Conditional Confidence of Improved Variance Intervals

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ABSTRACT

The conditional performance of improved confidence intervals for the variance of normal distribution is evaluated, and the relation of the intervals to Bayes procedures is investigated. Using the theory of relevant betting procedures, it is shown that the intervals have extremely good conditional properties. In particular if the reported post-data (conditional) confidence is the frequentist confidence coefficient, there are no positively or negatively biased relevant procedures. This implies that the conditional coverage probabilities (conditional on any subset of the sample space) cannot be uniformly bounded away from the confidence coefficient, showing that the procedures provide coherent inferences.

1. Introduction.

Improvements upon existing confidence procedures are usually in terms of frequentist optimality criteria, such as coverage probability and volume of the confidence set. For the problem of estimating the variance of the normal distribution, confidence intervals that are optimal according to the above criteria, were produced by Cohen (1972), Shorrock (1987) and Goutis and Casella (1989).

However, frequentist confidence procedures are constructed unconditionally, but there is often a temptation to interpret them conditionally. Quite often a post-data (or conditional) confidence statement can be different from a pre-data (or unconditional) one, raising disturbing questions about the unconditional statement. Evaluation of conditional properties of confidence procedures have become of increasing interest. The idea of conditional evaluation was first suggested by Fisher (1956). Buehler (1959) defined desired conditional properties by introducing a betting based theory which was formalized and extended by Robinson (1979).

In Robinson's formalization we observe a random variable X distributed according to a known density $f(x|\theta)$. Given X we construct a confidence region for θ, $C(X)$, and we quote confidence coefficient $1-\alpha(X)$. The confidence procedure consists of the pair $\langle C(X), 1-\alpha(X) \rangle$. After observing X and $C(X)$ we must be willing to accept bets at odds $1-\alpha(X):\alpha(X)$ as to whether $C(X)$ covers the parameter θ. A bettor places bets according to a strategy $k(X)$, which can be any bounded function. If $k(X) > 0$ the bettor places a bet of size $k(X)$ that $\theta \in C(X)$ while if $k(X) < 0$ the bettor places a bet of size $-k(X)$ that $\theta \notin C(X)$. If $k(X) = 0$ then no bets are placed.

The conditional properties of any confidence procedure $\langle C(X), 1-\alpha(X) \rangle$ can be evaluated by determining whether the bettor can find a winning strategy $k(X)$. If $\mathbb{I}$ is the indicator function of a set, the bettor's expected gain can be written as

$$E_\theta\left[\left\{ I_{C(X)}(\theta) - [1-\alpha(X)] \right\} k(X)\right]. \tag{1.1}$$

We are interested in two types of betting strategies: relevant and semirelevant. We define a strategy k(X) to be

(i) *semirelevant* if

$$E_\theta\left[\left\{ I_{C(X)}(\theta) - [1-\alpha(X)] \right\} k(X)\right] \geq 0 \tag{1.2}$$

for all θ with strict inequality for some θ, and

(ii) *relevant* if, for some $\epsilon > 0$

$$E_\theta\left[\left\{ I_{C(X)}(\theta) - [1-\alpha(X)] \right\} k(X)\right] \geq \epsilon\, E_\theta |\, k(X) | \tag{1.3}$$

for all θ.

There is little restriction on the form of the betting strategy k(X). The only major requirement is that k(X) is bounded since, as Robinson (1979) pointed out, unbounded strategies are not of statistical interest. We will be concerned with betting procedures that have a statistical interpretation.

Define a *positively biased* strategy as one in which $k(X) \geq 0$. A positively biased strategy corresponds to always betting that the interval covers θ. Similarly, when $k(X) \leq 0$, we define a *negatively biased* strategy, which corresponds to always betting against coverage. If k(X) is a signed indicator function of a subset S of the sample space $\mathfrak{X}$ then we have a straightforward statistical interpretation. The existence of a positively biased relevant betting procedure implies that

$$P_\theta(\, \theta \in C(X) \mid X \in S\,) \geq 1-\alpha+\epsilon \tag{1.4}$$

for all θ and some positive ϵ while the existence of a negatively biased procedure implies

$$P_\theta(\, \theta \in C(X) \mid X \in S\,) \leq 1-\alpha-\epsilon \tag{1.5}$$

for all θ and some positive ϵ. A confidence procedure that allows positively biased strategies can be thought to be conservative, in the sense that we are reporting a confidence coefficient which is too small. On the other hand, existence of negatively biased relevant betting procedures means that the conditional coverage probability can be bounded strictly below the nominal level.

We would consider a confidence procedure to have good conditional properties if it does not allow relevant betting procedures. Nonexistence of semirelevant strategies seem too strict a requirement, as it would eliminate most common statistical procedures, some of which are quite good. In particular, both the t-interval (Buehler 1959) or Scheffé's simultaneous intervals (Olshen 1973) suffer from semirelevant sets.

Existence of winning betting procedures are closely related with the interval being Bayes. Generally speaking, proper Bayes rules do not allow semirelevant betting procedures and limits of Bayes procedures tend to be free of relevant betting strategies.

Conditional performance of some intervals for the variance of the normal distribution with unknown mean have already been studied. Maatta and Casella (1987) investigated the conditional properties of interval estimators of the form

$$C(s^2) = (\, s^2/b_n,\, s^2/a_n \,) \tag{1.6}$$

where $s^2 = \sum(X_i - \overline{X})^2$ and a_n and b_n satisfy $P\{\, a_n < \chi^2_n < b_n \,\} = 1-\alpha$ where χ^2_n is a chi squared random variable with n degrees of freedom. They found that there are no relevant betting procedures against the procedures $\langle\, C(s^2), 1-\alpha \,\rangle$ and hence they are free from major conditional defects. For the minimum length and shortest unbiased intervals, given by Tate and Klett (1959), the result is actually stronger. When the reported confidence is $1-\alpha$, the latter intervals do not allow negatively biased semirelevant procedures.

Note that intervals of the form (1.6) depend on the data only through the sample variance. More recent work (Cohen (1972), Shorrock (1987) Goutis and Casella (1989)) show that we can improve upon intervals of the form (1.6), allowing a more explicit dependency on $\overline{X}$. It is these improved intervals that we focus on here, exploring their conditional properties. Before proceeding to more details it would be useful to state the distributional assumptions and the notation that will be used.

Let $\underset{\sim}{X} = (\underset{\sim}{X}_1, \underset{\sim}{X}_2)$ be a $(n+p)\times 1$ vector so that $\underset{\sim}{X}_1 = (X_1, X_2, \ldots, X_n)$ and $\underset{\sim}{X}_2 = (X_{n+1}, \ldots, X_{n+p})$. We assume that $\underset{\sim}{X}$ is a random variable from a multivariate normal distribution. The mean of the distribution is the vector $(\underset{\sim}{0}, \underset{\sim}{\mu})$ where $\underset{\sim}{0}$ is of order n and $\underset{\sim}{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)$ is unknown. The covariance matrix is σ^2 times the identity matrix of order $n+p$.

