ESSAYS ON THE MATHEMATICS OF MARKET EFFICIENCY

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MARKET EFFICIENCY

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In 1970, Fama defined an efficient market as one where prices always 'fully reflect' available information, but so far a rigorous definition has been lacking. This thesis addresses this issue by providing a definition based on economic equilibria. Efficiency is then characterized in terms of Merton's No Dominance condition together with absence of arbitrage in the sense of No Free Lunch With Vanishing Risk, as well as the existence of an equivalent (true) martingale measure for the discounted price process. The stability of the efficiency property with respect to changes in the information set is investigated. In particular, efficiency is preserved under information reduction, but not necessarily under information expansion. Next, checkable necessary and sufficient conditions for efficiency are provided for a large class of high dimensional stochastic volatility models. Finally, information reduction is studied in the inefficient setting. This leads to new results on filtration shrinkage for strict local martingales.

BIOGRAPHICAL SKETCH

Martin studied Engineering Mathematics at the Lund Institute of Technology, Sweden, from 2004 to 2007, leading to a Master of Science degree. From 2007 to 2008 he was in Zurich, Switzerland, obtaining a Master of Advanced Studies in Finance, jointly at the University of Zurich and the Swiss Federal Institute of Technology in Zurich. He started his PhD at Cornell in 2008.

Denna avhandling tillägnar jag mina föräldrar och min bror.

Det är verkligen det minsta jag kan göra.

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CHAPTER 1

INTRODUCTION

This thesis concerns three fundamental concepts in financial economics: Market efficiency, economic equilibrium, and absence of arbitrage. All three concepts have been thoroughly studied both empirically and theoretically, but nonetheless the relationship between them is not fully understood. To a large extent, this is due to the lack of a workable definition of market efficiency—indeed, the original definition, given by Fama [27] on p. 383 of his seminal paper, is:

A market in which prices always 'fully reflect' available information is called 'efficient'.

A major goal of this thesis is to establish a mathematically rigorous definition of market efficiency, consistent with its treatment in the literature. Briefly stated, the definition we propose is that there should exist some economic equilibrium consistent with the observed prices. Once the definition is available, market efficiency can be characterized in terms of absence of arbitrage, appropriately defined, and the existence of a martingale probability measure for the underlying price process. More specifically, a market is efficient if and only if there is an equivalent measure under which the discounted price process is a martingale. This in turn is true if and only if Merton's No Dominance condition holds, and there is no arbitrage opportunities (in the sense of the classical No Free Lunch With Vanishing Risk (NFLVR) condition.) The No Dominance condition postulates that it is not possible to outperform liquidly traded assets solely through dynamic investment in those assets. This condition, introduced in [54] but subsequently receiving little attention (two exceptions being [43] and

[44]), plays a central role in our results. Armed with the characterization theorem one can study the properties of models for efficient and inefficient markets. In this work, we focus on two key issues.

Firstly, we study the effect of perturbations to the agents' information set. Specifying an appropriate information set is of central importance in the literature on market efficiency. This is no surprise, since the investors' ability to carry out effective trading strategies crucially depends on the information available to them. The classical literature considers three distinct types: weak form, semi-strong, and strong form efficiency, distinguished by the available information. Understanding this distinction in the context of our definition of efficiency is a key motivation for studying changes in the information set.

Secondly, we investigate a class of stochastic volatility models for large markets, providing necessary and sufficient conditions for efficiency. This is feasible thanks to the characterization theorem, which allows us to study efficiency without specifying any particular equilibrium model. This is crucial in the context of testing: How does one accept or reject market efficiency through statistical tests? A major difficulty is the *joint hypothesis problem*: tests of efficiency so far have been contingent on specifying a particular equilibrium model. Testing the validity of such a model is, however, notoriously difficult. As a result, there is no way to exclude that a rejection of efficiency is really just a rejection of the assumed equilibrium model. Our characterization of efficiency makes it possible to circumvent this problem, and by providing checkable conditions within a concrete modeling framework, we take a first step toward new testing procedures.

