DESIGNS WITH ONE-WAY ELEMINATION OF HETEROGENEITY

BU-117-M W. T. Federer, Cornell University* May, 1960

The generalized analysis of variance is developed below for all designs with one-way elemination of heterogeneity. Such designs as the randomized complete block and incomplete block designs which are resolvable (the v treatments occur together in a complete block) and which are non-resolvable (v treatments occur in b incomplete blocks of size k, for $k < v$), are considered.

To be completely general, let the ith one of the v treatments be replicated r_i times in the b incomplete blocks of size $n_{\cdot jh}$. Let the yield of the ijhth observation be expressed by

$$Y_{ijh}=n_{ijh}(\mu+\tau_i+\rho_j+\beta_{jh}+\epsilon_{ijh}) \quad , \tag{1}$$

where $i=1,2,\cdots,v$ = number of treatments; $j=1,2,\cdots,r$ = number of complete blocks; $h=1,2,\cdots,k_j$ = number of incomplete blocks in the jth complete block; n_{ijh} = 1 if ith treatment occurs in the hth incomplete block of the jth complete block and zero otherwise; $n_{\cdot jh}$ = number of treatments in hth incomplete block of the jth complete block; $n_{\cdot j\cdot} = v_j$ = number of treatments in the jth complete block; $n_{i\cdot\cdot} = r_i$ = number of replicates of the ith treatment; $n_{\cdot\cdot\cdot}=\sum_{i=1}^{v} r_i=\sum_{j=1}^{r} v_j=\sum_{i=1}^{v}\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{ijh}$; μ = a general mean effect $= \frac{1}{v}\sum_{i=1}^{v}\mu_{i\cdot\cdot} = \frac{1}{r}\Sigma\mu_{\cdot j\cdot}$ $= \frac{1}{n_{\cdot\cdot\cdot}}\sum\sum\sum_{ijh} n_{ijh}\mu_{ijh} = \frac{1}{b=\Sigma k_j}\sum_{y=1}^{r}\sum_{h=1}^{k_j}\mu_{\cdot jh}$ where k_j = number of incomplete blocks in the jth complete block; $\tau_i=\mu_{i\cdot\cdot}-\mu$ = a treatment effect; $\rho_j = \mu_{\cdot j\cdot} - \mu$ = a complete block effect; $\beta_{jh} = \mu_{\cdot jh} - \mu_{\cdot j\cdot}$ = an incomplete block effect; and ϵ_{ijh} are random independent effects with mean zero and common variance σ_ϵ^2. These definitions imply $\sum_{i=1}^{v}\tau_i=\sum_{j=1}^{r}\rho_j=\sum_{h=1}^{k_j}\beta_{jh} = 0$. Other definitions for the effects are permissible.

Intrablock Analysis

The least squares estimates of effects for the above linear model (intrablock analysis) are obtained by minimizing the residual sum of squares:

*Biometrics Unit, Plant Breeding Department, Cornell University

$$\sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{ijh}\epsilon^2_{ijh} = \sum\sum\sum_{ijh} n_{ijh}(Y_{ijh}-\mu-\tau_i-\rho_j-\beta_{jh})^2 \quad .$$

Equating to zero each of the partial derivatives of the above residual sum of squares with respect to μ, τ_i, ρ_j, and β_{jh} results in the following normal equations:

$$\mu \;:\; n_{\cdot\cdot\cdot}\hat{\mu} + \sum_{i=1}^{v} \hat{\tau}_i n_{i\cdot\cdot} + \sum_{j=1}^{r} \hat{\rho}_j n_{\cdot j\cdot} + \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{\cdot jh}\hat{\beta}_{jh} = Y_{\cdot\cdot\cdot} = \text{grand total}$$

$$\tau_i \;:\; n_{i\cdot\cdot}(\hat{\mu}+\hat{\tau}_i) + \sum_{j=1}^{r} n_{ij\cdot}\hat{\rho}_j + \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{ijh}\hat{\beta}_{jh} = Y_{i\cdot\cdot} = i\text{th treatment total}$$

$$\rho_j \;:\; n_{\cdot j\cdot}(\hat{\mu}+\hat{\rho}_j) + \sum_{i=1}^{v} n_{ij\cdot}\hat{\tau}_i + \sum_{h=1}^{k_j} n_{\cdot jh}\hat{\beta}_{jh} = Y_{\cdot j\cdot} = j\text{th complete block total}$$

$$\beta_{jh} \;:\; n_{\cdot jh}(\hat{\mu}+\hat{\rho}_j+\hat{\beta}_{jh}) + \sum_i n_{ijh}\hat{\tau}_i = Y_{\cdot jh} = j\text{th incomplete block total}$$

In the τ_g equation, substitute for $\hat{\mu}+\hat{\rho}_j+\hat{\beta}_{jh}$ from the β_{jh} equations to obtain:

$$n_{g\cdot\cdot}\hat{\tau}_g + \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{gjh}(\hat{\mu}+\hat{\rho}_j+\hat{\beta}_{jh})$$

$$= n_{g\cdot\cdot}\hat{\tau}_g - \sum\sum_{jh} \frac{n_{gjh}}{n_{\cdot jh}}\left(\sum_{i=1}^{v} n_{ijh}\hat{\tau}_i - Y_{\cdot jh}\right) = Y_{g\cdot\cdot}$$

Rewriting we obtain:

$$n_{g\cdot\cdot}\hat{\tau}_g - \sum\sum_{jh} \frac{n_{gjh}}{n_{\cdot jh}} \sum_i n_{ijh}\hat{\tau}_i = Y_{g\cdot\cdot} - \sum\sum_{jh} n_{gjh}\bar{y}_{\cdot jh} = Q_{g\cdot\cdot} \qquad (2)$$

when $n_{g\cdot\cdot}=r$ and $n_{\cdot jh}=k$ the above equation becomes:

$$r\hat{\tau}_g - \frac{1}{k}\sum_i \hat{\tau}_i \sum\sum_{jh} n_{gjh}n_{ijh} = Q_{g\cdot\cdot}$$

where $\sum\sum_{jh} n_{gjh}n_{ijh} = \lambda_{gi}$ = number of times the gth treatment occurs with the ith treatment in all incomplete blocks. For balanced lattices $\lambda_{gi}=\lambda$ a constant for $i \neq g$ and the solution for $\hat{\tau}_g$ is (for $\Sigma\hat{\tau}_i$=zero):

$$\hat{\tau}_g = kQ_{g\cdot\cdot}/(kr-r+\lambda) = kQ_{g\cdot\cdot}(v-1)/rv(k-1)$$

as given by formula (XIII-12) in Federer where $Q_{\cdot j/k}=Q_{g\cdot\cdot}$. For the general case we can add $d_g\Sigma\hat{\tau}_i=0$, where $d_g\neq 0$ for at least one value of g, to each of the v equations in the $\hat{\tau}_i$; adding $d_g\Sigma\hat{\tau}_i$ to each of the equations in (2) results in:

$$n_{g\cdot\cdot}\hat{\tau}_g-\sum_{jh}\frac{n_{gjh}}{n_{\cdot jh}}\sum_i\hat{\tau}_i(n_{ijh}-d_g)=Q_{g\cdot\cdot}$$