Let $s^2 = \underset{\sim}{X}_1{}'\underset{\sim}{X}_1$, $y^2 = \underset{\sim}{X}_2{}'\underset{\sim}{X}_2$ and $t = y^2/s^2$. By sufficiency the data can be reduced to $(s^2, \underset{\sim}{X}_2)$. With the normality assumption we have that

$$\frac{s^2}{\sigma^2} \sim \chi^2_n \tag{1.7}$$

and

$$\frac{y^2}{\sigma^2} \sim \chi^2_p(\lambda), \quad \lambda = \frac{\underset{\sim}{\mu}'\underset{\sim}{\mu}}{\sigma^2}, \tag{1.8}$$

a central and noncentral chi squared distribution, the latter with noncentrality parameter λ. The noncentral chi squared density with n degrees of freedom and noncentrality parameter λ will be denoted by $f_n(x\,;\,\lambda)$. If $\lambda = 0$ we will omit λ from the notation and $f_n(x)$ becomes the central chi squared density. The respective cumulative distribution functions will be denoted by $F_n(x\,;\,\lambda)$ and $F_n(x)$.

The minimum length confidence interval based on s^2 alone were tabulated by Tate and Klett (1959) and have the form

$$C_U(s^2) = \left(\frac{1}{b_n}s^2, \frac{1}{a_n}s^2\right) \tag{1.9}$$

where a_n and b_n satisfy

$$\int_{a_n}^{b_n} f_n(x)\,dx = 1-\alpha \tag{1.10}$$

and

$$f_{n+4}(a_n) = f_{n+4}(b_n). \tag{1.11}$$

The interval constructed by Shorrock (1987) has fixed length equal to $c_0 = (1/a_n) - (1/b_n)$. By fixing the length he guaranteed that the interval is not longer than $C_U(s^2)$. Furthermore he proved that his interval has uniformly higher coverage probability. The interval depends on the data through s^2 and t and has the form

$$C_S(s^2, t) = (\phi_S(t)s^2, (\phi_S(t)+c_0)s^2) \tag{1.12}$$

where $\phi_S(t)$ satisfies the equation

$$f_{n+4}\left\{\frac{1}{\phi_S(t)}\right\} F_p\left\{\frac{t}{\phi_S(t)}\right\} = f_{n+4}\left\{\frac{1}{\phi_S(t)+c_0}\right\} F_p\left\{\frac{t}{\phi_S(t)+c_0}\right\}. \tag{1.13}$$

Goutis and Casella (1989) constructed intervals which improve upon $C_U(s^2)$ both in terms of coverage probability and length. Their intervals have the form

$$C_\tau(s^2, t) = (\phi_1(t)s^2, \phi_2(t)s^2) \tag{1.14}$$

where the endpoints are defined by the following equations

$$\frac{d\phi_1(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\} F_p\left\{\frac{t}{\phi_1(t)}\right\} = \frac{d\phi_2(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_2(t)}\right\} F_p\left\{\frac{t}{\phi_2(t)}\right\} \tag{1.15}$$

$$f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\} F_p\left\{\frac{\tau(t)}{\phi_1(t)}\right\} = f_{n+4}\left\{\frac{1}{\phi_2(t)}\right\} F_p\left\{\frac{\tau(t)}{\phi_2(t)}\right\} \tag{1.16}$$

with initial conditions

$$\lim_{t\to\infty} \phi_1(t) = \frac{1}{b_n} \text{ and } \lim_{t\to\infty} \phi_2(t) = \frac{1}{a_n}. \tag{1.17}$$

where $\tau(t)$ is a continuous increasing function such that $\tau(t) > t$. In the limiting case when $\tau(t) = t$ the interval $C_\tau(s^2, t)$ reduces to Shorrock's interval.

In this paper we evaluate the conditional properties of the intervals which belong to the family defined by (1.15) and (1.16), with various reported post-data confidence. We show that the confidence procedures are free of major conditional disadvantages. In particular, $\langle C_\tau(s^2, t), 1-\alpha \rangle$ does not admit positively biased relevant betting strategies. We were also able to prove that no negatively biased relevant betting procedures exist against $\langle C_\tau(s^2, t), 1-\alpha \rangle$ in cases of small noncentral degrees of freedom. (We suspect that the result is always true but our proof does not extend to the case $p > 2$.) These two results taken together imply that, if $p \leq 2$, then the coverage probability of $C_\tau(s^2, t)$ cannot be bounded uniformly away from $1-\alpha$ by any statistically meaningful conditioning set, showing the good post-data properties of this procedure.

In order to show the nonexistence of relevant betting strategies, we first investigate the relation of $C_\tau(s^2, t)$ with Bayes procedures. We show that the intervals are generalized Bayes, in the sense that they are highest posterior density regions. However, these priors are difficult to work with, and we use different priors to prove that the procedures do not admit relevant betting strategies. The general interval (1.14) is Generalized Bayes against a prior that we can only define by its moments. We end up investigating the conditional properties of the procedure $\langle C_\tau(s^2, t), \gamma_\tau^N(s^2, t) \rangle$, where $\gamma_\tau^N(s^2, t)$ is the posterior probability of $C_\tau(s^2, t)$ defined by (3.1). We not only show that this procedure is free of conditional defects, but also that the inequality, $\gamma_\tau^N(s^2, t) > 1-\alpha$ holds for all values of s^2 and t. Thus, this procedure has a legitimate frequentist interpretation while allowing a post-data (data dependent) confidence assessment that is free of major conditional defects.

2. Generalized Bayes intervals.

In order to find a prior against which the intervals are generalized Bayes we exploit the group structure of the problem and Zidek's (1969) characterization of Bayes invariant rules. The decision problem remains invariant under the group G of transformations (k, Γ) that map

$$(s^2, \underset{\sim}{X}_2) \rightarrow (k^2 s^2, k\Gamma \underset{\sim}{X}_2)$$

$$(\sigma^2, \underset{\sim}{\mu}) \rightarrow (k^2\sigma^2, k\Gamma\underset{\sim}{\mu})$$

$$\delta(s^2, \underset{\sim}{X}_2) \rightarrow k^2\, \delta(s^2, \underset{\sim}{X}_2)$$

where $k > 0$ is a positive real number and Γ is a $p \times p$ real orthogonal matrix. Under this group the invariant procedures have the form $\phi(t)s^2$. Following Kiefer (1957 example iv) we can write each point of the sample space as $x = (s^2, t)$ so that the group acts upon s^2 and t

is the maximal invariant. Similarly we can decompose the parameter space, writing each parameter value as $\theta = (\sigma^2, \lambda)$. Therefore we can apply Zidek's (1969) Theorem 3.1 and conclude that every Bayes invariant rule is actually a Bayes rule with respect to a prior of the form

$$\Pi(\sigma^2, \lambda) = \frac{1}{\sigma^2}\, \pi(\lambda)\, d\sigma^2\, d\lambda. \tag{2.1}$$

In order for a $1-\alpha$ confidence interval to be a Bayes interval it must minimize the posterior expected length. Thus, in particular, there must exist a prior density such that the interval is highest posterior density region. Using (1.16) and restricting our attention only to the class of invariant Bayes procedures we must seek a function $\pi(\lambda)$ such that

$$\pi(\sigma^2 \mid s^2, t) \propto f_{n+4}\left(\frac{s^2}{\sigma^2}\right) F_p\left(\frac{\tau(t)s^2}{\sigma^2}\right) \tag{2.2}$$

is posterior density with respect to a prior of the form (2.1). Note that

$$f(s^2, t \mid \sigma^2, \lambda) = \frac{s^2}{\sigma^2}\, f_p\left(\frac{s^2 t}{\sigma^2}; \lambda\right) \frac{1}{\sigma^2}\, f_n\left(\frac{s^2}{\sigma^2}\right). \tag{2.3}$$

