Two-Hop Interference Channels: Impact of Linear Time-Varying Schemes

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***Abstract*—We consider the two-hop interference channel (IC) with constant channel coefficients, which consists of two source-destination pairs, separated by two relays. We analyze the achievable degrees of freedom (DoF) of such network when relays are restricted to perform scalar amplify-forward (AF) operations, with possibly time-varying coefficients. We show that, somewhat surprisingly, by providing the flexibility of choosing time-varying AF coefficients at the relays, it is possible to achieve 4/3 sum-DoF. We also develop a novel outerbound that matches our achievability, hence characterizing the sum-DoF of two-hop interference channels with time-varying AF relaying strategies.**

I. Introduction

Multi-hopping is typically viewed as an effective approach to extend the coverage range of wireless networks, by bridging the gap between the sources and destinations via relays. However, it has also the potential to significantly impact network capacity by enabling new interference management techniques (see, e.g., [1]–[3]). In particular, from the degrees of freedom (DoF) perspective that is the focus of this paper, authors in [4] considered a two-hop interference channel (IC) consisting of two sources, two relays, and two destinations, and by introducing a new scheme called aligned-interference-neutralization, they showed that the sum-DoF of this network is 2 (i.e., twice the sum-DoF of a single-hop IC). Also, more recently in [5], two-hop interference networks with K sources, K relays and K destinations have been considered, and by developing a new scheme named aligned-network-diagonalization, it has been shown that relays have the potential to asymptotically cancel the interference between all source-destination pairs, hence achieving K-sum DoF (i.e., the cut-set bound).

While the aforementioned results essentially demonstrate that significant DoF gains can be achieved by carefully designing the interference management strategies in multi-hop interference networks, they often require complicated relaying strategies (such as, utilizing rational dimensions for neutralizing the interference when the channels are not time-varying). In this paper, we take a complementary approach and ask how much of these DoF gains can be realized if we limit the operation of relays to simple *scalar linear* strategies?

We focus on two-hop interference channels with constant channel coefficients (i.e., slow fading), and assume that the relays are only allowed to perform scalar amplify-forward (AF) operations, with possibly *time-varying* AF coefficients. It is easy to see that if AF coefficients of the relays remain constant during the course of the scheme, then the problem will induce to a single-hop IC, in which the sum-DoF is at most 1. However, we show that, somewhat surprisingly, by providing the flexibility of choosing time-varying AF coefficients at the relays, a sum-DoF of $4/3$ is achievable.

The key idea behind the achievability strategy is that the flexibility of choosing the relay AF factors allows canceling, in any specific time slot, one source signal from one destination. So, we use this flexibility to guarantee that, for each destination, at most one third of its received symbols are distinct interference symbols, which allows it to achieve $2/3$ DoF.

To derive the outerbound, we break the end-to-end mutual information achieved by any scheme into five different groups, based on five distinct states that scalar linear schemes can create at each time-step. We then proceed to prove three outerbounds that effectively capture the tension between these groups. Analyzing the three bounds yields that the sum-DoF is upper bounded by $4/3$ almost surely.

II. Problem Setting & Main Result

As illustrated in Figure 1, we consider the two-hop IC, consisting of two sources, two relays, and two destinations.

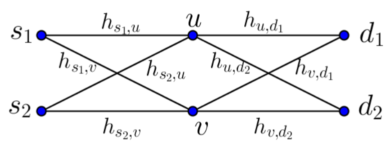


Fig. 1. Two-hop IC.

We denote the two sources by s_1 and s_2, the two relays by u and v, and the destinations by d_1 and d_2. Each source s_i has a message W_i intended for d_i ($i \in \{1,2\}$), and $W_1 \perp\!\!\!\perp W_2$.

Let $\mathbf{H_1} = \begin{bmatrix} h_{s_1,u} & h_{s_2,u} \\ h_{s_1,v} & h_{s_2,v} \end{bmatrix}$ and $\mathbf{H_2} = \begin{bmatrix} h_{u,d_1} & h_{v,d_1} \\ h_{u,d_2} & h_{v,d_2} \end{bmatrix}$ be the channels of the first and second hop, respectively. We assume that the channel gains are real-valued and drawn from a continuous distribution, fixed during the course of communication, and known at all nodes.

The transmit signal of s_i and relay r at time k are respectively denoted by $X_{i,k} \in \mathbb{R}$ and $X_{r,k} \in \mathbb{R}$, $i \in \{1,2\}$ and $r \in \{u,v\}$. The received signal of relay r at time k is

$$Y_{r,k} = h_{s_1,r}X_{1,k} + h_{s_2,r}X_{2,k} + Z_{r,k}, \qquad r \in \{u,v\},\ k \in \mathbb{N},$$

and for destination d_i, the received signal at time k is

$$Y_{i,k} = h_{u,d_i} X_{u,k} + h_{v,d_i} X_{v,k} + Z_{d_i,k}, \qquad i \in \{1,2\},\ k \in \mathbb{N},$$

where $Z_{r,k}$'s and $Z_{d_i,k}$'s are i.i.d (over time and with respect to each other) noise terms distributed as $\sim \mathcal{N}(0,1)$, which are also independent of the messages $\{W_1, W_2\}$. We will use X^n to denote a random column vector $[X_1\ X_2\ \ldots\ X_n]^T$.Also, for any $\mathcal{S} \subseteq \{1,2,\ldots,n\}$, we let $X^{\mathcal{S}}$ denote $\{X_k | k \in \mathcal{S}\}$.

