and C and their interactions to represent row effects and three other effects, say D, E, and F, and their interactions, to represent column effects. Superimposing a cyclic latin square of order eight in standard form upon the 8×8 square, we obtain:

	Columns							
Rows	000	001	010	011	100	101	110	111
000	000000 1	000001 2	000010 3	000011 4	000100 5	000101 6	000110 7	000111 8
001	001000 2	001001 3	001010 4	001011 5	001100 6	001101 7	001110 8	001111 1
010	010000 3	010001 4	010010 5	010011 6	010100 7	010101 8	010110 1	010111 2
011	011000 4	011001 5	011010 6	011011 7	011100 8	011101 1	011110 2	011111 3
100	100000 5	100001 6	100010 7	100011 8	100100 1	100101 2	100110 3	100111 4
101	101000 6	101001 7	101010 8	101011 1	101100 2	101101 3	101110 4	101111 5
110	110000 7	110001 8	110010 1	110011 2	110100 3	110101 4	110110 5	110111 6
111	111000 8	111001 1	111010 2	111011 3	111100 4	111101 5	111110 6	111111 7

where the numbers 1, 2, $\cdots$, 8 refer to the treatments in the cyclic latin square of order eight. The relationships of the treatments to the remaining $63 - 14 = 49$ factorial effects are as follows:

Factorial Effect	Treatment 1	2	3	4	5	6	7	8	Unconfounded
AD	2:6	u	6:2	0	6:2	u	2:6	1	
AE	u	u	u	u	u	u	u	u	X
AF	u	u	u	u	u	u	u	u	X
BD	u	u	u	u	u	u	u	u	X
BE	u	0	u	1	u	0	u	1	
BF	u	u	u	u	u	u	u	u	X
CD	u	u	u	u	u	u	u	u	X
CE	u	u	u	u	u	u	u	u	X
CF	0	1	0	1	0	1	0	1	
ABD	6:2	0	6:2	u	2:6	1	2:6	u	
ABE	u	u	u	u	u	u	u	u	X
ABF	u	u	u	u	u	u	u	u	X
ACD	6:2	u	6:2	u	2:6	u	2:6	u	
ACE	u	u	u	u	u	u	u	u	X
ACE	u	u	u	u	u	u	u	u	X
BCD	u	u	u	u	u	u	u	u	X
BCE	0	u	1	u	0	u	1	u	
BCF	u	u	u	u	u	u	u	u	X
ABCD	6:2	u	2:6	u	2:6	u	6:2	u	
ABCE	u	u	u	u	u	u	u	u	X
ABCF	u	u	u	u	u	u	u	u	X
ADE	6:2	0	6:2	u	2:6	1	2:6	u	
ADF	6:2	u	6:2	u	2:6	u	2:6	u	
AEF	u	u	u	u	u	u	u	u	X
BDE	u	u	u	u	u	u	u	u	X
BDF	u	u	u	u	u	u	u	u	X
BEF	0	u	1	u	0	u	1	u	
CDE	u	u	u	u	u	u	u	u	X
CDF	u	u	u	u	u	u	u	u	X
CEF	u	u	u	u	u	u	u	u	X
ADEF	6:2	u	2:6	u	2:6	u	6:2	u	
BDEF	u	u	u	u	u	u	u	u	X
CDEF	u	u	u	u	u	u	u	u	X
ABDE	6:2	u	2:6	1	2:6	u	6:2	0	
ABDF	6:2	u	2:6	u	2:6	u	6:2	u	
ABEF	u	u	u	u	u	u	u	u	X
ACDE	6:2	u	2:6	u	2:6	u	6:2	u	
ACDF	2:6	u	6:2	1	6:2	u	2:6	0	
ACEF	u	u	u	u	u	u	u	u	X
BCDE	u	u	u	u	u	u	u	u	X
BCDF	u	u	u	u	u	u	u	u	X
BCEF	u	1	u	0	u	1	u	0	
ABCDE	2:6	u	2:6	u	6:2	u	6:2	u	
ABCDF	6:2	1	6:2	u	2:6	0	2:6	u	
ABCEF	u	u	u	u	u	u	u	u	X
ABDEF	2:6	u	2:6	u	6:2	u	6:2	u	
ACDEF	6:2	1	6:2	u	2:6	0	2:6	u	
BCDEF	u	u	u	u	u	u	u	u	X
ABCDEF	6:2	u	2:6	0	2:6	u	6:2	1	

