

# KAUFFMAN BRACKET SKEIN MODULES AND THE QUANTUM TORUS

A Dissertation

Presented to the Faculty of the Graduate School  
of Cornell University

in Partial Fulfillment of the Requirements for the Degree of  
Doctor of Philosophy

by

Peter Curtis Samuelson

August 2012

© 2012 Peter Curtis Samuelson

ALL RIGHTS RESERVED

# KAUFFMAN BRACKET SKEIN MODULES AND THE QUANTUM TORUS

Peter Curtis Samuelson, Ph.D.

Cornell University 2012

If  $M$  is a 3-manifold, the *Kauffman bracket skein module* is a vector space  $K_q(M)$  functorially associated to  $M$  that depends on a parameter  $q \in \mathbb{C}^*$ . If  $F$  is a surface, then  $K_q(F \times [0, 1])$  is an algebra, and  $K_q(M)$  is a module over  $K_q((\partial M) \times [0, 1])$ . One motivation for the definition is that if  $L \subset S^3$  is a knot, then the (colored) Jones polynomials  $J_n(L) \in \mathbb{C}[q^{\pm 1}]$  can be computed from  $K_q(S^3 \setminus L)$ .

It was shown in [14] that  $K_q(T^2 \times [0, 1]) \cong A_q^{\mathbb{Z}_2}$ , the subalgebra of the quantum torus  $XY = q^2YX$  which is invariant under the involution  $X \mapsto X^{-1}$ ,  $Y \mapsto Y^{-1}$ . Our starting point is the observation that the category of  $A_q^{\mathbb{Z}_2}$ -modules is equivalent to the category of modules over a simpler algebra, the crossed product  $A_q \rtimes \mathbb{Z}_2$ . We write  $M_L$  for the image of  $K_q(S^3 \setminus L)$  under this equivalence. Theorem 5.2.1 gives a simple formula showing  $J_n(L)$  can be computed from  $M_L$ , and Corollary 5.3.3 shows a recursion relation for  $J_n(L)$  can be computed from  $M_L$  (if  $M_L$  is f.g. over  $\mathbb{C}[X^{\pm 1}]$ ). In Chapter 6 we give an explicit description of  $M_L$  when  $L$  is the trefoil. Conjecture 4.3.4 conjectures the general structure of  $M_L$  for torus knots.

The algebra  $A_q \rtimes \mathbb{Z}_2$  is the  $t = 1$  subfamily of the double affine Hecke algebra  $\mathcal{H}_{q,t}$  of type  $A_1$ . In Chapter 8 we give a new skein-theoretic realization of the spherical subalgebra  $\mathcal{H}_{q,t}^+$ , and we also give a construction associating an  $\mathcal{H}_{q,t}^+$ -module  $M'_L(t)$  to each knot  $L$ . In Chapter 9 we construct algebraic deformations of the skein module  $M_L$  to a family of modules  $M_L(t)$  over  $\mathcal{H}_{q,t}$ . In the case when  $L$  is the trefoil, we use these deformations to give example calculations of 2-variable polynomials  $J_n(q, t)$  that specialize to the colored Jones polynomials when  $t = 1$ .

## BIOGRAPHICAL SKETCH

Peter was born in 1983 to Bruce and Beth Samuelson in Khartoum, Sudan. When he was 6 months old his parents (who were missionaries) had some trouble with the authorities, and after a long, complicated series of events, they were deported. Peter spent the rest of his childhood in Cedar Hill, Texas, which is distinguished by being the town which appears most often on Wikipedia’s list of “manmade structures between 450m and 500m.” (It appears 8 times.) During his childhood he learned how to code in Visual Basic, juggle, and clap with one hand, and in high school he was a percussionist in the marching, jazz, and concert bands. He was valedictorian because the would-be valedictorian graduated a year early, and through a shocking turn of events he somehow was elected Homecoming King. Two of his summers were spent working for a trim carpenter in new houses, during which he shot approximately 80,000 nails, including one through his finger (which luckily missed the bone). After this he went to college at Caltech in Pasadena, California, and had many fun experiences, including making ice cream with liquid nitrogen, going to Dr. Beach, blowing up kegs in the desert, and bungee jumping. There was also a little bit<sup>1</sup> of math here and there. During high school and college he managed to rebuff all the coaches’ attempts to get him to play basketball. He attended graduate school at Cornell from 2005 to 2012. He spent much of this time in the basement of Malott, in Yuri’s office, outdoors, and he also occasionally visited such places as CTB and The Chapter House. During the winter breaks he joined several friends from college in visiting a new place for each New Year’s. So far the locations have included Prague, Rio, Berlin, Cancun, Edinburgh, Costa Rica, and Lisbon, and he hopes this tradition continues for many more years. After he finishes this thesis, he is going to the University of Toronto for a math postdoc.

---

<sup>1</sup>This is a lie, there was a whole lot of math and other work.

This thesis is dedicated to my family - Bruce, Beth, and John.

## ACKNOWLEDGEMENTS

The first thing that I would like to acknowledge is that math is hard.

Moving on to more traditional acknowledgements, I first would like to thank my advisor, Yuri Berest. He spent unreasonable amounts of time patiently and energetically explaining many different mathematical concepts, both simple and complicated. He was very gracious with his time and in life, and I will truly miss being able to go up to his office and get quick, insightful answers to any question I can ask. Most importantly, he has (in my opinion) very good mathematical taste, and I hope that I have been able to acquire a small amount of this from him.

I also would like to thank all the professors that have taught the classes that I have taken. Their hard work has been an important component of the completion of this thesis. In particular I would like to thank Yuri Berest and Allen Knutson for each teaching several very interesting classes that have significantly broadened my mathematical horizons. They both put a lot of effort into careful and insightful explanations which were very helpful and interesting.

I have also had very helpful discussions and emails with people outside of Cornell, including Dylan Thurston, Adam Sikora, Nathan Dunfield, Thang Lê, Paul Smith, and Tony Licata. Thank you for spending time to answer my many questions. I have learned a lot from fellow graduate students at Cornell, and especially from my “mathematical brother” Greg Muller, who spent a lot of time explaining many things to me, and who was also very encouraging. Also, I’d like to thank Peter Luthy and Jason Anema for helping me prove Lemma 6.4.7.

Finally, I would like to thank my family and friends, both from Caltech and Cornell. Instead of making a long list which would be impossible to complete, I’ll just say that if you happen to be reading this, you know who you are. You have given me years of happy memories that I will cherish forever.

## TABLE OF CONTENTS

Biographical Sketch . . . . .	iii
Dedication . . . . .	iv
Acknowledgements . . . . .	v
Table of Contents . . . . .	vi
List of Figures . . . . .	viii
<b>1 Introduction</b>	<b>1</b>
1.1 The Kauffman bracket skein module . . . . .	3
1.2 The quantum torus and Morita equivalence . . . . .	5
1.3 The double affine Hecke algebra . . . . .	10
1.3.1 A topological approach . . . . .	11
1.3.2 An algebraic approach . . . . .	13
<b>2 The Kauffman bracket skein module</b>	<b>15</b>
2.1 Basic knot theory . . . . .	16
2.2 Representation varieties . . . . .	19
2.3 Character varieties for $SL_2(\mathbb{C})$ . . . . .	22
2.4 The Kauffman bracket skein module . . . . .	24
2.4.1 The definition and formal properties . . . . .	26
2.4.2 $q = -1$ and the character variety . . . . .	29
2.4.3 Deformation and the Poisson bracket . . . . .	32
2.4.4 The sphere and the solid torus . . . . .	33
2.4.5 The quantum torus and $K_q(T^2)$ . . . . .	34
2.4.6 Complements of 2-bridge knots . . . . .	37
2.5 Knot complements and the colored Jones polynomials . . . . .	38
<b>3 Morita theory</b>	<b>43</b>
3.1 Basic definitions . . . . .	44
3.2 Lifting pairings . . . . .	47
<b>4 The quantum torus</b>	<b>49</b>
4.1 Basic properties and representations . . . . .	50
4.2 Invariant subalgebra . . . . .	52
4.3 The category $\mathcal{O}^{\text{tw}}$ and the filtration conjecture . . . . .	54
<b>5 The unknot complement and recursion relations</b>	<b>59</b>
5.1 The lifted unknot module . . . . .	60
5.2 Jones polynomials from lifted skein modules . . . . .	62
5.3 Recursion relations for the colored Jones polynomials . . . . .	66

<b>6</b>	<b>The trefoil complement</b>	<b>69</b>
6.1	Review about extensions . . . . .	70
6.2	The module $M$ . . . . .	73
6.3	Comparison with Gelca's description . . . . .	77
6.4	The colored Jones polynomials . . . . .	81
6.5	A recursion relation for the colored Jones polynomials . . . . .	86
<b>7</b>	<b>The double affine Hecke algebra</b>	<b>88</b>
7.1	Basic properties of $\mathcal{H}_{q,t}$ . . . . .	88
7.2	Induced representations . . . . .	92
<b>8</b>	<b>A skein-theoretic realization of the DAHA</b>	<b>96</b>
8.1	Modified KBSM for surfaces . . . . .	96
8.2	The deformation for 3-manifolds . . . . .	101
<b>9</b>	<b>Algebraic deformations of skein modules</b>	<b>103</b>
9.1	Deforming $A_q \rtimes \mathbb{Z}_2$ modules . . . . .	104
9.2	The trefoil revisited . . . . .	107
	<b>Bibliography</b>	<b>113</b>

## LIST OF FIGURES

2.1	Reidemeister moves . . . . .	18
2.2	Kauffman bracket skein relations . . . . .	27
2.3	Multiplying links by stacking . . . . .	28
2.4	Kauffman bracket skein versus trace relations, case 1 . . . . .	31
2.5	Kauffman bracket skein versus trace relations, case 2 . . . . .	31
2.6	Sample computation in $K_q(T^2)$ . . . . .	36
2.7	Generators of $K_q(S^3 \setminus L(m, n))$ . . . . .	38
2.8	A short skein calculation . . . . .	41
2.9	The unknot complement . . . . .	42
5.1	The unknot complement . . . . .	61
8.1	Kauffman bracket skein relation for loops . . . . .	98
8.2	Skein relation for special strands . . . . .	98
8.3	The generators for the punctured torus . . . . .	99
8.4	The embedding of the graph $G$ for a knot complement . . . . .	102

# CHAPTER 1

## INTRODUCTION

Since the discovery of the Jones polynomial in [23], there have been significant interactions between the fields of noncommutative algebra and low dimensional topology. The Jones polynomial  $J(L) \in \mathbb{C}[q^{\pm 1}]$  of a knot  $L \subset S^3$  was originally constructed using traces on a certain family of (noncommutative) operator algebras. A few years later, Reshetikhin and Turaev used the representation theory of quantum groups to construct an infinite family of polynomial knot invariants  $J(L, \mathfrak{g}, V)$ , where  $V$  is a finite dimensional representation of  $\mathcal{U}_q(\mathfrak{g})$  (see [37]). This construction reproduces the Jones polynomial when  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$  and  $V$  is the 2 dimensional irreducible representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$ . When  $V_n$  is the  $n$ -dimensional irreducible  $\mathcal{U}_q(\mathfrak{sl}_2)$ -module, the polynomial  $J(L, \mathfrak{sl}_2, V_n) \in \mathbb{C}[q^{\pm 1}]$  is called the  $n^{\text{th}}$  *colored Jones polynomial*.

A third construction of the (colored) Jones polynomials uses the *Kauffman bracket skein module*, which is a vector space<sup>1</sup>  $K_q(M)$  functorially associated to an (oriented) 3-manifold  $M$ . (The morphisms for which  $K_q(-)$  is functorial are oriented embeddings of 3-manifolds.) For  $M = S^3$  this was introduced by Kauffman, and a more general definition was given independently by Przytycki and Turaev. The vector space  $K_q(M)$  is a quotient of the vector space spanned by isotopy classes of links in  $M$ , and the projection associates to a knot  $L \subset S^3$  an element  $\alpha(L) \in K_q(S^3)$ . There is a natural isomorphism  $K_q(S^3) \cong \mathbb{C}$ , and the image of  $\alpha(L)$  in  $\mathbb{C}$  depends polynomially on  $q \in \mathbb{C}^*$ . We can therefore think of  $\alpha(L)$  as a regular function on  $\mathbb{C}^*$ , and this (Laurent) polynomial is the Jones polynomial. In this thesis we study the functor  $K_q(-)$  from an algebraic point of view.

---

<sup>1</sup>Actually,  $K_q(M)$  is a family of vector spaces depending on the parameter  $q \in \mathbb{C}^*$ .

The functor  $K_q(-)$  has two important properties. First, if  $M = F \times [0, 1]$  is the cylinder over a surface  $F$ , then  $K_q(F \times [0, 1])$  has a natural algebra structure (which is typically noncommutative). Second, if  $M$  has a nonempty boundary, then  $K_q(M)$  is a (left) module over the algebra  $K_q(\partial M \times [0, 1])$ . In the following we study the question “What kind of algebras and modules arise from this construction?”

We will mainly be interested in the modules  $K_q(S^3 \setminus L)$ , where  $L \subset S^3$  is a knot. The knot complement  $S^3 \setminus L$  has a torus as its boundary, and therefore  $K_q(S^3 \setminus L)$  is a module over  $K_q(T^2 \times [0, 1])$ . In [14] it was proved that  $K_q(T^2 \times [0, 1])$  is isomorphic to the invariant subalgebra  $A_q^{\mathbb{Z}_2}$  of the quantum torus

$$A_q := \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / (XY = q^2 YX)$$

(The generator of  $\mathbb{Z}_2$  acts on  $A_q$  by inverting  $X$  and  $Y$ , and  $A_q^{\mathbb{Z}_2}$  is the subalgebra of elements invariant under this action.) This algebra is one of the fundamental examples of a noncommutative algebra, and from an algebraic point of view, modules over this algebra coming “from nature” should be interesting. The module  $K_q(S^3 \setminus L)$  is also of interest topologically because the (colored) Jones polynomials  $J(L, n)$  (for  $n \in \mathbb{N}$ ) can be extracted from  $K_q(S^3 \setminus L)$ .

This thesis naturally splits into two parts. In Chapters 2 through 6 we study the  $A_q^{\mathbb{Z}_2}$  modules  $K_q(S^3 \setminus L)$  themselves. A key fact that we use is that the algebra  $A_q^{\mathbb{Z}_2}$  is Morita equivalent to the crossed product<sup>2</sup>  $A_q \rtimes \mathbb{Z}_2$ , which means that the categories of left modules over  $A_q^{\mathbb{Z}_2}$  and  $A_q \rtimes \mathbb{Z}_2$  are equivalent. This allows us ‘lift’ modules from  $A_q^{\mathbb{Z}_2}$  to  $A_q \rtimes \mathbb{Z}_2$ , and the fact that this is an equivalence of categories means that we don’t ‘lose any information.’ One of the goals of this first part of the thesis is to demonstrate that computations are much easier and more transparent at the level of  $A_q \rtimes \mathbb{Z}_2$ -modules.

---

<sup>2</sup>As a vector space,  $A_q \rtimes \mathbb{Z}_2 \cong A_q \otimes \mathbb{C}\mathbb{Z}_2$ , and the multiplication is  $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$ .

In the second part we study deformations of these modules. More precisely, we observe that the algebra  $A_q^{\mathbb{Z}_2}$  is part of a 2-parameter family of algebras  $H_{q,t}^+$ , with  $\mathcal{H}_{q,t=1}^+ \cong A_q^{\mathbb{Z}_2}$ . The algebras  $\mathcal{H}_{q,t}^+$  are the (spherical subalgebras of the) *double affine Hecke algebras* of type  $A_1$  introduced by Cherednik (see [9] and [10]). In Chapters 7 through 9 we discuss the question “Is there a natural family of modules  $M_{q,t}$  over  $\mathcal{H}_{q,t}^+$  such that  $M_{q,t=1} \cong K_q(S^3 \setminus L)$ ?”

We now give a more detailed description of the background and our results.

## 1.1 The Kauffman bracket skein module

We begin with an ahistorical motivation of the definition of the Kauffman bracket skein module. If  $M$  is a 3-manifold with boundary, then the inclusion  $\partial M \hookrightarrow M$  induces a map  $\pi_1(\partial M) \rightarrow \pi_1(M)$  of fundamental groups, and Waldhausen showed in [41] that this map is a complete invariant for a large class of 3-manifolds (including knot complements). One way to study the map between these groups is to study the induced map on their representation spaces. For an algebraic (e.g. matrix) group  $G$ , the set  $\text{Rep}(M) := \text{Hom}(\pi_1(M), G)$  has a natural structure of an algebraic variety equipped with a  $G$ -action by conjugation. The inclusion  $\partial M \rightarrow M$  induces a morphism of character varieties  $\text{Rep}(M) // G \rightarrow \text{Rep}(\partial M) // G$ , or equivalently an algebra map  $\mathcal{O}(\text{Rep}(\partial M))^G \rightarrow \mathcal{O}(\text{Rep}(M))^G$  between the rings of  $G$ -invariant regular functions on  $\text{Rep}(\partial M)$  and  $\text{Rep}(M)$ . This last map has been studied extensively, especially for  $G = SL_2(\mathbb{C})$ . For example, for a hyperbolic 3-manifold  $M$ , Culler and Shalen [11] used Thurston’s theorem that  $\text{Rep}(M) // SL_2$  has positive dimension to construct a splitting of  $\pi_1(M)$ , which can produce incompressible surfaces in  $M$ .

For a surface  $F$ , the algebra  $\mathcal{O}(\text{Rep}(F))^G$  has a Poisson structure due to Goldman [20]. One of the lessons learned from noncommutative algebra/geometry is that a naturally arising Poisson structure on a commutative algebra  $A$  is often induced from a deformation  $A_h$  of  $A$ . (See Section 2.4.3 for a more precise statement.) From this point of view, a fundamental question is “Does the map  $\mathcal{O}(\text{Rep}(\partial M))^G \rightarrow \mathcal{O}(\text{Rep}(M))^G$  deform?” In other words, is this algebra map the  $q = 1$  case of a more general family of maps depending on a parameter  $q$ ?

As explained in Section 2.4 below, the work of Frohman, Przytycki, Sikora, and others shows that a positive answer to this question is given by the Kauffman bracket skein module. In particular, for the special value  $q = -1$ , the vector space  $K_{q=-1}(M)$  is a commutative algebra (for *any* 3-manifold  $M$ ), and in [35] Przytycki and Sikora show that  $K_{q=-1}(M)$  is isomorphic to  $\mathcal{O}(\text{Rep}(M))^{SL_2(\mathbb{C})}$  (see also [3]). Furthermore, in [4] it is shown that if  $F$  is a surface, then  $K_q(F \times [0, 1])$  is a (typically noncommutative) algebra which is a deformation of the commutative algebra  $K_{q=-1}(F \times [0, 1])$  in the direction of Goldman’s Poisson bracket.

As mentioned above, the colored Jones polynomials  $J(L, n)$  of a knot  $L \subset S^3$  can be extracted from the  $A_q^{\mathbb{Z}_2}$ -module  $K_q(S^3 \setminus L)$ . In particular, a tubular neighborhood  $N_L$  of  $L$  is a right module over  $K_q(T^2 \times [0, 1])$ , and the embedding  $N_L \sqcup (S^3 \setminus N_L) \hookrightarrow S^3$  induces a pairing

$$\langle -, - \rangle : K_q(S^1 \times D^2) \otimes_{K_q(T^2 \times [0, 1])} K_q(S^3 \setminus N_L) \rightarrow K_q(S^3) \cong \mathbb{C} \quad (1.1)$$

(We have identified  $N_L$  with the solid torus  $S^1 \times D^2$ .) There is a vector space isomorphism  $\mathbb{C}[u] \rightarrow K_q(S^1 \times D^2)$ , and a theorem of Kirby and Melvin shows<sup>3</sup>

$$J(L, n) = \langle S_{n-1}(u), 1_c \rangle$$

---

<sup>3</sup>In this identification of the Jones polynomial, we view  $q$  as a variable instead of a formal parameter.

Here we have written  $1_c$  for the empty link in  $S^3 \setminus N_L$  (which is a canonical element in  $K_q(S^3 \setminus N_L)$ ). Also, the  $S_n$  are the Chebyshev polynomials with initial conditions  $S_0 = 1$ ,  $S_1 = x$ , and  $S_{n+1} = xS_n - S_{n-1}$ .

Finally, the starting point for this thesis is a theorem of Frohman and Gelca in [14]. They show that  $K_q(T^2 \times [0, 1])$  is isomorphic to the subalgebra  $A_q^{\mathbb{Z}_2}$  of  $\mathbb{Z}_2$ -invariants of the quantum torus

$$A_q := \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / (XY = q^2 YX) \quad (1.2)$$

(Here  $\mathbb{Z}_2$  acts by inverting  $X$  and  $Y$ .) As explained above, this implies that if  $L \subset S^3$  is a knot, then  $K_q(S^3 \setminus L)$  is an  $A_q^{\mathbb{Z}_2}$ -module. Our main focus in the next section will be studying the modules and pairings that arise in this way.

## 1.2 The quantum torus and Morita equivalence

In this section we begin the algebraic study of the modules  $K_q(S^3 \setminus L)$ . We first remark that when  $q = 1$ , a module over  $A_q^{\mathbb{Z}_2}$  is the same thing as a (quasi-coherent) sheaf on  $X := (\mathbb{C}^* \times \mathbb{C}^*) // \mathbb{Z}_2$ , where  $\mathbb{Z}_2$  acts by inverting each component. This sheaf can be pulled back along the projection  $\mathbb{C}^* \times \mathbb{C}^* \rightarrow X$ , and the result is a  $\mathbb{Z}_2$ -equivariant sheaf on  $\mathbb{C}^* \times \mathbb{C}^*$ . A key observation (which makes most of this thesis possible) is that this procedure works for arbitrary  $q$ . More precisely, there is a natural functor between the module categories of  $A_q^{\mathbb{Z}_2}$  and  $A_q \rtimes \mathbb{Z}_2$ , and this functor is an equivalence of categories when  $q \in \mathbb{C}^*$  is not a root of unity.

Recall that two algebras  $A$  and  $B$  are said to be **Morita equivalent** if their categories of (left) modules are equivalent. This definition is only useful for non-commutative algebras because two commutative algebras  $A$  and  $B$  are Morita

equivalent if and only if they are isomorphic. In other words, an affine scheme  $X$  is determined by the category of (quasi-coherent) sheaves over  $X$ . (One heuristic reason for this is that closed points of  $X$  are in bijection with skyscraper sheaves over  $X$ , and these are the simple objects in the category of sheaves over  $X$ .)

To see this Morita equivalence, we first remark that if  $\mathbf{e} = (1 + s)/2 \in A_q \rtimes \mathbb{Z}_2$  is the symmetrizing idempotent, then there is an isomorphism  $A_q^{\mathbb{Z}_2} \rightarrow \mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  given by  $a \mapsto \mathbf{e}a\mathbf{e}$ . The functor between  $A_q \rtimes \mathbb{Z}_2$ -modules and  $\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$ -modules is given by  $M \mapsto \mathbf{e}M$ , and the equality  $(A_q \rtimes \mathbb{Z}_2)\mathbf{e}(A_q \rtimes \mathbb{Z}_2) = A_q \rtimes \mathbb{Z}_2$  implies this is functor is an equivalence of categories (see Section 3.1 and Theorem 4.1.2.)

One of the main themes of the first part of this thesis is that this Morita equivalence is very useful, since the formulas for the action of  $A_q \rtimes \mathbb{Z}_2$  on  $M$  are typically much simpler and more transparent than the formulas for the action of  $\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  on  $\mathbf{e}M$ . This is analogous to the fact that if  $G$  is a group acting on  $X$ , then  $G$ -equivariant sheaves on  $X$  are often easier to deal with than the induced sheaves on the quotient  $X // G$ .

Our first illustration of this theme is given by a formula extracting the colored Jones polynomials from the ‘lifted’ skein modules. In more detail, if  $V$  and  $M_L$  are the (unique)  $A_q \rtimes \mathbb{Z}_2$  modules with  $\mathbf{e}V \cong K_q(S^1 \times D^2)$  and  $\mathbf{e}M_L \cong K_q(S^3 \setminus L)$ , then we show that there is a unique pairing

$$\langle -, - \rangle : V \otimes_{A_q \rtimes \mathbb{Z}_2} M_L \rightarrow \mathbb{C}$$

extending the pairing (1.1). Furthermore,  $V$  is isomorphic (as a  $\mathbb{C}[X^{\pm 1}]$ -module) to  $\mathbb{C}[X^{\pm 1}]$ , and the following theorem shows how the colored Jones polynomials can be extracted from  $M_L$ .

**Theorem (5.2.1).** *For  $n \in \mathbb{N}$ , the colored Jones polynomial is given by*

$$J(L, n) = 2\langle X^n, 1_L \rangle$$

This theorem also allows us to give a slightly more transparent explanation of a result of Garoufalidis and Lê. In [17] the authors extend the colored Jones polynomial to a function  $J(L, -) : \mathbb{Z} \rightarrow \mathbb{C}$  via the rule  $J(L, -n) = -J(L, n)$ . There is a right action of  $A_q \rtimes \mathbb{Z}_2$  on the vector space  $\text{Hom}(\mathbb{Z}, \mathbb{C})$  of such functions:

$$(X \cdot f)(n) = f(n+1), \quad (Y \cdot f)(n) = -q^{-2n}f(n), \quad (s \cdot f)(n) = -f(-n)$$

The main theorem of [17] is that there is an element  $a \in A_q$  with  $J(L, -) \cdot a = 0$ . In other words, the Jones polynomials satisfy a linear recursion relation.

The appearance of the action of  $A_q \rtimes \mathbb{Z}_2$  on  $\text{Hom}(\mathbb{Z}, \mathbb{C})$  in [17] seems somewhat mysterious. However, in Section 5.3 we show that this action arises naturally from the action of  $A_q \rtimes \mathbb{Z}_2$  on  $V$ . Furthermore, this observation makes the following corollary essentially obvious.

**Corollary (5.3.3).** *If  $a \in A_q$  satisfies  $a \cdot 1_L = 0$ , then  $J(L, -) \cdot \phi(a) = 0$ .*

(Here  $\phi : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  is a certain anti-involution preserving  $A_q$ , and  $1_L \in M_L$  is the canonical element corresponding to the empty link.) This corollary was also proved in [16], but we hope that our proof is slightly more transparent.

In [18], Gelca gave enough information to completely determine the module structure of  $K_q(S^3 \setminus L)$  when  $L \subset S^3$  is the trefoil. In Chapter 6 we use his description to give a completely explicit description of the lifted module  $M_L$ . With this description in hand, we give a short calculation of all the colored Jones polynomials and also produce an element  $a \in A_q$  with  $J(L, -) \cdot a = 0$ . We summarize these results here.

**Theorem.** *Let  $M_L$  be the  $A_q \rtimes \mathbb{Z}_2$ -module with  $\mathfrak{e}M_L \cong K_q(S^3 \setminus L)$ . Then*

1. *As a  $\mathbb{C}[X^{\pm 1}]$ -module,  $M_L \cong \mathbb{C}[x^{\pm 1}]^{\oplus 2}$ , with  $A_q \rtimes \mathbb{Z}_2$ -action determined by*

$$X \cdot (f(x), g(x)) = (xf(x), xg(x))$$

$$Y \cdot (1, 0) = (-1, 0)$$

$$Y \cdot (0, 1) = (q^2x^{-1} - q^6x^{-5}, q^6x^{-6})$$

$$s \cdot (1, 0) = (-1, 0)$$

$$s \cdot (0, 1) = (0, 1)$$

2.  $J(L, n) = \frac{1}{q^2 - q^6} \sum_{i=1}^n (-1)^i q^{6(n^2 - i^2)} (q^{10i-4} - q^{2i})$

3. *The colored Jones polynomials  $J(L, n)$  of the trefoil satisfy*

$$\begin{aligned} J(L, n+3) &= (q^{30+12n} - q^2 - q^{10}) J(L, n+2) \\ &\quad + ((q^{16} + q^8)q^{12(n+1)} - q^{12}) J(L, n+1) \\ &\quad + q^{18+12n} J(L, n) \end{aligned}$$

(We have written the formulas for the action of  $Y$  and  $s$  on  $(1, 0)$  and  $(0, 1)$  instead of writing the action on an arbitrary element  $(f(x), g(x))$ . However, the commutation relations in  $A_q \rtimes \mathbb{Z}_2$  allow one to compute the action of  $Y$  and  $s$  on an arbitrary element using the formulas we have given.)

It isn't immediately obvious from these formulas, but the first copy of  $\mathbb{C}[x^{\pm 1}]$  is an  $A_q \rtimes \mathbb{Z}_2$ -submodule of  $M_L$  that is isomorphic to the lift of  $K_q(S^3 \setminus \text{unknot})$ . In the future we hope to be able to give a topological interpretation of this fact.

For the sake of comparison, we also give Gelca's formulas for the action of  $A_q^{\mathbb{Z}_2}$  on  $\mathfrak{e}M_L$ . We use the notation  $x = X + X^{-1}$  and  $y = Y + Y^{-1}$  and  $z =$

$q^{-1}(XY + X^{-1}Y^{-1})$ . Also, we use the fact that as a  $\mathbb{C}[x]$ -module,  $\mathfrak{e}M_L$  is freely generated by elements  $v$  and  $w$ .

$$\begin{aligned} y \cdot v &= q^6 T_6(x)v + (q^6 S_4(x) - q^2)w \\ z \cdot v &= q^5 T_5(x)v + q^5 S_3(x)w \\ y \cdot w &= -(q^2 + q^{-2})w \\ z \cdot w &= -q^{-3}xw \end{aligned}$$

Here  $T_n$  and  $S_n$  are two different versions of Chebyshev polynomials (they have the same recursion relation but different initial conditions). Also, the action of  $y$  and  $z$  has only been written on the  $\mathbb{C}[x]$ -generators  $v$  and  $w$ , but the commutation relations allow one to compute the effect of  $y$  and  $z$  on arbitrary elements. However, these computations are less straightforward than the analogous computations for the lifted module  $M_L$ .

Last but not least, in Section 4.3 we propose a general conjecture about the structure of the lifted module  $M_L$  for a torus knot  $L \subset S^3$ . (A torus knot is a knot that can be embedded in the boundary of a tubular neighborhood of the unknot.) The trivial and sign representations of  $\mathbb{Z}_2$  induce two natural  $A_q \rtimes \mathbb{Z}_2$ -module structures on  $\mathbb{C}[x^{\pm 1}]$ :

$$(X \cdot f)(x) = xf(x), \quad (Y \cdot f)(x) = \pm f(q^{-2}x), \quad (s \cdot f)(x) = \pm f(x^{-1})$$

(Here the signs  $+$  and  $-$  correspond to the trivial and sign representations, respectively.) We write  $P^\pm$  for these modules. We also define an algebra automorphism  $\tau : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  using

$$\tau(X) = X, \quad \tau(Y) = q^{-1}XY, \quad \tau(s) = s$$

Since  $\tau$  fixes  $s$ , this automorphism restricts to  $A_q^{\mathbb{Z}_2}$ , and this automorphism is the same as the automorphism of  $K_q(T^2 \times [0, 1])$  induced by ‘one Dehn twist around

the meridian.’ We write  $P_k^\pm$  for the module  $P^\pm$  with action twisted by  $\tau^k$ , i.e.  $a \cdot f(x) := \tau^k(a)f(x)$ . We then conjecture

**Conjecture (4.3.4).** *If  $L \subset S^3$  is a torus knot, the lift  $M_L$  of the  $A_q^{\mathbb{Z}_2}$ -module  $K_q(S^3 \setminus L)$  has a finite filtration whose successive quotients are isomorphic to one of the  $P_k^\pm$ .*

This conjecture seems somewhat difficult to prove, since the computations in [18] for the torus are already quite involved. In Section 4.3 we discuss two corollaries of the conjecture which are known to be true, at least for certain classes of knots. The conjecture holds for the unknot because of the isomorphism  $eP_0^- \cong K_q(S^3 \setminus \text{unknot})$  (see Section 5.1). Finally, our results for the trefoil show that  $M_{\text{trefoil}}$  satisfies the conjecture because it fits into a short exact sequence

$$0 \rightarrow P_0^- \rightarrow M_L \rightarrow P_{-6}^+ \rightarrow 0$$

**Remark 1.2.1.** The restriction to torus knots is essentially because they have particularly simple  $A$ -polynomials. This connection was realized after this thesis was (mostly) written, and we hope to give a more detailed explanation elsewhere.

### 1.3 The double affine Hecke algebra

The algebra  $A_q^{\mathbb{Z}_2}$  is part of a family of algebras that depend on a second parameter  $t$ , as we now discuss. Motivated by earlier work of Dunkl [12], Cherednik [9] introduced the family of double affine Hecke algebras  $\mathcal{H}_{q,t}$  (or DAHAs) associated to a root system. He used special properties of the algebras  $\mathcal{H}_{q,t}$  to prove the Macdonald conjectures, which involved a family of multivariate orthogonal polynomials associated to the root system. The orthogonal polynomials were shown to

be eigenvectors of certain elements of  $\mathcal{H}_{q,t}$ , and symmetries of  $\mathcal{H}_{q,t}$  translated into nontrivial symmetries for the polynomials.