Definition 1. *An (n, R_1, R_2)-scheme with power constraint P on the two-hop IC consists of the following:*

1) *A message set $\mathcal{W}_i = \{1, 2, \ldots, 2^{nR_i}\}$ at s_i ($i \in \{1,2\}$).*
2) *An encoding function f_i: $\mathcal{W}_i \to \mathcal{X}_i^n$ for each source s_i, $i \in \{1,2\}$, such that $X_i^n = f_i(W_i)$, and every codeword x_i^n satisfies the power constraint $\sum_{k=1}^n x_{i,k}^2 \le nP$.*
3) *A relaying function $f_{r,k}$: $\mathcal{Y}_r^{k-1} \to \mathcal{X}_r$ at r for each $r \in \{u,v\}$ and each $k \in \{1,2,\ldots,n\}$, such that $X_{r,k} = f_{r,k}(Y_r^{k-1})$. In addition, every codeword x_r^n should satisfy the power constraint $\sum_{k=1}^n x_{r,k}^2 \le nP$.*
4) *A decoding function g_i: $\mathcal{Y}_i^n \to \mathcal{W}_i$ for destination d_i, $i \in \{1,2\}$, such that $\hat{W}_i = g_i(Y_i^n)$.*
5) *The error probability P_e^n of the scheme is defined as $P_e^n = \mathbf{Pr}\left(\bigcup_{i=1}^2 \{W_i \neq \hat{W}_i\}\right)$, where each W_i is chosen independently and uniformly at random from $\{1,2,\ldots,2^{nR_i}\}$, $i \in \{1,2\}$.*

Definition 2. (Time-varying AF scheme) *Let $\mathcal{U}$ and $\mathcal{V}$ be two finite subsets of $\mathbb{R}$. An (n, R_1, R_2)-scheme on the two-hop IC is called a time-varying AF on $(\mathcal{U},\mathcal{V})$ if there exist $\{\mu_k \in \mathcal{U}\}_{k=1}^n$ and $\{\lambda_k \in \mathcal{V}\}_{k=1}^n$ such that, for each $k \in \{1,2,\ldots,n\}$, $f_{u,k}(Y_u^{k-1}) = \mu_k Y_{u,k-1}$ and $f_{v,k}(Y_v^{k-1}) = \lambda_k Y_{v,k-1}$.*

Definition 3. *A rate pair (R_1, R_2) is time-varying-AF-achievable on $(\mathcal{U},\mathcal{V})$ if there exists a sequence of (n, R_1, R_2)-schemes that are time-varying AF on $(\mathcal{U},\mathcal{V})$, s.t. $\lim_{n\to\infty} P_e^n = 0$.*

Definition 4. *The sum-DoF achievable by time-varying AF, denoted by $\mathcal{D}$, is defined by*

$$\mathcal{D} = \sup_{\mathcal{U},\mathcal{V}} \limsup_{P\to\infty} \left\{ \frac{R_1+R_2}{\frac{1}{2}\log_2 P} \,\middle|\, \begin{array}{l}(R_1,R_2) \text{ is time-varying-}\\ \text{AF-achievable on } (\mathcal{U},\mathcal{V})\end{array} \right\}.$$

The main result of the paper is the following theorem.

Theorem 1. *The sum-DoF of two-hop IC with time-varying AF schemes is $4/3$ for almost all values of channel gains.*

In particular, the channel gain conditions needed for Theorem 1 to yield 4/3 sum-DoF are as follows:

(c-1) All channel gains are non-zero.

(c-2) rank$(\mathbf{H_i}) = 2,\ i \in \{1,2\}$.

(c-3) rank $\left(\mathbf{H}^i \triangleq \begin{bmatrix} h_{u,d_1} h_{s_i,u} & h_{v,d_1} h_{s_i,v} \\ h_{u,d_2} h_{s_{\bar{i}},u} & h_{v,d_2} h_{s_{\bar{i}},v} \end{bmatrix}\right) = 2,$
$i \in \{1,2\}, \bar{i} = 3 - i.$ (1)

It is easy to see that almost all values of channel gains satisfy the above conditions. In the rest of the paper, in which we prove Theorem 1, we assume that conditions (c-1)-(c-3) hold.

III. Achieving 4/3 Sum-DoF by Time-Varying AF

The achievability scheme consists of three phases, during which each source sends two distinct symbols, and at the end of the three phases each receiver is able to reconstruct an interference free, but noisy, version of its desired symbols.

