MIXING FRAMEWORK FOR SOCIAL/SEXUAL BEHAVIOR

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Abstract

In this paper, we continue the numerical and analytical investigation of our recently developed method for incorporating preference into one-sex mixing models with continuously distributed characteristics. Preference is incorporated through a mixing function which determines the proportion of partnerships (sexual or social) formed by individuals of specified sexual/social activity with individuals of all other sexual/social activities. Our method allows for the specification of preference through an arbitrary preference function with well-understood properties. Our illustrations concentrate on the effects that the mean sexual activity of the population, the variance in the preference function (which affects partner selectivity), and the distribution of sexual activity have on the shape of the mixing function.

1. Introduction

Determining who is mixing with whom is an important theoretical question in the study of social dynamics, and it has become a central question in the study of the dynamics of the HIV (human immunodeficiency virus). It is important because a better knowledge of the heterogeneities involved in human interactions is crucial in order to increase our understanding of disease transmission and in the evaluation of preventive measures. We need to determine the mixing function, that is, the function that specifies, for each activity level (new partners per unit time), the fraction of the partners corresponding to all other activity levels. Knowledge of this function will help determine the effects of social/sexual mixing on the relative rates of spread of HIV, and therefore will also help in the construction of a qualitative

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picture of the spread of HIV in highly heterogeneous populations. Furthermore, estimates of the mixing function can be fed into dynamic models to reduce the level of uncertainty of mid- and long-term predictions of HIV and AIDS incidence. Dynamic models not only provide us with a reasonable approach to evaluating the effects of behavioral changes on these predictions, but also with a systematic approach to studying the effects of possible intervention plans on different subpopulations. We also note that the alternative to general theories such as the one provided in this article and recently extended in Busenberg and Castillo-Chavez (1989a, b) is to plug huge numbers of data points into complicated models. This last approach, however, usually violates the constraints inherent in mixing, particularly that of partnership conservation (see condition 2, below).

The absence of a fully-developed theory of mathematical epidemiology for non-randomly mixing populations capable of generating testable hypotheses, and the lack of empirical studies that challenge the reliability of current theories has, in our opinion, seriously limited the scientific study of social/sexual interactions and their relationship to disease dynamics. This is quite surprising, particularly given the example of population genetics (see Crow and Kimura 1970; Nagylaki 1977) where there has always been a strong emphasis on the study of the effects of heterogeneity in gene flow dynamics and hence in the heterogeneities induced by mating systems. A glance at the literature in mathematical epidemiology, however, shows that most models for the spread of infectious diseases have assumed that populations mix at random (proportionate mixing). Nold (1980) modified this assumption and introduced a form of biased mixing. Hethcote and Yorke (1984) also introduced this type of mixing in their important work on gonorrhea and were able to obtain a good fit to data on gonorrhea incidence.

Sattenspiel (1987a), motivated by her background in sociology and anthropology and her interest in the spread of hepatitis A in day care centers in New Mexico, was one of the first researchers to seriously question the use of proportionate mixing in epidemiological models of disease transmission. She emphasized the importance of non-homogeneous mixing both within and between subpopulations while concentrating on the effects of social and nonsocial behaviors on the dynamics of disease. Although her formulation was motivated by a specific disease and a specific spatial distribution, she has always been aware (see Sattenspiel 1987a,1989; Sattenspiel and Simon 1988) that her framework can be applied to other situations, in particular to the study of the spread of sexually-transmitted diseases (STD's). Hyman and Stanley (1988,1989), partially motivated by Sattenspiel's work, have used a general type of biased mixing that fits into the framework of this paper. Their simulations of a model for a single population with heterogeneously distributed sexual activity allow them to contrast the effects of biased mixing vs. random mixing on the initial dynamics of HIV. Their simulations clearly point out the dangers of using proportionate mixing. Jacquez et al. (1988) have also shown the dangers of using proportionate mixing on models for the spread of

HIV/AIDS.

In this paper, we illustrate the role of preference and show, through numerical simulations, the effects of several factors on the shape of the mixing function. These factors include: the mean sexual activity of the population under consideration, the shape of the preference function (more specifically its width or variance), and the distribution of sexual activity of the population. Although the mixing function described in this article does not provide us with the general solution to the mixing problem, it comes very close to being so (see Section 4 and Busenberg and Castillo-Chavez 1989a, b). The main body of this paper consists of two sections. Section 2 introduces our mixing framework and illustrates a method of constructing very general mixing functions. Section 3 provides a specific example for which an approximate expression for the mixing function can be calculated and offers a series of numerical simulations for a variety of like-with-like preference functions. We conclude this paper, in Section 4, with some general remarks regarding our results. We also indicate some future directions for research.