Concerning the mathematics, the contributions of this thesis lie in the realm of semimartingale theory, in particular the distinction between true martingales and strict local martingales, as well as the theory of filtration expansion and filtration shrinkage. Our results on high dimensional market models are among the few in the literature attempting to characterize the strict local martingale property for high dimensional diffusions. The results on filtration shrinkage for strict local martingales contributes to the understanding of the structure of optional projections of such processes. In particular, there are interesting connections to the so-called Föllmer measure, and to predictable compensators of stopping times.

This thesis consists of three chapters, excluding this Introduction. Market efficiency is defined and characterized in Chapter 2, which also includes a discussion of how efficiency is affected by changes in the information set. Chapter 3 is devoted to studying the efficiency property of a class of high dimensional stochastic volatility models. Thanks to the characterization theorem obtained in Chapter 2, this task is reduced to finding probability measures that turn the price process into a (true) martingale. Whether or not this is possible turns out to depend crucially on the correlation structure of the price process vis-a-vis that of the underlying volatility process. The material in Chapter 2 and part of Chapter 3 has appeared in [45]. Chapter 4 is concerned with the mathematical problem of filtration shrinkage in the inefficient setting, continuing the discussion on different information sets initiated in Chapter 2.

Throughout this thesis, standard notational conventions from probability theory and mathematical finance will be adhered to. For the relevant notions from semimartingale theory and stochastic integration, we refer to the books by Jacod [39], Jacod and Shiryaev [41], and Protter [59]. A basic reference on the theory of diffusion processes used in Chapters 3 and 4 is the book by Rogers and Williams [60]. The value of a stochastic process X at time t will interchangeably be denoted X_t or $X(t)$—both have advantages depending on the context. Unless explicitly stated otherwise, relations between random variables, such a equalities and inequalities, are to be understood up to almost sure equivalence. Relations involving processes are to be understood up to evanescence. Some exceptions to these conventions occur in Chapter 4, but they will be pointed out clearly.

CHAPTER 2

MARKET EFFICIENCY: DEFINITION AND CHARACTERIZATION

As mentioned in the Introduction, Fama's original definition of market efficiency is:

A market in which prices always 'fully reflect' available information is called 'efficient'.

In quantifying this definition, for its use in testing market efficiency, it is commonly believed (see, for example, [7] and [29]) that one must first specify an equilibrium model. This is called the *joint-hypothesis* or the *bad-model* problem. Indeed, Fama states ([28], p. 1575 and [29], p. 285):

The joint-hypothesis problem is more serious. Thus, market efficiency per se is not testable. It must be tested jointly with some model of equilibrium, an asset pricing model. This point, the theme of the 1970 review (Fama (1970)), says that we can only test whether information is properly reflected in prices in the context of a pricing model that defines the meaning of 'properly'.

Market efficiency must be tested jointly with a model for expected (normal) returns, and all models show problems describing average returns. The bad-model problem is ubiquitous, but it is more serious in long-term returns.

In contrast, we quantify the original definition in such a manner that one can test market efficiency without specifying *a particular* equilibrium model. As such, our formulation overcomes the bad-model problem in the existing tests. We prove this assertion below. Our claim has precedence in the literature where it is well understood that the existence of an arbitrage opportunity rejects market efficiency (see, for example, [46]). And, of course, identifying an arbitrage opportunity does not require the specification of a particular equilibrium model.

More generally, the purpose of this chapter is to revisit the meaning of market efficiency to rectify various misconceptions in the literature and to develop new theorems related to market efficiency. As such, one can then better understand the implications of an efficient market for empirical testing, profitable trading strategies, and the properties of asset price processes. This analysis is facilitated by our accumulated understanding of martingale pricing methods and their application to equilibrium models (for a review see [25]).

To start, we first provide an analytic definition of an efficient market with respect to an information set that is consistent with the existing definition but independent of a particular equilibrium asset pricing model. Next, we provide two alternative characterizations of this definition that facilitate both theorem proving and empirical testing. The first characterization relates to the existence of an equivalent probability measure making the normalized asset price processes martingales (sometimes called risk neutral measures). The second characterization relates to no arbitrage (in the sense of No Free Lunch with Vanishing Risk (NFLVR)) and No Dominance (ND). This latter characterization formalizes the notion that an efficient market has ”no profitable” trading strategies (see [46]).