Now apply Bayes Theorem with the densities $f(s^2, t \mid \sigma^2, \lambda)$ and $\Pi(\sigma^2, \lambda)$, writing the noncentral chi squared density as a weighted sum of central chi squared densities, with weights equal to Poisson probabilities with mean $\lambda/2$. Ignoring constants independent of σ^2 and λ we have

$$\pi(\sigma^2, \lambda \mid s^2, t) \propto \left(\frac{1}{\sigma^2}\right)^3 f_n\left(\frac{s^2}{\sigma^2}\right) \sum_{j=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{(\lambda/2)^j}{j!} f_{p+2j}\left(\frac{s^2 t}{\sigma^2}\right) \pi(\lambda). \tag{2.4}$$

Integrating out λ and using the explicit formula for the chi squared density we get

$$\pi(\sigma^2 \mid s^2, t) \propto f_{n+4}\left(\frac{s^2}{\sigma^2}\right) e^{-\frac{s^2 t}{2\sigma^2}} \sum_{j=0}^{\infty} \frac{\left(\frac{s^2 t}{2\sigma^2}\right)^{\frac{p+2j}{2}}}{\Gamma(\frac{p}{2}+j)\, j!} \int_0^{\infty} e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^j \pi(\lambda)\, d\lambda. \tag{2.5}$$

Using the identity

$$F_p(x) = e^{-\frac{x}{2}} \sum_{j=0}^{\infty} \frac{1}{\Gamma(\frac{p}{2}+j+1)} \left(\frac{x}{2}\right)^{\frac{p+2j}{2}} \tag{2.6}$$

we must look for a $\pi(\lambda)$ satisfying

$$\exp\{-\frac{s^2 t}{2\sigma^2}\} \sum_{j=0}^{\infty} \frac{(\frac{s^2 t}{2\sigma^2})^{\frac{p+2j}{2}}}{\Gamma(\frac{p}{2}+j) j!} \int_0^{\infty} e^{-\frac{\lambda}{2}} (\frac{\lambda}{2})^j \pi(\lambda)\, d\lambda$$

$$= \exp\{-\frac{s^2 \tau(t)}{2\sigma^2}\} \sum_{j=0}^{\infty} \frac{(\frac{s^2 \tau(t)}{2\sigma^2})^{\frac{p+2j}{2}}}{\Gamma(\frac{p}{2}+j+1)}. \qquad (2.7)$$

Bringing $\exp\{-s^2 t/2\sigma^2\}$ to the other side, using the Taylor series expansion of $\exp\{-s^2\{\tau(t)-t\}/2\sigma^2\}$ and collecting terms, we need $\pi(\lambda)$ to satisfy

$$\sum_{j=0}^{\infty} \frac{1}{\Gamma(\frac{p}{2}+j) j!} (\frac{s^2 t}{2\sigma^2})^j \int_0^{\infty} e^{-\frac{\lambda}{2}} (\frac{\lambda}{2})^j \pi(\lambda)\, d\lambda \qquad (2.8)$$

$$= \sum_{j=0}^{\infty} (\frac{s^2 t}{2\sigma^2})^j \left\{ \sum_{i=0}^{j} \frac{1}{i!} (1-\frac{\tau(t)}{t})^i \frac{1}{\Gamma(\frac{p}{2}+j+1-i)} (\frac{\tau(t)}{t})^{\frac{p+2j}{2}-i} \right\}.$$

In general we do not know if there exists such a $\pi(\lambda)$. However in the special case when $\tau(t) = Ct$ for some constant C the RHS of (2.8) is a power series of $x = s^2 t/2\sigma^2$. Defining the coefficients of the series

$$c_j = \frac{1}{\Gamma(\frac{p}{2}+j)\, j!} \int_0^{\infty} e^{-\frac{\lambda}{2}} (\frac{\lambda}{2})^j \pi(\lambda)\, d\lambda, \qquad (2.9)$$

$$d_j = \sum_{i=0}^{j} \frac{(1-C)^i}{i!} \frac{1}{\Gamma(\frac{p}{2}+j+1-i)} C^{\frac{p+2j}{2}-i} \qquad (2.10)$$

we see that c_j and d_j are constants independent of x. Hence equation (2.8) is satisfied if and only if $c_j = d_j$ for every j, that is, if and only if

$$\int_0^{\infty} e^{-\frac{\lambda}{2}} \lambda^j \pi(\lambda)\, d\lambda = 2^j j!\, \Gamma(\frac{p}{2}+j) \sum_{i=0}^{j} \frac{(1-C)^i}{i!\, \Gamma(\frac{p}{2}+j+1-i)} C^{\frac{p+2j}{2}-i}. \qquad (2.11)$$

The last relation defines the moments of the function $e^{-\frac{\lambda}{2}} \pi(\lambda)$. Therefore we can implicitly define $\pi(\lambda)$ by its moments. Note that we do not know if there is a unique $\pi(\lambda)$

satisfying (2.11). However, even if there is more than one prior, they all yield the same posterior. If C = 1 (2.11) simplifies considerably since all but the first terms of the sum in the RHS disappear. For C = 1 the intervals reduce to Shorrock's interval and, of course, by (2.11) we are led to Shorrock's prior:

$$\pi(\sigma^2, \lambda) = \frac{1}{\sigma^2}\, \lambda^{\frac{p}{2}-1} \int_0^\infty e^{-\frac{\lambda u}{2}}\, \frac{u^{\frac{p}{2}-1}}{u+1}\, du. \tag{2.12}$$

Note that it is the same prior used by Brewster and Zidek (1974) for their point estimator of the variance, which is analogous to Shorrock's interval estimator. We can reparameterize $\pi(\sigma^2, \lambda)$ in terms of $\underset{\sim}{\mu}$ and σ^2, in which case it becomes

$$\pi(\sigma^2, \mu_1, \mu_2, \ldots, \mu_p) = \left(\frac{1}{\sigma^2}\right)^{\frac{p}{2}} \int_0^\infty \exp\left(-\frac{u \sum \mu_i^{\,2}}{2\sigma^2}\right)\, \frac{u^{\frac{p}{2}-1}}{u+1}\, du. \tag{2.13}$$

There are several versions of this prior depending on the setup and the parameterization of the problem. With either parameterization, the posterior density is

$$\pi(\sigma^2 \mid s^2, t) = \frac{f_{n+4}\left(\frac{s^2}{\sigma^2}\right) F_p\left(\frac{s^2 t}{\sigma^2}\right)}{\frac{s^2}{n(n+2)}\, F_{p,n}\left(\frac{n}{p}\, t\right)}, \tag{2.14}$$

where $F_{p,n}$ denotes the F cumulative distribution function with p and n degrees of freedom.

3. Conditional Confidence Properties.

As mentioned earlier good conditional properties are associated with the interval being Bayes. By reporting post-data confidence equal to the posterior probability with respect to some prior, we expect not to have major conditional defects. We will use the prior (2.13) and the one given later by (3.9) to examine conditional properties of the interval $C_\tau(s^2, t)$ of (1.15)−(1.16). Recall that the priors are not the ones against which the intervals are highest posterior density regions, but they are tractable.