First note that, for time-varying AF strategies, the received signals at the destinations at each time k can be written as

$$\begin{aligned}\begin{bmatrix} Y_{1,k} \\ Y_{2,k} \end{bmatrix} &= \mathbf{H_2} \begin{bmatrix} \mu_k & 0 \\ 0 & \lambda_k \end{bmatrix} \mathbf{H_1} \begin{bmatrix} X_{1,k-1} \\ X_{2,k-1} \end{bmatrix} + \begin{bmatrix} \tilde{Z}_{1,k} \\ \tilde{Z}_{2,k} \end{bmatrix} \\ &= \mathbf{G}_k \begin{bmatrix} X_{1,k-1} \\ X_{2,k-1} \end{bmatrix} + \begin{bmatrix} \tilde{Z}_{1,k} \\ \tilde{Z}_{2,k} \end{bmatrix},\end{aligned} \qquad (2)$$

where μ_k and λ_k are the AF coefficients at time k, $\tilde{Z}_{i,k} = h_{u,d_i}\mu_k Z_{u,k-1} + h_{v,d_i}\lambda_k Z_{v,k-1} + Z_{d_i,k}$ is the effective noise at destination d_i, $i \in \{1,2\}$, and $\mathbf{G}_k = \mathbf{H_2} \begin{bmatrix} \mu_k & 0 \\ 0 & \lambda_k \end{bmatrix} \mathbf{H_1}$ is the equivalent end-to-end channel matrix given by

$$\mathbf{G}_k = \qquad (3)$$
$$\begin{bmatrix} \mu_k h_{u,d_1} h_{s_1,u} + \lambda_k h_{v,d_1} h_{s_1,v} & \mu_k h_{u,d_1} h_{s_2,u} + \lambda_k h_{v,d_1} h_{s_2,v} \\ \mu_k h_{u,d_2} h_{s_1,u} + \lambda_k h_{v,d_2} h_{s_1,v} & \mu_k h_{u,d_2} h_{s_2,u} + \lambda_k h_{v,d_2} h_{s_2,v} \end{bmatrix}.$$

For notational convenience, let $\mathbf{G}_k = \begin{bmatrix} \alpha_{1,k} & \beta_{1,k} \\ \alpha_{2,k} & \beta_{2,k} \end{bmatrix}$. Also, we will only need $\tilde{Z}_{i,k}$ for our analysis; so we will drop the tilde and write $Z_{i,k}$. Then, the received signal at destination d_i, $i \in \{1,2\}$, at time k is

$$Y_{i,k} = \alpha_{i,k} X_{1,k} + \beta_{i,k} X_{2,k} + Z_{i,k}, \quad k \in \{1,2,\ldots,n\}. \quad (4)$$

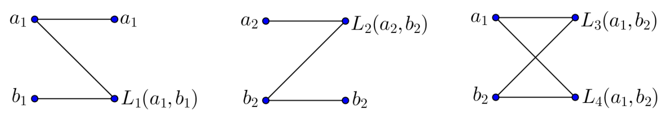
Note that the variance of $Z_{i,k}$ depends only on channel coefficients and amplifying factors (chosen from $(\mathcal{U},\mathcal{V})$), therefore it does not scale with P.

We will now describe the three phases of our time-varying AF achievability scheme in details. Set $\mathcal{U} = \{c\}$, and $\mathcal{V} = \{0, -ch_{u,d_1}h_{s_2,u}/h_{v,d_1}h_{s_2,v}, -ch_{u,d_2}h_{s_1,u}/h_{v,d_2}h_{s_1,v}\}$, where the constant $c \in \mathbb{R}$ is chosen to satisfy the power constraint P at the relays. More specifically, $c = \min\left\{\sqrt{1/(h_{s_1,u}^2 + h_{s_2,u}^2 + 1)}, l\sqrt{1/(h_{s_1,v}^2 + h_{s_2,v}^2 + 1)}\right\}$, where $l = \min\{|h_{v,d_1}h_{s_2,v}/h_{u,d_1}h_{s_2,u}|, |h_{v,d_2}h_{s_1,v}/h_{u,d_2}h_{s_1,u}|\}$. Note that the denominators are non-zero by condition (c-1).

Phase 1. In this phase, s_1 and s_2 send two symbols a_1 and b_1, respectively ($a_1^2, b_1^2 \le P$). We choose the AF factors at the relays such that the interference from s_2 is canceled at d_1. More specifically, we set $\mu_1 = c$ and $\lambda_1 = -ch_{u,d_1}h_{s_2,u}/h_{v,d_1}h_{s_2,v}$. By inserting this choice of λ_1 and μ_1 in (4), d_1 and d_2 will respectively receive

$$y_{1,1} = \alpha_{1,1} a_1 + z_{1,1}, \text{ and } y_{2,1} = \underbrace{\alpha_{2,1} a_1 + \beta_{2,1} b_1}_{L_1(a_1,b_1)} + z_{2,1}, \quad (5)$$

where $\alpha_{1,1} \neq 0$ and $\beta_{2,1} \neq 0$ (due to conditions (c-1), (c-2), and (c-3) in (1)), and $L_1(a_1, b_1)$ indicates a linear equation in a_1 and b_1. Thus, as shown in Figure 2(a), d_1 and d_2 now respectively have noisy versions of a_1 and $L_1(a_1, b_1)$.



(a) Phase 1 (b) Phase 2 (c) Phase 3

Fig. 2. Illustration of achievability scheme. At each phase, the transmit symbols by sources are shown on the left. The received signals at destinations are given on the right, where the noise is dropped and $L(x, y)$ denotes a linear combination of x and y.