For all parameters,  $\mathcal{H}_{q,t}$  is linearly isomorphic to  $\mathbb{C}P \otimes \mathbb{C}P^\vee \otimes H_t$ , where  $P$  and  $P^\vee$  are the weight and coweight lattices and  $H_t$  is the (finite) Hecke algebra. Each term is a subalgebra, and the vector space isomorphism is induced by the multiplication in  $\mathcal{H}_{q,t}$ , which depends on the parameters  $q, t \in \mathbb{C}^\times$ . In type  $A_1$  both  $P$  and  $P^\vee$  are isomorphic to  $\mathbb{Z}$  and  $H_q$  is a 2-dimensional algebra which is a deformation of the group algebra of the Weyl group  $\mathbb{Z}_2$ . Precisely, the **double affine Hecke algebra**  $\mathcal{H}_{q,t}$  is generated by  $X^{\pm 1}, Y^{\pm 1}$ , and  $T$  with relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad XY = q^2YXT^2, \quad (T - t)(T + t^{-1}) = 0 \quad (1.3)$$

If  $t \neq \pm i$ , the algebra  $\mathcal{H}_{q,t}$  contains the idempotent  $\mathbf{e} := (1 + tT)/(1 + t^2)$ , and the **spherical subalgebra** of  $\mathcal{H}_{q,t}$  is the (nonunital) subalgebra  $\mathcal{H}_{q,t}^+ := \mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$ . If  $t = 1$ , the last relation simplifies to  $T^2 = 1$ , which implies  $\mathcal{H}_{q,1} \cong A_q \rtimes \mathbb{Z}_2$  and  $\mathcal{H}_{q,1}^+ \cong A_q^{\mathbb{Z}_2}$ ; in this way,  $\mathcal{H}_{q,t}^+$  is a deformation of the invariant subalgebra  $A_q^{\mathbb{Z}_2}$ .

In the second half of the thesis we attempt to answer the question “Is there a natural family of  $H_{q,t}$ -modules  $M_t$  such that  $\mathbf{e}M_{t=1} \cong K_q(S^3 \setminus L)$ ?” In Chapters 8 and 9 we approach this question topologically and algebraically, respectively.

### 1.3.1 A topological approach

The naïve topological approach to this problem would be to try to insert a second parameter in the Kauffman skein relations so that the skein module of the torus would have two parameters instead of just one. However, these skein relations are quite rigid, and there seems to be “no room” for a second parameter.

The key insight is that  $\mathcal{H}_{q,t}^+$  is a quotient of the Kauffman bracket skein module of the *punctured* torus  $T'$ . The embedding  $T' \rightarrow T^2$  induces a surjective algebra map  $K_q(T') \rightarrow K_q(T^2)$ , and the kernel of this map is generated by the element  $\delta + q^2 + q^{-2}$ , where  $\delta$  is a loop around the puncture. It turns out that  $\mathcal{H}_{q,t}^+$  is also a quotient of  $K_q(T')$  by the ideal generated by  $\delta + c_t$ , where  $c_t$  is a constant depending on  $t$  (and  $q$ ).

We use this fact to define a new skein module  $K_{q,t}(F)$  associated to each surface  $F$ . We replace the puncture with a ‘special strand,’ which is a vertical arc in  $F \times [0, 1]$  which begins at  $p \times \{0\}$  and ends at  $p \times \{1\}$  (where  $p \in f$  is a fixed point). The special strand doesn’t satisfy the Kauffman skein relations, but it does satisfy a new skein relation which imposes the condition that a loop around the special strand is equal to  $c_t \in \mathbb{C}$ .

**Corollary (8.1.7).** *The algebras  $e\mathcal{H}_{q,t}e$  and  $K_{q,t}(T^2)$  are isomorphic.*

One curious fact about this construction is that it is only well-behaved for the torus  $T^2$ . In particular, there is a natural set of loops in  $T^2$  that is a linear basis for both  $K_q(T^2)$  and  $K_{q,t}(T^2)$ , and we can therefore view  $K_{q,t}(T^2)$  as a deformation of  $K_q(T^2)$ . However, for surfaces of genus at least 2, the algebra  $K_{q,t}(F)$  is ‘much bigger’ than the algebra  $K_q(F)$ . In particular, there is a natural surjective map  $K_{q,t=1}(F) \rightarrow K_q(F)$  which forgets the special strand, but this map seems to only be injective for  $F = T^2$ .

In Section 8.2 we extend this construction to produce a 2-parameter skein module  $K_{q,t}(M, f)$  for a 3-manifold  $M$  with nonempty boundary. We include the letter  $f$  in the notation because in general this construction depends on an extra piece of data. However, we show that if  $M = S^3 \setminus L$  is a knot complement, then there is a uniform choice for the data  $f$ , which gives us the following theorem.

**Theorem (8.2.2).** *The vector space  $K_{q,t}(S^3 \setminus L)$  is a module over  $\mathcal{H}_{q,t}^+ \cong K_{q,t}(T^2)$ .*

Unfortunately, calculations with the spaces  $K_{q,t}(S^3 \setminus L)$  will require further topological insight than we have presently attained. In general these modules are ‘much bigger’ than their counterparts  $K_q(S^3 \setminus L)$ , and this makes calculations more difficult. However, it seems that when  $q = 1$  this space should be related to the  $SL_2(\mathbb{C})$  character variety of  $S^3 \setminus L$  (or to the representation variety), and in the future we intend to investigate this further. There is also a possibility of vague connections with recent work of Kronheimer and Mrowka [27], but this is as precise a statement as we can make at this time.

### 1.3.2 An algebraic approach

In the final chapter we take an algebraic approach to deforming skein modules. We work exclusively with the ‘lifted’ skein module, i.e. the (unique)  $A_q \rtimes \mathbb{Z}_2$ -module  $M$  with  $\mathbf{e}M \cong K_q(S^3 \setminus L)$ . For this approach we assume that Conjecture 4.3.4 holds, i.e. that  $M$  has a finite filtration whose successive quotients are of the form  $P_k^\pm$ . (These  $A_q \rtimes \mathbb{Z}_2$ -modules are discussed in Section 4.3.) The question of deformations of knots more complicated than torus knots will be more approachable once a concrete example has been computed.

**Theorem (9.1.2).** *Assume  $M$  has a filtration  $0 \subset M_1 \subset \cdots \subset M_n = M$  such that each quotient  $M_i/M_{i-1}$  is isomorphic to  $P_k^\pm$  for some  $k \in \mathbb{Z}$ . Then there is a family of  $\mathcal{H}_{q,t}$ -modules  $M_t$  with  $M_{t=1} \cong M$ .*

The first step in our construction is to deform the modules  $P_k^\pm$ , and this was essentially done by Cherednik. (For completeness we give an explicit deformation

(with proofs) in Section 7.2.) Second, we show that if  $A_X \subset A_q \rtimes \mathbb{Z}_2$  is the subalgebra generated by  $X^{\pm 1}$  and  $s$ , then the filtration on  $M$  ‘splits over  $A_X$ ’, in the sense that it gives an  $A_X$ -module isomorphism  $M \cong \bigoplus P_{k_i}^{\pm}$ . (As one should expect, this splitting is far from unique.) This allows  $M$  to be deformed to an  $\mathcal{H}_X$ -module  $M_t$  (where  $\mathcal{H}_X \subset \mathcal{H}_{q,t}$  is the subalgebra generated by  $X^{\pm 1}$  and  $T$ ). Finally, a clever trick used by Cherednik (in a slightly different context) allows the  $\mathcal{H}_X$ -module structure on  $M_t$  to be extended to an  $\mathcal{H}_{q,t}$ -module structure. In particular, if we define

$$Y_t := Y_1 T_1 T_t$$

then a formal calculation shows the operators  $X, T_t, Y_t : M \rightarrow M$  satisfy the relations of  $\mathcal{H}_{q,t}$ .

Unfortunately, the deformations this theorem produces are not unique. We can make a rough analogy to a situation where this non-uniqueness is obvious. If  $E \rightarrow X$  is a vector bundle over a smooth manifold  $X$  and  $x \in X$  is a point, then one can ask if a choice of an element in the fiber of  $x$  can be extended to a section over all of  $X$ . The answer is obviously yes, and this extension is obviously very far from unique. However, if we pick a flat connection on  $E$ , then the choice of a section at a point extends *uniquely* to a flat global section. We believe in the future it would be interesting to try to find an analogous statement in our situation.

## CHAPTER 2

### THE KAUFFMAN BRACKET SKEIN MODULE

In this chapter we introduce our main object of study, the *Kauffman bracket skein module*, which is a vector space  $K_q(M)$  associated to an oriented 3-manifold  $M$ . This was first introduced when  $M = S^3$  by Kauffman, and the more general definition was given independently by Turaev and Przytycki. We will mainly be concerned with the case where  $M$  is the complement of a knot in  $S^3$ , so we begin the chapter with a very brief introduction to classical knot theory.

A more modern point of view is that  $K_q(M)$  is a  $q$ -deformation of the ring of functions on the  $SL_2(\mathbb{C})$  *character variety* of  $M$ . This is the variety that parameterizes isomorphism classes of semi-simple representations of  $\pi_1(M)$  into  $SL_2(\mathbb{C})$ . We briefly discuss representation varieties and character varieties. We then give the definition of  $K_q(M)$  and explain some of its formal properties. In particular, if  $M = F \times [0, 1]$  for some surface  $F$ , then the vector space  $K_q(F \times [0, 1])$  has a natural algebra structure.

We then give several examples of explicit calculations of  $K_q(M)$ . We explain the structure of  $K_q(S^3)$  and  $K_q(S^1 \times D^2)$ , both of which are used several times throughout the thesis. We then recall a theorem of Frohman and Gelca from [14] which shows that the algebra  $K_q(T^2 \times [0, 1])$  is isomorphic to a subalgebra of the quantum torus. This result is the starting point of this thesis.

One of the reasons that  $K_q(M)$  is interesting topologically is that the colored Jones polynomials  $J(L, n)$  of a knot  $L \subset S^3$  can be computed using the module  $K_q(S^3 \setminus L)$ . We explain this in the final section of this chapter and recall a result from [28] that give a vector space basis for  $K_q(S^3 \setminus L)$  when  $L$  is a 2-bridge knot.

## 2.1 Basic knot theory

In this section we give some basic definitions and give a *very* brief historical introduction to knot theory. All facts mentioned can be found in Chapters 1 and 3 of [7].

**Definition 2.1.1.** The three most basic definitions we use are the following:

1. Two embeddings  $f_0, f_1 : X \rightarrow Y$  are **isotopic** if there is a path  $\gamma : [0, 1] \rightarrow \text{Diffeo}(Y)$  such that  $\gamma(0) = \text{Id}_Y$  and  $f_1 = \gamma(1) \circ f_0$ .
2. A **knot** is a smooth embedding  $S^1 \hookrightarrow S^3$ , and a **link** is a smooth embedding  $\sqcup_{i=1}^n S^1 \hookrightarrow S^3$ . ( $n = 0$  is allowed.)
3. A **framed knot** is a smooth embedding  $S^1 \times [0, 1] \hookrightarrow S^3$ , and a **framed link** is a smooth embedding  $\sqcup_{i=1}^n (S^1 \times [0, 1]) \hookrightarrow S^3$ .

Equivalently, a framed knot is a knot together with a smoothly varying normal vector at each point of the knot, and can therefore be thought of as an infinitesimal thickening of a knot. The definition of the Kauffman bracket skein module requires framed links, but the framing is a technical necessity which we suppress from the notation as much as possible.

Knots are typically identified with their image in  $S^3$ . Furthermore, they are (almost always) only considered up to ambient isotopy. The intuition is that if you are holding a loop of string and are allowed to move it but not cut it, then the possible positions of the string are exactly the embeddings inside one ambient isotopy class of embeddings  $S^1 \hookrightarrow \mathbb{R}^3$ . The technical definition of isotopy may seem strange, since when you move a string around the ambient space does not follow the string. A more accurate mental model of isotopy is to imagine that the

the string is embedded in a large pool of honey, which is viscous enough to ‘follow the string.’

The most common way of describing a knot in  $S^3$  is to project the image onto a plane in  $\mathbb{R}^3$ . A generic projection will give a map  $S^1 \rightarrow \mathbb{R}^2$  where only finitely many image points have 2 inverse images, the rest of the image points have one inverse image, and intersections are transverse. When this projection is drawn, the piece that goes “underneath” an intersection is drawn with a small break. Conversely, such a diagram can be turned into a knot in  $S^3$  in an obvious way.

## Distinguishing Knots

Given two knot diagrams, one can ask if they represent the same (ambient isotopy class of a) knot. One way to answer this question was given in 1926 by Reidemeister [36] using what are now called Reidemeister moves. The diagrams below (and most other diagrams in these notes) represent pieces of knots (or links), and the changes are local (so that the parts of the knots that are not drawn are not changed).

**Theorem 2.1.2** (Reidemeister). *Two knot diagrams depict equivalent knots if and only if one diagram can be changed into the other by a sequence of the three Reidemeister moves in Figure 2.1 (and ambient isotopies of  $\mathbb{R}^2$  that preserve the crossing data).*

**Remark 2.1.3.** A *framed* knot can also be represented by a planar diagram, with the convention that the framing is always perpendicular to the paper. Reidemeister’s theorem for framed knots says that two planar diagrams represent the same framed knot if and only if one can be changed into the other by a sequence of Reidemeister moves 2 and 3. (Reidemeister move 1 changes the framing of the knot by adding a twist.)

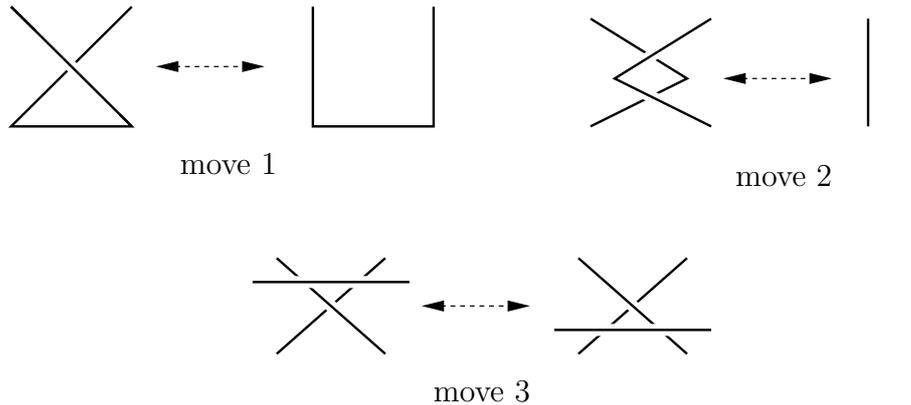


Figure 2.1: Reidemeister moves

There is an exponential bound for the number of Reidemeister moves needed:

**Theorem 2.1.4.** [21, Thm 1.1] *The number of Reidemeister moves needed to unknot a diagram with  $n$  crossings representing the unknot is at most  $2^{(10^{11})n}$ .*

The constant  $10^{11}$  is certainly not optimal, but it seems that no significantly better bound is known. This shows that Reidemeister moves give, in some sense, an unsatisfactory procedure for determining whether two knots are the same, or even whether a diagram represents the unknot. A large part of knot theory is devoted to coming up with knot invariants that can distinguish (some) knots.

One of the most classical knot invariants is the **knot group**  $\pi_1(S^3 \setminus L)$ , which is the fundamental group of the complement of a knot  $L$ . To describe the strength of this invariant, we first give a homological definition/lemma. (This lemma will be important later for our study of skein modules, so we state it precisely.) Let  $L \subset S^3$  be a knot,  $V \subset S^3$  be a (closed) tubular neighborhood of  $L$ , and  $C \subset S^3$  be the closure of the complement of  $V$ .

**Lemma 2.1.5.** *There exist two simple closed curves  $m, l$  on  $\partial V$  with the properties below. These properties uniquely determine the curves  $m, l$  (up to isotopy).*

1.  $m$  and  $l$  intersect at one point.
2. in  $V$ , the loop  $m$  is nullhomologous and  $l$  is homologous to  $L$ ,
3. in  $C$ , the loop  $l$  is nullhomologous,
4. in  $S^3$ , the linking numbers  $(m, L)$  and  $(l, L)$  are 1 and 0, respectively.

The curves  $m$  and  $l$  will be called the **meridian** and **longitude**, respectively.

This lemma has the following corollary which will be useful later.

**Corollary 2.1.6.** *Up to ambient isotopy, there is a canonical identification  $S^1 \times S^1 \xrightarrow{\sim} \partial V$  for each knot  $L$ , which is determined by identifying  $S^1 \times \{0\}$  with the meridian and  $\{0\} \times S^1$  with the longitude.*

The following theorem was essentially proved by Waldhausen in 1968 in [41].

**Theorem 2.1.7.** *Let  $L_1, L_2$  be two knots with meridian and longitude  $m_i$  and  $l_i$ , and let  $p_i = m_i \cap l_i$ . Then  $m_i, l_i \in \pi_1(C_i, p_i)$  are distinguished elements of the knot groups. Then  $L_1$  is ambient isotopic to  $L_2$  if and only if there is an isomorphism  $\phi : \pi_1(C_1) \rightarrow \pi_1(C_2)$  with  $\phi(m_1) = m_2$  and  $\phi(l_1) = l_2$ .*

We end this section with an interesting fact about the fundamental groups.

**Theorem 2.1.8.** *The map  $\pi_1(T^2) \rightarrow \pi_1(C)$  is injective unless  $L$  is the unknot.*

## 2.2 Representation varieties

The discussion in the previous section showed that the fundamental group  $\pi$  of a knot complement is a very powerful invariant of a knot. To study  $\pi$  it is useful

to study its *representation varieties*, which are varieties parameterizing representations of the (discrete) group  $\pi$  into a complex algebraic group  $G$ . In this section we define these varieties and give a construction for  $G = SL_2(\mathbb{C})$ .

Let  $\pi$  be a finitely presented (discrete) group

$$\pi = \langle g_1, \dots, g_m \mid r_1, \dots, r_n \rangle$$

If  $G$  is a complex algebraic group, the **representation variety** is the variety whose underlying set of closed points is

$$\text{Rep}(\pi, G) := \text{Hom}_{\mathbf{Grp}}(\pi, G)$$

The scheme structure on  $\text{Rep}(\pi, G)$  is usually described by realizing it as the closed subscheme of  $\text{Rep}(\hat{\pi}, G)$  consisting of representations which descend to  $\pi$  (where  $\hat{\pi}$  is the free group generated by the  $g_i$ ). The set  $\text{Rep}(\hat{\pi}, G)$  is naturally isomorphic to  $G^m$  since a representation of a free group is uniquely determined by the images of a set of generators, and  $G^m$  is naturally a scheme. It is not immediately obvious that this definition is independent of the choice of presentation for  $\pi$ , but this is the case (see, e.g. [29]). (In [1] there is an elegant proof of independence for  $G = GL_n(\mathbb{C})$  using Morita theory, but discussing this would take us too far afield.)

We give a more concrete description of the representation variety when  $G = SL_2(\mathbb{C})$ , using [2] as a reference. Define  $A(\hat{\pi}) := \mathbb{C}[\{a_{ij}^k\}_{1 \leq i, j \leq 2, 1 \leq k \leq m}]/I$ , where  $I$  is the ideal generated by the polynomial equations  $\det(a_{ij}^k) = 1$  for all  $k$ . We can identify  $A(\hat{\pi})$  with the ring of functions on  $SL_2(\mathbb{C})^m$  in the manner suggested by the notation - in other words, if  $\rho \in \text{Rep}(\hat{\pi}, SL_2(\mathbb{C}))$ , then  $a_{ij}^k(\rho)$  is the  $i, j$  matrix entry of  $\rho(g_k)$ . Therefore, the closed points of  $\text{Spec}(A(\hat{\pi}))$  are in bijection with representations  $\text{Rep}(\hat{\pi}, SL_2(\mathbb{C}))$ . The scheme structure of  $\text{Rep}(\hat{\pi}, SL_2(\mathbb{C}))$  is defined via this bijection. (We will see later that this scheme structure doesn't depend on the choice of generators for  $\hat{\pi}$ .)

The algebra  $A(\hat{\pi})$  comes with a *universal representation*  $\rho_{\hat{\pi}} : \hat{\pi} \rightarrow SL_2(A(\hat{\pi}))$ , which is defined on the generators  $g_k$  via

$$\rho_{\hat{\pi}}(g_k) = \begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix}$$

Now let  $J \subset A(\hat{\pi})$  be the ideal generated by all the entries of the matrices  $\text{Id} - \rho_{\hat{\pi}}(r_i) \in SL_2(A(\hat{\pi}))$ , and define

$$A(\pi) := A(\hat{\pi})/J, \quad \rho_{\pi} : \pi \rightarrow SL_2(A(\pi)), \quad \rho_{\pi}(g_i) = \begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix}$$

(We have abused notation by writing  $a_{ij}^k$  and  $g_i$  for their images in  $A(\pi)$  and  $\pi$ , respectively.) The algebra  $A(\pi)$  and representation  $\rho_{\pi}$  satisfy the following universal property:

**Lemma 2.2.1.** [2, Prop 8.1] *If  $B$  is a commutative  $\mathbb{C}$ -algebra and  $\rho : \pi \rightarrow SL_2(B)$  is a representation, then there exists a unique  $\mathbb{C}$ -algebra homomorphism  $\phi : A(\pi) \rightarrow B$  such that  $\rho = \tilde{\phi} \circ \rho_{\pi} : \pi \rightarrow SL_2(A(\pi)) \rightarrow SL_2(B)$ , where  $\tilde{\phi} : SL_2(A(\pi)) \rightarrow SL_2(B)$  is induced by applying  $\phi : A(\pi) \rightarrow B$  to matrix entries.*

In particular, closed points of  $\text{Spec}(A(\pi))$  are (by definition) in bijection with algebra homomorphisms  $A(\pi) \rightarrow \mathbb{C}$ , and by the lemma these are in bijection with group homomorphisms  $\pi \rightarrow SL_2(\mathbb{C})$ . This provides the scheme structure for  $\text{Rep}(\pi, SL_2(\mathbb{C}))$ , and by the following lemma this does not depend on the choice of presentation for  $\pi$ .

**Lemma 2.2.2.** [2, Prop 8.2] *A group homomorphism  $\pi' \rightarrow \pi$  functorially induces a  $\mathbb{C}$ -algebra homomorphism  $A(\pi') \rightarrow A(\pi)$ , so the algebra  $A(\pi)$  only depends on the isomorphism type of the group  $\pi$ .*

## 2.3 Character varieties for $SL_2(\mathbb{C})$

In this section we describe the  $SL_2(\mathbb{C})$  *character variety* of a discrete group  $\pi$ , which parameterizes representations of  $\pi$  up to isomorphism (instead of parameterizing all representations). We state Procesi's Theorem that the coordinate ring of the character variety is generated by traces, and we also recall an explicit description of the coordinate ring of this variety from [2]. Finally, we state Goldman's Theorem that if  $\pi$  is the fundamental group of a surface, then (the smooth locus of) the character variety is symplectic.

There is an action of  $SL_2(\mathbb{C})$  on  $A(\hat{\pi})$  given by simultaneous conjugation of each 'matrix'  $(a_{ij}^k)$ , and this action descends to an action on the quotient  $A(\pi)$ . The **character variety** of  $\pi$  is the spectrum of the invariant subalgebra:

$$\text{Char}(\pi) := \text{Spec} \left( A(\pi)^{SL_2(\mathbb{C})} \right) = \text{Rep}(\pi, SL_2(\mathbb{C})) // SL_2(\mathbb{C})$$

We write  $\mathcal{O}\text{Char}(\pi)$  for the coordinate ring of  $\text{Char}(\pi)$ , which is the algebra of conjugation-invariant functions on  $\text{Rep}(\pi, SL_2(\mathbb{C}))$ . One obvious source of invariant functions is given by taking traces of representations. Precisely, if  $g \in \pi$ , then the following is an invariant function on  $\text{Rep}(\pi, SL_2(\mathbb{C}))$ :

$$\text{Tr}_g : \text{Rep}(\pi, SL_2(\mathbb{C})) \rightarrow \mathbb{C}, \quad \text{Tr}_g(\rho) := \text{Tr}(\rho(g))$$

(Here we have written  $\text{Tr}$  for the standard trace on matrices in  $SL_2(\mathbb{C})$ .) The following proposition was (essentially) proved by Procesi in [33].

**Proposition 2.3.1.** [2, Prop 9.2] *The elements  $\text{Tr}_g \in \mathcal{O}\text{Char}(\pi)$  generate  $\mathcal{O}\text{Char}(\pi)$  as an algebra.*

**Remark 2.3.2.** There are many relations between the generators  $\text{Tr}_g$ . In particular, the following relation (valid for any  $M, N \in SL_2(\mathbb{C})$ ) will play an important

role in the next section:

$$\mathrm{Tr}(M)\mathrm{Tr}(N) = \mathrm{Tr}(MN) + \mathrm{Tr}(MN^{-1})$$

We recall one of the main theorems of [2] which gives a more useful characterization of  $\mathcal{O}\mathrm{Char}(\pi)$ . First, we remark that if  $B$  is a (commutative)  $\mathbb{C}$ -algebra, then any representation  $\pi \rightarrow SL_2(B)$  extends uniquely to an algebra map  $\mathbb{C}\pi \rightarrow M_2(B)$ , and certain elements of  $\mathbb{C}\pi$  are always in the kernel of such a map. For example, if  $M \in SL_2(B)$ , then  $M + M^{-1} = \mathrm{Tr}(M)\mathrm{Id}$ , so the image (under any representation) of every element  $g + g^{-1} \in \pi$  is central. Motivated by this, in [2] Brumfiel and Hilden define

$$H(\pi) := \mathbb{C}\pi / \{h(g + g^{-1}) - (g + g^{-1})h \mid \forall g, h \in \pi\}$$

At first this may look somewhat arbitrary, but the first part of the following theorem implies that the ideal  $(h(g + g^{-1}) - (g + g^{-1})h) \subset \mathbb{C}\pi$  is exactly the set of elements that must vanish in any representation  $\mathbb{C}\pi \rightarrow SL_2(B)$ . This indicates that this definition is indeed natural. For the statement of the theorem, we note that there is an anti-involution  $\iota : H(\pi) \rightarrow H(\pi)$  determined by  $\iota(g) = g^{-1}$ , and we write  $H^+(\pi)$  for the subalgebra fixed by this anti-involution. (This subalgebra is the span of elements of the form  $g + g^{-1}$ , and is therefore commutative.)

**Theorem 2.3.3.** [2, Prop 9.1] *There is a natural map  $\hat{\rho}_\pi : H(\pi) \rightarrow M_2(A(\pi))$  such that*

- $\hat{\rho}_\pi$  is injective,
- the image of  $H^+(\pi)$  coincides with the image of  $A(\pi)^{SL_2(\mathbb{C})}$  under the diagonal embedding  $A(\pi) \rightarrow M_2(A(\pi))$ .

(The second statement refers to  $A(\pi)^{GL_2(\mathbb{C})}$  in [2] because they work over an arbitrary base ring, but over  $\mathbb{C}$ , invariance under  $SL_2(\mathbb{C})$  is the same as invariance under  $GL_2(\mathbb{C})$ .)

**Corollary 2.3.4.** *The algebra  $H^+(\pi)$  is isomorphic to the coordinate ring  $\mathcal{O}\text{Char}(\pi)$ .*

Finally, we give a very brief discussion of Goldman's symplectic form on (the smooth part of)  $\text{Char}(\pi_1(F))$  for a topological surface  $F$  of genus at least 2. Let  $G$  be an algebraic group with Lie algebra  $\mathfrak{g}$ , and suppose there is a  $G$ -equivariant isomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ .

**Theorem 2.3.5.** [20] *There is a natural symplectic form on the smooth points of  $X := \text{Hom}(\pi_1(F), G)//G$ .*

*Proof.* (Very brief sketch.) We write  $\pi = \pi_1(F)$ . First, if  $\rho : \pi \rightarrow G$  is a smooth point in  $X$ , one computes that the Zariski tangent space to  $X$  at  $\rho$  is  $H^1(\pi, \mathfrak{g}_{\text{Ad}\rho})$ , the group cohomology of  $\pi$  with coefficients in the module  $\mathfrak{g}$  (where the module structure is given by composing  $\rho$  with the adjoint representation of  $G$ ). Second, there is a cup product pairing

$$H^1(\pi, \mathfrak{g}_{\text{Ad}\rho}) \times H^1(\pi, \mathfrak{g}_{\text{Ad}\rho}) \rightarrow H^2(\pi, \mathbb{C}) = \mathbb{C}$$

where the coefficient pairing is given by the isomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ . One then shows that this form is skew-symmetric, nondegenerate, and closed.  $\square$

## 2.4 The Kauffman bracket skein module

In the previous section we mentioned Goldman's symplectic structure on  $\text{Char}(\pi_1(F))$  for a surface  $F$ . This induces a Poisson structure on the coor-

dinate ring  $\mathcal{O}\text{Char}(\pi_1(F))$ . From the point of view of noncommutative algebra/geometry, it is instinctive to ask whether there is a naturally occurring deformation of  $\mathcal{O}\text{Char}(\pi_1(F))$  in the direction of this Poisson bracket.

A positive answer to this question is given by the *Kauffman bracket skein module*, which is a vector space  $K_q(M)$  associated to each oriented 3-manifold  $M$ . (The association  $M \mapsto K_q(M)$  is functorial with respect to oriented embeddings, so ‘Kauffman functor’ would probably be a more appropriate name, but we will use the conventional name.) This was first introduced in [24] by Kauffman for  $M = S^3$ , and later a more general definition was given by Przytycki in [34] and independently by Turaev [39].

In this section we discuss formal properties of this functor, and we will briefly explain why it provides a deformation of character varieties of surfaces. We also explain the structure of the Kauffman bracket skein module for the sphere, solid torus, and fattened torus ( $T^2 \times [0, 1]$ ). In particular,  $K_q(T^2 \times [0, 1])$  is an algebra which is isomorphic to a subalgebra of the quantum torus by [14]. Finally, we briefly describe a result of Le in [28] that partially determines the structure of these modules for 2-bridge knots.

We first give a brief motivation for the definition (which will be helpful in Section 2.4.2). In the previous section we defined the trace map  $\text{Tr} : \pi \rightarrow \mathcal{O}\text{Char}(\pi)$  by  $g \mapsto \text{Tr}_g$  (where  $\pi$  is a discrete group and  $\mathcal{O}\text{Char}(\pi)$  is the coordinate ring of the character variety). This map assigns to the element  $g \in \pi$  the function  $\text{Tr}_g$  defined by  $\text{Tr}_g(\rho) = \text{Tr}(\rho(g))$ . If we write  $CC(\pi)$  for the set of conjugacy classes in  $\pi$ , it is clear that the trace descends to a map of sets  $\text{Tr} : CC(\pi) \rightarrow \mathcal{O}\text{Char}(\pi)$ . If we write  $\mathbb{C}[CC(\pi)]$  for the (infinitely generated) polynomial ring with generators the elements of  $CC(\pi)$ , we can upgrade the trace map to a map of commutative

algebras  $\text{Tr} : \mathbb{C}[CC(\pi)] \rightarrow \mathcal{O}\text{Char}(\pi)$ , and Procesi's theorem implies this algebra map is surjective.

It is instructive to give this map a more topological definition following [35]. Let  $\pi = \pi_1(M)$  for some manifold  $M$  (with no restrictions on the dimension). Conjugacy classes in  $\pi$  are exactly the same thing as free homotopy classes of loops in  $M$ . (A 'free homotopy' is a homotopy that is not required to respect basepoints.) We can also give the algebra  $\mathbb{C}[CC(\pi)]$  a topological realization. First, let  $S^1(n)$  be the disjoint union of  $n$  copies of  $S^1$ , and let  $HL(M)$  be the set of homotopy classes of maps  $S^1(n) \rightarrow M$ , for all  $n$ . (We allow  $n = 0$ , so  $S^1(0)$  is the empty set.) The set  $HL(M)$  has a natural monoid structure: if  $f : S^1(m) \rightarrow M$  and  $g : S^1(n) \rightarrow M$ , then we define  $fg := [f \sqcup g : S^1(m+n) \rightarrow M]$ . Then  $\mathbb{C}HL(M)$  is an algebra which is naturally isomorphic to  $\mathbb{C}[CC(\pi)]$ . The trace map assigns to a map  $f : S^1(n) \rightarrow M$  and representation  $\rho : \pi \rightarrow SL_2(\mathbb{C})$  the product of the traces of  $\rho$  along the  $n$  loops in the domain of  $f$ .