Out of the 49 effects listed above, there are 28 effects which are completely unconfounded (orthogonal) to the eight treatments in the cyclic latin squares. These are marked with an X in the last column of the above and may be used to construct 28 pairwise orthogonal F(8; 4, 4)-squares. The remaining problem is to construct 14 F(8; 4, 4)-squares to obtain a complete set of 42 orthogonal F-squares. There are 21 effects and hence 21 single degrees of freedom. Seven of these 21 are used up for the treatments in the latin square. One effect, CF, is completely confounded with the latin square treatments); the other 20 effects are partially confounded. As for the 4×4 latin square, these 20 effects where they are unconfounded with latin square treatments may be used to construct the remaining 14 F(8; 4, 4)-squares.

Grouping the 20 effects that are partially confounded with some treatments in the latin square, we obtain:

Effect	Treatment 1	2	3	4	5	6	7	8	F-square Even nos.
BE	u	0	u	1	u	0	u	1	F_{29}
BCE	0	u	1	u	0	u	1	u	
BEF	0	u	1	u	0	u	1	u	F_{30}
BCEF	u	1	u	0	u	1	u	0	
AD	2:6	u	6:2	0	6:2	u	2:6	1	
ABDE	6:2	u	2:6	1	2:6	u	6:2	0	
ACDF	2:6	u	6:2	1	6:2	u	2:6	0	
ABCDEF	6:2	u	2:6	0	2:6	u	6:2	1	F_{31-4}
ABD	6:2	0	6:2	u	2:6	1	2:6	u	
ADE	6:2	0	6:2	u	2:6	1	2:6	u	
ABCDF	6:2	1	6:2	u	2:6	0	2:6	u	
ACDEF	6:2	1	6:2	u	2:6	0	2:6	u	
ACD	6:2	u	6:2	u	2:6	u	2:6	u	F_{35}
ADF	6:2	u	6:2	u	2:6	u	2:6	u	F_{36}
ABCD	6:2	u	2:6	u	2:6	u	6:2	u	F_{37}
ADEF	6:2	u	2:6	u	2:6	u	6:2	u	F_{38}
ABDF	6:2	u	2:6	u	2:6	u	6:2	u	F_{39}
ACDE	6:2	u	2:6	u	2:6	u	6:2	u	F_{40}
ABCDE	2:6	u	2:6	u	6:2	u	6:2	u	F_{41}
ABDEF	2:6	u	2:6	u	6:2	u	6:2	u	F_{42}

Two F(8; 4, 4)-squares, numbers 29 and 30, may be completed using one scheme for even-numbered treatments and another for odd. For the remaining F-squares and the even-numbered treatments, use levels of the following effects to obtain the zeros and ones in the 14 F(8; 4, 4)-squares:

F_{29}: BE for odd treatments and BCE for even.
F_{30}: BCEF for odd treatments and BEF for even.
F_{31}: Levels of AD for 2 and 6 and levels of ABD for 4 and 8.
F_{32}: Levels of ABDE for 2 and 6 and levels of ADE for 4 and 8.
F_{33}: Levels of ACDF for 2 and 6 and levels of ABCDF for 4 and 8.
F_{34}: Levels of ABCDEF for 2 and 6 and levels of ACDEF for 4 and 8.
F_{35}: Levels of ACD for 2, 4, 6, and 8.
F_{36}: Levels of ADF for 2, 4, 6, and 8.
F_{37}: Levels of ABCD for 2, 4, 6, and 8.
F_{38}: Levels of ADEF for 2, 4, 6, and 8.
F_{39}: Levels of ABDF for 2, 4, 6, and 8.
F_{40}: Levels of ACDE for 2, 4, 6, and 8.
F_{41}: Levels of ABCDE for 2, 4, 6, and 8.
F_{42}: Levels of ABDEF for 2, 4, 6, and 8.