We have v equations and v unknowns and the problem is to solve the equations. In matrix notation the v+1 equations from (2) plus the equation $\Sigma\hat{\tau}_i=0$ is:

$$\begin{bmatrix} n_{1\cdot\cdot}-\Sigma\Sigma\frac{n^2_{1jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{1jh}n_{2jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{1jh}n_{3jh}}{n_{\cdot jh}} & \cdots & -\Sigma\Sigma\frac{n_{1jh}n_{vjh}}{n_{\cdot jh}} & 1 \\ -\Sigma\Sigma\frac{n_{2jh}n_{1jh}}{n_{\cdot jh}} & n_{2\cdot\cdot}-\Sigma\Sigma\frac{n^2_{2jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{2jh}n_{3jh}}{n_{\cdot jh}} & \cdots & -\Sigma\Sigma\frac{n_{2jh}n_{vjh}}{n_{\cdot jh}} & 1 \\ -\Sigma\Sigma\frac{n_{3jh}n_{1jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{3jh}n_{2jh}}{n_{\cdot jh}} & n_{3\cdot\cdot}-\Sigma\Sigma\frac{n^2_{3jh}}{n_{\cdot jh}} & \cdots & -\Sigma\Sigma\frac{n_{3jh}n_{vjh}}{n_{\cdot jh}} & 1 \\ \vdots & & & & & \\ -\Sigma\Sigma\frac{n_{vjh}n_{1jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{vjh}n_{2jh}}{n_{\cdot jh}} & -\Sigma\Sigma\frac{n_{vjh}n_{3jh}}{n_{\cdot jh}} & \cdots & n_{v\cdot\cdot}-\Sigma\Sigma\frac{n^2_{vjh}}{n_{\cdot jh}} & 1 \\ 1 & 1 & 1 & & 1 & 0 \end{bmatrix}$$

$$\cdot\begin{bmatrix}\hat{\tau}_1\\ \hat{\tau}_2\\ \hat{\tau}_3\\ \\ \hat{\tau}_v\\ 0\end{bmatrix}=\begin{bmatrix}Q_{1\cdot\cdot}=Y_{1\cdot\cdot}-\Sigma\Sigma n_{1jh}\bar{y}_{\cdot jh}\\ Q_{2\cdot\cdot}=Y_{2\cdot\cdot}-\Sigma\Sigma n_{2jh}\bar{y}_{\cdot jh}\\ Q_{3\cdot\cdot}=Y_{3\cdot\cdot}-\Sigma\Sigma n_{3jh}\bar{y}_{\cdot jh}\\ \vdots\\ Q_{v\cdot\cdot}=Y_{v\cdot\cdot}-\Sigma\Sigma n_{vjh}\bar{y}_{\cdot jh}\\ 0\end{bmatrix}$$

Again in matrix notation the solution for the $\hat{\tau}_i$ are obtained as:

$$\begin{bmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \\ \vdots \\ \hat{\tau}_v \end{bmatrix} = \begin{bmatrix} n^{11} & n^{12} & n^{13} & \cdots & n^{1v} \\ n^{21} & n^{22} & n^{23} & \cdots & n^{2v} \\ n^{31} & n^{32} & n^{33} & \cdots & n^{3v} \\ \vdots & & & & \\ n^{v1} & n^{v2} & n^{v3} & \cdots & n^{vv} \end{bmatrix} \cdot \begin{bmatrix} Q_{1..} \\ Q_{2..} \\ Q_{3..} \\ \vdots \\ Q_{v..} \end{bmatrix}$$

where n^{ig} are the elements of the inverse matrix. The solution for $\hat{\tau}_g$ is

$$\hat{\tau}_g = \sum_{i=1}^{v} n^{gi} Q_{i..} \quad ,$$

the variance of $\hat{\tau}_g$ is

$$V(\hat{\tau}_g) = n^{gg} \sigma_\epsilon^2 \quad ;$$

and the variance of a difference between $\hat{\tau}_{g'}$ and $\hat{\tau}_g$ is

$$V(\hat{\tau}_{g'} - \hat{\tau}_g) = \sigma_\epsilon^2 (n^{g'g'} + n^{gg} - n^{g'g} - n^{gg'}) \quad .$$

In the ρ_j equations, substitute for $\hat{\mu}+\hat{\tau}_i$ from $\hat{\tau}_i$ equation, thus for j=f:

$$n_{.f.}\hat{\rho}_f + \sum_i n_{if.}(\hat{\tau}_i + \hat{\mu}) + \sum_{h=1}^{k_j} n_{.fh}\hat{\beta}_{fh} = Y_{.f.} \quad ,$$

$$n_{.f.}\hat{\rho}_f + \sum_i \frac{n_{if.}}{n_{i..}}(Y_{i..} - \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{ijh}(\hat{\rho}_j + \hat{\beta}_{jh})) + \sum_{i=1}^{v} \sum_{h=1}^{k_j} n_{ifh}\hat{\beta}_{fh} = Y_{.f.} \quad ,$$

$$n_{.f.}\hat{\rho}_f - \sum_i \frac{n_{if.}}{n_{i.}} \sum_j \hat{\rho}_j n_{ij.} - \sum_i \frac{n_{if.}}{n_{i.}} \sum\sum_{jh} n_{ijh}\hat{\beta}_{jh} + \sum_h n_{.fh}\hat{\beta}_{fh}$$

$$= Y_{.f.} - \sum_i n_{if.}\bar{y}_{i..} = Q_{.f.} \quad .$$

The solutions for the ρ_j and β_{jh} must be obtained jointly since they are not orthogonal and since three sets of unknowns are present. Equations involving the τ_i only are possible here because the incomplete blocks in the complete blocks can be considered as b incomplete blocks. For the solution we proceed to the β_{jh} equations and substitute for $\hat{\mu}+\hat{\tau}_i$ to obtain (for j=f and h=e):

$$n_{\cdot fe}(\hat{\rho}_f+\hat{\beta}_{\cdot fe})-\sum_i \frac{n_{ife}}{n_{i\cdot\cdot}} \sum\sum_{jh} n_{ijh}(\hat{\rho}_j+\hat{\beta}_{jh})$$