Let  $J$  be the kernel of the algebra map  $\mathbb{C}HL(M) \rightarrow \mathcal{O}\text{Char}(\pi_1(M))$ . The Kauffman bracket skein module is a family of vector spaces  $K_q(M)$  such that at  $q = -1$  there is a natural isomorphism  $\mathbb{C}HL(M)/J \rightarrow K_{q=-1}(M)$ . The construction of  $K_q(M)$  for general  $q$  is more delicate and requires  $M$  to be an oriented 3-manifold, as we explain in the next section.

### 2.4.1 The definition and formal properties

Recall that two maps  $f, g : M \rightarrow N$  of manifolds are **ambiently isotopic** if they are in the same orbit of the identity component of the diffeomorphism group of  $N$ . A **framed link** is an *embedding* of a disjoint union of annuli  $S^1 \times [0, 1]$  into a

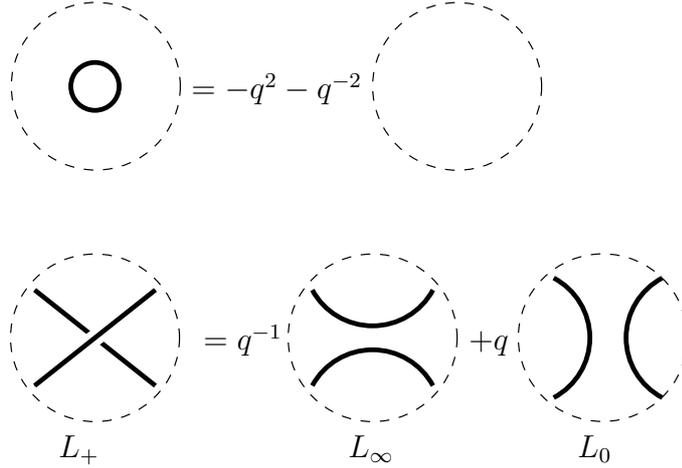


Figure 2.2: Kauffman bracket skein relations

oriented 3-manifold  $M$ . (The framing is a technical detail that will be swept under the rug where possible.) Define

$$\mathcal{L}(M) := \mathbb{C}\{\text{ambient isotopy classes of framed unoriented links in } M\}$$

$$\mathcal{L}'(M) := \mathbb{C}\{L_+ - qL_0 - q^{-1}L_\infty, L \sqcup \bigcirc + (q^2 + q^{-2})L\}$$

The links  $L_+$ ,  $L_0$ , and  $L_\infty$  are identical outside of a small 3-ball (embedded as an oriented manifold), and inside the 3-ball they appear as in Figure 2.2. The second relation in  $\mathcal{L}'(M)$  says that the disjoint union of a link  $L$  with a contractible loop is equal to  $-(q^2 + q^{-2})L$ . (All pictures drawn in this thesis will have blackboard framing. In other words, a line on the page represents a strip  $[0, 1] \times [0, 1]$  in a tubular neighborhood of the page, and the strip is always perpendicular to the paper (the intersection with the paper is  $[0, 1] \times \{0\}$ .)

**Definition 2.4.1.** The **Kauffman bracket skein module** is the vector space

$$K_q(M) := \mathcal{L}/\mathcal{L}'$$

In general  $K_q(M)$  is just a vector space (or more formally, a family of vector

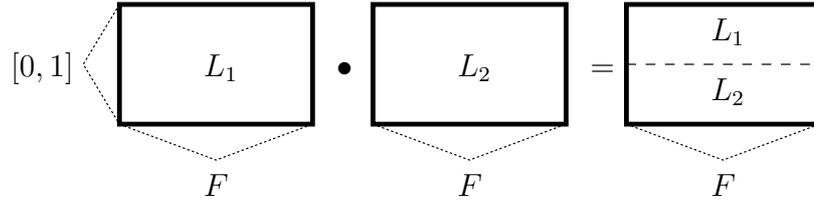


Figure 2.3: Multiplying links by stacking

spaces depending on a parameter  $q \in \mathbb{C}^*$ ). However, if  $M$  has extra structure, then  $K_q(M)$  also has extra structure. In particular,

1. If  $M = F \times [0, 1]$  for some surface  $F$ , then  $K_q(M)$  is an algebra. The multiplication is given by “stacking links” as in Figure 2.3.
2. If  $M$  is a manifold with boundary  $F$ , then  $K_q(M)$  is a module over  $K_q(F \times [0, 1])$ . The multiplication is given by “pushing links from the boundary into the manifold.” The module structure depends on the identification of  $F$  with a boundary component of  $M$ , but this identification will be implicit and will not appear in the notation.
3. An oriented embedding  $M \hookrightarrow N$  induces a linear map  $K_q(M) \rightarrow K_q(N)$ , so this construction is functorial with respect to embeddings.

**Remark 2.4.2.** If  $M = F \times [0, 1]$ , we will often write  $K_q(F)$  instead of  $K_q(M)$ . This saves space and also indicates that  $K_q(F)$  is an algebra with algebra structure coming from the decomposition  $M = F \times [0, 1]$ . (There exist manifolds  $M$  with two different decompositions of the form  $F_i \times [0, 1]$  with the two decompositions inducing non-isomorphic algebra structures on  $K_q(M)$ .)

**Remark 2.4.3.** A quick calculation with the skein relation (Figure 2.2) shows that Reidemeister move 1 (Figure 2.1) is equivalent to multiplying a link by either

$-q^3$  or  $-q^{-3}$ , depending on whether the crossing in the Reidemeister move has a positive or negative crossing.

**Remark 2.4.4.** There are two types of unoriented crossings, the one on the left of Figure 2.2 and its rotation by 90 degrees. For the definition of  $K_q$  (using the relations in Figure 2.2) to make sense, these two crossings must give different links in general. (In other words, switching from one crossing type to another should change a link). This is the reason the definition of  $K_q$  requires framed links, isotopy classes of links (instead of homotopy classes), and a 3-dimensional manifold. (In dimension 4 and higher, ambient isotopy of links is the same as homotopy of links, and in general the two crossing types are homotopic, which means changing the crossing type doesn't change the homotopy type of a link.)

## 2.4.2 $q = -1$ and the character variety

If  $q = \pm 1$ , then  $q = q^{-1}$  and the skein relation in Figure 2.2 is symmetric under rotation of each diagram by 90 degrees. This implies that a basis for  $K_q(M)$  is given by *homotopy* classes of framed links, modulo the relation that twisting the framing of a component by 1 multiplies the link by  $-q^{\pm 3}$ . (See Remark 2.4.3.) In particular, if  $q = -1$ , then a basis for  $K_q(M)$  is given by homotopy classes of *unframed* links. We therefore have an isomorphism of vector spaces  $K_{q=-1}(M) \cong \mathbb{C}HL(M)$ , so  $K_{q=-1}(M)$  is a commutative algebra (with the product of two links being the disjoint union of their images, which is well-defined up to homotopy). The algebra  $K_{q=-1}(M)$  comes with a natural algebra map  $\text{Tr} : K_{q=-1}(M) \rightarrow \mathcal{O}\text{Char}(\pi_1(M))$ , with a loop  $\gamma$  being sent to the function  $\rho \mapsto -\text{Tr}(\rho(\gamma))$ .

**Remark 2.4.5.** We have inserted a minus sign from the previous definition of  $\text{Tr}$ , which is important technically in the following theorem.

**Theorem 2.4.6.** [35], [3] *The algebra map  $\text{Tr} : K_{q=-1}(M) \rightarrow \mathcal{O}\text{Char}(\pi_1(M))$  is an isomorphism.*

*Proof.* (Sketch.) The theorem is essentially saying that trace relations in  $SL_2(\mathbb{C})$  are the same as the Kauffman bracket skein relations at  $q = -1$ . We won't prove this, but we will give a sketch of an explanation of why the ideal of skein relations vanishes in  $\mathcal{O}\text{Char}(\pi_1(M))$ .

One skein relation says that removing a contractible loop is the same as multiplying by  $-q^2 - q^{-2} = -2$ . The image of a contractible loop under the (negated) trace map is the constant function  $-2$  (because a contractible loop represents the conjugacy class of the identity in  $\pi_1(M)$  and the trace of the identity is always 2). Therefore this skein relation is contained in the kernel of the map  $\text{Tr} : K_{q=-1}(M) \rightarrow \mathcal{O}\text{Char}(\pi_1(M))$ .

The other skein relation is more subtle. First, we remark that if  $A \in SL_2(\mathbb{C})$ , then  $\text{Tr}(A) = \text{Tr}(A^{-1})$ , so if  $\gamma$  is a loop in  $M$  and  $g \in \pi_1(M)$  is a representative of  $\gamma$ , then  $\text{Tr}_g = \text{Tr}_{g^{-1}}$ . Next, we need to draw 3 links, one for each of the links in the skein relation, pick representatives in  $\pi_1(M)$  for the components of the links, and see how traces of these representatives are related. There are two cases, depending on the connectivity of the diagrams.

Case 1 is depicted in Figure 2.4. We let  $A, B \in \pi_1(M)$  be (oriented) representatives of the two loops in the left diagram. Then diagram on the left gets sent (under the trace map) to  $(-\text{Tr}_A)(-\text{Tr}_B)$  (the product of traces is taken because there are two components in the link). The middle diagram gets sent to  $-\text{Tr}_{AB}$  (there is just one trace since there is one component, and the trace of  $AB$  is taken because the chosen orientations of the two loops agree). Finally, the right link is

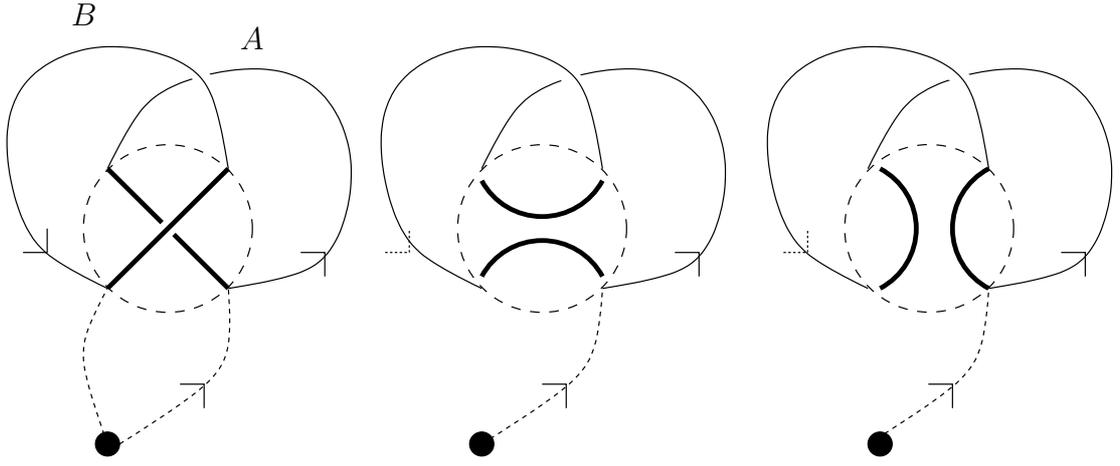


Figure 2.4: Kauffman bracket skein versus trace relations, case 1

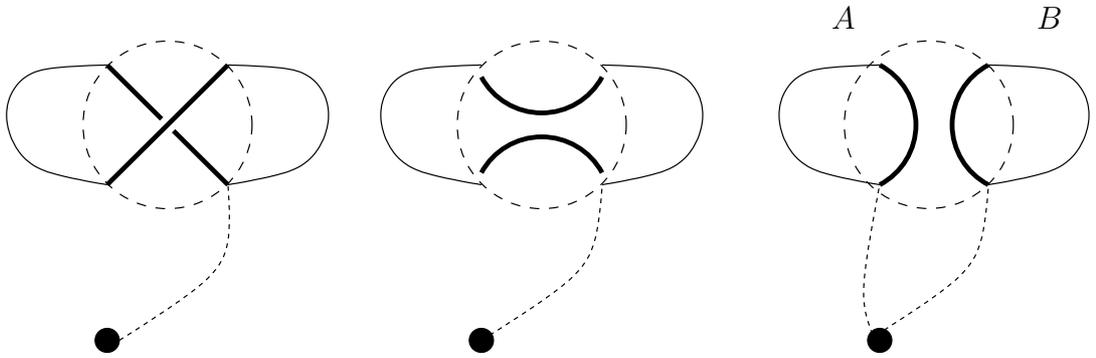


Figure 2.5: Kauffman bracket skein versus trace relations, case 2

sent to  $-\text{Tr}_{AB^{-1}}$ . Since  $q = -1$ , the skein relation says  $\text{Tr}_A \text{Tr}_B = \text{Tr}_{AB} + \text{Tr}_{AB^{-1}}$ , which is exactly the trace relation mentioned in Remark 2.3.2. We have include a diagram of case 2 in Figure 2.5, and similar reasoning shows that in this case the skein relation also gets sent to the trace relation  $\text{Tr}_A \text{Tr}_B = \text{Tr}_{AB} + \text{Tr}_{AB^{-1}}$ .  $\square$

### 2.4.3 Deformation and the Poisson bracket

A *star product* on a commutative algebra  $A$  is a formal deformation of the product in  $A$ . A star product always induces a Poisson bracket on  $A$ , and a theorem in [4] that says that if  $F$  is a surface, the Poisson bracket on  $K_{q=-1}(F)$  coming from the star product in  $K_{q=-e^h}(F)$  is the same as the Poisson bracket induced by Goldman's symplectic structure on  $K_{q=-1}(F) \cong \mathcal{O}\text{Char}(\pi_1(F))$ . In this section we make these ideas more precise.

Let  $A$  be a commutative algebra,  $\hbar$  a parameter,  $\mathbb{C}[[\hbar]]$  the ring of formal power series in  $\hbar$ , and  $A[[\hbar]] = A \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$ . A **star product** on  $A$  is a  $\mathbb{C}[[\hbar]]$ -linear map

$$* : A[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} A[[\hbar]] \rightarrow A[[\hbar]]$$

which is associative and which satisfies  $a * b \equiv ab \pmod{\hbar}$ . Given a star product, we can define a bracket

$$\{-, -\} : A \otimes A \rightarrow A, \quad \{a, b\} := \frac{a * b - b * a}{\hbar}$$

The definition makes sense because  $A$  is commutative, so the 0 order term in  $a * b - b * a$  is  $ab - ba = 0$ .

**Lemma 2.4.7.** *The bracket  $\{-, -\}$  is a Poisson bracket.*

*Proof.* The commutator on the algebra  $A[[\hbar]]$  is a Lie bracket, and the identities satisfied by a Lie bracket quickly imply the identities required of a Poisson bracket. □

Now we temporarily replace the coefficient field  $\mathbb{C}$  in the definition of  $K_q(F)$  with the coefficient ring  $\mathbb{C}[[\hbar]]$ , and we let  $q = -e^{\hbar/4} \in \mathbb{C}[[\hbar]]$ . Then  $K_{q=-e^h}(F)$  is an

algebra over  $\mathbb{C}[[\hbar]]$ , and  $K_{q=-e\hbar}(F)/\hbar \cong K_{q=-1}(F) \cong \mathcal{O}\text{Char}(\pi_1(F))$ . We can then view the product in  $K_{q=-e\hbar}(F)$  as a star product on  $K_{q=-1}(F) \cong \mathcal{O}\text{Char}(\pi_1(F))$ .

**Theorem 2.4.8.** [4] *The Poisson bracket on  $\mathcal{O}\text{Char}(\pi_1(F))$  induced by the product in  $K_{q=-e\hbar}(F)$  is the same as the Poisson bracket associated to Goldman's symplectic form.*

#### 2.4.4 The sphere and the solid torus

In this section we briefly discuss the structure of  $K_q(S^3)$  and  $K_q(S^1 \times D^2)$ . There is an isomorphism of vector spaces

$$K_q(S^3) \cong \mathbb{C}$$

The original motivation for the definition of  $K_q(M)$  is that under this isomorphism, the image of a link is the Jones polynomial of the link (up to a factor of  $\pm q^{3k}$ ). (In this interpretation,  $q$  is regarded as a variable instead of a formal parameter.) To describe the map, fix a plane in  $S^3$ . Then any link in generic position can be projected to the plane, and this gives a diagram consisting of loops with crossings. Then the skein relations in Figure 2.2 can be used to remove crossings and trivial loops until the diagram has been reduced to a multiple of the empty link  $\emptyset$ . This shows that the vector space map  $\mathbb{C} \rightarrow K_q(S^3)$  sending  $\lambda \mapsto \lambda \cdot \emptyset$  is surjective. Showing it is injective is equivalent to showing the Jones polynomial of a link is well-defined, which is non-obvious. However, this follows from the following lemma:

**Lemma 2.4.9.** [6, Prop 1.1(5)] *A linear basis for the vector space  $K_q(F)$  is given by the set of noncrossing links in  $F$  with no contractible components, including the empty link.*

Similarly, if  $F = S^1 \times [0, 1]$  is the annulus, then there is an isomorphism of vector spaces  $K_q(F) \cong \mathbb{C}[u]$ , where  $u^n$  is sent to the link consisting of  $n$  parallel copies of the unique nontrivial simple closed curve in  $F$ . The product structure on  $K_q(F)$  is given by stacking links, and it is easy to see that this product is the standard product on the vector space  $\mathbb{C}[u]$ .

### 2.4.5 The quantum torus and $K_q(T^2)$

In this section we recall the description of the algebra  $K_q(T^2)$  due to Frohman and Gelca in [14], which is the starting point for this thesis. We first use Lemma 2.4.9 to give a vector space basis for  $K_q(T^2)$  and then we state their theorem describing the multiplication. The proof is quite technical so we will not include it, but we will include an example calculation which is hopefully enlightening.

For  $r, s \in \mathbb{Z}$  relatively prime, let  $(r, s)$  be the unoriented simple closed curve in  $T^2$  that is the image of the line of slope  $r/s$  in  $\mathbb{R}^2$  under the natural projection. (Since we are dealing with unoriented curves, we have the equality  $(r, s) = (-r, -s)$ .) For arbitrary  $r, s \in \mathbb{Z}$  with  $d = \gcd(r, s)$ , write  $(r, s)$  for  $d$  parallel copies of the  $(r/d, s/d)$  curve. Let  $T_n$  be the Chebyshev polynomials, where  $T_0 = 2$ ,  $T_1(x) = x$ , and  $T_{n+1} = xT_n - T_{n-1}$ . Also, let  $(r, s)_T$  be the evaluation of  $T_d$  on the  $(r/d, s/d)$  curve, where  $d = \gcd(r, s)$ .

**Lemma 2.4.10.** *The set  $\{(r, s)_T\}_{r, s \in \mathbb{Z}} / \sim$  is a linear basis for  $K_q(T^2)$  (where  $(r, s)_T \sim (-r, -s)_T$ ).*

*Proof.* It is easy to see that any simple closed curve in  $T^2$  is homotopic to some  $(r, s)$ . Furthermore, if we cut the torus along the  $(r, s)$  curve, the resulting space

is homeomorphic to an annulus  $S^1 \times [0, 1]$ , and any disjoint link in  $S^1 \times [0, 1]$  is homotopic to a disjoint union of boundary-parallel curves. Therefore, any non-crossing link in  $T^2$  is homotopic to a disjoint union of  $(r, s)$  curves (for some  $(r, s)$  only depending on the link). Therefore,  $\mathbb{Z}^2/\sim$  is a linear basis for  $K_q(T^2)$ , where  $(r, s) \sim (-r, -s)$ . Since Chebyshev polynomials form a basis for  $\mathbb{C}[x]$ , this shows that the set  $(r, s)_T$  also forms a basis for  $K_q(T^2)$ .  $\square$

To state the theorem of Frohman and Gelca we define the **quantum torus** to be the algebra  $A_q$  generated by  $X, Y, X^{-1}, Y^{-1}$  with relation  $XY = q^2YX$ . (We discuss this algebra in more detail in Chapter 4.) There is an involution  $\theta : A_q \rightarrow A_q$  defined by  $\theta(X) = X^{-1}$  and  $\theta(Y) = Y^{-1}$ , and we write  $A_q^{\mathbb{Z}_2}$  for the subalgebra of elements invariant under this involution.

**Theorem 2.4.11.** [14, Thm. 4.3] *The linear map  $F : K_q(T^2) \rightarrow A_q^{\mathbb{Z}_2}$  given by  $(r, s)_T \mapsto q^{-rs}(Y^{-r}X^s + Y^rX^{-s})$  is an isomorphism of algebras.*

**Remark 2.4.12.** To fit in with later notation, we have chosen to twist the isomorphism of [14] by composing it with an automorphism of  $A_q$ . We will comment on this choice more later, but the short reason is that we want ‘multiplying by the meridian’ to be the same as ‘multiplying by  $X + X^{-1}$ ,’ and the convention seems to be that the curve  $(0, 1)_T$  is the meridian.

**Corollary 2.4.13.** *If  $L \subset S^3$  is a knot, then the vector space  $K_q(S^3 \setminus L)$  is naturally a left module over  $A_q^{\mathbb{Z}_2}$ .*

**Remark 2.4.14.** A priori, the  $A_q^{\mathbb{Z}_2}$ -module structure of  $K_q(S^3 \setminus L)$  depends on the identification of  $T^2$  with the boundary of  $S^3 \setminus L$ . However, Lemma 2.1.5 implies that such an identification can be chosen canonically (up to isotopy), and since links in  $M$  are considered up to isotopy, this gives the vector space  $K_q(S^3 \setminus L)$  a *canonical*  $A_q^{\mathbb{Z}_2}$ -module structure.

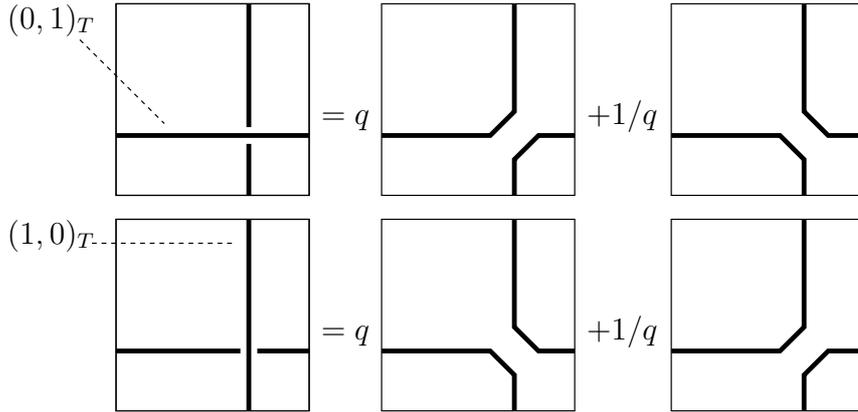


Figure 2.6: Sample computation in  $K_q(T^2)$

We now give an example calculation which might make the isomorphism of Theorem 2.4.11 more understandable. First, if we write  $[a, b]_q := qab - q^{-1}ba$ , a straightforward computation gives

$$[X + X^{-1}, Y + Y^{-1}]_q = (q^2 - q^{-2})q^{-1}(XY + X^{-1}Y^{-1}) \quad (2.1)$$

Under the isomorphism in Theorem 2.4.11 this corresponds to the identity

$$[(0, 1)_T, (1, 0)_T]_q = (q^2 - q^{-2})(-1, 1)_T \quad (2.2)$$

(The  $q^{-1}$  on the right hand side of (2.1) is absorbed into the power  $q^{-rs}$  in the statement of the theorem.) In Figure 2.6 we give a diagrammatic computation of  $(0, 1)_T(1, 0)_T$  and  $(1, 0)_T(0, 1)_T$  in the top and bottom line, respectively. Multiplying the top line by  $q$  and the bottom line by  $q^{-1}$  and then taking their difference causes the cancellation of two of the four pictures on the right, which gives the identity (2.2).

## 2.4.6 Complements of 2-bridge knots

In general it seems that very little is known about the  $A_q^{\mathbb{Z}_2}$ -module  $K_q(S^3 \setminus L)$ . In fact, it is even difficult to find a vector space basis for  $K_q(S^3 \setminus L)$ . In this section we briefly recall a theorem from [28] that seems to be the strongest result in this direction at the time this thesis was written. In Section 4.3 we give a conjecture about the general structure of  $K_q(S^3 \setminus L)$  as an  $A_q^{\mathbb{Z}_2}$ -module when  $L$  is a torus knot. This conjecture implies the corollary that  $K_q(S^3 \setminus L)$  is free as a module over the subalgebra  $\mathbb{C}[\text{meridian}]$ , and for 2-bridge knots, this corollary is also implied by the theorem below from [28].

It is a classical theorem that any knot in  $S^3$  can be separated into two balls  $B_1$  and  $B_2$  such that inside each  $B_i$  the knot is in “standard position.” In other words, inside each  $B_i$  the knot is isotopic to  $n$  vertical strands. Such a separation is called a *bridge decomposition*, and in this viewpoint, all the complexity of the knot is encoded in the gluing map between the boundaries of the  $B_i$ . (This can be contrasted with a braid presentation of a knot, which splits the knot into two balls  $B_i$  such that the knot is in standard position in  $B_1$ , is a braid in  $B_2$ , and the gluing between the boundaries of  $B_1$  and  $B_2$  is the “standard gluing.” In this viewpoint, all the complexity of the knot is contained in the braid.) The **bridge number** of  $L$  is the minimum number of strands in the  $B_i$ , where the minimum is taken over all bridge decompositions of  $L$ .

Any 2-bridge knot  $L(m, n)$  is determined by the choice of two odd integers  $m, n \in \mathbb{N}$  with  $n > m \geq 1$ . Furthermore,  $L(m, n)$  is isotopic to  $L(m', n)$  if  $mm' \equiv 1$  modulo  $n$ . (For our immediate purposes it isn't important what  $m, n$  represent.) In Figure 2.7 we have given a diagram of  $B_1$  for a 2-bridge knot  $L(m, n)$ . The two dotted vertical strands represent the portion of  $L(m, n)$  that is inside  $B_1$ ,

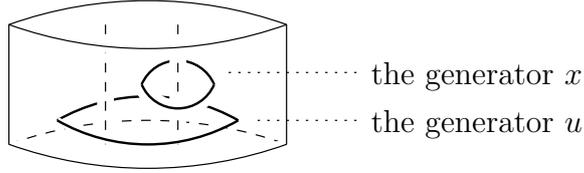


Figure 2.7: Generators of  $K_q(S^3 \setminus L(m, n))$

and  $x, u$  represent elements of  $K_q(S^3 \setminus L(m, n))$ . We can view  $B_1 \setminus L(m, n)$  as a twice-punctured disc crossed with an interval, and therefore  $K_q(B_1 \setminus L(m, n))$  is an algebra. The inclusion of  $B_1 \setminus L(m, n) \rightarrow S^3 \setminus L(m, n)$  induces a linear map of the corresponding skein modules<sup>1</sup>, and we write  $x^n$  and  $u^n$  for the images of  $x^n \in B_1 \setminus L(m, n)$  and  $u^n \in B_1 \setminus L(m, n)$  under this map. More concretely,  $x^n$  and  $u^n$  are unions of  $n$  parallel copies of  $x$  and  $u$ , respectively.

**Theorem 2.4.15.** [28, Thm. 2] *The skein module  $K_q(S^3 \setminus L(m, n))$  is linearly spanned by the elements  $\{x^i u^j \mid 0 \leq j \leq (n-1)/2\}$ .*

**Remark 2.4.16.** It is also proved that the meridian acts on this basis by multiplication by  $x$ . In particular, as a module over  $\mathbb{C}[\text{meridian}]$ , the module  $K_q(S^3 \setminus L(m, n))$  is free of rank  $(n-1)/2$ .

## 2.5 Knot complements and the colored Jones polynomials

Let  $L \subset S^3$  be a knot. The **colored Jones polynomials** are a family of (Laurent) polynomials  $J(L, n) \in \mathbb{C}[q, q^{-1}]$  depending on a natural number  $n$ . They were initially defined in [37] combinatorially using representations of the braid group obtained from irreducible representations of quantum groups. (Their construction works for general quantum groups, but we only deal with the colored Jones

<sup>1</sup>The skein module  $B_q \setminus L(m, n)$  is an algebra, but  $K_q(S^3 \setminus L(m, n))$  is *not*.

polynomials associated to  $\mathcal{U}_q(\mathfrak{sl}_2)$ , and in this case irreducible representations are naturally indexed by their dimension.)

In this section we describe how the colored Jones polynomials of  $L$  can be extracted from a natural pairing

$$\langle -, - \rangle : K_q(D^2 \times S^1) \otimes_{K_q(T^2)} K_q(S^3 \setminus L) \rightarrow K_q(S^3) = \mathbb{C}$$

Informally, this pairing is induced by gluing a solid torus to the complement of a tubular neighborhood of  $L$  to obtain  $S^3$ . More precisely, let  $N_L \subset S^3$  be a closed tubular neighborhood of  $L$ , and let  $N_c$  be the closure of its complement in  $S^3$ . Then  $N_L \cap N_c$  is a torus  $T$ , and we let  $N_T$  be a closed tubular neighborhood of  $T$ . (We have used the convention that  $N_X$  is a closed neighborhood of  $X$ .)

As described in Lemma 2.1.5, this data distinguishes a unique (up to isotopy) pair of curves in  $T$  called the *meridian* and *longitude*. The meridian/longitude pair gives both  $K_q(N_L)$  and  $K_q(N_c)$  a canonical  $A_q^{\mathbb{Z}_2}$ -module structure. More precisely, if we identify  $N_T$  with  $T \times [0, 1]$ , then the embedding  $T \times [0, 1] \hookrightarrow S^3$  gives  $K_q(N_c)$  and  $K_q(N_L)$  a left and right  $K_q(T^2)$ -module structure<sup>2</sup> (respectively). The meridian and longitude give a unique (up to isotopy) identification of the standard torus  $S^1 \times S^1$  with  $T$ , which leads to an identification of  $A_q^{\mathbb{Z}_2} \cong K_q(S^1 \times S^1)$  with  $K_q(T) \cong K_q(S^1 \times S^1)$ .

It is easy to see that  $K_q(N_L \sqcup N_c) \cong K_q(N_L) \otimes_{\mathbb{C}} K_q(N_c)$ , and the functoriality of  $K_q(-)$  shows that the inclusion  $N_L \sqcup N_c \hookrightarrow S^3$  induces a linear map

$$\langle -, - \rangle : K_q(N_L) \otimes_{\mathbb{C}} K_q(N_c) \rightarrow K_q(S^3) = \mathbb{C}$$

If  $\alpha \subset N_T$  is a link, it can be isotoped to a link inside  $N_L$  or a link inside  $N_c$ ,

---

<sup>2</sup>The asymmetry between left and right comes from the definition of multiplication in  $F \times [0, 1]$ : the product  $ab$  means “stack  $a$  on top of  $b$ .” Since the tori  $T^2 \times \{0\}$  and  $T^2 \times \{1\}$  are glued to  $N_c$  and  $N_L$ , the spaces  $K_q(N_c)$  and  $K_q(N_L)$  are a left and right  $K_q(T^2)$ -modules (respectively).

and inside  $S^3$  both these links are isotopic. Since these isotopies define the module structure of  $K_q(N_L)$  and  $K_q(N_c)$ , the pairing  $\langle -, - \rangle$  descends to

$$\langle -, - \rangle : K_q(D^2 \times S^1) \otimes_{K_q(T^2)} K_q(S^3 \setminus L) \rightarrow \mathbb{C} \quad (2.3)$$

(To ease notation for later reference, in this formula we have identified  $N_L$  with the solid torus  $D^2 \times S^1$  and written  $S^3 \setminus L$  for  $N_c$ .)

**Lemma 2.5.1.** *The right  $A_q^{\mathbb{Z}_2}$ -module  $K_q(D^2 \times S^1)$  in the pairing does not depend on the knot  $L$ .*

*Proof.* As a vector space we can identify  $K_q(D^2 \times S^1)$  with  $\mathbb{C}[u]$  by identifying the image of the longitude in  $D^2 \times S^1$  with  $u$ . (This is possible because the meridian and longitude generate the first homology of  $N_T$ , their images in  $N_L$  generate the first homology, and the image of the meridian is homologous to 0 in  $N_L$ .) By Lemma 4.2.3, it suffices to check that the action of the meridian on 1 and  $u$  is independent of the knot  $L$  (where 1 is the empty link in  $D^2 \times S^1$ ). If  $m$  is the meridian, then  $m \cdot 1$  is a trivial link in  $N_L$ , so  $m \cdot 1 = -q^2 - q^{-2}$  is independent of  $L$ . Similarly, the skein computation in Figure 2.8 shows that  $m \cdot u = (-q^4 - q^{-4})u$ , which is also independent of  $L$ .  $\square$

We define an anti-involution<sup>3</sup>  $\phi : A_q \rightarrow A_q$  via

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(s) = s$$

Since  $\phi$  fixes  $s$ , it restricts to an anti-involution  $\phi : A_q^{\mathbb{Z}_2} \rightarrow A_q^{\mathbb{Z}_2}$ . If  $M$  is a right  $A_q^{\mathbb{Z}_2}$ -module, write  $\phi(M)$  for the left module structure on the same vector space where the action is twisted by  $\phi$  (in other words,  $a \cdot_{\phi} m := m\phi(a)$ ).