The above effects are unconfounded with the even-numbered latin square treatments as indicated above.

To complete the F-squares for the odd-numbered treatments, we proceed as follows. For a pair of effects such as AD and ABDE, note that treatments 1 and 7 are confounded with levels $(AD)_1$ and $(ABDE)_0$ and vice versa for 3 and 5. Therefore, these effects are unconfounded at the other level of the effect. For treatment number 1, the following levels are:

	00000	001111	010110	011101	100100	101011	110010	111001
AD	0X	1X	1	1	0X	1X	1	1
ABDE	0	0	1X	0X	0	0	1X	0X

The zero or one marked X indicates the level selected. Note that for the four ones for AD, there are two zeros and two ones of ABDE marked X and likewise for the zeros of ABDE. Proceeding in the same manner for the other three odd-numbered treatments, we obtain F_{31} as:

0	0	1	0	1	1	0	1
0	0	0	0	1	1	1	1
1	1	1	0	0	0	0	1
1	0	0	1	0	1	1	0
1	1	0	1	0	0	1	0
1	1	1	1	0	0	0	0
0	0	0	1	1	1	1	0
0	1	1	0	1	0	0	1

Proceeding in this same fashion, F-squares 32 to 42 may be completed using other pairs of effects in such a manner as to use all effects. The pairs that were used are:

F_{32}:	ACDF	and	ABCDEF
F_{33}:	ABD	and	ABCDF
F_{34}:	ADE	and	ACDEF
F_{35}:	ACD	and	ABCDE
F_{36}:	ADF	and	AD
F_{37}:	ABCD	and	ACDF
F_{38}:	ADEF	and	ABCDE
F_{39}:	ABDF	and	ABDEF
F_{40}:	ACDE	and	AD
F_{41}:	ABCDE	and	ABDE
F_{42}:	ABDEF	and	ABCDEF

Note the pairing is not unique and that many sets of 42 F-squares are possible by selecting various combinations for F-squares 29 to 42.

Note that the method of construction used in this and in the previous section uses factorial and confounding theory. This method differs from the one used by Schwager, Federer, and Mandeli (1984).

4. Complete Sets of F-squares For a Cyclic Latin Square of Order 9

There are $9^2 = 81$ row-column intersections or combinations for a 9×9 square. Denote these 81 combinations as coming from a 3^4 factorial. Select two effects, say A and B, and their interactions, AB and AB^2, to represent row effects. Select two different effects, say C and D, and their interactions, CD and CD^2, to represent column effects. Superimposing a cyclic latin square of order 9 in standard order upon the square, we obtain the following (the numbers 1, 2, $\cdots$, 9 represent the treatments in the latin square):

	00	01	02	10	11	12	20	21	22
00	0000 1	0001 2	0002 3	0010 4	0011 5	0012 6	0020 7	0021 8	0022 9
01	0100 2	0101 3	0102 4	0110 5	0111 6	0112 7	0120 8	0121 9	0122 1
02	0200 3	0201 4	0202 5	0210 6	0211 7	0212 8	0220 9	0221 1	0222 2
10	1000 4	1001 5	1002 6	1010 7	1011 8	1012 9	1020 1	1021 2	1022 3
11	1100 5	1101 6	1102 7	1110 8	1111 9	1112 1	1120 2	1121 3	1122 4
12	1200 6	1201 7	1202 8	1210 9	1211 1	1212 2	1220 3	1221 4	1222 5
20	2000 7	2001 8	2002 9	2010 1	2011 2	2012 3	2020 4	2021 5	2022 6
21	2100 8	2101 9	2102 1	2110 2	2111 3	2112 4	2120 5	2121 6	2122 7
22	2200 9	2201 1	2202 2	2210 3	2211 4	2212 5	2220 6	2221 7	2222 8

There are 40 effects plus the mean in a 3^4 factorial. The confounding relationships between the 32 factorial effects not confounded with rows and columns in the latin square are given below, where u means unconfounded.