These two characterizations enable us to obtain some new insights and to prove some new theorems regarding efficient markets. First we show that to test for an efficient market, one only needs to show that there are no arbitrage opportunities nor dominated securities with respect to an information set. These tests are both necessary and sufficient. Surprisingly, when restricted to discrete trading economies, market efficiency is in fact equivalent only to the notion of no arbitrage (NFLVR). This is especially relevant because most of the existing empirical studies of market efficiency are based on discrete time models (see [27, 28, 29] and [46] for reviews). Because such empirical tests do not require the specification of a particular equilibrium model, this confirms our claim that market efficiency can be tested without the joint model hypothesis.

Three information sets have been considered when discussing efficient markets[1]: (i) historical prices (weak form efficiency), (ii) publicly available information (semi-strong efficiency), and (iii) private information (strong form efficiency). A market may or may not be efficient with respect to each of these information sets.[2] In order to account for this distinction, which is well established in the literature, we study how information expansion and reduction affects market efficiency. As is well known, we show that information reduction preserves with market efficiency, while information expansion need not. In other words, if the market is semi-strong form efficient, then it is weak-form efficient; but, if the market is semi-strong form efficient, it need not be strong-form efficient. Theorems and examples illustrate these statements. With respect to information expansion, we also study the question: if the market is semi-strong form efficient and it is impossible to produce arbitrage in the sense of NFLVR with respect to inside information, then is the market strong-form efficient? In general the answer is no, but we provide sufficient conditions for its validity—if the market is either: (i) discrete time, (ii) complete, or (iii) the H-hypothesis holds. The H-hypothesis is a mathematical condition often used in the area of

[1]This partitioning of the information sets is attributed to Harry Roberts, unpublished paper presented at the Seminar of the Analysis of Security Prices, U. of Chicago, May 1967 (see Fama (1970)).

[2]Market efficiency is closely related to the notion of a Rational Expectations Equilibrium (REE) where equilibrium prices reveal private information. A fully revealing REE is one where prices reveal all private information, analogous to a market that is strong-form efficient. A partially revealing REE is one where prices only partially reveal all private information, corresponding to semi-strong form efficiency (see [48] and [1] for reviews). This relationship is discussed further in Section 2 below.

credit risk pricing and hedging (see [26] and [5]). Our analysis thus provides an economic interpretation of the H-hypothesis relating to market efficiency.

2.1 The Model

We consider a continuous time and continuous trading economy on an infinite horizon. There are a finite number of traders in the economy. Securities markets are assumed to be competitive and frictionless.

2.1.1 The Market

We are given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ on $[0, \infty)$ that satisfies the usual conditions of right-continuity and P-completeness. P is the statistical probability measure. The traded securities consist of a locally riskless money market account together with d risky securities whose market prices at time t, given in units of the money market account, are $S(t) = (S^1(t), \ldots, S^d(t))$. We let security S^0 correspond to the locally riskless money market account with $S^0(t) \equiv 1$. To simplify the presentation we assume that the securities have no cash flows. We also make the following assumption:

$$S^i(t) \geq 0 \text{ a.s. for all } t \text{ and } i = 1, ..., d.$$

$S = (S(t))_{t \geq 0}$ denotes a vector stochastic process, and we let $\mathbb{F}^S$ denote the natural filtration of S, made right-continuous and augmented with the P-null sets. The process S is assumed to be a (not necessarily locally bounded) semimartingale with respect to $\mathbb{F}^S$. We assume that $\mathbb{F}$ contains $\mathbb{F}^S$ and that S is a semimartingale with respect to $\mathbb{F}$. Although we do not require that $\mathcal{F}_0$ be P-trivial, we do assume that S_0 is a.s. constant.

For a given filtration $\mathbb{F}$, we refer to the pair $(\mathbb{F}, S)$ as a *market*.