---

<sup>3</sup>An anti-involution is a linear map  $\phi : A \rightarrow A$  satisfying  $\phi(ab) = \phi(b)\phi(a)$  and  $\phi^2 = \text{Id}_A$ .

$$\begin{aligned}
& \text{Diagram 1} = q \text{ Diagram 2} + q^{-1} \text{ Diagram 3} \\
& = q^2 \text{ Diagram 4} + 2 \text{ Diagram 5} + q^{-2} \text{ Diagram 6} \\
& = -q^4 - q^{-4} \text{ Diagram 7}
\end{aligned}$$

Figure 2.8: A short skein calculation

In later chapters it will be convenient to twist  $K_q(D^2 \times S^1)$  by the anti-involution  $\phi$  and to make the pairing contravariant with respect to this anti-involution. Precisely, the form  $\langle -, - \rangle : K_q(D^2 \times S^1) \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus L) \rightarrow \mathbb{C}$  is the same thing as a form

$$\langle -, - \rangle : \phi(K_q(D^2 \times S^1)) \otimes K_q(S^3 \setminus L) \rightarrow \mathbb{C}, \quad \text{satisfying } \langle au, v \rangle = \langle u, \phi(a)v \rangle$$

One reason this will be convenient is given in the following lemma.

**Lemma 2.5.2.** *Let  $M$  be the right  $A_q^{\mathbb{Z}_2}$ -module  $K_q(D^2 \times S^1)$  (with module structure given by the embedding of  $D^2 \times S^1$  as a tubular neighborhood of a knot  $L$ ), and let  $U$  be the left  $A_q^{\mathbb{Z}_2}$ -module  $K_q(S^3 \setminus \text{unknot})$ . Then  $\phi(M) \cong U$ .*

*Proof.* The meridian and longitude in  $K_q(T^2)$  are identified with  $X + X^{-1}$  and  $Y + Y^{-1}$ , respectively, so the anti-involution  $\phi$  switches the meridian and longitude. It is clear that the meridian can be taken to be the generator  $u$  in the identification  $K_q(S^3 \setminus \text{unknot}) \cong \mathbb{C}[u]$  (see Figure 2.9). Finally, if  $l$  and  $m$  are the longitude and meridian and  $1_U$  is the empty link in the unknot complement, it is clear that

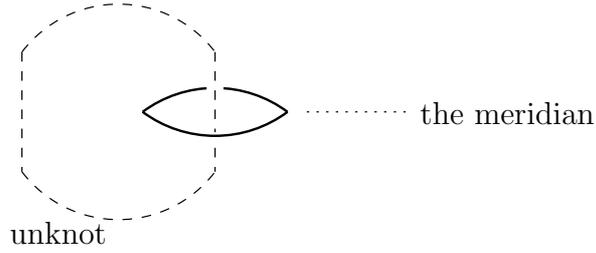


Figure 2.9: The unknot complement

$l \cdot 1_U = -q^2 - q^{-2}$  and  $l \cdot u = (-q^4 - q^{-4})u$  (this uses the same skein computation in Figure 2.8). Since  $m, l$  act on  $1_M$  in the same way that  $l, m$  act on  $1_U$ , respectively, and  $\phi$  is an anti-involution switching  $m$  and  $l$ , Lemma 4.2.3 finishes the proof.  $\square$

We can now give the relation between the pairing and the colored Jones polynomials. First, let  $S_n(x)$  be the Chebyshev polynomials, i.e. the family of polynomials  $S_n$  with initial conditions  $S_0 = 1$  and  $S_1 = x$  and recursion relation  $S_{n+1} = xS_n - S_{n-1}$ . Also, let  $1_L$  be the empty link in  $S^3 \setminus L$  and  $u$  be the image of the the longitude in  $K_q(\text{solid torus})$ .

**Theorem 2.5.3.** [25] *The colored Jones polynomial  $J(L, n)$  is equal to the evaluation of the pairing  $\langle S_{n-1}(u), 1_L \rangle$ .*

**Remark 2.5.4.** Since  $S_{-1} = 0$  and  $S_0 = 1$ , we have  $J(L, 0) = 0$  for all knots  $L$ . Furthermore, since  $J(L, 1) = \emptyset \in K_q(S^3)$  is the empty link in  $S^3$ , and under the isomorphism  $K_q(S^3) \rightarrow \mathbb{C}$  the empty link is sent to 1, we also have  $J(L, 1) = 1$  for all  $L$ . Because of this, some authors reindex  $J(L, n)$  to ignore  $J(L, 0)$  and sometimes  $J(L, 1)$ . It may seem that this is not important, but in Section 5.3 we extend  $J(L, -)$  to all integers using  $J(L, -n) = -J(L, n)$  and then discuss recursion relations satisfied by this sequence of polynomials. Our choice of indexing is very important during this discussion, since removing some terms from a sequence can easily destroy any recursion relation satisfied by the sequence.

CHAPTER 3  
MORITA THEORY

One general principle in geometry is that studying  $G$ -equivariant objects over a  $G$ -space  $X$  is often easier than studying the induced objects over the quotient space  $X/G$ . Algebraically, this says that if  $G$  acts on a commutative algebra  $A$ , then modules over<sup>1</sup>  $A \rtimes G$  are often easier to understand than modules over  $A^G$ , and in nice situations nothing is lost by working over  $A \rtimes G$ . This algebraic principle usually holds if  $A$  is noncommutative, and this makes possible most of the results of this thesis. In particular, we explained in the previous chapter that to each knot  $L \subset S^3$  one can associate a module  $K_q(L)$  over an invariant subalgebra  $A_q^{\mathbb{Z}_2}$  (where  $A_q$  is the quantum torus). In this situation, the key fact is that the category of modules over  $A_q^{\mathbb{Z}_2}$  is equivalent to the category of modules over  $A_q \rtimes \mathbb{Z}_2$ .

If  $A$  and  $B$  are two algebras whose categories of (left) modules are equivalent, then  $A$  and  $B$  are called **Morita equivalent**. In the first section of this chapter we provide a brief introduction to the theory of Morita equivalence. This theory isn't immediately visible geometrically because two commutative algebras  $A$  and  $B$  are Morita equivalent if and only if they are isomorphic, as we see below.

One of the reasons that the module  $K_q(L)$  is of interest topologically is that the colored Jones polynomials  $J(L, n)$  of  $L$  can be computed from  $K_q(L)$ . In the second section we show that the colored Jones polynomials can also be computed from the  $A \rtimes \mathbb{Z}_2$ -module which is the 'lift' of the  $A^{\mathbb{Z}_2}$ -module  $K_q(L)$ . This follows from an easy technical lemma which must be well known, but which we could not locate in the literature.

---

<sup>1</sup>We recall that as a vector space,  $A \rtimes G \cong A \otimes \mathbb{C}G$ , with the multiplication given by  $(a \otimes g)(a' \otimes g') = ag(a') \otimes gg'$ .

### 3.1 Basic definitions

In this section we give a brief overview of Morita theory using [30] and [19] as a reference. The theory has a geometric motivation - if  $X$  is a topological space, it is natural to ask how much of the geometry of  $X$  is captured by the category of sheaves on  $X$ . If  $X$  is an affine scheme, this is equivalent to asking whether a commutative algebra  $A$  is determined by its category of modules. As we will see below, the answer to this is always yes. However, if the algebra  $A$  is noncommutative, then there are many nonisomorphic algebras  $B$  whose module categories are equivalent to the module category of  $A$ .

The basic example of Morita equivalent algebras is given by an algebra  $A$  with an idempotent  $\mathbf{e}^2 = \mathbf{e} \in A$  such that  $A\mathbf{e}A = A$ . In this case, the algebras<sup>2</sup>  $A$  and  $\mathbf{e}A\mathbf{e}$  are Morita equivalent, with inverse equivalences given by tensoring by the bimodules  $\mathbf{e}A$  and  $A\mathbf{e}$ .

As a specific example, let  $A = M_n(\mathbb{C})$  and let  $\mathbf{e}$  be the matrix whose only nonzero entry is a 1 in the upper left corner. Then  $\mathbf{e}A\mathbf{e}$  is the (nonunital) subalgebra of matrices which are zero with the possible exception of the upper-left entry. This algebra is just  $\mathbb{C}$ , so the category of  $M_n(\mathbb{C})$ -modules is equivalent to the category of vector spaces. (The equivalence is given by tensoring with the column vector  $\mathbb{C}^n$ , which is a left  $M_n(\mathbb{C})$ -module and a right  $\mathbb{C}$ -module.) Since any vector space splits as a sum of 1-dimensional subspaces, the categorical equivalence implies that if  $M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  is a unital algebra map, then  $m$  divides  $n$ .

It will be useful recall that an  $A$ -module  $P$  is **projective** if and only if one of the following equivalent conditions hold:

---

<sup>2</sup>One minor but slightly subtle point worth mentioning is that  $\mathbf{e}A\mathbf{e}$  is a unital algebra whose identity element is  $\mathbf{e}$ , and therefore is not a *unital* subalgebra of  $A$ .

1.  $P$  is a summand of a free module, i.e. there is a  $Q$  with  $P \oplus Q \cong A^I$  for some index set  $I$ ,
2. for any  $f : P \rightarrow M/M'$  there is a lift  $F : P \rightarrow M$  with  $f = \pi \circ F$ ,
3.  $\text{Hom}_A(P, -)$  is exact,
4. there exist  $p_i \in P$  and  $g_i : P \rightarrow A$  such that  $g_i(p_i) = 0$  for all but finitely many  $i$  and  $\sum_i g_i(p)p_i = p$  for all  $p \in P$ .

**Remark 3.1.1.** The fourth item is perhaps less well known - it is the ‘Dual Basis Lemma’ of [30, Lemma 3.5.2]. (The name is slightly imprecise, since it does not mean that the  $g_i$  and  $p_i$  are bases on their own. Instead, the pair  $\{g_i\}, \{p_i\}$  is a dual-basis.)

We also recall the dual definition of a **generator**, which is an  $A$ -module  $M$  satisfying one of the equivalent properties

1.  $A$  is a direct summand of  $M^{\oplus n}$  for some  $n$
2.  $M^*M = A$ .

If a module  $M$  is finitely generated, projective, and a generator in  $A\text{-mod}$ , then it is called a **progenerator**. For example,  $A$  itself, viewed as a left module, is clearly a progenerator. Also, if  $A = M_n(\mathbb{C})$ , then  $\mathbb{C}^n$  is a progenerator. To see projectivity, we can identify  $\mathbb{C}^n$  with the (left) submodule of matrices in  $M_n(\mathbb{C})$  whose only nonzero column is the first column - then a complement is the submodule of matrices whose first column is zero. Similarly,  $M_n(\mathbb{C}) \cong \bigoplus_{i=1}^n \mathbb{C}^n$  as left modules, so  $\mathbb{C}^n$  is a generator.

**Lemma 3.1.2.** *Two rings  $A$  and  $B$  are called **Morita equivalent** if the following equivalent conditions hold:*

1. there are bimodules  ${}_A P_B$  and  ${}_B Q_A = P^* = \text{Hom}_A(P, A)$  such that  $P \otimes_B Q \cong A$  as  $A$ -bimodules and  $Q \otimes_A P \cong B$  as  $B$ -bimodules,
2.  $\text{Mod}(A)$  and  $\text{Mod}(B)$  are equivalent categories,
3. there is an  $A$ -module  $P$  which is a progenerator and  $B \cong \text{End}_A(P)$ ,
4. there exists an integer  $n \in \mathbb{N}$  and idempotent  $\mathbf{e} \in M_n(A)$  such that  $B \cong \mathbf{e}M_n(A)\mathbf{e}$  and  $M_n(A)\mathbf{e}M_n(A) = M_n(A)$ .

*Proof.* We give some indications of how some proofs proceed. To show (1) implies (2), the functor  $F : \text{Mod}(A) \rightarrow \text{Mod}(B)$  is defined as  $F(M) = {}_B Q_A \otimes_A M$ , and the assumed isomorphisms formally imply  $F$  is an equivalence of categories (with inverse given by  ${}_A P_B \otimes_B -$ ). Conversely, suppose  $F : \text{Mod}(A) \rightarrow \text{Mod}(B)$  is an equivalence of categories, and let  $Q = F(A)$ . Since  $F$  is an equivalence of categories,  $Q$  is a progenerator, and it is clearly a left  $B$ -module. Since  $F$  is a functor, right multiplication by  $a \in A$  is carried to a  $B$ -linear endomorphism of  $Q$ , which implies  $Q$  is a  $B$ - $A$ -bimodule. To show that  $F$  is isomorphic to the functor  ${}_A M \mapsto {}_B Q_A \otimes_A {}_A M$ , one constructs a map  $Q \otimes_A M \rightarrow F(M)$  as follows. An element  $m \in M$  induces an  $A$ -linear map  $f_m : A \rightarrow M$ , which is determined by  $f_m(1) = m$ . We then define  $\Phi : Q \times M \rightarrow F(M)$  via  $\Phi(q, m) = F(f_m)(q)$ . One then checks that this descends to a map  $Q \otimes_A M \rightarrow F(M)$ , that this is functorial in  $M$ , and that this is an isomorphism.  $\square$

**Remark 3.1.3.** The fourth condition shows that the case  $B = M_n(A)$  is a typical example of Morita equivalent rings, but it does not exhaust all examples. More precisely, there are many examples of Morita equivalent algebras  $A$  and  $B$  with  $B$  not isomorphic to  $M_n(A)$  for any  $n$ .

The first condition shows that if  $A$  and  $B$  are Morita equivalent, then the func-

tor  $M \mapsto {}_B Q_A \otimes_A M \otimes_A P_B$  from  $A$ -bimodules to  $B$ -bimodules is an equivalence.

**Corollary 3.1.4.** *If  $A$  and  $B$  are Morita equivalent, then their centers  $Z(A)$  and  $Z(B)$  are isomorphic.*

*Proof.* (Sketch.) Let  $f : A \rightarrow A$  be an  $A$ -bimodule isomorphism. Then  $f(1)r = f(r) = rf(1)$ , which shows that  $f(1) \in Z(A)$ . Then  $Z(A) \cong \text{End}_{A \otimes A^{\text{op}}}(A)$ , and since the equivalence between  $A$ -bimodules and  $B$ -bimodules takes  $A$  to  $B$ , this proves the result.  $\square$

**Corollary 3.1.5.** *If  $e \in A$  is an idempotent with  $AeA = A$  and  $B \cong eAe$ , then  $A$  and  $B$  are Morita equivalent, with  $P = Ae$  and  $Q = eA$  in the notation of the lemma above.*

## 3.2 Lifting pairings

In this section we assume  $A$  is a  $\mathbb{C}$ -algebra containing an idempotent  $e^2 = e$  such that  $AeA = A$ . Our goal is to relate pairings between modules over  $eAe$  and modules over  $A$ . More precisely, if we have a pairing

$$\langle -, - \rangle_e : Me \otimes_{eAe} eN \rightarrow \mathbb{C}$$

we would like to show that there exists a unique pairing

$$\langle -, - \rangle : M \otimes_A N \rightarrow \mathbb{C}$$

that lifts the original pairing, in the sense that

$$\langle me, en \rangle_e = \langle me, en \rangle$$

(This condition makes sense because we can view  $Me$  and  $eN$  as subspaces of  $M$  and  $N$ , respectively.) This follows directly from the following lemma.

**Lemma 3.2.1.** *Let  $M_A$  and  ${}_A N$  be right and left  $A$ -modules, respectively, and assume  $\mathbf{e} \in A$  is an idempotent with  $A\mathbf{e}A = A$ . Then the ( $\mathbb{C}$ -linear) inclusion  $M\mathbf{e} \otimes_{\mathbb{C}} \mathbf{e}N \hookrightarrow M \otimes_{\mathbb{C}} N$  descends to an isomorphism of vector spaces*

$$M\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} \mathbf{e}N \xrightarrow{\sim} M \otimes_A N$$

*Proof.* By Lemma 3.1.5, the functors  $A\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} -$  and  $\mathbf{e}A \otimes_A -$  are inverse equivalences, so the natural map  $A\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} \mathbf{e}A \otimes_A A \rightarrow A$  given by  $a\mathbf{e} \otimes \mathbf{e}b \otimes c \mapsto a\mathbf{e}bc$  is an isomorphism. We then have the following isomorphisms of vector spaces:

$$\begin{aligned} M\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} \mathbf{e}N &\cong (M \otimes_A A\mathbf{e}) \otimes_{\mathbf{e}A\mathbf{e}} (\mathbf{e}A \otimes_A N) \\ &\cong M \otimes_A (A\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} \mathbf{e}A) \otimes_A N \\ &\cong M \otimes_A A \otimes_A N \\ &\cong M \otimes_A N \end{aligned}$$

Under this chain of isomorphisms the element  $m\mathbf{e} \otimes \mathbf{e}n$  is mapped to

$$\begin{aligned} m\mathbf{e} \otimes \mathbf{e}n &\mapsto (m \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes n) \\ &\mapsto m \otimes (\mathbf{e} \otimes \mathbf{e}) \otimes n \\ &\mapsto m \otimes \mathbf{e} \otimes n \\ &\mapsto m \otimes \mathbf{e}n = m\mathbf{e} \otimes \mathbf{e}n \end{aligned}$$

□

CHAPTER 4  
THE QUANTUM TORUS

In this chapter we define the algebra  $A_q$  called the *quantum torus* and give several of its basic properties. There is an action of  $\mathbb{Z}_2$  on  $A_q$ , and we also give basic properties of the algebras  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$ . In particular, we show that  $A_q$  is simple (i.e. has no 2-sided ideals), which is the key step in the proof that  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are Morita equivalent.

In the second section we give a presentation for the invariant subalgebra  $A_q^{\mathbb{Z}_2}$  which is a very useful tool for computations. We also prove two lemmas which give sufficient conditions for two  $A_q^{\mathbb{Z}_2}$ -modules to be isomorphic.

In the final section we define a certain category  $\mathcal{O}^{\text{tw}}$  of representations of  $A_q \rtimes \mathbb{Z}_2$ . As in Lie theory, many interesting modules of  $A_q \rtimes \mathbb{Z}_2$  are induced from subalgebras of  $A_q \rtimes \mathbb{Z}_2$ . In particular, the trivial and sign representations of  $\mathbb{Z}_2$  induce representations of  $A_q \rtimes \mathbb{Z}_2$  on the space of Laurent polynomials  $\mathbb{C}[X^{\pm 1}]$ . A module  $M$  is in  $\mathcal{O}^{\text{tw}}$  if it has a filtration whose successive quotients are certain twists of these two induced representations.

From the previous two chapters we know that we can associate a module  $M_L$  over  $A_q \rtimes \mathbb{Z}_2$  to each knot  $L \subset S^3$ , and we conjecture that if  $L$  is a torus knot, then  $M_L$  is an object in  $\mathcal{O}^{\text{tw}}$ . In the next two chapters we confirm this conjecture for the unknot and trefoil. However, the structure of  $M_L$  for more complicated knots still seems mysterious at this point.

## 4.1 Basic properties and representations

In this section we review some basic properties of the quantum torus

$$A_q := \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / (XY - q^2 YX)$$

For this entire chapter we assume  $q \in \mathbb{C}^\times$  is generic, in the sense that

$$q^n \neq 1, \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

One of the most important facts about  $A_q$  is the PBW property, which is analogous to the Poincaré-Birkhoff-Witt property of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of a semisimple Lie algebra  $\mathfrak{g}$ .

**Lemma 4.1.1.** *For all  $q \in \mathbb{C}^*$ , the following composition is a vector space isomorphism:*

$$\mathbb{C}[X^{\pm 1}] \otimes \mathbb{C}[Y^{\pm 1}] \xrightarrow{\iota_X \otimes \iota_Y} A_q \otimes A_q \xrightarrow{m} A_q$$

where  $m$  is the multiplication map of  $A_q$  and  $\iota_X$  and  $\iota_Y$  are the natural inclusion of subalgebras.

*Proof.* This follows from [30, Sec 1.2.3] and is true for all  $q$ . □

We recall that an algebra is called *simple* if it has no proper two-sided ideals, and it is called *hereditary* if every submodule of a projective module is projective.

**Lemma 4.1.2.** *If  $q^n \neq 1$  for all nonzero  $n$ , then  $A_q$  is a simple hereditary Noetherian domain and  $K_0(A_q) \cong \mathbb{Z}$ .*

*Proof.* The fact that  $A_q$  is a simple Noetherian domain follows from [30, 1.8.7 Ex. 2] (there are many other references for this). However, we indicate one proof that

$A_q$  has no two-sided ideals. First, note that  $X(X^i Y^j)X^{-1} = q^{2j}(X^i Y^j)$ . Suppose that  $I \subset A_q$  is a two-sided ideal containing a nonzero  $f \in A_q$ . Assume  $f$  has terms with different  $Y$ -degrees. If the terms with highest  $Y$ -degree have the form  $X^i Y^j$ , then  $g := XfX^{-1} - q^{2j}f$  has no terms of  $Y$ -degree  $j$ . Since  $q^n \neq 1$  for all  $n$ , the kernel of the operator  $f \mapsto XfX^{-1} - q^{2j}f$  is spanned by monomials of the form  $X^i Y^j$ , and since  $f$  had terms of other  $Y$ -degree,  $g$  is nonzero. By induction, we can now assume that all terms in  $f$  have  $Y$ -degree  $j$ . Now we repeat the same argument with conjugation by  $X$  replaced by conjugation by  $Y$ , and by induction we obtain a nonzero monomial. Since monomials are units, we see that  $I$  contains a unit, which implies  $I = A_q$ .

Since  $A_q$  is simple, [30, Thm 7.5.5] shows that  $A_q$  has (right) global dimension 1, i.e. is hereditary. To compute  $K_0(A_q)$ , define  $R = \mathbb{C}[y]$ , and define an algebra automorphism  $\sigma : R \rightarrow R$  by  $\sigma(y^j) = q^j y^j$ . Since  $R$  is a right Noetherian right regular ring with  $K_0(R) \cong \mathbb{Z}$ , [30, 12.3.3(iv)] implies  $K_0(R[x, x^{-1}; \sigma]) \cong \mathbb{Z}$ . Since  $A_q$  is the localization of  $R[x, x^{-1}; \sigma]$  at the multiplicative set  $\{y^i\}$ , [30, 12.1.12(ii)] shows that  $K_0(A_q) \cong \mathbb{Z}$ .  $\square$

There is an automorphism

$$\theta : A_q \rightarrow A_q, \quad \theta(X) = X^{-1}, \quad \theta(Y) = Y^{-1}$$

and this gives an action of  $\mathbb{Z}_2$  on  $A_q$ . We write  $A_q \rtimes \mathbb{Z}_2$  for the semidirect product, which is the algebra whose underlying vector space is  $A_q \otimes \mathbb{C}\mathbb{Z}_2$  and whose multiplication is  $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$ , where  $a, b \in A_q$  and  $g, h \in \mathbb{Z}_2$ . We can also give a presentation for  $A_q \rtimes \mathbb{Z}_2$ :

$$A_q \rtimes \mathbb{Z}_2 = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, s \rangle / (XY - q^2 YX, s^2 - 1, sXsX^{-1}, sYsY^{-1})$$

**Lemma 4.1.3.** *If  $q^n \neq 1$  for all nonzero  $n$ , then  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are Morita equivalent and simple,  $A_q$  is a finitely generated projective  $A_q^{\mathbb{Z}_2}$  module, and*

$$A_q \rtimes \mathbb{Z}_2 \cong \text{End}_{A_q^{\mathbb{Z}_2}}(A_q)$$

*Proof.* By Lemma 4.1.2,  $A_q$  is simple. Furthermore,  $\theta$  is clearly an outer automorphism (since any inner automorphism acts on each monomial as a constant), and  $|\mathbb{Z}_2|^{-1} = 1/2 \in A_q$ . The claims follow from [31, Thm 2.3, 2.5] (also see [30, Thm 7.8.12]).  $\square$

We remark that  $\mathbf{e} := (1 + s)/2 \in A_q \rtimes \mathbb{Z}_2$  is an idempotent, and there is a natural isomorphism

$$\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e} \cong A_q^{\mathbb{Z}_2}, \quad \mathbf{e}(X^i Y^j)\mathbf{e} \mapsto X^i Y^j + X^{-i} Y^{-j}$$

(This defines the automorphism because  $X^i Y^j s \mathbf{e} = X^i Y^j \mathbf{e}$ .) The vector space  $(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  is an  $(A_q \rtimes \mathbb{Z}_2) - A_q^{\mathbb{Z}_2}$  bimodule, and as a bimodule we can identify

$$(A_q \rtimes \mathbb{Z}_2)\mathbf{e} \cong A_q, \quad X^i Y^j \cdot (X^k Y^l) = q^{-2jk} X^{i+k} Y^{j+l}, \quad s \cdot (X^k Y^l) = X^{-k} Y^{-l}$$

where the left  $A_q \rtimes \mathbb{Z}_2$ -module structure of  $A_q$  is given by the formulas on the right, and the right  $A_q^{\mathbb{Z}_2}$ -module structure is by multiplication via the identification  $A_q^{\mathbb{Z}_2} \cong \mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$ . Corollary 3.1.5 shows that the Morita equivalences between  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2} \cong \mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  are given by tensoring with  $A_q$ .

## 4.2 Invariant subalgebra

In this section we give a presentation for the invariant subalgebra  $A_q^{\mathbb{Z}_2}$ . Geometrically, for  $q = 1$  this presentation realizes the quotient of the (algebraic) torus by

the elliptic involution as a cubic hypersurface in  $\mathbb{C}^3$ , and the presentation below is just a deformation of this realization. We also prove two technical lemmas that give useful sufficient conditions for two  $A_q^{\mathbb{Z}_2}$ -modules to be isomorphic.

Let  $B'_q$  be the algebra generated by  $x, y, z$  modulo the following relations:

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x \quad (4.1)$$

Of course, when  $q = 1$  these relations just say that  $x, y, z$  commute. Also, define  $B_{q,t}$  to be the quotient of  $B'_q$  by the additional relation

$$q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz = \left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2 \quad (4.2)$$

**Theorem 4.2.1.** *There is an isomorphism  $B_{q,t=1} \xrightarrow{\sim} A_q^{\mathbb{Z}_2}$  given by*

$$x \mapsto X + X^{-1}, \quad y \mapsto Y + Y^{-1}, \quad z \mapsto q^{-1}(XY + X^{-1}Y^{-1})$$

*Proof.* There are at least two proofs in the literature. The first is a topological proof using skein algebras - one just compares the description of the algebra  $K_q(T^2 \times [0, 1])$  in [14] with the description in [6, Theorem 2.3]. There is also a purely algebraic proof in [26].  $\square$

We now give two useful criterion for finding isomorphisms between  $A_q^{\mathbb{Z}_2}$ -modules. Suppose  $M$  and  $N$  are  $A_q^{\mathbb{Z}_2}$ -modules that are generated over  $\mathbb{C}[x]$  by elements  $\{m_i\} \subset M$  and  $\{n_k\} \subset N$ . Also, suppose  $f : M \rightarrow N$  is an isomorphism of  $\mathbb{C}[x]$ -modules such that  $f(m_i) = n_i$ .

**Lemma 4.2.2.** *With the above assumptions, if  $f(y \cdot m_i) = y \cdot f(m_i)$  and  $f(z \cdot m_i) = z \cdot f(m_i)$ , then  $f$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* By linearity of  $f$  and the assumption that  $M$  is generated over  $\mathbb{C}[x]$  by the  $m_i$ , to show  $f(y m) = y f(m)$  and  $f(z m) = z f(m)$  for any  $m \in M$ , it is sufficient to

show  $f(yx^k m_k) = yf(x^k m_k)$  and  $f(zx^k m_k) = zf(x^k m_k)$ . We proceed by induction on  $k$ , so the base case  $k = 0$  is part of the hypotheses. Assume  $f(yx^j m) = yf(x^j m)$  and  $f(zx^j m) = zf(x^j m)$  for all  $j < k$ . Then using the commutation relations,

$$\begin{aligned}
f(yx^k m) &= f((yx)x^{k-1}m) \\
&= f((q^2xy + (q^{-1} - q^3)z)x^{k-1}m) \\
&= (q^2xy + (q^{-1} - q^3)z)f(x^{k-1}m) \\
&= (yx)f(x^{k-1}m) \\
&= yf(x^k m)
\end{aligned}$$

(The step from the second to third line uses the fact that  $f$  is  $\mathbb{C}[x]$ -linear along with the inductive assumption.) A similar calculation for  $z$  completes the proof.  $\square$

**Lemma 4.2.3.** *With the above assumptions, if  $f(y \cdot m_i) = y \cdot f(m_i)$  and  $f(y \cdot x m_i) = yx \cdot f(m_i)$ , then  $f$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* The following calculation allows us to apply the previous lemma:

$$\begin{aligned}
f(zm_i) &= f((q^2 - q^{-2})(qxy - q^{-1}yx)m_i) \\
&= (q^2 - q^{-2})(qxy - q^{-1}yx)f(m_i) \\
&= zf(m_i)
\end{aligned}$$

$\square$

### 4.3 The category $\mathcal{O}^{\text{tw}}$ and the filtration conjecture

In this section we define a certain category  $\mathcal{O}^{\text{tw}}$  of representations of  $A_q \rtimes \mathbb{Z}_2$ . We prove two basic lemmas about objects in this category and then state a conjecture

that the lifted skein module  $A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus L)$  of a knot  $L \subset S^3$  is an object in  $\mathcal{O}^{\text{tw}}$  if  $L$  is a torus knot (i.e.  $L$  can be embedded into the boundary of a tubular neighborhood of the unknot in  $S^3$ ). In Chapters 5 and 6 we confirm the conjecture for the unknot and trefoil, respectively. At the moment this conjecture seems difficult to prove in general. However, as we discuss at the end of the section, the conjecture implies several statements that are already known to be true, at least for some classes of knots.

As in the case of a universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ , many interesting representations of  $A_q \rtimes \mathbb{Z}_2$  are induced from subalgebras in the PBW decomposition. Let  $A_Y$  be the subalgebra of  $A_q \rtimes \mathbb{Z}_2$  generated by  $Y^{\pm 1}$  and  $s$ . Suppose that  $V$  is a finite dimensional representation of  $\mathbb{C}\mathbb{Z}_2$ . We can extend this to a representation of the subalgebra  $A_Y$  by defining  $Y \cdot v := s \cdot v$  for all  $v \in V$ . We will write  $\mathbb{C}_{\pm}$  for the  $A_Y$ -module obtained in this way from the  $\mathbb{Z}_2$ -module  $\mathbb{C}$  with action  $s \cdot m = \pm m$ , for  $m \in \mathbb{C}$ .

We will also need to define an automorphism  $\tau : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$ :

$$\tau(X) = X, \quad \tau(s) = s, \quad \tau(Y) = q^{-1}XY$$

Given any automorphism  $\sigma : A \rightarrow A$  and an  $A$ -module  $M$ , we write  $\sigma(M)$  for the  $A$ -module  $M$  with action twisted by  $\sigma$ . Precisely, we define a new action on the vector space  $M$  via  $a \cdot m := \sigma(a)m$ , where the action on the left hand side is the new action and the action on the right hand side is the old one.

**Definition 4.3.1.** For  $k \in \mathbb{Z}$  we define

$$P_k^{\pm} := \tau^k \left( \text{Ind}_{A_Y}^{A_q \rtimes \mathbb{Z}_2} (\mathbb{C}_{\pm}) \right)$$

The definition of this module is somewhat abstract, but using the PBW property and the commutation relations in  $A_q \rtimes \mathbb{Z}_2$  the action can be made very explicit.

In particular, the PBW property implies that we can identify  $P_k^\pm \cong \mathbb{C}[X^{\pm 1}]$  as a  $\mathbb{C}[X^{\pm 1}]$ -module, and the commutation relations imply

$$X \cdot f(X) = Xf(X), \quad Y \cdot f(X) = \pm q^{-k} X^k f(q^{-2}X), \quad s \cdot f(X) = \pm f(X^{-1})$$

We remark that the representation  $P_0^+$  of  $A_q \rtimes \mathbb{Z}_2$  is very similar in spirit to the action of the Weyl algebra on  $\mathbb{C}[X]$ . However, in this case  $Y$  acts by an algebra automorphism which rescales  $X$ , whereas in the Weyl algebra the generator  $Y$  acts as the derivation sending  $X$  to 1.