2. One-sex mixing framework

For one-sex models with heterogeneous social/sexual activity, the interactions between individuals of different activity levels are partially described by the mixing function. The *mixing function* $\rho(s, r,t)$ (Blythe and Castillo-Chavez 1989) is such that

$$\int_{r}^{r+\Delta r} \rho(s,u,t)du$$

denotes the fraction of partnerships that a person with activity level s (s new partners per unit time) has with persons with activity levels in the interval $(r,r+\Delta r)$ at time t. It therefore satisfies the following constraints for all s, r, and t:

$$\rho(s, r,t) \geq 0, \tag{1}$$

$$\int_0^{\infty} \rho(s, r,t)dr = 1, \tag{2}$$

$$\rho(s, r,t)sT(s,t) = \rho(r, s,t)rT(r,t), \tag{3}$$

where $T(r,t)\Delta r$ denotes the number of individuals in the population with activities in the interval $(r,r+\Delta r)$. Conditions (1) and (2) are obvious and allow us to interpret $\rho(s,r,t)$ as a probability density function. Condition (3) represents a conservation law as it expresses the principle that the total number of partnerships of s-people with r-people must equal the total number of partnerships of r-people with s-people. Since properties (1)-(3) have to be satisfied for all positive time, in the remainder of this paper we omit the variable t. We further note that there is no loss of generality in looking for solutions of the form $rT(r)B(s,r)$, and that condition (3) implies that B(s,r) = B(r,s).

During the completion of this volume, Busenberg and Castillo-Chavez (1989a,b) found an expression for the general solution (also in the presence of age-structure) of the above functional relationships (1-3). This representation formula is given in terms of the preference function described in Blythe and Castillo-Chavez (1989a) and is motivated by the general family of solutions generated in Blythe and Castillo-Chavez (1989) and by the fact (shown in Busenberg and Castillo-Chavez 1989a) that proportionate mixing is the only separable solution of system (1)-(3). In the framework described in this article, the standard mixing model for proportionate mixing is given by

$$\rho(s, r) = \frac{rT(r)}{\int_0^\infty uT(u)du} \quad . \tag{4}$$

We note that ρ(s,r) is independent of s. Proportionate mixing corresponds to a situation where the fraction of partnerships formed by any individual in the population with individuals of activity r is proportional to the total number of partnerships formed by all r-people. It is obvious that (4) satisfies conditions (1)-(3).

An example of a biased additive solution, that is, a solution for which individuals mix in a biased form but where this bias is additive, is given by the so-called preferred or biased mixing function, namely:

$$\rho(s, r) = (1\text{-}\alpha) \frac{rT(r)}{\int_0^\infty uT(u)du} + \alpha\, \delta(s\text{-}r), \tag{5}$$

where δ(s-r) is a Dirac delta function and the constant α represents the bias or preference towards partners of the same activity level. A discrete version of this model can be found in the work of Jacquez et al. (1988). In general, we note that any convex linear combination of mixing functions is a mixing function, that is, if $\delta_i(s,r)$ are mixing functions (i = 1,$\cdots$,N) and the constants $\alpha_i > 0$ (i = 1,$\cdots$,N) are such that $\Sigma_{i=1}^N \alpha_i = 1$, then $\delta(s,r) = \Sigma_{i=1}^N \alpha_i \delta_i(s,r)$ is also a mixing function.

A general example of a biased multiplicative solution, a solution for which individuals mix in a biased form (as determined by a preference function) but where the bias is multiplicative, is given by the following mixing function:

$$\rho(s, r) = f(r)\left\{ \frac{P(r)P(s)}{\int_0^\infty f(u)P(u)du} + \frac{\phi(s\text{-}r)}{A} \right\}, \tag{6}$$

where

$$f(r) = rT(r), \tag{7}$$

the preference function ϕ is such that ϕ(s-r) = ϕ(r-s) and

$$\int_{-\infty}^{+\infty} \phi(y)dy = 1, \tag{8}$$

and

$$P(x) = 1 - \frac{1}{A}\int_0^\infty f(u)\phi(x - u)du. \tag{9}$$

A is an appropriately chosen constant guaranteeing that P(x) is strictly positive. It is now trivial to show that the mixing function defined by (6) satisfies conditions (1)-(3). We note that the general solution to the mixing constraints, Equations (1)-(3), consists of multiplicative perturbations similar to those described by Equation (6) (for more details see Section 4). In the next section, we proceed to look numerically at the effects of the preference function in the shape of the mixing function. We observe that the mixing function determines the nonlinearity in the dynamics of classical epidemiological models for the spread of sexually transmitted diseases (see Castillo-Chavez et al. 1989).

3. Like-with-like preference mixing functions

This section provides some specific examples of mixing functions of the multiplicative type. Below, a simple example is computed directly, and some numerical examples are provided using three preference functions.