Let $\gamma_\tau^N(s^2, t)$ be the posterior probability of $C_\tau(s^2, t)$ with respect to (2.13), that is,

$$\gamma_\tau^N(s^2, t) = \int_{\phi_1(t)s^2}^{\phi_2(t)s^2} \frac{f_{n+4}(\frac{s^2}{\sigma^2})\, F_p(\frac{s^2 t}{\sigma^2})}{\frac{s^2}{n(n+2)}\, F_{p,n}(\frac{n}{p}t)}\, d\sigma^2. \tag{3.1}$$

Note that $C_\tau(s^2, t)$, $\gamma_\tau^N(s^2, t)$ and $\pi(\sigma^2 \,|\, s^2, t)$ are functions of the sample through s^2 and t. We will interchangeably denote them by $C_\tau(s^2, \underset{\sim}{X}_2)$, $\gamma_\tau^N(s^2, \underset{\sim}{X}_2)$ and $\pi(\sigma^2 \,|\, s^2, \underset{\sim}{X}_2)$. Now we can prove the following theorem.

Theorem 3.1. Let $C_\tau(s^2, t)$ be the confidence interval $(\phi_1(t)s^2, \phi_2(t)s^2)$ where $\phi_1(t)$ and $\phi_2(t)$ are defined by (1.15) and (1.16) and $\gamma_\tau^N(s^2, t)$ given by (3.1). There exist no relevant betting procedures for the confidence procedure $\langle C_\tau(s^2, t), \gamma_\tau^N(s^2, t) \rangle$.

Proof. We will use proof by contradiction. Suppose that a relevant betting procedure exists. Then there is a betting strategy $k(s^2, \underset{\sim}{X}_2)$ and an $\epsilon > 0$ such that

$$E_\theta\left[\left\{ I_{C_\tau(s^2,\underset{\sim}{X}_2)}(\sigma^2) - \gamma_\tau^N(s^2,\underset{\sim}{X}_2) \right\} k(s^2,\underset{\sim}{X}_2)\right] > \epsilon\, E_\theta |\, k(s^2,\underset{\sim}{X}_2) \,| \tag{3.2}$$

for all $\theta = (\sigma^2, \underset{\sim}{\mu})$. Multiply both sides of (3.2) by the prior distribution $\pi(\sigma^2, \underset{\sim}{\mu})$ given by (3.2) and integrate with respect to $\sigma^2, \mu_1, \mu_2, \ldots, \mu_p$. If $k(s^2, \underset{\sim}{X}_2)$ is a relevant betting procedure then

$$\int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty E_\theta\left[\left\{ I_{C_\tau(s^2,\underset{\sim}{X}_2)}(\sigma^2) - \gamma_\tau^N(s^2,\underset{\sim}{X}_2) \right\} k(s^2,\underset{\sim}{X}_2)\right] \pi(\sigma^2, \underset{\sim}{\mu}) \prod_{i=1}^p d\mu_i d\sigma^2$$

$$> \epsilon \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty E_\theta \,|\, k(s^2,\underset{\sim}{X}_2) \,|\, \pi(\sigma^2, \underset{\sim}{\mu}) \prod_{i=1}^p d\mu_i d\sigma^2. \tag{3.3}$$

Working as in Theorem 2.1 of Maatta and Casella (1987) we are able to interchange the order of integration and, using (3.1), show that the LHS is identically equal to zero which produces the needed contradiction. □

Theorem 3.1 establishes that if we report post-data confidence $\gamma_\tau^N(s^2, t)$, the procedures are free from conditional defects. However, if we are interested in making preexperimental confidence statements our confidence coefficient should not depend on the data. In that case we would quote the pre-data confidence coefficient which is the minimum unconditional coverage probability. We would like our confidence intervals to have nice conditional properties with respect to the unconditional confidence coefficient $1-\alpha$. The following theorem shows that the procedure $\langle\, C_\tau(s^2, t), 1-\alpha \,\rangle$ has acceptable conditional properties for small values of the noncentral degrees of freedom p.

Theorem 3.2. Let $C_\tau(s^2, t)$ be the confidence interval $(\phi_1(t)s^2, \phi_2(t)s^2)$ where $\phi_1(t)$ and $\phi_2(t)$ are defined by (1.15) and (1.16) and $p \le 2$. Then no negatively biased relevant betting procedures exist for the confidence procedure $\langle\, C_\tau(s^2, t), 1-\alpha \,\rangle$.

Proof. Suppose that there is a negatively biased relevant betting procedure. That is, there is a negative function $k(s^2, \underset{\sim}{X}_2)$ such that

$$E_\theta\left[\left\{ I_{C_\tau(s^2,\underset{\sim}{X}_2)}(\sigma^2) - (1-\alpha) \right\} k(s^2,\underset{\sim}{X}_2)\right] > \epsilon\, E_\theta |\, k(s^2,\underset{\sim}{X}_2)\,| \tag{3.4}$$

for all $\theta = (\sigma^2, \underset{\sim}{\mu})$. As in Theorem 3.1 we can multiply both sides by the prior distribution $\pi(\sigma^2, \underset{\sim}{\mu})$ given by (2.13) and integrate with respect to $\sigma^2, \mu_1, \mu_2, \ldots, \mu_p$. If $k(s^2, \underset{\sim}{X}_2)$ is a negatively biased relevant betting procedure then

$$\int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty E_\theta\left[\left\{ I_{C_\tau(s^2,\underset{\sim}{X}_2)}(\sigma^2) - (1-\alpha)\right\} k(s^2,\underset{\sim}{X}_2)\right] \pi(\sigma^2,\underset{\sim}{\mu}) \prod_{i=1}^p d\mu_i d\sigma^2$$

$$> \epsilon \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty E_\theta\, |\, k(s^2,\underset{\sim}{X}_2)|\, \pi(\sigma^2,\underset{\sim}{\mu}) \prod_{i=1}^p d\mu_i d\sigma^2. \tag{3.5}$$

Working as before we can show that the LHS of (3.5) can be written as

$$\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \int_0^\infty \left\{ \gamma_\tau^N(s^2, \underset{\sim}{X}_2) - (1-\alpha) \right\} k(s^2,\underset{\sim}{X}_2)\, m(s^2,\underset{\sim}{X}_2)\, ds^2 \prod_{i=n+1}^{n+p} dX_i. \tag{3.6}$$

If we can show that $\gamma_\tau^N(s^2, t) \ge 1-\alpha$ for every s^2, t then the last expression will be negative since $k(s^2,\underset{\sim}{X}_2) \le 0$. Since the RHS of (3.5) is positive, we will have the desired contradiction and conclude that there is no negatively biased relevant betting procedure against

$\langle\, C_\tau(s^2, t), 1-\alpha \,\rangle$. Lemmas A.1 – A.4 of the appendix establish the fact $\gamma_\tau^N(s^2, t) \geq 1-\alpha$.