Phase 2. In this phase, s_1 and s_2 send two new symbols a_2 and b_2 ($a_2^2, b_2^2 \leq P$). However, this time, we cancel the effect of s_1 at d_2, by letting $\mu_2 = c$ and $\lambda_2 = -ch_{u,d_2}h_{s_1,u}/h_{v,d_2}h_{s_1,v}$. Then d_1 and d_2 will respectively receive

$$y_{1,2} = \underbrace{\alpha_{1,2}a_2 + \beta_{1,2}b_2}_{L_2(a_2,b_2)} + z_{1,2}, \text{ and } y_{2,1} = \beta_{2,2}b_2 + z_{2,2}, \quad (6)$$

where $\alpha_{1,2} \neq 0$ and $\beta_{2,2} \neq 0$ (due to conditions (c-1), (c-2), and (c-3) in (1)), and $L_2(a_2, b_2)$ indicates a linear equation in a_2 and b_2. Thus, as shown in Figure 2(b), d_1 and d_2 now respectively have noisy versions of $L_2(a_2, b_2)$ and b_2.

Phase 3. Now notice that, if, at phase 3, destination d_1 receives a linear combination of a_1 and b_2 ($L_3(a_1, b_2)$), then it can solve for (a noisy version of) a_2 given equations (5) and (6). Similarly, if d_2 receives $L_4(a_1, b_2)$ then it can also solve for (a noisy version of) b_1 given equations (5) and (6). Thus, as shown in Figure 2(c), in phase 3, s_1 sends a_1, s_2 sends b_2, and we choose $\mu_3 = c$, and $\lambda_3 = 0$, so that d_1 and d_2 receive

$$y_{1,3} = \underbrace{\alpha_{1,3}a_1 + \beta_{1,3}b_2}_{L_3(a_1,b_2)} + z_{1,3}, \quad (7)$$

and

$$y_{2,3} = \underbrace{\alpha_{2,3}a_1 + \beta_{2,3}b_2}_{L_4(a_1,b_2)} + z_{2,3}, \quad (8)$$

where $\beta_{1,3} \neq 0$, and $\alpha_{2,3} \neq 0$ (due to condition (c-1) in (1)). Therefore, after the three phases, d_1 can construct

$$y_1^{a_1} = a_1 + z_{1,1}/\alpha_{1,1}, \quad (9)$$

and

$$y_1^{a_2} = a_2 + \frac{1}{\alpha_{1,2}}z_{1,2} - \frac{\beta_{1,2}}{\alpha_{1,2}\beta_{1,3}}z_{1,3} + \frac{\alpha_{1,3}\beta_{1,2}}{\alpha_{1,1}\alpha_{1,2}\beta_{1,3}}z_{1,1}. \quad (10)$$

from $(y_{1,1}, y_{1,2}, y_{1,3})$. Let σ_1^2 and σ_2^2 be the variances of the noise terms in equations (9) and (10). Note that they depend only on channel coefficients and AF factors. Hence, they are constants that do not scale with P. Then, by using a proper outercode, we can achieve a rate of

$$R_1 = \frac{1}{6}\left(\log\left(1 + \frac{P}{\sigma_1^2}\right) + \log\left(1 + \frac{P}{\sigma_2^2}\right)\right) \geq \frac{1}{3}\log\frac{P}{\sigma_1\sigma_2}.$$

So d_1 can achieve $2/3$ DoF. Similarly, d_2 can also achieve $2/3$ DoF, hence achieving a total of $4/3$ sum-DoF.

IV. Outerbounds on DoF of Time-Varying AF

Consider a time-varying AF (n, R_1, R_2)-scheme $\mathcal{C}$ with power constraint P, and error probability P_e^n such that $P_e^n \to 0$ as $n \to \infty$. We will prove that $R_1 + R_2 \leq (2/3)\log P + o(\log P)$. Let μ_k and λ_k denote the amplifying factors of $\mathcal{C}$ at time k of relays u and v, respectively. Consider the end-to-end channel matrix $\mathbf{G}_k$ (defined in (3)) created by scheme $\mathcal{C}$ at time k. Note that the i-th column (row) of $\mathbf{G}_k$ ($i \in \{1, 2\}$) corresponds to a linear combination of columns of $\mathbf{H_1}$ ($\mathbf{H_2}$) with coefficients $\mu_k h_{s_i,u}$ and $\lambda_k h_{s_i,v}$ ($\mu_k h_{u,d_i}$ and $\lambda_k h_{v,d_i}$). Also, the entries of the main diagonal are linear combinations of the columns of $\mathbf{H^1}$ (defined in (1)) with coefficients μ_k and λ_k; similarly, the entries of the counterdiagonal are linear combinations of the columns of $\mathbf{H^2}$ (defined in (1)) with coefficients μ_k and λ_k. Since by conditions (c-1), (c-2), and (c-3), specified in (1), all channel coefficients are non-zero, and $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H^1}$, and $\mathbf{H^2}$ have full rank, then no pair of entries in $\mathbf{G}_k$ can be zero, unless $\lambda_k = \mu_k = 0$. Therefore, at each time k either $\mathbf{G}_k$ has at most one zero entry or $\mathbf{G}_k = \mathbf{0}$. As a result, if $\mathbf{G}_k$ is non-zero at any time k, then it belongs to one of the states shown in Figure 3. Asterisks denote non-zero entries. We denote the collective state (C_1, C_2, C_3) by C.

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \quad \begin{bmatrix} * & * \\ * & * \end{bmatrix} \quad \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \quad \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$$

(a) State A (b) State B (c) State C_1 (d) State C_2 (e) State C_3

Fig. 3. If at a time k, $\max\{|\mu_k|, |\lambda_k|\} > 0$, then the end-to-end channel matrix $\mathbf{G}_k$ is in one of the above states. Asterisks denote non-zero entries.