**Lemma 4.3.2.** *If  $q$  is not a root of unity, then the  $A_q \rtimes \mathbb{Z}_2$ -module  $P_k^\pm$  is simple (in the category of all  $A_q \rtimes \mathbb{Z}_2$ -modules).*

*Proof.* It clearly suffices to prove this for  $k = 0$ . The proof is similar to but simpler<sup>1</sup> than the proof of Lemma 4.1.2. Precisely, assume  $f(x) \in M \subset P_0^\pm$  is a nonzero element. If  $f(x)$  is a monomial  $x^k$  then  $X^{-k}x^k = 1$ , and  $M = P_0^\pm$ . Since  $q$  is not a root of unity, the kernel of the operator  $Y \mp q^{-2k}$  is spanned by the monomial  $x^k$ . Therefore, if  $f(x)$  is not a monomial and has highest degree term  $x^k$ , the element  $(Y \mp q^{-2k}) \cdot f(x)$  is nonzero and has highest degree term  $x^j$  for some  $j < k$ . Induction shows  $M$  contains a monomial, which shows  $M = P_0^\pm$ .  $\square$

We now give a conjecture about the structure of skein modules of complements of torus knots. Let  $\text{Mod}(A)$  be the category of finitely generated  $A$ -modules.

**Definition 4.3.3.** Let  $\mathcal{O}^{\text{tw}}$  be the full subcategory of  $\text{Mod}(A_q \rtimes \mathbb{Z}_2)$  whose objects are the modules that admit a finite filtration whose successive quotients are isomorphic to one of the  $P_{k,\epsilon}$ .

**Conjecture 4.3.4.** For each torus knot  $L \subset S^3$ , the lifted skein module  $A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus L)$  is an object in  $\mathcal{O}^{\text{tw}}$ .

---

<sup>1</sup>This ‘pun’ is intentional.

For a cryptic explanation of the restriction of the conjecture to torus knots, see Remark 1.2.1. Also, the appearance of the automorphism  $\tau$  may seem somewhat artificial and deserves some comments. First, since  $\tau$  preserves  $s$ , it restricts to an automorphism of  $A_q^{\mathbb{Z}_2}$ , and under the identification  $A_q^{\mathbb{Z}_2} \xrightarrow{\sim} K_q(T^2)$ , this automorphism has a natural topological interpretation as follows. It is clear that a diffeomorphism  $T^2 \rightarrow T^2$  induces an isomorphism  $K_q(T^2) \rightarrow K_q(T^2)$ . A *Dehn twist* of  $T^2$  is a diffeomorphism given by cutting  $T^2$  along a simple closed curve, twisting one end once, and then regluing to obtain  $T^2$ .

**Lemma 4.3.5.** *The automorphism  $K_q(T^2) \rightarrow K_q(T^2)$  induced by the Dehn twist along the meridian is the same as the automorphism given by restricting  $\tau$  to  $A_q^{\mathbb{Z}_2}$ .*

*Proof.* It is easy to check  $\tau(x) = x$  and  $\tau(y) = q^{-1}z$ . The Dehn twist along the meridian fixes the meridian (which is  $x$ ) and takes the  $(1, 0)$  curve to the  $(1, 1)$  curve (which are  $y$  and  $q^{-1}z$ , respectively). Since  $x$  and  $y$  generate  $A_q^{\mathbb{Z}_2}$  (for generic  $q$ ), the two automorphisms are the same.  $\square$

**Remark 4.3.6.** It seems that many other automorphisms of  $A_q \rtimes \mathbb{Z}_2$  that preserve  $s$  also have a topological interpretation as Dehn twists. However, if we allow twists by automorphisms that don't preserve  $X$ , then the modules we obtain will not be free over  $\mathbb{C}[X^{\pm 1}]$ . Since (lifted) skein modules of 2-bridge knots are always free over  $\mathbb{C}[X^{\pm 1}]$  (see Section 2.4.6), it seems reasonable to only allow twists that preserve  $X$  in the definition of  $\mathcal{O}^{\text{tw}}$ .

The conjecture has some corollaries that are known to be true, at least for some classes of knots.

**Proposition 4.3.7.** *If  $L$  is a knot whose lifted skein module  $M_L := A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(L)$  is an object of category  $\mathcal{O}^{\text{tw}}$ , then*

- $M_L$  is free over  $\mathbb{C}[X^{\pm 1}]$ ,
- the sequence of Jones polynomials  $J(L, n)$  satisfy a linear recursion relation.

*Proof.* For the first, since the  $P_k^{\pm}$  are free over  $\mathbb{C}[X^{\pm 1}]$ , a filtration on  $M$  whose successive quotients are  $P_k^{\pm}$  “splits” over  $\mathbb{C}[X^{\pm 1}]$ , which shows  $M$  is free as a  $\mathbb{C}[X^{\pm 1}]$ -module. The second statement is Corollary 5.3.4.  $\square$

The first conclusion is known to be true for 2-bridge knots (see Section 2.4.6). The second is true for all knots by a theorem of Garafoulidis and Lê [17]. In Section 5.3 we discuss their work and its relation to ours in detail.

We now give a technical lemma which we will use in Section 5.3.

**Lemma 4.3.8.** *If  $M$  is finitely generated over  $\mathbb{C}[X^{\pm 1}]$ , then for any  $m \in M$  there is an  $a \in A_q$  with  $a \cdot m = 0$ . In particular, the conclusion holds if  $M$  is an object of  $\mathcal{O}^{\text{tw}}$ .*

*Proof.* As a  $\mathbb{C}[X^{\pm 1}]$ -module,  $A_q$  is isomorphic to  $\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[X^{\pm 1}]Y^i$ , and in particular is a countably generated free module. Let  $m \in M$  and let  $f_m : A_q \rightarrow M$  be the  $\mathbb{C}[X^{\pm 1}]$ -module map  $f_m(X^j Y^k) = X^j Y^k \cdot m$ . If  $M$  is finitely generated over  $\mathbb{C}[X^{\pm 1}]$ , then it is Noetherian as a  $\mathbb{C}[X^{\pm 1}]$  module, so any  $\mathbb{C}[X^{\pm 1}]$ -module map from an infinite rank free  $\mathbb{C}[X^{\pm 1}]$ -module must have a nonzero kernel.

Each object in  $\mathcal{O}^{\text{tw}}$  (by definition) has a finite filtration whose successive quotients are  $P_k^{\pm}$ , and each  $P_k^{\pm}$  is a cyclic  $\mathbb{C}[X^{\pm 1}]$ -module. This shows that each object in  $\mathcal{O}^{\text{tw}}$  is finitely generated over  $\mathbb{C}[X^{\pm 1}]$ .  $\square$

## CHAPTER 5

### THE UNKNOT COMPLEMENT AND RECURSION RELATIONS

In Section 2.5 we explained that for each knot  $L \subset S^3$  there is a pairing

$$\langle -, - \rangle_{\mathbf{e}} : K_q(\text{unknot}) \otimes K_q(S^3 \setminus L) \rightarrow \mathbb{C}, \quad \text{with } \langle au, v \rangle_{\mathbf{e}} = \langle u, \phi(a)v \rangle_{\mathbf{e}}$$

(Recall  $\phi : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  is the anti-involution that sends  $X$ ,  $Y$ , and  $s$  to  $Y^{-1}$ ,  $X^{-1}$ , and  $s$ , respectively.) The results of the previous two chapters imply that there are unique  $A_q \rtimes \mathbb{Z}_2$  modules  $V$  and  $M_K$  such that  $\mathbf{e}V \cong K_q(\text{unknot})$  and  $\mathbf{e}M_L \cong K_q(L)$  (where  $\mathbf{e} = (1 + s)/2 \in A_q \rtimes \mathbb{Z}_2$  is the symmetrizing idempotent). Furthermore, there is a unique pairing

$$\langle -, - \rangle : V \otimes M_L \rightarrow \mathbb{C}, \quad \text{with } \langle au, v \rangle = \langle u, \phi(a)v \rangle$$

such that the restriction of  $\langle -, - \rangle$  to  $\mathbf{e}V \otimes \mathbf{e}M_L$  gives the pairing  $\langle -, - \rangle_{\mathbf{e}}$ . There is a fixed  $\mathbb{C}$ -basis of  $\mathbf{e}V$  given by the Chebyshev polynomials  $S_n$ , and the colored Jones polynomial  $J(L, n)$  is equal to  $\langle S_{n-1}, 1_L \rangle_{\mathbf{e}}$ , where  $1_L$  is the empty link in  $K_q(S^3 \setminus L)$ . In this chapter we explain that as a vector space,  $V \cong \mathbb{C}[X^{\pm 1}]$ , and the colored Jones polynomials can also be computed with the formula

$$J(L, n) = 2\langle X^n, 1_L \rangle = -2\langle X^{-n}, 1_L \rangle$$

This formula shows one advantage of working with the lifted modules  $V$  and  $M$  instead of  $\mathbf{e}V$  and  $\mathbf{e}M$ , since the monomials  $X^n$  are much simpler than the Chebyshev polynomials  $S_n(u)$ .

This computation also allows us to reinterpret a result of Garoufalidis and Lê in [17]. We can view  $J$  as an element of  $\text{Hom}(\mathbb{Z}, \mathbb{C})$ . This vector space is a right  $A_q$ -module with action given by

$$(Xf)(n) = f(n + 1), \quad (Yf)(n) = -q^{-2n}f(n)$$

In [17], Garoufalidis and Lê showed that there is a nonzero element  $a \in A_q$  such that  $J \cdot a = 0$ . (They twisted the action by the anti-automorphism that inverts  $Y$  and fixes  $X$  and  $s$  to obtain a left action.) In other words, the colored Jones polynomials satisfy a linear recursion relation.

In the last section we use the identity  $J(L, n) = 2\langle X^n, 1_L \rangle$  to give a conceptual explanation of why the colored Jones polynomials behave nicely with respect to the  $A_q$  action on  $\text{Hom}(\mathbb{Z}, \mathbb{C})$ . In particular, the existence of such an  $a \in A_q$  is implied by the assumption that there is a  $b \in A_q$  such that  $b \cdot 1_L = 0$ , and by Lemma 4.3.8, Conjecture 4.3.4 implies the existence of such a  $b \in A_q$ . (Garoufalidis already showed in [16] that the existence of such a  $b \in A_q$  implies the existence of such an  $a \in A_q$ , but the explanation in the last section might be slightly more transparent than the one in [16].)

## 5.1 The lifted unknot module

In this section we give an explicit computation of the  $A_q \rtimes \mathbb{Z}_2$ -module  $V$  with  $\mathbf{e}V \cong K_q(S^3 \setminus \text{unknot})$  (where  $\mathbf{e} = (1+s)/2 \in A_q^{\mathbb{Z}_2}$  is the symmetrizing idempotent). We first recall that we can identify  $K_q(S^3 \setminus \text{unknot}) \cong \mathbb{C}[u]$  as vector spaces, where  $u$  is a generator of the homology of the unknot complement (which is a solid torus). (See Figure 5.1.) Let  $m, l \in A_q^{\mathbb{Z}_2}$  be the meridian and longitude of the unknot. In the proof of Lemma 2.5.2, we showed that the action satisfies

$$m \cdot f(u) = uf(u), \quad l \cdot 1 = (-q^2 - q^{-2}), \quad l \cdot u = (-q^4 - q^{-4})u \quad (5.1)$$

Since 1 generates  $\mathbb{C}[u]$  over  $\mathbb{C}[m]$  and  $m$  and  $l$  are identified with  $X + X^{-1}$  and  $Y + Y^{-1}$ , respectively, in  $A_q^{\mathbb{Z}_2}$ , Lemma 4.2.3 shows that these formulas completely determine the module structure of  $\mathbb{C}[u]$ .

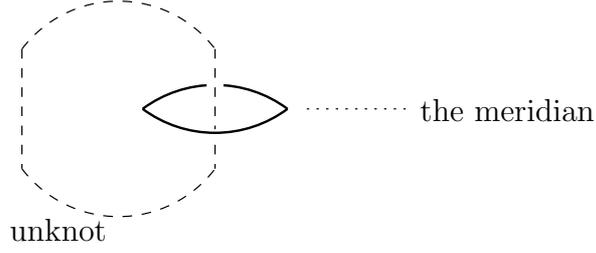


Figure 5.1: The unknot complement

Let  $V = P_0^-$  (see Section 4.3) with  $\delta = X - X^{-1} \in V$ . As a vector space, we can identify  $V \cong \mathbb{C}[X^{\pm 1}]$ , and the action of  $A_q \rtimes \mathbb{Z}_2$  is given by

$$X \cdot f(X) = Xf(X), \quad Y \cdot f(X) = -f(q^{-2}X), \quad s \cdot f(X) = -f(X^{-1})$$

**Lemma 5.1.1.** *As a  $\mathbb{C}[X + X^{-1}]$ -module,  $\mathfrak{e}V \cong \mathbb{C}[X + X^{-1}]\delta$ .*

*Proof.* The invariant subspace  $\mathfrak{e}V$  is spanned by eigenvectors of  $\mathfrak{e}$  with eigenvalue 1. Since  $s \cdot X^n = -X^{-n}$ , as a vector space we can identify

$$\mathfrak{e}V \cong_{\mathbb{C}} \{f(X) - f(X^{-1}) \mid f(X) \in V\}$$

We recall that any anti-symmetric polynomial can be written uniquely in the form  $f(X) - f(X^{-1}) = g(X + X^{-1})(X - X^{-1})$ .  $\square$

**Lemma 5.1.2.** *The action of  $A_q^{\mathbb{Z}_2}$  on  $\mathfrak{e}V$  is determined by*

$$x \cdot f(x)\delta = xf(x)\delta, \quad y \cdot \delta = -(q^2 + q^{-2})\delta, \quad yx \cdot \delta = (-q^4 - q^{-4})x\delta \quad (5.2)$$

*Proof.* We first check that these formulas hold:

$$\begin{aligned} y \cdot \delta &= (Y + Y^{-1}) \cdot (X - X^{-1}) \\ &= q^{-2}XY - q^2X^{-1}Y + q^2XY^{-1} - q^{-2}X^{-1}Y^{-1} \\ &= -(q^2 + q^{-2})X + (q^2 + q^{-2})X^{-1} \\ &= -(q^2 + q^{-2})\delta \end{aligned}$$

Similarly, we compute

$$\begin{aligned}
yx \cdot \delta &= (Y + Y^{-1})(X + X^{-1})(X - X^{-1}) \\
&= (Y + Y^{-1})(X^2 - X^{-2}) \\
&= YX^2 + Y^{-1}X^2 - YX^{-2} - Y^{-1}X^{-2} \\
&= -q^{-4}X^2 - q^4X^2 - q^4X^{-2} - q^{-4}X^{-2} \\
&= (-q^4 - q^{-4})(X^2 - X^{-2}) \\
&= (-q^4 - q^{-4})x\delta
\end{aligned}$$

Since we already know  $\mathbf{e}V$  is an  $A_q^{\mathbb{Z}_2}$ -module, Lemma 4.2.2 shows that these formulas determine the module structure.  $\square$

**Corollary 5.1.3.** *The assignment  $\delta \mapsto 1$  extends to an  $A_q^{\mathbb{Z}_2}$ -module isomorphism  $\mathbf{e}V \rightarrow \mathbb{C}[u]$ .*

*Proof.* Since  $\mathbf{e}V$  and  $\mathbb{C}[u]$  are free  $\mathbb{C}[x]$ -modules of rank 1 generated by  $\delta$  and 1, respectively, the assignment  $\delta \mapsto 1$  extends to a  $\mathbb{C}[x]$ -module isomorphism. Since the action of  $y$  and  $yx$  on  $\delta$  and 1 are the same (see (5.2) and (5.1)), Lemma 4.2.2 shows that this is actually a  $A_q^{\mathbb{Z}_2}$ -linear map.  $\square$

## 5.2 Jones polynomials from lifted skein modules

Let  $L \subset S^3$  be a knot and let  $V$  and  $M$  be the unique  $A_q \rtimes \mathbb{Z}_2$ -modules with  $\mathbf{e}V \cong K_q(S^3 \setminus \text{unknot})$  and  $\mathbf{e}M \cong K_q(S^3 \setminus L)$ . In Section 2.5 we explained that the embeddings  $L \hookrightarrow S^3$  and  $(S^3 \setminus L) \hookrightarrow S^3$  induce a pairing

$$\langle -, - \rangle_{\mathbf{e}} : K_q(S^3 \setminus \text{unknot}) \otimes K_q(S^3 \setminus L) \rightarrow \mathbb{C}, \quad \text{with } \langle au, v \rangle_{\mathbf{e}} = \langle u, \phi(a)v \rangle_{\mathbf{e}}$$

(We recall  $\phi : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  is the anti-involution given the following formula.)

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(s) = s \quad (5.3)$$

In Section 3.2 we showed that the pairing above lifts to a pairing

$$\langle -, - \rangle : V \otimes M \rightarrow \mathbb{C}, \quad \text{with } \langle au, v \rangle = \langle u, \phi(a)v \rangle$$

(When we say ‘lift’ we mean the new pairing satisfies  $\langle \mathbf{e}u, \mathbf{e}v \rangle_{\mathbf{e}} = \langle \mathbf{e}u, \mathbf{e}v \rangle$ .)

We recall Theorem 2.5.3, which says that the colored Jones polynomial  $J(L, n)$  of the knot  $L$  can be computed using the pairing  $\langle -, - \rangle_{\mathbf{e}}$ . Let  $S_n$  and  $T_n$  be the Chebyshev polynomials, i.e. the family of polynomials  $S_n$  and  $T_n$  determined by

$$S_0 = 1, \quad T_0 = 2, \quad T_1 = S_1 = x, \quad S_{n+1} = xS_n - S_{n-1}, \quad T_{n+1} = xT_n - T_{n-1}$$

Also, let  $1_L$  be the empty link in  $S^3 \setminus L$  and let  $u$  be the image of the the longitude in  $K_q(\text{solid torus})$ . Theorem 2.5.3 gives the equality

$$J(L, n) = \langle S_{n-1}(u), 1_L \rangle_{\mathbf{e}}$$

**Theorem 5.2.1.** *For  $n \in \mathbb{N}$ , the colored Jones polynomial is given by*

$$J(L, n) = 2\langle X^n, 1_L \rangle$$

*Proof.* First we claim that  $J(L, n) = \langle X^n - X^{-n}, 1_L \rangle$ . It suffices to show  $S_{n-1}(u) = X^n - X^{-n} \in V$ . We recall that the identification  $\mathbb{C}[u] \rightarrow \mathbf{e}V$  is determined by sending  $1 \mapsto \delta = X - X^{-1}$  and  $u \mapsto (X + X^{-1})\delta$ . The claim follows from the identity  $(X - X^{-1})S_{n-1}(X + X^{-1}) = X^n - X^{-n}$ , which is proved in the following lemma.

To see that this implies the result, we note that since  $1_L \in \mathbf{e}M$ , we have  $(1 - s)1_L = (1 - s)\mathbf{e}1_L = 0$ . By the contravariance of the pairing, we obtain

$$\langle X^n, (1 - s)1_L \rangle = \langle (1 - s)X^n, 1_L \rangle = \langle X^n + X^{-n}, 1_L \rangle = 0$$

(Remember that  $V$  is the induced sign representation, so  $s \cdot X^n = -X^{-n}$ .) This implies  $\langle X^n, 1_L \rangle = -\langle X^{-n}, 1_L \rangle$ , which finishes the proof.  $\square$

For completeness we prove the following identities which we use several times.

**Lemma 5.2.2.** *The Chebyshev polynomials satisfy  $S_{-n} = -S_{n-2}$  and  $T_{-n} = T_n$ . They also satisfy the following identities:*

1.  $T_n = S_n - S_{n-2}$
2.  $S_n = (1/2) \sum_{i=-n/2}^{n/2} T_{2i}$
3.  $T_n(X + X^{-1}) = X^n + X^{-n}$
4.  $(X - X^{-1})S_n(X + X^{-1}) = X^{n+1} - X^{-n-1}$

*Proof.* (We remark that the index  $i$  in the sum in the second identity is always a half-integer if  $n$  is odd.) For the base case for the first identity, we note that the polynomials for  $S_{-2} \cdots S_2$  are  $-x, -1, 0, 1, x$ , and it is easy to check these satisfy the recursion relation. We then assume  $n \geq 3$  and define  $S_{-n}$  via the formula  $S_{-n} := -S_{n-2}$  (which implies  $S_{-1} = 0$ ). We show that the  $S_{-n}$  satisfy the same recursion relation as the  $S_n$ :

$$\begin{aligned}
 S_{-n-1} &= -S_{n-1} \\
 &= -(xS_{n-2} - S_{n-3}) \\
 &= -(-xS_{-n} + S_{-n+1}) \\
 &= xS_{-n} - S_{-n+1}
 \end{aligned}$$

For the base case for the second identity, the polynomials  $T_{-1}, T_0, T_1$  are  $x, 2, x$ , and it is easy to see these satisfy the recursion relation. Now assume  $n \geq 1$  and

define  $T_{-n} = T_n$ . Then we compute

$$\begin{aligned}
T_{-n+1} &= T_{n-1} \\
&= xT_n - T_{n+1} \\
&= xT_{-n} - T_{-n-1}
\end{aligned}$$

We use these to show the remaining 4 identities.

1. For the base case,  $T_0 = 2 = 1 - (-1) = S_0 - S_{-2}$  and  $T_1 = x - 0 = S_1 - S_{-1}$ .

Suppose  $T_n = S_n - S_{n-2}$ . Then

$$T_{n+1} = xT_n - T_{n-1} = x(S_n - S_{n-2}) - (S_{n-1} - S_{n-3}) = S_{n+1} - S_{n-2}$$

2. For the base case,  $S_0 = 1 = (1/2)(2)$  and  $S_1(x) = x = (1/2)(x + x) = (1/2)(T_1(x) + T_{-1}(x))$ . Suppose  $S_n = (1/2) \sum_{i=-n/2}^{n/2} T_i$ . Then

$$\begin{aligned}
2S_{n+1} &= 2xS_n - 2S_{n-1} \\
&= x \left( \sum_{i=-n/2}^{n/2} T_{2i} \right) - \sum_{i=(-n+1)/2}^{(n-1)/2} T_{2i} \\
&= \sum_{i=-n/2}^{n/2} (T_{2i+1} + T_{2i-1}) - \sum_{i=(-n+1)/2}^{(n-1)/2} T_{2i} \\
&= \sum_{i=-n/2+1/2}^{n/2+1/2} T_{2i} + \sum_{i=-n/2-1/2}^{n/2-1/2} T_{2i} - \sum_{i=(-n+1)/2}^{(n-1)/2} T_{2i} \\
&= \sum_{i=-n/2-1/2}^{n/2+1/2} T_{2i}
\end{aligned}$$

3. For the base case,  $T_0 = 2 = X^0 + X^0$  and  $T_1(X + X^{-1}) = X + X^{-1}$ . Suppose  $T_n(X + X^{-1}) = X^n + X^{-n}$ . Then

$$\begin{aligned}
T_{n+1}(X + X^{-1}) &= (X + X^{-1})T_n(X + X^{-1}) - T_{n-1}(X + X^{-1}) \\
&= (X + X^{-1})(X^n + X^{-n}) - X^{n-1} - X^{-n+1} \\
&= X^{n+1} + X^{-n-1}
\end{aligned}$$

4. This follows from the previous two identities:

$$\begin{aligned}
(X - X^{-1})S_n(X + X^{-1}) &= (X - X^{-1})(1/2) \sum_{-n/2}^{n/2} T_{2i}(X + X^{-1}) \\
&= (X - X^{-1}) \sum_{-n/2}^{n/2} X^{2i} \\
&= X^{n+1} - X^{-n-1}
\end{aligned}$$

□

### 5.3 Recursion relations for the colored Jones polynomials

In [17], Garoufalidis and Lê proved that the colored Jones polynomials satisfy a linear recursion relation. In particular, if we define the colored Jones polynomials for negative integers via  $J(L, -n) = -J(L, n)$ , then we can think of  $J$  as an element of the vector space  $\text{Hom}(\mathbb{Z}, \mathbb{C})$ . There is a *right* action<sup>1</sup> of  $A_q \rtimes \mathbb{Z}_2$  on  $\text{Hom}(\mathbb{Z}, \mathbb{C})$  given by

$$(Xf)(n) = f(n + 1), \quad (Yf)(n) = -q^{-2n}f(n), \quad (s \cdot f)(n) = -f(-n)$$

and the main theorem of [17] is that there exists an  $a \in A_q$  such that  $J(L, -) \cdot a = 0$ . In [17] it isn't immediately obvious why this action should be related to the colored Jones polynomials, or why the extension  $J(L, -n) = -J(L, n)$  is used (instead of, e.g. a symmetric extension). In this section we explain how this action and extension arise naturally from the identity from Theorem 5.2.1

$$J(L, n) = 2\langle X^n, 1_L \rangle$$

---

<sup>1</sup>This is twisted from the action of [17] by the anti-automorphism of  $A_q \rtimes \mathbb{Z}_2$  sending  $Y \mapsto -Y^{-1}$  and fixing the other generators, but this doesn't affect any of the results described.

This identity already explains the occurrence of the extension  $J(L, -n) = -J(L, n)$ , since the proof of Lemma 5.2.1 shows  $\langle X^n, 1_L \rangle = -\langle X^{-n}, 1_L \rangle$ . Essentially, the sign in  $J(L, -n) = -J(L, n)$  comes from the fact that the lift of  $K_q(\text{unknot})$  is isomorphic to the sign representation of  $\mathbb{C}[Y^{\pm 1}] \rtimes \mathbb{Z}_2$ .

**Example 5.3.1.** To explain why the stated theorem can be interpreted as a recursion relation, suppose  $f : \mathbb{Z} \rightarrow \mathbb{C}$  and  $f \cdot (X^2 - XY^3 - Y^{-1}) = 0$ . Equivalently,

$$f(n+2) = q^{-6n}f(n+1) + q^{2n}f(n)$$

which can be viewed as a linear recursion relation for  $f$ .

We now explain how the action of  $A_q$  on  $\text{Hom}(\mathbb{Z}, \mathbb{C})$  arises naturally from skein theory. Let  $V$  and  $M_L$  be the lifts of  $K_q(S^3 \setminus \text{unknot})$  and  $K_q(S^3 \setminus L)$ , and let  $\langle -, - \rangle : V \otimes M_L \rightarrow \mathbb{C}$  be the pairing from Section 5.2.

**Lemma 5.3.2.** *We have  $(J(L, -) \cdot a)(n) = 2\langle aX^n v_-, 1_L \rangle$ .*

*Proof.* Recall from Section 5.2 that  $J(L, n) = 2\langle X^n v_-, 1_L \rangle$ . We then compute

$$\begin{aligned} (J \cdot X^j Y^k)(n) &= (-1)^k q^{-2kn} J(L, j+n) \\ &= 2(-1)^k q^{-2kn} \langle X^{j+n} v_-, 1_L \rangle \\ &= 2\langle X^j Y^k X^n, 1_L \rangle \end{aligned}$$

A similar computation for  $X^j Y^k s$  completes the proof. □

**Corollary 5.3.3.** *If  $a \in A_q$  annihilates  $1_L \in M_L$ , then  $J(L, -) \cdot \phi(a) = 0$ .*

*Proof.* The pairing  $\langle -, - \rangle$  is contravariant with respect to the anti-automorphism  $\phi : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  (see Section 2.5). We therefore have

$$(J(L, -) \cdot \phi(a))(n) = 2\langle \phi(a)X^n, 1_L \rangle = 2\langle X^n, a \cdot 1_L \rangle = 0$$

□

A very similar corollary appeared in [16], but the proof there seems slightly more involved than the proof here.

Let  $L \subset S^3$  be a knot and let  $M$  be the  $A_q \rtimes \mathbb{Z}_2$ -module such that  $\mathfrak{e}M \cong K_q(S^3 \setminus L)$ .

**Corollary 5.3.4.** *If  $K_q(S^3 \setminus L)$  is finitely generated as a module over  $\mathbb{C}[\text{meridian}]$ , then  $J(L, n)$  satisfies a linear recursion relation. In particular, if  $M$  is an object in  $\mathcal{O}^{\text{tw}}$ , then  $J(L, n)$  satisfies a linear recursion relation.*

*Proof.* If  $\mathfrak{e}M$  is finitely generated over  $\mathbb{C}[X + X^{-1}]$ , then  $M$  is finitely generated over  $\mathbb{C}[X]$ . By Lemma 4.3.8, for every  $m \in M$  there is an  $a \in A_q$  with  $a \cdot m = 0$ . Therefore the previous corollary applies.  $\square$

CHAPTER 6  
THE TREFOIL COMPLEMENT

In this chapter we give a detailed study of the skein module  $K_q(L) := K_q(S^3 \setminus L)$  of the complement of the trefoil  $L \subset S^3$ . In particular, the results in Chapter 4 imply that there is a unique  $A_q \rtimes \mathbb{Z}_2$ -module  $M$  such that the  $A_q^{\mathbb{Z}_2}$ -module  $\mathbf{e}M$  is isomorphic to the skein module  $K_q(L)$ . It turns out that  $M$  is the middle term of a (non-split) short exact sequence, and after reviewing a technique for describing non-split extensions, we give a completely explicit description of  $M$ . We recall a description of  $K_q(L)$  given by Gelca in [18], and we then check that  $\mathbf{e}M$  and  $K_q(L)$  are in fact isomorphic.

One general principle is that calculations at the level of  $A_q \rtimes \mathbb{Z}_2$  are much easier than at the level of  $A_q^{\mathbb{Z}_2}$ . To illustrate this, we give a short calculation of a closed formula for all the colored Jones polynomials of the trefoil, and we find a recursion relation satisfied by the colored Jones polynomials. (Both of these have been done before, but the calculations in our setting are very short.)

We also remark that finding a complete description of the lifted module  $M$  involved a lot of calculations by Y. Berest and the author. However, once this explicit description of  $M$  is found, it is relatively easy to show it is correct (i.e. to show that  $\mathbf{e}M \cong K_q(L)$ ). We take the shorter path of defining  $M$  and proving  $\mathbf{e}M \cong K_q(L)$ , and we omit the calculations leading to the description of  $M$ .

Finally, a result of the computations in this chapter is that the skein module  $K_q(U)$  of the unknot  $U$  is an  $A_q \rtimes \mathbb{Z}_2$  submodule of  $K_q(L)$ . We suspect that this has a nice topological interpretation, but we unfortunately do not provide one in this thesis.

## 6.1 Review about extensions

In this section we give a short review of the data required to specify an extension of (left) modules  $N, N'$  of an algebra  $A$ . We give an elementary summary of the facts which we need in the later sections (this ends at Formula (6.3)). For completeness, we then give a more precise statement using homological algebra.

Suppose that there is an exact sequence of left  $A$ -modules

$$0 \rightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} N' \rightarrow 0 \quad (6.1)$$

An  $A$ -module map  $\rho : N' \rightarrow M$  is called a *section* of  $\pi$  if  $\pi \circ \rho : N' \rightarrow N'$  is the identity map. In this case we say the short exact sequence *splits*, and the choice of a section determines an isomorphism  $M \cong N \oplus N'$ , where  $A$  acts on the vector space  $N \oplus N'$  diagonally. Of course, such a section  $\rho$  might not exist (for example, not all matrices are conjugate to diagonal matrices). However, we can pick a  $\mathbb{C}$ -linear section  $\rho_{\mathbb{C}} : N' \rightarrow M$ , which gives an isomorphism of vector spaces  $M \cong_{\mathbb{C}} N \oplus N'$ . We then want to answer the question “how can we describe the  $A$ -module structure on the vector space  $N \oplus N'$ ?”

Given the notation above, let  $\Psi : A \otimes_{\mathbb{C}} N' \rightarrow M$  be the  $\mathbb{C}$ -linear map  $\Psi(a, n') := a \cdot \rho_{\mathbb{C}}(n') - \rho_{\mathbb{C}}(a \cdot n')$ . Rearranging terms, we have the equality

$$a \cdot \rho_{\mathbb{C}}(n') = \rho_{\mathbb{C}}(a \cdot n') + \Psi(a, n')$$

We also claim that  $\Psi(a, n') \in \iota(N)$ . Precisely,

$$\pi(\Psi(a, n')) = \pi(a \cdot \rho_{\mathbb{C}}(n') - \rho_{\mathbb{C}}(a \cdot n')) = a \cdot \pi(\rho_{\mathbb{C}}(n')) - \pi(\rho_{\mathbb{C}}(a \cdot n')) = a \cdot n' - a \cdot n' = 0$$

Since the sequence (6.1) is exact, this shows that  $\Psi(a, n') \in \ker(\pi) = \iota(N)$ . Summarizing, the  $A$ -module structure on the vector space  $N \oplus N'$  is given by

$$a \cdot (n, n') = (a \cdot n + \Psi(a, n'), a \cdot n') \quad (6.2)$$

It is convenient to use the Hom-Tensor adjunction to view  $\Psi$  as a linear map  $\Psi : A \rightarrow \text{Hom}_{\mathbb{C}}(N', N)$ . The target is an  $A$ -bimodule using the rule  $(afb)(n') = af(bn')$ , and it is easy to compute how  $\Psi$  behaves with respect to the product in  $A$ :

$$\begin{aligned} \Psi(ab, n') &= ab \cdot \rho_{\mathbb{C}}(n') - \rho_{\mathbb{C}}(ab \cdot n') \\ &= a(b \cdot \rho_{\mathbb{C}}(n') - \rho_{\mathbb{C}}(b \cdot n')) + (a\rho_{\mathbb{C}}(b \cdot n') - \rho_{\mathbb{C}}(a(b \cdot n'))) \\ &= a\Psi(b, n') + \Psi(a, n')b \end{aligned}$$

In other words,  $\Psi : A \rightarrow \text{Hom}_{\mathbb{C}}(N', N)$  is a derivation. Furthermore, if  $\Psi : A \rightarrow \text{Hom}_{\mathbb{C}}(N', N)$  is any derivation, this computation in reverse shows that formula (6.2) gives an action of  $A$  on  $N \oplus N'$  (the Leibnitz rule shows this action is associative). We have therefore described a surjective map (of sets)

$$\text{Der}(A, \text{Hom}_{\mathbb{C}}(N', N)) \rightarrow \{\text{isomorphism classes of extensions of } N' \text{ by } N\} \quad (6.3)$$

For the sake of completeness, we now describe the homological algebra underlying this construction.

