JACOBI STRUCTURES AND DIFFERENTIAL FORMS ON CONTACT QUOTIENTS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2012

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Cornell University 2012

In the first part of this thesis, we generalize the notion of a Jacobi bracket on the algebra of smooth functions on a manifold to the notion of a Jacobi bracket on an abstract commutative algebra. We also prove certain useful properties of the Jacobi structure on a contact manifold.

In the second part of this thesis, we develop a de Rham model for stratified spaces resulting from contact reduction. We show that the contact form induces a form on the quotient, and investigate the properties of the reduced contact form. We also describe a Jacobi bracket on the algebra of 0-forms on the singular contact quotient.

BIOGRAPHICAL SKETCH

Fatima Mahmood was born on December 28, 1984, in Islamabad, Pakistan, to Durray-Shahwar Iqbal Mahmood and Sajjad Mahmood. Her family lived in Islamabad; Durham, England; and Canberra, Australia, before immigrating to the USA in 1997. They moved to Clifton Park, New York, where Fatima attended Shenendehowa High School, graduating in 2002.

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Reyer Sjamaar, for all his help and guidance through the years. He has been very generous with his ideas and patient with me, which I deeply appreciate.

I am also grateful to Tara Holm, for leading several enjoyable classes in symplectic geometry and related topics during my time at Cornell. I learned a lot from these.

I would also like to thank Yuri Berest, whose NSF Grant DMS-0901570 partially supported my graduate work.

Finally, many thanks to the Cornell University Department of Mathematics for the Teaching Assistantship that supported me during graduate school.

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CHAPTER 1

INTRODUCTION

1.1 Notation

We set some basic notation used throughout this thesis. Let M be a smooth, finite-dimensional manifold.

We denote the tangent bundle of M by TM, and the cotangent bundle of M by T^*M. We denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M. That is, $\mathfrak{X}(M)$ is the space of smooth sections of TM. For every nonnegative integer k, we write $\Omega^k(M)$ for the space of differential k-forms on M. That is, $\Omega^k(M)$ is the space of smooth sections of $\bigwedge^k T^*M$. The de Rham complex of differential forms on M is denoted by $\Omega(M) = \bigoplus_{k\geq 0} \Omega^k(M)$. The space of smooth real-valued functions on M is denoted by $C^\infty(M) = \Omega^0(M)$.

Let $\Xi \in \mathfrak{X}(M)$. If $\beta \in \Omega^k(M)$, then we let $i_\Xi\beta$ denote the interior product of β with Ξ. We use the convention that $(i_\Xi\beta)(\Xi_1, \ldots, \Xi_{k-1}) = \beta(\Xi, \Xi_1, \ldots, \Xi_{k-1})$, for all vector fields $\Xi_1, \ldots, \Xi_{k-1}$. We denote the Lie derivative of tensor fields along Ξ by $\mathcal{L}_\Xi$.

Let $f\colon M \to N$ be a smooth map between manifolds. We let $f^*\colon T^*N \to T^*M$ denote the pullback by f of covectors, and we let $f^*\colon \Omega(N) \to \Omega(M)$ denote the pullback by f of differential forms. We let $f_*\colon TM \to TN$ denote the pushforward by f of tangent vectors.

Let G be a Lie group. The Lie algebra of G is denoted by $\mathfrak{g}$. The dual of the Lie algebra of G is denoted by $\mathfrak{g}^*$.

1.2 Contact Manifolds

Contact geometry is considered the odd-dimensional analogue of symplectic geometry, which arose as the mathematical language of Hamiltonian mechanics. The even-dimensional phase space of a mechanical system has the structure of a symplectic manifold, and the odd-dimensional extended phase space has the structure of a contact manifold. The origins of contact geometry lie in Sophus Lie's work on partial differential equations from the late nineteenth century, in which he introduced the notion of a contact transformation. The subject has since grown to become an important branch of differential geometry and topology on its own. In this thesis, we study certain singular spaces that arise from compact Lie group actions on contact manifolds.

As we set up the relevant definitions and background results of contact geometry that will be referenced throughout this thesis, we also mention some basics of symplectic geometry in order to display the relationship between the two. References for the following definitions and results are [5], [7], and [10].

A ***symplectic manifold*** is a differentiable manifold M equipped with a closed non-degnerate de Rham 2-form ω called the ***symplectic form***. Since ω is non-degenerate, the symplectic manifold M is necessarily even-dimensional. On the other hand, a ***contact manifold*** is an odd-dimensional manifold equipped with a smooth field of tangent hyperplanes, each of which is a symplectic vector space. We develop the preliminary notions of contact geometry carefully.

Let M be a smooth manifold.

Definition 1.1. A ***contact structure*** on M is a smooth field of tangent hyperplanes $\mathcal{H} \subset TM$, such that for any 1-form α which locally satisfies $\mathcal{H} = \ker \alpha$, $d\alpha|_{\mathcal{H}}$ is nondegenerate. The pair $(M, \mathcal{H})$ is a ***contact manifold***.

Every smooth field of tangent hyperplanes $\mathcal{H} \subset TM$ can locally be obtained as the kernel of a 1-form α ([7], Lemma 1.1.1). If M is a contact manifold, then the fact that $d\alpha_x$ is nondegenerate on $\mathcal{H}_x$ implies that $\mathcal{H}_x$ is a symplectic vector space with $\omega = d\alpha_x$. So $\mathcal{H}_x$ is an even-dimensional subspace of T_xM of codimension 1, implying that M is odd-dimensional. The contact structure $\mathcal{H}$ on M need not be defined as the kernel of a global 1-form, but it is if and only if the quotient line bundle $TM/\mathcal{H}$ is orientable ([7], Lemma 1.1.1). This inspires the following definition.

Definition 1.2. The contact structure $\mathcal{H}$ is ***coorientable*** if there exists a 1-form $\alpha \in \Omega(M)$ which satisfies $\mathcal{H} = \ker \alpha$ globally. Such a form α is called a ***contact form***.

Assuming that $\mathcal{H}$ is coorientable, M has dimension $2n+1$, and α is a contact form on M, then $\alpha \wedge (d\alpha)^n$ is a volume form on M. Notice that for any nowhere-vanishing smooth function $f \in C^\infty(M)$, we have $\ker \alpha = \ker (f\alpha)$, so that $f\alpha$ is also a contact form. If f is a positive function, we may replace α with $f\alpha$ without changing the contact structure or the sign of the corresponding volume form. So a choice of coorientation of $\mathcal{H}$ is actually a choice of a conformal class $[\alpha] = \{f\alpha : f \in C^\infty(M) \text{ and } f > 0\}$. We call the contact structure $\mathcal{H}$ ***cooriented*** if a choice of conformal class of global contact form has been made.

In this thesis, unless otherwise indicated, M will denote a $(2n+1)$-dimensional manifold equipped with a cooriented contact structure $\mathcal{H}$ and a contact form α. When a choice of global contact form is not necessary, we just refer to the contact manifold $(M, \mathcal{H})$. However, when a choice of global contact form is essential, we refer to the ***strict contact manifold*** $(M, \mathcal{H}, \alpha)$. In the literature, a strict contact manifold is sometimes referred to as a Pfaffian manifold, as Lichnerowicz does in [13].

A few examples of contact manifolds follow.

Example 1.3. Consider $M = \mathbb{R}^{2n+1}$ with coordinates $x_1, y_1, \ldots, x_n, y_n, z$. The 1-form $\alpha_0 = \sum_{j=1}^{n} x_j dy_j + dz$ is a global contact form, whose kernel is the ***standard contact structure***

$$\mathcal{H} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial z}, \ldots, \frac{\partial}{\partial y_n} - x_n\frac{\partial}{\partial z}\right\} \tag{1.1}$$

on $\mathbb{R}^{2n+1}$.

Example 1.4. Consider $M = S^{2n+1}$. Using the inclusion $i\colon S^{2n+1} \to \mathbb{R}^{2n+2}$, where the coordinates on $\mathbb{R}^{2n+2}$ are $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$, a global contact form on M is $\alpha = i^*\left(\frac{1}{2}\sum_{j=1}^{n+1}(x_j dy_j - y_j dx_j)\right)$. The kernel of this form is the ***standard contact structure on the unit sphere***.

Example 1.5. Consider $T^*M \times \mathbb{R}$ for any manifold M. The cotangent space T^*M carries a canonical symplectic form $\omega = -d\alpha$, where α is the tautological one-form on T^*M ([5], p. 10). Define $\alpha = \frac{1}{2} i_L\omega + dt$, where L is the Liouville vector field on T^*M obtained as the derivative at the identity of the action of $\mathbb{R}$ on T^*M by scalar multiplication, and t is the coordinate on $\mathbb{R}$. (We are actually using the pullbacks of $i_L\omega$ and dt under the standard projections $T^*M \times \mathbb{R} \to T^*M$ and $T^*M \times \mathbb{R} \to \mathbb{R}$, respectively, but we leave out the extra notation.) Then α is a contact form on $T^*M \times \mathbb{R}$.

If the manifold M in Example 1.5 is the configuration space of a mechanical system, then the cotangent bundle T^*M is the ***phase space*** of the system. Each point $(p, \xi) \in T^*M$ consists of position coordinates p and momentum coordinates ξ. The manifold $T^*M \times \mathbb{R}$ is the ***extended phase space*** of the system, in which there is an extra coordinate for time.

Example 1.6. Let (V, ω) be a symplectic vector space. Consider $W = V \times \mathbb{R}$. Define a 1-form $\alpha = \frac{1}{2} i_R \omega + dt$, where R is the radial vector field $R(v) := v$ on V and t is the coordinate on $\mathbb{R}$. Then α is a contact form on W, and we refer to (W, α) as a ***contact vector space***. It is the ***contactization*** of the symplectic vector space (V, ω).

We consider Example 1.6 to be the definition of a contact vector space. This example ends up playing a very important role in the proof of our de Rham theorem for contact quotients in Chapter 3.

Examples 1.5 and 1.6 illustrate the contactization of symplectic manifolds. There is an analogous notion going in the opposite direction. Let $(M, \mathcal{H}, \alpha)$ be a strict contact manifold. The ***symplectization*** of M is the manifold $\widetilde{M} = M \times \mathbb{R}$ equipped with symplectic form $\omega = d(e^t \pi^* \alpha)$, where t is the coordinate on $\mathbb{R}$ and $\pi \colon \widetilde{M} \to M$ is the projection $(p, t) \mapsto p$. This is another way in which contact geometry and symplectic geometry are closely related.

A diffeomorphism from one symplectic manifold to another which preserves the symplectic forms under pullback is called a ***symplectomorphism***. A diffeomorphism from one contact manifold to another which preserves the contact structures under pushforward is also given a special name.

Definition 1.7. A diffeomorphism f from one contact manifold $(M, \mathcal{H})$ to another $(M', \mathcal{H}')$ is called a ***contactomorphism*** if $f_* \mathcal{H} = \mathcal{H}'$. A contactomorphism f is called a ***cooriented contactomorphism*** if it also preserves the coorientation of the contact structures.

Under the pullback, a contactomorphism preserves the respective contact forms up to multiplication by a nowhere-vanishing smooth function. A cooriented contactomorphism preserves the respective contact forms up to multiplication by a positive function.

The Darboux theorem for symplectic manifolds states that every symplectic manifold (M, ω) of dimension $2n$ is locally symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \sum_{j=1}^{2n} dx_j \wedge dy_j$. There is an analogous theorem for contact manifolds, which states that every contact manifold of dimension $2n+1$ is locally contactomorphic to $\mathbb{R}^{2n+1}$ with the standard contact structure.

Theorem 1.8. *Let $(M, \mathcal{H}, \alpha)$ be a $(2n+1)$-dimensional strict contact manifold and let $p \in M$. Then there exists a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ centered at p such that*

$$\alpha|_U = \sum_{j=1}^{n} x_j dy_j + dz \tag{1.2}$$

This is ([7], Theorem 2.5.1) and a weaker version is ([5], Theorem 10.4). A complete proof can be found in [7]. One of the ideas mentioned in [5] is to use the symplectic Darboux theorem on the symplectization of the contact manifold.

Each contact form on a contact manifold gives rise to a particular vector field, which proves useful in many contexts.

Definition 1.9. The ***Reeb vector field*** E corresponding to a contact form α is the vector field on M uniquely determined by the following two equations:

$$i_E d\alpha = 0 \tag{1.3}$$

$$i_E \alpha = 1 \tag{1.4}$$

We check that these two conditions do indeed uniquely determine E. Let $p \in M$. First, because $d\alpha_p$ is nondegenerate on the tangent hyperplane $\mathcal{H}_p \subset T_pM$, it must have a one-dimensional kernel in T_pM. Since α is smooth, $\mathcal{E} = \ker(d\alpha)$ is a line-bundle over M. So any section E of $\mathcal{E}$ satisfies the first condition. By definition α is nonzero on $\mathcal{E}$, so $i_E\alpha$ is a nowhere-vanishing function. Therefore, the second condition serves to normalize our vector field and result in a unique vector field E.

Example 1.10. The Reeb vector field on $\mathbb{R}^{2n+1}$ corresponding to the contact form α_0 given in Example 1.3 is $\frac{\partial}{\partial z}$.

Let $\mathcal{E}$ denote the line bundle spanned by E. Then the tangent bundle of M can be written as

$$TM = \mathcal{H} \oplus \mathcal{E}. \tag{1.5}$$

Let $\mathcal{H}^\vee$ be the annihilator of E in T^*M and let $\mathcal{E}^\vee$ be the line bundle spanned by α. Then the cotangent bundle of M can be written as

$$T^*M = \mathcal{H}^\vee \oplus \mathcal{E}^\vee. \tag{1.6}$$

The restriction of $d\alpha$ to $\mathcal{H}$ is a symplectic form on each vector space $\mathcal{H}_p \subset T_pM$. Call it ω. So, from the maps at each point, there is an isomorphism

$$\begin{aligned} \omega^\sharp \colon \mathcal{H} &\to \mathcal{H}^\vee \\ V &\mapsto i_V d\alpha. \end{aligned} \tag{1.7}$$

There is also an isomorphism

$$\begin{aligned} q\colon \mathcal{E} &\to \mathcal{E}^\vee \\ fE &\mapsto f\alpha, \end{aligned} \tag{1.8}$$

for all $f \in C^\infty(M)$. Together, $\omega^\sharp$ and q give rise to an isomorphism

$$TM \to T^*M. \tag{1.9}$$

On the level of sections, this results in the isomorphism

$$\begin{aligned} \flat\colon \mathfrak{X}(M) &\to \Omega^1(M) \\ V &\mapsto V^\flat := i_V d\alpha + i_V\alpha \cdot \alpha. \end{aligned} \tag{1.10}$$

We refer to $\flat$ as the flat map. Let $\sharp$ denote the inverse of $\flat$. That is, if $\eta \in \Omega^1(M)$, then $\eta^\sharp$ is the vector field in $\mathfrak{X}(M)$ that gets sent to η by the flat map. We refer to $\sharp$ as the sharp map.

Note that the flat map $\flat$ sends the Reeb vector field E to the contact form α. That is, $E^\flat = \alpha$ and $\alpha^\sharp = E$.

Contact manifolds provide examples of Jacobi algebras, which are defined and discussed in Chapter 2. Contact manifolds equipped with an action of a compact Lie group give rise to singular quotients, on which we define differential forms in Chapter 3.

1.3 Contact Vector Fields

A vector field Ξ on a symplectic manifold (M, ω) is a ***symplectic vector field*** if it preserves ω under the Lie derivative, that is, $\mathcal{L}_\Xi \omega = 0$. A vector field Ξ on a symplectic manifold (M, ω) is a ***hamiltonian vector field*** if there exists a function

f, called the ***hamiltonian function*** of Ξ, such that $i_\Xi \omega = df$. Every hamiltonian vector field is symplectic, but every symplectic vector field is not necessarily hamiltonian. Every function $f \in C^\infty(M)$ determines a unique hamiltonian vector field Ξ_f due to the nondegeneracy of ω. Similar ideas arise in contact geometry by considering vector fields which preserve the contact structure. A reference for these ideas is Geiges's book [7].

Let $(M, \mathcal{H}, \alpha)$ be a strict contact manifold. Let Ξ be a vector field on M and let $\psi_t \colon M \to M$ denote its local flow. This is the unique family of smooth maps satisfying

$$\begin{aligned} \psi_0(p) &= p \\ \frac{d}{dt}\psi_t(p) &= \Xi(\psi_t(p)), \end{aligned} \tag{1.11}$$

for all $t \in \mathbb{R}$ and all $p \in M$ for which the terms are defined. Note that for a fixed nonzero value of t, the map ψ_t is not necessarily defined on all of M.

Definition 1.11. The vector field Ξ is a ***contact vector field***, or an ***infinitesimal automorphism of*** $\mathcal{H}$, if $\psi_{t*}(\mathcal{H}) = \mathcal{H}$ for all $t \in \mathbb{R}$ and all points on M for which the relevant terms are defined.

So the local flow ψ_t of a contact vector field Ξ preserves the contact form α up to multiplication by a positive function $f_t \in C^\infty(M)$, and $f_0 = 1$. Denote the set of contact vector fields on M by $\mathfrak{X}_{\text{con}}(M)$. Contact vector fields may also be described in terms of the Lie derivative of the contact form.

Proposition 1.12. *The vector field Ξ is a contact vector field if and only if*

$$\mathcal{L}_\Xi \alpha = \mu_\Xi \alpha \tag{1.12}$$

for some function $\mu_\Xi \in C^\infty(M)$.

Proof. Assume Ξ is a contact vector field, so that its flow ψ_t preserves the contact structure $\mathcal{H}$. This is the same as saying $\psi_t^*\alpha = f_t\alpha$ for some family of positive functions $f_t \in C^\infty(M)$ in which $f_0 = 1$. Then

$$\begin{aligned}
\mathcal{L}_\Xi \alpha &= \frac{d}{dt}(\psi_t^*\alpha)\bigg|_{t=0} \\
&= \frac{d}{dt}(f_t\alpha)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}f_t\bigg|_{t=0}\right)\alpha.
\end{aligned} \tag{1.13}$$

So setting $\mu_\Xi = \frac{d}{dt}f_t\big|_{t=0}$ yields the result.

For the other direction, assume there exists a function μ_Ξ such that $\mathcal{L}_\Xi\alpha = \mu_\Xi\alpha$. Then we have

$$\begin{aligned}
\frac{d}{dt}\psi_t^*\alpha &= \psi_t^*\mathcal{L}_\Xi\alpha \\
&= \psi_t^*(\mu_\Xi\alpha) \\
&= (\mu_\Xi \circ \psi_t)\psi_t^*\alpha.
\end{aligned} \tag{1.14}$$

So by integrating with respect to t, we obtain

$$\psi_t^*\alpha = \exp\left\{\int_0^t (\mu_\Xi \circ \psi_s)\, ds\right\}\alpha. \tag{1.15}$$

So setting $f_t = \exp\left\{\int_0^t (\mu_\Xi \circ \psi_s)\, ds\right\}$ yields the result. □

Notice that the function μ_Ξ is determined by Ξ. Plugging the Reeb vector field E into the left hand side of Equation 1.12 and using a defining property of E, we have

$$\begin{aligned}
\mathcal{L}_\Xi\alpha(E) &= i_\Xi d\alpha(E) + di_\Xi\alpha(E) \\
&= i_E i_\Xi d\alpha + i_E di_\Xi\alpha \\
&= -i_\Xi i_E d\alpha + i_E di_\Xi\alpha \\
&= i_E di_\Xi\alpha.
\end{aligned} \tag{1.16}$$

Plugging E into the right hand side of Equation 1.12 and using a defining property of E, we have

$$\begin{aligned} \mu_\Xi \alpha(E) &= \mu_\Xi i_E \alpha \\ &= \mu_\Xi . \end{aligned} \tag{1.17}$$

Therefore,

$$\mu_\Xi = i_E d i_\Xi \alpha. \tag{1.18}$$

Example 1.13. The Reeb vector field E on M corresponding to the contact form α is a contact vector field. We check this using Cartan's magic formula. Observe that

$$\begin{aligned} \mathcal{L}_E \alpha &= d i_E \alpha + i_E d\alpha \\ &= d(1) + 0 \\ &= 0 \cdot \alpha. \end{aligned} \tag{1.19}$$

Contact vector fields are the analogue of symplectic vector fields. We now define the notion of a hamiltonian vector field in the contact case.

Definition 1.14. A vector field Ξ on $(M, \mathcal{H}, \alpha)$ is a ***contact hamiltonian vector field*** if there exists a function $f \in C^\infty(M)$ such that the following two equations hold:

$$i_\Xi \alpha = f \tag{1.20}$$

$$i_\Xi d\alpha = i_E df \cdot \alpha - df \tag{1.21}$$

The function f is called the ***hamiltonian function*** of Ξ. The vector field Ξ is called the ***hamiltonian vector field*** of f.