2.1.2 Trading Strategies

The economy is populated by a finite number of investors each of whom have the beliefs P_k and the information filtration $\mathbb{F}$ where the probability beliefs P_k are assumed to be equivalent to P. Due to the competitive markets assumption, traders act as price takers. Given frictionless markets (no transaction costs nor restrictions on trade), the trading strategies available to an investor are modeled by $\mathbb{F}$-*admissible strategies* H. That is, H is an $\mathbb{F}$ predictable and S-integrable process which is $(\mathbb{F}, a)$-*admissible* for some $a \in \mathbb{R}$, meaning that $H \cdot S \geq -a$. Here,

$$(H \cdot S)_t = \int_{0+}^{t} \sum_{i=0}^{d} H^i(s) dS^i(s)$$

corresponds to a vector stochastic integral, see [59] and [39]. Note that the left endpoint is not included, so that $(H \cdot S)_0 = 0$.

We require that the admissible trading strategies be *self-financing*, meaning that there are no cash flows generated by the trading strategy. That is, letting $V(t) = \sum_{i=0}^{d} H^i(t) S^i(t)$ denote the time t value of the trading strategy, the self-financing condition is that $V(t) = V(0) + (H \cdot S)_t$ for all t. A variant of the self-financing condition will be discussed later in the context of endowment and consumption streams.

2.1.3 No Arbitrage (NFLVR)

Our no arbitrage condition is the classical No Free Lunch with Vanishing Risk (NFLVR) due to [17, 20]. NFLVR means that there is no sequence $f_n = (H^n \cdot S)_\infty$, where each H^n is admissible and $(H^n \cdot S)_\infty$ exists, such that $\|\max(-f_n, 0)\|_\infty \to 0$ and $f_n \to f$ a.s. for some $f \geq 0$ with $P(f > 0) > 0$. In our context, we will need to impose NFLVR on specific time intervals. We therefore make the following definition (note that taking $T = \infty$ yields the usual definition of NFLVR).

Definition 2.1.1 *A market $(\mathbb{F}, S)$ satisfies NFLVR on [0,T] if the stopped process S^T, together with the filtration $\mathbb{F}$, satisfies NFLVR.*

The Fundamental Theorem of Asset Pricing (see Delbaen and Schachermayer [17, 20]) states that in our setting NFLVR is equivalent to the existence of an equivalent local martingale measure[3]. In other words, a market $(\mathbb{F}, S)$ satisfies NFLVR on $[0, T]$ if and only if the set

$$\mathcal{M}_{loc}(\mathbb{F}, S, T) = \{Q : Q \sim P \text{ and } S \text{ is an } (\mathbb{F}, Q) \text{ local martingale on } [0, T]\}$$

is non-empty. When there is no risk of confusion, we will sometimes simply write $\mathcal{M}_{loc}$, $\mathcal{M}_{loc}(\mathbb{F})$, etc.

2.1.4 No Dominance (ND)

The notion of No Dominance (ND) was introduced by [54] to study the properties of option prices. Merton's definition can be formalized as follows.

[3]Notice that we do not have to distinguish between local martingales and sigma martingales since prices are nonnegative. This follows from the definition of a sigma martingale and the Ansel-Stricker theorem.

Definition 2.1.2 (No Dominance) *the i^{th} security $S^i = (S^i(t))_{t\geq 0}$ is undominated on $[0, T]$ if there is no admissible strategy H such that*

$$S^i(0) + (H \cdot S)_T \geq S^i(T) \text{ a.s.} \qquad \text{and} \qquad P\{S^i(0) + (H \cdot S)_T > S^i(T)\} > 0.$$

A market $(\mathbb{F}, S)$ satisfies ND on $[0, T]$ if each S^i, $i = 0, \ldots, d$, is undominated on $[0, T]$.

In words, ND states that it is not possible to find a trading strategy that generates a set of payoffs at time T that dominate the payoffs to any traded security. ND has been used recently in the literature by [43, 44] for the study of asset price bubbles. Moreover, a closely related notion known as "Relative Arbitrage" has been recently studied by several authors; see for instance [31], [30] and [62].