Similarly to equation (4), we will write the vector of n received signals at destination d_i (with an abuse of notation)

$$Y_i^n = \alpha_i^n X_1^n + \beta_i^n X_2^n + Z_i^n, \quad i \in \{1, 2\}, \quad (11)$$

where α_i^n and β_i^n are understood as $n \times n$ diagonal matrices, where the j^{th} entries of the diagonals are respectively $\alpha_{i,j}$, and $\beta_{i,j}$ ($i \in \{1, 2\}, j \in \{1, \ldots, n\}$). Similarly, for any $L \subset \{1, 2, \ldots, n\}$, we write

$$Y_i^L = \alpha_i^L X_1^L + \beta_i^L X_2^L + Z_i^L, \quad (12)$$

where α_i^L and β_i^L are $|L| \times |L|$ diagonal matrix, whose diagonal entries are respectively $\{\alpha_{i,l}\}_{l \in L}$ and $\{\beta_{i,l}\}_{l \in L}$ ($i \in \{1, 2\}$).

Now, for any code $\mathcal{C}$ (with AF coefficients λ_k and μ_k, $k = 1, \ldots, n$), we define the set $A_{\mathcal{C}}$ as

$$A_{\mathcal{C}} = \{k \in \{1, 2, \ldots, n\} : \mu_k h_{u,d_2}h_{s_1,u} + \lambda_k h_{v,d_2}h_{s_1,v} = 0\}.$$

Similarly, let $B_{\mathcal{C}}$, $C_{\mathcal{C}}$, $C_{1,\mathcal{C}}$, $C_{2,\mathcal{C}}$, and $C_{3,\mathcal{C}}$ be the sets of time indices corresponding to states B, C, C_1, C_2, and C_3, respectively. Also, let $S_{\mathcal{C}} = A_{\mathcal{C}} \cup B_{\mathcal{C}} \cup C_{\mathcal{C}}$. So that $S_{\mathcal{C}}^c = \{k \in \{1, 2, \ldots, n\} : \mu_k = \lambda_k = 0\}$. Note that, since the channel gains are fixed, and we are considering a specific scheme $\mathcal{C}$ which fixes μ_k and λ_k for all k, then the previously defined sets are deterministic, and thus well-defined. Also, for ease of notation, we will drop the subscript $\mathcal{C}$ in the rest of this section, and refer to those sets as A, B, C, C_1, C_2, C_3, and S. We now state our main lemma which yields $\mathcal{D} \leq 4/3$.

Lemma 1. *For any time-varying AF (n, R_1, R_2)-scheme, $\mathcal{C}$, with power constraint P and associated sets A, B, C, as defined above, we have*

$$R_1 + R_2 \leq \frac{1}{2}\left(1 + \frac{|C|}{n}\right)\log_2 P + \tau_1, \qquad \text{(Bound 1)}$$

$$R_1 + R_2 \leq \frac{1}{2}\left(1 + \frac{|B|}{n}\right)\log_2 P + \tau_2, \qquad \text{(Bound 2)}$$

$$R_1 + R_2 \leq \frac{1}{2}\left(1 + \frac{|A|}{n}\right)\log_2 P + \tau_3, \qquad \text{(Bound 3)}$$

where τ_1, τ_2, and τ_3 are constants that do not depend on P.

Before proving Lemma 1, we first demonstrate how it yields $\mathcal{D} \leq 4/3$. Suppose that the Lemma is true. Then, by taking the minimum of the three bounds, we get

$$R_1 + R_2 \leq \min_{L \in \{A,B,C\}} \frac{1}{2}(1 + \frac{|L|}{n})\log_2 P + \tau \leq \frac{2}{3}\log_2 P + \tau,$$
$$\Rightarrow \mathcal{D} \leq 4/3, \qquad (13)$$

where $\tau = \max(\tau_1, \tau_2, \tau_3)$, the second inequality follows from the fact that $\min\{|A|, |B|, |C|\} \leq n/3$ since $|A| + |B| + |C| \leq n$. We will now go back to proving the bounds in Lemma 1.

Proof of Bound (1) in Lemma 1

Recall that $S = A \cup B \cup C$. We claim that $I(X_1^n; Y_1^n) = I(X_1^S; Y_1^S)$, since:

$$\begin{aligned}
&I(X_1^n; Y_1^n) - I(X_1^S; Y_1^S) \\
&= I(X_1^{S^c}; Y_1^S | X_1^S) + I(X_1^n; Y_1^{S^c} | Y_1^S) \\
&\overset{(a)}{=} I(X_1^{S^c}; Y_1^S | X_1^S) = h(Y_1^S | X_1^S) - h(Y_1^S | X_1^n) \\
&= h(\beta_1^S X_2^S + Z_2^S | X_1^S) - h(\beta_1^S X_2^S + Z_2^S | X_1^n) \\
&\overset{(b)}{=} h(\beta_1^S X_2^S + Z_2^S) - h(\beta_1^S X_2^S + Z_2^S) = 0, \qquad (14)
\end{aligned}$$