When the context is clear, we will refer to contact hamiltonian vector fields as hamiltonian vector fields. Notice that if Ξ_f is the hamiltonian vector field of

$f \in C^\infty(M)$, then by Cartan's magic formula, we have

$$\begin{aligned} \mathcal{L}_{\Xi_f}\alpha &= i_{\Xi_f} d\alpha + d i_{\Xi_f}\alpha \\ &= i_E df \cdot \alpha - df + df \\ &= i_E df \cdot \alpha. \end{aligned} \tag{1.22}$$

So by Proposition 1.12, every contact hamiltonian vector field is a contact vector field. Unlike in the symplectic case, it turns out that all contact vector fields are hamiltonian. This is a result of the following, which is ([7], Theorem 2.3.1).

Theorem 1.15. *There is a one-to-one correspondence between contact vector fields* Ξ *and smooth functions* $f\colon M \to \mathbb{R}$*, given by the following two maps:*

$$\Xi \mapsto f_\Xi = i_\Xi \alpha \tag{1.23}$$

$$f \mapsto \Xi_f, \text{ defined uniquely by } \begin{cases} i_{\Xi_f}\alpha = f \\ i_{\Xi_f} d\alpha = i_E df \cdot \alpha - df \end{cases} \tag{1.24}$$

We refer to this correspondence between contact vector fields and smooth functions as the ***hamiltonian correspondence***. It will be discussed in more detail in Section 2.3, in which we will formulate the isomorphism from $C^\infty(M)$ to $\mathfrak{X}_{\text{con}}(M)$ explicitly and relate it to the Jacobi bracket.

1.4 Contact Quotients

Let M be a manifold, not necessarily contact or symplectic, and let G be a Lie group. Assume that we have a map from the manifold to the dual of the Lie algebra of G, denoted by

$$\Phi\colon M \to \mathfrak{g}^*. \tag{1.25}$$

Let $\xi \in \mathfrak{g}$. The component of Φ in the direction of ξ is the map $\Phi^\xi \colon M \to \mathbb{R}$ defined by

$$\Phi^\xi(p) = \langle \Phi(p), \xi \rangle, \tag{1.26}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Now assume G acts smoothly on M. Then for any $\xi \in \mathfrak{g}$, there is a vector field ξ_M on M induced by the action of the one parameter subgroup $\{\exp(t\xi) : t \in \mathbb{R}\} \subset G$. It is defined by

$$\xi_M = \frac{d}{dt} \exp(t\xi) \Big|_{t=0}. \tag{1.27}$$

Finally, we denote the diffeomorphism of M arising from $g \in G$ by $p \mapsto gp$, for all $p \in M$.

A smooth action of a Lie group G on a symplectic manifold (M, ω) which consists of symplectomorphisms is called a ***symplectic action***. Assume there exists a map $\Phi \colon M \to \mathfrak{g}^*$ satisfying the following two conditions:

(i) The component of Φ in the direction of $\xi \in \mathfrak{g}$ is a hamiltonian function for $\xi_M \in \mathfrak{X}(M)$. That is,

$$d\Phi^\xi = i_{\xi_M} \omega. \tag{1.28}$$

(ii) The map Φ is equivariant with respect to the G-action on M and the coadjoint action of G on $\mathfrak{g}^*$. That is,

$$\Phi \circ g = \mathrm{Ad}_g^* \circ \Phi \tag{1.29}$$

for all $g \in G$.

Then the action is called a ***hamiltonian action***, and the map Φ is called a ***moment map*** for the action. Given a hamiltonian action of a compact Lie group

G on a symplectic manifold (M, ω), the ***symplectic quotient*** of M by G is the topological space

$$M /\!\!/ G = \Phi^{-1}(0)/G. \tag{1.30}$$

The process of obtaining this quotient is called ***symplectic reduction***. The Marsden-Weinstein-Meyer theorem states that if G acts freely on $\Phi^{-1}(0)$, then the reduced space $M /\!\!/ G$ is a symplectic manifold [15] [16]. More generally, $M /\!\!/ G$ is a singular space.

Analogous notions exist in contact geometry. From now on, let $(M, \mathcal{H}, \alpha)$ be a strict contact manifold.

Definition 1.16. A smooth action of a Lie group G on M is called a ***contact action*** if it preserves the contact structure, that is, $g_* \mathcal{H} = \mathcal{H}$ for all $g \in G$.

Note that this means for every $g \in G$, the diffeomorphism $g \colon M \to M$ is a contactomorphism. So for every $g \in G$, there exists a nowhere-vanishing function $f_g \in C^\infty(M)$ such that

$$g^* \alpha = f_g \alpha. \tag{1.31}$$

A contact manifold equipped with a contact G-action is called a ***contact G-manifold***.

Now let G be a compact connected Lie group with a contact action on M. The fact that every group element corresponds to a nowhere-vanishing function through Equation 1.31 gives a homomorphism from G to the subset $C^\infty(M)^\times$ of nowhere-vanishing functions in $C^\infty(M)$. Composing this homomorphism with the evaluation of functions at a point yields another homomorphism

$$G \to C^\infty(M)^\times \to \mathbb{R}^\times, \tag{1.32}$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Since G is compact and connected, its image under this map must be compact and connected in $\mathbb{R}^\times$. This image is forced to be $\{1\} \subset \mathbb{R}^\times$, so that $f_g = 1$ for all $g \in G$. So α is G-invariant.

Unlike symplectic actions, it turns out that every contact action is hamiltonian in the sense that there exists a contact moment map. This map is defined as follows.

Definition 1.17. The ***contact moment map*** for the G-action on M is the map $\Phi \colon M \to \mathfrak{g}^*$ defined by

$$\Phi^\xi(p) = \alpha_p(\xi_M(p)) \tag{1.33}$$

for all $\xi \in \mathfrak{g}$ and all $p \in M$.

When the context is clear, we will refer to the contact moment map as just a moment map. We show that Φ is G-equivariant.

Proposition 1.18. *The contact moment map Φ is equivariant with respect to the G-action on M and the coadjoint action of G on $\mathfrak{g}^*$. That is,*

$$\Phi \circ g = \mathrm{Ad}_g^* \circ \Phi \tag{1.34}$$

for all $g \in G$.

Proof. Let $\xi \in \mathfrak{g}$, $g \in G$, and $p \in M$. Observe that

$$\begin{aligned} \xi_M(gp) &= \frac{d}{dt} \exp(t\xi)(gp)\Big|_{t=0} \\ &= \frac{d}{dt} \left(gg^{-1}\exp(t\xi)g\right)(p)\Big|_{t=0} \\ &= g_{*p}\left(\frac{d}{dt}\left(g^{-1}\exp(t\xi)g\right)(p)\Big|_{t=0}\right) \\ &= g_{*p}\left((\mathrm{Ad}_{g^{-1}}\xi)_M(p)\right). \end{aligned} \tag{1.35}$$

Using Equation 1.35 and the G-invariance of α, we have

$$\begin{aligned}\Phi^{\xi}(gp) &= \alpha_{gp}\left(\xi_M(gp)\right)\\ &= \alpha_{gp}\left(g_{*p}\left((\mathrm{Ad}_{g^{-1}}\xi)_M(p)\right)\right)\\ &= (g^*\alpha)_p\left((\mathrm{Ad}_{g^{-1}}\xi)_M(p)\right)\\ &= \alpha_p\left((\mathrm{Ad}_{g^{-1}}\xi)_M(p)\right)\\ &= \langle \Phi(p), \mathrm{Ad}_{g^{-1}}\xi\rangle\\ &= \langle \mathrm{Ad}_g^*(\Phi(p)), \xi_M(p)\rangle\\ &= (\mathrm{Ad}_g^* \circ \Phi)^{\xi}(p).\end{aligned} \tag{1.36}$$

□

The G-equivariance of Φ ensures that $\Phi^{-1}(0)$ is G-invariant. We denote the zero level set of the contact moment map by $Z = \Phi^{-1}(0)$.

Definition 1.19. The ***contact quotient*** of M by G is the topological space

$$X = Z/G. \tag{1.37}$$

Sometimes it is useful to denote this space by $M/\!\!/G$.

The process of taking this quotient is called ***contact reduction***. It is known that if G acts freely on Z, then zero is a regular value of the moment map $\Phi|_Z$ and X is a contact manifold ([7], Theorem 7.7.5). The contact form α on M descends to a contact form α_X on X. That is, $\pi^*\alpha_X = \iota^*\alpha$, where $\pi\colon Z \to X$ is the projection and $\iota\colon Z \to M$ is the inclusion. If the G-action is not free, then X is a singular space.

Thus arises the question of how to define a "differential form" on the singular space X. We answer this question in Chapter 3, generalizing the work done by Sjamaar in the symplectic case in [17]. We will show how to define a complex of differential forms on X for which the corresponding cohomology ring is isomorphic to the singular cohomology ring of X with real coefficients.

CHAPTER 2

JACOBI STRUCTURES

2.1 The Schouten-Nijenhuis Bracket

In this section we review the Schouten-Nijenhuis bracket, which is a generalization of the Lie bracket of vector fields. The Schouten-Nijenhuis bracket is used in the description of Jacobi structures on manifolds. For the following material, we refer to Koszul's paper [8]. Also informative are [6] and [14].

Let M be a manifold and let $\mathcal{T}(M) = \bigoplus_{k \geq 0} \mathcal{T}^k(M)$ be the graded algebra of multivector fields on M. Note that $\mathcal{T}^0(M) = C^\infty(M)$ and $\mathcal{T}^1(M) = \mathfrak{X}(M)$, which is the space of sections of TM. In general, $\mathcal{T}^k(M)$ is the space of sections of $\bigwedge^k(TM)$. The ***Schouten-Nijenhuis bracket*** on $\mathcal{T}(M)$ is an $\mathbb{R}$-bilinear mapping

$$[\cdot,\cdot] : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) \tag{2.1}$$

of degree -1 which is characterized by the following three properties:

(i) For all $\Xi \in \mathcal{T}^1(M) = \mathfrak{X}(M)$, the map $[\Xi,\cdot] : \mathcal{T}(M) \to \mathcal{T}(M)$ is the Lie derivative $\mathcal{L}_\Xi(\cdot)$ with respect to Ξ.

(ii) For all $\Xi \in \mathcal{T}^k(M)$ and $\Upsilon \in \mathcal{T}^p(M)$, we have $[\Xi,\Upsilon] = -(-1)^{(k-1)(p-1)}\,[\Upsilon,\Xi]$.

(iii) For all $\Xi \in \mathcal{T}^k(M)$, the map $[\Xi,\cdot] : \mathcal{T}(M) \to \mathcal{T}(M)$ is a derivation of degree $k-1$.

The formula found on the top of ([8], p. 266) describes how to evaluate the interior product of a tensor field with the Schouten-Nijenhuis bracket of two multivector fields. This formula states that if Ξ and Υ are multivector fields, then

$$i_{[\Xi,\Upsilon]} = [[i_\Xi, d], i_\Upsilon], \tag{2.2}$$

where the brackets on the right hand side are graded commutators. This formula is also discussed in ([14], Proposition 4.1 and Remark 4.2).

We clarify the meaning of the right hand side of Equation 2.2. For a multivector field $\Xi \in \mathcal{T}^k(M)$, observe that the interior product $i_\Xi(\cdot)\colon \Omega(M) \to \Omega(M)$ is of degree $-k$. Also, the exterior derivative $d\colon \Omega(M) \to \Omega(M)$ is of degree 1. So

$$[i_\Xi, d] = i_\Xi \circ d - (-1)^{-k} d \circ i_\Xi, \tag{2.3}$$

which is an endomorphism of $\Omega(M)$ of degree $1-k$. Now, assuming $\Upsilon \in \mathcal{T}^p(M)$, we have

$$\begin{aligned}
[[i_\Xi, d], i_\Upsilon] &= [i_\Xi, d] \circ i_\Upsilon - (-1)^{-p(1-k)} i_\Upsilon \circ [i_\Xi, d] \\
&= \big(i_\Xi \circ d - (-1)^{-k} d \circ i_\Xi\big) \circ i_\Upsilon - (-1)^{-p(1-k)} i_\Upsilon \circ \big(i_\Xi \circ d - (-1)^{-k} d \circ i_\Xi\big) \\
&= i_\Xi \circ d \circ i_\Upsilon - (-1)^{-k} d \circ i_\Xi \circ i_\Upsilon \\
&\qquad - (-1)^{-p(1-k)} i_\Upsilon \circ i_\Xi \circ d + (-1)^{-p(1-k)-k} i_\Upsilon \circ d \circ i_\Xi \\
&= i_\Xi \circ d \circ i_\Upsilon + (-1)^{-k} d \circ i_{\Xi \wedge \Upsilon} \\
&\qquad - (-1)^{-p(1-k)} i_{\Xi \wedge \Upsilon} \circ d + (-1)^{-p(1-k)-k} i_\Upsilon \circ d \circ i_\Xi.
\end{aligned} \tag{2.4}$$

2.2 Jacobi Manifolds

Let M be a manifold. A Poisson structure on M is a bilinear bracket on $C^\infty(M)$ that is skew-symmetric, satisfies the Jacobi identity, and is a derivation in each argument [12] [19]. Every symplectic manifold (M, ω) is a Poisson manifold, with the Poisson bracket defined by sending pairs of functions $f, h \in C^\infty(M)$ to the function $\omega(\Xi_f, \Xi_h) \in C^\infty(M)$, where Ξ_f and Ξ_h are the hamiltonian vector fields of f and h. The notion of a Jacobi structure generalizes the notion of a Poisson structure. The following definitions come from Lichnerowicz's paper [13]. Another informative reference is [3].

Definition 2.1. A ***Jacobi structure*** on M is a pair (Λ, E), where Λ is a multivector field of degree 2 and E is a vector field satisfying the following two equations:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \tag{2.5}$$

$$[E, \Lambda] = 0 \tag{2.6}$$

A manifold M equipped with a Jacobi structure is called a ***Jacobi manifold***.

The set of vector fields $\mathfrak{X}(M)$ on M is the Lie algebra of derivations of $C^\infty(M)$ ([4], Proposition 3.5.3). So we can view $E \in \mathcal{T}^1(M) = \mathfrak{X}(M)$ as a derivation of the algebra $C^\infty(M)$. It is defined by $E(f) = i_E df$ for all $f \in C^\infty(M)$. It is also enough to think of $\Lambda \in \mathcal{T}^2(M)$ as a skew-symmetric biderivation of the algebra $C^\infty(M)$. That is, Λ is a bilinear, skew-symmetric map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that $\Lambda(f, \cdot)\colon C^\infty(M) \to C^\infty(M)$ is a derivation. It is defined by the formula $\Lambda(f, h) = i_\Lambda(df \wedge dh)$ for all $f, h \in C^\infty(M)$. We take this point of view from now on, and our notation will reflect it.

Lichnerowicz uses the Jacobi structure on M to define a bracket on $C^\infty(M)$ as follows [13].

Definition 2.2. The ***Jacobi bracket*** is a bilinear operator

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \tag{2.7}$$

defined by

$$\{f, g\} = \Lambda(f, g) + fE(g) - gE(f) \tag{2.8}$$

for all $f, g \in C^\infty(M)$.

The Jacobi bracket is a generalization of the Poisson bracket, in the sense that it is skew-symmetric, it satisfies the Jacobi identity, and in a special case it is a derivation in each argument. We prove these claims below.

Proposition 2.3. *The Jacobi bracket $\{\cdot,\cdot\}$ is skew-symmetric.*

Proof. This follows from the skew-symmetry of $\Lambda(\cdot,\cdot)$. In detail, we see that

$$\begin{aligned}\{f,h\} &= \Lambda(f,h) + fE(h) - hE(f)\\ &= -\Lambda(h,f) - hE(f) + fE(h)\\ &= -\Big(\Lambda(h,f) + hE(f) - fE(h)\Big)\\ &= -\{h,f\}.\end{aligned} \tag{2.9}$$

□

To show that the Jacobi bracket satisfies the Jacobi identity, we first obtain some useful formulas. We begin by finding a formula for the interior product of the wedge of three 1-forms with the multivector field $[\Lambda,\Lambda] \in \mathcal{T}^3(M)$. Let $f, h, k \in C^\infty(M)$ and consider the 3-form $\eta = df \wedge dh \wedge dk$. Note that η is closed and is killed by interior products with multivector fields of degree greater than 3. So when we use Equations 2.2 and 2.4 with $\Xi = \Upsilon = \Lambda$, we get

$$\begin{aligned}i_{[\Lambda,\Lambda]}\eta &= i_\Lambda d i_\Lambda \eta + (-1)^{-2} d i_{\Lambda\wedge\Lambda}\eta\\ &\qquad - (-1)^{-2(1-2)} i_{\Lambda\wedge\Lambda} d\eta + (-1)^{-2(1-2)-2} i_\Lambda d i_\Lambda \eta\\ &= i_\Lambda d i_\Lambda \eta + (-1)^{2-2} i_\Lambda d i_\Lambda \eta\\ &= 2 i_\Lambda d i_\Lambda \eta.\end{aligned} \tag{2.10}$$

To get a better understanding of the right hand side of Equation 2.10, we describe how to take the interior product of η with the bivector field Λ. The following formula can be derived by evaluating the interior product of η with the wedge of two vector fields. Since Λ is necessarily a sum of such bivector fields, the technique can be generalized to obtain

$$\begin{aligned} i_\Lambda \eta &= i_\Lambda(df \wedge dh \wedge dk) \\ &= i_\Lambda(df \wedge dh) \cdot dk - i_\Lambda(df \wedge dk) \cdot dh + i_\Lambda(dh \wedge dk) \cdot df \\ &= \Lambda(f,h) \cdot dk - \Lambda(f,k) \cdot dh + \Lambda(h,k) \cdot df. \end{aligned} \tag{2.11}$$

So taking the exterior derivative results in

$$di_\Lambda \eta = d\left(\Lambda(f,h)\right) \wedge dk - \Lambda\left(\Lambda(f,k)\right) \wedge dh + \Lambda\left(\Lambda(h,k)\right) \wedge df. \tag{2.12}$$

Then taking the interior product with Λ gives us that

$$i_\Lambda d i_\Lambda \eta = \Lambda\left(\Lambda(f,h),k\right) - \Lambda\left(\Lambda(f,k),h\right) + \Lambda\left(\Lambda(h,k),f\right). \tag{2.13}$$

So a useful form of Equation 2.10 is:

$$\frac{1}{2}\left[\Lambda,\Lambda\right](f,h,k) = \Lambda\left(\Lambda(f,h),k\right) - \Lambda\left(\Lambda(f,k),h\right) + \Lambda\left(\Lambda(h,k),f\right). \tag{2.14}$$

Similarly to how we derived the interior product of η with Λ, we can find an expression for $i_{E\wedge\Lambda}\eta = (E \wedge \Lambda)(f,h,k)$. It turns out that

$$(E \wedge \Lambda)(f,h,k) = E(f)\Lambda(h,k) - E(h)\Lambda(f,k) + E(k)\Lambda(f,h). \tag{2.15}$$

We also note that since E is a vector field and Λ is a multivector field of degree 2, then $[E,\Lambda] = \mathcal{L}_E\Lambda$. So by the definition of the Lie derivative of a multivector field, we have that for all $f,h \in C^\infty(M)$,

$$\begin{aligned} \left[E,\Lambda\right](f,h) &= (\mathcal{L}_E\Lambda)(f,h) \\ &= E\left(\Lambda(f,h)\right) - \Lambda(\mathcal{L}_E f,h) - \Lambda(f,\mathcal{L}_E h) \\ &= E\left(\Lambda(f,h)\right) - \Lambda\left(E(f),h\right) - \Lambda\left(f,E(h)\right). \end{aligned} \tag{2.16}$$

Using Equations 2.14, 2.15, and 2.16 we can find an expression for the Jacobiator of $\{\cdot,\cdot\}$ in terms of Λ, E, and the Schouten-Nijenhuis bracket. The formula matches that in ([13], Equation 1.6).