$$=Y_{\cdot fe}-\sum n_{ife}\bar{y}_{i\cdot\cdot}=Q_{\cdot fe}$$

From these b equations solutions for $\widehat{\rho_j+\beta_{jh}}$ are obtained. Summing over h, solutions for the $\hat{\rho}_j$ are obtained since $\sum_{h=1}^{k_j} \hat{\beta}_{jh}=0$. If b is less than v, then solve for $\widehat{\rho_j+\beta_{jh}}$; if not, solve for the $\hat{\tau}_i$. Of course, as a check one could obtain solutions for both sets of effects and the results must check by satisfying the normal equations. Also,

$$\hat{\mu}=\frac{1}{b}\left\{\sum_{j=1}^{r}\sum_{h=1}^{k_j}\bar{y}_{\cdot jh}-\sum_{j=1}^{r}\sum_{h=1}^{k_j}\frac{1}{n_{\cdot jh}}\sum_{i=1}^{v} n_{ijh}\hat{\tau}_i\right\}$$

$$=\frac{1}{v}\left\{\sum_{i=1}^{v}\bar{y}_{i\cdot\cdot}-\sum_i \frac{1}{n_{i\cdot\cdot}}\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{ijh}(\widehat{\rho_j+\beta_{jh}})\right\} .$$

The analysis of variance may be put in the form:

Source of variation	df	Sum of squares	Mean squares
Total (uncorrected)	$n_{\cdot\cdot\cdot}$	$\sum\sum\sum n_{ijh}Y^2_{ijh}$	--
Correction for mean=CF	1	$Y^2_{\cdot\cdot\cdot}/n_{\cdot\cdot\cdot}$	--
Among incomplete blocks (ignoring treatments and complete blocks)	b-1	$\sum\sum_{jh}\frac{Y^2_{\cdot jh}}{n_{\cdot jh}}-\frac{Y^2_{\cdot\cdot\cdot}}{n_{\cdot\cdot\cdot}}$	--
Among treatments (eliminating complete and incomplete block effects)	v-1	$\sum_i\hat{\tau}_iQ_{i\cdot\cdot}$	T'
Intrablock error	$n_{\cdot\cdot\cdot}$-b-v+1	subtraction	E_e
Complete blocks (eliminating treatments, ignoring incomplete blocks)	r-1	$\sum_j\rho^*_jQ_{\cdot j\cdot}$	--
Incomplete blocks (eliminating treatments and complete blocks)	b-r	$\sum\sum_{jh}(\widehat{\rho_j+\beta_{jh}})Q_{\cdot jh}-\sum_j\rho^*_jQ_{\cdot j\cdot}$	E_b
Among complete and incomplete blocks (eliminating treatments)	b-1	$\sum\sum_{jh}(\widehat{\rho_j+\beta_{jh}})Q_{\cdot jh}$	E'_b

The ρ_j^* in the above table are obtained from the r equations

$$\rho_f^* n_{\cdot f\cdot} - \sum_{i=1}^{v} \frac{n_{ij\cdot}}{n_{i\cdot\cdot}} \sum_{j=1}^{r} n_{ij\cdot}\rho_j^* = Y_{\cdot f\cdot} - \sum_{i=1}^{v} n_{if\cdot}\bar{y}_{i\cdot\cdot} = Q_{\cdot f\cdot}$$

plus the equation $\sum_{j=1}^{r} \rho_j^* = 0$. These equations are obtained from the normal equations for μ, τ_i, and ρ_j setting each $\beta_{jh}=0$.

The expected value of E_e is σ_ϵ^2. The expected value of $E_b' = \sum_{j=1}^{r} \sum_{h=1}^{k_j} (\widehat{\rho_j+\beta_{jh}}) Q_{\cdot jh} = \sum_{j=1}^{r} \sum_{h=1}^{k_j} Q_{\cdot jh} \sum_{f=1}^{r} \sum_{g=1}^{k_f} k^{jhfg} Q_{\cdot fg}$ (where k^{jhfg} are the elements of the inverse matrix in the solution of the $\widehat{\rho_j+\beta_{jh}}$) is:

$$E[E_b'] = \sum_{j=1}^{r} \sum_{h=1}^{k_j} \sum_{f=1}^{r} \sum_{g=1}^{k_f} k^{jhfg} (Y_{\cdot jh} - \sum_{i=1}^{v} n_{ijh}\bar{y}_{i\cdot\cdot})(Y_{\cdot fg} - \sum_{d=1}^{v} n_{dfg}\bar{y}_{d\cdot\cdot})]$$

$$= \sum_{j=1}^{r} \sum_{h=1}^{k_j} \sum_{f=1}^{r} \sum_{g=1}^{k_f} k^{jhfg} [n_{\cdot jh}(\rho_j+\beta_{jh}) + \sum_i n_{ijh}\epsilon_{ijh}$$

$$- \sum_{i=1}^{v} \frac{n_{ijh}}{n_{i\cdot\cdot}} (\sum_{e=1}^{r} n_{ie\cdot}\rho_e + \sum_{e=1}^{r} \sum_{c=1}^{k_e} n_{iec}(\beta_{ec}+\epsilon_{iec})]\cdot[n_{\cdot fg}(\rho_f+\beta_{fg})$$

$$+ \sum_{d=1}^{v} n_{dfg}\epsilon_{dfg} - \sum_{d=1}^{v} \frac{n_{dfg}}{n_{d\cdot\cdot}} (\sum_{o=1}^{r} n_{do\cdot}\rho_o + \sum_{o=1}^{r} \sum_{p=1}^{k_o} n_{dop}(\beta_{op}+\epsilon_{dop}))]$$

$$= \sigma_\rho^2 \Big\{ \sum_{j=1}^{r} \sum_{h=1}^{k_j} k^{jhjh} [n_{\cdot jh}^2 - 2n_{\cdot jh} \sum_{i=1}^{v} \frac{n_{ijh} n_{ij\cdot}}{n_{i\cdot\cdot}}$$

$$+ \sum_{i=1}^{v} \frac{n_{ijh}}{n_{i\cdot\cdot}} \sum_{d=1}^{v} \frac{n_{djh}}{n_{d\cdot\cdot}} \sum_{e=1}^{r} n_{de\cdot} n_{ie\cdot}]$$

$$+ \sum_{j=1}^{r} \sum_{\substack{h=1 \\ h\neq g}}^{k_j} \sum_{g=1}^{k_j} k^{jhjg} [n_{\cdot jh} n_{\cdot jg} - n_{\cdot jh} \sum_{i=1}^{v} \frac{n_{ijg} n_{ij\cdot}}{n_{i\cdot\cdot}}$$