**Lemma 6.1.1.** *If  $N'$  and  $N$  are left  $A$ -modules, then*

$$\text{Ext}_A^n(N', N) \cong \text{Ext}_{A \otimes A^{\text{op}}}^n(A, \text{Hom}_{\mathbb{C}}(N', N))$$

*Proof.* We recall that to compute  $\text{Ext}_A^n(N', N)$ , one picks a projective resolution

$$\begin{array}{ccccccc} \cdots & \rightarrow & B_1 & \rightarrow & B_0 & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \rightarrow & 0 & \rightarrow & N' & & \end{array} \quad (6.4)$$

where the  $B_i$  are projective left  $A$ -modules and the vertical map gives a quasi-isomorphism of complexes. Then the Ext groups are the cohomology of the top

complex, i.e.  $\text{Ext}_A^n(N', N) \cong H^n(\text{Hom}_A(B_\bullet, N))$ . One way to obtain a resolution<sup>1</sup> of  $N'$  (as a left  $A$ -module) is to take a fixed resolution  $B(A)$  of  $A$  (as an  $A$ -bimodule), and then the tensor product  $B(A) \otimes_A N'$  is a resolution of  $N'$  (as a left module). Once such a resolution is chosen, the general version of the Hom-Tensor adjunction shows<sup>2</sup>

$$\text{Hom}_A(B_\bullet \otimes_A N', N) \cong \text{Hom}_{A \otimes A^{\text{op}}}(B_\bullet, \text{Hom}_C(N', N))$$

This completes the proof. □

To finish our goal of understanding the surjection (6.3) we recall the short exact sequence of  $A$ -bimodules defining the  $A$ -bimodule  $\Omega^1(A)$ :

$$0 \rightarrow \Omega^1(A) \rightarrow A \otimes A \xrightarrow{\text{mult}} A \rightarrow 0 \quad (6.5)$$

In other words, the bimodule  $\Omega^1(A)$  is the kernel of the multiplication map. (It is often called ‘non-commutative 1-forms,’ in analogy with the commutative case.) We also recall the following proposition from [19]:

**Proposition 6.1.2.** *For every  $A$ -bimodule  $M$ , there is a canonical isomorphism*

$$\text{Der}(A, M) \cong \text{Hom}_{A \otimes A^{\text{op}}}(\Omega^1(A), M)$$

If  $M$  is an  $A$ -bimodule, let  $Z(M)$  be the *center* of  $M$ , i.e. the submodule of elements  $m \in M$  satisfying  $am = ma$  for all  $a \in A$ .

**Lemma 6.1.3.** *For an  $A$ -bimodule  $M$ , there is an exact sequence*

$$0 \rightarrow Z(M) \rightarrow M \rightarrow \text{Der}(A, M) \rightarrow \text{Ext}_{A \otimes A^{\text{op}}}^1(A, M) \rightarrow 0$$

---

<sup>1</sup>It turns out that there is a very natural choice of a resolution  $B(A)$  of  $A$  called the *bar resolution*, which is important for many calculations. However, for our specific purpose it isn’t needed, so we won’t define it.

<sup>2</sup>We recall that  $A^{\text{op}}$  is the opposite algebra of  $A$ , i.e. the same vector space with the multiplication reversed. Using this notation, an  $A$ -bimodule is the same thing as a left  $A \otimes A^{\text{op}}$ -module.

*Proof.* Given an  $A$ -bimodule  $M$ , we apply the functor  $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(-, M)$  to the defining sequence (6.5) to obtain the long exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, M) &\rightarrow \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A \otimes A, M) \rightarrow \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(\Omega^1(A), M) \\ &\rightarrow \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^1(A, M) \rightarrow \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^1(A \otimes A, M) \rightarrow \cdots \end{aligned}$$

We can simplify most terms of this sequence. The first term is equal to  $Z(M)$  since an  $A$ -bimodule map from  $A$  is determined by the image of 1, and the identity  $a1 = 1a$  shows that the image of  $1 \in A$  must satisfy  $am = ma$  for all  $a$ . Since  $A \otimes A$  is a free rank 1  $A$ -bimodule, the second term is just  $M$  and the last term is zero. We apply Proposition 6.1.2 to the third term to finish the proof.  $\square$

The proof of Proposition 6.1.2 shows that the map  $M \rightarrow \mathrm{Der}(A, M)$  assigns the derivation  $a \mapsto am - ma$  to the element  $m \in M$ . Applying this to  $M = \mathrm{Hom}_{\mathbb{C}}(N', N)$  and combining this with Lemma 6.1.1 above, we obtain

**Lemma 6.1.4.** *There is a canonical map  $\mathrm{Der}(A, \mathrm{Hom}_{\mathbb{C}}(N', N)) \rightarrow \mathrm{Ext}_A^1(N', N)$  whose kernel is the subspace of inner derivations  $a \mapsto \mathrm{ad}_a(f)$ , where  $f \in \mathrm{Hom}_{\mathbb{C}}(N', N)$  and  $\mathrm{ad}_a(f)(n') = af(n') - f(an')$ .*

## 6.2 The module $M$

As mentioned above, the lift  $M$  of the skein module of the trefoil fits into a nonsplit short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow N' \rightarrow 0$$

We first describe  $N$  and  $N'$  and then give a derivation  $\Psi : A_q \rtimes \mathbb{Z}_2 \rightarrow \mathrm{Hom}_{\mathbb{C}}(N', N)$  which determines the module structure on the vector space  $N \oplus N'$  via the formula

$$a \cdot (n, n') := (an + \Psi(a, n'), an')$$

We write  $\Psi(a, n')$  instead of  $\Psi(a)(n')$ , and with this notation the Leibniz identity becomes  $\Psi(ab, n') = a\Psi(b, n') + \Psi(a, bn')$ .

**Remark 6.2.1.** Since the sequence is non-split and picking the derivation  $\Psi$  involves picking a  $\mathbb{C}$ -linear splitting, it will be important to distinguish between the action of  $A_q \rtimes \mathbb{Z}_2$  on the module  $N'$  and on its image in  $M$ . In this chapter we will exclusively write  $a \cdot_M m$  for the action of  $A_q \rtimes \mathbb{Z}_2$  on  $M$  whenever confusion would change the result. (For example, the splitting we pick to obtain  $\Psi$  is linear over  $\mathbb{C}[X^{\pm 1}]$ , so we will never need to write  $X \cdot_M m$  because  $X \cdot_M m = X \cdot m$ . Similarly, if  $n \in N$ , then  $a \cdot_M n = a \cdot n$  since  $N$  is an  $A_q \rtimes \mathbb{Z}_2$ -submodule of  $M$ .)

Using the notation of Section 4.3, we define  $N = P_0^-$  and  $N' = P_{-6}^+$ . We recall the explicit description: as  $\mathbb{C}[X^{\pm 1}]$ -modules we can identify  $N \cong \mathbb{C}[X^{\pm 1}]n_-$  and  $N' \cong \mathbb{C}[X^{\pm 1}]n'_+$ , with the action of  $s$  and  $Y$  given by

$$\begin{aligned} Y \cdot f(X)n_- &= -f(q^{-2}X)n_-, & s \cdot f(X)n_- &= -f(X^{-1})n_-, \\ Y \cdot f(X)n'_+ &= q^6 X^{-6} f(q^{-2}X)n'_+, & s \cdot f(X)n'_+ &= f(X^{-1})n'_+ \end{aligned}$$

**Remark 6.2.2.** A short calculation shows the action of  $Y^k$  is given by

$$Y^k \cdot f(X)n'_+ = q^{6k^2} X^{-6k} f(q^{-2k}X)n'_+$$

We give this explicit formula because it is useful later, and because it caused some confusion when these calculations were first performed.

**Definition 6.2.3.** We define  $\Psi : A_q \rtimes \mathbb{Z}_2 \rightarrow \text{Hom}_{\mathbb{C}}(N', N)$  using the formulas

$$\Psi(X, -) = \Psi(s, -) = 0, \quad \Psi(Y, X^k n'_+) = (q^{-2k} X^k)(q^2 X^{-1} - q^6 X^{-5})n_- \quad (6.6)$$

The module  $M$  is the extension  $0 \rightarrow N \rightarrow M \rightarrow N' \rightarrow 0$  determined by the derivation  $\Psi$ .

Using Lemma 6.2.4 below, these formulas extend uniquely to a derivation if  $(s + q^6 X^6 Y^{-1})\Psi(Y, n'_+) = 0$ . This is easy to check:

$$(s + q^6 X^6 Y^{-1})(q^2 X^{-1} - q^6 X^{-5}) = (-q^2 X + q^6 X^5) + q^6 X^6(-X^{-1} + q^{-4} X^{-5}) = 0$$

We also note that the Leibnitz rule implies  $\Psi(Y^{-1}, n') = -Y^{-1}\Psi(Y, Y^{-1}n')$ , and using this identity, a short calculation shows

$$\Psi(Y^{-1}, n'_+) = (q^6 X^5 - q^2 X)n_-$$

We now give a lemma which gives sufficient conditions for constructing an extension of an  $A_q \rtimes \mathbb{Z}_2$  module  $Q$  by  $P_k^\pm$ . (We work in this generality since the proof of the specific result we need is identical to the general proof.) We recall the module  $P_k^\pm$  is isomorphic to  $\mathbb{C}[X^{\pm 1}]$  as a  $\mathbb{C}[X^{\pm 1}]$ -module, with  $A_q \rtimes \mathbb{Z}_2$ -module structure determined by

$$Y \cdot f(X) = \pm q^{-k} X^k f(q^{-2} X), \quad s \cdot f(X) = \pm f(X^{-1})$$

We write  $p_\pm$  for the image of 1 under the  $\mathbb{C}[X^{\pm 1}]$ -linear isomorphism  $\mathbb{C}[X^{\pm 1}] \xrightarrow{\sim} P_k^\pm$ .

**Lemma 6.2.4.** *Fix  $k \in \mathbb{Z}$ . If  $Q$  is an  $A_q \rtimes \mathbb{Z}_2$ -module and  $m \in Q$  satisfies*

$$(s + q^{-k} X^{-k} Y^{-1}) \cdot m = 0$$

*then the formulas*

$$\Psi(X, -) = \Psi(s, -) = 0, \quad \Psi(Y, X^n p_\pm) = (q^{-2n} X^n) m \tag{6.7}$$

*uniquely extend to a derivation  $\Psi : A_q \rtimes \mathbb{Z}_2 \rightarrow \text{Hom}_{\mathbb{C}}(P_k^\pm, Q)$ .*

*Proof.* It is clear that such a derivation must be unique because it is defined on generators of the algebra. To prove that the formulas extend to a derivation, let

$\bar{X}, \bar{Y}, \bar{s}$  generate a free algebra  $F$ , and use the formulas (6.7) to define a derivation  $\bar{\Psi} : F \rightarrow \text{Hom}_{\mathbb{C}}(P_k^{\pm}, Q)$ , where the target is given an  $F$ -module structure through the projection  $F \rightarrow A_q \rtimes \mathbb{Z}_2$  which “removes bars.” Since  $F$  is free, these formulas extend to a derivation. To check that (6.7) defines a derivation on  $A_q \rtimes \mathbb{Z}_2$ , we need to check that the correct relations are satisfied by  $\bar{\Psi}$ . More precisely, if  $I \subset F$  is the kernel of the projection  $F \rightarrow A_q \rtimes \mathbb{Z}_2$ , we need to check  $\bar{\Psi}(I, -) = 0$ . For notational convenience we have left off the “bars” in this computation.

$$\begin{aligned}
\Psi(XY, X^n p_{\pm}) &= X\Psi(Y, X^n p_{\pm}) + \Psi(X, YX^n p_{\pm}) \\
&= X(q^{-2n} X^n)\Psi(Y, p_{\pm}) + 0 \\
&= q^2(q^{-2n-2} X^{n+1})\Psi(Y, p_{\pm}) \\
&= q^2\Psi(Y, X^{n+1} p_{\pm}) \\
&= q^2\Psi(YX, X^n p_{\pm}) + 0
\end{aligned}$$

In other words,  $\Psi(XY, -) = q^2\Psi(YX, -)$ , which shows the relation  $XY - q^2YX$  is satisfied. The two relations involving  $X$  and  $s$  are trivial, but the relation  $sY = Y^{-1}s$  is nontrivial. We first note that the inverse function rule implies  $\Psi(Y^{-1}, p) = -Y^{-1}\Psi(Y, Y^{-1}p)$  for all  $p \in P_k^{\pm}$ , and then compute

$$\begin{aligned}
\Psi(sY, X^n p_{\pm}) &= s\Psi(Y, X^n p_{\pm}) \\
&= sq^{-2n} X^n \Psi(Y, p_{\pm}) \\
&= q^{-2n} X^{-n} s\Psi(Y, p_{\pm})
\end{aligned}$$

We also compute

$$\begin{aligned}
\Psi(Y^{-1}s, X^n p_{\pm}) &= \Psi(Y^{-1}, sX^n p_{\pm}) \\
&= -Y^{-1}\Psi(Y, Y^{-1}sX^n p_{\pm}) \\
&= -Y^{-1}\Psi(Y, q^{-2n}X^{-n}q^{-k}X^{-k}p_{\pm}) \\
&= -Y^{-1}q^k X^{-n-k}\Psi(Y, p_{\pm}) \\
&= -q^{-2n}X^{-n}q^{-k}X^{-k}Y^{-1}\Psi(Y, p_{\pm})
\end{aligned}$$

Therefore, for the relation  $sY = Y^{-1}s$  to be satisfied, we require

$$(s + q^{-k}X^{-k}Y^{-1})\Psi(Y, p_{\pm}) = 0$$

This completes the proof because the assumption of the lemma was that the element  $\Psi(Y, p_{\pm}) = m \in Q$  satisfies this equation.  $\square$

### 6.3 Comparison with Gelca's description

In this section we recall Gelca's description of  $K_q(L)$  from [18] and then show that  $\mathfrak{e}M \cong K_q(L)$ . Before we do this we need to set up some notation. First we recall a presentation of  $A_q^{\mathbb{Z}_2}$  given in Section 4.2. We defined  $B_q$  to be the algebra generated by  $x, y, z$  with relations

$$\begin{aligned}
[x, y]_q &= (q^2 - q^{-2})z \\
[z, x]_q &= (q^2 - q^{-2})y \\
[y, z]_q &= (q^2 - q^{-2})x \\
qxyz &= q^2x^2 + q^{-2}y^2 + q^2z^2 - \left(\frac{1}{q} - q\right)^2 - \left(q + \frac{1}{q}\right)^2
\end{aligned}$$

There is an isomorphism  $B_q \rightarrow A_q^{\mathbb{Z}_2}$  given by

$$x \mapsto X + X^{-1}, \quad y \mapsto Y + Y^{-1}, \quad z \mapsto q^{-1}(XY + X^{-1}Y^{-1})$$

Gelca uses two versions of Chebyshev polynomials,  $S_n$  and  $T_n$ , which satisfy

$$S_0 = 1, \quad T_0 = 2, \quad T_1 = S_1 = x, \quad S_{n+1} = xS_n - S_{n-1}, \quad T_{n+1} = xT_n - T_{n-1}$$

### Gelca's results

In [18], it was shown that the skein module  $K_q(L)$  of the complement of the trefoil is isomorphic (as a  $\mathbb{C}[x]$ -module) to  $\mathbb{C}[x] \oplus \mathbb{C}[x]v$ , where  $x$  acts by multiplication. (This was also shown in [28] and [5].) Since the module is free over  $\mathbb{C}[x]$ , Lemma 4.2.2 shows that the module structure of  $K_q(L)$  is uniquely determined by the following formulas from [18], translated into our notation<sup>3</sup>:

$$\begin{aligned} y \cdot 1 &= q^6 S_6(x) + q^4 S_4(x)v - q^2 S_0(x)(1 + q^{-2}v) \\ z \cdot 1 &= q^5 S_5(x) + q^3 S_3(x)v \\ y \cdot v &= -S_0(x)(1 + q^{-2}v) - q^8 S_6(x) - q^6 S_4(x)v \\ z \cdot v &= -q^{-1} S_1(x)(1 + q^{-2}v) - q^7 S_5(x) - q^5 S_3(x)v \end{aligned}$$

These formulas seem unwieldy, at best. However, they can be massaged into a form that is much more manageable.

The first step is to use a slightly different  $\mathbb{C}[x]$ -basis for the module  $K_q(S^3 \setminus L)$ .

We define  $w := 1 + q^{-2}v$  and then compute

$$\begin{aligned} y \cdot 1 &= q^6 T_6(x) + (q^6 S_4(x) - q^2)w \\ z \cdot 1 &= q^5 T_5(x) + q^5 S_3(x)w \\ y \cdot w &= -(q^2 + q^{-2})w \\ z \cdot w &= -q^{-3} S_1(x)w \end{aligned}$$

---

<sup>3</sup>The first two formulas are Lemma 3, and the last two are Lemma 7 for  $q = 0$  and  $q = -1$ , respectively (our  $q$  is his  $t$ , and his  $q$  is an integer index, completely unrelated to our  $q$ ).

We also remark that  $1, w$  form a  $\mathbb{C}[x]$ -basis for  $K_q(L)$ . The main reason this new basis is useful is that the formulas make it obvious that  $w$  generates a (proper) submodule of  $K_q(L)$ . This submodule will correspond (under the Morita equivalence  $M \mapsto \mathbf{e}M$ ) to the submodule  $N \hookrightarrow M$ .

### The comparison

First, we remark that the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow N' \rightarrow 0$$

is sent (under the Morita equivalence  $M \mapsto \mathbf{e}M$ ) to a short exact sequence

$$0 \rightarrow \mathbf{e}N \rightarrow \mathbf{e}M \rightarrow \mathbf{e}N' \rightarrow 0$$

Since the derivation  $\Psi : A_q \rtimes \mathbb{Z}_2 \rightarrow \text{Hom}_{\mathbb{C}}(N', N)$  is linear over the subalgebra  $\mathbb{C}[s]$ , it restricts to a derivation  $\Psi : A_q^{\mathbb{Z}_2} \rightarrow \text{Hom}_{\mathbb{C}}(\mathbf{e}N', \mathbf{e}N)$ . Since it is also linear over  $\mathbb{C}[X^{\pm 1}]$ , the restricted derivation is linear over  $\mathbb{C}[x] = \mathbb{C}[X + X^{-1}]$ .

The  $\mathbb{C}[x]$  module structure of  $\mathbf{e}N$  is  $\mathbf{e}N \cong \mathbb{C}[x]\mathbf{e}(X - X^{-1})n_-$ , which is a free module of rank 1. (Since  $s \cdot n_- = -n_-$ , the subspace  $\mathbf{e}N$  is the subspace of anti-symmetric polynomials (times  $n_-$ ), and this subspace is freely generated over  $\mathbb{C}[x]$  by  $(X - X^{-1})n_-$ .) Similarly,  $\mathbf{e}N' \cong \mathbb{C}[x]n'_+$  as  $\mathbb{C}[x]$ -modules. Therefore, as a  $\mathbb{C}[x]$ -module,  $\mathbf{e}M$  is freely generated by the elements  $\mathbf{e}(X - X^{-1})n_-$  and  $\mathbf{e}n'_+$ .

**Theorem 6.3.1.** *The  $\mathbb{C}[x]$ -module isomorphism  $\mathbf{e}M \rightarrow K_q(L)$  determined by*

$$\mathbf{e}(X - X^{-1})n_- \mapsto w, \quad \mathbf{e}n'_+ \mapsto 1$$

*is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* By Lemma 4.2.2, it suffices to show that the generators  $y, z$  act on  $\mathbf{e}(X - X^{-1})n_-$  and  $\mathbf{e}n_+$  in the same way that they act on  $w$  and  $1$ , respectively. We first compute

$$\begin{aligned}
y \cdot \mathbf{e}(X - X^{-1})n_- &= (Y + Y^{-1})(X - X^{-1})n_- \\
&= (q^{-2}(XY - X^{-1}Y^{-1}) + q^2(XY^{-1} - X^{-1}Y))n_- \\
&= -(q^2 + q^{-2})(X - X^{-1})n_- \\
z \cdot \mathbf{e}(X - X^{-1})n_- &= q^{-1}(XY + X^{-1}Y^{-1})(X - X^{-1})n_- \\
&= q^{-1}(q^{-2}(X^2Y - X^{-2}Y^{-1}) + q^2(Y^{-1} - Y))n_- \\
&= -q^{-3}(X + X^{-1})(X - X^{-1})n_-
\end{aligned}$$

We next compute the action of the generators on  $\mathbf{e}n'_0$ :

$$\begin{aligned}
y \cdot_M \mathbf{e}n'_+ &= (Y + Y^{-1}) \cdot_M n'_+ \\
&= \Psi(Y + Y^{-1}, n'_+) + (q^6 X^6 + q^6 X^{-6})n'_+ \\
&= (q^2 X^{-1} - q^6 X^{-5} + q^6 X^5 - q^2 X)n_- + q^6 T_6(X + X^{-1})n'_+ \\
&= (q^6 S_4(X + X^{-1}) - q^2)(X - X^{-1})n_- + q^6 T_6(X + X^{-1})n'_+
\end{aligned}$$

(In this computation we made use of the relations for Chebyshev polynomials proved in Lemma 5.2.2. In particular, we used  $T_n(X + X^{-1}) = X^n + X^{-n}$  and  $(X - X^{-1})S_n(X + X^{-1}) = X^{n+1} - X^{-n-1}$ .) Finally,

$$\begin{aligned}
z \cdot_M \mathbf{e}n'_+ &= q^{-1}(XY + X^{-1}Y^{-1}) \cdot_M \mathbf{e}n'_+ \\
&= q^{-1}(X\Psi(Y, n'_+) + X^{-1}\Psi(Y^{-1}, n'_+) + q^6(X^5 + X^{-5})n'_+) \\
&= q^{-1}(q^2 - q^6 X^{-4} + q^6 X^4 - q^2)n_- + q^5 T_5(X + X^{-1})n'_+ \\
&= q^5 S_3(X + X^{-1})(X - X^{-1})n_- + q^5 T_5(X + X^{-1})n'_+
\end{aligned}$$

□

## 6.4 The colored Jones polynomials

In this section we compute all the colored Jones polynomials  $J(L, n)$  of the trefoil. Let  $V$  be the lift of  $K_q(S^3 \setminus \text{unknot})$ . We identify  $V$  with  $\mathbb{C}[X^{\pm 1}]v_-$  as a  $\mathbb{C}[X^{\pm 1}]$  module, and recall from Section 5.1 that the  $A_q \rtimes \mathbb{Z}_2$ -module structure is given by  $X \cdot f(X)v_- = Xf(X)v_-$ ,  $Y \cdot f(X)v_- = -f(q^{-2}X)v_-$ ,  $s \cdot f(X)v_- = -f(X^{-1})v_-$

We also recall from Section 5.2 that there is a topologically induced pairing

$$\langle -, - \rangle : V \otimes M \rightarrow \mathbb{C}, \quad \text{with } \langle av, m \rangle = \langle v, \phi(a) \cdot_M m \rangle$$

(Recall that  $\phi$  is the anti-involution of  $A_q \rtimes \mathbb{Z}_2$  fixing  $s$  which is determined by  $\phi(X) = Y^{-1}$  and  $\phi(Y) = X^{-1}$ .) Furthermore, Lemma 5.2.1 says that this pairing determines the Jones polynomials:

$$J(L, n) = 2\langle X^n, 1_L \rangle$$

Therefore, to compute  $J(L, n)$ , we must determine the pairing. The contravariance of the pairing (with respect to  $\phi$ ) makes this straightforward.

**Lemma 6.4.1.** *The values  $\langle v_-, n_- \rangle$  and  $\langle v_-, n'_+ \rangle$  uniquely determine the pairing  $\langle -, - \rangle : V \otimes_{\mathbb{C}} M \rightarrow \mathbb{C}$ .*

*Proof.* In this computation we use the fact that  $V$  is (freely) generated over  $\mathbb{C}[X^{\pm 1}]$  by  $v_0$ . The idea is to move powers of  $X$  “to the other side” to turn them into powers of  $Y$ , which then get “absorbed.” We compute

$$\begin{aligned} \langle f(X)v_-, g(X)n_- \rangle &= \langle g(Y^{-1})f(X)v_-, n_- \rangle \\ &= \langle f(g(q^2)X)g(-1)v_-, n_- \rangle \\ &= g(-1)\langle v_-, f(g(q^2)Y^{-1})n_- \rangle \\ &= g(-1)f(-g(q^2))\langle v_-, n_- \rangle \end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\langle f(X)v_-, g(X)n'_+ \rangle &= \langle g(Y^{-1})f(X)v_-, n'_+ \rangle \\
&= g(-1)\langle f(q^2X)v_-, n'_+ \rangle \\
&= g(-1)\langle v_-, f(q^2Y^{-1}) \cdot_M n'_+ \rangle \\
&= g(-1)\langle v_-, \Psi(f(q^2Y^{-1}), n'_+) + F(X)n'_+ \rangle \\
&= g(-1)\langle v_-, G(X)n_- + F(X)n'_+ \rangle \\
&= g(-1)\langle G(Y^{-1})v_-, n_- \rangle + g(-1)\langle F(Y^{-1})v_-, n'_+ \rangle \\
&= g(-1)G(-1)\langle v_-, n_+ \rangle + g(-1)F(-1)\langle v_-, n'_+ \rangle
\end{aligned}$$

Here  $F(X)$  and  $G(X)$  are some Laurent polynomials in  $X$  that are hard to write explicitly in general. However, these calculations are enough to show that the pairing  $\langle v, m \rangle$  of any two elements can be written as a linear combination of  $\langle v_-, n_- \rangle$  and  $\langle v_-, n'_+ \rangle$ .  $\square$

**Remark 6.4.2.** This lemma is equivalent to the statement that the vector space  $V_0^- \otimes M$  has dimension at most 2, and that in particular  $v_- \otimes n_-$  and  $v_- \otimes n'_+$  span  $V_0^- \otimes M$ . The next two lemmas show that the dimension is exactly 1.

**Lemma 6.4.3.** *If  $L \subset S^3$  is a knot with  $1_L \in K_q(S^3 \setminus L)$  the empty link, and  $Q$  is the  $A_q \rtimes \mathbb{Z}_2$ -module with  $\mathfrak{e}Q \cong K_q(S^3 \setminus L)$ , then*

$$\langle v_-, 1_L \rangle = 0$$

*In particular,  $\langle v_-, n'_+ \rangle = 0$ .*

*Proof.* Since  $1_L \in \mathfrak{e}M$ , we have  $s \cdot 1_L = 1_L$ . Since  $s \cdot v_- = -v_-$ , we have

$$\langle -v_-, 1_L \rangle = \langle sv_-, 1_L \rangle = \langle v_-, s1_L \rangle = \langle v_-, 1_L \rangle$$

$\square$

**Lemma 6.4.4.** *We have  $\langle v_-, n_- \rangle = \frac{1}{2(q^2 - q^6)}$ .*

*Proof.* For this proof we must use a smidgen of topology. We recall that if  $1_V$  and  $1_M$  are the empty links in the solid torus and trefoil complement, then their pairing is the empty link in the sphere. Under our isomorphism  $K_q(S^3) \rightarrow \mathbb{C}$ , the empty link gets sent to 1. Since the empty link in  $V$  is  $(X - X^{-1})v_-$  and the empty link in  $M$  is  $n'_+$ , we have the evaluation

$$\begin{aligned}
1 &= \langle 1_V, 1_M \rangle_{\mathbf{e}} \\
&= \langle (X - X^{-1})v_-, n'_+ \rangle \\
&= \langle v_-, (Y^{-1} - Y) \cdot_M n'_+ \rangle \\
&= \langle v_-, (q^6 X^5 - q^2 X - q^2 X^{-1} + q^6 X^{-5})n_- + F(X)n'_+ \rangle \\
&= \langle (q^6(Y^5 + Y^{-5}) - q^2(Y + Y^{-1}))v_-, n_- \rangle \\
&= 2(q^2 - q^6)\langle v_-, n_- \rangle
\end{aligned}$$

Going from the fourth to the fifth line we have used the previous lemma, which allows us to avoid the computation of the exact value of  $F(X)$  (even though this computation is easy).  $\square$

We can now finally compute the colored Jones polynomials:

**Theorem 6.4.5.** *The colored Jones polynomial  $J(L, n)$  of the trefoil is*

$$\frac{1}{q^2 - q^6} \sum_{i=1}^n (-1)^i q^{6(n^2 - i^2)} (q^{10i-4} - q^{2i}) \tag{6.8}$$

**Remark 6.4.6.** Note that the difference between the powers of  $q$  in the polynomial  $q^{10i-4} - q^{2i}$  is  $8i - 4$ , which is divisible by 4. This implies each term in the sum is indeed a Laurent polynomial.

*Proof.* We use the fact that  $\Psi$  is a  $\mathbb{C}[X^{\pm 1}, s]$ -linear derivation to compute

$$\begin{aligned}
J_n &= 2\langle X^n v_-, n'_+ \rangle \\
&= 2\langle v_-, Y^{-n} \cdot_M n'_+ \rangle \\
&= 2\langle v_-, \Psi(Y^{-n}, n'_+) + (q^{6n^2} X^{6n}) n'_+ \rangle \\
&= 2 \sum_{i=1}^n \langle v_-, Y^{1-i} \Psi(Y^{-1}, Y^{i-n} n'_+) \rangle \\
&= 2 \sum_{i=1}^n q^{6(n-i)^2} \langle v_-, Y^{1-i} \Psi(Y^{-1}, X^{6(n-i)} n'_+) \rangle \\
&= 2 \sum_{i=1}^n q^{6(n-i)^2} q^{12i(n-i)} \langle v_-, Y^{1-i} \Psi(Y^{-1}, n'_+) \rangle \\
&= 2 \sum_{i=1}^n q^{6(n^2-i^2)} \langle v_-, Y^{1-i} (q^6 X^5 - q^2 X) n_- \rangle \\
&= 2 \sum_{i=1}^n q^{6(n^2-i^2)} \langle v_-, (-1)^{i-1} (q^{10i-4} X^5 - q^{2i} X) n_- \rangle \\
&= \frac{1}{q^2 - q^6} \sum_{i=1}^n (-1)^i q^{6(n^2-i^2)} (q^{10i-4} - q^{2i})
\end{aligned}$$

□

We now recall a formula for the colored Jones polynomial  $J(s, t, n)$  of the  $(s, t)$  torus knot from Hikami [22, Prop. 2] which was recalled from Morton [32]. To match our conventions, we replace their  $q$  by  $q^{-4}$ , multiply by  $(q^{2n} - q^{-2n}) / (q^2 - q^{-2})$  (which is the colored Jones polynomial of the unknot), and then multiply by  $(-1)^n$ . After doing this, we obtain

$$J(s, t, n) = (-1)^n \frac{q^{st(n^2-1)}}{q^2 - q^{-2}} \sum_{r=-(n-1)/2}^{(n-1)/2} q^{-4(str^2 - (s+t)r + 1/2)} - q^{-4(str^2 - (s-t)r - 1/2)}$$

Specializing to the case  $(s, t) = (2, 2p + 1)$ , we obtain

$$J(p, n) = (-1)^n \frac{q^{2(2p+1)(n^2-1)}}{q^2 - q^{-2}} \sum_{r=-(n-1)/2}^{(n-1)/2} q^{-8(2p+1)r^2 + 4(2p+3)r - 2} - q^{-8(2p+1)r^2 + 4(1-2p)r + 2}$$

Using some computations involving  $(2, 2p+1)$  torus knots that were done after this thesis was written and that will appear in another document, we define

$$J'(p, n) = \frac{q^{2(2p+1)n^2}}{q^2 - q^{-2}} \sum_{i=1}^n (-1)^{i+1} q^{-2(2p+1)i^2} (q^{-4p-4+4ip+6i} - q^{-4p+4ip-2i}) \quad (6.9)$$

**Lemma 6.4.7.** *There is an equality  $J(p, n) = J'(p, n)$ . In particular, the formula in Theorem 6.4.5 agrees (up to rescaling) with Morton's formula in [32].*

*Proof.* Equality will be established by a careful term-by-term comparison of the two formulas. Towards this end, we define

$$\begin{aligned} R_a(k) &:= q^{p(-16k^2+8k-4)-8k^2+12k-4} \\ R_b(k) &:= -q^{p(-16k^2-8k-4)-8k^2+4k} \\ I_a(k) &:= (-1)^{k+n} q^{p(-4k^2+4k-4)-2k^2+6k-4} \\ I_b(k) &:= (-1)^{k+n+1} q^{p(-4k^2+4k-4)-2k^2-2k} \end{aligned}$$

After some high school algebra, we see

$$\begin{aligned} J(p, n) &= (-1)^n \frac{q^{2(2p+1)(n^2)}}{q^2 - q^{-2}} \sum_{r=-(n-1)/2}^{(n-1)/2} (R_a(r) + R_b(r)) \\ J'(p, n) &= (-1)^n \frac{q^{2(2p+1)(n^2)}}{q^2 - q^{-2}} \sum_{i=1}^n (I_a(i) + I_b(i)) \end{aligned}$$

Therefore, it suffices to find a bijection between the sets  $\{R_a(r), R_b(r)\}$  and  $\{I_a(i), I_b(i)\}$ . After some more high school algebra, we obtain the equalities

$$\begin{aligned} R_a(r) &= (-1)^{2r+n+1} I_a(2r) \\ R_b(r) &= (-1)^{2r+n+1} I_a(2r+1) \\ R_a(-r) &= (-1)^{2r+n+1} I_b(2r+1) \\ R_b(-r) &= (-1)^{2r+n+1} I_b(2r) \end{aligned}$$

First, we assume  $n$  is even, and note that the index  $r$  in the sum for  $J(p, n)$  ranges over half-integers, so  $2r + n + 1$  is always even. Now as  $r$  ranges over the set  $\{r = (n - 1)/2 + k \mid k \in \mathbb{Z}, 0 < r \leq (n - 1)/2\}$ , the two values  $i = 2r$  and  $i = 2r + 1$  range over the set  $\{1, \dots, n\}$ . Therefore, each term  $I_a(i)$  is equal to exactly one of the  $R_a(r)$  or  $R_b(r)$  for positive  $r$ . Similarly, each  $I_b(i)$  is equal to exactly one of the  $R_a(r)$  when  $r$  is negative. If  $n$  is even, then  $r$  is never 0, and this shows  $J'(p, n) = J(p, n)$  when  $n$  is even.