Lemma 2.4. *For all $f, h, k \in C^\infty(M)$, we have*

$$\begin{aligned}\{\{f,h\},k\} + \{\{h,k\},f\} + \{\{k,f\},h\} &= \left(\frac{1}{2}[\Lambda,\Lambda] - (E\wedge\Lambda)\right)(f,h,k) \\ &- \Big(f\cdot[E,\Lambda](h,k) + h\cdot[E,\Lambda](k,f) + k\cdot[E,\Lambda](f,h)\Big).\end{aligned} \tag{2.17}$$

Proof. Using the definition of the Jacobi bracket, we find that

$$\begin{aligned}\{\{f,h\},k\} &= \Lambda(\{f,h\},k) + \{f,h\}E(k) - kE(\{f,h\}) \\ &= \Lambda\Big(\Lambda(f,h) + fE(h) - hE(f), k\Big) \\ &\quad + \Big(\Lambda(f,h) + fE(h) - hE(f)\Big)E(k) \\ &\quad - kE\Big(\Lambda(f,h) + fE(h) - hE(f)\Big) \\ &= \Lambda(\Lambda(f,h),k) + \Lambda(fE(h),k) - \Lambda(hE(f),k) \\ &\quad + \Lambda(f,h)E(k) + fE(h)E(k) - hE(f)E(k) \\ &\quad - kE(\Lambda(f,h)) - kE(fE(h)) + kE(hE(f)) \\ &= \Lambda(\Lambda(f,h),k) + f\Lambda(E(h),k) + E(h)\Lambda(f,k) \\ &\quad - h\Lambda(E(f),k) - E(f)\Lambda(h,k) \\ &\quad + \Lambda(f,h)E(k) + fE(h)E(k) - hE(f)E(k) \\ &\quad - kE(\Lambda(f,h)) - kfE(E(h)) - kE(h)E(f) \\ &\quad + khE(E(f)) + kE(f)E(h).\end{aligned} \tag{2.18}$$

Therefore, when the cyclic permutations of $\{\{f,h\},k\}$ are added together, some terms cancel each other out and we are left with

$$\begin{aligned}
&\{\{f,h\},k\}+\{\{h,k\},f\}+\{\{k,f\},h\}\\
&\quad=\Lambda(\Lambda(f,h),k)+f\Lambda(E(h),k)+E(h)\Lambda(f,k)\\
&\qquad-h\Lambda(E(f),k)-kE(\Lambda(f,h))\\
&\qquad+\Lambda(\Lambda(h,k),f)+h\Lambda(E(k),f)+E(k)\Lambda(h,f)\\
&\qquad-k\Lambda(E(h),f)-fE(\Lambda(h,k))\\
&\qquad+\Lambda(\Lambda(k,f),h)+k\Lambda(E(f),h)+E(f)\Lambda(k,h)\\
&\qquad-f\Lambda(E(k),h)-hE(\Lambda(k,f)).
\end{aligned} \tag{2.19}$$

Using the skew-symmetry of Λ and grouping relevant terms together, we have

$$\begin{aligned}
&\{\{f,h\},k\}+\{\{h,k\},f\}+\{\{k,f\},h\}\\
&\quad=\Lambda(\Lambda(f,h),k)-\Lambda(\Lambda(f,k),h)+\Lambda(\Lambda(h,k),f)\\
&\qquad-(E(f)\Lambda(h,k)-E(h)\Lambda(f,k)+E(k)\Lambda(f,h))\\
&\qquad-f\Big(E(\Lambda(h,k))-\Lambda(E(h),k)-\Lambda(h,E(k))\Big)\\
&\qquad-h\Big(E(\Lambda(k,f))-\Lambda(E(k),f)-\Lambda(k,E(f))\Big)\\
&\qquad-k\Big(E(\Lambda(f,h))-\Lambda(E(f),h)-\Lambda(f,E(h))\Big).
\end{aligned} \tag{2.20}$$

Finally, Equations 2.14, 2.15, and 2.16 imply the result. □

Proposition 2.5. *The Jacobi bracket satisfies the Jacobi identity. That is, for all $f,h,k\in C^\infty(M)$, we have*

$$\{\{f,h\},k\}+\{\{h,k\},f\}+\{\{k,f\},h\}=0. \tag{2.21}$$

Proof. This follows by using the facts that $[\Lambda,\Lambda]=2E\wedge\Lambda$ and $[E,\Lambda]=0$ in the statement of Lemma 2.4. □

Note that Propositions 2.3 and 2.5 imply that the Jacobi bracket gives $C^\infty(M)$ the structure of a Lie algebra.

To finish showing that the Jacobi bracket is a generalization of the Poisson bracket, it remains to show that in certain cases it is a Poisson bracket. It turns out that if the vector field E vanishes, the Jacobi bracket defines a Poisson bracket on M. We prove this below.

Proposition 2.6. *If E vanishes, then the Jacobi bracket $\{\cdot,\cdot\}$ is a Poisson bracket.*

Proof. We have already shown that the bilinear Jacobi bracket is skew-symmetric and satisfies the Jacobi identity. So it remains to show that if E vanishes, $\{\cdot,\cdot\}$ is a derivation in each argument. By skew-symmetry, it suffices to show this in just one argument. Let $f, h, k \in C^\infty(M)$. Since E is a derivation and Λ is a biderivation, observe that

$$\begin{aligned}\{f, hk\} &= \Lambda(f, hk) + fE(hk) - hkE(f)\\ &= h\Lambda(f,k) + k\Lambda(f,h) + f(hE(k) + kE(h)) - hkE(f)\\ &= h\Lambda(f,k) + k\Lambda(f,h) + fhE(k) + fkE(h) - hkE(f)\\ &= h\Big(\Lambda(f,k) + fE(k) - kE(f)\Big) + k\Big(\Lambda(f,h) + fE(h) - hE(f)\Big) + hkE(f)\\ &= h\{f,k\} + k\{f,h\} + hkE(f).\end{aligned} \tag{2.22}$$

If E vanishes, then we have

$$\{f, hk\} = h\{f,k\} + k\{f,h\}. \tag{2.23}$$

In other words, $\{f,\cdot\} : C^\infty(M) \to C^\infty(M)$ is a derivation. □

Equation 2.22 suggests that the Jacobi bracket determines the pair (Λ, E). Indeed, if $h = k = 1$, then we get

$$\{f, 1\} = \{f,1\} + \{f,1\} + E(f), \tag{2.24}$$

for all $f \in C^\infty(M)$. Solving for $E(f)$, we have

$$E(f) = -\{f, 1\}, \tag{2.25}$$

for all $f \in C^\infty(M)$. Using this in the definition of the Jacobi bracket and solving for $\Lambda(f, h)$ results in

$$\Lambda(f, h) = \{f, h\} + f\{h, 1\} - h\{f, 1\}, \tag{2.26}$$

for all $f, h \in C^\infty(M)$. Equation 2.25 defines E as a map $C^\infty(M) \to C^\infty(M)$. Equation 2.26 defines Λ as a map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. In Section 2.4 we will define an abstract Jacobi bracket $\{\cdot, \cdot\}$ so that the above equations define E and Λ as a derivation and a skew-symmetric biderivation, respectively.

Example 2.7. A contact manifold is a Jacobi manifold [3]. Let $(M, \mathcal{H}, \alpha)$ be a $(2n+1)$-dimensional strict contact manifold. Define the biderivation Λ by

$$\Lambda(f, h) = d\alpha(df^\sharp, dh^\sharp) \tag{2.27}$$

for all $f, h \in C^\infty(M)$, where $\sharp$ is the inverse of the isomorphism $\flat \colon \mathfrak{X}(M) \to \Omega^1(M)$ described in Equation 1.10. (Equivalently, the bivector field Λ may be defined by $i_\Lambda(\eta \wedge \beta) = d\alpha(\eta^\sharp, \beta^\sharp)$ for all $\eta, \beta \in \Omega^1(M)$.) Define E to be the Reeb vector field with respect to α. Then the pair (Λ, E) satisfies the properties in the definition of a Jacobi structure.

2.3 Relationship to Contact Vector Fields

In this section, $(M, \mathcal{H}, \alpha)$ is a strict contact manifold with Jacobi structure (E, Λ) as given in Example 2.7. First we explore the structure on the set of contact vector fields $\mathfrak{X}_{\mathrm{con}}(M)$, then we relate the Jacobi structure to the hamiltonian correspondence.

Proposition 2.8. *The family of contact vector fields $\mathfrak{X}_{\text{con}}(M)$ is a Lie algebra, with the usual Lie bracket on vector fields.*

Proof. Let Ξ and Υ be contact vector fields on M. So by Proposition 1.12, there exist $\mu_\Xi, \mu_\Upsilon \in C^\infty(M)$ such that $\mathcal{L}_\Xi\alpha = \mu_\Xi\alpha$ and $\mathcal{L}_\Upsilon\alpha = \mu_\Upsilon\alpha$. We check that the Lie bracket $[\Xi, \Upsilon]$ of Ξ and Υ is also a contact vector field by calculating the Lie derivative of α with respect to it. Observe that

$$\begin{aligned}
\mathcal{L}_{[\Xi,\Upsilon]}\alpha &= \left(\mathcal{L}_\Xi\mathcal{L}_\Upsilon - \mathcal{L}_\Upsilon\mathcal{L}_\Xi\right)\alpha \\
&= \mathcal{L}_\Xi\left(\mu_\Upsilon\alpha\right) - \mathcal{L}_\Upsilon\left(\mu_\Xi\alpha\right) \\
&= \left(\mathcal{L}_\Xi\mu_\Upsilon\right)\alpha + \mu_\Upsilon\mathcal{L}_\Xi\alpha - \left(\mathcal{L}_\Upsilon\mu_\Xi\right)\alpha - \mu_\Xi\mathcal{L}_\Upsilon\alpha \\
&= \left(\mathcal{L}_\Xi\mu_\Upsilon - \mathcal{L}_\Upsilon\mu_\Xi\right)\alpha + \mu_\Upsilon\mu_\Xi\alpha - \mu_\Xi\mu_\Upsilon\alpha \\
&= \left(\mathcal{L}_\Xi\mu_\Upsilon - \mathcal{L}_\Upsilon\mu_\Xi\right)\alpha.
\end{aligned} \tag{2.28}$$

Since $\left(\mathcal{L}_\Xi\mu_\Upsilon - \mathcal{L}_\Upsilon\mu_\Xi\right) \in C^\infty(M)$, we have shown that $[\Xi, \Upsilon]$ is a contact vector field. □

Recall that $C^\infty(M)$ is a Lie algebra, with the Lie bracket given by the Jacobi bracket. It turns out that the hamiltonian correspondence between $\mathfrak{X}_{\text{con}}(M)$ and $C^\infty(M)$ given in Theorem 1.15 is a Lie algebra isomorphism ([11], Proposition 13.5).

To study the hamiltonian correspondence further, we first obtain an explicit way to express it.

Proposition 2.9. *The hamiltonian vector field of $f \in C^\infty(M)$ is given by*

$$\Xi_f = (-df + (f + i_E df)\alpha)^\sharp. \tag{2.29}$$

Proof. Let Ξ_f denote the hamiltonian vector field of f. We apply the flat map to it and utilize the hamiltonian correspondence to obtain

$$\begin{aligned}(\Xi_f)^\flat &= i_{\Xi_f} d\alpha + i_{\Xi_f}\alpha \cdot \alpha \\ &= i_E df \cdot \alpha - df + i_{\Xi_f}\alpha \cdot \alpha \\ &= i_E df \cdot \alpha - df + f\alpha \\ &= -df + (f + i_E df)\alpha.\end{aligned} \tag{2.30}$$

This is precisely $((-df + (f + i_E df)\alpha)^\sharp)^\flat$, since $\flat$ and $\sharp$ are inverses. Since $\flat$ is an isomorphism, this proves the proposition. □

The explicit formula for the hamiltonian vector field of a function allows us to come up with other useful formulas.

Lemma 2.10. *For all $f \in C^\infty(M)$, we have*

$$i_{df^\sharp}\alpha = i_E df. \tag{2.31}$$

Proof. We distribute the sharp map in the explicit hamiltonian correspondence from Proposition 2.9 to get

$$\Xi_f = -df^\sharp + (f + i_E df)E. \tag{2.32}$$

Solving for $df^\sharp$, we obtain

$$df^\sharp = -\Xi_f + (f + i_E df)E. \tag{2.33}$$

So, by using the defining properties of Ξ_f and E, we have

$$\begin{aligned}i_{df^\sharp}\alpha &= i_{-\Xi_f + (f + i_E df)E}\alpha \\ &= -i_{\Xi_f}\alpha + (f + i_E df) i_E\alpha \\ &= -f + f + i_E df \\ &= i_E df.\end{aligned} \tag{2.34}$$

□

Lemma 2.11. *For all $f \in C^\infty(M)$, we have*

$$i_{df^\sharp} d\alpha = -i_{\Xi_f} d\alpha. \tag{2.35}$$

Proof. Using Equation 2.33 and the definition of the Reeb vector field, we have

$$\begin{aligned} i_{df^\sharp} d\alpha &= i_{-\Xi_f + (f + i_E df)E} d\alpha \\ &= -i_{\Xi_f} d\alpha + (f + i_E df) i_E d\alpha \\ &= -i_{\Xi_f} d\alpha. \end{aligned} \tag{2.36}$$

□

Lemma 2.11 allows us to evaluate Λ in two new ways. The first way involves the hamiltonian vector fields of the input functions.

Lemma 2.12. *For all $f, h \in C^\infty(M)$, we have*

$$\Lambda(f, h) = -i_{\Xi_f} i_{\Xi_h} d\alpha. \tag{2.37}$$

Proof. Using Lemma 2.11, we have

$$\begin{aligned} \Lambda(f, h) &= i_{dh^\sharp} i_{df^\sharp} d\alpha \\ &= -i_{dh^\sharp} i_{\Xi_f} d\alpha \\ &= i_{\Xi_f} i_{dh^\sharp} d\alpha \\ &= -i_{\Xi_f} i_{\Xi_h} d\alpha. \end{aligned} \tag{2.38}$$

□

The second new way to evaluate Λ involves the Reeb vector field and the hamiltonian vector field of one of the input functions.

Lemma 2.13. *For all $f, h \in C^\infty(M)$, we have*

$$\Lambda(f, h) = -fE(h) + \Xi_f(h). \tag{2.39}$$

Proof. Using Lemma 2.12 and the hamiltonian correspondence, we have

$$\begin{aligned}\Lambda(f,h) &= -i_{\Xi_f} i_{\Xi_h} d\alpha \\ &= -i_{\Xi_f}(i_E dh \cdot \alpha - dh) \\ &= -i_{\Xi_f} i_E dh \cdot \alpha + i_{\Xi_f} dh \\ &= -i_E dh \cdot i_{\Xi_f}\alpha + i_{\Xi_f} dh \\ &= -f i_E dh + i_{\Xi_f} dh \\ &= -fE(h) + \Xi_f(h).\end{aligned} \tag{2.40}$$

□

Notice that an alternate way to write this formula is

$$\Lambda(f,h) = hE(f) - \Xi_h(f). \tag{2.41}$$

The final relationship between contact vector fields and the Jacobi structure that we will present in this section is that the hamiltonian vector field of a function $f \in C^\infty(M)$ may in fact be written in terms of the Jacobi bracket.

Proposition 2.14. *The hamiltonian vector field Ξ_f of $f \in C^\infty(M)$ is defined as a derivation by*

$$\Xi_f(h) = \{f,h\} - h\{f,1\}, \tag{2.42}$$

for all $h \in C^\infty(M)$.

Proof. Let Ξ_f be the hamiltonian vector field of $f \in C^\infty(M)$. For all $h \in C^\infty(M)$, let $\Theta_f(h) = \{f,h\} - h\{f,1\}$. We aim to show $\Theta_f = \Xi_f$. We calculate $\Theta_f(h)$ using the definition of the Jacobi bracket on $C^\infty(M)$ and Equation 2.25. Observe that

$$\begin{aligned}\Theta_f(h) &= \{f,h\} - h\{f,1\} \\ &= \Lambda(f,h) + fE(h) - hE(f) + hE(f) \\ &= \Lambda(f,h) + fE(h).\end{aligned} \tag{2.43}$$

The map $h \mapsto \Lambda(f, h)$ is described in Lemma 2.13. We use this to obtain

$$\begin{aligned} \Theta_f(h) &= -fE(h) + \Xi_f(h) + fE(h) \\ &= \Xi_f(h). \end{aligned} \tag{2.44}$$

So indeed $\Theta_f = \Xi_f$. It is easy to check that the map given in the proposition is a derivation by using Equation 2.22. □

Note one can also write this relationship as

$$\Xi_f(h) = \{f, h\} + hE(f). \tag{2.45}$$

2.4 Jacobi Algebras

It is useful to define a Jacobi bracket on an abstract algebra, instead of just on the specific one of smooth functions on a manifold. An abstract definition of the Jacobi bracket allows us to define Jacobi structures on singular contact quotients.

Definition 2.15. Let R be a commutative ring. A commutative R-algebra A is a ***Jacobi algebra*** if it has a binary operation $\{\cdot, \cdot\} : A \times A \to A$ that satisfies the following four axioms:

(i) It is R-bilinear.

(ii) It is skew-symmetric.

(iii) It satisfies the Jacobi identity

$$\{\{f, h\}, k\} + \{\{h, k\}, f\} + \{\{k, f\}, h\} = 0, \tag{2.46}$$

for all $f, h, k \in A$.

(iv) It satisfies the equation

$$\{f, hk\} = h\{f, k\} + k\{f, h\} - hk\{f, 1\}, \tag{2.47}$$

for all $f, h, k \in A$.

We call such a bracket $\{\cdot, \cdot\}$ a ***Jacobi bracket*** on the algebra A. We will see that it is analogous to our prior notion of Jacobi bracket, but this one is defined on an abstract algebra. Note that bilinearity implies that $\{f, 0\} = 0$ for all $f \in C^\infty(M)$. Note that skew-symmetry implies that $\{f, f\} = 0$ for all $f \in C^\infty(M)$. Also note that property (iv) serves as a product rule for the map $\{f, \cdot\} : A \to A$, which is not a derivation unless f is a constant function.

Let M be a manifold and consider $A = C^\infty(M)$. Assume that A is a Jacobi algebra, that is, it carries a Jacobi bracket in the sense of Definition 2.15. In order to see that our abstract definition of Jacobi bracket makes sense and is compatible with the definition of a Jacobi manifold, we check that the Jacobi bracket on A determines a Jacobi structure on M. As in Equations 2.25 and 2.26, we define E and Λ by:

$$E(f) = -\{f, 1\} = \{1, f\} \tag{2.48}$$

and

$$\Lambda(f, h) = \{f, h\} + f\{h, 1\} - h\{f, 1\} \tag{2.49}$$

for all $f, h \in A$.

We check that the axioms in Definition 2.15 are sufficient for (Λ, E) to be a Jacobi structure on M. First we show that E is derivation of A and Λ is a skew-symmetric biderivation of A.

Lemma 2.16. *E is a derivation of A.*

Proof. Note that E is a linear map since the Jacobi bracket is bilinear. Let $f, h \in C^\infty(M)$. We see that E is a derivation by using the product rule for the Jacobi bracket and finding that

$$\begin{aligned} E(fh) &= -\{fh, 1\} \\ &= -f\{h,1\} - h\{f,1\} + fh\{1,1\} \\ &= fE(h) + hE(f). \end{aligned} \tag{2.50}$$

□

Lemma 2.17. *Λ is a skew-symmetric biderivation of A.*

Proof. Note that Λ is a bilinear skew-symmetric map since the Jacobi bracket is bilinear and skew-symmetric. Let $f, h, k \in C^\infty(M)$. We see that Λ is a biderivation by using the product rule for the Jacobi bracket and finding that

$$\begin{aligned} \Lambda(f, hk) &= \{f, hk\} + f\{hk, 1\} - hk\{f, 1\} \\ &= h\{f,k\} + k\{f,h\} - hk\{f,1\} + f(h\{k,1\} + k\{h,1\}) - hk\{f,1\} \\ &= h\Big(\{f,k\} + f\{k,1\} - k\{f,1\}\Big) + k\Big(\{f,h\} + f\{h,1\} - h\{f,1\}\Big) \\ &= h\Lambda(f,k) + k\Lambda(f,h). \end{aligned} \tag{2.51}$$

□

Now we check that the pair (E, Λ) satisfies the defining properties of a Jacobi structure on M.

Lemma 2.18. $[E, \Lambda] = 0$.

Proof. Let $f, h \in C^\infty(M)$. We use Equation 2.16 and the definitions of E and Λ

in terms of the Jacobi bracket to get

$$\begin{aligned}
[E,\Lambda](f,h) &= E(\Lambda(f,h)) - \Lambda(E(f),h) - \Lambda(f,E(h)) \\
&= -\{\Lambda(f,h),1\} - \Lambda(-\{f,1\},h) - \Lambda(f,-\{h,1\}) \\
&= -\{\{f,h\} + f\{h,1\} - h\{f,1\},1\} \\
&\quad + \{\{f,1\},h\} + \{f,1\}\{h,1\} - h\{\{f,1\},1\} \\
&\quad + \{f,\{h,1\}\} + f\{\{h,1\},1\} - \{h,1\}\{f,1\} \\
&= -\{\{f,h\},1\} - \{f\{h,1\},1\} + \{h\{f,1\},1\} \\
&\quad + \{\{f,1\},h\} - h\{\{f,1\},1\} \\
&\quad + \{f,\{h,1\}\} + f\{\{h,1\},1\}.
\end{aligned} \tag{2.52}$$

By the product rule for the Jacobi bracket, we have

$$\begin{aligned}
[E,\Lambda](f,h) &= -\{\{f,h\},1\} - f\{\{h,1\},1\} - \{h,1\}\{f,1\} \\
&\quad + h\{\{f,1\},1\} + \{f,1\}\{h,1\} \\
&\quad + \{\{f,1\},h\} - h\{\{f,1\},1\} \\
&\quad + \{f,\{h,1\}\} + f\{\{h,1\},1\} \\
&= -\{\{f,h\},1\} + \{\{f,1\},h\} + \{f,\{h,1\}\}.
\end{aligned} \tag{2.53}$$

Finally, we use the Jacobi identity to obtain

$$\begin{aligned}
[E,\Lambda](f,h) &= -\Big(\{\{f,h\},1\} + \{\{1,f\},h\} + \{\{h,1\},f\} \Big) \\
&= 0.
\end{aligned} \tag{2.54}$$

□

Lemma 2.19. $[\Lambda,\Lambda] = 2E \wedge \Lambda$.

Proof. By the definitions of E and Λ in terms of the Jacobi bracket, we know that $\{f,g\} = \Lambda(f,g) + fE(g) - gE(f)$. This is the same way the Jacobi bracket was defined on a Jacobi manifold. So the formula obtained in Lemma 2.4 is valid here.