$$- n_{\cdot jg} \sum_{d=1}^{v} \frac{n_{djh} n_{dj\cdot}}{n_{d\cdot\cdot}} + \sum_{i=1}^{v} \frac{n_{ijh}}{n_{i\cdot\cdot}} \sum_{d=1}^{v} \frac{n_{djg}}{n_{d\cdot\cdot}} \sum_{e=1}^{r} n_{ie\cdot} n_{de\cdot}]$$

$$- \sum_{j=1}^{r} \sum_{h=1}^{k_j} \sum_{f=1}^{r} \sum_{g=1}^{k_f} k^{jhfg} [n_{\cdot jh} \sum_{i=1}^{v} \frac{n_{ifg} n_{ij\cdot}}{n_{i\cdot\cdot}} + n_{\cdot fg} \sum_{d=1}^{v} \frac{n_{djh} n_{df\cdot}}{n_{d\cdot\cdot}}$$

$$jh\neq fg \quad j\neq f \qquad - \sum_{i=1}^{v} \frac{n_{ijh}}{n_{i\cdot\cdot}} \sum_{d=1}^{v} \frac{n_{djh}}{n_{d\cdot\cdot}} \sum_{e=1}^{r} n_{ie\cdot} n_{de\cdot}] \Big\}$$

$$+\sigma_\beta^2\left\{\sum_{j=1}^{r}\sum_{h=1}^{k_j}k^{jhjh}\left[n_{\cdot jh}^2-2n_{\cdot jh}\sum_{i=1}^{v}\frac{n_{ijh}}{n_{i\cdot\cdot}}\right.\right.$$

$$\left.+\sum_{k=1}^{v}\frac{n_{ijh}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{djh}}{n_{d\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_e}n_{iec}n_{dec}\right]$$

$$-\sum_{j=1}^{r}\sum_{\substack{h=1\\ jh\neq fg}}^{k_j}\sum_{f=1}^{r}\sum_{g=1}^{k_f}k^{jhfg}\left[n_{\cdot jh}\sum_{d=1}^{v}\frac{n_{dfg}n_{djh}}{n_{d\cdot\cdot}}+n_{\cdot fg}\sum_{i=1}^{v}\frac{n_{ifg}n_{djh}}{n_{i\cdot\cdot}}\right.$$

$$\left.\left.-\sum_{i=1}^{v}\frac{n_{ijh}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{dfg}}{n_{d\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_e}n_{iec}n_{dec}\right]\right\}$$

$$+\sigma_\epsilon^2(b-1)=K_1\sigma_\rho^2+K_2\sigma_\beta^2+(b-1)\sigma_\epsilon^2 \quad .$$

The expected value of $\sum_{j=1}^{r}\rho_j^* Q_{\cdot j\cdot}$ is

$$E\left[\sum_{j=1}^{r}Q_{\cdot j\cdot}\sum_{f=1}^{r}a^{jf}Q_{\cdot f\cdot}=\sum_{j=1}^{r}(Y_{\cdot j\cdot}-\sum_{i=1}^{v}n_{ij\cdot}\bar{y}_{i\cdot\cdot})\sum_{f=1}^{r}a^{jf}(Y_{\cdot f\cdot}-\sum_{i=1}^{v}n_{if\cdot}\bar{y}_{i\cdot\cdot})\right.$$

$$=\sum_{j=1}^{r}\sum_{f=1}^{r}a^{jf}(n_{\cdot j\cdot}(\mu+\rho_j)+\sum_{i=1}^{v}n_{ij\cdot}\tau_i+\sum_{h=1}^{k_j}n_{\cdot jh}\beta_{jh}$$

$$+\sum_{i=1}^{v}\sum_{h=1}^{k_j}n_{ijh}\epsilon_{ijh}-\sum_i n_{ij\cdot}(\mu+\tau_i+\frac{1}{n_{i\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_e}n_{iec}(\rho_e$$

$$+\beta_{ec}+\epsilon_{iec}))(n_{\cdot f\cdot}(\mu+\rho_f)+\sum_{d=1}^{v}n_{df\cdot}\tau_d+\sum_{g=1}^{k_f}n_{\cdot fg}\beta_{fg}$$

$$+\sum_{d=1}^{v}\sum_{g=1}^{k_f}n_{dfg}\epsilon_{dfg}-\sum_{d=1}^{v}n_{df\cdot}(\mu+\tau_d+\frac{1}{n_{d\cdot\cdot}}\sum_{o=1}^{r}\sum_{p=1}^{k_o}n_{dop}(\rho_o+\beta_{op}+\epsilon_{dop})))$$

$$=\sum_{j=1}^{r}\sum_{f=1}^{r}a^{jf}(n_{\cdot j\cdot}\rho_j+\sum_{h=1}^{k_j}n_{\cdot jh}\beta_{jh}+\sum_{i=1}^{v}\sum_{h=1}^{k_j}n_{ijh}\epsilon_{ijh}$$

$$-\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_e}n_{iec}(\rho_e+\beta_{ec}+\epsilon_{iec}))(n_{\cdot f\cdot}\rho_f+\sum_{h=1}^{k_f}n_{\cdot fg}\beta_{fg}$$

$$+\sum_{d=1}^{v}\sum_{g=1}^{k_f} n_{dfg}\epsilon_{dfg}-\sum_{d=1}^{v}\frac{n_{df\cdot}}{n_{d\cdot\cdot}}\sum_{o=1}^{r}\sum_{p=1}^{k_o} n_{dop}(\rho_o+\beta_{op}+\epsilon_{dop}))]$$

$$=\sigma_\rho^2\left\{\sum_{j=1}^{r} a^{jj}\left(n^2_{\cdot j\cdot}-2n_{\cdot j\cdot}\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}+\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{dj\cdot}}{n_{d\cdot\cdot}}\sum_{e=1}^{r} n_{ie\cdot}n_{de\cdot}\right)\right.$$

$$-\sum_{i=1}^{r}\sum_{\substack{f=1\\ j\neq f}}^{r} a^{jf}\left(n_{\cdot j\cdot}\sum_{i=1}^{v}\frac{n_{if\cdot}n_{ij\cdot}}{n_{i\cdot\cdot}}+n_{\cdot f\cdot}\sum_{d=1}^{v}\frac{n_{df\cdot}n_{dj\cdot}}{n_{d\cdot\cdot}}\right.$$

$$\left.\left.-\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{dj\cdot}}{n_{d\cdot\cdot}}\sum_{e} n_{ie\cdot}n_{de\cdot}\right)\right\}$$