where (a) follows from the fact that $Y_1^{S^c} = Z_1^{S^c}$ which is independent of messages and noise at other time steps, and (b) follows from the fact that X_1^S is a function of W_1, hence independent of W_2 (and X_2^S) and of noise. Similarly, $I(X_2^n; Y_2^n) = I(X_2^S; Y_2^S)$. Now, using Fano and (14), we get

$$\begin{aligned}
&n(R_1 + R_2) \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon_n \\
&= I(X_1^S; Y_1^S) + I(X_2^S; Y_2^S) + n\epsilon_n \\
&\overset{(S = A \cup B \cup C)}{=} I(X_1^S; Y_1^A, Y_1^B) + I(X_2^S; Y_2^A, Y_2^B) + n\epsilon_n \\
&+ I(X_1^S; Y_1^C | Y_1^A, Y_1^B) + I(X_2^S; Y_2^C | Y_2^A, Y_2^B), \qquad (15)
\end{aligned}$$

where $\epsilon_n \to 0$, as $P_e^n \to 0$. Now, we bound the last two terms:

$$\begin{aligned}
I(X_i^S; Y_i^C | Y_i^A, Y_i^B) &\leq h(Y_i^C) - h(Y_i^C | Y_i^A, Y_i^B, X_i^S, X_{\bar{i}}^C) \\
&= h(Y_i^C) - h(Z_i^C), \qquad (16)
\end{aligned}$$

where $i \in \{1, 2\}$, $\bar{i} = 3 - i$, and the equality follows from the fact that noise is independent of $\{W_1, W_2\}$ and of noise terms at other time steps. Now, to bound the first two terms in (15), consider the following chain of inequalities.

$$\begin{aligned}
&I(X_1^S; Y_1^A, Y_1^B) + I(X_2^S; Y_2^A, Y_2^B) \\
&\leq h(Y_1^A) + h(Y_1^B) - h(Y_1^A, Y_1^B | X_1^S) \\
&+ h(Y_2^A) + h(Y_2^B) - h(Y_2^A, Y_2^B | X_2^S) \\
&= h(Y_1^A) + h(Y_1^B) - h(\beta_1^A X_2^A + Z_1^A, Z_1^B | X_1^S) \\
&+ h(Y_2^A) + h(Y_2^B) - h(\alpha_2^B X_1^B + Z_2^B, Z_2^A | X_2^S) \\
&\overset{(a)}{=} h(Y_1^A) + h(Y_2^B) - h(Z_1^B) - h(Z_2^A) \\
&+ \left[h(\alpha_1^B X_1^B + Z_1^B) - h(\alpha_2^B X_1^B + Z_2^B)\right] \\
&+ \left[h(\beta_2^A X_2^A + Z_2^A) - h(\beta_1^A X_2^A + Z_1^A)\right], \qquad (17)
\end{aligned}$$

where (a) follows from the fact that W_1 and W_2 are independent, noise and (W_1, W_2) are independent, and noise terms at different time steps are independent. Now, consider the following lemma.

Lemma 2. *Let X, Y, Z be two random vectors of size n, such that $X \perp\!\!\!\perp (Y, Z)$. Let $\mathbf{M}$ and $\mathbf{M}'$ be two $n \times n$ constant invertible matrices. Then*

$$\begin{aligned}
&h(\mathbf{M}X + Y) - h(\mathbf{M}'X + Z) \leq \\
&h(\mathbf{M}'\mathbf{M}^{-1}Y - Z) - h(Z|Y) - \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right|.
\end{aligned}$$

Proof:

$$\begin{aligned}
&h(\mathbf{M}X + Y) - h(\mathbf{M}'X + Z) \\
&= h(\mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y) - h(\mathbf{M}'X + Z) \\
&- \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&\leq h(\mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y) - h(\mathbf{M}'X + Z | \mathbf{M}'\mathbf{M}^{-1}Y - Z) \\
&- \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&= -h(\mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y | \mathbf{M}'\mathbf{M}^{-1}Y - Z) \\
&+ h(\mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y) - \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&= I(\mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y; \mathbf{M}'\mathbf{M}^{-1}Y - Z) \\
&- \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&= h(\mathbf{M}'\mathbf{M}^{-1}Y - Z) - h(\mathbf{M}'\mathbf{M}^{-1}Y - Z | \mathbf{M}'X + \mathbf{M}'\mathbf{M}^{-1}Y) \\
&- \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&\leq h(\mathbf{M}'\mathbf{M}^{-1}Y - Z) - h(\mathbf{M}'\mathbf{M}^{-1}Y - Z | X, Y) \\
&- \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right| \\
&\leq h(\mathbf{M}'\mathbf{M}^{-1}Y - Z) - h(Z|Y) - \log\left|\det\left(\mathbf{M}'\mathbf{M}^{-1}\right)\right|.
\end{aligned}$$