That is, we know that

$$\begin{aligned}\{\{f,h\},k\}+\{\{h,k\},f\}+\{\{k,f\},h\} &= \left(\frac{1}{2}[\Lambda,\Lambda]-(E\wedge\Lambda)\right)(f,h,k)\\ &\quad -\Big(f\cdot[E,\Lambda](h,k)+h\cdot[E,\Lambda](k,f)+k\cdot[E,\Lambda](f,h)\Big),\end{aligned} \tag{2.55}$$

for all $f,h,k \in A$. The left hand side of this equation is 0 by the Jacobi identity, and according to Lemma 2.18, $[E,\Lambda]=0$. This forces

$$\frac{1}{2}[\Lambda,\Lambda]-(E\wedge\Lambda)=0, \tag{2.56}$$

which implies the result. □

2.5 Jacobi Subalgebras and Jacobi Ideals

In this section, R is still a commutative ring, and A is a Jacobi algebra in the sense of Definition 2.15. Note that A has a usual multiplication operation and a Jacobi bracket $\{\cdot,\cdot\}$.

Definition 2.20. A submodule $B \subset A$ is a ***Jacobi subalgebra*** of A if B is closed under both multiplication and the Jacobi bracket.

Let G be a group acting on A. We denote the endomorphism of A corresponding to $g \in G$ by $f \mapsto g \cdot f$ for all $f \in A$. We determine conditions under which the subspace of invariants $A^G = \{f \in A : g\cdot f = f \text{ for all } g \in G\}$ is a Jacobi subalgebra.

Proposition 2.21. *The subspace of invariants A^G is a Jacobi subalgebra of A if for all $g \in G$ and all $f,h \in A$, the following conditions are satisfied:*

$$g\cdot(fh)=(g\cdot f)(g\cdot h) \tag{2.57}$$

$$\{g\cdot f, g\cdot h\} = g\cdot\{f,h\} \tag{2.58}$$

Proof. Let $g \in G$ and $f, h \in A^G$, and assume the two given conditions. Equation 2.57 guarantees the closure of A^G under multiplication. Equation 2.58 guarantees the closure of A^G under the Jacobi bracket. □

Note that the second condition in Proposition 2.21 is the same as saying that the Jacobi bracket is G-equivariant.

Now let I be an ideal of A with respect to multiplication. The quotient $\bar{A} = A/I$ is a commutative R-algebra. We find conditions on I under which $\bar{A}$ is a Jacobi algebra.

Definition 2.22. An ideal I of A with respect to multiplication is a ***Jacobi ideal*** if for all $f \in A$ and all $h \in I$, $\{f, h\} \in I$.

Proposition 2.23. *The algebra $\bar{A}$ inherits a Jacobi bracket $\{\cdot,\cdot\}_{\bar{A}}$ if and only if I is a Jacobi ideal.*

Proof. Let I be a Jacobi ideal. We define a bracket on $\bar{A}$ descending from the Jacobi bracket on A. Let $[f] = f + I$ and $[h] = h + I$ denote elements in $\bar{A}$. Define

$$\{[f], [h]\}_{\bar{A}} = [\{f, h\}]. \tag{2.59}$$

This bracket is well-defined because if f and f' are in the same equivalence class, and h and h' are in the same equivalence class, then $f' = f + b$ and $h' = h + c$ for some $b, c \in I$, so

$$\{f', h'\} = \{f + b, h + c\} = \{f, h\} + \{f, c\} + \{b, h\} + \{b, c\}. \tag{2.60}$$

Since I is a Jacobi ideal, then $\{f, c\} + \{b, h\} + \{b, c\} \in I$. Therefore,

$$[\{f', h'\}] = [\{f, h\}]. \tag{2.61}$$

The definition of $\{\cdot,\cdot\}_{\bar{A}}$ in Equation 2.59 satisfies all the axioms of a Jacobi bracket because $\{\cdot,\cdot\}$ does, and $\bar{A}$ is a quotient ring and a quotient module.

Now assume that the bracket $\{\cdot,\cdot\}_{\bar{A}}$ defined by Equation 2.59 is a Jacobi bracket. Let $f \in A$ and $h \in I$. To show that I is a Jacobi ideal, we need to show that $\{f, h\} \in I$. It is enough to show that $[\{f, h\}] = [0] \in \bar{A}$. Using the fact that $[h] = [0]$, we see that

$$[\{f, h\}] = \{[f], [h]\}_{\bar{A}} = \{[f], [0]\}_{\bar{A}} = [0]. \tag{2.62}$$

□

Propositions 2.21 and 2.23 are used later on in the context of contact reduction, to show that the algebra of “smooth functions” on a singular contact quotient carries a Jacobi bracket.

2.6 Examples of Jacobi Algebras

Let $(M, \mathcal{H}, \alpha)$ be a strict contact manifold. We have already seen that M is a Jacobi manifold. In other words, the algebra $C^\infty(M)$ of smooth real-valued functions on M is a Jacobi algebra. The Jacobi bracket on this algebra is given by

$$\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f), \tag{2.63}$$

where E is the Reeb vector field associated to α and Λ is defined as in Example 2.7.

Now consider the contact action of a compact connected Lie group G on M. For any $g \in G$, we denote the corresponding contactomorphism $g\colon M \to M$ by $p \mapsto gp$, for all $p \in M$. In this section, we show that the submodule $C^\infty(M)^G$

of smooth real-valued functions on M that are invariant under the G-action is a Jacobi subalgebra of $C^\infty(M)$.

First, note that an action of G on M induces actions on vector fields $\mathfrak{X}(M)$ and differential forms $\Omega(M)$. There is a left-action on $\mathfrak{X}(M)$ given by $X \mapsto g_*X$ for all $g \in G$. There is a left-action of G on $\Omega(M)$ given by $\beta \mapsto (g^{-1})^*\beta$ for all $g \in G$. In particular, the G-action on $C^\infty(M) = \Omega^0(M)$ is given by $f \mapsto (g^{-1})^*f$ for all $g \in G$. This is the action of G on the Jacobi algebra $C^\infty(M)$.

To prove that $C^\infty(M)^G$ is a Jacobi subalgebra of $C^\infty(M)$, we check the two conditions in Proposition 2.21. The first condition is met automatically, since by the definition of function multiplication and pullback, we have

$$g^*(fh) = (g^*f)(g^*h), \tag{2.64}$$

for all $g \in G$ and all $f, h \in C^\infty(M)$. So it remains to check the G-equivariance of the Jacobi bracket.

In order to do this, we need a lemma about the flat map. As a preliminary, we recall how to take pullbacks of forms which are themselves interior products. Let $q\colon N \to \widetilde{N}$ denote a diffeomorphism between manifolds. Let $\beta \in \Omega(\widetilde{N})$. Let Ξ be a vector field on $\widetilde{N}$, and let Υ be a multivector field on N. Then by the definition of pullback and our interior product convention, we have

$$\begin{aligned}(q^*(i_\Xi\beta))(\Upsilon) &= i_\Xi\beta(q_*\Upsilon)\\ &= \beta(\Xi \wedge q_*\Upsilon)\\ &= \beta\left(q_*\left(q_*^{-1}\Xi \wedge \Upsilon\right)\right)\\ &= (q^*\beta)\left(q_*^{-1}\Xi \wedge \Upsilon\right)\\ &= i_{q_*^{-1}\Xi}(q^*\beta)(\Upsilon).\end{aligned} \tag{2.65}$$

Now we are in a position to prove that the flat map is G-equivariant.

Lemma 2.24. *The flat map* $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ *defined by* $\Xi^\flat = i_\Xi d\alpha + i_\Xi\alpha \cdot \alpha$ *is equivariant with respect to the actions on* $\mathfrak{X}(M)$ *and* $\Omega(M)$ *induced by the* G*-action on* M*.*

Proof. Let $g \in G$ and $\Xi \in \mathfrak{X}(M)$. Using Equation 2.65, the fact that pullbacks commute with the exterior derivative, and the fact that α is G-invariant, we have

$$\begin{aligned} \left(g^{-1}\right)^* \left(\Xi^\flat\right) &= \left(g^{-1}\right)^* i_\Xi d\alpha + \left(g^{-1}\right)^* \left(i_\Xi\alpha \cdot \alpha\right) \\ &= i_{g_*\Xi} d\left(\left(g^{-1}\right)^* \alpha\right) + i_{g_*\Xi}\left(\left(g^{-1}\right)^* \alpha\right) \cdot \left(\left(g^{-1}\right)^* \alpha\right) \\ &= i_{g_*\Xi} d\alpha + i_{g_*\Xi}\alpha \cdot \alpha \\ &= \left(g_*\Xi\right)^\flat. \end{aligned} \tag{2.66}$$

Note that another way to write this equivariance is

$$g_*\left(\eta^\sharp\right) = \left(\left(g^{-1}\right)^* \eta\right)^\sharp \tag{2.67}$$

for all $\eta \in \Omega^1(M)$. $\square$

Now we can prove that the Jacobi bracket on $C^\infty(M)$ is G-equivariant.

Proposition 2.25. *For all* $g \in G$ *and all* $f, h \in C^\infty(M)$*, we have*

$$g^*\{f, h\} = \{g^*f, g^*h\}. \tag{2.68}$$

Proof. Let $f, h \in C^\infty(M)$. Let $g \in G$. Then by Lemma 2.24, the fact that $d\alpha$ is G-invariant, and the fact that pullback is compatible with exterior derivative, we have

$$
\begin{aligned}
g^*\left(\Lambda(f,h)\right) &= g^*(i_{dh^\sharp} i_{df^\sharp} d\alpha) \\
&= i_{g_*^{-1}(dh^\sharp)}(g^* i_{df^\sharp} d\alpha) \\
&= i_{g_*^{-1}(dh^\sharp)} i_{g_*^{-1}(df^\sharp)}(g^* d\alpha) \\
&= i_{(g^*dh)^\sharp} i_{(g^*df)^\sharp}(d\alpha) \\
&= i_{(d(g^*h))^\sharp} i_{(d(g^*f))^\sharp}(d\alpha) \\
&= \Lambda(g^*f, g^*h).
\end{aligned}
\tag{2.69}
$$

Also, since E is G-invariant, we have

$$
\begin{aligned}
g^*(fE(h)) &= (g^*f)\left(g^*(E(h))\right) \\
&= (g^*f)\left(\left({g_*}^{-1}E\right)(g^*h)\right) \\
&= (g^*f)\left(E(g^*h)\right).
\end{aligned}
\tag{2.70}
$$

Using Equations 2.69 and 2.70, we obtain

$$
\begin{aligned}
g^*\{f,h\} &= g^*(\Lambda(f,h)) + g^*(fE(h)) - g^*(hE(f)) \\
&= \Lambda(g^*f, g^*h) + (g^*f)(E(g^*h)) - (g^*h)(E(g^*f)) \\
&= \{g^*f, g^*h\}.
\end{aligned}
\tag{2.71}
$$

□

Proposition 2.26. *The smooth G-invariant functions $C^\infty(M)^G$ form a Jacobi subalgebra of $C^\infty(M)$.*

Proof. Since both conditions in Proposition 2.21 are met, we can conclude that $C^\infty(M)^G$ is a Jacobi subalgebra of $C^\infty(M)$. □

CHAPTER 3

A DE RHAM THEOREM FOR CONTACT QUOTIENTS

3.1 Introduction

Let G be a compact Lie group which acts on a connected symplectic manifold M, with equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$. Then the Marsden-Meyer-Weinstein symplectic quotient $X = \Phi^{-1}(0)/G$ is in general a singular space, with a stratification into symplectic manifolds [18]. Let $\Omega(X)$ denote the de Rham complex of differential forms on X, defined to be those forms on the principal stratum of X which are induced by forms on M. Sjamaar proved that the de Rham cohomology ring $H(\Omega(X))$ is naturally isomorphic to the (Čech or singular) cohomology ring of X with real coefficients $H(X, \mathbb{R})$ [17].

Suppose instead that M is a manifold with a cooriented contact structure $\mathcal{H}$, a G-action that preserves $\mathcal{H}$, and a global G-invariant contact form α. The G-action has an equivariant contact moment map $\Phi\colon M \to \mathfrak{g}^*$ and is the analogue of a hamiltonian action in the symplectic setting. The corresponding reduced space is the contact quotient $X = \Phi^{-1}(0)/G$, as defined by Lerman and Willett [10]. This is not a smooth manifold unless G acts freely on $\Phi^{-1}(0)$. However, analogously to the symplectic case, it has a stratification into contact manifolds [21].

In this chapter, we extend Sjamaar's de Rham model for symplectic quotients to a de Rham model for contact quotients. We define the notion of a differential form on the stratified contact quotient X similar to that defined on a stratified symplectic quotient and get a definition for the de Rham complex $\Omega(X)$. Then, using a local normal form theorem for contact quotients, we prove a local Poincaré lemma which allows us to conclude that the de Rham cohomology ring $H(\Omega(X))$ is isomorphic to the (Čech or singular) cohomology ring of X with real coefficients $H(X, \mathbb{R})$.

An example of a differential form on the contact quotient X is the reduced contact form α_X. We show that this is a contact form on each stratum and, with the help of a Stokes' theorem for contact quotients, that it gives rise to a nonzero top-dimensional cohomology class.

3.2 Stratification of Contact Quotients

We adopt the set up of Section 1.4. That is, $(M, \mathcal{H}, \alpha)$ is a strict contact manifold with a contact action of a compact connected Lie group G and associated moment map $\Phi \colon M \to \mathfrak{g}^*$. Recall that because G is compact and connected, the contact form α is G-invariant. The zero fiber of the moment map is denoted by $Z = \Phi^{-1}(0)$ and the contact quotient by $X = Z/G$. Since X is in general a singular space, we proceed as in [10], summarized below.

For any $p \in M$, denote the stabilizer subgroup of p by $G_p = \{g \in G : gp = p\}$. Recall that for a subgroup H of G, the set of points of ***orbit type*** H is

$$M_{(H)} = \{p \in M : \exists g \in G \text{ such that } gG_pg^{-1} = H\}. \tag{3.1}$$

The ***canonical partition*** of X is given by the connected components of sets of the form

$$\frac{Z \cap M_{(H)}}{G} \tag{3.2}$$

for all conjugacy classes (H) of G. The canonical partition of Z is given by the connected components of sets of the form

$$Z \cap M_{(H)} \tag{3.3}$$

for all conjugacy classes (H) of G.

By ([10], Theorem 1), the canonical partition of X is a ***stratification*** of X, in the following sense. For every subgroup H of G, each connected component S of $(Z \cap M_{(H)})/G$ is a manifold. These manifolds partition X. For every component S and every point $p \in S$, there exist an open neighborhood U of p, an open ball B in S around p, a compact stratified space L, called the ***link*** of p, and an isomorphism $\rho\colon B \times \mathring{c}(L) \to U$ of partitioned spaces, where $\mathring{c}(L)$ is the cone on L. If $L = \emptyset$, then U is required to be homeomorphic to the ball B. Each component S is called a ***stratum*** of X. We call the canonical partition of X the ***orbit type stratification*** of X. The canonical partition of Z is a stratification of Z, and we refer to it as the orbit type stratification of Z.

The contact quotient X is not necessarily connected, because Z is not necessarily connected [2]. Since the cohomology of the whole space is the direct product of the cohomologies of its connected components [9], we work with connected components of X. Let X_a, for a in an indexing set $\mathcal{A}$, denote the strata of X. Note that each connected component of X is also a stratified space, with strata given by the X_a that are contained in the connected component. Let Z_a, for $a \in \mathcal{A}$, denote the strata of Z. Note that $X_a = Z_a/G$ for all $a \in \mathcal{A}$, since G is connected.

Since G acts properly on M preserving the contact form α, then by ([10], Theorem 3.9), each connected component of X has a unique connected open dense stratum, which we call the ***principal*** or ***top*** stratum of that connected component. This is the stratum corresponding to the maximal element under the partial ordering defined by $a_1 \leq a_2$ if $X_{a_1} \subset \overline{X}_{a_2}$. Denote the principal strata by X_b, where

b is in an indexing set $\mathcal{B}$. Taking the disjoint union of all the principal strata gives us the ***principal*** stratum X_{prin} of X. That is,

$$X_{\text{prin}} = \coprod_{b \in \mathcal{B}} X_b. \tag{3.4}$$

Correspondingly, each connected component of Z has a principal stratum Z_b, where $b \in \mathcal{B}$, and the disjoint union of these is the principal stratum Z_{prin} of Z. That is,

$$Z_{\text{prin}} = \coprod_{b \in \mathcal{B}} Z_b. \tag{3.5}$$

Note that $X_{\text{prin}} = Z_{\text{prin}}/G$.

For any stratum X_a of X, we have an inclusion map

$$\iota_a \colon Z_a \to M \tag{3.6}$$

and an orbit map

$$\pi_a \colon Z_a \to X_a. \tag{3.7}$$

From these we get the maps

$$\iota_{\text{prin}} \colon Z_{\text{prin}} \to M \tag{3.8}$$

and

$$\pi_{\text{prin}} \colon Z_{\text{prin}} \to X_{\text{prin}}. \tag{3.9}$$

The pullbacks of these maps have an essential role in defining differential forms on the singular contact quotient.

3.3 Forms on a Contact Quotient

Since the contact quotient X is in general a singular space, we need to define the concept of a differential form on it.

Definition 3.1. A ***differential form*** on X is a differential form β on the principal stratum X_{prin} such that there exists a differential form $\tilde{\beta}$ on M satisfying

$$\pi_{\text{prin}}^* \beta = \iota_{\text{prin}}^* \tilde{\beta}.$$

In other words, the appropriate pullbacks of β and $\tilde{\beta}$ agree on Z_{prin}. We say that $\tilde{\beta}$ ***induces*** β.

We may assume $\tilde{\beta}$ to be G-invariant, because if it is not we can average it over G with respect to the Haar measure. We denote the collection of differential forms on X by $\Omega(X)$. Note that our definition implies that $\Omega(X)$ is a subcomplex of $\Omega(X_{\text{prin}})$, with the usual exterior derivative d. Also, we can check that $\Omega(X)$ is closed under the wedge product.

Forms on X are all induced by forms on M. We now show which forms on M descend to forms on X. A form $\beta \in \Omega(M)$ is called ***basic*** with respect to the G-action if it is G-invariant and G-horizontal, that is, killed by all interior products i_{ξ_M} for $\xi \in \mathfrak{g}$. We denote the set of basic forms on M by $\Omega_{\text{bas}}(M)$. The notion of a form whose restriction to Z_{prin} is G-horizontal will be important for us.

Definition 3.2. A differential form $\beta \in \Omega(M)$ is Φ***-basic*** if it is G-invariant and $\iota_{\text{prin}}^* \beta \in \Omega(Z_{\text{prin}})$ is G-horizontal.

We denote the set of Φ-basic forms on M by $\Omega_\Phi(M)$. Note that $\Omega_\Phi(M)$ is a subcomplex of $\Omega(M)$, with the usual exterior derivative d.

Lemma 3.3. *If a G-invariant differential form $\tilde{\beta} \in \Omega^l(M)$ induces a form $\beta \in \Omega^l(X)$, then $\tilde{\beta}$ is Φ-basic.*

Proof. Let $\tilde{\beta} \in \Omega^l(M)$ be a G-invariant form. Assume $\tilde{\beta}$ induces $\beta \in \Omega^l(X)$. That is, $\beta \in \Omega^l(X_{\text{prin}})$ and $\iota_{\text{prin}}^* \tilde{\beta} = \pi_{\text{prin}}^* \beta$. To see that $\iota_{\text{prin}}^* \tilde{\beta} \in \Omega(Z_{\text{prin}})$ is G-horizontal, let $\xi \in \mathfrak{g}$. Since π_{prin} sends each G-orbit in Z_{prin} to a point in X_{prin}, then the pushforward π_{prin_*} of the tangent to a G-orbit in Z_{prin} is the zero vector in the tangent space of X_{prin}. In particular, $\pi_{\text{prin}_*}(\xi_{Z_{\text{prin}}}) = 0$. Because of this, we have

$$\begin{aligned} i_{\xi_{Z_{\text{prin}}}}(\iota^*_{\text{prin}}\tilde{\beta})(\Xi_1, \ldots, \Xi_{l-1}) &= i_{\xi_{Z_{\text{prin}}}}(\pi^*_{\text{prin}}\beta)(\Xi_1, \ldots, \Xi_{l-1}) \\ &= (\pi^*_{\text{prin}}\beta)(\xi_{Z_{\text{prin}}}, \Xi_1, \ldots, \Xi_{l-1}) \\ &= \beta(\pi_{\text{prin}_*}(\xi_{Z_{\text{prin}}}), \pi_{\text{prin}_*}(\Xi_1), \ldots, \pi_{\text{prin}_*}(\Xi_{l-1})) \qquad (3.10) \\ &= \beta(0, \pi_{\text{prin}_*}(\Xi_1), \ldots, \pi_{\text{prin}_*}(\Xi_{l-1})) \\ &= 0, \end{aligned}$$

for all vector fields $\Xi_1, \ldots, \Xi_{l-1}$ on Z_{prin}. Since $\tilde{\beta}$ is G-invariant and $\iota^*_{\text{prin}}\tilde{\beta}$ is G-horizontal, then $\tilde{\beta}$ is Φ-basic. □

The following theorem is useful for us. Its proof is a generalization of the well-known analagous statement for free actions.

