A NOTE ON VANDERMONDE MATRICES AND CRAIG'S THEOREM

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Abstract

Vandermonde matrices simplify Read and Driscoll's proof of the necessity condition of the extension of Craig's theorem to the non-central case.

The Theorem

The extension of Craig's (1943) Theorem to the non-central case, i.e., for $\mathbf{y} \sim \mathcal{N}(\mu, \mathbf{V})$, is that $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are stochastically independent if and only if $\mathbf{AVB} = \mathbf{0}$.

Sufficiency is easily proven, but an accessible proof of necessity has long been elusive (see the history by Driscoll and Gundberg, 1986). Happily, such proof now exists, thanks to Read and Driscoll (1988). But one of its more difficult arguments can be simplified by using Vandermonde matrices.

Vandermonde Matrices

A Vandermonde matrix of order n is a square matrix $\mathbf{W}_n = \left\{\lambda_j^{i-1}\right\}$ for i, j $= 1,\cdots,n$; e.g., for n = 3

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}. \tag{1}$$

With the λs all different, $\mathbf{W}_n$ is non-singular: Since

$$|\mathbf{W}_n| = p_n \text{ for } p_n = \prod_{t=2}^{n} \prod_{j=1}^{t-1} (\lambda_t - \lambda_j) , \tag{2}$$

p_n is then clearly non-zero and so $|\mathbf{W}_n| \neq 0$.

A proof of $|\mathbf{W}_n| = p_n$ readily available to statisticians is Theorem 8.12.2 in Graybill (1983). But a simpler and shorter proof comes from noting the form of $\mathbf{W}_n$, e.g., (1), that $|\mathbf{W}_n|$ is homogeneous in the λs, of degree $\sum_{i=1}^{n}(i-1) = \frac{1}{2}n(n-1)$. So also is p_n of (2), the product of all $\frac{1}{2}n(n-1)$ differences $\lambda_j - \lambda_{j'}$ for $j > j'$. Therefore $|\mathbf{W}_n| = \alpha p_n$ for some α; and equating coefficients gives $\alpha = 1$.

The Simplification for Read and Driscoll (1988)

Read and Driscoll's (1988) equation (6) introduces a square matrix Λ of order 2k, the non-singularity of which is salient to their main result. They establish this by viewing determinants as polynomials and considering their roots. But the Vandermonde properties evident in Λ yield an easier development. Since the first k columns of Λ are $c_j = \lambda_j[1\ \lambda_j\ \lambda_j^2 \cdots \lambda_j^{2k-1}]'$ for $j = 1,\cdots,k$, they constitute a matrix $T = U\Delta$ where U is the first k columns of W_{2k} and $\Delta = \text{diag}\{\lambda_1\ \lambda_2 \cdots \lambda_k\}$ is a diagonal matrix of the λs. These λs, in Read and Driscoll (1988), are all different. Therefore U has full column rank, Δ is non-singular and so T has full column rank, k.

Similarly, the last k columns of Λ are Dc_j for $j = 1,\cdots,k$, where $D = \text{diag}\ \{1, 2, \cdots, 2k\}$. Therefore Λ=[T DT] with each of T and DT having full column rank, k. Hence Λ is non-singular if columns of DT are linearly independent of those of T. Assume otherwise. Then

$$TL = DT \tag{3}$$

for some L. Let $T_1L = D_1T$ be the first k rows of (3). Because D is diagonal $D_1T = [D_1^* \quad 0]\begin{bmatrix}T_1\\T_2\end{bmatrix} = D_1^*T_1$ where $D_1^* = \text{diag}\{1,\cdots,k\}$ and T_1 is k x k. Therefore $T_1L = D_1T$ is $T_1L = D_1^*T_1$. Taking determinants gives

$$|L| = |D_1^*| = k!$$

Similarly, the last k rows of (3) give $T_2L = D_2^*T_2$ where T_2 is k x k and $D_2^* = \text{diag}\{k+1,\cdots,2k\}$. Again take determinants, and get

$$|L| = |D_2^*| = (2k)!/k!\ .$$

Since |L| cannot have the two values k! and (2k)!/k!, there is no L such that (3) is true. Therefore Λ has full rank and so is non-singular.

References

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