■

Then we can apply Lemma 2 on the bracketed terms in equation (17), where for the first term $\{X = X_1^B, Y = Z_1^B, Z = Z_2^B, \mathbf{M} = \alpha_1^B, \text{ and } \mathbf{M}' = \alpha_2^B\}$, and for the second term $\{X = X_2^A, Y = Z_2^A, Z = Z_1^A, \mathbf{M} = \beta_2^A, \text{ and } \mathbf{M}' = \beta_1^A\}$. So by setting $\mathbf{M_1} = (\alpha_2^B)(\alpha_1^B)^{-1}$, $\mathbf{M_2} = (\beta_1^A)(\beta_2^A)^{-1}$, we get

$$\begin{aligned}
&I(X_1^S; Y_1^A, Y_1^B) + I(X_2^S; Y_2^A, Y_2^B) \\
&\leq h(Y_1^A) + h(Y_2^B) - h(Z_1^B) - h(Z_2^A) \\
&+ h\left(\mathbf{M_1}Z_1^B - Z_2^B\right) - h\left(Z_2^B | Z_1^B\right) - \log|\det(\mathbf{M_1})| \\
&+ h\left(\mathbf{M_2}Z_2^A - Z_1^A\right) - h\left(Z_1^A | Z_2^A\right) - \log|\det(\mathbf{M_2})| \\
&\leq h(Y_1^A) + h(Y_2^B) + \gamma_1 n, \qquad (18)
\end{aligned}$$

where γ_1 is a constant that does not depend on P. Now, by equations (15), (16), and (18), we get

$$n(R_1+R_2) \leq h(Y_1^A)+h(Y_2^B)+h(Y_1^C)+h(Y_2^C)+\gamma_2 n, \quad (19)$$

where γ_2 is a constant that does not depend on P. Now, we bound $h(Y_1^A)$ by

$$h(Y_1^A) - |A|\log_2(2\pi e)/2 \leq \sum_{k\in A} h(Y_{1,k}) - |A|\log_2(2\pi e)/2$$
$$\overset{(a)}{\leq} \sum_{k\in A} \frac{1}{2}\log_2(\alpha_{1,k}^2 \mathbf{E}[X_{1,k-1}^2] + \beta_{1,k}^2 \mathbf{E}[X_{2,k-1}^2] + \mathbf{E}[Z_{1,k}^2]),$$

where (a) is true because Gaussian distribution maximizes differential entropy. Define $M_{i,j}$ $(i,j \in \{1,2\})$, and M as

$$\begin{aligned} M_{i,j} &= \max_{\mu\in\mathcal{U},\lambda\in\mathcal{V}} (\mu h_{u,d_i} h_{s_j,v} + \lambda h_{v,d_i} h_{s_j,v})^2, \\ M &= \max_{j\in\{1,2\}} \max_{i\in\{1,2\}} (M_{i,j}). \end{aligned} \quad (20)$$

Recall $\alpha_{1,k} = \mu_k h_{u,d_1} h_{s_1,u} + \lambda_k h_{v,d_1} h_{s_1,v}$. Then $\alpha_{1,k}^2 \leq M$, $\forall k$. Similarly, $\beta_{1,k}^2 \leq M$, $\forall k$. Also, define N

$$N = \max_{i\in\{1,2\}} \left(\max_{\mu\in\mathcal{U},\lambda\in\mathcal{V}} (h_{u,d_i}^2 \mu^2 + h_{v,d_i}^2 \lambda^2) \right) + 1. \quad (21)$$

Then $\mathbf{E}[Z_{1,k}^2] = ((h_{u,d_1}\mu_k)^2 + (h_{v,d_1}\lambda_k)^2 + 1) \leq N$, $\forall k$. Thus

$$\begin{aligned} &h(Y_1^A) - |A|\log_2(2\pi e)/2 \\ &\leq \sum_{k\in A} \frac{1}{2}\log_2\left(M\mathbf{E}[X_{1,k-1}^2] + M\mathbf{E}[X_{2,k-1}^2] + N\right) \\ &\overset{(a)}{\leq} \frac{|A|}{2}\log_2\left(N + M\frac{\sum_{k\in A}(\mathbf{E}[X_{1,k-1}^2] + \mathbf{E}[X_{2,k-1}^2])}{|A|}\right) \\ &\overset{(b)}{\leq} \frac{|A|}{2}\log_2\left(N + 2MnP/|A|\right) \\ &\leq \frac{|A|}{2}\log_2 P + \frac{|A|}{2}\log_2(N + 2Mn/|A|), \end{aligned} \quad (22)$$

where (a) follows from Jensen's inequality, and (b) follows from the power constraint P. Now, notice that

$$\begin{aligned} (N + 2Mn/|A|)^{|A|} &= N^{|A|}\left(1 + \frac{2Mn/N}{|A|}\right)^{|A|} \\ &\overset{(a)}{\leq} N^{|A|}(1 + 2M/N)^n, \end{aligned}$$

where (a) follows from the fact that the sequence $(1+x/m)^m$ is monotonically increasing in m, when $x > 0$. Then

$$|A|\log_2(N + 2Mn/|A|) \leq n\log_2(1 + 2M/N) + |A|\log_2 N,$$

which, combined with equation (22), yields

$$h(Y_1^A) \leq \frac{|A|}{2}\log_2 P + \gamma_3 n, \quad (23)$$

where γ_3 is a constant that does not depend on P. Similarly

$$h(Y_2^B) \leq \frac{|B|}{2}\log_2 P + \gamma_4 n, \quad (24)$$

$$h(Y_i^C) \leq \frac{|C|}{2}\log_2 P + \gamma_{5,i} n, \qquad i \in \{1,2\}, \quad (25)$$

where $\gamma_4, \gamma_{5,1}$, and $\gamma_{5,2}$ are constants that do not depend on P. So, from equations (19), (23), (24), and (25) we get

$$\begin{aligned} n(R_1 + R_2) &\leq \frac{1}{2}(|S| + |C|)\log_2 P + \tau_1 n \\ &\leq \frac{n}{2}(1 + \frac{|C|}{n})\log_2 P + \tau_1 n, \end{aligned}$$

where τ_1 is a constant that does not depend on P. ■

Proof of Bound (2) in Lemma 1

Define the set $E = C_1 \cup C_2$, and consider the following.