Effect	Treatment 1	2	3	4	5	6	7	8	9
AC	3:0:6	6:0:3	0	6:3:0	3:6:0	1	0:6:3	0:3:6	2
AC^2	u	u	u	u	u	u	u	u	u
AD	u	u	u	u	u	u	u	u	u
AD^2	u	u	u	u	u	u	u	u	u
BC	u	u	u	u	u	u	u	u	u
BC^2	u	u	u	u	u	u	u	u	u
BD	0	1	2	0	1	2	0	1	2
BD^2	u	u	u	u	u	u	u	u	u
ACD	6:3:0	3:6:0	u	0:6:3	0:3:6	u	3:0:6	6:0:3	u
ACD^2	6:3:0	6:0:3	u	0:6:3	3:6:0	u	3:0:6	0:3:6	u
AC^2D	u	u	u	u	u	u	u	u	u
AC^2D^2	u	u	u	u	u	u	u	u	u
BCD	u	u	u	u	u	u	u	u	u
BCD^2	u	u	u	u	u	u	u	u	u
BC^2D	u	u	u	u	u	u	u	u	u
BC^2D^2	u	u	u	u	u	u	u	u	u
ABC	6:3:0	3:6:0	u	0:6:3	0:3:6	u	3:0:6	6:0:3	u
ABC^2	u	u	u	u	u	u	u	u	u
AB^2C	6:3:0	6:0:3	u	0:6:3	3:6:0	u	3:0:6	0:3:6	u
AB^2C^2	u	u	u	u	u	u	u	u	u
ABD	u	u	u	u	u	u	u	u	u
ABD^2	u	u	u	u	u	u	u	u	u
AB^2D	u	u	u	u	u	u	u	u	u
AB^2D^2	u	u	u	u	u	u	u	u	u
ABCD	3:0:6	3:6:0	2	6:3:0	0:3:6	0	0:6:3	6:0:3	1
$ABCD^2$	6:3:0	0:3:6	u	0:6:3	6:0:3	u	3:0:6	3:6:0	u
ABC^2D	u	u	u	u	u	u	u	u	u
ABC^2D^2	u	u	u	u	u	u	u	u	u
AB^2CD	6:3:0	0:3:6	u	0:6:3	6:0:3	u	3:0:6	3:6:0	u
AB^2CD^2	3:0:6	0:3:6	1	6:3:0	6:0:3	2	0:6:3	3:6:0	0
AB^2C^2D	u	u	u	u	u	u	u	u	u
$AB^2C^2D^2$	u	u	u	u	u	u	u	u	u

In the above, 3:0:6 for a treatment means that level zero occurred three times with that treatment, level one did not occur, and level two occurred six times. In order to be a u the ratio needed to be 3:3:3, i.e., each level occurred with equal frequency with a treatment meaning that the effect was orthogonal, i.e., unconfounded with the treatment. When only one number occurs in the above ratio, this means there were nine occurrence of a given level. For the number 2, say, the ratio would be 0:0:9, meaning the treatment was completely confounded with that level of the factorial effect. There are 22 effects which are unconfounded with latin square treatments. These may be used to construct 22 F(9; 3, 3, 3)-squares which are pairwise orthogonal and orthogonal to the latin square.

There is one effect, BD, which is completely confounded with latin square treatments. The remaining nine effects, AC, ACD, ACD^2, ABC, AB^2C, ABCD, $ABCD^2$, AB^2CD, and AB^2CD^2, are partially confounded with treatments. Levels of these effects, where they are unconfounded, may be used to construct the remaining six F(9; 3, 3, 3)-squares to complete the set. Note that for six of the above nine effects, treatments 3, 6, and 9 are unconfounded with effects ACD, ACD^2, ABC, AB^2C, $ABCD^2$, and AB^2CD. Where these treatments occur in the square, replace the treatment with levels of the effect to obtain a part of the six F-squares F_{23} to F_{28}. For example, F_{23} for effect ACD is:

		2			0			1
	1			2			0	
0			1			2		
		0			1			2
	2			0			1	
1			2			0		
		1			2			0
	0			1			2	
2			0			1		