To compute a mixing function directly, we choose a family of preference functions given by a delta sequence. Since $\phi(x)$ is a non-negative, locally integrable function for which

$$\int_{-\infty}^{\infty} \phi(x)dx = 1, \tag{10}$$

then with $\alpha > 0$, the family

$$\phi_\alpha(x) \equiv \frac{1}{\alpha}\, \phi(\tfrac{x}{\alpha}) \tag{11}$$

generates a sequence of delta functions. This family of functions $\{\phi_\alpha$: α a positive parameter$\}$ satisfies the following properties:

$$\int_{-\infty}^{\infty} \phi_\alpha(x)dx = 1, \tag{12}$$

$$\lim_{\alpha \to 0} \int_{|x|>A} \phi_\alpha(x)dx = 0 \quad \text{for each } A > 0, \text{ and} \tag{13}$$

$$\lim_{\alpha \to 0} \int_{|x|<A} \phi_\alpha(x)dx = 1 \quad \text{for each } A > 0. \tag{14}$$

Hence, for small positive α, $\phi_\alpha(x)$ is highly concentrated about $x = 0$ in such a way that the total strength of this distributed source is one. Families of this type are potentially useful for modeling like-with-like preference. It can then be easily shown for very general functions $\pi(x)$ that

$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} \phi_\alpha(x)\pi(x)dx = \pi(0). \tag{15}$$

In this case (and for some of the expressions below), convergence means convergence in distribution, as defined in the study of generalized functions (see Stakgold 1979).

If we substitute a delta sequence of this type into Equation (6), we get

$$\rho(s, r) = f(r)\left\{ \frac{P(r)P(s)}{\int_0^\infty f(u)P(u)du} + \frac{\phi_\alpha(s\text{-}r)}{A}\right\} \tag{16}$$

with

$$P(x) = 1 - \frac{1}{A}\int_0^\infty f(u)\phi_\alpha(x - u)du \quad . \tag{17}$$

If we now let $\alpha \to 0$, we get the following explicit expression for the mixing function:

$$\rho(s, r) = f(r)\left\{ \frac{(1 - \frac{f(r)}{A})(1 - \frac{f(s)}{A})}{\int_0^\infty f(u)(1 - \frac{f(u)}{A})du} + \frac{\delta(s\text{-}r)}{A}\right\}. \tag{18}$$

If we take

$$\phi_\alpha(s\text{-}r) = \begin{cases} 0 & \text{if} \quad |s - r| > \frac{1}{2h} \\ k & \text{if} \quad |s - r| < \frac{1}{2h} \end{cases} , \tag{19}$$

where $\alpha = 1/2h$, then for sufficiently small α, we can compute the following expression for the mixing function:

$$\rho(s, r) \simeq f(r)\left[\frac{(1 - 2\alpha \frac{f(r) + f(s)}{A})}{\int_0^\infty f(u)du - 2\alpha \frac{\int_0^\infty [f(u)]^2 du}{\int_0^\infty f(u)du}} + \frac{f(r)}{h\int_0^\infty f(u)du}\right]. \tag{20}$$

A further approximation (which is justified if α is sufficiently small) is given by

$$\rho(s, r) \simeq \frac{f(r)}{\int_0^\infty f(u)du}\left\{1 - \frac{2\alpha}{\int_0^\infty f(u)du}\left[f(s) + f(r) - \frac{\int_0^\infty [f(u)]^2 du}{\int_0^\infty f(u)du}\right] + z(s,r)\right\}, \tag{21}$$

where

$$z(s,r) = \begin{cases} \dfrac{f(r)}{\int_0^\infty f(u)du} & \text{if} \quad s - \alpha < r < s + \alpha \\ 0 & \text{elsewhere} \end{cases} . \tag{22}$$

We note that the above expressions have only been derived for the purpose of illustrating our mixing framework, and hence, we do not suggest that real populations have such simple functional forms. If, furthermore, we let (as in Blythe and Castillo-Chavez 1989) $T(s) = Lke^{-ks}$, where $T(s)$ is the distribution of sexual activity in the population, L is the total population size and $1/k$ is the mean sexual activity; we find that

$$\rho(s,r) \simeq k^2 re^{-kr}\left[1 - 2k\alpha\{kse^{-ks} + kre^{-kr}\}\right] + \sigma(s,r), \tag{23}$$

where

$$\sigma(s,r) = \begin{cases} k^2 re^{-kr} & \text{if} \quad s - \alpha < r < s + \alpha \\ 0 & \text{elsewhere} \end{cases} . \tag{24}$$

All the following numerical examples use the above distribution of sexual activity. This distribution was used in Blythe and Castillo-Chavez (1989) where it was fitted to the data of Carne and Weller (partners per month of homosexual men attending STD clinics in London) as reported by Hyman and Stanley (1989). Figure 1 illustrates the proportionate mixing case: here ρ is in fact independent of s. Figure 2 illustrates the case where the preference function $\phi(s,r)$ is a narrow rectangular with width 2α; $\alpha = 0.1$. The approximate form is a ridge along s=r superimposed on a proportionate mixing surface. We note that, as the width of the preference function α increases, the shape of the mixing function becomes closer to that of proportionate mixing.

This example can be easily generalized to include preference functions of the form

$$\phi(s,r) = \beta_{m+1}\delta(s - r) + \sum_{i=1}^{m} \beta_i[\delta(s - a_i r) + \delta(sa_i - r)], \tag{25}$$

where the β_i's are properly chosen (for details see Blythe and Castillo-Chavez 1989). This function describes the preference of individuals with activity s for individuals with activities $s/a_1,\ldots,s/a_{2m+1}$. If, for example, we let

(a) $\Lambda \;\; = \sup_x f(x), \qquad \Lambda < \infty$ and

(b) $Q(s) = \int_0^\infty \phi(s,r)f(r)dr,$

then $\rho(s,r)$ is given by the expression

$$\rho(s, r) = f(r)\left\{ \frac{(1 - \frac{Q(r)}{\Lambda})(1 - \frac{Q(s)}{\Lambda})}{\int_0^\infty f(u)(1 - \frac{Q(u)}{\Lambda})du} + \frac{f(r)}{\Lambda}\phi(s,r)\right\} \tag{26}$$

which can almost be "read" off Equation (18).