$$+\sigma_\beta^2\left\{\sum_{j=1}^{r} a^{jj}\left(\sum_{h=1}^{k_j} n^2_{\cdot jh}-2\sum_{h=1}^{k_j} n_{\cdot jh}\sum_{i=1}^{v}\frac{n_{ijh}}{n_{i\cdot\cdot}}\right.\right.$$

$$\left.+\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{dj\cdot}}{n_{d\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_e} n_{iec}n_{dec}\right)$$

$$-\sum_{j=1}^{r}\sum_{\substack{f=1\\ j\neq f}}^{r} a^{jf}\left(\sum_{h=1}^{k_j} n_{\cdot jh}\sum_{i=1}^{v}\frac{n_{if\cdot}n_{ijh}}{n_{i\cdot\cdot}}+\sum_{g=1}^{k_f} n_{\cdot fg}\sum_{i=1}^{v}\frac{n_{ij\cdot}n_{ifg}}{n_{i\cdot\cdot}}\right.$$

$$\left.\left.-\sum_{i=1}^{v}\frac{n_{ij\cdot}}{n_{i\cdot\cdot}}\sum_{d=1}^{v}\frac{n_{df\cdot}}{n_{d\cdot\cdot}}\sum_{e=1}^{r}\sum_{c=1}^{k_c} n_{iec}n_{dec}\right)\right\}$$

$$+\sigma_\epsilon^2(r-1)=K_3\sigma_\rho^2+K_4\sigma_\beta^2+(r-1)\sigma_\epsilon^2 \quad .$$

Now, $\sum_{j=1}^{r}\sum_{h=1}^{k_j}(\widehat{\rho_j+\beta_{jh}})Q_{\cdot jh}-\sum_{j=1}^{r}\rho_j^{*}Q_{\cdot j\cdot}$ is the sum of squares for incomplete blocks within complete blocks eliminating treatment effects and has the expectation:

$$(K_2-K_4)\sigma_\beta^2+(b-r)\sigma_\epsilon^2$$

By definition the coefficient of σ_ρ^2 must be zero; hence, $K_1=K_3$; no proof of the equality is given here. It should be possible to simplify the coefficient for σ_β^2

since there is a relationship between the a^{jf} and the k^{jhfg}. Perhaps this should be done prior to programming for high speed computers.

The treatment mean adjusted for incomplete and complete block effects is $\hat{\mu}+\hat{\tau}_i$. Only intrablock information is utilized in obtaining the adjusted means. The variances of adjusted means and for differences between adjusted means are given above.

Recovery of Interblock Information

The sum of squares to be minimized is

$$w \sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{k_j} n_{ijh}(Y_{ijh}-\mu-\rho_j-\tau_i-\beta_{jh})^2$$

$$+w' \sum_{j=1}^{r} \sum_{h=1}^{k_j} (Y_{\cdot jh}-n_{\cdot jh}(\mu+\rho_j)-\sum_i n_{ijh}\tau_i)^2/n_{\cdot jh}$$

where the true weights are $\omega=1/\sigma_\epsilon^2$ and $\omega'_{jh}=1/(\sigma_\epsilon^2+n_{\cdot jh}\sigma_\beta^2)$ for β_{jh} independently distributed with mean zero and variance σ_β^2. Instead of using a different weight for each incomplete block, an average coefficient is utilized and is given below. This actual weight, w'_{jh}, should be utilized if there is sizeable disparity among the $n_{\cdot jh}$. The resulting normal equations for $w'_{jh}=w'$ are:

$$\mu: \quad (w+w')\left\{\mu n_{\cdot\cdot\cdot}+\sum_{j=1}^{r} n_{\cdot j\cdot}\rho_j+\sum_{i=1}^{v} n_{i\cdot\cdot}\tau_i\right\}+w\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{\cdot jh}\beta_{jh}=(w+w')Y_{\cdot\cdot\cdot} \quad \text{or}$$

$$n_{\cdot\cdot\cdot}\mu+\Sigma n_{\cdot j\cdot}\rho_j+\Sigma n_{i\cdot\cdot}\tau_i+\frac{w}{w+w'}\sum_{jh}\sum n_{\cdot jh}\beta_{jh}=Y_{\cdot\cdot\cdot}$$

$$\tau_g: \quad wn_{g\cdot\cdot}\tau_g+w'\sum_{i=1}^{v}\tau_i\sum_{j=1}^{r}\sum_{h=1}^{k_j}\frac{n_{ijh}n_{gjh}}{n_{\cdot jh}}+(w+w')\sum_{j=1}^{r} n_{gj\cdot}(\mu+\rho_j)+w\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{gjh}\beta_{jh}$$

$$=wY_{g\cdot\cdot}+w'\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{gjh}\bar{y}_{\cdot jh}$$

$$\rho_j: \quad (w+w')\left\{n_{\cdot j\cdot}(\mu+\rho_j)+\sum_{i=1}^{v} n_{ij\cdot}\tau_i+w\sum_{h=1}^{k_j} n_{\cdot jh}\beta_{jh}=(w+w')Y_{\cdot j\cdot}\right\} \quad \text{or}$$

$$n_{\cdot j\cdot}(\mu+\rho_j)+\sum_{i=1}^{v} n_{ij\cdot}\tau_i+\frac{w}{w+w'}\sum_{j=1}^{k_j} n_{\cdot jh}\beta_{jh}=Y_{\cdot j\cdot}$$

$$\beta_{jh}:\ w\left\{n_{\cdot jh}(\mu+\rho_j+\beta_{jh})+\sum_{i=1}^{v} n_{ijh}\tau_i = Y_{\cdot jh}\right\}$$

Substituting for β_{jh} from the last equation in the previous three equations results in

$$\mu:\quad n_{\cdot\cdot\cdot}\mu+\sum_{j=1}^{r} n_{\cdot j\cdot}\rho_j+\sum_{i=1}^{v} n_{i\cdot\cdot}\tau_i = Y_{\cdot\cdot\cdot}$$

$$\rho_j:\quad n_{\cdot j\cdot}(\mu+\rho_j)+\sum_{i=1}^{v} n_{ij\cdot}\tau_i = Y_{\cdot j\cdot}$$

$$\tau_g:\quad wn_{g\cdot\cdot}\tau_g-(w-w')\sum_{i=1}^{v}\tau_i\sum_{j=1}^{r}\sum_{h=1}^{k_j}\frac{n_{ijh}n_{gjh}}{n_{\cdot jh}}-w'\sum_j\frac{n_{gj\cdot}}{n_{\cdot j\cdot}}\sum_i n_{ig\cdot}\tau_i$$

$$=wY_{g\cdot\cdot}-(w-w')\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{gjh}\bar{y}_{\cdot jh}-w'\sum_{j=1}^{r} n_{gj\cdot}\bar{y}_{\cdot j\cdot}$$

$$=w\left\{Y_{g\cdot\cdot}-\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{gjh}\bar{y}_{\cdot jh}\right\}+w'\left\{\sum_{j=1}^{r}\sum_{h=1}^{k_j} n_{gjh}\bar{y}_{\cdot jh}-\sum_{j=1}^{r} n_{gj\cdot}\bar{y}_{\cdot j\cdot}\right\}=Z_{g\cdot\cdot}$$