Theorem 3.4. *Suppose G acts smoothly on a manifold P and all points of P are of the same orbit type (G_x and G_y are conjugate subroups for all $x, y \in P$). Then $\pi\colon P \to P/G$ induces an isomorphism $\pi^*\colon \Omega(P/G) \to \Omega_{\text{bas}}(P)$.*

By this theorem, we get the following useful fact.

Lemma 3.5. *The projection $\pi_a\colon Z_a \to X_a$ induces an isomorphism*

$$\pi^*_a\colon \Omega(X_a) \to \Omega_{\text{bas}}(Z_a), \qquad (3.11)$$

for every $a \in \mathcal{A}$.

In particular, we know that $\pi_b\colon Z_b \to X_b$ induces an isomorphism

$$\pi^*_b\colon \Omega(X_b) \to \Omega_{\text{bas}}(Z_b), \qquad (3.12)$$

for each connected component Z_b of Z_{prin}. Putting these together gives us the following result.

Lemma 3.6. *The map*

$$\pi_{\mathrm{prin}}^*\colon \Omega(X_{\mathrm{prin}}) \to \Omega_{\mathrm{bas}}(Z_{\mathrm{prin}}) \tag{3.13}$$

is an isomorphism.

So there is an inverse map $(\pi_{\mathrm{prin}}^*)^{-1}\colon \Omega_{\mathrm{bas}}(Z_{\mathrm{prin}}) \to \Omega(X_{\mathrm{prin}})$. By composing this with $\iota_{\mathrm{prin}}^*\colon \Omega_\Phi(M) \to \Omega_{\mathrm{bas}}(Z_{\mathrm{prin}})$, we obtain a natural map

$$\begin{aligned} s\colon \Omega_\Phi(M) &\to \Omega(X_{\mathrm{prin}}) \\ \tilde{\beta} &\mapsto (\pi_{\mathrm{prin}}^*)^{-1}(\iota_{\mathrm{prin}}^*\tilde{\beta}). \end{aligned} \tag{3.14}$$

Note that by the definiton of $\Omega(X)$, the image of s is actually contained in $\Omega(X)$.

Proposition 3.7. *The map $s\colon \Omega_\Phi(M) \to \Omega(X)$ is a surjection.*

Proof. If $\beta \in \Omega(X)$, then by definition, there exists a G-invariant form $\tilde{\beta} \in \Omega(M)$ such that $\pi_{\mathrm{prin}}^*\beta = \iota_{\mathrm{prin}}^*\tilde{\beta}$. By Lemma 3.3 , $\tilde{\beta}$ is Φ-basic. Utilizing Lemma 3.6, we see that $(\pi_{\mathrm{prin}}^*)^{-1}(\iota_{\mathrm{prin}}^*\tilde{\beta}) = \beta$. □

The kernel of s consists precisely of all the forms $\tilde{\beta} \in \Omega_\Phi(M)$ such that $\iota_{\mathrm{prin}}^*\tilde{\beta} = 0$. This is the ideal

$$I_\Phi(M) = \{\tilde{\beta} \in \Omega(M)^G : \iota_{\mathrm{prin}}^*\tilde{\beta} = 0\}. \tag{3.15}$$

Therefore, we have an isomorphism

$$\Omega(X) \cong \Omega_\Phi(M)/I_\Phi(M). \tag{3.16}$$

Notice that we can define $\Omega_\Phi(O)$ and $I_\Phi(O)$ for any G-invariant neighborhood O containing Z in M, since O would be a contact G-manifold itself. In particular, the above isomorphism would still be valid if we replaced M with O.

The following lemma is stated for symplectic quotients as ([17], Lemma 3.3). The proof remains valid in our case.

Lemma 3.8. *(a) Let $\tilde{\beta} \in \Omega_\Phi(M)$. Then $\iota_a^*\tilde{\beta}$ is a horizontal form on Z_a for all a.*

(b) Let $\tilde{\beta} \in I_\Phi(M)$. Then $\iota_a^*\tilde{\beta} = 0$ for all a.

(c) There is a well-defined restriction map $\Omega(X) \to \Omega(X_a)$ for each stratum X_a.

It is worthwhile to describe the restriction map $\Omega(X) \to \Omega(X_a)$ referred to in this lemma, because we will use it later on to compare forms on each stratum X_a to forms on the quotient X. Let $\tilde{\beta} \in \Omega_\Phi(M)$. By part (a) of the lemma, the pullback $\iota_a^*\tilde{\beta} \in \Omega(Z_a)$ is a horizontal form. So it descends to a form $\tilde{\beta}_a \in \Omega(X_a)$. For every a, define a homomorphism

$$\begin{aligned} \Omega_\Phi(M) &\to \Omega(X_a) \\ \tilde{\beta} &\mapsto \tilde{\beta}_a. \end{aligned} \tag{3.17}$$

By part (b) of the lemma, this map is 0 on $I_\Phi(M)$. So, by the isomorphism 3.16, it yields the restriction map $\Omega(X) \to \Omega(X_a)$.

3.4 Contact Induction

In the proof of the de Rham theorem for symplectic quotients, symplectic induction and reduction in stages is used to demonstrate an isomorphism between the de Rham complexes of certain isomorphic symplectic quotients [17]. Here we provide an analogous summary of contact induction, which will be relevant in the proof of our Poincaré lemma.

Let H be a closed subgroup of G and let $(N, \mathcal{H}_N, \alpha_N)$ be a strict contact H-manifold. For any $h \in H$, denote the action by $n \mapsto hn$, for all $n \in N$. Denote the corresponding contact moment map by $\Psi\colon N \to \mathfrak{h}^*$. Let $Z_N = \Psi^{-1}(0)$ be the zero fiber of the moment map.

Now let $P = T^*G \times N$. Note that P is also a contact manifold, with contact form $\alpha_{T^*G} + \alpha_N$, where α_{T^*G} is the tautological 1-form on the cotangent bundle of G. We turn P into a contact H-manifold by letting H act on T^*G by the lift of right multiplication and using the given action of H on N. We identify

$$T^*G \times N \cong G \times \mathfrak{g}^* \times N \tag{3.18}$$

via the left trivialization

$$\begin{aligned} T^*G &\to G \times \mathfrak{g} \\ (g, \xi) &\mapsto (g, L_g^*\xi), \end{aligned} \tag{3.19}$$

where L_g^* is the dual map of left multiplication by $g \in G$, restricted to T_g^*G. Under this identification, the contact action of H on P is given by

$$h \cdot (g, \eta, n) = (g, \mathrm{Ad}_h^*\eta, hn) \tag{3.20}$$

for all $h \in H$. Choose an H-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{m}^*$ are H-invariant splittings. Then the moment map $\mu\colon P \to \mathfrak{h}^*$ for the H-action on P is given by

$$\mu(g, \eta, n) = -\eta|_{\mathfrak{h}^*} + \Psi(n) \tag{3.21}$$

for all $(g, \eta, n) \in P$.

Let G act on P by the lift of left multiplication on T^*G and by the trivial action on N. Via the chosen trivialization 3.19, this contact action is given by

$$k \cdot (g, \eta, n) = (kg, \eta, n) \tag{3.22}$$

for all $k \in G$. The moment map $\nu\colon P \to \mathfrak{g}^*$ for the G-action on P is given by

$$\nu(g, \eta, n) = \mathrm{Ad}_g^*(\eta) \tag{3.23}$$

for all $(g, \eta, n) \in P$. Then P is a contact $(G \times H)$-manifold, with the corresponding moment map being the sum of the G and H moment maps.

We show that the contact quotient of P by G is N. The zero fiber of the G-moment map on P is $\nu^{-1}(0) = G \times \{0\} \times N$. Since the action of G on itself is transitive, then

$$P /\!\!/ G := \frac{\nu^{-1}(0)}{G} = N. \tag{3.24}$$

Since the action of G on P is free, then $P /\!\!/ G$ is indeed a contact manifold, with a contact form descending from the one on P. This is precisely the given contact form on N.

Let M be the contact quotient of P by H. Since H acts freely on P, then M is a smooth contact manifold, with a contact form descending from the one on P. The G-action on P also descends to a G-action on M. Note that the zero fiber of the H-moment map on P is

$$\begin{aligned} \mu^{-1}(0) &= \{(g, \eta, n) : -\eta|_{\mathfrak{h}^*} + \Psi(n) = 0\} \\ &= \{(g, \gamma + \Psi(n), n) | \gamma \in \mathfrak{m}^*\}. \end{aligned} \tag{3.25}$$

So we can identify $G \times \mathfrak{m}^* \times N$ with $\mu^{-1}(0)$ under the map

$$\begin{aligned} G \times \mathfrak{m}^* \times N &\to P \cong G \times \mathfrak{g}^* \times N \\ (g, \gamma, n) &\mapsto (g, \gamma + \Psi(n), n). \end{aligned} \tag{3.26}$$

Since this is an H-equivariant diffeomorphism, the respective H-orbit spaces are G-equivariantly diffeomorphic. That is,

$$M \cong (G \times \mathfrak{m}^* \times N)/H. \tag{3.27}$$

The right hand side is a homogeneous vector bundle over G/H with fiber $\mathfrak{m}^* \times M$. Under this identification, a point in M is written as $[g, \gamma, n]$, where $g \in G$, $\gamma \in \mathfrak{m}^*$, and $n \in N$. The G-action on P descends to the contact G-action on M defined by

$$k \cdot [g, \gamma, n] = [kg, \gamma, n]. \tag{3.28}$$

for all $k \in G$. The moment map $\Phi \colon M \to \mathfrak{g}^*$ for this action is given by

$$\Phi([g, \gamma, n]) = \mathrm{Ad}_g^*(\gamma + \Psi(n)) \tag{3.29}$$

for all $[g, \gamma, n] \in M$. Let $Z = \Phi^{-1}(0)$ be the zero fiber of the moment map.

Lemma 3.9. *The zero fiber of the moment map Φ for the action of G on M is*

$$Z = G \times_H Z_N. \tag{3.30}$$

Proof. We know $[g, \gamma, n] \in Z$ if and only if

$$0 = \mathrm{Ad}_g^*(\gamma + \Psi(n)). \tag{3.31}$$

This means that for all $X \in \mathfrak{g}$, we have

$$\langle \mathrm{Ad}_g^*(\gamma + \Psi(n)), X \rangle = 0. \tag{3.32}$$

But

$$\langle \mathrm{Ad}_g^*(\gamma + \Psi(n)), X \rangle = \langle \gamma + \Psi(n), \mathrm{Ad}_{g^{-1}} X \rangle, \tag{3.33}$$

so that

$$\langle \gamma + \Psi(n), \mathrm{Ad}_{g^{-1}} X \rangle = 0. \tag{3.34}$$

Since $X \in \mathfrak{g}$ is arbitrary and $\mathrm{Ad}_{g^{-1}} \colon \mathfrak{g} \to \mathfrak{g}$ is invertible, then

$$\gamma + \Psi(n) = 0. \tag{3.35}$$

Since we are using the splitting $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$, and the inclusion of $\mathfrak{h}^*$ into $\mathfrak{g}^*$ is injective, then we must have $\gamma = 0$ and $\Psi(n) = 0$. So $[g, \gamma, n] \in Z$ if and only if $\gamma = 0$ and $n \in Z_N$.

Therefore, we have

$$\begin{aligned} Z &= \{[g, 0, n] \in M : \Psi(n) = 0\} \\ &= G \times_H (\{0\} \times Z_N). \end{aligned} \tag{3.36}$$

By sending $[g, 0, n]$ to $[g, n]$, we arrive at

$$Z = G \times_H Z_N. \tag{3.37}$$

□

Now we embed N into M via the map

$$\begin{aligned} f \colon N &\to M \\ n &\mapsto [1, 0, p]. \end{aligned} \tag{3.38}$$

It is straightforward to check that f is H-equivariant and that $\Phi \circ f = \mathrm{pr}^* \circ \Psi$, where pr^* is the dual of the projection $\mathrm{pr} \colon \mathfrak{g} \to \mathfrak{h}$. Therefore, f maps Z_N into Z. So f descends to a map from the contact quotient $Y = Z_N/H$ to the contact quotient $X = Z/G$. Denote this map by $r \colon Y \to X$. The theory of reduction in stages tells us that r is an isomorphism, that is, a stratification-preserving homeomorphism. In fact, we can directly check that r restricts to a contactomorphism on each stratum.

The discussion so far can be summarized in the following commutative diagram, in which the arrows indicate contact reduction by the relevant groups.

$$\begin{array}{ccc} P & \overset{G}{\dashrightarrow} & N \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ M & \underset{G}{\dashrightarrow} & X \cong Y \end{array}$$

Since f is homeomorphic onto its image, then f maps the principal stratum $(Z_N)_{\mathrm{prin}}$ of Z_N into the principal stratum Z_{prin} of Z. In fact, by Lemma 3.9, we know that $Z = G \times_H Z_N$ and $Z_{\mathrm{prin}} = G \times_H (Z_N)_{\mathrm{prin}}$. So the pullback map

$f^*\colon \Omega(M) \to \Omega(N)$ sends $\Omega_\Phi(M)$ to $\Omega_\Psi(N)$ and $I_\Phi(M)$ to $I_\Psi(N)$. Therefore, f^* descends to a map $r^*\colon \Omega(X) \to \Omega(Y)$.

Proposition 3.10. *The map $r^*\colon \Omega(X) \to \Omega(Y)$ is an isomorphism.*

Proof. The proof is analogous to the proof of ([17], Proposition 4.2). □

3.5 A de Rham Theorem

The proof of our de Rham theorem relies upon some sheaf theory. In order to obtain a sheaf from the de Rham complex $\Omega(X)$, we clarify the concept of a differential form on an open subset of X. Let U be an open subset of X. The stratification of $X = \coprod_{a\in\mathcal{A}} X_a$ induces a stratification on U, in which the strata of U are given by the connected components of $U_a = X_a \cap U$, for all $a \in \mathcal{A}$. Note that $U_{\text{prin}} = X_{\text{prin}} \cap U$ is the principal stratum of U. Furthermore, the stratification of U induces one on any open subset of U, in which the strata are given by intersecting the strata of U with the open subset.

Definition 3.11. A ***differential form*** on U is a differential form β on U_{prin} such that for all $x \in U$ there exist $\beta' \in \Omega(X)$ and an open neighborhood U' of x in U such that $\beta = \beta'$ on U'_{prin}.

The collection of differential forms on U is denoted by $\Omega(U)$. As in ([17], Section 5), the presheaf $\Omega\colon U \mapsto \Omega(U)$ is a sheaf, and its global sections constitute the de Rham complex $\Omega(X)$.

The de Rham complex of a symplectic quotient is shown to be an acyclic resolution of the constant sheaf in ([17], Section 5). Using the existence of smooth partitions of unity on X, one can similarly check that the sheaf $\Omega\colon U \mapsto \Omega(U)$ is acyclic. One of the goals of this chapter is to show that the de Rham complex of X is in fact an acyclic resolution of the constant sheaf. Then results in sheaf theory will imply the following main theorem.

Theorem 3.12. *The de Rham cohomology ring $H(\Omega(X))$ is naturally isomorphic to the (Čech or singular) cohomology ring of X with real coefficients $H(X, \mathbb{R})$.*

This is a de Rham theorem for contact quotients analogous to the one for symplectic quotients ([17], Theorem 5.5).

3.6 The Poincaré Lemma

We prove the following local Poincaré lemma.

Lemma 3.13. *Every $x \in X$ has a basis of open neighborhoods U such that the sequence*

$$0 \to \mathbb{R} \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots$$

is exact.

This lemma implies that the de Rham complex of X is an acyclic resolution of the constant sheaf. The proof utilizes the work of Sjamaar in the symplectic case in [17] and a local normal form theorem for contact quotients due to Lerman and Willet in [10].

Adopting the set up from Section 3.2, we additionally let $(M', \mathcal{H}', \alpha')$ be a strict contact G-manifold with contact moment map $\Phi'\colon M' \to \mathfrak{g}^*$. The zero fiber of the moment map is $Z' = (\Phi')^{-1}(0)$ and the contact quotient is $X' = Z'/G$. We have stratifications of $X' = \coprod_{a' \in \mathcal{A}'} X'_a$ and $Z' = \coprod_{a' \in \mathcal{A}'} Z'_a$ analogous to those for X and Z. We also define $X'_{\text{prin}} = \coprod_{b' \in \mathcal{B}'} X_{b'}$ and $Z'_{\text{prin}} = \coprod_{b' \in \mathcal{B}'} Z_{b'}$ be the disjoint unions of the top strata of the connected components of X and Z respectively.

Definition 3.14. A map $f \colon M \to M'$ is ***allowable*** if it satisfies the following conditions:

(i) f is smooth and G-equivariant.

(ii) $f(Z) \subseteq Z'$.

(iii) $f_*(T_z Z_{\text{prin}}) \subseteq T_{f(z)} Z'_{a(z)}$ for all $z \in Z_{\text{prin}}$, where $Z'_{a(z)} \subseteq Z'$ is the stratum of $f(z)$.

Example 3.15. Let (W, α) be a contact vector space, obtained as the contactization of a symplectic vector space (V, ω) as in Example 1.6. That is, $W = V \times \mathbb{R}$ and $\alpha = \frac{1}{2} i_R \omega + dt$, where R is the radial vector field on V and t is the variable on $\mathbb{R}$. Assume that G acts linearly and symplectically on (V, ω). Then the action of G preserves the radial vector field R, thereby making the 1-form $\frac{1}{2} i_R \omega$ G-invariant. Let G act on $\mathbb{R}$ via the identity map. Then the pushforward g_* for $g \in G$ is also the identity, making $g^* dt = dt$. So the G-action preserves the contact form and is a contact action. The symplectic moment map $\Phi_V \colon V \to \mathfrak{g}^*$ for the action of G on V is given by

$$\langle \Phi_V(v), \xi \rangle = \frac{1}{2} \omega(v, \xi v) \tag{3.39}$$

for all $v \in V$ and all $\xi \in \mathfrak{g}$, where $\xi \in \mathfrak{g}$ acts on V via the infinitesimal representation $\mathfrak{g} \to \mathfrak{sp}(V)$. Since the action of G on $\mathbb{R}$ is trivial, then the contact moment map $\Phi_W \colon W \to \mathfrak{g}^*$ of the action of G on W is given by

$$\langle \Phi_W(v, t), \xi \rangle = \langle \Phi_V(v), \xi \rangle \tag{3.40}$$

for all $w = (v,t) \in W$ and all $\xi \in \mathfrak{g}$. In detail, observe that

$$\begin{aligned}
\langle \Phi_W(v,t), \xi\rangle &= \alpha_w(\xi_W(v,t)) \\
&= \left(\frac{1}{2}(i_R\omega)_v + dt\right)(\xi_V(v), \xi_{\mathbb{R}}(t)) \\
&= \left(\frac{1}{2}(i_R\omega)_v + dt\right)(\xi_V(v), 0) \\
&= \frac{1}{2}(i_R\omega)_v(\xi_V(v)) + dt(0) \\
&= \frac{1}{2}\omega(v, \xi_V(v)) \\
&= \langle \Phi_V(v), \xi\rangle.
\end{aligned} \tag{3.41}$$

In particular, $(v,t) \in \Phi_W^{-1}(0)$ if and only if $v \in \Phi_V^{-1}(0)$.

Let $r \in \mathbb{R}$ and let $f\colon W \to W$ be the dilation $f(v,t) = (rv, rt)$. This map is clearly smooth and G-equivariant. If $(v,t) \in \Phi_W^{-1}(0)$, then $v \in \Phi_V^{-1}(0)$ and

$$\begin{aligned}
\langle \Phi_V(rv), \xi\rangle &= \frac{1}{2}\omega(rv, \xi_V(rv)) \\
&= \frac{1}{2}r^2\omega(v, \xi_V(v)) \\
&= r^2\langle \Phi_V(v), \xi\rangle \\
&= r^2 \cdot 0 \\
&= 0.
\end{aligned} \tag{3.42}$$

Thus $rv \in \Phi_V^{-1}(0)$ and therefore $(rv, rt) \in \Phi_W^{-1}(0)$. So f preserves $Z = \Phi_W^{-1}(0)$ and Z is contractible. For $r \neq 0$, the stabilizer of $f(v,t) = (rv, rt)$ is equal to the stabilizer of (v,t), because the action of G is linear. Therefore $f(v,t)$ and (v,t) are in the same connected component of $Z \cap W_{(H)}$, for some subgroup H of G. That is, they are in the same stratum of Z. So $f(Z_{\text{prin}}) \subseteq Z_{\text{prin}}$. For $r = 0$, $f(Z_{\text{prin}}) = 0$. In both cases, f maps Z_{prin} into a single stratum of Z, so that $f_*(T_w Z_{\text{prin}}) \subseteq T_{f(w)} Z_{\text{prin}}$ for all $w \in Z_{\text{prin}}$. Therefore, f is allowable. By the same arguments, if $|r| \leq 1$ and B is a G-invariant open ball about the origin in W, the restriction of f is an allowable map from B to itself.