F_{24} to F_{28} may be started as above with effects ACD^2, ABC, AB^2C, $ABCD^2$, and AB^2CD. Treatment number 1 is associated with the following nine combinations and levels of effects AC and ACD:

	0000	0122	0221	1020	1112	1211	2010	2102	2201
AC	0	2	2	0	2	2	0	2	2
ACD	0	1	0	0	1	0	0	1	0

Where 0000, 1020, and 2010 occur insert a zero from AC in F_{23}; where 0122, 1112, and 2102 occur insert a one from ACD; and where 0221, 1211, and 2201 occur insert a two from AC. Proceed in a similar manner for treatments 2, 4, 5, 7, and 8 to complete F_{23}. Following a similar process and using all the partially confounded effects, F_{24} to F_{28} may be obtained to complete the set. Again note that the set is not unique in that there are several options for selecting effects.

The S9 matrix used in GAUSS to obtain the above confounding relations, together with the corresponding factorial effect, is (see Federer, 1992):

$$S9_{32\times 4} = \begin{bmatrix} 1&0&1&0\\ 1&0&2&0\\ 1&0&0&1\\ 1&0&0&2\\ 0&1&1&0\\ 0&1&2&0\\ 0&1&0&1\\ 0&1&0&2\\ 1&0&1&1\\ 1&0&1&2\\ 1&0&2&1\\ 1&0&2&2\\ 0&1&1&1\\ 0&1&1&2\\ 0&1&2&1\\ 0&1&2&2\\ 1&1&1&0\\ 1&1&2&0\\ 1&2&1&0\\ 1&2&2&0\\ 1&1&0&1\\ 1&1&0&2\\ 1&2&0&1\\ 1&2&0&2\\ 1&1&1&1\\ 1&1&1&2\\ 1&1&2&1\\ 1&1&2&2\\ 1&2&1&1\\ 1&2&1&2\\ 1&2&2&1\\ 1&2&2&2 \end{bmatrix} \qquad \begin{array}{c} \underline{\text{Factorial Effect}} \\ AC\\ AC^2\\ AD\\ AD^2\\ BC\\ BC^2\\ BD\\ BD^2\\ ACD\\ ACD^2\\ AC^2D\\ AC^2D^2\\ BCD\\ BCD^2\\ BC^2D\\ BC^2D^2\\ ABC\\ ABC^2\\ AB^2C\\ AB^2C^2\\ ABD\\ ABD^2\\ AB^2D\\ AB^2D^2\\ ABCD\\ ABCD^2\\ ABC^2D\\ ABC^2D^2\\ AB^2CD\\ AB^2CD^2\\ AB^2C^2D\\ AB^2C^2D^2 \end{array}$$

The nine combinations (columns) for treatment one from the latin square forms the following Y1 matrix:

$$Y1_{4\times 9} = \begin{bmatrix} 0&0&0&1&1&1&2&2&2\\ 0&1&2&0&1&2&0&1&2\\ 0&2&2&2&1&1&1&0&0\\ 0&2&1&0&2&1&0&2&1 \end{bmatrix}.$$

Then $S9 * Y1$, modulo 3, produces the confounding relationships of levels of effects with treatment one. The first rows of the product of these matrices are:

$$
S9*Y1 = \begin{bmatrix}
0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 \\
0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 2 \\
0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 \\
0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & &
\end{bmatrix}
\qquad
\begin{array}{c}
\underline{\text{Confounding}} \\
3:0:6 \\
u = 3:3:3 \\
u = 3:3:3 \\
u = 3:3:3 \\
u = 3:3:3 \\
u = 3:3:3 \\
0 = 9:0:0
\end{array}
$$

5. Complete Sets of F-squares for a Pair of Orthogonal Cyclic Latin Squares of Order 9

A cyclic latin square of order nine which is orthogonal to the one in the previous section is:

	00	01	02	10	11	12	20	21	22
00	0000 1	0001 2	0002 3	0010 4	0011 5	0012 6	0020 7	0021 8	0022 9
01	0100 3	0101 4	0102 5	0110 6	0111 7	0112 8	0120 9	0121 1	0122 2
02	0200 5	0201 6	0202 7	0210 8	0211 9	0212 1	0220 2	0221 3	0222 4
10	1000 7	1001 8	1002 9	1010 1	1011 2	1012 3	1020 4	1021 5	1022 6
11	1100 9	1101 1	1102 2	1110 3	1111 4	1112 5	1120 6	1121 7	1122 8
12	1200 2	1201 3	1202 4	1210 5	1211 6	1212 7	1220 8	1221 9	1222 1
20	2000 4	2001 5	2002 6	2010 7	2011 8	2012 9	2020 1	2021 2	2022 3
21	2100 6	2101 7	2102 8	2110 9	2111 1	2112 2	2120 3	2121 4	2122 5
22	2200 8	2201 9	2202 1	2210 2	2211 3	2212 4	2220 5	2221 6	2222 7

The confounding relationships of the 32 effects unconfounded with rows and column to treatments is:

Effect	Treatment 1	2	3	4	5	6	7	8	9
AC	u	u	u	u	u	u	u	u	u
AC^2	u	3:6:0	6:3:0	u	6:0:3	3:0:6	u	0:3:6	0:6:3
AD	u	u	u	u	u	u	u	u	u
AD^2	u	u	u	u	u	u	u	u	u
BC	u	u	u	u	u	u	u	u	u
BC^2	u	u	u	u	u	u	u	u	u
BD	u	u	u	u	u	u	u	u	u
BD^2	0	2	1	0	2	1	0	2	1
ACD	u	u	u	u	u	u	u	u	u
ACD^2	u	u	u	u	u	u	u	u	u
AC^2D	u	3:6:0	3:0:6	u	6:0:3	0:6:3	u	0:3:6	6:3:0
AC^2D^2	0	0:3:6	6:3:0	2	3:6:0	3:0:6	1	6:0:3	0:6:3
BCD	u	u	u	u	u	u	u	u	u
BCD^2	u	u	u	u	u	u	u	u	u
BC^2D	u	u	u	u	u	u	u	u	u
BC^2D^2	u	u	u	u	u	u	u	u	u
ABC	u	u	u	u	u	u	u	u	u
ABC^2	u	6:0:3	6:3:0	u	0:3:6	3:0:6	u	3:6:0	0:6:3
AB^2C	u	u	u	u	u	u	u	u	u
AB^2C^2	0	6:0:3	3:0:6	2	0:3:6	0:6:3	1	3:6:0	6:3:0
ABD	u	u	u	u	u	u	u	u	u
ABD^2	u	u	u	u	u	u	u	u	u
AB^2D	u	u	u	u	u	u	u	u	u
AB^2D^2	u	u	u	u	u	u	u	u	u
ABCD	u	u	u	u	u	u	u	u	u
$ABCD^2$	u	u	u	u	u	u	u	u	u
ABC^2D	0	3:6:0	0:6:3	2	6:0:3	6:3:0	1	0:3:6	3:0:6
ABC^2D^2	u	6:0:3	0:6:3	u	0:3:6	6:3:0	u	3:6:0	3:0:6
AB^2CD	u	u	u	u	u	u	u	u	u
AB^2CD^2	u	u	u	u	u	u	u	u	u
AB^2C^2D	u	0:3:6	3:0:6	u	3:6:0	0:6:3	u	6:0:3	6:3:0
$AB^2C^2D^2$	u	0:3:6	0:6:3	u	3:6:0	6:3:0	u	6:0:3	3:0:6

As with the square in the previous section, nine of the 32 effects are partially confounded with treatments and one effect, BD^2, is completely confounded. Treatments 1, 4, and 7 will take the role of treatments 3, 6, and 9 from the previous section to start forming the six F(9; 3, 3, 3)-squares to complete the orthogonal set.