The motivation of Lemmas A.1 – A.4 comes from the construction of the interval itself, that the construction of $C_\tau(s^2, t)$ is a limit of intervals based on a finite number of cutoff points. We show that in each step the posterior probability of the resulting interval has the desired bound. The expression of $\gamma_\tau^N(s^2, t)$, as a function of t, is rather intractable because, in (3.1), t appears in both the limits of integration and the integrand. The lemmas show that $\gamma_\tau^N(s^2, t)$ is bounded below by the posterior probability of the usual interval $C_U(s^2)$. This posterior probability, denoted by $\gamma_U(s^2, t)$, is given by

$$\gamma_U(s^2, t) = \int_{\frac{1}{b_n}s^2}^{\frac{1}{a_n}s^2} \frac{f_{n+4}(\frac{s^2}{\sigma^2})\, F_p(\frac{s^2 t}{\sigma^2})}{\frac{s^2}{n(n+2)}\, F_{p,n}(\frac{n}{p}t)}\, d\sigma^2. \tag{3.7}$$

Next, the function $\gamma_U(s^2, t)$ is shown to be bounded below by $1-\alpha$. Therefore, we establish $\gamma_\tau^N(s^2, t) \geq \gamma_U(s^2, t) > 1-\alpha$, proving the theorem. □

The proof of Lemma A.4 fails for $p > 2$ because we cannot show inequality (A.21) to be true and because

$$\lim_{p\to\infty} \int_{a_n}^{b_n} f_{n+p}(x)\, dx = 0, \tag{3.8}$$

which implies that, for sufficiently large p, the limit cannot be greater than $1-\alpha$. Hence, we know that the posterior probability of the interval $C_U(s^2)$ is below $1-\alpha$ for small t and large p. However, we believe that Theorem 3.2 is always true because for small t and large p the endpoints of $C_\tau(s^2, t)$ are far from the endpoints of $C_U(s^2)$ and we expect their respective posterior probabilities to differ substantially.

Remark. The interval constructed by Cohen has also posterior probability greater than the usual minimum length interval. Cohen's interval, by construction, has the same length as the usual interval, but it maximizes the area under the curve $f_{n+4}(1/x)\, F_p(K/x)$. Hence Theorem 3.2 applies for that interval, too.

Now we proceed to show that the procedure $\langle C_\tau(s^2, t),\ 1-\alpha \rangle$ does not admit positively biased relevant betting strategies. The technique of proof is similar to the one used for negatively biased relevant betting strategies. We will use the prior

$$\pi(\sigma^2, \mu_1, \mu_2, \ldots, \mu_p) = \frac{1}{\sigma^2}. \tag{3.9}$$

and we will show that the posterior probability with respect to the prior is bounded above by $1-\alpha$.

The posterior density with respect to (3.9), after integrating out $\underset{\sim}{\mu}$, is

$$\pi(\sigma^2 \mid s^2, t) = \frac{n(n+2)}{s^2} f_{n+4}\left(\frac{s^2}{\sigma^2}\right). \tag{3.10}$$

Let $\gamma_\tau^{\mathrm{P}}(s^2, t)$, also denoted by $\gamma_\tau^{\mathrm{P}}(s^2, \underset{\sim}{X}_2)$, be the respective posterior probability:

$$\gamma_\tau^{\mathrm{P}}(s^2, t) = \int_{\phi_1(t)s^2}^{\phi_2(t)s^2} \frac{n(n+2)}{s^2} f_{n+4}\left(\frac{s^2}{\sigma^2}\right) d\sigma^2. \tag{3.11}$$

The following theorem is analogous to Theorem 3.1.

Theorem 3.3. Let $C_\tau(s^2, t)$ be the confidence interval $(\phi_1(t)s^2, \phi_2(t)s^2)$ where $\phi_1(t)$ and $\phi_2(t)$ are defined by (1.15) and (1.16) and $\gamma_\tau^{\mathrm{P}}(s^2, t)$ given by (3.11). There exist no relevant betting procedures for the confidence procedure $\langle C_\tau(s^2, t), \gamma_\tau^{\mathrm{P}}(s^2, t) \rangle$.

Proof. Same as Theorem 3.1. □

If we report post-data confidence $\gamma_\tau^{\mathrm{P}}(s^2, t)$ then the procedure is free from conditional defects. If we establish that the confidence procedure $\langle C_\tau(s^2, t),\ 1-\alpha \rangle$ admits no positively biased betting strategies then, combined with the results of Theorem 3.2, we will have shown the confidence interval $C_\tau(s^2, t)$ to have desirable conditional properties. The following theorem establishes this.

Theorem 3.4. Let $C_\tau(s^2, t)$ be the confidence interval $(\phi_1(t)s^2, \phi_2(t)s^2)$ where $\phi_1(t)$ and $\phi_2(t)$ are defined by (1.15) and (1.16). Then no positively biased relevant betting procedures exist for the confidence procedure $\langle\, C_\tau(s^2, t),\, 1-\alpha \,\rangle$.

Proof. Working as in the proof of Theorem 3.2 we will reach a contradiction. Suppose that there is a positive function $k(s^2, \underset{\sim}{X}_2)$ such that

$$E_\theta\left[\left\{ I_{C_\tau(s^2, \underset{\sim}{X}_2)}(\sigma^2) - (1-\alpha) \right\} k(s^2, \underset{\sim}{X}_2)\right] > \epsilon\, E_\theta |\, k(s^2, \underset{\sim}{X}_2)\, | \tag{3.12}$$

for all $\theta = (\sigma^2, \underset{\sim}{\mu})$. Multiply both sides by the prior distribution $\pi(\sigma^2, \underset{\sim}{\mu})$ given by (3.9) and integrate with respect to $\sigma^2, \mu_1, \mu_2, \ldots, \mu_p$. Using (3.11), the defining equation of $\gamma_\tau^P(s^2, t)$, we get

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ \gamma_\tau^P(s^2, \underset{\sim}{X}_2) - (1-\alpha) \right\} k(s^2, \underset{\sim}{X}_2) m(s^2, \underset{\sim}{X}_2)\, ds^2 \prod_{i=n+1}^{n+p} dX_i$$

$$> \epsilon \int_0^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E_\theta\, |\, k(s^2, \underset{\sim}{X}_2)\, |\, \pi(\sigma^2, \underset{\sim}{\mu}) \prod_{i=1}^{p} d\mu_i\, d\sigma^2. \tag{3.13}$$

If we can show that $\gamma_\tau^P(s^2, t) \le 1-\alpha$ for every s^2, t then we will have a contradiction. Making the transformation $x = s^2/\sigma^2$ in (3.11) we have

$$\gamma_\tau^P(s^2, t) = \int_{\frac{1}{\phi_2(t)}}^{\frac{1}{\phi_1(t)}} f_n(x)\, dx. \tag{3.14}$$

Now if we differentiate with respect to t, applying Leibniz' rule we get

$$\frac{d\,\gamma_\tau^P(s^2, t)}{dt} = \frac{d\phi_2(t)}{dt} \left(\frac{1}{\phi_2(t)}\right)^2 f_n\!\left(\frac{1}{\phi_2(t)}\right) - \frac{d\phi_1(t)}{dt} \left(\frac{1}{\phi_1(t)}\right)^2 f_n\!\left(\frac{1}{\phi_1(t)}\right). \tag{3.15}$$