We need the following lemma, which for the symplectic case is ([17], Lemma 6.3). Its proof in the contact case, as in the symplectic case, follows from Lemma 3.8.

Lemma 3.16. *Let $f\colon M \to M'$ be allowable. Then the pullback homomorphism $f^*\colon \Omega(M') \to \Omega(M)$ sends $\Omega_{\Phi'}(M')$ to $\Omega_\Phi(M)$ and $I_{\Phi'}(M')$ to $I_\Phi(M)$, and therefore induces a homomorphism $f^*\colon \Omega(X') \to \Omega(X)$.*

We review the way homotopies induce chain homotopies on the de Rham complex, in the manner of ([17], Section 6). Let $F\colon M \times [0,1] \to M'$ be a smooth homotopy and let r denote the coordinate on $[0,1]$. Set $F_r = F|_{M\times\{r\}}$. For $\gamma \in \Omega(M')$, define

$$\kappa_F\gamma = \int_0^1 i_{\frac{\partial}{\partial r}}(F^*\gamma)dr. \tag{3.43}$$

Note that κ_F lowers the degree of γ by 1 and

$$F_1^*\gamma - F_0^*\gamma = \kappa_F d\gamma + d\kappa_F\gamma, \tag{3.44}$$

which by definition means that κ_F is a chain homotopy. Assume that F is equivariant with respect to the given actions of G on M and M' and the trivial action on $[0,1]$. We can can check that for all $g \in G$,

$$\kappa_F \circ g^* = g^* \circ \kappa_F. \tag{3.45}$$

Also, for all $\xi \in \mathfrak{g}$,

$$\kappa_F \circ i_{\xi_{M'}} = -i_{\xi_M} \circ \kappa_F. \tag{3.46}$$

Definition 3.17. The homotopy F is ***allowable*** if it satisfies the following conditions:

(i) F is smooth and G-equivariant.

(ii) $F_r \colon M \to M'$ is allowable for almost all $r \in [0,1]$.

(iii) $(F_*)_{(z,r)}\left(\frac{\partial}{\partial r}\right) \in T_{F(z,r)}Z'_{a(z,r)}$ for almost all $r \in [0,1]$ and for all $z \in Z_{\text{prin}}$, where $Z'_{a(z,r)} \subseteq Z'$ is the stratum of $F(z,r)$.

Example 3.18. Let (W,α) be a contact vector space with a contact G-action as in Example 3.15. The radial contraction $F \colon W \times [0,1] \to W$ defined by $F((v,t),r) = (rv, rt)$ is smooth and G-equivariant. For every $r \in [0,1]$, F_r is like the map f in Example 3.15, so it is allowable. Furthermore, $F_r(Z_{\text{prin}}) \subseteq Z_{\text{prin}}$ for $r \neq 0$, which implies property (iii) in Definition 3.17. Hence F is an allowable homotopy. Similarly, the restriction of F gives an allowable homotopy $B \times [0,1] \to B$ for any G-invariant open ball B about the origin in W.

The following lemma is the same as ([17], Lemma 6.7), and its proof works for the contact case as well.

Lemma 3.19. *Let $F \colon M \times [0,1] \to M'$ be an allowable homotopy. Then the homotopy operator $\kappa_F \colon \Omega(M') \to \Omega(M)$ sends $\Omega_{\Phi'}(M')$ to $\Omega_\Phi(M)$ and $I_{\Phi'}(M')$ to $I_\Phi(M)$, and therefore induces a homotopy $\kappa_F \colon \Omega(X') \to \Omega(X)$.*

Example 3.20. Consider the radial contraction F of the previous example. Lemma 3.19 gives us the homotopy operator $\kappa_F \colon \Omega(X') \to \Omega(X)$, which satisfies the same formulas that $\kappa_F \colon \Omega(M') \to \Omega(M)$ does. Let $\beta \in \Omega^l(X)$. Then

$$F_1^*\beta - F_0^*\beta = \kappa_F(d\beta) + d(\kappa_F\beta), \tag{3.47}$$

where F_0^* is the zero map and F_1^* is the identity map. So the identity map is homotopic to the zero map, which shows that the de Rham complex of the contact quotient of W by G is homotopically trivial. If $Y = (B \cap Z)/G$ is the contact quotient of any G-invariant open ball B about the origin, then $\Omega(Y)$ is homotopically trivial.

We use a local model of contact quotients developed by Lerman and Willett in [10] in order to generalize our examples to arbitrary contact quotients.

Since $(M, \mathcal{H}, \alpha)$ is a contact G-manifold with a G-invariant contact form α, then the vector bundle $\mathcal{H} \subset TM$ is symplectic with symplectic form $\omega = d\alpha$ and an induced action of G. So we can choose a G-invariant almost complex structure J on $\mathcal{H}$ which is compatible with ω. This means $\sigma(x, y) = \omega(x, Jy)$ is a positive-definite inner product on each fiber of $\mathcal{H}$. We extend σ to give a smoothly varying positive-definite inner product on TM. Let $E \in \mathfrak{X}(M)$ denote the Reeb vector field on M associated to α. Declare the length of E to be 1 and declare $TM = \mathcal{H} \oplus \mathbb{R}E$ to be an orthogonal sum, with respect to σ. Now σ provides a positive-definite inner product on the tangent spaces, and so it is a Riemannian metric on M.

Choose $z \in Z$ mapping to $x \in X$. Let $H = G_z$ be the stabilizer of z and let $G \cdot z$ denote the orbit of z. Let $\mathfrak{h}$ denote the Lie algebra of H, and let $\mathfrak{h}^0$ denote the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Choose an H-equivariant splitting $\mathfrak{g}^* = \mathfrak{h}^0 \oplus \mathfrak{h}^*$, with corresponding injection $i \colon \mathfrak{h}^* \to \mathfrak{g}^*$.

Now, consider the symplectic subspace $T_z(G \cdot z) \oplus J(T_z(G \cdot z))$ of the symplectic vector space $(\mathcal{H}_z, \omega_z)$. Let V be the σ-orthogonal complement to this subspace. Note that V is the symplectic slice $(T_z(G \cdot z))^\omega / T_z(G \cdot z)$ at z. The almost complex structure on $\mathcal{H}$ induces one on V. The vector space V also inherits a symplectic action of H, and extending it trivially to $\mathbb{R}$ yields a contact action of H on the contactization $W = V \times \mathbb{R}$. Denote the corresponding moment map by $\Phi_W \colon W \to \mathfrak{h}^*$.

Consider the vector bundle

$$\mathcal{Y} = (G \times_H (\mathfrak{h}^0 \times W)) \to G/H. \tag{3.48}$$

This vector bundle inherits a contact form ϵ from its identification, via the splitting

$\mathfrak{g}^* = \mathfrak{h}^0 \oplus \mathfrak{h}^*$, with the contact quotient $(T^*G \times W)/\!\!/H$ seen in Section 3.4. Define a contact action of G on $\mathcal{Y}$ by $g \cdot [a, \eta, w] = [ga, \eta, w]$. Then ([10], Theorem 4.1) tells us that there exist a G-invariant neighborhood U of z in M, a G-invariant neighborhood U_0 of the zero section on the vector bundle $\mathcal{Y}$, and a G-equivariant contactomorphism $\phi\colon U_0 \to U$ such that $\phi([1,0,0]) = x$ and

$$(\Phi \circ \phi)([g, \eta, w]) = f([g, \eta, w])\mathrm{Ad}^*_g(\eta + i(\Phi_W(w))), \tag{3.49}$$

where f is a positive function.

It is this contactomorphism ϕ that will allow us to see that contact quotients locally look like quotients of contact vector spaces. We study it further.

Lemma 3.21. *The map ϕ is allowable.*

Proof. Note that ϕ is smooth and G-equivariant from the local normal form theorem. Recall $Z = \Phi^{-1}(0) \subseteq M$ and let $Z' = (\Phi \circ \phi)^{-1}(0) \subseteq \mathcal{Y}$. Let $p \in Z' \cap U_0$. Then

$$\begin{aligned} \Phi(\phi(p)) &= (\Phi \circ \phi)(p) \\ &= 0. \end{aligned} \tag{3.50}$$

So $\phi(p) \in Z \cap U$. So $\phi(Z' \cap U_0) \subseteq Z \cap U$. To conclude that ϕ is allowable, it suffices to show that for all $p \in Z'_{\mathrm{prin}}$, $\phi(p)$ has the same stabilizer as p. But this is automatic, since ϕ is a diffeomorphism. □

Now by Lemma 3.16, the pullback homomorphism $\phi^*\colon \Omega(U) \to \Omega(U_0)$ sends $\Omega_\Phi(U)$ to $\Omega_{\Phi\circ\phi}(U_0)$ and $I_\Phi(U)$ to $I_{\Phi\circ\phi}(U_0)$, and therefore induces a homomorphism

$$\phi^*\colon \Omega((Z \cap U)/G) \to \Omega((Z' \cap U_0)/G). \tag{3.51}$$

Since $\phi\colon U \to U_0$ is a diffeomorphism, then $\phi^*\colon \Omega(U) \to \Omega(U_0)$ is an isomorphism.

Work similar to that in Lemma 3.21 shows that $\phi^{-1}\colon U \to U_0$ is allowable. Then by Lemma 3.16, the pullback homomorphism $(\phi^{-1})^* : \Omega(U_0) \to \Omega(U)$ sends $\Omega_{\Phi\circ\phi}(U_0)$ to $\Omega_\Phi(U)$ and $I_{\Phi\circ\phi}(U_0)$ to $I_\Phi(U)$, and therefore induces a homomorphism

$$\left(\phi^{-1}\right)^* : \Omega((Z' \cap U_0)/G) \to \Omega((Z \cap U)/G). \tag{3.52}$$

Note that $(\phi^{-1})^* = (\phi^*)^{-1}$ is an inverse for the homomorphism ϕ^* in (3.51). Thus we have proved the following result.

Lemma 3.22. *The map ϕ^* in (3.51) is an isomorphism.*

We would like to conclude that the de Rham complex of $(Z \cap U)/G$ is isomorphic to the de Rham complex of the quotient of a contact vector space. By Lemma 3.22, it suffices to show that the de Rham complex of $(Z' \cap U_0)/G$ is isomorphic to the de Rham complex of the quotient of a contact vector space. Replacing M with $\mathcal{Y}$ and N with W in the Section 3.4 shows that $\Omega(\mathcal{Y}/\!\!/G)$ is isomorphic to $\Omega(W/\!\!/H)$.

Since the de Rham complex of $W/\!\!/H$ is homotopically trivial, then so is the de Rham complex of $\mathcal{Y}/\!\!/G$. Since U_0 is a G-invariant neighborhood of the zero section on $\mathcal{Y}$, then $\Omega((Z' \cap U_0)/G)$ is homotopically trivial. Since $\Omega((Z' \cap U_0)/G)$ is isomorphic to $\Omega((Z \cap U)/G)$ by Lemma 3.22, then the de Rham complex of $(Z \cap U)/G$ is homotopically trivial.

By taking an open ball B sitting inside U and applying the above argument, we find that the de Rham complex of $(B \cap Z)/G$ is homotopically trivial. Letting B shrink to a point gives us a collection of such neighborhoods, which is a basis of the topology at x. This concludes the proof of the Poincaré Lemma.

3.7 The Reduced Contact Form and Integration

In this section, we first show that the contact form $\alpha \in \Omega(M)$ induces a form $\alpha_X \in \Omega(X)$, which restricts to a contact form on each stratum of X. Then we prove a Stokes' theorem for contact quotients, from which we are able to conclude that the cohomology class of the volume form $\alpha_X \wedge (d\alpha_X)^k$ is nonzero.

By Proposition 3.7 and the work following it, all nonzero differential forms on X are induced by Φ-basic forms on M whose restrictions to Z_{prin} do not vanish. We show that $\alpha \in \Omega(M)$ is such a differential form. First, note that because α is G-invariant, then so is $\iota^*_{\text{prin}}\alpha \in \Omega(Z_{\text{prin}})$. Next, let $\xi \in \mathfrak{g}$ and let $\xi_{Z_{\text{prin}}}$ denote the vector field induced (on each component) of Z_{prin}. Since $\xi_{Z_{\text{prin}}}$ is the restriction of ξ_M to Z_{prin}, then by the definition of the contact moment map,

$$\begin{aligned} i_{\xi_{Z_{\text{prin}}}} \left(\iota^*_{\text{prin}}\alpha\right)_z &= \langle \Phi(z), \xi_M \rangle \\ &= 0 \end{aligned} \tag{3.53}$$

for all $z \in Z_{\text{prin}}$. So $\iota^*_{\text{prin}}\alpha$ is G-horizontal. Therefore, $\alpha \in \Omega_\Phi(M)$. Since $i_E\alpha = 1$, where E is the Reeb vector field, then $\alpha \notin I_\Phi(M)$. So $\alpha \in \Omega(M)$ induces a nonzero 1-form $\alpha_X \in \Omega(X)$, given by

$$\alpha_X = s(\alpha) = (\pi^*_{\text{prin}})^{-1}(i^*_{\text{prin}}\alpha). \tag{3.54}$$

We call α_X the ***reduced contact form***, since it descends from the contact form α on M.

Proposition 3.23. *The form $\alpha_X \in \Omega(X)$ is a contact form when restricted to each stratum of X. It is the same contact form as that found by Willett in [21].*

Proof. Let H be a stabilizer subgroup of G. We will consider the strata X_a of X which are connected components of $(Z \cap M_{(H)})/G$. Let $N(H)$ denote the normalizer of H in G, $L = N(H)/H$, and $M_H = \{p \in M : G_p = H\}$. Then by ([21], Proposition 4.1), M_H is a contact manifold, for which α restricted to M_H is a global contact form. By the same proposition, L acts freely on M_H, with moment map the restriction of Φ. So $M_H /\!\!/ L$ is a contact manifold. Let $\iota_H \colon Z \cap M_H \to Z \cap M_{(H)} \subset M$ denote inclusion and $\pi_H \colon Z \cap M_H \to M_H /\!\!/ L$ denote the projection. Then the form $\beta \in \Omega(M_H /\!\!/ L)$ such that

$$\pi_H{}^* \beta = \iota_H{}^* \alpha \tag{3.55}$$

is the contact form on $M_H /\!\!/ L$. Willett further shows that the inclusion ι_H gives rise to a diffeomorphism

$$\theta \colon M_H /\!\!/ L \to (Z \cap M_{(H)})/G \tag{3.56}$$

defined by sending the L-orbit $[p]_L$ of p in M_H to the G-orbit $[p]_G$ of p. Note that θ is a stratified map, in that it sends connected components to connected components. As shown in ([21], Theorem 3), the contact structure on each connected component of $(Z \cap M_{(H)})/G$ is obtained as the pushforward under θ of the contact structure on $M_H /\!\!/ L$.

In light of these results, it suffices to show that $\beta \in \Omega(M_H /\!\!/ L)$ is the pullback under θ of the restriction of $\alpha_X \in \Omega(X)$ to $(Z \cap M_{(H)})/G$. This follows from the commutative diagram of stratified maps

$$\begin{array}{ccc} Z \cap M_H & \xrightarrow{\iota_H} & Z \cap M_{(H)} \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi} \\ M_H /\!\!/ L & \xrightarrow[\theta]{} & (Z \cap M_{(H)})/G \end{array}$$

from which we know $\pi_H = \theta^{-1} \circ \pi \circ \iota_H$. Let X_a be a connected component of $(Z \cap M_{(H)})/G$. Since α_X is induced by α, then by Lemma 3.8, the restriction of α_X to X_a is the form $\alpha_a \in \Omega(X_a)$ such that $\pi_a^* \alpha_a = \iota_a^* \alpha$. Since $\pi_a^* \colon \Omega(X_a) \to \Omega_{\mathrm{bas}}(Z_a)$

is an isomorphism by Lemma 3.5, then $\alpha_a = (\pi_a^*)^{-1}(\iota_a^*\alpha)$. Observe that

$$\begin{aligned} \pi_H^*(\theta^*\alpha_a) &= \pi_H^*\left(\left(\theta^* \circ (\pi_a^*)^{-1} \circ \iota_a^*\right)\alpha\right) \\ &= \left(\left(\iota_H^* \circ \pi_a^* \circ (\theta^{-1})^*\right) \circ \left(\theta^* \circ (\pi_a^*)^{-1} \circ \iota_a^*\right)\right)\alpha \\ &= \iota_H^*(\iota_a^*\alpha). \end{aligned} \tag{3.57}$$

So recalling Equation 3.55, we see that the restriction of β to the connected component of $M_H /\!\!/ L$ that corresponds to X_a is indeed $\theta^*\alpha_a$. This shows not only that α_a is a contact form on X_a, but also that it is the same contact form on X_a as the one found by Willet in [21]. It also shows that α_X is a contact form on X_{prin}, and that α_X is the analogue of a contact form on X. □

We now turn to discussing integration on the singular contact quotient X. We do not assume X to be compact, but instead show that it has locally finite volume. The argument rests on lifting to the symplectization and using an analogous result there.

Recall that the symplectization of $(M, \mathcal{H}, \alpha)$ is the symplectic manifold $(\widetilde{M}, \omega)$, where $\widetilde{M} = M \times \mathbb{R}$ and $\omega = d(e^t\alpha)$, where t is the coordinate on $\mathbb{R}$. Assume from now on that the dimension of M is $2n+1$. The volume element on M is $(\alpha \wedge (d\alpha)^n)/(n!)$. Using local coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ so that the contact form α is of the form $\sum_{j=1}^n x_j dy_j + dz$, one can check that

$$\frac{1}{(n+1)!}\omega^{(n+1)} = \frac{1}{n!}(e^t)^n dt \wedge \alpha \wedge (d\alpha)^n. \tag{3.58}$$

So the measure induced by the volume element $\omega^{n+1}/((n+1)!)$ on $\widetilde{M}$ corresponds to the product of the measure on M and the measure $(e^t)^n dt$ on $\mathbb{R}$.

We can extend the contact G-action on M to a hamiltonian action on $\widetilde{M}$ by letting G act trivially on $\mathbb{R}$. Then the symplectic quotient of $\widetilde{M}$ by G is isomorphic to $X \times \mathbb{R}$, where X is the contact quotient of M by G. Working with a connected component of X instead yields a stratification-preserving homeomorphism from the symplectic quotient of $\widetilde{M}$ by G to the corresponding connected component of $X \times \mathbb{R}$.

Lemma 3.24. *Every $x \in X$ has an open neighborhood U such that* $\mathrm{vol}(U_{\mathrm{prin}})$ *is finite. Hence X_{prin} has finite volume if X is compact.*

Proof. Given $x \in X$, take an open neighborhood U containing x such that $\overline{U}$ is compact. Then $\overline{U} \times [0,1]$ is a compact neighborhood in $X \times \mathbb{R}$ containing the point $(x, 1/2)$. Since $X \times \mathbb{R}$ is a symplectic quotient by a compact Lie group action, then by ([17], Lemma 7.1), it has locally finite volume. Since $\overline{U} \times [0,1]$ is compact, then $\mathrm{vol}((\overline{U} \times [0,1])_{\mathrm{prin}})$ is finite. Since $U \times (0,1) \subseteq \overline{U} \times [0,1]$, then $\mathrm{vol}((U \times (0,1))_{\mathrm{prin}})$ is finite. Finally, by Fubini's theorem, $\mathrm{vol}(U_{\mathrm{prin}}) = \mathrm{vol}((U \times (0,1))_{\mathrm{prin}})$ is finite. □

An alternate proof of the lemma relies on introducing a metric on X_{prin} and using the local normal form theorem for contact quotients from [10]. Some of the elements in that argument are useful for proving a Stokes' theorem, so we develop what is needed here.

Recall from the discussion of the local normal form theorem in Section 3.6 that we have a compatible triple consisting of the symplectic form ω, the almost complex structure J, and the positive-definite inner product σ on $\mathcal{H}$. Since the triple is compatible, then $\omega^n/(n!)$ is equal to the volume form induced by σ on $\mathcal{H}$. Further, recall that σ can be extended to a Riemannian metric on M. The compatibility condition ensures that we can pick a Darboux basis on each tangent space of M that is orthonormal with respect to σ, which implies that the volume element determined by σ is the same as the volume form $(\alpha \wedge (d\alpha)^n)/(n!)$ on M. The almost complex structure J and the Riemannian metric σ descend to each stratum of the contact quotient. Let the dimension of X be $2k+1$. Then

$$\mu = \frac{1}{k!}\alpha_X \wedge (d\alpha_X)^k \tag{3.59}$$

is the volume element of X_{prin}.

Choose $z \in Z$ mapping to $x \in X$ and let $H = G_z$. By ([10], Theorem 4.1), we may take U to be the H-contact quotient of an H-invariant neighborhood B of the origin in a contact vector space $W = V \times \mathbb{R}$. Recall that V is the symplectic slice $(T_z(G \cdot z))^{d\alpha}/T_z(G \cdot z)$ at z, with an almost complex structure induced from that on $\mathcal{H} = \ker \alpha$ and an inherited H-action which is extended trivally to $\mathbb{R}$.