Now we assume  $n$  is odd, and note that  $2r + n + 1$  is even since in this case  $r$  is an integer. If  $r = 0$ , then  $R_b(0) + R_a(0) = I_a(1) + I_b(1)$ . Now as  $r$  ranges over  $\{1, \dots, (n - 1)/2\}$ , the indices  $i = 2r$  and  $i = 2r + 1$  range over  $\{2, \dots, n\}$ , so the  $I_a(i)$  terms for  $i \geq 2$  are in bijection with the  $R_a(r)$  and  $R_b(r)$  terms when  $r > 0$ . Similarly, the  $I_b(i)$  terms for  $i \geq 0$  are in bijection with the  $R_a(r)$  and  $R_b(r)$  terms when  $r < 0$ , which shows  $J'(p, n) = J(p, n)$ .

The final claim in the statement follows from the observation that the trefoil is the  $(2, 3)$  torus knot and the fact that  $J'(p, n)$  defined in (6.9) simplifies to the expression in (6.8) when  $p = 1$ . □

## 6.5 A recursion relation for the colored Jones polynomials

In this section we compute a recursion relation for the colored Jones polynomials of the trefoil. We recall from Section 5.3 that to produce such a recursion relation it suffices to find an element  $a \in A_q$  such that  $a \cdot 1_L = 0$  (where  $1_L$  is the empty link in the complement of  $L$ ). With our explicit description of  $M$  this is actually quite easy. The idea is to multiply by an element  $a \in A_q$  such that  $a \cdot 1_L$  is contained in a proper submodule of  $A_q 1_L$  and to repeat this until the proper submodule is 0.

In the case of the trefoil, there is exactly one proper submodule of  $M$ , so this idea only takes two steps.

**Lemma 6.5.1.** *The colored Jones polynomial  $J(L, n)$  of the trefoil satisfies the recursion relation*

$$\begin{aligned} J(L, n + 3) &= (q^{30+12n} - q^2 - q^{10}) J(L, n + 2) \\ &\quad + ((q^{16} + q^8)q^{12(n+1)} - q^{12}) J(L, n + 1) \\ &\quad + q^{18+12n} J(L, n) \end{aligned}$$

*Proof.* First we compute

$$(Y^{-1} - q^6 X^6) \cdot n'_+ = (q^6 X^5 - q^2 X)n_- + q^6 X^6 n'_+ - q^6 X^6 n'_+ = (q^6 X^5 - q^2 X)n_-$$

We then note that since  $Y \cdot n_- = -n_-$ , we have  $(Y^{-1} + q^{2k})X^k n_- = 0$ . Therefore, the element  $a = (Y^{-1} + q^2)(Y^{-1} + q^{10})(Y^{-1} - q^6 X^6)$  satisfies  $a \cdot 1_L = 0$ . Corollary 5.3.3 then implies  $\phi(a) = (X - q^6 Y^{-6})(X + q^{10})(X + q^2)$  annihilates  $J(L, n)$ . Expanding this element out and writing the action on  $J(L, n)$  explicitly, we obtain the claim.  $\square$

**Remark 6.5.2.** As a sanity check, this recursion relation has been experimentally verified in Mathematica for  $-50 < n < 50$ .

## CHAPTER 7

### THE DOUBLE AFFINE HECKE ALGEBRA

In the 90's Cherednik introduced **double affine Hecke algebras**, or DAHAs, which are a family of algebras attached to each root system. These algebras have found applications in many areas of mathematics. In particular Cherednik used special properties of these algebras to prove the Macdonald conjectures, which involved families of symmetric polynomials attached to root systems. In this chapter we recall some basic facts about the double affine Hecke algebra  $\mathcal{H}_{q,t}$  of type  $A_1$ .

This is a 2-parameter family of algebras that is closely related to the quantum torus. In particular, if  $t = 1$  then there is an isomorphism  $\mathcal{H}_{q,t=1} \cong A_q \rtimes \mathbb{Z}_2$ , so  $\mathcal{H}_{q,t}$  can be viewed as a deformation of  $A_q \rtimes \mathbb{Z}_2$ . Many of the useful properties of  $A_q \rtimes \mathbb{Z}_2$  survive the deformation, and we describe some of them below. For brevity we will restrict the discussion to properties of  $\mathcal{H}_{q,t}$  that we will use in the subsequent two chapters.

#### 7.1 Basic properties of $\mathcal{H}_{q,t}$

**Definition 7.1.1.** Let  $q, t \in \mathbb{C}^\times$ , with  $q$  not a root of unity and  $t \neq \pm i$ . Define  $\mathcal{H}_{q,t}$  to be the algebra generated by  $X^{\pm 1}$ ,  $Y^{\pm 1}$ , and  $T$  subject to the relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad XY = q^2YXT^2, \quad (T - t)(T + t^{-1}) = 0 \quad (7.1)$$

There are several remarks/observations that are appropriate to mention here. First, we have replaced the  $q$  that is standard in the third relation with  $q^2$  to agree with our conventions for the quantum torus and the standard conventions for the

skein relations in Figure 8.1. Second, even though we have not required that  $T$  be invertible, its invertibility is a formal consequence of the relations. Indeed, it is easy to check (using the fourth relation) that  $T^{-1} = T + t^{-1} - t$ .

We also point out that there is not a typo in the first two relations - there is no  $T^{-1}$  in either relation. (This is worth mentioning because the first two relations look similar to relations that occur in the semidirect product of an algebra with a group that acts on that algebra. However, the elements  $1, T \in \mathcal{H}_{q,t}$  form a group only when  $t = \pm 1$ .) Finally, and most importantly for our purposes, if we set  $t = 1$ , then the fourth relation reduces to  $T^2 = 1$  and the third relation becomes  $XY = q^2 YX$ . These imply that  $\mathcal{H}_{q,1} \cong A_q \rtimes \mathbb{Z}_2$ , where the generator of  $\mathbb{Z}_2$  acts on  $A_q$  via the automorphism  $\theta$  which inverts  $X$  and  $Y$  (as before).

### Spherical algebra

If  $t \neq \pm i$ , the algebra  $\mathcal{H}_{q,t}$  contains the idempotent  $\mathbf{e} := (1 + tT)/(1 + t^2)$  (the identity  $\mathbf{e}^2 = \mathbf{e}$  is equivalent to the last relation in (9.1)). We define the *spherical subalgebra* of  $\mathcal{H}_{q,t}$  by

$$\mathcal{H}_{q,t}^+ := \mathbf{e} \mathcal{H}_{q,t} \mathbf{e} \quad (7.2)$$

Note that  $\mathcal{H}_{q,t}^+$  inherits its additive and multiplicative structure from  $\mathcal{H}_{q,t}$ , but the identity element of  $\mathcal{H}_{q,t}^+$  is  $\mathbf{e}$ , which is different from  $1 \in \mathcal{H}_{q,t}$ .

At least in the two cases: (a)  $q^n \neq 1$  and  $t \neq \pm i$ , and (b)  $q = 1$  and  $t \notin \{\pm 1, \pm i\}$ , [40, Thm. 3.6.9] that  $\mathcal{H}_{q,t}^+$  is Morita equivalent to  $\mathcal{H}_{q,t}$ ; the mutually inverse equivalences are given by

$$\text{Mod } \mathcal{H}_{q,t} \rightarrow \text{Mod } \mathcal{H}_{q,t}^+, \quad M \mapsto \mathbf{e}M \quad (7.3)$$

$$\text{Mod } \mathcal{H}_{q,t}^+ \rightarrow \text{Mod } \mathcal{H}_{q,t}, \quad M \mapsto \mathcal{H}_{q,t} \mathbf{e} \otimes_{\mathcal{H}_{q,t}^+} M \quad (7.4)$$

In the case  $t = 1$ , there is an isomorphism  $A_q^{\mathbb{Z}_2} \cong \mathcal{H}_{q,1}^+$  given by  $w \mapsto \mathbf{e}w\mathbf{e}$ , where  $w \in A_q^{\mathbb{Z}_2}$  is a symmetric word in  $X, Y$ . For later use we will need a presentation of  $\mathcal{H}_{q,t}^+$  which we give here. Let  $B'_q$  be the algebra generated by  $x, y, z$  modulo the following relations (where we have written  $[a, b]_q = qab - q^{-1}ba$ ):

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x \quad (7.5)$$

Also, define  $B_{q,t}$  to be the quotient of  $B'_q$  by the additional relation

$$q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz = \left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2 \quad (7.6)$$

**Remark 7.1.2.** The element on the left hand side of (8.3) is central in  $B'_q$ . This can be verified by a (tedious) algebraic calculation or by a shorter topological calculation (see Corollary 8.1.5).

**Theorem 7.1.3.** *There is an algebra isomorphism  $f : B_{q,t} \rightarrow \mathcal{H}_{q,t}^+$  defined by the following formulas:*

$$f(x) = (X + X^{-1})\mathbf{e}, \quad f(y) = (Y + Y^{-1})\mathbf{e}, \quad f(z) = q^{-1}(t^{-2}XY + YX)\mathbf{e} \quad (7.7)$$

*Proof.* This is proved in [26] and in [13]. However, for future reference we give some of the identities that are involved in showing that this is an algebra map. All of these identities are straightforward computations which we omit.

$$\begin{aligned} [\mathbf{e}, X + X^{-1}] &= 0 \\ [\mathbf{e}, Y + Y^{-1}] &= 0 \\ YX\mathbf{e} &= q^{-2}XYt^{-2}\mathbf{e} \\ Y^{-1}X^{-1}\mathbf{e} &= q^{-2}(X^{-1}Y^{-1} + (t^{-2} - 1)XY)\mathbf{e} \\ Y^{-1}X\mathbf{e} &= q^2(XY^{-1} + (1 - t^{-2})XY)\mathbf{e} \\ X^{-1}Y\mathbf{e} &= (q^{-2}YX^{-1} + q^{-4}(t^{-2} - 1)XY)\mathbf{e} \end{aligned}$$

Combining all these computations leads to the identity

$$[(X + X^{-1})\mathbf{e}, (Y + Y^{-1})\mathbf{e}]_q = (q^2 - q^{-2}) (q^{-1}t^{-2}XY + q^{-1}X^{-1}Y^{-1}) \mathbf{e}$$

This shows that the map  $f : B_{q,t} \rightarrow \mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$  satisfies the first relation of  $B_q$ , and it also shows that the element  $t^{-2}XY + X^{-1}Y^{-1}$  commutes with  $\mathbf{e}$ .

□

### The Poincarè-Birkhoff-Witt property

One of the key properties of  $\mathcal{H}_{q,t}$  is the so-called PBW property, which says that, for all  $q, t$ , the multiplication map on  $\mathcal{H}_{q,t}$  yields a linear isomorphism

$$\mathbb{C}[X^{\pm 1}] \otimes \frac{\mathbb{C}[T]}{(T-t)(T+t^{-1})} \otimes \mathbb{C}[Y^{\pm 1}] \xrightarrow{\sim} \mathcal{H}_{q,t} .$$

Another way of stating this property is: the elements  $\{X^n T^\varepsilon Y^m : m, n \in \mathbb{Z}, \varepsilon = 0, 1\}$  form a linear basis in  $\mathcal{H}_{q,t}$ . See [10], Theorem 2.5.6.

**Remark 7.1.4.** From the commutation relations, it is obvious that we can move  $T$  to the right of  $X$ , since we have the relation  $TX = X^{-1}T^{-1}$ . However, it is slightly more subtle to move  $T$  past  $X^{-1}$ . To do this, we use the fact that we can move  $T^{-1}$  to the right of  $X^{-1}$ . Specifically, write  $T = T^{-1} + t - t^{-1}$  and then compute  $TX^{-1} = (T^{-1} + t - t^{-1})X^{-1} = XT + (t - t^{-1})X^{-1}$ . A similar remark holds for  $Y$ .

### Standard representations

Our definition of  $\mathcal{H}_{q,t}$  was in terms of generators and relations. However,  $\mathcal{H}_{q,t}$  can also be viewed as a family of subalgebras of  $\text{End}_{\mathbb{C}}(\mathbb{C}[x, x^{-1}])$  using the construction

we describe in the next section. (The explicit embedding is given in formula (7.9).) The double affine Hecke algebra  $\mathcal{H}_{q,t}$  contains two canonical subalgebras  $\mathcal{H}_Y$  and  $\mathcal{H}_X$ , which are isomorphic to the affine Hecke algebra of  $\mathbb{Z} \rtimes \mathbb{Z}_2$ : namely,

$$\mathcal{H}_Y := \mathbb{C}\langle Y^{\pm 1}, T \rangle / \{TY^{-1}T = Y, (T-t)(T+t^{-1}) = 0\},$$

and similarly for  $\mathcal{H}_X$ . A representation  $V$  of  $\mathcal{H}_Y$  induces a representation of  $\mathcal{H}_{q,t}$ :

$$M_{q,t}(V) := \text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(V) = \mathcal{H}_{q,t} \otimes_{\mathcal{H}_Y} V \quad (7.8)$$

If  $V$  is a (finite-dimensional) irreducible representation of  $\mathcal{H}_Y$ , the corresponding induced representation (7.8) is usually called *standard*. Such representations play a fundamental role in all applications of  $\mathcal{H}_{q,t}$ . In the next section we compute explicit formulas for two of the more important standard modules.

## 7.2 Induced representations

In this section we give explicit formulas for the action of  $\mathcal{H}_{q,t}$  on the induced representations  $\text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_t)$  and  $\text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_{-t^{-1}})$ . These formulas were found by Cherednik, but we give proofs for completeness.

Let  $\epsilon \in \{1, -1\}$ , and suppose  $\hat{s}, \hat{y} : \mathbb{C}[X^{\pm 1}] \rightarrow \mathbb{C}[X^{\pm 1}]$  are defined by

$$\hat{s}(g(X)) = \epsilon g(X^{-1}), \quad \hat{y}(g(X)) = \epsilon g(q^{-2}X)$$

Define the operators  $Y, T : \mathbb{C}[X^{\pm 1}] \rightarrow \mathbb{C}[X^{\pm 1}]$  via

$$T := t^\epsilon \hat{s} + \frac{t-t^{-1}}{X^2-1}(\epsilon \hat{s} - 1), \quad Y := \hat{y} \hat{s} T \quad (7.9)$$

We remark that the given formula for  $T$  is naturally an operator on rational functions  $\mathbb{C}(X)$ . However,  $T$  preserves the subspace  $\mathbb{C}[X^{\pm 1}] \hookrightarrow \mathbb{C}(X)$  - indeed, we

have  $(\epsilon\hat{s} - 1)(f(X)) = f(X^{-1}) - f(X)$  and this is always divisible (in  $\mathbb{C}[X^{\pm 1}]$ ) by  $X^2 - 1$ . We also remark that the operators  $X, \hat{s}, \hat{y}$  give a representation of  $A_q \rtimes \mathbb{Z}_2$ .

**Lemma 7.2.1.** *The operators  $X, Y, T : \mathbb{C}[X^{\pm 1}] \rightarrow \mathbb{C}[X^{\pm 1}]$  satisfy the relations of the  $\mathcal{H}_{q,t}$ . Furthermore, the  $\mathcal{H}_{q,t}$ -module  $\mathbb{C}[X^{\pm 1}]$  is isomorphic to  $\text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_{\epsilon t^\epsilon})$ .*

*Proof.* First we assume that the formulas (7.9) give  $\mathbb{C}[X^{\pm 1}]$  an  $\mathcal{H}_{q,t}$ -module structure. Let  $P = \text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_{\epsilon t^\epsilon})$ , with  $1_P \in P$  a generator of  $\mathbb{C}_{\epsilon t^\epsilon}$ . By the PBW property, there is a  $\mathbb{C}[X^{\pm 1}]$ -module isomorphism  $\mathbb{C}[X^{\pm 1}] \rightarrow P$  induced by  $1 \mapsto 1_P$ . Since  $Y$  and  $T$  act on  $1$  and  $1_P$  in the same way, Lemma 4.2.2 shows  $\mathbb{C}[X^{\pm 1}]$  and  $P$  are isomorphic as  $\mathcal{H}_{q,t}$ -modules. (Technically Lemma 4.2.2 only applies to the spherical subalgebra, but the same proof gives the same statement for  $\mathcal{H}_{q,t}$ .)

Now we show that the formulas (7.9) give an  $\mathcal{H}_{q,t}$ -module structure. First, we note that  $\hat{s}$  and  $1$  commute with the operator  $X + X^{-1}$ , and therefore commute with elements of the form  $f(X + X^{-1})$ . Since any Laurent polynomial  $f(X)$  can be written in the form  $g(X + X^{-1}) + h(X + X^{-1})(X - X^{-1})$ , it suffices to check the claimed identities on the elements  $1$  and  $X - X^{-1}$  in  $\mathbb{C}[X^{\pm 1}]$ .

The relation  $(T - t)(T + t^{-1})(1) = 0$  is clear because  $T(1) = t^\epsilon \hat{s}(1) = \epsilon t^\epsilon$ . To check the second, we compute:

$$\begin{aligned}
TXT(1) &= \epsilon t^\epsilon TX(1) \\
&= \epsilon t^\epsilon T(X) \\
&= \epsilon t^\epsilon (\epsilon t^\epsilon X^{-1} + (t - t^{-1})(-X^{-1})) \\
&= \epsilon t^\epsilon (\epsilon t^\epsilon - t + t^{-1})(X^{-1}) \\
&= \epsilon t^\epsilon (\epsilon t^{-\epsilon})(X^{-1}) \\
&= X^{-1}
\end{aligned}$$

(The second to last equality is easiest to check by checking for  $\epsilon = 1$  and  $\epsilon = -1$  separately.) Now to check the identities applied to  $X - X^{-1}$ , we compute

$$(T + t^{-1})(X - X^{-1}) = t^{-1}(X - X^{-1}) + \epsilon t^\epsilon(X^{-1} - X) + 2(t - t^{-1})X^{-1}$$

If  $\epsilon = 1$ , the RHS is equal to  $(t - t^{-1})(X + X^{-1})$ . Since  $T$  commutes with symmetric polynomials and  $(T - t)(1) = 0$  (if  $\epsilon = 1$ ), we have proved  $(T - t)(T + t^{-1})(X - X^{-1}) = 0$  if  $\epsilon = 1$ . Similarly

$$(T - t)(X - X^{-1}) = (t^{-1} - t)(X + X^{-1}) \quad \text{if } \epsilon = -1$$

and in this case,  $(T + t^{-1})(1) = 0$ . We have therefore proved  $(T - t)(T + t^{-1}) = 0$  on all of  $\mathbb{C}[X^{\pm 1}]$ .

This relation formally implies  $T$  is invertible and  $T^{-1} = T + t^{-1} - t$ . Indeed, we can check

$$T(T + t^{-1} - t) = T^2 + (t^{-1} - t)T = (t - t^{-1})T + 1 + (t^{-1} - t)T = 1$$

Therefore, we can slightly simplify the last identity we have to check by multiplying on the right by  $T^{-1}$  and applying both sides to  $X - X^{-1}$ :

$$\begin{aligned} TX(X - X^{-1}) &= T(X^2 - 1) \\ &= \epsilon t^\epsilon(X^{-2} - 1) + (t - t^{-1})(X^{-2} - X^2)/(X^2 - 1) \\ &= \epsilon t^\epsilon(X^{-2} - 1) + (t - t^{-1})(-1 - X^{-2}) \\ &= -(\epsilon t^\epsilon + t - t^{-1}) + \epsilon t^{-\epsilon}X^{-2} \end{aligned}$$

We now compare this to

$$\begin{aligned}
X^{-1}T^{-1}(X - X^{-1}) &= X^{-1}(T + t^{-1} - t)(X - X^{-1}) \\
&= (t^{-1} - t)(1 - X^{-2}) + X^{-1}T(X - X^{-1}) \\
&= (t^{-1} - t)(1 - X^{-2}) + \epsilon t^\epsilon(X^{-2} - 1) + \frac{2(t - t^{-1})(X^{-2} - 1)}{(X^2 - 1)} \\
&= (t - t^{-1} + \epsilon t^\epsilon)(X^{-2} - 1) - 2(t - t^{-1})X^{-2} \\
&= -(t - t^{-1} + \epsilon t^\epsilon) + \epsilon t^{-\epsilon}X^{-2}
\end{aligned}$$

The relations for  $Y$  now follow formally from the straightforward fact that the assignments  $Y \mapsto \hat{y}$  and  $s \mapsto \hat{s}$  give a representation of  $A_q \rtimes \mathbb{Z}_2$ . Indeed, we have

$$\begin{aligned}
TY^{-1}T &= T(T^{-1}\hat{s}^{-1}\hat{y}^{-1})T \\
&= \hat{s}\hat{y}^{-1}T \\
&= \hat{y}\hat{s}T \\
&= Y
\end{aligned}$$

Similarly, we check

$$\begin{aligned}
XY &= X\hat{y}\hat{s}T \\
&= q^2\hat{y}X\hat{s}T \\
&= q^2\hat{y}\hat{s}X^{-1}T \\
&= q^2\hat{y}\hat{s}(X^{-1}T^{-1})T^2 \\
&= q^2\hat{y}\hat{s}T XT^2 \\
&= q^2Y XT^2
\end{aligned}$$

□

## CHAPTER 8

### A SKEIN-THEORETIC REALIZATION OF THE DAHA

In Chapter 2 we described the Kauffman bracket skein module, which associates the algebra  $A_q^{\mathbb{Z}_2}$  to the torus, and associates to each knot in  $S^3$  a (left) module over  $A_q^{\mathbb{Z}_2}$ . In Chapter 7 we described  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$ , a 2-parameter family of algebras originally defined by Cherednik that can be viewed as a deformation of  $A_q^{\mathbb{Z}_2}$ . More precisely, when  $t$  is specialized to 1, there is an isomorphism  $A_q^{\mathbb{Z}_2} \xrightarrow{\sim} \mathbf{e}\mathcal{H}_{q,t=1}\mathbf{e}$ , so we can think of the family  $A_q^{\mathbb{Z}_2}$  as a 1-parameter subfamily of  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$ . This leads to the natural questions

- Is there a topological realization of the algebra  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$ ?
- Is there a way to associate a module over  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$  to each knot in  $S^3$ ?

Our goal in this chapter is to give a positive answer to both questions. Unfortunately, it seems that any further calculations involving these modules will require further insight than we have presently obtained. One difficulty is that the module  $K_{q,t}(S^3 \setminus L)$  associated to a knot complement seems to be much more complicated than the classical Kauffman bracket skein module  $K_q(S^3 \setminus L)$ . In particular,  $K_{q,t=1}(S^3 \setminus L)$  has  $K_q(S^3 \setminus L)$  as a quotient, but in general is much bigger, which makes computations more difficult.

#### 8.1 Modified KBSM for surfaces

Our goal in this section is to define a 2-parameter skein module  $K_{q,t}(F)$  for a (connected) surface  $F$ . As we describe below, the algebra  $K_{q,t}(F)$  will be a quotient

of  $K_q(F \setminus \{p\})$  by an ideal depending on  $t$ . For general surfaces, the specialization  $K_{q,t=1}(F)$  is ‘bigger’ than  $K_q(F)$ , in the sense that the set of links with non-crossing, nontrivial components forms a (linear) basis of  $K_q(F)$  but does not span  $K_{q,t=1}(F)$ . However, if  $F$  is the torus  $T^2$ , then this set is a linear basis for  $K_{q,t}(T^2)$  (for all  $t$ ) and also for  $K_q(T^2)$ . Therefore,  $K_{q,t}(T^2)$  can be viewed as a deformation of  $K_q(T^2)$ . Our goal is to show that it is the same deformation as the algebraic deformation given by the spherical subalgebra  $e\mathcal{H}_{q,t}e$  described in the previous chapter.

**Definition 8.1.1.** We fix a point  $p \in F$  and define

1. a **vertical special strand** is a homeomorphism  $[0, 1] \rightarrow \{p\} \times [0, 1] \subset F \times [0, 1]$ ,
2. a **framed link with a vertical special strand** is an isotopy class of embeddings  $[0, 1] \sqcup (\sqcup_n S^1 \times [0, 1]) \hookrightarrow F \times [0, 1]$ , where  $n \geq 0$  and the isotopy class contains a representative whose embedding of  $[0, 1]$  is a vertical special strand.

Let  $\mathcal{L}_{q,t}$  be the vector space spanned by framed links with a vertical special strand, and let  $\mathcal{L}'_{q,t} \subset \mathbb{C}\mathcal{L}$  be the subspace generated by elements of the form

$$L_+ - qL_0 - q^{-1}L_\infty, \quad L \sqcup \bigcirc + q^2 + q^{-2}, \quad \text{and } L_s + (q^2t^{-2} + q^{-2}t^2)L' \quad (8.1)$$

where  $L_0, L_+, L_\infty$  are links that are identical outside of a ball, and inside the ball appear as in the terms of Figure 8.1, and  $L_s, L'$  are identical outside a ball, and inside a ball appear as in the left of Figure 8.2. (The dotted lines in Figure 8.2 represent the special strand.)

**Definition 8.1.2.** The *modified Kauffman bracket skein module*  $K_{q,t}(F \times [0, 1])$  is the algebra  $\mathbb{C}\mathcal{L}/\mathcal{L}'$ .

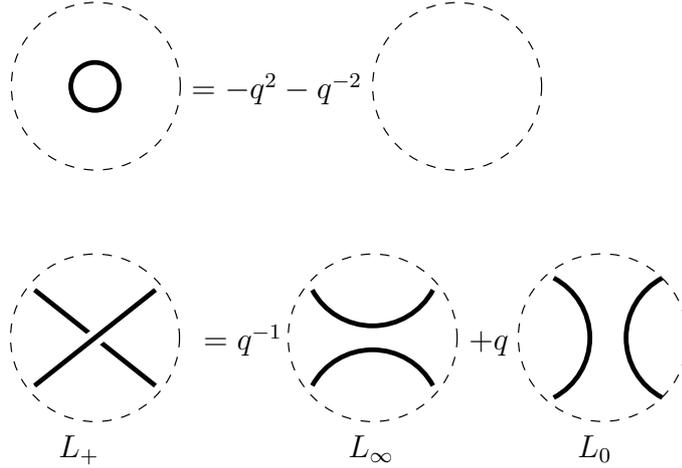


Figure 8.1: Kauffman bracket skein relation for loops

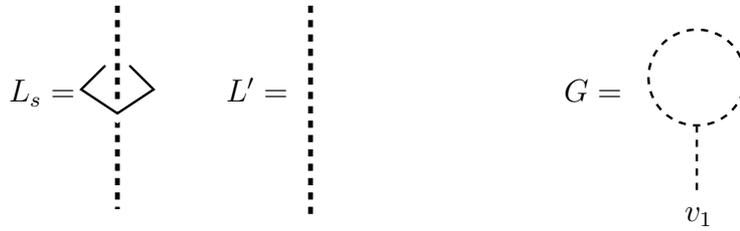


Figure 8.2: Skein relation for special strands

The algebra structure is given by “stacking links” as before. More precisely, given two links  $L_1, L_2 \subset F \times [0, 1]$ , we take two copies of  $F \times [0, 1]$ , each containing one of the  $L_i$ , and glue  $F \times \{1\}$  in one to  $F \times \{0\}$  in the other via the identity map. Since each special strand begins and ends at  $p$ , the special strands glue together. The identity element of this algebra is the link with one special strand and no other components. We also remark that we do not include the point  $p \in F$  in the notation because different choices of  $p$  lead to isomorphic algebras.

We now compute the algebra structure of  $K_{q,t}(T^2)$ . We first recall a useful result from [6]. Let  $T'$  be the torus with one puncture, and let  $x'$  and  $y'$  be simple closed curves that intersect once, and let  $z'$  be the (simple closed) curve with the

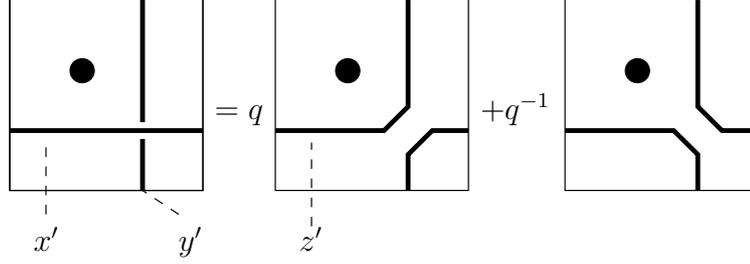


Figure 8.3: The generators for the punctured torus

coefficient  $q$  in the resolution of the product  $x'y'$ . (See Figure 8.3.)

We also need the algebras  $B'_q$  and  $B_{q,t}$  from Section 4.2. We recall that  $B'_q$  is the algebra generated by  $x, y, z$  modulo the following relations:

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x \quad (8.2)$$

and that  $B_{q,t}$  is the quotient of  $B'_q$  by the additional relation

$$q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz = \left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2 \quad (8.3)$$

**Theorem 8.1.3.** [6, Thm 2.1] *There is an algebra isomorphism  $B'_q \rightarrow K_q(T')$  induced by the assignments  $x \mapsto x'$ ,  $y \mapsto y'$ , and  $z \mapsto z'$ .*

**Remark 8.1.4.** It is a general fact that if  $F$  is any surface and  $\partial$  is a curve parallel to a boundary component of  $F$ , then  $\partial$  is a central element in  $K_q(F)$ . This is true because if  $\partial$  is ‘on top of’ a link  $L$ , then  $\partial$  can be shrunk to be very close to the boundary, slid down the boundary until it is below  $L$ , and then expanded. In other words, the link  $\partial L$  is isotopic to the link  $L\partial$ .

**Corollary 8.1.5.** *The element  $w = q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz$  is central in  $B'_q$ .*

*Proof.* A short calculation with the skein relations shows that the element  $-w + q^2 + q^{-2}$  is the loop parallel to the boundary of  $T'$ , so the considerations in the previous remark apply.  $\square$

It is clear that there is a surjection  $K_q(T') \rightarrow K_{q,t}(T^2)$  induced topologically by ‘filling in the puncture with the special strand.’ This means that we can consider the loops  $x', y', z'$  as elements of  $K_{q,t}(T^2)$ .

**Corollary 8.1.6.** *There is an algebra isomorphism  $B_{q,t} \rightarrow K_{q,t}(T^2)$  induced by the assignments  $x \mapsto x', y \mapsto y',$  and  $z \mapsto z'$ .*

*Proof.* First we check that the composition  $f : B'_q \rightarrow K_q(T') \rightarrow K_{q,t}(T^2)$  factors through the quotient  $B'_q \rightarrow B_{q,t}$ . To do this we need to check that relation (8.3) holds, i.e. that  $f(w) = \left(\frac{t}{q} - \frac{q}{t}\right)^2 + \left(q + \frac{1}{q}\right)^2$ . If  $\partial \in K_q(T')$  is the loop around the puncture in  $T'$ , then  $\partial = -w + q^2 + q^{-2} \in K_q(T')$  (see the proof of Corollary 8.1.5). Then the third relation in (8.1) implies  $f(\partial) = -q^2t^{-2} - q^{-2}t^2$ , and the calculation  $(t/q - q/t)^2 + (q + q^{-1})^2 = q^2 + q^{-2} + q^2t^{-2} + q^{-2}t^2$  shows the claim.