The above v equations plus an additional equation, e.g., $\Sigma\tau_i^*=0$, results in unique solutions for the τ_i^*; thus

$$\begin{pmatrix}\tau_1^*\\ \tau_2^*\\ \tau_3^*\\ \tau_4^*\\ \tau_v^*\end{pmatrix}=\begin{pmatrix}c^{11} & c^{12} & c^{13} & \cdots & c^{1v}\\ c^{21} & c^{22} & c^{23} & \cdots & c^{2v}\\ c^{31} & c^{32} & c^{33} & \cdots & c^{3v}\\ \vdots & & & & \\ c^{v1} & c^{v2} & c^{v3} & \cdots & c^{vv}\end{pmatrix}\begin{pmatrix}Z_{1\cdot\cdot}\\ Z_{2\cdot\cdot}\\ Z_{3\cdot\cdot}\\ \vdots\\ Z_{v\cdot\cdot}\end{pmatrix}$$

where the original equations were in the form:

$$\begin{pmatrix} n_{11} & n_{12} & n_{13} & \cdots & n_{1v} & 1 \\ n_{21} & n_{22} & n_{23} & \cdots & n_{2v} & 1 \\ n_{31} & n_{32} & n_{33} & \cdots & n_{3v} & 1 \\ \vdots & & & & & \\ n_{v1} & n_{v2} & n_{v3} & \cdots & n_{vv} & 1 \\ 1 & 1 & 1 & & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \tau_1^* \\ \tau_2^* \\ \tau_3^* \\ \vdots \\ \tau_v^* \\ 0 \end{pmatrix} = \begin{pmatrix} Z_{1\cdot\cdot} \\ Z_{2\cdot\cdot} \\ Z_{3\cdot\cdot} \\ \vdots \\ Z_{v\cdot\cdot} \\ 0 \end{pmatrix}$$

and where the c^{iu} are the elements of the inverse of the matrix of coefficients for the τ_i^*.

The variances of a difference between two adjusted means recovering inter-block information, say $\mu^*+\tau_1^*$ and $\mu^*+\tau_2^*$, is

$$V(\tau_1^*-\tau_2^*)=c^{11}+c^{22}-c^{12}-c^{21}$$

and the variance of τ_i^* is:

$$V(\tau_i^*)=c^{ii} \; .$$

It should be remembered the σ_ϵ^2 and σ_β^2 appear in the expected values of the $Z_{i\cdot\cdot}$.

Now let us return to the calculation of the weight $w'=\dfrac{1}{\sigma_\epsilon^2+\bar{n}\sigma_\beta^2}$. Now $\bar{n}$ is determined from a nested classification in which there are no treatment effects. The expectation is obtained from page 106 of "Experimental Design -- theory and application." In the present notation the expected value of the mean square for among incomplete blocks within complete block in the absence of treatment effects is:

$$\sigma_\epsilon^2+\frac{1}{b-r}\left\{n_{\cdot\cdot\cdot}-\sum_{j=1}^{r}\sum_{h=1}^{k_j}\frac{n_{\cdot jh}^2}{n_{\cdot j\cdot}}\right\}\sigma_\beta^2$$

and hence

$$\bar{n}=\left\{n_{\cdot\cdot\cdot}-\sum_{j=1}^{r}\sum_{h=1}^{k_j}\frac{n_{\cdot jh}^2}{n_{\cdot j\cdot}}\right\}/(b-r)$$

$(b=\sum_{j=1}^{r} k_j)$.

The weight w' for the jh-th incomplete block will be

$$w'_{jh}=\frac{1}{\hat{\sigma}_\epsilon^2+n_{\cdot jh}\hat{\sigma}_\beta^2} \quad .$$

The expected value of the mean square, E_b, for incomplete blocks within complete blocks eliminating treatment and complete block effects is:

$$\sigma_\epsilon^2+\sigma_\beta^2(K_2-K_4)/(b-r)=\sigma_\epsilon^2+\bar{m}\sigma_\beta^2 \quad .$$

The average amount of interblock information is estimated as follows:

$$w'=\frac{\bar{m}}{\bar{n}E_b-(\bar{n}-\bar{m})E_e}$$

and the intrablock information is estimated as

$$w=1/E_e \quad .$$

With these weights it is now possible to obtain solutions for the τ_i^*.

Incomplete Blocks Not Arranged in Complete Blocks

The previous results may be used directly for an incomplete block design for which the b incomplete blocks are completely randomized. To apply the formulae set $\rho_j=0$, $j=0$, $k_j=b$, $n_{ij}=0$ or 1, and $r=0$. The equations then become:

$$Y_{ij}=n_{ih}(\mu+\tau_i+\beta_h+\epsilon_{ih})$$

$$Y_{\cdot\cdot}=n_{\cdot\cdot}\mu+\sum_{i=1}^{v} n_{i\cdot}\tau_i+\sum_{h=1}^{b}\beta_h$$

$$Y_{i\cdot}=n_{i\cdot}(\mu+\tau_i)+\sum_{h=1}^{b} n_{ih}\beta_h$$

$$Y_{\cdot h}=n_{\cdot h}(\mu+\beta_h)+\sum_{i=1}^{v} n_{ih}\tau_i$$

$$n_{f\circ}\beta_f - \sum_{i=1}^{v} \frac{n_{if}}{n_{i\cdot}} \sum_{h=1}^{b} n_{ih}\beta_h = Y_{\cdot f} - \sum_{i=1}^{v} n_{if}\bar{y}_i$$

$$n_{g\circ}\tau_g - \sum_{j=1}^{b} \frac{n_{gh}}{n_{\cdot h}} \sum_{i=1}^{v} n_{ih}\tau_i = Y_{g\circ} - \sum_{h=1}^{b} n_{gh}\bar{y}_{\cdot h}$$

and the expected value for blocks eliminating treatment effects sum of squares becomes:

$$(b-1)\sigma_\epsilon^2+\sigma_\beta^2\left\{ \sum_{h=1}^{b} k^{hh}(n_{\cdot h}^2-2n_{\cdot h} \sum_i \frac{n_{ih}}{n_{i\cdot}} + \sum_{i=1}^{v} \frac{n_{ih}}{n_{i\circ}} \sum_{d=1}^{v} \frac{n_{dh}}{n_{d\cdot}} \sum_{e=1}^{b} n_{ie}n_{de}) \right.$$

$$- \sum_{h=1}^{b} \sum_{\substack{f=1 \\ h\neq f}}^{b} k^{hf}(n_{\cdot h} \sum_{d=1}^{v} \frac{n_{df}n_{dj}}{n_{d\cdot}} + n_{\cdot f} \sum_{i=1}^{v} \frac{n_{if}n_{ih}}{n_{i\cdot\cdot}}$$

$$\left. - \sum_{i=1}^{v} \frac{n_{ih}}{n_{i\cdot}} \sum_d \frac{n_{df}}{n_{d\cdot}} \sum_{e=1}^{b} n_{ie}n_{de}) \right\}$$

$=(b-1)\sigma_\epsilon^2+(b-1)\bar{m}\sigma_\beta^2$. Utilizing these results, the analysis goes through in much the same manner as for the experimental design in which the incomplete blocks are arranged in complete blocks.