Denote the subspace of invariants of W by $W^H = \{w \in W : hw = w\}$. Note $W^H = V^H \times \mathbb{R}$, where V^H is a complex subspace of V. Let $W^\perp$ be the symplectic orthogonal in of V^H in V. We can consider it to be the orthogonal complement of W^H in W. Since V^H is a complex subspace of V, then $W^\perp$ is a symplectic vector space. Recall that the moment map of the H-action on W is given by $\langle \Phi_W(w), \xi \rangle = \langle \Phi_V(v), \xi \rangle$, where $w = (v,t)$ and Φ_V is the quadratic symplectic moment map of the H-action on V. Since Φ_V is constant along V^H, then Φ_W is constant along W^H. So $Z_W = W^H \times Z_{W^\perp}$, where $Z_W = \Phi_W^{-1}(0)$ and $Z_{W^\perp} = \Phi_W^{-1}(0) \cap W^\perp$. Let $B = B_1 \times B_2$, where B_1 is an open ball about the origin in W^H and B_2 is an open ball about the origin in $W^\perp$. Then the product of the flat metrics on W^H and $W^\perp$ gives a metric on B. Therefore, $(Z_W)_{\text{prin}} = W^H \times (Z_{W^\perp})_{\text{prin}}$ and the quotient

$$\begin{aligned} U_{\text{prin}} &= (B_1 \times (B_2 \cap (Z_{W^\perp})_{\text{prin}}))/H \\ &= B_1 \times (B_2 \cap (Z_{W^\perp})_{\text{prin}})/H \end{aligned} \tag{3.60}$$

each also has a metric. As in ([17], p. 162), note that $B_2 \cap (Z_{W^\perp})_{\text{prin}}$ is a metric cone over the manifold $\partial B_2 \cap (Z_{W^\perp})_{\text{prin}}$. After taking quotients, $(B_2 \cap (Z_{W^\perp})_{\text{prin}})/H$ is a metric cone over $(\partial B_2 \cap (Z_{W^\perp})_{\text{prin}})/H$.

Before proving Stokes' theorem, we also need to show that the relevant integrals converge. The Riemannian metric σ on M induces metrics on $\Omega^l(M) \cong \Lambda^l(TM)$ for all l. Let $|\cdot|$ denote the pointwise norm of differential forms over either M or X_{prin}, as appropriate. For any $\tilde{\gamma} \in \Omega_\Phi(M)$, $|\tilde{\gamma}|$ is a G-invariant continuous function on M. If $\gamma \in \Omega(X)$ is induced by $\tilde{\gamma} \in \Omega_\Phi(M)$, then

$$\pi^*_{\text{prin}}|\gamma| \leq \iota^*_{\text{prin}}|\tilde{\gamma}|. \tag{3.61}$$

The support of γ is the closure in X of the support of γ on X_{prin}. Assume that $K = \text{supp}(\gamma) \subset X$ is compact. Since G is a compact group acting on M, then the action is proper and the orbit map $\pi\colon Z \to X$ is proper. So $\pi^{-1}(K)$ is compact in Z. Now, take an open neighborhood around $\pi^{-1}(K)$ in Z and define a bump function f which is 1 on this neighborhood. Then $f\tilde{\gamma}$ induces the same form as $\tilde{\gamma}$. So replace $\tilde{\gamma}$ with $f\tilde{\gamma}$. Observe that now $|\tilde{\gamma}|$ is a continuous function with compact support S in M, which contains $\pi^{-1}(K)$. So by the extreme value theorem, $|\tilde{\gamma}|$ attains a maximum on S. By the estimate 3.61, this maximum serves as a bound for the pointwise norm $|\gamma|$ on X_{prin}. Then Lemma 3.24 implies that $\int_{X_{\text{prin}}} |\gamma|\mu = \int_{K\cap X_{\text{prin}}} |\gamma|\mu$ is finite. In particular, if α is of degree $2k+1$ then $\int_{X_{\text{prin}}} \gamma$ is absolutely convergent.

This sets us up to prove Stokes' theorem for contact quotients. The main idea is that since all the strata of the singular contact quotient X are contact manifolds, then the dimension of each stratum is at most $\dim X - 2 = 2k - 1$. This means that there are no singularities in X of dimension $2k$. So there is no boundary of X_{prin}, and so there are no boundary terms when integrating.

Proposition 3.25. *$\int_{X_{\text{prin}}} d\gamma = 0$ if $\gamma \in \Omega^{2k}(X)$ has compact support.*

Proof. Assume that γ has compact support in an open subset U of the form

$U = B_1 \times (B_2 \cap Z_{W^\perp})/H$. It suffices to do this since the use of partitions of unity implies the general case. Let $2m$ be the dimension of the symplectic quotient $Z_{W^\perp}/H$. Since all the singularities in the quotient U come from $W^\perp$, then $m = 0$ implies that U is nonsingular. In this case the usual Stokes' theorem applies, so that

$$\int_{X_{\text{prin}}} d\gamma = \int_U d\gamma = \int_{\partial U} \gamma = 0.$$

Assume $m \geq 1$. Let $\chi\colon [0,\infty] \to [0,1]$ be a smooth function such that $\chi(t) = 0$ for t near 0 and $\chi(t) = 1$ for $t \geq 1$. For all positive integers j, define H-invariant functions $\tilde{\chi}_j\colon W \to [0,1]$ by $\tilde{\chi}_j(w) = \chi(j|\mathrm{pr}_{W^\perp} w|)$, where $\mathrm{pr}_W\colon W \to W^\perp$ is the orthogonal projection. Notice that $\tilde{\chi}_j(w) = 1$ only when $|\mathrm{pr}_{W^\perp} w| \geq 1/j$. These functions descend to smooth functions $\chi_j\colon U \to [0,1]$. Consider the functions $(1-\chi_j)\colon U \to [0,1]$. Observe that

$$S_j = \mathrm{supp}(1-\chi_j) \subseteq B_1 \times B_{1/j}(0) \cap Z_{W^\perp}, \tag{3.62}$$

where $B_{1/j}(0)$ is the ball of radius $1/j$ about the origin in $W^\perp$. As j increases, the radius of these balls decreases, so that the S_j form a decreasing sequence. Furthermore,

$$\bigcap_j S_j = B_1 \times \{0 \bmod H\}, \tag{3.63}$$

where $\{0 \bmod H\}$ is the vertex, or image of origin, in the singular symplectic quotient $Z_{W^\perp}/H$. This is the most singular stratum of U, since it is contained in the closure of all other strata. Therefore, the restriction to the principal stratum $\bigcap_j (S_j)_{\text{prin}}$ is empty. Because $(1-\chi_j)d\gamma$ is of top degree, there exists an upper bound C for $|(1-\chi_j)d\gamma|$. Then

$$\left| \int_{X_{\text{prin}}} d\gamma - \int_{X_{\text{prin}}} \chi_j d\gamma \right| = \left| \int_{X_{\text{prin}}} (1-\chi_j) d\gamma \right| \leq C \cdot \mathrm{vol}((S_j)_{\text{prin}}). \tag{3.64}$$

Taking the limit of each side of this inequality as j approaches infinity, and utilizing the fact that taking volume is continuous, we get

$$\lim_{j\to\infty}\left|\int_{X_{\text{prin}}} d\gamma - \int_{X_{\text{prin}}} \chi_j d\gamma\right| = 0. \tag{3.65}$$

Therefore,

$$\int_{X_{\text{prin}}} d\gamma = \lim_{j\to\infty}\int_{X_{\text{prin}}} \chi_j d\gamma. \tag{3.66}$$

Our goal is to show that this limit is zero.

The product rule, applied to $d(\chi_j\gamma)$, will be useful in the form

$$\int_{X_{\text{prin}}} \chi_j d\gamma = \int_{X_{\text{prin}}} d(\chi_j\gamma) - \int_{X_{\text{prin}}} d\chi_j \wedge \gamma. \tag{3.67}$$

Since $(1-\chi_j)$ for all j are supported near the most singular stratum, then χ_j for all j are supported away from it. So we can assume by induction on the depth of the stratification of X that

$$\int_{X_{\text{prin}}} d(\chi_j\gamma) = 0. \tag{3.68}$$

Let D be an upper bound for $|\gamma|$. Then

$$\left|\int_{X_{\text{prin}}} d\chi_j \wedge \gamma\right| \leq \int_{X_{\text{prin}}} |d\chi_j||\gamma|\mu \leq D\int_{(S_j)_{\text{prin}}} |d\chi_j|\mu. \tag{3.69}$$

For each j, let $\tilde{\rho}_j\colon W^\perp \to W^\perp$ be the dilation $\tilde{\rho}_j(w) = jw$. Consider the induced map $\rho_j\colon Z_W/H \to Z_W/H$. Then $\chi_j = \chi_1 \circ \rho_j$ and $S_j = \rho_j^{-1}(S_1)$. Equation 3.60 expresses U_{prin} as the product of a ball and a metric cone. Using this characterization, we see that $\text{vol}((S_j)_{\text{prin}}) = j^{-2m}\text{vol}((S_1)_{\text{prin}})$. So,

$$\left|\int_{X_{\text{prin}}} d\chi_j \wedge \gamma\right| \leq Dj^{1-2m}\int_{(S_1)_{\text{prin}}} |d\chi_1|\mu. \tag{3.70}$$

Taking limits, we find that

$$\lim_{j\to\infty}\left|\int_{X_{\text{prin}}} d\chi_j \wedge \gamma\right| = 0. \tag{3.71}$$

Now putting Equations 3.67, 3.68, and 3.71 together, we conclude that

$$\lim_{j\to\infty}\int_{X_{\text{prin}}} \chi_j d\gamma = 0. \tag{3.72}$$

□

Stokes' theorem allows us to obtain more information about the reduced contact form α_X. In particular, if X is compact, then $\alpha_X \wedge (d\alpha_X)^k$ is a volume form which satisfies

$$0 < \int_{X_{\text{prin}}} \alpha_X \wedge (d\alpha_X)^k < \infty. \tag{3.73}$$

So by the contrapositive of Stokes' theorem, $\alpha_X \wedge (d\alpha_X)^k \neq d\gamma$ for any compactly supported form $\gamma \in \Omega^{2k}(X)$. That is, $\alpha_X \wedge (d\alpha_X)^k$ is not exact and therefore its cohomology class $H^{2k+1}(\Omega(X))$ is nonzero.

Corollary 3.26. *If X is compact with dimension $2k+1$, then the cohomology class of $\alpha_X \wedge (d\alpha_X)^k$ in $H^{2k+1}(\Omega(X))$ is nonzero.*

This is another way in which the reduced contact form on a singular quotient is analogous to a contact form on a smooth manifold.

3.8 The Jacobi Structure on a Contact Quotient

In this section we define a Jacobi structure on the contact quotient X, utilizing the abstract definition of Jacobi algebra. Since each stratum of X is a contact manifold, each stratum is already a Jacobi manifold. We show that the Jacobi structure on the whole quotient is compatible with the Jacobi structure present on each stratum.

Recall Φ is the contact moment map of the G-action on M and Z is its zero fiber. Denote the G-invariant functions that vanish on Z by $I(Z)^G$. That is,

$$I(Z)^G = \left\{ f \in C^\infty(M)^G : f|_Z = 0 \right\}. \tag{3.74}$$

This set is clearly an ideal of $C^\infty(M)^G$ under function multiplication.

Now, recall that the de Rham complex of differential forms on X has a useful identification with the complex of Φ-basic forms modulo the ideal of those forms whose restrictions to Z_{prin} vanish. This is the isomorphism 3.16. In degree zero, we use the notation $C^\infty(X) = \Omega^0(X)$. Since every G-invariant function on M is trivially killed by all interior products, then $C^\infty(M)^G = \Omega^0_\Phi(M)$. Since every smooth function which restricts to the zero function on an open dense subset of Z must be zero on all of Z, then $I(Z)^G = I^0_\Phi(M)$. So in degree zero, the isomorphism 3.16 states that

$$C^\infty(X) \cong C^\infty(M)^G / I(Z)^G. \tag{3.75}$$

We take this to be the definition of ***smooth functions*** on the singular space X. It matches the definition in the symplectic case given by Arms, Cushman, and Gotay in [1], and the definition given in the context of subcartesian spaces by Watts in [20].

We will show that the quotient algebra $C^\infty(X)$ is actually a Jacobi algebra. We have already shown in Proposition 2.26 that $C^\infty(M)^G$ is a Jacobi algebra. We will now show that $I(Z)^G$ is a Jacobi ideal. This requires a few preliminary results.

Lemma 3.27. *For any $\xi \in \mathfrak{g}^*$, we have*

$$d\Phi^\xi = -i_{\xi_M} d\alpha. \tag{3.76}$$

Proof. By the definition of the contact moment map, $\Phi^\xi = i_{\xi_M}\alpha$. Since $\mathcal{L}_{\xi_M}\alpha = 0$, then by Cartan's magic formula, we have

$$di_{\xi_M}\alpha + i_{\xi_M}d\alpha = 0. \tag{3.77}$$

Therefore, we have

$$di_{\xi_M}\alpha = -i_{\xi_M}d\alpha, \tag{3.78}$$

which is the result. □

Now we can prove that the derivation given by the Reeb vector field E sends each component Φ^ξ of the moment map to the zero function. This shows that the flow of the Reeb vector field preserves the zero level set Z of the moment map.

Lemma 3.28. *For any $\xi \in \mathfrak{g}^*$, we have*

$$E(\Phi^\xi) = 0. \tag{3.79}$$

Proof. We use Lemma 3.27 and the defining properties of the Reeb vector field E to get

$$\begin{aligned} E(\Phi^\xi) &= i_E d\Phi^\xi \\ &= -i_E i_{\xi_M} d\alpha \\ &= i_{\xi_M} i_E d\alpha \\ &= 0. \end{aligned} \tag{3.80}$$

□

The following is also a useful fact.

Lemma 3.29. *The Reeb vector field E associated with α is G-invariant. That is, $g_*E = E$ for all $g \in G$.*

Proof. Recall that α and $d\alpha$ are both G-invariant. We observe that

$$\begin{aligned} i_{g_*E}d\alpha &= \left(g^{-1}\right)^* \left(i_E \left(\left(g^{-1}\right)^* d\alpha\right)\right) \\ &= \left(g^{-1}\right)^* \left(i_E d\alpha\right) \\ &= \left(g^{-1}\right)^* (0) \\ &= 0. \end{aligned} \tag{3.81}$$

Also, we have

$$\begin{aligned} i_{g_*E}\alpha &= \left(g^{-1}\right)^* \left(i_E \left(\left(g^{-1}\right)^* \alpha\right)\right) \\ &= \left(g^{-1}\right)^* \left(i_E \alpha\right) \\ &= \left(g^{-1}\right)^* (1) \\ &= 1. \end{aligned} \tag{3.82}$$

Since g_*E satisfies the two conditions of being the Reeb vector field associated with α, then $g_*E = E$. □

The G-invariance of E helps to show that the hamiltonian correspondence identifies G-invariant functions with G-invariant hamiltonian vector fields.

Lemma 3.30. *The function f is G-invariant if and only if its hamiltonian vector field Ξ_f is G-invariant.*

Proof. Let $g \in G$ be arbitrary. We use the conditions satisfied by f and Ξ_f under the hamiltonian correspondence.

Assume that f is G-invariant. Because α is also G-invariant, we have that

$$\begin{aligned} i_{g_*\Xi_f}\alpha &= \left(g^{-1}\right)^* \left(i_{\Xi_f} \left(\left(g^{-1}\right)^* \alpha\right)\right) \\ &= \left(g^{-1}\right)^* \left(i_{\Xi_f} \alpha\right) \\ &= \left(g^{-1}\right)^* f \\ &= f. \end{aligned} \tag{3.83}$$

Furthermore, since E and df are also G-invariant, then

$$\begin{aligned}
i_{g_*\Xi_f} d\alpha &= \left(g^{-1}\right)^* \left(i_{\Xi_f} \left(\left(g^{-1}\right)^* d\alpha\right)\right) \\
&= \left(g^{-1}\right)^* \left(i_{\Xi_f} d\alpha\right) \\
&= \left(g^{-1}\right)^* \left(i_E df \cdot \alpha - df\right) \\
&= \left(\left(g^{-1}\right)^* \left(i_E df\right)\right) \cdot \left(\left(g^{-1}\right)^* \alpha\right) - \left(g^{-1}\right)^* df \\
&= i_{g_* E} \left(\left(g^{-1}\right)^* df\right) \cdot \alpha - \left(g^{-1}\right)^* df \\
&= i_E df \cdot \alpha - df.
\end{aligned} \tag{3.84}$$

So $g_*\Xi_f$ is identified with f under the hamiltonian correspondence. Therefore, $g_*\Xi_f = \Xi_f$ and so Ξ_f is G-invariant.

Assume that Ξ_f is G-invariant. Because α is also G-invariant, we have that

$$\begin{aligned}
g^* f &= g^*(i_\Xi \alpha) \\
&= i_{g_*^{-1}\Xi}(g^*\alpha) \\
&= i_\Xi \alpha \\
&= f.
\end{aligned} \tag{3.85}$$

That is, f is G-invariant. □

Now we explore what the derivation given by a G-invariant hamiltonian vector field does to each component Φ^ξ of the moment map.

Proposition 3.31. *Let $f \in C^\infty(M)^G$ and Ξ_f be its hamiltonian vector field. Then*

$$\Xi_f(\Phi^\xi) = i_E df \cdot \Phi^\xi \tag{3.86}$$

for all $\xi \in \mathfrak{g}$.

Proof. Recall that $\Xi_f = (-df + (f + i_E df)\alpha)^\sharp$ by Proposition 2.9. Let $\xi \in \mathfrak{g}$.

Distributing the sharp map and invoking Lemma 3.28, we have

$$\begin{aligned}\Xi_f(\Phi^\xi) &= -df^\sharp(\Phi^\xi) + f\alpha^\sharp(\Phi^\xi) + i_E df \cdot \alpha^\sharp(\Phi^\xi) \\ &= -df^\sharp(\Phi^\xi) + fE(\Phi^\xi) + i_E df \cdot E(\Phi^\xi) \\ &= -df^\sharp(\Phi^\xi).\end{aligned} \tag{3.87}$$

We continue by using Lemma 3.27 and Equation 3.78 to get

$$\begin{aligned}\Xi_f(\Phi^\xi) &= -df^\sharp(\Phi^\xi) \\ &= -i_{df^\sharp} d\Phi^\xi \\ &= -i_{df^\sharp} d i_{\xi_M}\alpha \\ &= i_{df^\sharp} i_{\xi_M} d\alpha \\ &= -i_{\xi_M} i_{df^\sharp} d\alpha.\end{aligned} \tag{3.88}$$

Now we use Lemma 2.11 and the hamiltonian correspondence to get

$$\begin{aligned}\Xi_f(\Phi^\xi) &= -i_{\xi_M} i_{df^\sharp} d\alpha \\ &= i_{\xi_M} i_{\Xi_f} d\alpha \\ &= i_{\xi_M}(i_E df \cdot \alpha - df) \\ &= i_{\xi_M}(i_E df \cdot \alpha) - i_{\xi_M} df.\end{aligned} \tag{3.89}$$

Since f is G-invariant, then $i_{\xi_M} df = \xi_M(f) = 0$, so we are left with

$$\begin{aligned}\Xi_f(\Phi^\xi) &= i_{\xi_M}(i_E df \cdot \alpha) \\ &= i_E df \cdot i_{\xi_M}\alpha \\ &= i_E df \cdot \Phi^\xi.\end{aligned} \tag{3.90}$$

□

Note that since $\Phi^\xi\big|_Z = 0$ this proposition implies that for all $z \in Z$,

$$\Xi_f(\Phi^\xi)(z) = i_E df(z) \cdot \Phi^\xi(z) = 0. \tag{3.91}$$

This means that the flow of Ξ_f preserves the zero level set Z of the moment map. Furthermore, recall that by Equation 2.45,

$$\Xi_f(\Phi^\xi) = \left\{f, \Phi^\xi\right\} + i_E df \cdot \Phi^\xi. \tag{3.92}$$

But because $\Xi_f(\Phi^\xi) = i_E df \cdot \Phi^\xi$ by Proposition 3.31, then

$$\left\{f, \Phi^\xi\right\} = 0. \tag{3.93}$$

Now we can prove that $I(Z)^G$ is a Jacobi ideal.

Proposition 3.32. *If $f \in C^\infty(M)^G$ and $h \in I(Z)^G$, then $\{f, h\} \in I(Z)^G$. In other words, $I(Z)^G$ is a Jacobi ideal.*

Proof. Let $z \in Z$. We know that $\{f, h\} = \Xi_f(h) - i_E df \cdot h$. So, using the fact that $h|_Z = 0$, we have

$$\begin{aligned} \{f, h\}(z) &= \Xi_f(h)(z) - (i_E df \cdot h)(z) \\ &= \Xi_f(h)(z) - i_E df(z) \cdot h(z) \\ &= \Xi_f(h)(z) \end{aligned} \tag{3.94}$$

Therefore, to conclude that $\{f, h\}|_Z = 0$, it remains to show $\Xi_f(h)(z) = 0$.