Using equation (1.15) we can see that

$$\frac{d\phi_2(t)}{dt} f_{n+4}\!\left(\frac{1}{\phi_2(t)}\right) > \frac{d\phi_1(t)}{dt} f_{n+4}\!\left(\frac{1}{\phi_1(t)}\right) \tag{3.16}$$

since $F_p(t/\phi_2(t)) < F_p(t/\phi_1(t))$. Therefore the derivative is positive which implies that the function $\gamma_\tau^P(s^2, t)$ is increasing in t. But $\lim_{t\to\infty} \phi_1(t) = 1/b_n$ and $\lim_{t\to\infty} \phi_2(t) = 1/a_n$ so

$$\lim_{t \to \infty} \gamma_\tau^{P}(s^2, t) = \int_{a_n}^{b_n} f_n(x)\, dx = 1-\alpha. \tag{3.17}$$

For every t we have $\gamma_\tau^{P}(s^2, t) \leq 1-\alpha$ which completes the proof. □

4. Discussion.

Acceptable conditional confidence statements can be attached to the interval $C_\tau(s^2, t)$. In particular, the confidence procedure $\langle C_\tau(s^2, t), 1-\alpha \rangle$ allows no positively or negatively biased winning betting strategies. Note that $1-\alpha$ is the pre-data, frequentist confidence coefficient, so the procedure $\langle C_\tau(s^2, t), 1-\alpha \rangle$ is acceptable both conditionally and unconditionally.

For the confidence procedures $\langle C_\tau(s^2, t), \gamma_\tau^{N}(s^2, t) \rangle$ and $\langle C_\tau(s^2, t), \gamma_\tau^{P}(s^2, t) \rangle$ we saw that there is no relevant betting strategy. It is interesting to note that although $\gamma_\tau^{N}(s^2, t) > 1-\alpha > \gamma_\tau^{P}(s^2, t)$ and the frequentist confidence coefficient is $1-\alpha$, the procedures $\langle C_\tau(s^2, t), \gamma_\tau^{N}(s^2, t) \rangle$ or $\langle C_\tau(s^2, t), \gamma_\tau^{P}(s^2, t) \rangle$ are conditionally preferable to $\langle C_\tau(s^2, t), 1-\alpha \rangle$. The first two are free from all relevant betting procedures, but by quoting post-data confidence $1-\alpha$ we are guaranteed that there are no positively or negatively biased winning strategies, which is a weaker condition. However, we have not established any formal frequentist optimality of $\gamma_\tau^{N}(s^2, t)$ and $\gamma_\tau^{P}(s^2, t)$, so the use of one of these data-dependent confidence reports is only justified on conditional grounds.

We actually know that $\gamma_\tau^{P}(s^2, t)$ is strictly less than the coverage probability, so, although we cannot win by betting against $\langle C_\tau(s^2, t), \gamma_\tau^{P}(s^2, t) \rangle$, $\gamma_\tau^{P}(s^2, t)$ is too conservative. Its use has really been to establish the conditional properties of $\langle C_\tau(s^2, t), 1-\alpha \rangle$, and is not a recommended procedure. On the other hand we know that the coverage probability of $C_\tau(s^2, t)$ is greater than $1-\alpha$, with strict inequality for all values of λ except $\lambda = 0$ (see Goutis and Casella (1989) for details). In that sense $1-\alpha$, being a lower bound, is conservative and the procedure $\langle C_\tau(s^2, t), \gamma_\tau^N(s^2, t) \rangle$ is quite attractive. It is free from all conditional defects, and allows a confidence report, $\gamma_\tau^N(s^2, t)$, that is greater than $1-\alpha$ when we are sure that the coverage probability is greater than $1-\alpha$. This should give it some acceptability to frequentists.

Another interesting question, that bears on frequentist acceptability, is to find a function $\gamma(s^2, t)$, which, in some sense, would be closer to the coverage probability. One way of measuring this would be, for example, to try to estimate the indicator function $\mathbb{I}_{C_\tau(s^2, \underset{\sim}{X}_2)}(\sigma^2)$ (Robinson 1979). Then, using squared error loss as a measure of distance, we would seek a $\gamma(s^2, t)$ such that

$$E_\theta\left[\left\{\mathbb{I}_{C_\tau(s^2,t)}(\sigma^2) - \gamma(s^2,t)\right\}^2\right] \le E_\theta\left[\left\{\mathbb{I}_{C_\tau(s^2,t)}(\sigma^2) - (1-\alpha)\right\}^2\right] \tag{4.1}$$

with strict inequality for some θ. Establishing such an inequality, however, seems to be quite difficult.

Note that the Theorems of Section 3 are also valid for Shorrock's interval, which can be derived in two ways. One way is to fix the difference of endpoints and maximize the area under the posterior density given by (2.14). Such a derivation would directly show the posterior probability $\gamma_\tau^N(s^2, t)$ is greater than $1-\alpha$, and, consequently, there are no negatively biased relevant betting strategies against the confidence procedure $\langle C_S(s^2, t), 1-\alpha \rangle$. Another way to obtain $C_S(s^2, t)$ is by using equations (1.15) and (1.16) and taking the limiting case $\tau(t) = t$. The requirement $\tau(t) > t$ is used to guarantee that the intervals $C_\tau(s^2, t)$ have length less than c_0, but the proofs of the Theorems 3.1 – 3.4 use essentially only equation (1.15) and the construction of the intervals by successively shifting the endpoints towards zero. Hence the conditional performance of $C_S(s^2, t)$ is the same as the performance of $C_\tau(s^2, t)$.

Finally, we note that since the coverage probability of the intervals considered here is greater than the confidence coefficient $1-\alpha$, we immediately have a semirelevant betting strategy against a procedure like $\langle\, C_\tau(s^2, t),\ 1-\alpha \,\rangle$. The strategy is to always bet for coverage. This existence of a positively biased semirelevant strategy should not be that troubling, however, as it merely shows us here that we are too conservative in our confidence assessment of the shorter interval. Therefore, a procedure of the form $\langle\, C_\tau(s^2, t),\ \gamma_\tau^N(s^2, t) \,\rangle$ allows a more realistic confidence assessment, and should be considered an acceptable improvement over the usual frequentist intervals.

APPENDIX

Lemma A.1. Let the interval $I_1\,(s^2, t, K)$ be defined by

$$I_1\,(s^2, t, K) = \begin{cases} (\frac{1}{b_n}s^2, \frac{1}{a_n}s^2) & \text{if} \quad t > K \\ (\phi_1(K)s^2, \phi_2(K)s^2) & \text{if} \quad t \le K \end{cases}$$

where $\phi_1(K)$ and $\phi_2(K)$ are determined from the following equations:

$$\int_{\phi_1(K)}^{\phi_2(K)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K}{x})\, dx = \int_{\frac{1}{b_n}}^{\frac{1}{a_n}} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K}{x})\, dx \tag{A.1}$$

and

$$f_{n+4}\Big\{\frac{1}{\phi_1(K)}\Big\}\, F_p\Big\{\frac{\tau(K)}{\phi_1(K)}\Big\} = f_{n+4}\Big\{\frac{1}{\phi_2(K)}\Big\}\, F_p\Big\{\frac{\tau(K)}{\phi_2(K)}\Big\}. \tag{A.2}$$

Then the posterior probability of $I_1\,(s^2, t, K)$ with respect to the prior (2.13) is greater than the posterior probability of the interval $C_U(s^2)$ for every t. The inequality is strict when $t < K$.