Example 1 -- Using a randomized complete block design for v treatments and r replicates on each treatment as an example to illustrate the use of the expectation on page 8, we note that

$$\begin{pmatrix} \rho_1^* \\ \rho_2^* \\ \vdots \\ \rho_r^* \end{pmatrix} = \frac{1}{rv} \begin{pmatrix} r-1 & -1 & \cdots & -1 \\ -1 & r-1 & \cdots & -1 \\ \vdots & & & \\ -1 & -1 & \cdots & r-1 \end{pmatrix} \begin{pmatrix} Y_{\cdot 1}-v\bar{y}=Q_{\cdot 1} \\ Y_{\cdot 2}-v\bar{y}=Q_{\cdot 2} \\ \vdots \\ Y_{\cdot 3}-v\bar{y}=Q_{\cdot r} \end{pmatrix}$$

and that the expected value of $\Sigma\rho_j^* Q_{\cdot j}$ is:

$$\sigma_\rho^2 \Big\{ r \frac{(r-1)}{rv} (v^2 - \frac{2v^2}{r} + \frac{v^2}{r})$$

$$- r(r-1)(-\frac{1}{rv})(\frac{2v^2}{r} - \frac{v^2}{r}) \Big\} + (r-1)\sigma_\epsilon^2$$

$$= \sigma_\rho^2 v(r-1) + (r-1)\sigma_\epsilon^2$$

$$= (r-1)(\sigma_\epsilon^2 + v\sigma_\rho^2) \quad ,$$

which is the expected values of the replicate sum of squares with r-1 degrees of freedom in the randomized complete block design.

Example 2 -- For the resolvable balanced lattice design with $v=k^2$ treatments in incomplete blocks of size k and with $r=k+1$, we may use the formula on pages 6-7 to obtain the expected value of the sum of squares among incomplete blocks (eliminating treatment effects) with $b-1=k(k+1)-1$ degrees of freedom. The solutions for the $\rho_j+\beta_{jh}$ (with $\sum_{jh}\sum(\widehat{\rho_j+\beta_{jh}})=0$) are obtained as follows:

$$\begin{pmatrix} \widehat{\rho_1+\beta_{11}} \\ \widehat{\rho_1+\beta_{12}} \\ \vdots \\ \widehat{\rho_1+\beta_{1k}} \\ \widehat{\rho_2+\beta_{21}} \\ \vdots \\ \widehat{\rho_2+\beta_{2k}} \\ \vdots \\ \widehat{\rho_{k+1}+\beta_{k+1,k}} \end{pmatrix} = \frac{1}{k^3} \begin{pmatrix} k^2+k-1 & -1 & \cdots & -1 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & k^2+k-1 & \cdots & -1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ -1 & -1 & \cdots & k^2+k-1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & k^2+k-1 & \cdots & -1 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & 0 & -1 & \cdots & k^2+k-1 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & k^2+k-1 \end{pmatrix} \begin{pmatrix} Q_{\cdot 11} \\ Q_{\cdot 12} \\ \vdots \\ Q_{\cdot 1k} \\ Q_{\cdot 21} \\ \vdots \\ Q_{\cdot 2k} \\ \vdots \\ Q_{\cdot k+1,k} \end{pmatrix}$$

where

$$Q_{\cdot jh}=Y_{\cdot jh}-\sum_{i=1}^{k^2} n_{ijh}\bar{y}_{i\cdot\cdot}$$

Also, $n_{\cdot jh}=k$, $n_{i\cdot\cdot}=k+1$, $n_{ij\cdot}=1$, and $n_{ijh}=0$ or 1. Substituting these results in the general equation we obtain:

$$\sigma_\rho^2 \Big\{ (k+1)(k)\Big(\frac{k^2+k-1}{k^3}\Big)\Big(k^2-\frac{2k^2}{k+1}+\frac{k^2}{k+1}\Big)$$

$$+(k+1)(k)(k-1)\Big(-\frac{1}{k^3}\Big)\Big(k^2-\frac{2k^2}{k+1}+\frac{k^2}{k+1}\Big)-0 \Big\}$$

$$+\sigma_\beta^2 \Big\{ (k+1)(k)\Big(\frac{k^2+k-1}{k^3}\Big)\Big(k^2-\frac{2k^2}{k+1}+\frac{2k^2}{(k+1)^2}\Big)$$

$$-(k+1)(k)(k-1)\Big(-\frac{1}{k^3}\Big)\Big(0-\frac{k^2}{(k+1)^2}\Big) \Big\}$$

$$+\sigma_\epsilon^2(b-1)=\sigma_\rho^2 k^3+\sigma_\beta^2 k^3+\sigma_\epsilon^2(k^2+k-1) \quad ,$$

which agrees with coefficient for σ_β^2, $bk-v=k(k+1)(k)-k^2=k^3$, given on page 422 of Experimental Design -- Theory and Application.

The solution for the ρ_j setting each β_{jh} equal to zero are obtained as:

$$\begin{pmatrix} \rho_1^* \\ \rho_2^* \\ \rho_3^* \\ \vdots \\ \rho_{k+1}^* \end{pmatrix} = \frac{1}{k^2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_{\cdot 1 \cdot} = Y_{\cdot 1 \cdot} - k^2\bar{y} \\ Q_{\cdot 2 \cdot} = Y_{\cdot 2 \cdot} - k^2\bar{y} \\ Q_{\cdot 3 \cdot} = Y_{\cdot 3 \cdot} - k^2\bar{y} \\ \vdots \\ Q_{\cdot k+1 \cdot} = Y_{\cdot k+1 \cdot} - k^2\bar{y} \end{pmatrix}$$

also, $n_{\cdot j \cdot}=k^2$, $n_{ij\cdot}=1$, and $n_{i\cdot\cdot}=k+1$.