Let $\gamma(t)$ be the hamiltonian trajectory of f through z. That is, $\gamma(t)$ is the solution to the following initial value problem

$$\begin{aligned} \gamma(0) &= z \\ \gamma'(t) &= \Xi_f(\gamma(t)). \end{aligned} \tag{3.95}$$

Let $\xi \in \mathfrak{g}$ be arbitrary. We know $\Phi^\xi(\gamma(t)) = 0$ when $t = 0$. We want to show that $\Phi^\xi(\gamma(t)) = 0$ for all t. The derivative of $\Phi^\xi \circ \gamma$ with respect to t is

$$\begin{aligned} \frac{d}{dt}\Phi^\xi(\gamma(t)) &= d\Phi^\xi_{\gamma(t)}(\gamma'(t)) \\ &= d\Phi^\xi_{\gamma(t)}(\Xi_f(\gamma(t))) \\ &= \Xi_f(\Phi^\xi)(\gamma(t)) \\ &= i_E df(\gamma(t)) \cdot \Phi^\xi(\gamma(t)). \end{aligned} \tag{3.96}$$

Let $\phi(t) = \Phi^\xi(\gamma(t))$ and $\psi(t) = i_E df(\gamma(t))$. Both are smooth functions of $\mathbb{R}$. So the above work is summarized by the initial value problem

$$\begin{aligned} \frac{d\phi}{dt} &= \psi\phi \\ \phi(0) &= 0. \end{aligned} \tag{3.97}$$

The zero function $\phi(t) = 0$ is indeed a solution to this problem. By the existence and uniqueness theorem, the zero solution $\phi(t) = 0$ must be the only solution. In other words, $\Phi^\xi(\gamma(t)) = 0$ for all t. This implies $\gamma(t) \in Z$ for all t. So we have that

$$h(\gamma(t)) = 0 \tag{3.98}$$

for all t. Differentiating this equation at $t = 0$ results in

$$\Xi_f(h)(z) = 0, \tag{3.99}$$

as desired. □

Note that the argument here is slightly more subtle than in the symplectic case. In the symplectic case, the moment map Φ is a global constant of motion of f. But in the contact case, the moment map Φ is a constant of motion of f only inside Z. If $\gamma(t) \notin Z$, then $\frac{d}{dt}\Phi^\xi(\gamma(t))$ is not necessarily zero.

Theorem 3.33. *The quotient algebra $C^\infty(X) = C^\infty(M)^G/I(Z)^G$ is a Jacobi algebra.*

Proof. This follows from Propositions 3.32 and 2.23. □

The quotient algebra $C^\infty(X)$ inherits a Jacobi bracket from $C^\infty(M)^G$ as described in Proposition 2.23. Denote the Jacobi bracket on $C^\infty(X)$ by $\{\cdot,\cdot\}_X$. Let $[f]$ denote the equivalence class of $f \in C^\infty(M)^G$ in $C^\infty(X)$. Then for all $f, g \in C^\infty(M)^G$, we have

$$\{[f],[g]\}_X = [\{f,g\}], \tag{3.100}$$

where the bracket on the right is the Jacobi bracket on $C^\infty(M)$. We can think of the bracket $\{\cdot,\cdot\}_X$ as an abstract Jacobi structure on the singular contact quotient X.

Now recall that the orbit type stratification of X consists of strata which are themselves contact manifolds, and therefore are Jacobi manifolds. We now turn our attention to showing that the abstract Jacobi structure on X defined above is compatible with the Jacobi structure on each stratum.

From now on, let $X_a = Z_a/G$ be an arbitrary stratum of X. Recall that by Lemma 3.23, the contact form α_a on X_a is the restriction of α_X under the map in Lemma 3.8. Denote the Reeb vector field on X_a with respect to α_a by E_a. Define the biderivation Λ_a of $C^\infty(X_a)$ by $\Lambda_a(f,h) = d(\alpha_a)(df^\sharp, dh^\sharp)$, for all $f,h \in C^\infty(X_a)$. Then the pair (Λ_a, E_a) is Jacobi structure on the manifold X_a. Denote the Jacobi bracket on X_a arising from its contact structure (Λ_a, E_a) by $\{\cdot,\cdot\}_{X_a}$. That is,

$$\{f,h\}_{X_a} = \Lambda_a(f,h) + fE_a(h) - hE_a(f) \tag{3.101}$$

for all $f,h \in C^\infty(X_a)$.

Our goal is to show that the Jacobi bracket $\{\cdot,\cdot\}_X$ on $C^\infty(X)$ is compatible with the Jacobi bracket $\{\cdot,\cdot\}_{X_a}$ on $C^\infty(X_a)$. Since the Jacobi bracket on $C^\infty(X)$ descends from the one on $C^\infty(M)$, we start by relating the contact structure (Λ_a, E_a) on X_a to the contact structure (Λ, E) on M.

We require a preliminary result about vector fields and pushforwards by a projection map.

Proposition 3.34. *Suppose G acts smoothly on a manifold P and all points of P are of the same orbit type. Let $\pi\colon P \to P/G$ be the projection. If Ξ is a G-*

invariant vector field on P, then the pushforward $\pi_*\Xi$ is a well-defined vector field on the manifold P/G.

Proof. Let $x \in P/G$ and $p, q \in \pi^{-1}(x) \subset P$. So there is an element $g \in G$ such that $p = gq$. Denote the differential of g at q by $g_{*q}\colon T_qP \to T_pP$. Since Ξ is G-invariant, then $\Xi_p = g_{*q}(\Xi_{g^{-1}p}) = g_{*q}(\Xi_q)$. Using this fact, the global chain rule, and the fact that $\pi \circ g = \pi$, we have

$$\begin{aligned} \pi_{*p}(\Xi_p) &= \pi_{*p}(g_{*q}(\Xi_q)) \\ &= (\pi_{*p} \circ g_{*q})(\Xi_q) \\ &= (\pi \circ g)_{*q}(\Xi_q) \\ &= \pi_{*q}(\Xi_q). \end{aligned} \tag{3.102}$$

So the vector $\pi_{*p}(\Xi_p) \in T_x(P/G)$ does not depend on the choice of $p \in \pi^{-1}(x)$. Thus, $\pi_*\Xi$ is a well-defined vector field on P/G. □

By this proposition, any G-invariant vector field on Z_a pushes forward by the projection $\pi_a\colon Z_a \to X_a$ to a vector field on X_a. It turns out that every G-invariant hamiltonian vector field on M restricts to a G-invariant hamiltonian vector field on Z_a.

Lemma 3.35. *If Ξ is a G-invariant vector field on M, then the local flow ψ_t of Ξ is G-equivariant.*

Proof. Let $g \in G$. We would like to show that $\psi_t \circ g = g \circ \psi_t$. It suffices to show that $g^{-1} \circ \psi_t \circ g = \psi_t$. We know that for any fixed t, the smooth map ψ_t is the unique solution the initial value problem 1.11. So it suffices to show that $g^{-1} \circ \psi_t \circ g$ is a solution to the same initial value problem. Observe that for all $p \in M$,

$$\begin{aligned} (g^{-1} \circ \psi_0 \circ g)(p) &= g^{-1}(gp) \\ &= p. \end{aligned} \tag{3.103}$$

Using the chain rule and the fact that Ξ is G-invariant, we have that

$$\begin{aligned}\frac{d}{dt}(g^{-1} \circ \psi_t \circ g)(p) &= g_*^{-1}\frac{d}{dt}(\psi_t(gp)) \\ &= g_*^{-1}\Xi(\psi_t(gp)) \\ &= \Xi((g^{-1} \circ \psi_t \circ g)(p)),\end{aligned} \tag{3.104}$$

for all $t \in \mathbb{R}$ and all $p \in M$ for which the terms are defined. □

Lemma 3.36. *If Ξ is a G-invariant hamiltonian vector field on M, then Ξ is tangent to Z_a.*

Proof. Recall that Z_a is a connected component of $Z \cap M_{(H)}$, for some conjugacy class (H) of stabilizer subgroups of G. Denote the local flow of Ξ by ψ_t. Since Ξ is G-invariant, then ψ_t is G-equivariant by Lemma 3.35. Now let $p \in M_{(H)}$. We may pick the representative H to be the stabilizer G_p of p. Let $g \in H$. We show that g is also in the stabilizer $G_{\psi_t(p)}$ of $\psi_t(p)$. Observe that

$$\begin{aligned}g(\psi_t(p)) &= \psi_t(gp) \\ &= \psi_t(p).\end{aligned} \tag{3.105}$$

Conversely, if $g \in G_{\psi_t(p)}$, then we get the same equation, and plugging in $t = 0$ yields $gp = p$. So $H = G_{\psi_t(p)}$, which implies that the stabilizers of points are preserved along trajectories of Ξ. This means that $\psi_t(p)$ remains in $M_{(H)}$ for all $p \in M_{(H)}$ and $t \in \mathbb{R}$ for which it is defined. That is, Ξ is tangent to $M_{(H)}$.

Now let f be the hamiltonian function of Ξ. Since Ξ is G-invariant, then f is G-invariant by Lemma 3.30. By Proposition 3.31, the vector field Ξ preserves the zero level set Z of the moment map. So Ξ is tangent to Z.

Since Ξ is tangent to both $M_{(H)}$ and Z, then it must be tangent to the connected component Z_a of $M_{(H)} \cap Z$. □

This lemma implies that every G-invariant hamiltonian vector field Ξ restricts to a vector field $\Xi|_{Z_a}$ on the submanifold Z_a. Since Ξ is G-invariant, then so is $\Xi|_{Z_a}$. Then by Proposition 3.34, the pushforward $\pi_{a*}(\Xi|_{Z_a})$ is a well-defined vector field on X_a.

In particular, the Reeb vector field E restricts to a G-invariant vector field $E|_{Z_a}$ on Z_a, and the pushforward $\pi_{a*}(E|_{Z_a})$ is a well-defined vector field on X_a. It is easy to check that $E|_{Z_a}$ is the Reeb vector field associated to the contact form $\iota_a^*\alpha$ on Z_a. Now we show that the Reeb vector field E_a on X_a descends from the Reeb vector field E on M.

Proposition 3.37. *The Reeb vector field E_a associated to α_a on X_a is the vector field $\pi_{a*}(E|_{Z_a})$.*

Proof. The Reeb vector field is uniquely determined by two conditions. We check these two conditions here, using the definition of α_a and the fact that $E|_{Z_a}$ is the Reeb vector field associated to $\iota_a^*\alpha$ on Z_a. First, observe that

$$\begin{aligned} i_{\pi_{a*}(E|_{Z_a})}d(\alpha_a) &= i_{\pi_{a*}(E|_{Z_a})}d((\pi_a^*)^{-1}\iota_a^*\alpha) \\ &= (\pi_a^*)^{-1}(i_{E|_{Z_a}}d(\iota_a^*\alpha)) \\ &= (\pi_a^*)^{-1}(0) \\ &= 0. \end{aligned} \tag{3.106}$$

Second, observe that

$$\begin{aligned} i_{\pi_{a*}(E|_{Z_a})}(\alpha_a) &= i_{\pi_{a*}(E|_{Z_a})}((\pi_a^*)^{-1}\iota_a^*\alpha) \\ &= (\pi_a^*)^{-1}(i_{E|_{Z_a}}(\iota_a^*\alpha)) \\ &= (\pi_a^*)^{-1}(1) \\ &= 1. \end{aligned} \tag{3.107}$$

Therefore, $E_a = \pi_{a*}(E|_{Z_a})$. □

Lemma 3.36 also allows us to prove that for every $f \in C^\infty(M)^G$, the G-invariant hamiltonian vector field Ξ_f on M descends to a hamiltonian vector field on X_a. As a preliminary step, note that $\Xi_f|_{Z_a}$ is the hamiltonian vector field for $\iota_a^* f \in C^\infty(Z_a)$. Also, recall the map 3.17 used to define the restriction map in Lemma 3.8. We adopt the same notation as was used there, so that every function $f \in C^\infty(M)^G$ is sent to to a function $f_a \in C^\infty(X_a)$. Since $\iota_a^* f \in \Omega_{\text{bas}}(Z_a)$, then we know by Lemma 3.5 that $f_a = (\pi_a^*)^{-1}(\iota_a^* f)$.

Proposition 3.38. *For all $f \in C^\infty(M)^G$, the hamiltonian vector field $\Xi_{f_a} \in \mathfrak{X}(X_a)$ of $f_a \in C^\infty(X_a)$ is $\pi_{a*}(\Xi_f|_{Z_a})$.*

Proof. We check the two conditions which uniquely determine the hamiltonian vector field of $f_a \in C^\infty(X_a)$. We use the fact that $\Xi_f|_{Z_a}$ is the hamiltonian vector field of the function $\iota_a^* f \in C^\infty(Z_a)$. First, observe that

$$\begin{aligned}
i_{\pi_{a*}(\Xi_f|_{Z_a})}(\alpha_a) &= i_{\pi_{a*}(\Xi_f|_{Z_a})}((\pi_a^*)^{-1}\iota_a^*\alpha) \\
&= (\pi_a^*)^{-1}(i_{\Xi_f|_{Z_a}}(\iota_a^*\alpha)) \\
&= (\pi_a^*)^{-1}\iota_a^* f \\
&= f_a.
\end{aligned} \tag{3.108}$$

Second, observe that

$$\begin{aligned}
i_{\pi_{a*}(\Xi_f|_{Z_a})}d(\alpha_a) &= i_{\pi_{a*}(\Xi_f|_{Z_a})}d((\pi_a^*)^{-1}\iota_a^*\alpha) \\
&= (\pi_a^*)^{-1}(i_{\Xi_f|_{Z_a}}d(\iota_a^*\alpha)) \\
&= (\pi_a^*)^{-1}(i_{E|_{Z_a}}d(\iota_a^* f)\cdot(\iota_a^*\alpha) - d(\iota_a^* f)) \\
&= i_{\pi_{a*}(E|_{Z_a})}d((\pi_a^*)^{-1}\iota_a^* f)\cdot((\pi_a^*)^{-1}\iota_a^*\alpha) - d((\pi_a^*)^{-1}\iota_a^* f) \\
&= i_{E_a}d(f_a)\cdot\alpha_a - d(f_a)
\end{aligned} \tag{3.109}$$

Therefore, $\Xi_{f_a} = \pi_{a*}(\Xi_f|_{Z_a})$. ∎

Now that we have an understanding of the Reeb vector field E_a and and hamiltonian vector fields Ξ_{f_a} on X_a, we can describe the biderivation Λ_a on X_a.

Proposition 3.39. *For all $f, h \in C^\infty(M)^G$, we have*

$$\Lambda_{X_a}(f_a, h_a) = -f_a E_a(h_a) + \Xi_{f_a}(h_a). \tag{3.110}$$

Proof. This follows from Lemma 2.13, and Propositions 3.37 and 3.38. □

The following lemma is useful for characterizing Λ_a further.

Lemma 3.40. *Let Ξ be a G-invariant hamiltonian vector field on M. Then, for all functions $h \in C^\infty(M)^G$, we have $(\Xi(h))_a = (\pi_{a*}(\Xi|_{Z_a}))(h_a)$.*

Proof. Since Ξ and h are both G-invariant, then the function $\iota_a^*(i_\Xi dh)$ is a basic form on Z_a. So by Lemmas 3.5 and 3.36, we have

$$\begin{aligned}(\Xi(h))_a &= (\pi_a^*)^{-1}\iota_a^*(i_\Xi dh)\\ &= (\pi_a^*)^{-1}(i_{\Xi|_{Z_a}}(\iota_a^* dh))\\ &= i_{\pi_{a*}(\Xi|_{Z_a})}((\pi_a^*)^{-1}\iota_a^* dh)\\ &= i_{\pi_{a*}(\Xi|_{Z_a})}(d((\pi_a^*)^{-1}\iota_a^* h))\\ &= i_{\pi_{a*}(\Xi|_{Z_a})}d(h_a)\\ &= (\pi_{a*}(\Xi|_{Z_a}))(h_a).\end{aligned} \tag{3.111}$$

□

Notice that the vector field E on M is the hamiltonian vector field corresponding to the zero function. So by Lemma 3.40, $(E(h))_a = E_a(h_a)$ for all $h \in C^\infty(M)^G$.

Lemma 3.41. *For all $f, h \in C^\infty(M)^G$, we have*

$$(\Lambda(f,h))_a = \Lambda_a(f_a, h_a). \tag{3.112}$$

Proof. Using Lemmas 2.13 and 3.40, and Propositions 3.38 and 3.39, we have

$$\begin{aligned}(\Lambda(f,h))_a &= (-fE(h)+\Xi_f(h))_a \\ &= (-fE(h))_a + (\Xi_f(h))_a \\ &= -f_a(E(h))_a + (\pi_{a*}(\Xi_f|_{Z_a}))(h_a) \\ &= -f_aE_a(h_a) + \Xi_{f_a}(h_a) \\ &= \Lambda_a(f_a,h_a).\end{aligned} \tag{3.113}$$

□

Finally, we can show that the Jacobi brackets $\{\cdot,\cdot\}_X$ and $\{\cdot,\cdot\}_{X_a}$ are compatible. The main idea is to restrict the Jacobi bracket of two functions $[f],[h] \in C^\infty(X)$ to a stratum X_a, and then show that this is equal to the function on X_a defined by the Jacobi bracket of $f_a, h_a \in C^\infty(X_a)$.

Theorem 3.42. *Let X_a be an arbitrary stratum of X. For all $f,h \in C^\infty(M)^G$, we have*

$$(\{[f],[h]\}_X)_a = \{f_a,h_a\}_{X_a}. \tag{3.114}$$

Proof. Lemmas 3.40 and 3.41 together give us the result. We also use the fact that $[\{f,h\}] \in C^\infty(X)$ descends from $\{f,h\} \in C^\infty(M)^G$. Observe that

$$\begin{aligned}(\{[f],[h]\}_X)_a &= ([\{f,h\}])_a \\ &= \{f,h\}_a \\ &= (\Lambda(f,h)+fE(h)-hE(f))_a \\ &= (\Lambda(f,h))_a + (fE(h))_a - (hE(f))_a \\ &= \Lambda_a(f_a,h_a) + f_aE_a(h_a) - h_aE_a(f_a) \\ &= \{f_a,h_a\}_{X_a}.\end{aligned} \tag{3.115}$$

□

BIBLIOGRAPHY

[1] J. M. Arms, R. H. Cushman, and M. J. Gotay. A Universal Reduction Procedure for Hamiltonian Group Actions. In *The Geometry of Hamiltonian Systems (Berkeley, CA, 1989)*, volume 22 of *Mathematical Sciences Research Institute Publications*, pages 33–51. Springer, New York, 1991.

[2] R. Chiang and Y. Karshon. Convexity Package for Momentum Maps on Contact Manifolds. *Algebraic and Geometric Topology*, 10(2):925–977, 2010.

[3] D. Chinea, J. C. Marrero, and M. de León. A Canonical Differential Complex for Jacobi Manifolds. *Michigan Mathematics Journal*, 45:547–579, 1998.

[4] L. Conlon. *Differentiable Manifolds.* Birkhäuser Advanced Texts. Birkhäuser, Boston, 2001.

[5] A. Cannas da Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2001.

[6] S. Evens and J.-H. Lu. Poisson Harmonic Forms, Kostant Harmonic Forms, and the S^1-Equivariant Cohomology of K/T. *Advances in Mathematics*, 142:171–220, 1999.

[7] H. Geiges. *An Introduction to Contact Topology*, volume 109 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2008.

[8] J.-L. Koszul. Crochet de Schouten-Nijenhuis et Cohomologie. *Astérisque*, (Numero Hors Serie):257–271, 1985. The Mathematical Heritage of Élie Cartan (Lyon, 1984).

[9] J. M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer, 2002.

[10] E. Lerman and C. Willett. The Topological Structure of Contact and Symplectic Quotients. *International Mathematics Research Notices*, 1:33–52, 2001.

[11] P. Libermann and C.-M. Marle. *Symplectic Geometry and Analytical Mechanics.* Mathematics and its Applications. D. Reidel Publishing Company, Dordrecht, 1987.

[12] A. Lichnerowicz. Les Variétés de Poisson et leurs Algèbres de Lie Associées. *Journal of Differential Geometry*, 12(2):253–300, 1977.

[13] A. Lichnerowicz. Les Variétés de Jacobi et leurs Algèbres de Lie Associées. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 57(4):453–488, 1978.

[14] C.-M. Marle. The Schouten-Nijenhuis Bracket and Interior Products. *Journal of Geometry and Physics*, 23(3-4):350–359, 1997.

[15] J. Marsden and A. Weinstein. Reduction of Symplectic Manifolds with Symmetry. *Reports on Mathematical Physics*, 5(1):121–130, 1974.

[16] K. Meyer. Symmetries and Integrals in Mechanics. In *Dynamical Systems (Proceedings of a Symposium, University of Bahia, Salvador, 1971)*, pages 259–272, New York, 1973. Academic Press.

[17] R. Sjamaar. A de Rham Theorem for Symplectic Quotients. *Pacific Journal of Mathematics*, 220(1):153–166, 2005.

[18] R. Sjamaar and E. Lerman. Stratified Symplectic Spaces and Reduction. *Annals of Mathematics*, 134(2):375–422, 1991.

[19] I. Vaisman. *Lectures on the Geometry of Poisson Manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

[20] J. Watts. Differential Forms on Symplectic Quotients, 2011. `arXiv:1104.4084v2 [math.SG]`.

[21] C. Willett. Contact Reduction. *Transactions of the AMS*, 354(10):4245–4260, 2002.