To show that  $f : B_{q,t} \rightarrow K_{q,t}(T^2)$  is injective, it suffices to show that the kernel of the map  $K_q(T') \rightarrow K_{q,t}(T^2)$  is cyclic and is generated by  $\delta + q^2t^{-2} + q^{-2}t^2$ , where  $\delta$  is the loop in  $T'$  that encircles the puncture. This element is exactly the skein relation on the left of Figure 8.2, and any time this relation appears in a sum of links, we can slide the relation to the top of the manifold  $T' \times [0, 1]$ . In other words, any element  $a$  in the kernel of  $K_q(T') \rightarrow K_{q,t}(T^2)$  can be written in the form  $a = a'(\delta + q^2t^{-2} + q^{-2}t^2)$  for some  $a' \in K_q(T')$ , and this proves  $\delta + q^2t^{-2} + q^{-2}t^2$  generates the kernel, as desired.

□

**Corollary 8.1.7.** *The algebras  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$  and  $K_{q,t}(T^2)$  are isomorphic.*

*Proof.* Compose the isomorphism of Corollary 8.1.6 and Theorem 7.1.3. □

## 8.2 The deformation for 3-manifolds

We now define a deformed skein module  $K_{q,t}(M)$  for a 3-manifold  $M$  with a boundary component  $F$ , a connected surface. This definition will depend on additional data, but if  $M$  is a knot complement  $S^3 \setminus K$ , then we give a canonical choice for this data.

Let  $G$  be the ‘tadpole’ graph depicted on the right of Figure 8.2, and let  $f : G \rightarrow M$  be an embedding with  $f(v_1) = p \in F \subset \partial M$ . (As before, the definition doesn’t depend on the choice of  $p \in F$ , but it *does* depend on the choice of  $f$ .)

**Definition 8.2.1.** A **framed link compatible with  $f$**  is an isotopy class of embeddings  $G \sqcup (\sqcup_n S^1 \times [0, 1]) \hookrightarrow M$  containing a representative such that the embedding of  $G$  is given by the map  $f$ . The **modified Kauffman bracket skein module**  $K_{q,t}(M, f)$  is the vector space of framed links compatible with  $f$  modulo the relations in Equation (8.1).

If  $K$  is a knot in  $S^3$ , then there is a unique (up to isotopy) choice of a longitude and meridian in the boundary of a tubular neighborhood of  $K$ . There is therefore a unique (up to isotopy) choice of curve  $C$  in  $S^3 \setminus K$  that is parallel to the meridian  $m$ , i.e.  $C$  is a boundary component of an annulus whose other boundary component is the meridian of  $K$ . We pick a simple arc  $a$  in this annulus connecting  $C$  to the point  $p \in \partial(S^3 \setminus K)$ , and we define  $f : G \rightarrow S^3 \setminus K$  to be the map given by the union of the embeddings of  $C$  and  $a$ . (See Figure 8.4.) We will write  $K_{q,t}(K)$  or  $K_{q,t}(S^3 \setminus K)$  instead of  $K_{q,t}(S^3 \setminus K, f)$ . (There are many choices of an arc in an annulus connecting the two boundary components, but the definition does not depend on this choice because all such choices are isotopic via an isotopy that does not fix the embedding of  $C$ , and such isotopies are allowed in the definition of a

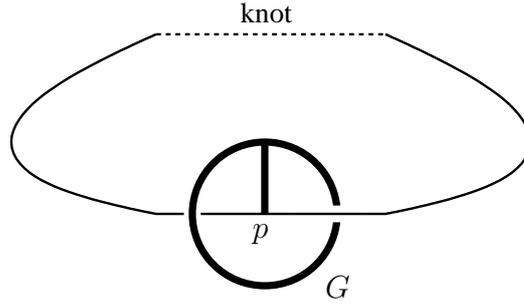


Figure 8.4: The embedding of the graph  $G$  for a knot complement

‘framed link compatible with  $f$ .’)

**Theorem 8.2.2.** *The vector space  $K_{q,t}(S^3 \setminus L)$  is a module over  $\mathfrak{e}\mathcal{H}_{q,t}\mathfrak{e} \cong K_{q,t}(T^2)$ .*

*Proof.* Since the skein relations in  $K_{q,t}(T^2)$  are the same as the ones in  $K_{q,t}(S^3 \setminus L)$ , the space  $K_{q,t}(S^3 \setminus L)$  is a module over the spherical subalgebra  $\mathcal{H}_{q,t}^+$ , with the module structure given by gluing  $T^2 \times [0, 1]$  to the boundary of  $S^3 \setminus L$ .  $\square$

However, the map  $K_{q,t=1}(S^3 \setminus K) \rightarrow K_q(S^3 \setminus K)$  is surjective but not an isomorphism, even for the unknot. This is somewhat reminiscent of the situation in Lie theory - finite dimensional modules only exist for integral weights, so they ‘don’t deform.’ However, Verma modules exist for all weights, and the ones associated to integral weights surject onto the finite dimensional modules.

## CHAPTER 9

### ALGEBRAIC DEFORMATIONS OF SKEIN MODULES

We recall that the double affine Hecke algebra  $\mathcal{H}_{q,t}$  is the family of algebras (depending on parameters  $q, t \in \mathbb{C}^*$ ) generated by  $X^{\pm 1}$ ,  $Y^{\pm 1}$ , and  $T$  subject to the relations

$$(T - t)(T + t^{-1}) = 0, \quad XY = q^2YXT^2, \quad TXT = X^{-1}, \quad TY^{-1}T = Y \quad (9.1)$$

If we specialize  $t = 1$ , then  $T^2 = 1$  and so  $\mathcal{H}_{q,t} \cong A_q \rtimes \mathbb{Z}_2$ . In this section we discuss deforming an  $A_q \rtimes \mathbb{Z}_2$  module  $M$  to a module over  $\mathcal{H}_{q,t}$  for arbitrary  $t$ . In other words, given an algebra morphism  $f : A_q \rtimes \mathbb{Z}_2 \rightarrow \text{End}_{\mathbb{C}}(M)$ , we would like to produce a (flat) family of algebra morphisms  $f_t : \mathcal{H}_{q,t} \rightarrow \text{End}_{\mathbb{C}}(M)$  such that  $f_{t=1} = f$ . Such a family  $f_t$  will be called a **deformation** of  $f$ , and if a deformation exists we will say that  $M$  **deforms**. Ideally, we would like a procedure to produce a *unique* such family  $f_t$ . If we are even more greedy, we would like such a family to be functorial (with respect to  $A_q \rtimes \mathbb{Z}_2$ -module maps  $M \rightarrow M'$ ).

We give a construction that shows that such deformations exist in a fairly general setting. In particular, if an  $A_q \rtimes \mathbb{Z}_2$  module  $M$  satisfies the conclusion of Conjecture 4.3.4, then  $M$  deforms. This construction does *not* produce a unique deformation, and in fact it makes it obvious that in general there are many deformations (which should be expected).

One reason the uniqueness of such a deformation could be interesting is that it could lead to 2-variable knot polynomials depending on  $q, t$  that specializes to the colored Jones polynomials. More precisely, let  $\mathbf{e}V \cong K_q(S^3 \setminus \text{unknot})$  and  $\mathbf{e}M \cong K_q(S^3 \setminus L)$ , and suppose  $V_t$  and  $M_t$  are deformations of  $V$  and  $M$ , respectively. If the pairing  $\langle -, - \rangle : V \otimes M \rightarrow \mathbb{C}$  deforms to a pairing  $\langle -, - \rangle_t : V_t \otimes M_t \rightarrow \mathbb{C}$ , then

we can define

$$J_{q,t}(L, n) = 2\langle X^n, 1_L \rangle_t$$

Here  $1_L$  is the empty link in  $S^3 \setminus L$ . (This expression makes sense because the deformations  $V_t$  and  $M_t$  are actions of  $\mathcal{H}_{q,t}$  on the *fixed* vector spaces  $V$  and  $M$ .) Then for each  $n \in \mathbb{N}$ , the number  $J_{q,t}(L, n)$  depends polynomially on  $q$  and  $t$ , and the (Laurent) polynomial  $J_{q,t}(L, n)$  specializes to  $J_q(L, n)$  when  $t = 1$ .

We emphasize again that we have not proved that  $J_{q,t}(L, n)$  is a knot invariant, since it seems to depend on the choice of deformation. However, for the sake of the curious, the second section in this chapter contains some Mathematica code that evaluates the first few  $J_{q,t}(L, n)$  for the trefoil, using a deformation produced in the first section. The pairing has been normalized so that  $\langle 1_V, 1_L \rangle_t = 1$  (where  $1_V$  and  $1_L$  are the empty links in the tubular neighborhood of the knot and the knot complement, respectively). However, we have not formally shown this pairing is well-defined.

There have been several very recent preprints ([8], [15], [38]) that produce 2-variable polynomials that seem similar to the polynomials we calculate for the trefoil. Comparisons between these various approaches will be left for later work.

## 9.1 Deforming $A_q \rtimes \mathbb{Z}_2$ modules

Let  $M$  be an  $A_q \rtimes \mathbb{Z}_2$  module, and let  $A_X$  and  $A_Y$  be the subalgebras

$$A_X := \mathbb{C}[\{X^{\pm 1}, T_1\}] \subset A_q \rtimes \mathbb{Z}_2, \quad A_Y := \mathbb{C}[\{Y^{\pm 1}, T_1\}] \subset A_q \rtimes \mathbb{Z}_2$$

We recall that we would like to define operators  $X_t, T_t, Y_t : M \rightarrow M$  so that the operators  $X_t, Y_t, T_t$  satisfy the relations of the DAHA  $\mathcal{H}_{q,t}$ , and so that the module

structure on  $M$  given by  $X_1, Y_1, T_1$  agrees with the given  $A_q \rtimes \mathbb{Z}_2$  module structure on  $M$ . We will further require that  $X_t$  is constant, i.e. that  $X_t = X_1$ . We will therefore omit the subscript  $t$  from the notation for  $X$ .

We recall from Section 4.3 the definition of the  $A_q \rtimes \mathbb{Z}_2$ -module

$$P_k^\pm := \tau^k(\text{Ind}_{A_Y}^{A_q \rtimes \mathbb{Z}_2}(\mathbb{C}_\pm))$$

for some  $k \in \mathbb{Z}$  (and  $\pm$  representing  $\pm 1$ ). As a reminder the module structure is

$$P_k^\pm \cong_{\mathbb{C}[X^{\pm 1}]} \mathbb{C}[X^{\pm 1}], \quad Y_1 \cdot f(X) = \pm q^{-k} X^k f(q^{-2}X), \quad T_1 \cdot f(X) = \pm f(X^{-1}) \quad (9.2)$$

As a  $\mathbb{C}[X^{\pm 1}]$ -module,  $P$  is freely generated by  $1 \in P$ .

**Lemma 9.1.1.** *Any short exact sequence*

$$0 \rightarrow N \rightarrow M \xrightarrow{\pi} P_k^\pm \rightarrow 0$$

of  $A_q \rtimes \mathbb{Z}_2$ -modules splits as a sequence of modules over  $A_X$ .

*Proof.* Let  $p = 1 \in P_k^\pm$  (using the identification in (9.2)) and write  $P = P_k^\pm$ . Let  $\bar{m} \in M$  such that  $\pi(\bar{m}) = p \in P$ , and define  $m := (\bar{m} + \epsilon T_1 \bar{m})/2$ . Then since  $\pi$  is  $T_1$ -linear and  $\epsilon^2 = 1$ , we see  $\pi(m) = p$ . Also,

$$T_1 \cdot m = T_1 \cdot \bar{m} + \epsilon \bar{m} = \epsilon(\epsilon T_1 \cdot \bar{m} + \bar{m}) = \epsilon m$$

Now define a  $\mathbb{C}[X^{\pm 1}]$ -linear splitting  $\rho : P \rightarrow M$  via  $\rho(f(X)p) = f(X)m$ . Then

$$\begin{aligned} T_1 \cdot \rho(f(X)p) &= T_1 \cdot f(X)\rho(p) \\ &= f(X^{-1})T_1 \cdot m \\ &= f(X^{-1})\epsilon m \\ &= \rho(f(X^{-1})\epsilon p) \\ &= \rho(T_1 \cdot (f(X)p)) \end{aligned}$$

In other words, the splitting  $\rho$  is  $A_X$ -linear. □

**Theorem 9.1.2.** *Assume  $M$  has a filtration  $0 \subset M_1 \subset \cdots \subset M_n = M$  such that each quotient  $M_i/M_{i-1}$  is isomorphic to  $P_k^\pm$  for some  $k \in \mathbb{Z}$ . Then  $M$  deforms via the operators  $T_t$  and  $Y_t$  defined in (9.3) and (9.4), respectively.*

*Proof.* Let  $P_i = M_i/M_{i-1}$  and let  $p_i = 1 \in P_i$  (using the identification (9.2)). Using the previous lemma and induction, we can pick a  $A_X$ -linear isomorphism  $\rho : \bigoplus P_i \rightarrow M$ . If we let  $m_i = \rho(p_i)$ , we can reformulate this statement as follows:

1. the  $m_i$  form a  $\mathbb{C}[X^{\pm 1}]$ -basis for  $M$ ,
2. each  $m_i$  is an eigenvector for  $T_1$ , i.e.  $T_1 \cdot m_i = \epsilon_i m_i$  with  $\epsilon_i \in \{\pm 1\}$ .

We first define a family of operators  $T_{t,i} : M \rightarrow M$  for each  $1 \leq i \leq n$  via the formula

$$T_{t,i} \cdot g(X)m_j := \begin{cases} 0 & \text{if } i \neq j \\ \left[ t^{\epsilon_i} T_{t=1} + \frac{t-t^{-1}}{x^2-1} (\epsilon_i T_{t=1} - 1) \right] g(X)m_i & \text{if } i = j \end{cases}$$

The relations  $(T_{t,i} - t)(T_{t,i} + t^{-1})g(X)m_i = 0$  and  $T_{t,i}X T_{t,i} = X^{-1}$  are straightforward (but tedious) calculations which are in Lemma 7.2.1. We now define  $T_t : M \rightarrow M$  as

$$T_t := \sum_{i=1}^n T_{t,i} \tag{9.3}$$

Because of the second assumption,  $T_t(g(X)m_i) \in \mathbb{C}[X^{\pm 1}]m_i$ , and since the operator  $X$  also “acts diagonally” (with respect to the basis  $\{m_i\}$ ), the relations  $(T_t - t)(T_t + t^{-1}) = 0$  and  $T_t X T_t = X^{-1}$  are now clear from the same relations for the  $T_{t,i}$ .

All that remains is to define  $Y_t : M \rightarrow M$  and check the relations  $XY_t = q^2 Y_t X T_t^2$  and  $T_t Y^{-1} T_t = Y_t$ . We define  $Y_t$  via the formula

$$Y_T := Y_1 T_1 T_t \tag{9.4}$$

The relations we need to check follow purely formally. Precisely,

$$\begin{aligned}
T_t Y_t^{-1} T_t &= T_t (T_t^{-1} T_1^{-1} Y_1^{-1}) T_t \\
&= T_1 Y_1^{-1} T_t \\
&= Y_1 T_1 T_t \\
&= Y_t
\end{aligned}$$

Similarly, we check

$$\begin{aligned}
XY_t &= XY_1 T_1 T_t \\
&= q^2 Y_1 X T_1 T_t \\
&= q^2 Y_1 T_1 X^{-1} T_t \\
&= q^2 Y_1 T_1 (X^{-1} T_t^{-1}) T_t^2 \\
&= q^2 Y_1 T_1 T_t X T_t^2 \\
&= q^2 Y_t X T_t^2
\end{aligned}$$

□

We remark that the formula (9.4) is (essentially) the same as (7.9), which was first used by Cherednik in a different context.

## 9.2 The trefoil revisited

Below we include Mathematica code that calculates 2-variable polynomials  $J(q, t, n)$  that specialize to the colored Jones polynomials  $J(q, n)$  when  $t = 1$  for the trefoil. We use the technique of the previous section to obtain a deformation  $M_t$  of the  $A_q \rtimes \mathbb{Z}_2$ -module  $M$  (where  $\mathfrak{e}M \cong K_q(S^3 \setminus L)$ ). We normalize the pairing

$\langle -, - \rangle_t : V_t \otimes M_t \rightarrow \mathbb{C}$  by requiring  $\langle 1_V, 1_M \rangle_t = 1$ , which determines the deformed pairing uniquely. (At this point, we haven't formally proved that this actually defines a pairing.)

To be more explicit, we write down enough information to determine the  $\mathcal{H}_{q,t}$ -module  $M_t$ . We fix the following identification of  $\mathbb{C}[X^{\pm 1}]$ -modules:

$$M_t \cong_{\mathbb{C}[X^{\pm 1}]} \mathbb{C}[X^{\pm 1}]n_- \oplus \mathbb{C}[X^{\pm 1}]n'_+ \quad (9.5)$$

**Definition 9.2.1.** Define operators  $s, T_t, Y_1, Y_t : M_t \rightarrow M_t$  via the following:

$$\begin{aligned} s \cdot f(x)n_- + g(x)n'_+ &= -f(x^{-1})n'_- + g(x^{-1})n'_+ \\ T_t \cdot f(x)n_- &= [t^{-1}s + \frac{t-t^{-1}}{X^2-1}(-s-1)] \cdot f(x)n_- \\ T_t \cdot g(x)n'_+ &= [ts + \frac{t-t^{-1}}{X^2-1}(s-1)] \cdot g(x)n'_+ \\ Y_1 \cdot n_- &= -n_- \\ Y_1 \cdot n'_+ &= (q^2X^{-1} - q^6X^{-5})n_- + q^6X^{-6}n'_+ \\ Y_t &= Y_1 \circ s \circ T_t \end{aligned}$$

The results of Chapter 6 show that  $M_{t=1}$  is the lift of the skein module of the trefoil complement, and the results of the previous section show that the operators  $X, T_t, Y_t$  give an action of  $\mathcal{H}_{q,t}$  on  $M_t$ .

We can also describe this module in terms of an extension prescribed by a derivation. The automorphism  $\tau : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  extends to  $\mathcal{H}_{q,t}$  via the same formulas:

$$\tau(X) = X, \quad \tau(T) = T, \quad \tau(Y) = q^{-1}XY$$

We will slightly abuse notation by writing

$$P_t^+ = \text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_t), \quad P_t^- = \text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}_{q,t}}(\mathbb{C}_{-t^{-1}})$$

As in the case  $t = 1$ , we define  $P_{k,t}^+ = \tau^k(P_t^+)$  and  $P_{k,t}^- = \tau(P_t^-)$ . Then there is an exact sequence

$$0 \rightarrow P_{0,t}^- \rightarrow M_t \rightarrow P_{-6,t}^+ \rightarrow 0$$

Furthermore, under the identification in (9.5), the  $\mathcal{H}_{q,t}$  action on  $M_t$  is given by a derivation  $\Psi : \mathcal{H}_{q,t} \rightarrow \text{Hom}_{\mathbb{C}}(P_{-6,t}^+, P_{0,t}^-)$  defined by

$$\Psi_t(X, -) = 0 = \Psi_t(T, -), \quad \Psi_t(Y, p_+) = t\Psi_1(Y, p_+) = t(q^2X^{-1} - q^6X^{-5})p_-$$

The fact that these formulas for  $\Psi$  extend to a derivation follows from the fact that the operators in Definition 9.2.1 give an action of  $\mathcal{H}_{q,t}$ , and this fact follows from the results in the first section of this chapter.

**Remark 9.2.2.** It is worth mentioning that in addition to fixing a  $\mathbb{C}[X^{\pm 1}]$ -module structure on  $M_t$ , we have also fixed a  $\mathbb{C}[X^{\pm 1}]$ -splitting of  $M_t$  as in (9.5). This does not conflict with the fact that  $\Psi_t(Y, p_+)$  changes as a function of  $t$  because the derivation  $\Psi_t$  is measuring how far the  $\mathbb{C}[X^{\pm 1}]$ -linear splitting is from being a  $\mathcal{H}_{q,t}$ -linear splitting.

The code below implements these explicit formulas, but it is probably hard to read at best. We have replaced the parameter  $q$  with the parameter  $v$  because  $v$  comes later in the alphabet than  $t$ , which seems to make Mathematica happier. The code first outputs the colored Jones polynomial and then outputs the two-variable polynomial that specializes to the Jones polynomial when  $t = 1$ . It is easy to see by inspection that the polynomials do in fact satisfy this specialization.

We again remind the reader that the polynomials below have not been proven to be invariants of the trefoil since they seem to depend on a choice of the deformation  $M_t$ .

```

NN = 4;
Print["Left Trefoil"]
Y1op =
Function[{f},
{(Evaluate[-f[[1]][q^(-2)#] + f[[2]][q^(-2)#](q^2#^(-1) - q^6#^(-5)))]&,
(Evaluate[q^6#^(-6)f[[2]][q^(-2)#]])]&};
Y1invop =
Function[{f},
{(Evaluate[-f[[1]][q^(2)#] + f[[2]][q^(2)#](-q^2# + q^6#^(5)))]&,
(Evaluate[q^6#^(6)f[[2]][q^(2)#]])]&};
EvalX = Function[{list}, Map[(Apply[#, {X}])&, list]];
Pairv0 = Function[{list}, list[[1]][-1](1/(2q^2 - 2q^6))];
Pairv0t = Function[{list}, list[[1]][-t](t^(-2)/(2(q/t)^2 - 2(q/t)^6))];
minefunc =
Function[n, 1/(q^(2) - q^(6))
Sum[(-1)^i q^(6(n^2 - i^2))(q^(10i - 4) - q^(2i)), {i, 1, n}]];
JonesN = Function[{n}, Pairv0[Nest[Y1invop, {0&, 1&}, n]] -
Pairv0[Nest[Y1op, {0&, 1&}, n]]];
Tt = Function[{list},
{(Evaluate[-t^(-1)list[[1]][#^(-1)] +
(t - t^(-1))/(#^2 - 1)(list[[1]][#^(-1)] - list[[1]][#])]&,
(Evaluate[tlist[[2]][#^(-1)] + (t - t^(-1))/(#^2 - 1)
(list[[2]][#^(-1)] - list[[2]][#])]&});
T1 = Function[{list}, {(Evaluate[-list[[1]][#^(-1)])&,
(Evaluate[list[[2]][#^(-1)])&});
Ttinv =

```

```

Function[{list}, {(Evaluate[Tt[list][[1]][#] + (t^(-1) - t)list[[1]][#]])&,
(Evaluate[Tt[list][[2]][#] + (t^(-1) - t)list[[2]][#]])&};
Y1opt = Function[{list}, Y1op[T1[Tt[list]]];
Y1invopt = Function[{list}, Ttinv[T1[Y1invop[list]]];
JonesNt =
Function[{n}, Pairv0t[Nest[Y1invopt, {0&, 1&}, n]] -
Pairv0t[Nest[Y1opt, {0&, 1&}, n]];
jn = FullSimplify[Expand[FullSimplify[Array[minefunc[#]&, NN]]];
jnt = FullSimplify[Array[Expand[FullSimplify[JonesNt[#]]&, NN]];
Together[Expand[jnt/.{t -> t, q -> v}]];
mya = Collect[%, v];
myb = Expand[jn]/.{q -> v};
Print["The v in the equations below is our q"]
Print["***Jones poly #1, 2-var Jones poly #0"]
Simplify[myb[[1]]]
Collect[Simplify[mya[[1]], v]
Simplify[mya[[1]]/.{t -> 1}]
Print["***Jones poly #1, 2-var Jones poly #1"]
myb[[2]]
mya[[2]]
Print["***Jones poly #2, 2-var Jones poly #2"]
myb[[3]]
mya[[3]]
Print["***Jones poly #3, 2-var Jones poly #3"]
myb[[4]]
mya[[4]]

```

Left Trefoil

The v in the equations below is our q

\*\*\*Jones poly #1, 2-var Jones poly #1

1

1

1

\*\*\*Jones poly #1, 2-var Jones poly #1

$$-v^2 - v^6 - v^{10} + v^{18}$$

$$-tv^2 - \frac{v^6}{t} - \frac{v^{10}}{t^3} + \frac{(-1+t^2)v^{14}}{t^5} + \frac{v^{18}}{t^5}$$

\*\*\*Jones poly #2, 2-var Jones poly #2

$$v^4 + v^8 + v^{12} + v^{16} + v^{20} - v^{32} - v^{36} - v^{40} + v^{48}$$

$$\frac{(t^8-t^{10}+t^{12})v^4}{t^{10}} + v^8 + \frac{(t^6-t^8+t^{10})v^{12}}{t^{10}} + \frac{v^{16}}{t^2} + \frac{(t^4-t^6+t^8)v^{20}}{t^{10}} + \frac{(t^6-t^8)v^{24}}{t^{10}} + \frac{(t^2-t^4)v^{28}}{t^{10}} - \frac{v^{32}}{t^4} + \frac{(1-2t^2)v^{36}}{t^{10}} - \frac{v^{40}}{t^{10}} + \frac{(-1+t^2)v^{44}}{t^{10}} + \frac{v^{48}}{t^{10}}$$

\*\*\*Jones poly #3, 2-var Jones poly #3

$$-v^6 - v^{10} - v^{14} - v^{18} - v^{22} - v^{26} - v^{30} + v^{46} + v^{50} + v^{54} + v^{58} + v^{62} - v^{74} - v^{78} - v^{82} + v^{90}$$

$$\frac{(-t^{14}+t^{16}-t^{18})v^6}{t^{15}} - \frac{v^{10}}{t^3} + \frac{(-t^{10}+t^{14}-t^{16})v^{14}}{t^{15}} + \frac{(-t^8+t^{10}-t^{12}+t^{14}-t^{16})v^{18}}{t^{15}} - \frac{v^{22}}{t} + \frac{(-t^{10}+t^{12}-t^{14})v^{26}}{t^{15}} + \frac{(-t^6+t^8-t^{14})v^{30}}{t^{15}} + \frac{(-t^{12}+t^{14})v^{34}}{t^{15}} + \frac{(-t^4+t^6)v^{42}}{t^{15}} + \frac{(t^6-t^8+t^{12})v^{46}}{t^{15}} + \frac{v^{50}}{t^5} + \frac{(-t^2+2t^4-t^6+t^8)v^{54}}{t^{15}} + \frac{(t^4-t^6+t^8)v^{58}}{t^{15}} + \frac{(t^2-t^4+t^6)v^{62}}{t^{15}} + \frac{(-1+2t^2-t^4+t^6-t^8)v^{66}}{t^{15}} + \frac{(1-t^4)v^{70}}{t^{15}} + \frac{(1-2t^2+t^4-t^6)v^{74}}{t^{15}} - \frac{v^{78}}{t^{13}} - \frac{v^{82}}{t^{15}} + \frac{(-1+t^2)v^{86}}{t^{15}} + \frac{v^{90}}{t^{15}}$$

## BIBLIOGRAPHY

- [1] Yuri Berest, George Khachatryan, and Ajay Ramadoss. Derived representation schemes and cyclic homology. *Preprint, 1112.1449*, 2011.
- [2] G. W. Brumfiel and H. M. Hilden.  $SL(2)$  representations of finitely presented groups, volume 187 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1995.
- [3] Doug Bullock. Rings of  $SL_2(\mathbf{C})$ -characters and the Kauffman bracket skein module. *Comment. Math. Helv.*, 72(4):521–542, 1997.
- [4] Doug Bullock, Charles Frohman, and Joanna Kania-Bartoszyńska. Understanding the Kauffman bracket skein module. *J. Knot Theory Ramifications*, 8(3):265–277, 1999.
- [5] Doug Bullock and Walter Lo Faro. The Kauffman bracket skein module of a twist knot exterior. *Algebr. Geom. Topol.*, 5:107–118 (electronic), 2005.
- [6] Doug Bullock and Józef H. Przytycki. Multiplicative structure of Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.*, 128(3):923–931, 2000.
- [7] Gerhard Burde and Heiner Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2003.
- [8] I. Cherednik. Jones polynomials of torus knots via daha. *ArXiv 1111.6195*, November 2011.
- [9] Ivan Cherednik. Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald’s operators. *Internat. Math. Res. Notices*, (9):171–180, 1992.
- [10] Ivan Cherednik. *Double affine Hecke algebras*, volume 319 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
- [11] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.
- [12] Charles F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311(1):167–183, 1989.

- [13] Alimjon Eshmatov and Farkhod Eshmatov. Private notes. 2011.
- [14] Charles Frohman and Răzvan Gelca. Skein modules and the noncommutative torus. *Trans. Amer. Math. Soc.*, 352(10):4877–4888, 2000.
- [15] H. Fuji, S. Gukov, and P. Sułkowski. Volume Conjecture: Refined and Cate-  
gorified. *ArXiv 1205.1515*, March 2012.
- [16] Stavros Garoufalidis. Difference and differential equations for the colored Jones function. *J. Knot Theory Ramifications*, 17(4):495–510, 2008.
- [17] Stavros Garoufalidis and Thang T. Q. Lê. The colored Jones function is  $q$ -holonomic. *Geom. Topol.*, 9:1253–1293 (electronic), 2005.
- [18] Răzvan Gelca. Non-commutative trigonometry and the  $A$ -polynomial of the trefoil knot. *Math. Proc. Cambridge Philos. Soc.*, 133(2):311–323, 2002.
- [19] Victor Ginzburg. Lectures on noncommutative geometry. *Preprint, 0506603*, 2011.
- [20] William M. Goldman. The symplectic nature of fundamental groups of sur-  
faces. *Adv. in Math.*, 54(2):200–225, 1984.
- [21] Joel Hass and Jeffrey C. Lagarias. The number of Reidemeister moves needed  
for unknotting. *J. Amer. Math. Soc.*, 14(2):399–428 (electronic), 2001.
- [22] Kazuhiro Hikami. Difference equation of the colored Jones polynomial for  
torus knot. *Internat. J. Math.*, 15(9):959–965, 2004.
- [23] V. F. R. Jones. Hecke algebra representations of braid groups and link poly-  
nomials. *Ann. of Math. (2)*, 126(2):335–388, 1987.
- [24] Louis H. Kauffman. State models and the Jones polynomial. *Topology*,  
26(3):395–407, 1987.
- [25] Robion Kirby and Paul Melvin. The 3-manifold invariants of Witten and  
Reshetikhin-Turaev for  $\mathfrak{sl}(2, \mathbf{C})$ . *Invent. Math.*, 105(3):473–545, 1991.
- [26] Tom H. Koornwinder. Zhedanov’s algebra  $AW(3)$  and the double affine Hecke  
algebra in the rank one case. II. The spherical subalgebra. *SIGMA Symmetry  
Integrability Geom. Methods Appl.*, 4:Paper 052, 17, 2008.

- [27] P. B. Kronheimer and T. S. Mrowka. Knot homology groups from instantons. *Preprint*.
- [28] Thang T. Q. Lê. The colored Jones polynomial and the  $A$ -polynomial of knots. *Adv. Math.*, 207(2):782–804, 2006.
- [29] Alexander Lubotzky and Andy R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985.
- [30] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small.
- [31] Susan Montgomery. *Fixed rings of finite automorphism groups of associative rings*, volume 818 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [32] H. R. Morton. The coloured Jones function and Alexander polynomial for torus knots. *Math. Proc. Cambridge Philos. Soc.*, 117(1):129–135, 1995.
- [33] C. Procesi. The invariant theory of  $n \times n$  matrices. *Advances in Math.*, 19(3):306–381, 1976.
- [34] Józef H. Przytycki. Skein modules of 3-manifolds. *Bull. Polish Acad. Sci. Math.*, 39(1-2):91–100, 1991.
- [35] Józef H. Przytycki and Adam S. Sikora. On skein algebras and  $\mathrm{Sl}_2(\mathbf{C})$ -character varieties. *Topology*, 39(1):115–148, 2000.
- [36] H. Reidemeister. Elementare begründung der knotentheorie. *Abh. Math. Sem. Univ. Hamburg*, 5:24–32, 1926.
- [37] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.
- [38] S. Shakirov. Beta Deformation and Superpolynomials of  $(n,m)$  Torus Knots. *ArXiv 1111.7035*, November 2011.
- [39] Vladimir G. Turaev. Skein quantization of Poisson algebras of loops on surfaces. *Ann. Sci. École Norm. Sup. (4)*, 24(6):635–704, 1991.
- [40] M. Varagnolo and E. Vasserot. Double affine Hecke algebras at roots of unity. *Represent. Theory*, 14:510–600, 2010.

- [41] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)*, 87:56–88, 1968.