There are 12 effects which are unconfounded with the treatments in either latin square. These are AD, AD^2, BC, BC^2, BCD, BCD^2, BC^2D, BC^2D^2, ABD, ABD^2, AB^2D, and AB^2D^2. These 12 effects may be used to form 12 F(9; 3, 3, 3)-squares which are mutually orthogonal to both cyclic latin squares. These 12 plus the two sets of six F(9; 3, 3, 3)-squares (F_{23} to F_{28}) formed above for each latin square, forms the complete set of orthogonal F-squares.

6. Discussion and Generalizations

The procedure discussed above may be generalized as follows. For $s = p^m$, p a prime number, and m an integer > 0, form an $s \times s$ square with row-column interactions corresponding to the combinations of a p^{2m} factorial. Assign $p^m / (p-1)$ effects (m effects and their interactions) to rows and another set to columns. There will be a total of $(p^{2m}-1)/(p-1)$ effects plus the mean in the $s^2 = p^{2m}$ combinations of the $s \times s$ square. Superimpose a cyclic latin square of order s upon the $s \times s$ square. There will be $(p^{2m}-1)/(p-1) - 2(p^m-1)/(p-1) = K$ effects associated with the row$\times$column interaction. For these K effects, form the matrix $S_{K \times 2m}$ corresponding to the superscripts of the 2m factors in an effect. Form the matrix $Y(i)_{2m \times s}$ from the combinations where treatment i, $i = 1, 2, \cdots, s$, occurs in the cyclic latin square and for the columns of Y(i), form the combinations of the factorial where treatment i occurs. Then, form the matrices $S * Y(i)$ for each i to determine the confounding scheme of effects and treatments in the latin square. From the set of effects unconfounded with any treatment, construct $F(s; p^{m-1}, p^{m-1}, \cdots, p^{m-1})$-squares, where there are p^{m-1} occurrences of a symbol in the F-square with p symbols. From the effects partially confounded with treatments, form the remaining F-squares to complete the set.

For each of the $p - 2$ cyclic latin squares of order s which are orthogonal to the first square above, repeat the process described in the preceding paragraph. The sets of F-squares formed from the partially confounded effects for each latin square, plus the remaining set which is unconfounded with treatments in any of the orthogonal latin squares, form the complete set of F-squares. These plus the set of orthogonal latin squares form an F-square geometry which is nonisomorphic to latin square projective geometries.

As pointed out in the examples, there are many sets of F-squares that could be constructed from the partially confounded set of effects. However, all of these may be permuted in such a manner to obtain the other set. The different sets merely represent different linear combinations of the same set of effects. Thus, these sets are all isomorphic and do not result in new F-square geometries.

These results lead to the following conjectured theorem:

Theorem: *For* $s = p^m$, p *a prime number and* m *an integer* > 0, *an F-square geometry may be formed from* $p - 1$, *mutually orthogonal cyclic latin squares of order* s *and the* $K - p^m + 1$ *F-squares with* p *symbols each repeated* p^{m-1} *times.*

Proof: Since the complete set of F-squares may be formed for any specified latin square of order s, the theorem must be true. □

An unresolved problem is the determination of a general formula for the number of effects which are unconfounded, the number of effects which are completely confounded, and the number of effects which are partially confounded with the treatments in the latin square. For the examples, they were:

		Confounding		
s	K	no	complete	partial
4	9	4	1	4
8	49	28	1	20
9	32	22	1	9

Note that a computer search could be used to complete the set of orthogonal F-squares rather than using the factorial confounding approach described above. The F-squares could be partially formed as for treatments 3, 6, and 9 for the first latin square of Section 4 and treatments 1, 4, and 7 for the F-squares in Section 6. Then, the F-squares may be completed using a computer search procedure as was done by Schwager, Federer, and Mandeli (1984) to obtain specific sets.

7. Literature Cited

Federer, W. T. (1992). Aliasing contrasts for a given fractional replicate. BU-1160-M in the Technical Report Series of the Biometrics Unit, Cornell University, Ithaca, NY 14853.

Schwager, S. J., W. T. Federer, and J. P. Mandeli (1984). Embedding cyclic Latin squares of order 2^n in a complete set of orthogonal F-squares. *J. of Statistical Planning and Inference* **10**: 207-218.