$$\begin{aligned} &n(R_1 + R_2) - n\epsilon_n \overset{(a)}{\leq} I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ &\overset{(b)}{=} I(X_1^S; Y_1^S) + I(X_2^S; Y_2^S) \overset{(c)}{\leq} I(X_1^S; Y_1^S) + I(X_2^S; Y_2^S | X_1^S) \\ &\leq h(Y_1^S) - h(\beta_1^A X_2^A + Z_1^A, Z_1^B, \beta_1^C X_2^C + Z_1^C) - h(Z_2^S) \\ &+ h(Y_2^B) + h(\beta_2^A X_2^A + Z_2^A, \beta_2^E X_2^E + Z_2^E, Z_2^{C_3}) \\ &\leq \left.\begin{matrix} [h(\beta_2^A X_2^A + Z_2^A, \beta_2^E X_2^E + Z_2^E) \\ -h(\beta_1^A X_2^A + Z_1^A, \beta_1^E X_2^E + Z_1^E)] \end{matrix}\right\} (T1) \\ &- h(Z_1^B, \beta_1^{C_3} X_2^{C_3} + Z_1^{C_3} | \beta_1^A X_2^A + Z_1^A, \beta_1^E X_2^E + Z_1^E) \ (T2) \\ &- h(Z_2^S) + h(Y_1^S) + h(Y_2^B) + h(Z_2^{C_3}), \end{aligned} \quad (26)$$

where (a) follows from Fano's inequality, (b) follows from equation (14), and (c) follows from the independence of W_1 and W_2. Now, we will bound the term $(T1)$. First, set $\mathbf{M_2} = (\beta_1^A)(\beta_2^A)^{-1}$, and $\mathbf{M_3} = (\beta_1^E)(\beta_2^E)^{-1}$. Then note

$$\begin{aligned} &h(\beta_2^A X_2^A + Z_2^A, \beta_2^E X_2^E + Z_2^E) \\ &- h(\beta_1^A X_2^A + Z_1^A, \beta_1^E X_2^E + Z_1^E) \leq \\ &h(\mathbf{M_2} Z_2^A - Z_1^A, \mathbf{M_3} Z_2^E - Z_1^E) - h(Z_1^A, Z_1^E | Z_2^A, Z_2^E) \\ &- \log|\det(\mathbf{M_2})\det(\mathbf{M_3})|, \end{aligned} \quad (27)$$

where the inequality follows from Lemma 2 and the fact that $h(\mathbf{M}X, Y) = h(X, Y) + \log|\det \mathbf{M}|$. Now, we bound $(T2)$:

$$\begin{aligned} &h(Z_1^B, \beta_1^{C_3} X_2^{C_3} + Z_1^{C_3} | \beta_1^A X_2^A + Z_1^A, \beta_1^E X_2^E + Z_1^E) \geq \\ &h(Z_1^B, Z_1^{C_3} | X_2^{C_3}, \beta_1^A X_2^A + Z_1^A, \beta_1^E X_2^E + Z_1^E) \geq \\ &h(Z_1^B) + h(Z_1^{C_3}). \end{aligned} \quad (28)$$

Then, by equations (26), (27), and (28), we get

$$\begin{aligned} &n(R_1 + R_2) \\ &\leq h(Y_1^S) + h(Y_2^B) - h(Z_1^B) - h(Z_1^{C_3}) \\ &+ h(\mathbf{M_2} Z_2^A - Z_1^A, \mathbf{M_3} Z_2^E - Z_1^E) \\ &- h(Z_1^A, Z_1^E | Z_2^A, Z_2^E) - \log|\det(\mathbf{M_2})\det(\mathbf{M_3})| \\ &\leq h(Y_1^S) + h(Y_2^B) + \gamma_6 n, \end{aligned} \quad (29)$$

where γ_6 is a constant that does not depend on P. Now, similarly to (23), we bound $h(Y_1^S)$ as

$$h(Y_1^S) \leq \frac{|S|}{2}\log_2 P + \gamma_7 n, \quad (30)$$

where γ_7 is a constant that does not depend on P. Then, from equations (29), (30), and (24), we get

$$n(R_1 + R_2) \leq \frac{1}{2}(|S| + |B|) \log_2 P + \tau_2 n,$$
$$\leq \frac{n}{2}(1 + \frac{|B|}{n}) \log_2 P + \tau_2 n,$$

where τ_2 is a constant that does not depend on P. ■

The proof of the third bound is similar, and thus omitted.

V. Conclusion

In this paper, we analyzed the sum-DoF of the two-hop IC with constant coefficients, when relays are restricted to perform time-varying AF schemes. We showed that $4/3$ sum-DoF is achievable using such schemes, as opposed to constant AF schemes that achieve at most 1. We also proved a matching outerbound, hence characterizing the sum-DoF of this network with time-varying AF strategies. This study can be extended in several directions. One direction could be to consider the impact of time-varying AF strategies in more general two-unicast networks, such as the layered networks studied in [6].

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