The following set of simulations use a Gaussian preference function; specifically,

$$\phi(s,r) = \frac{1}{\sigma(2\pi)^2} e^{-\left[\frac{(s-r)^2}{2\sigma^2}\right]}, \tag{27}$$

$$P(x) = 1 - \frac{k}{(2\pi)^2} e^{-\frac{x^2}{2\sigma^2}} e^{\left[\frac{\sigma^2}{4}\{k - \frac{x}{\sigma^2}\}\right]} D_{-1}\left[(k - \frac{x}{\sigma^2})2\sigma\right], \tag{28}$$

where

$$D_{-1}(z) = e^{-\frac{z^2}{4}} \int_0^\infty e^{-zx - \frac{x^2}{2}} x\, dx, \quad \text{(a parabolic function)}. \tag{29}$$

In our set of simulations, we used the parameters $\sigma = 0.25$ (Figure 3), $\sigma = 1.0$ (Figure 4) and $\sigma = 4.0$ (Figure 5). We then used a population with mean sexual activity $1/k = 2$ (new partners per month). In Figure 3, the preference function $\phi(s,r)$ is a Gaussian distribution with mean s=r. Since the variance is small, there is a well defined narrow ridge along the line s=r (compare with Figure 2). The background is essentially proportionate mixing. In Figure 4, we have kept the same preference function but have increased the variance. For intermediate variance, there is a significant deviation from the underlying (approximately) proportionate mixing surface, mainly in the vicinity of the line s=n r, and for r close to zero. In Figure 5, the variance of the Gaussian preference function is now significantly increased. For large variance, the surface is very similar to that of proportionate mixing. We note that significant increases or decreases in the mean sexual activity of the population $1/k$ can have a substantial effect on the shape of the mixing function. For further details, see Blythe and Castillo-Chavez (1989).

4. Conclusion

In this article, we have explored numerically and through some analytical approximations the effects of the preference function in the shape of the mixing function. Since our mixing functions have been generated through Equation 6, we have concentrated on the effects of like-with-like mixing. As pointed out in our earlier simulations (Blythe and Castillo-Chavez 1989), the fidelity of the transformation $\rho(s,r)$ to the underlying preference (neighborhood) function $\phi(s,r)$ depends upon the width of ϕ, the mean activity $1/k$, and (not explored in this paper) the value of s in relation to $1/k$. These results support some of the numerical experiments of Hyman and Stanley (1988,1989) regarding the width of the neighborhood preference function and its relationship to proportionate mixing. If, in addition, we assume that $rT(r)sT(s) = 0$

$\Rightarrow \rho(s,r) = 0$, then the general solution to the mixing problem is given by the following result:

Representation Theorem (Busenberg and Castillo-Chavez 1989a, b):

Let $\phi:\Re^2_+ \rightarrow \Re^+$ be a measurable and jointly symmetric function, and suppose that

$$\int_0^\infty \bar{\rho}(r)\phi(s,r)dr \leq 1 \text{ and}$$

$$\int_0^\infty \bar{\rho}(r)\left\{\int_0^\infty \bar{\rho}(r')\phi(r')dr'\right\}dr < 1.$$

Defining $\rho_1(s)$ by

$$\rho_1(s) = 1 - \int_0^\infty \bar{\rho}(r')\phi(s,r)dr, \tag{30}$$

we arrive at the following fundamental representation formula for a mixing function:

$$\rho(s,r) = \bar{\rho}(r)\left[\frac{\rho_1(s)\rho_1(r)}{\int_0^\infty \bar{\rho}(r)\rho_1(r)dr} + \phi(s,r)\right], \tag{31}$$

where

$$\bar{\rho}(r) = \frac{rT(r)}{\int_0^\infty r\,T(r)dr}\ . \tag{32}$$

The converse also holds; that is, for every mixing function ρ, there exists a preference function ϕ satisfying the hypotheses of the theorem such that ρ is given by (31) with ρ_1 defined by (32). A similar result holds when we allow ρ to be a generalized function (and include the possibility of delta functions or other distributions, see Busenberg and Castillo-Chavez 1989a, b).

This theorem allows us to look at situations other than like-with-like mixing. The mixing function plays a key role in disease dynamics as it determines the incidence rate ("force" of infection). A glance at the literature shows that several authors have used very different expressions for the "force" of infection. Our framework makes it clear that all of them were "correct," in the sense that they assumed (implicitly or explicitly) different types of mixing. This re-interpretation can be accomplished through the determination of all possible mixings, and this is what the above representation theorem is all about. The above framework (as was pointed out to us by Andrea Pugliese) is very general in the sense that it could easily incorporate a spatial distribution. All we have to do is consider the mixing of different types at a specific location (i.e., a localized mixing function) and then superimpose a movement/migration matrix like those of Sattenspiel (1987a,b and 1989) and Sattenspiel and Simon (1988). Although the use of a general framework of this type could be very useful in theoretical considerations, its applicability to specific situations is probably quite limited due to the tremendous number of parameters involved.