Proof. The intervals $C_U(s^2)$ and $I_1\,(s^2, t, K)$ are different only when $t \le K$. So we have to prove that

$$\int_{\phi_1(K)s^2}^{\phi_2(K)s^2} \pi(\sigma^2 \mid s^2, t)\, d\sigma^2 \ge \int_{\frac{1}{b_n}s^2}^{\frac{1}{a_n}s^2} \pi(\sigma^2 \mid s^2, t)\, d\sigma^2 \tag{A.3}$$

for $t \le K$. Substituting the expression for $\pi(\sigma^2 \mid s^2, t)$ from (2.14) and letting x be equal to σ^2/s^2 equation (A.3) is equivalent to

$$\int_{\phi_1(K)}^{\phi_2(K)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{t}{x})\, dx \ge \int_{\frac{1}{b_n}}^{\frac{1}{a_n}} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{t}{x})\, dx. \tag{A.4}$$

The proof of (A.4) follows the lines of the proof of Theorem 2.1 of Goutis and Casella (1989). For fixed γ and t, define for each w the function $g_{\gamma,t}(w)$ as the solution to:

$$\gamma = \int_{w}^{g_{\gamma,t}(w)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{t}{x})\, dx. \tag{A.5}$$

Let

$$\gamma_1 = \int_{\frac{1}{b_n}}^{\frac{1}{a_n}} f_{n+4}(\,\tfrac{1}{x}\,)\, F_p(\,\tfrac{K}{x}\,)\, dx \tag{A.6}$$

and

$$\gamma_2 = \int_{\frac{1}{b_n}}^{\frac{1}{a_n}} f_{n+4}(\,\tfrac{1}{x}\,)\, F_p(\,\tfrac{t}{x}\,)\, dx. \tag{A.7}$$

We establish (A.4) by showing $\phi_2 > g_{\gamma_2,t}(\phi_1)$.

Define $G(w) = g_{\gamma_1,K}(w) - g_{\gamma_2,t}(w)$ and observe that

$$G(\tfrac{1}{b_n}) = g_{\gamma_1,K}(\tfrac{1}{b_n}) - g_{\gamma_2,t}(\tfrac{1}{b_n}) = \tfrac{1}{a_n} - \tfrac{1}{a_n} = 0 \tag{A.8}$$

whereas $G(\phi_1) = \phi_2 - g_{\gamma_2,t}(\phi_1)$. Let x_0 be a point such that $G(x_0) = 0$ and let $y_0 = g_{\gamma_1,K}(x_0) = g_{\gamma_2,t}(x_0)$. Now for fixed γ and t we have

$$\frac{dg(w)}{dw} = \frac{f_{n+4}(\,\frac{1}{w}\,)\, F_p(\,\frac{t}{w}\,)}{f_{n+4}(\,\frac{1}{g(w)}\,)\, F_p(\,\frac{t}{g(w)}\,)} \tag{A.9}$$

therefore

$$\left.\frac{dG(w)}{dw}\right|_{w=x_0} = \frac{f_{n+4}(\,\frac{1}{x_0}\,)}{f_{n+4}(\,\frac{1}{y_0}\,)} \left\{ \frac{F_p(\,\frac{K}{x_0}\,)}{F_p(\,\frac{K}{y_0}\,)} - \frac{F_p(\,\frac{t}{x_0}\,)}{F_p(\,\frac{t}{y_0}\,)} \right\}. \tag{A.10}$$

Applying Lemma A.5 with $x_1 = K/x_0$, $x_2 = K/y_0$ and $\beta = t/K$ we conclude that the term in braces is less than zero so the derivative evaluated at x_0 is negative. Since $G(1/b_n) = 0$ and G meets the assumptions of Lemma A.6, for every number less than $1/b_n$ that G is positive. From Lemma A.2 of Goutis and Casella (1989) we know $\phi_1 < 1/b_n$, therefore $\phi_2 > g_{\gamma_2,\lambda}(\phi_1)$ which proves (A.4). □

Lemma A.2. Let $\underset{\sim}{K}_2 = (K_1, K_2)$, $K_2 < K_1$, and $I_2(s^2, t, \underset{\sim}{K}_2)$ be defined

$$I_2(s^2, t, \underset{\sim}{K}_2) = \begin{cases} (\frac{1}{b_n}s^2, \frac{1}{a_n}s^2) & \text{if} \quad t > K_1 \\ (\phi_1(K_1)s^2, \phi_2(K_1)s^2) & \text{if} \quad K_2 < t \le K_1 \\ (\phi_1(K_2)s^2, \phi_2(K_2)s^2) & \text{if} \quad t \le K_2, \end{cases}$$

where ϕ_1 and ϕ_2 satisfy the equations:

$$\int_{\phi_1(K_1)}^{\phi_2(K_1)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K_1}{x})\, dx = \int_{\frac{1}{b_n}}^{\frac{1}{a_n}} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K_1}{x})\, dx \tag{A.11}$$

$$\int_{\phi_1(K_2)}^{\phi_2(K_2)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K_2}{x})\, dx = \int_{\phi_1(K_1)}^{\phi_2(K_1)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{K_2}{x})\, dx \tag{A.12}$$

$$f_{n+4}\Big\{\frac{1}{\phi_1(K_i)}\Big\}\, F_p\Big\{\frac{\tau(K_i)}{\phi_1(K_i)}\Big\} = f_{n+4}\Big\{\frac{1}{\phi_2(K_i)}\Big\}\, F_p\Big\{\frac{\tau(K_i)}{\phi_2(K_i)}\Big\} \tag{A.13}$$

for $i = 1, 2$. Then $I_2(s^2, t, \underset{\sim}{K}_2)$ has posterior probability greater than the posterior probability of $C_U(s^2)$. The equality is strict when $t \le K_1$.

Proof. We first compare the posterior probabilities of $I_2(s^2, t, \underset{\sim}{K}_2)$ and $I_1(s^2, t, K_1)$, which is based on K_1 alone. $I_2(s^2, t, \underset{\sim}{K}_2)$ differs from $I_1(s^2, t, K_1)$ only when $t \le K_2$. Therefore it suffices to show

$$\int_{\phi_1(K_2)}^{\phi_2(K_2)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{t}{x})\, dx > \int_{\phi_1(K_1)}^{\phi_2(K_1)} f_{n+4}(\tfrac{1}{x})\, F_p(\tfrac{t}{x})\, dx \tag{A.14}$$

for $t < K_2$. If in the proof of Lemma A.1 we replace K, $1/b_n$ and $1/a_n$ by K_2, $\phi_2(K_1)$ and $\phi_1(K_1)$, respectively, we can show (A.14) to be true. An immediate consequence is that, when $t \le K_1$, the interval $I_2(s^2, t, \underset{\sim}{K}_2)$ has posterior probability strictly greater than $\gamma_U(s^2, t)$. □

Lemma A.3. Let $\phi_1(t)$ and $\phi_2(t)$ satisfy (1.15) and (1.16). Then for any $t < \infty$

$$\int_{\phi_1(t)s^2}^{\phi_2(t)s^2} \pi(\sigma^2 \mid s^2, t)\, d\sigma^2 \geq \int_{\frac{1}{b_n}s^2}^{\frac{1}{a_n}s^2} \pi(\sigma^2 \mid s^2, t)\, d\sigma^2. \tag{A.15}$$

Proof. In an obvious way we can generalize the results of Lemmas A.1 and A.2 and see that the posterior probability of any interval $I_m(s^2, t, \underset{\sim}{K}_m)$, based on a finite number of cutoff points $\underset{\sim}{K}_m = (K_{m,1}, \ldots, K_{m,m-1}, K_{m,m})$, is greater than $\gamma_U(s^2, t)$. By applying the Lebesgue Dominated Convergence Theorem we can establish (A.15). □

Lemma A.4. If $p \leq 2$, the posterior probability of the usual minimum length confidence interval $C_U(s^2)$ with respect to prior (2.13) is greater than $1-\alpha$.