Substituting these values in the general equation for the expected value of $\Sigma\rho_j^* Q_{\cdot j \cdot}$ on page 8 we obtain:

$$\sigma_\rho^2 \left\{ (k+1)\left(\frac{1}{k^2}\right)\left(k^4 - \frac{2k^2(k^2)}{(k+1)} + \frac{k^4}{(k+1)}\right) - 0 \right\}$$

$$+\sigma_\beta^2 \left\{ (k+1)\left(\frac{1}{k^2}\right)\left(k^3 - \frac{2k^3}{k+1} + \frac{k^3}{k+1}\right) - 0 \right\}$$

$$+\sigma_\epsilon^2(k+1-1)$$

$$=\sigma_\rho^2 k^3+\sigma_\beta^2 k^2+k\sigma_\epsilon^2$$

$$=k(\sigma_\epsilon^2+k\sigma_\beta^2+k^2\sigma_\rho^2)$$

Therefore, the expected value of the blocks within replicates (eliminating treatments) sum of squares with k^2-1 degrees of freedom is:

$$(k^2+k-1)\sigma_\epsilon^2+k^3\sigma_\beta^2+k^3\sigma_\rho^2$$

$$-k\sigma_\epsilon^2-k^2\sigma_\beta^2-k^3\sigma_\rho^2$$

$$=(k^2-1)\sigma_\epsilon^2+k^2(k-1)\sigma_\beta^2$$

$$=(k^2-1)(\sigma_\epsilon^2+\frac{k^2}{k+1}\sigma_\beta^2) \; .$$

Example 3 -- To further illustrate the use of the general formulae for obtaining expectation of the various sums of squares, consider the double lattice design where the $v=k^2$ treatments are in incomplete blocks of size k and where r=2. The ρ_j^* are obtained from the following equations:

$$\begin{pmatrix} \rho_1^* \\ \\ \rho_2^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2k^2} & -\frac{1}{2k^2} \\ \\ -\frac{1}{2k^2} & \frac{1}{2k^2} \end{pmatrix} \begin{pmatrix} Q_{.1.} = Y_{.1.} - k^2\bar{y} \\ \\ Q_{.2.} = Y_{.2.} - k^2\bar{y} \end{pmatrix}$$

The expected value of $\sum_{j=1}^{2} \rho_j^* Q_{.j.}$ with 1 degree of freedom is:

$$\sigma_\rho^2 \left\{2\left(\frac{1}{2k^2}\right)\left(k^4-k^3+\frac{k^2}{2}\right)-2\left(-\frac{1}{2k^2}\right)\left(2k^2\left(\frac{k}{2}\right)-\frac{k^2}{2}\right)\right\}$$

$$+\sigma_\beta^2 \left\{2\left(\frac{1}{2k^2}\right)\left(k^3-k^2+\frac{k^2}{2}\right)-2\left(-\frac{1}{2k^2}\right)\left(k^2-\frac{k^2}{2}\right)\right\}$$

$$+\sigma_\epsilon^2$$

$$=\sigma_\epsilon^2+k\sigma_\beta^2+k^2\sigma_\rho^2 \ ,$$

as given on page 374 of <u>Experimental Design</u>. In the above $a^{jj}=\frac{1}{2k^2}$, $a^{jf}=-\frac{1}{2k^2}$ (for $j\neq f$), $n_{ijh}=0$ or 1, $n_{.jh}=k$, $n_{i..}=2$, $n_{.j.}=k^2$, and $n_{ij.}=1$.

The $\widehat{\rho_j+\beta_{jh}}$ values may be obtained from the following equation:

$$\begin{pmatrix} \widehat{\rho_1+\beta_{11}} \\ \widehat{\rho_1+\beta_{12}} \\ \vdots \\ \widehat{\rho_1+\beta_{1k}} \\ \widehat{\rho_2+\beta_{21}} \\ \widehat{\rho_2+\beta_{22}} \\ \vdots \\ \widehat{\rho_2+\beta_{2k}} \end{pmatrix} = \frac{1}{k^2} \begin{pmatrix} 2k-1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2k-1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ -1 & -1 & \cdots & 2k-1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2k-1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2k-1 & \cdots & -1 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & 2k-1 \end{pmatrix} \begin{pmatrix} Q_{\cdot 11} \\ Q_{\cdot 12} \\ \vdots \\ Q_{\cdot 1k} \\ Q_{\cdot 21} \\ Q_{\cdot 22} \\ \vdots \\ Q_{\cdot 2k} \end{pmatrix}$$

where $Q_{\cdot jh}=Y_{\cdot jh}-\sum_{i=1}^{k^2} n_{ijh}\bar{y}_{i\cdot\cdot}$. Hence, $k^{jhjh}=\frac{2k-1}{k^2}$, $k^{jhjg}=-\frac{1}{k^2}$ for $h\neq g$, and $k^{jhfg}=0$ for $f\neq j$. With these solutions the expected value of the among incomplete block sum of squares (eliminating treatment effects) with 2k-1 degrees of freedom is:

$$E[\sum_{jh}\sum(\rho_j+\beta_{jh})Q_{\cdot jh}]$$

$$=\sigma_\rho^2\left\{2k\left(\frac{2k-1}{k^2}\right)\left(k^2-2k\left(\frac{k}{2}\right)+\frac{k^2}{2}\right)+2k(k-1)\left(-\frac{1}{k^2}\right)\left(k^2-k^2+\frac{k^2}{2}\right)-0\right\}$$

$$+\sigma_\beta^2\left\{2k\left(\frac{2k-1}{k^2}\right)\left(k^2-2k\left(\frac{k}{2}\right)+\frac{k^2}{2}\right)\right.$$

$$\left.-2k(k-1)\left(-\frac{1}{k^2}\right)\left(0+0-\frac{k^2}{2}\right)\right\}+(2k-1)\sigma_\epsilon^2$$

$$=(2k-1)\sigma_\epsilon^2+k^2\sigma_\beta^2+k^2\sigma_\rho^2 \quad .$$

Therefore, the expected value of the blocks within replicate sum of squares (eliminating treatment effects) with 2(k-1) degrees of freedom, is:

$$(2k-1)\sigma_\epsilon^2+k^2\sigma_\beta^2+k^2\sigma_\rho^2$$

$$-\sigma_\epsilon^2-k\sigma_\beta^2-k^2\sigma_\rho^2$$

$$=2(k-1)\sigma_\epsilon^2+k(k-1)\sigma_\beta^2$$

$$=2(k-1)(\sigma_\epsilon^2+\frac{k}{2}\sigma_\beta^2)$$

which agrees with the expectation given on page 374 of Experimental Design.