Some possible future directions for research involve the analytical and numerical explorations of more general mixings (i.e., other than like-with-like), the expansion of the above framework to include two sexes (see Castillo-Chavez and Busenberg 1989a), and the incorporation of the above framework into dynamic models for the spread of sexually transmitted diseases with and without age-structure (see Busenberg and Castillo-Chavez 1989a,b). We finally note that the above formulation allows an alternative approach to that of Dietz and Hadeler (1988) and hence to the study of models that follow pairs (see Castillo-Chavez and Busenberg 1989).

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Legends for figures

Fig. 1 Proportionate mixing with mean s = 2.0. The mixing function ρ(s,r) is independent of s..

Fig. 2 Graph of the mixing function ρ(s,r) for a narrow rectangular preference function ϕ with mean s = 2 and $\alpha = 0.1$.

Fig. 3 Graph of the mixing function ρ(s,r) for a gaussian preference function ϕ with $\sigma = 0.25$.

Fig. 4 Graph of the mixing function ρ(s,r) for a gaussian preference function ϕ with $\sigma = 1.0$.

Fif. 5 Graph of the mixing function ρ(s,r) for a gaussian preference function ϕ with $\sigma = 4.0$.

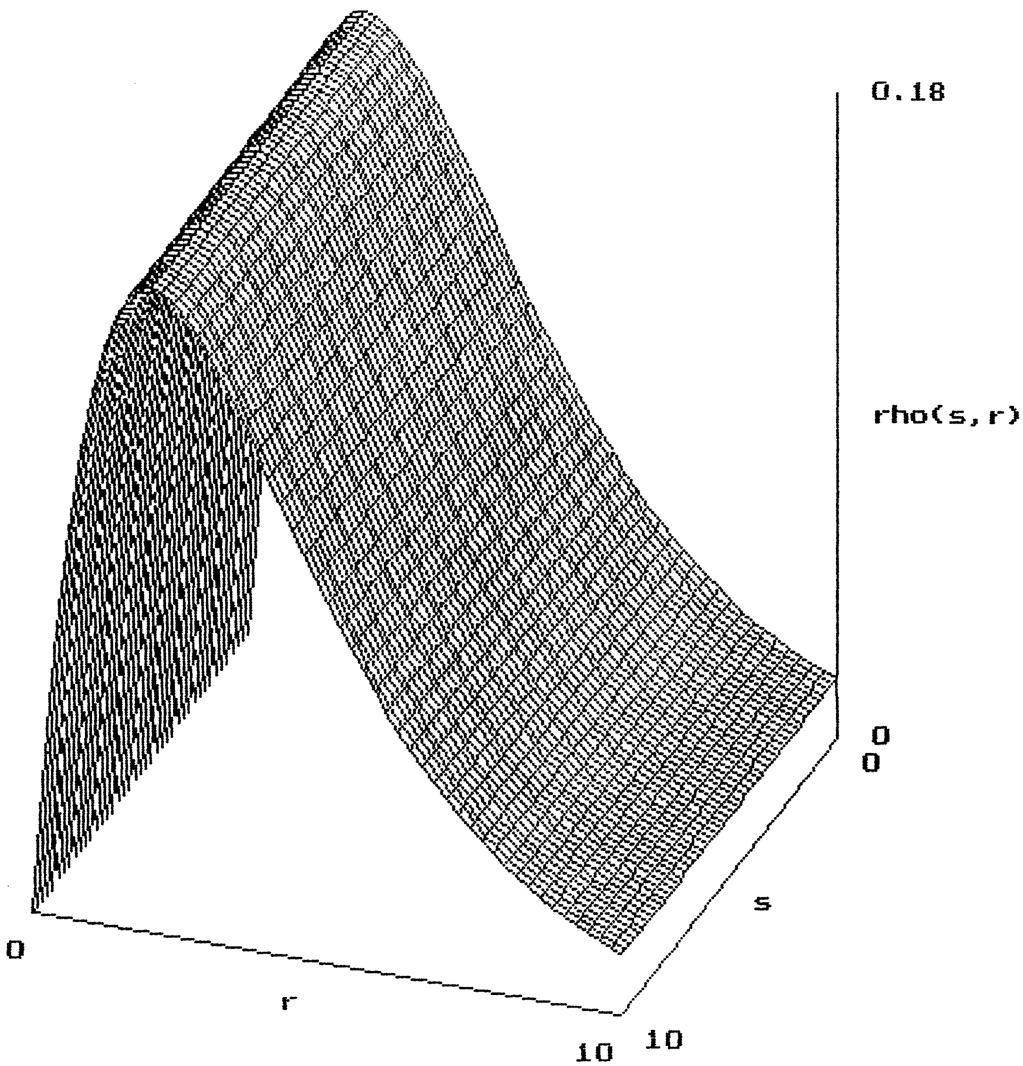

Figure 2

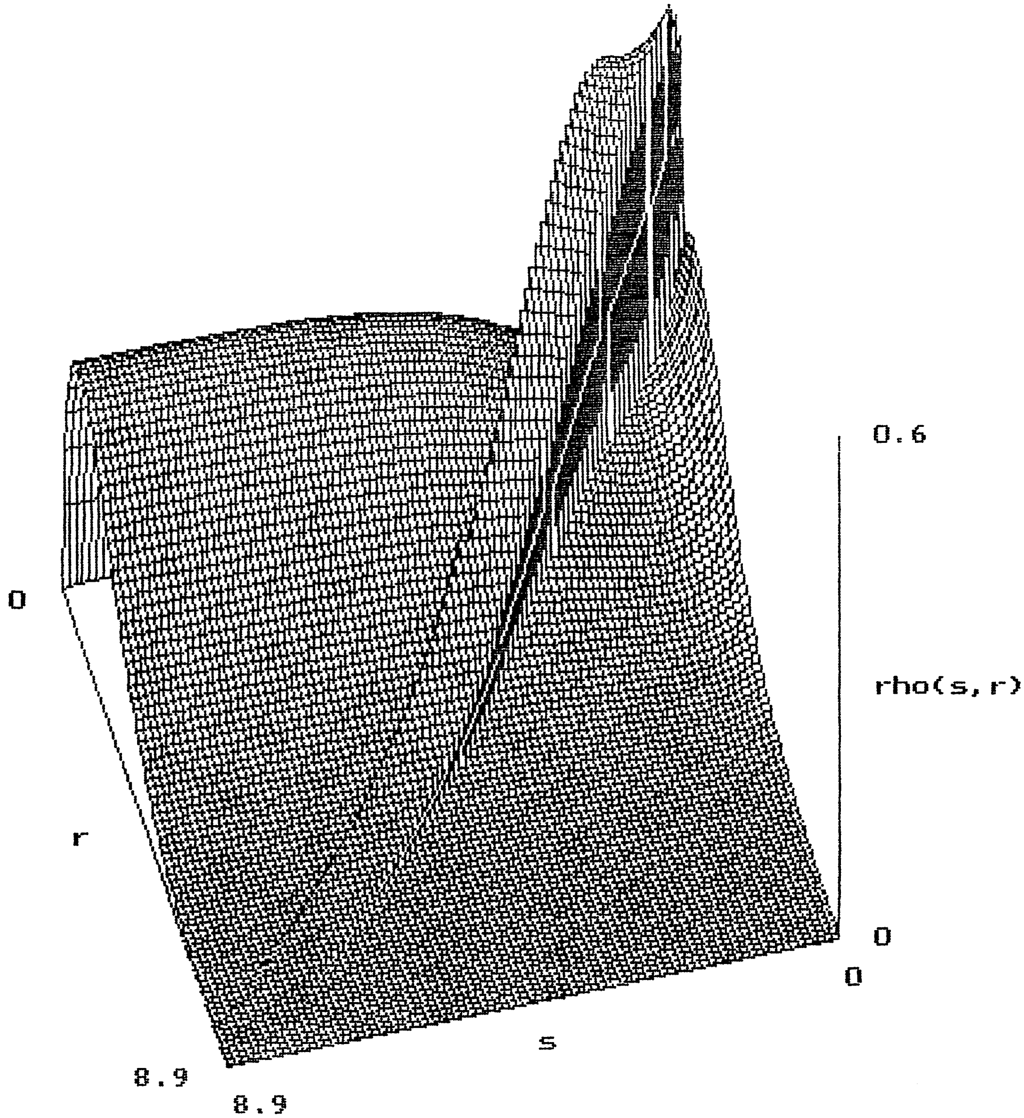

Figure 2

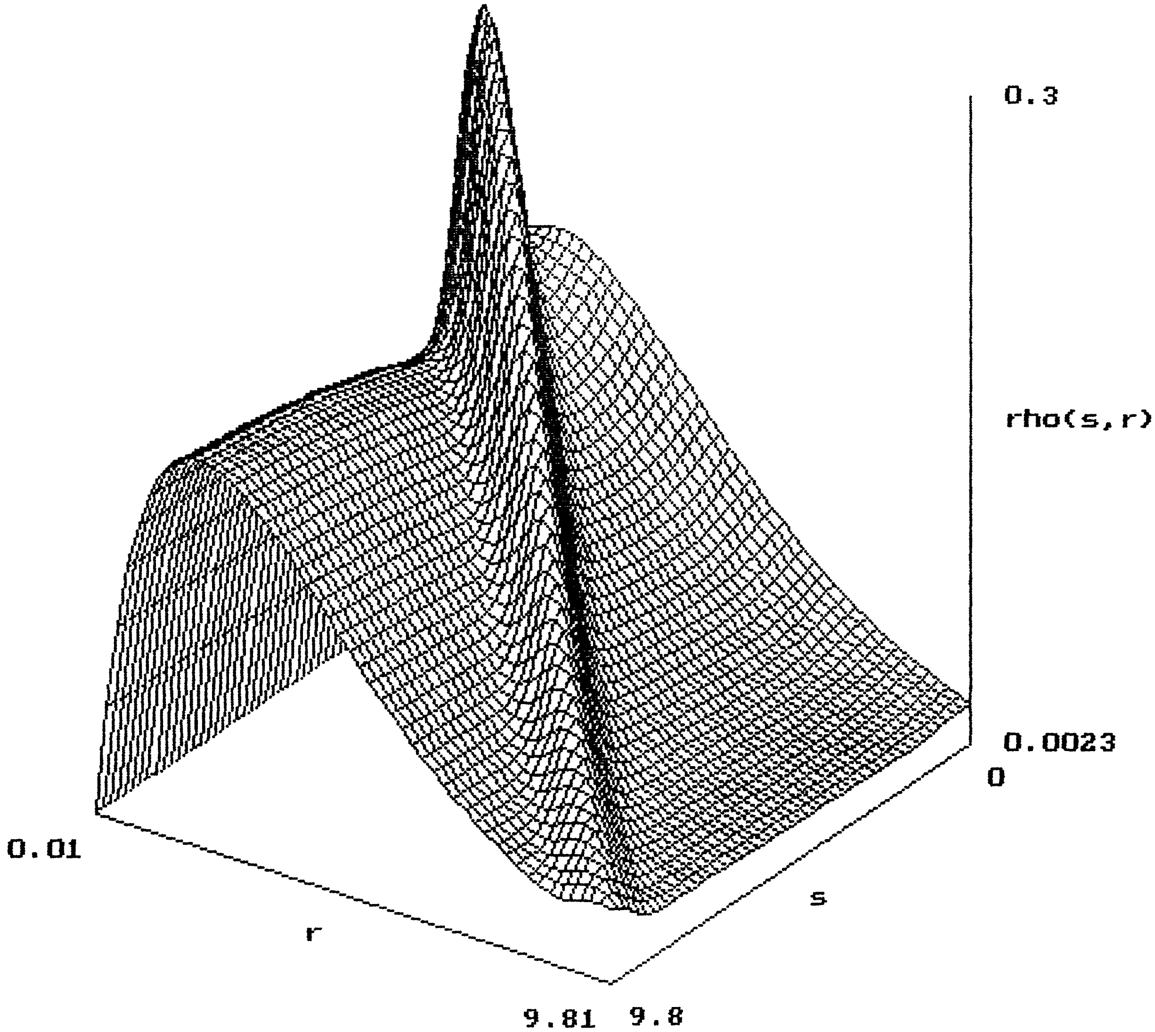

Figure 3

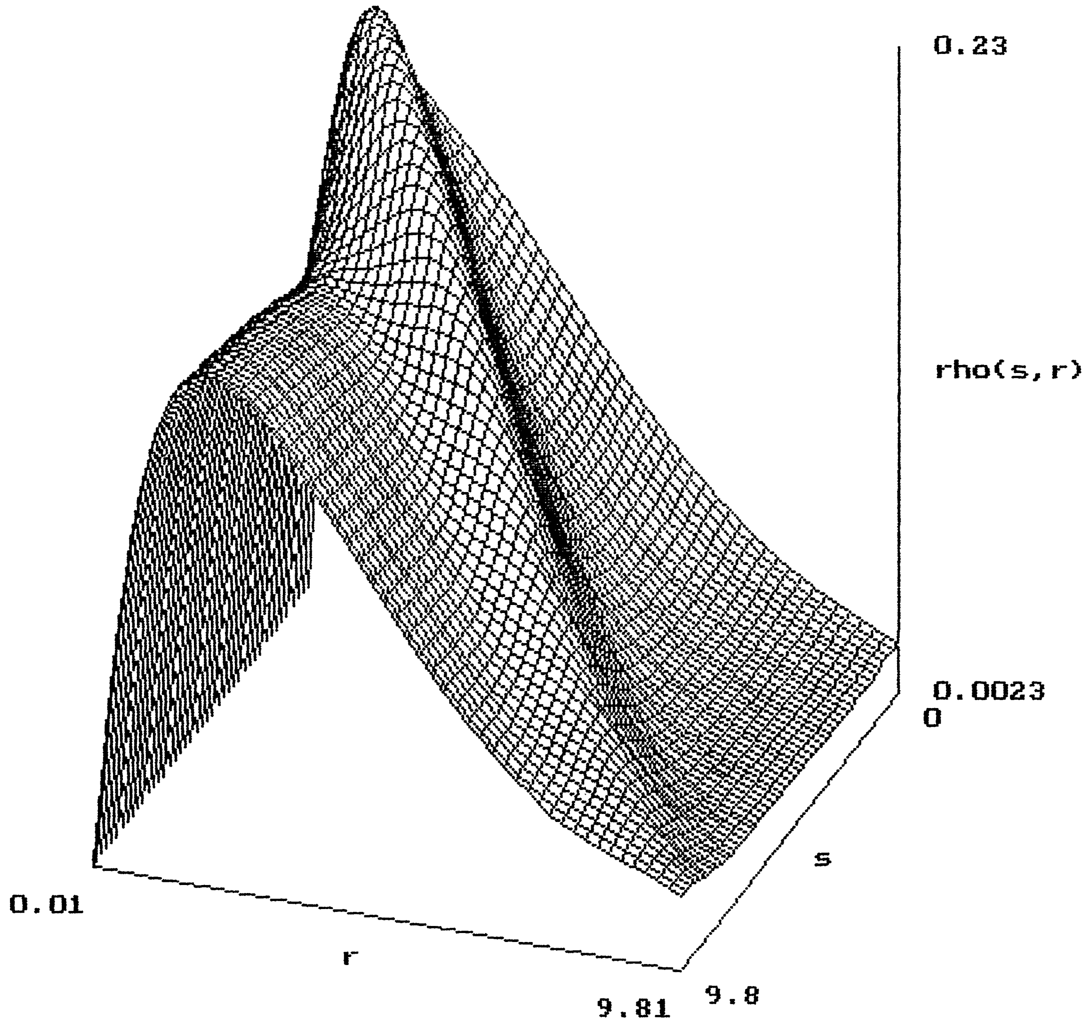

Fig. 4

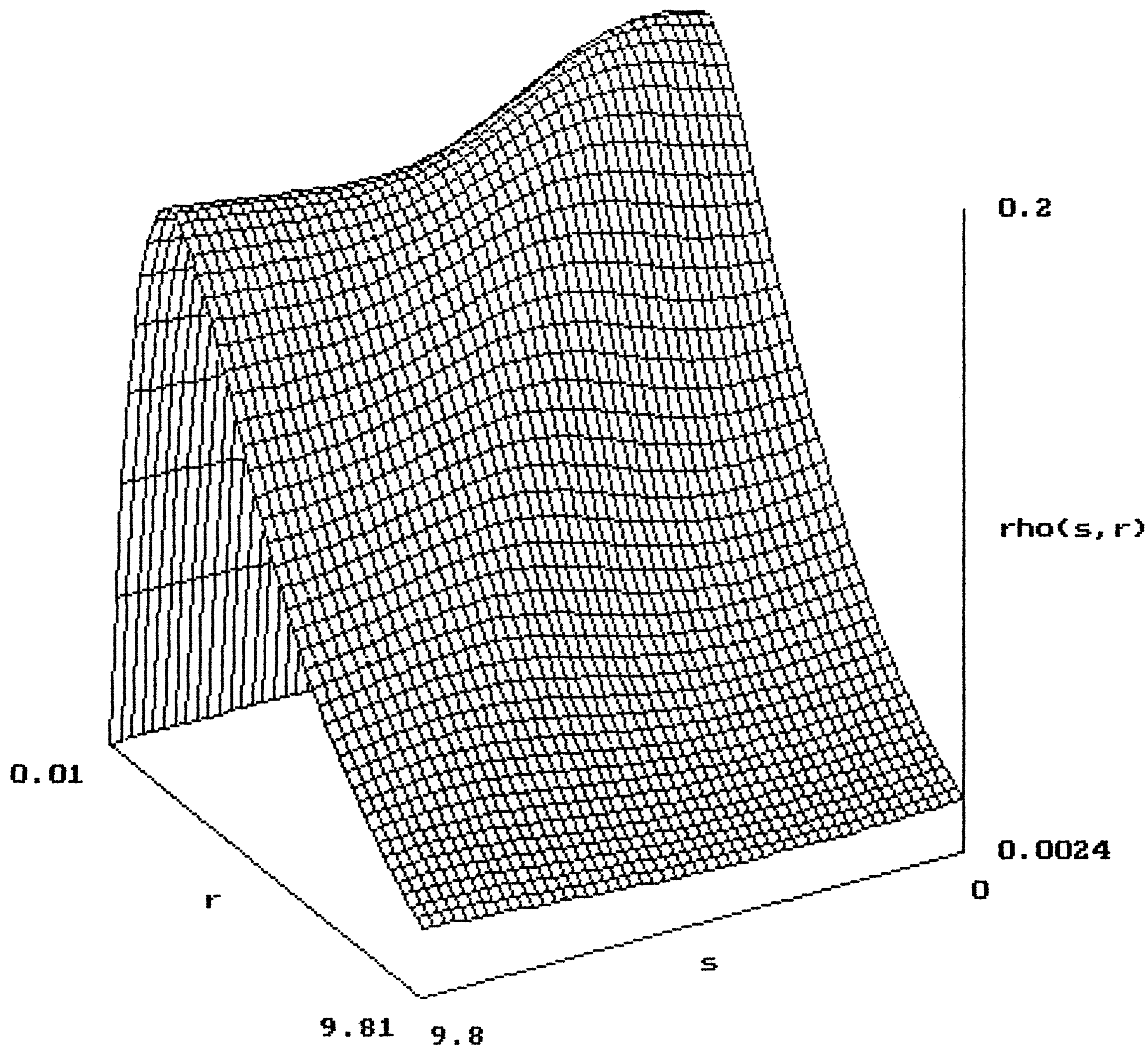

Figure 5