A SIMPLIFICATION OF THE FACTORING ARGUMENT FOR PROVING THE NON-SINGULARITY OF A MATRIX BASED ON VANDERMONDE MATRICES

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Abstract

A simplification is given of the factoring argument of Reid and Driscoll (1988) in proving the non-singularity of a matrix vital to their proof of the necessity condition of the non-central extension of Craig's Theorem.

The Theorem

The extension of Craig's (1943) Theorem to the non-central case, i.e., for $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V})$, is that $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are stochastically independent if and only if $\mathbf{AVB} = \mathbf{0}$.

Sufficiency is easily proven, but an accessible proof of necessity has long been elusive (see the history by Driscoll and Gundberg, 1986). Happily, such proof now exists, thanks to Reid and Driscoll (1988). But one of its more difficult arguments can be simplified by a direct application of the factoring argument used for evaluating the determinant of a Vandermonde matrix.

The Simplification for Reid and Driscoll (1988)

Reid and Driscoll's (1988) equation (6) introduces a square matrix $\boldsymbol{\Lambda}$ of order 2k, the non-singularity of which is salient to their main result. They establish this by viewing determinants as polynomials in a particular root. But the Vandermonde properties evident in $\boldsymbol{\Lambda}$ yield an easier development.

The first k columns of $\boldsymbol{\Lambda}$ are $c_j = \lambda_j[1\ \lambda_j\ \lambda_j^2 \cdots \lambda_j^{2k-1}]'$ for $j = 1, \cdots, k$. Therefore they constitute a matrix $\mathbf{U\Delta}$ where $\mathbf{U}$ consists of the first k columns of a Vandermonde matrix of order 2k, and $\boldsymbol{\Delta} = \text{diag}\{\lambda_1\ \lambda_2 \cdots \lambda_k\}$ is a diagonal matrix of the λs. These λs, in Reid and Driscoll (1988), are all different and non-zero. Therefore $\mathbf{U}$ has full column rank, and $\boldsymbol{\Delta}$ is non-singular.

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Similarly, the last k columns of $\mathbf{\Lambda}$ are $\mathbf{DU\Delta}$, where $\mathbf{D} = \text{diag}\{1, 2, \cdots, 2k\}$. Therefore

$$\mathbf{\Lambda} = [\mathbf{U\Delta} \quad \mathbf{DU\Delta}] = \mathbf{M}\begin{bmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta} \end{bmatrix} \quad \text{for} \quad \mathbf{M} = [\mathbf{U} \quad \mathbf{DU}] \; . \tag{1}$$

Hence

$$|\mathbf{\Lambda}| = |\mathbf{M}|\,|\mathbf{\Delta}|^2 = |\mathbf{M}| \prod_{i=1}^{k} \lambda_i^2 \; . \tag{2}$$

Consider $|\mathbf{M}|$ as a polynomial in $\lambda, \cdots, \lambda_k$. Since setting $\lambda_i = 0$ makes the i'th column of $\mathbf{U}$ and $\mathbf{DU}$ in $\mathbf{M}$ be the same, it makes $|\mathbf{M}| = 0$. Therefore λ_i is a factor of $|\mathbf{M}|$. Similarly $\lambda_i = \lambda_j$ for $i \neq j$ also makes two columns of $\mathbf{M}$ be identical and so $(\lambda_i - \lambda_j)$ is also a factor of $|\mathbf{M}|$. Therefore $|\mathbf{M}|$ has $\prod_{i=1}^{k} \lambda_i \prod_{i<j}^{k} (\lambda_i - \lambda_j)$ as a factor. But $|\mathbf{M}|$ by the very nature of $\mathbf{M}$ is a symmetric function of the λs, of order $0+1+2+ \cdots +2k-1 = k(2k-1)$. Hence for some real c and positive integers p and q,

$$|\mathbf{M}| = c \prod_{i=1}^{k} \lambda_i^q \prod_{i<j}^{k} (\lambda_i - \lambda_j)^p \; . \tag{3}$$

for all k. Equating the orders of the two sides of this equation gives

$$k(2k-1) = kq + k + \tfrac{1}{2}k(k-1)p$$

i.e.,

$$q = 2k - 1 - \tfrac{1}{2}(k-1)p$$

or

$$p = \frac{2(2k-1-q)}{k-1} = 4 - \frac{2(q-1)}{k-1}$$

Since this must be true for all positive integers k, p and q, it is clear that $q = 1$ and $p = 4$. Then equating coefficients of any particular term on both sides of (3) gives, depending on the value of k, $c = \pm 1$. Thus

$$|\mathbf{\Lambda}| = |\mathbf{M}| \prod_{i=1}^{k} \lambda_i^2 = (\pm 1) \prod_{i=1}^{k} \lambda_i^3 \prod_{i<j}^{k} (\lambda_i - \lambda_j)^4 \, ,$$

and so, since the λs are all different and non-zero, $|\mathbf{\Lambda}| \neq 0$ and hence $\mathbf{\Lambda}$ is non-singular.

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References

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