Proof. The proof is based on the observation that if a function has negative second derivative when the first derivative is zero then the only possible interior extremum is a maximum. Checking the values of the function in the boundary points will give us the lower bound of the function.

Letting $x = s^2/\sigma^2$ in (3.7) the posterior probability of $(\frac{1}{b_n}s^2, \frac{1}{a_n}s^2)$ is equal to

$$\gamma_U(s^2, t) = \int_{a_n}^{b_n} \frac{f_n(x)\, F_p(tx)}{F_{p,n}(\frac{n}{p}t)}\, dx. \tag{A.16}$$

Differentiating with respect to t the first derivative of $\gamma_U(s^2, t)$ is

$$\frac{d}{dt}\left[\gamma_U(s^2, t)\right] = \int_{a_n}^{b_n} \frac{f_n(x)xf_p(tx)\, F_{p,n}(\frac{n}{p}t) - f_n(x)\, F_p(tx)\frac{n}{p} f_{p,n}(\frac{n}{p}t)}{\left\{F_{p,n}(\frac{n}{p}t)\right\}^2}\, dx$$

$$= \frac{\frac{n}{p} f_{p,n}(\frac{n}{p}t)}{F_{p,n}(\frac{n}{p}t)} \left\{ \int_{a_n}^{b_n} \frac{f_n(x)xf_p(tx)}{\frac{n}{p} f_{p,n}(\frac{n}{p}t)} - \frac{f_n(x)\, F_p(tx)}{F_{p,n}(\frac{n}{p}t)}\, dx \right\} \tag{A.17}$$

where $f_{p,n}$ denotes the F density with p and n degrees of freedom. Using the explicit formulae for the chi squared densities and simplifying, we can see that

$$\frac{f_n(x)\,x\,f_p(tx)}{\frac{n}{p}\,f_{p,n}(\frac{n}{p}\,t)} = f_{n+p}\{(t+1)x\}. \tag{A.18}$$

Therefore substituting in (A.17) and making the transformation $w = (t+1)x$, the derivative becomes

$$\frac{\frac{n}{p}\,f_{p,n}(\frac{n}{p}\,t)}{F_{p,n}(\frac{n}{p}\,t)}\left\{\int_{a_n(t+1)}^{b_n(t+1)} f_{n+p}(w)\,dw - \int_{a_n}^{b_n}\frac{f_n(x)\,F_p(tx)}{F_{p,n}(\frac{n}{p}\,t)}\,dx\right\}. \tag{A.19}$$

The last expression is zero only when the term in braces is zero. Differentiating once more and ignoring the zero terms, the second derivative has the same sign as

$$\frac{d}{dt}\left[\int_{a_n(t+1)}^{b_n(t+1)} f_{n+p}(w)\,dw\right] = b_n\,f_{n+p}\{(t+1)\,b_n\} - a_n\,f_{n+p}\{(t+1)\,a_n\} \tag{A.20}$$

which is negative if and only if

$$\frac{a_n\,f_{n+p}\{(t+1)\,a_n\}}{b_n\,f_{n+p}\{(t+1)\,b_n\}} > 1. \tag{A.21}$$

Using the condition $f_{n+4}(a_n) = f_{n+4}(b_n)$, after substituting the formula for the chi squared density and simplifying the constants we have

$$\frac{a_n\,f_{n+p}\{(t+1)\,a_n\}}{b_n\,f_{n+p}\{(t+1)\,b_n\}} = \frac{a_n^{\frac{p}{2}-1}\,e^{-\frac{ta_n}{2}}}{b_n^{\frac{p}{2}-1}\,e^{-\frac{tb_n}{2}}} \tag{A.22}$$

which, since $a_n < b_n$, is greater than 1 for every positive t and $p \le 2$.

Now we need to check what happens at the boundary points of the interval of definition of $\gamma_U(s^2, t)$. Using (A.16) we have

$$\lim_{t\to\infty}\gamma_U(s^2,t) = \lim_{t\to\infty}\int_{a_n}^{b_n}\frac{f_n(x)\,F_p(tx)}{F_{p,n}(\frac{n}{p}\,t)}\,dx$$

$$= \int_{a_n}^{b_n} f_n(x)\,dx = 1-\alpha. \tag{A.23}$$

At the other endpoint, using L'Hôpital rule and (A.18)

$$\lim_{t \to 0} \gamma_U(s^2, t) = \int_{a_n}^{b_n} f_{n+p}(x)\, dx \tag{A.24}$$

which for p = 1, 2 can be shown to be greater than $1-\alpha$. If P_k denotes the probability $P\{ a_n < \chi^2_k < b_n \}$, where χ^2_k is a chi squared random variable with k degrees of freedom, integration by parts yields

$$P_k = P_{k+2} + 2\,\{f_{k+2}(b_n) - f_{k+2}(a_n)\}. \tag{A.25}$$

Since $f_{n+4}(b_n) = f_{n+4}(a_n)$ the last equation implies $P_{n+2} > P_n = 1-\alpha$. For p = 1, we use the variation reducing properties of the chi squared density (see Brown, Johnstone and MacGibbon (1981) for definitions and details). Suppose that P_{n+1} were less than or equal to $1-\alpha$. Since $\lim_{k \to \infty} P_k = 0$, for every $C \in [P_{n+1}, P_n]$, the maximum number of sign changes of the sequence $P_k - C$, counting zeros as either + or −, would be at least three. But

$$P_k - C = E\{ \mathbb{I}_{(a_n, b_n)}(\chi^2_k) - C \} \tag{A.26}$$

and, since chi squared densities belong to the exponential family, we know (example 3.1 of Brown *et al.* (1981)) that the number of sign changes of $P_k - C$ as a function of k cannot exceed the number of sign changes of $\mathbb{I}_{(a_n, b_n)}(x) - C$ as a function of x. Hence we must have $P_{n+1} > 1-\alpha$, which completes the proof. □

Lemma A.5. Let F_p be a chi squared distribution function with $p \geq 1$ degrees of freedom. If $\beta < 1$ and $x_1 > x_2$, then

$$\frac{F_p(\beta x_1)}{F_p(x_1)} > \frac{F_p(\beta x_2)}{F_p(x_2)}. \tag{A.27}$$

Proof. It follows from the fact that the gamma densities have monotone likelihood ratio in the scale parameter. (See also Lemma 4.2 of Cohen (1972)). □

Lemma A.6. If a differentiable function f(x) defined on the real line has $f'(x) < 0$ whenever $f(x) = 0$ and there is an x_0 such that $f(x_0) = 0$, then f(x) is positive for $x < x_0$ and negative for $x > x_0$.

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