Notice that the above definition also makes sense for $T = \infty$. The reason is that $(H \cdot S)_\infty$ exists for every admissible H, so in particular $S^i(0) + (H^i \cdot S)_\infty = S^i(\infty)$ exists for every i, where H^i is given by

$$H^i = (0, \ldots, 0, 1, 0, \ldots, 0), \tag{2.1}$$

with the one in position i. This shows that ND on $[0, \infty]$ is a well-defined notion in the presence of NFLVR. In addition, we point out that if S^i is undominated on $[0, T]$, it is also undominated at all earlier times $T' < T$. Indeed, if there were a dominating strategy H, one could apply the strategy $K(t) = H(t)\mathbf{1}_{\{\mathbf{t \leq T'}\}} + H^i(t)\mathbf{1}_{\{\mathbf{t > T'}\}}$ where H^i is as in (2.1). This corresponds to holding one unit of asset i up to the time horizon. The nonnegativity of S^i ensures that H^i is admissible. The strategy K satisfies

$$S^i(0) + (K \cdot S)_T = S^i(T) + S^i(0) + (H \cdot S)_{T'} - S^i(T') \geq S^i(T),$$

with positive probability of having a strict inequality. But, this is impossible since S^i is undominated on $[0, T]$.

NFLVR and ND are distinct conditions, but both imply the simpler No Arbitrage (NA) condition: there can be no admissible strategy H such that

$$(H \cdot S)_T \geq 0 \text{ a.s.} \qquad \text{and} \qquad P\{(H \cdot S)_T > 0\} > 0.$$

Indeed, since ND in particular implies that $S^0 \equiv 1$ is undominated, it follows that ND implies NA. And, it has been shown by [17] that a market $(\mathbb{F}, S)$ satisfies NFLVR if and only if it satisfies NA together with the condition that the set of payoffs of 1-admissible strategies with bounded support is bounded in probability.

2.1.5 Maximal Trading Strategies

Essential in proving many of our results in the notion of maximal trading strategies introduced by [20].

Definition 2.1.3 (Maximal Strategies) *A process H is called* $\mathbb{F}$-maximal on $[0, T]$ *if it is $\mathbb{F}$-admissible and for every $\mathbb{F}$-admissible strategy K such that $(K \cdot S)_T \geq (H \cdot S)_T$, it is true that $(K \cdot S)_T = (H \cdot S)_T$.*

If the filtration and/or the time horizon is clear from the context, we drop these qualifiers and simply call H *maximal*.

To understand the meaning of a maximal trading strategy H, one first fixes a time T payout generated by a trading strategy $(H \cdot S)_T$. Then, a maximal admissible trading strategy has the largest such fixed payoff possible starting at time 0 with zero investment. In terms of maximality, the No Dominance

(ND) condition can be phrased as requiring that all the strategies H^i in (2.1) are maximal.

We need two results from [20] concerning maximal strategies.

Lemma 2.1.1 *If S is a positive $\mathbb{F}$ semimartingale that satisfies NFLVR with respect to $\mathbb{F}$, then for any $\mathbb{F}$-admissible strategy H the following are equivalent:*

(*i*) *H is $\mathbb{F}$-maximal on $[0, T]$.*

(*ii*) *There is $Q \in \mathcal{M}_{loc}(\mathbb{F})$ such that $H \cdot S$ is an $(\mathbb{F}, Q)$ martingale on $[0, T]$.*

(*iii*) *There is $Q \in \mathcal{M}_{loc}(\mathbb{F})$ such that $E^Q[(H \cdot S)_T] = 0$.*

Proof. See [20], Theorem 5.12., while keeping in mind that local martingale measures and sigma martingale measures coincide in our setting where S is nonnegative. □

Lemma 2.1.2 *Finite sums of maximal strategies are again maximal.*

Proof. This follows from [19], Theorem 2.14, which is stated for the case where S is locally bounded. However, an examination of the proof of this theorem, and the results that it relies on (Lemma 2.11, Proposition 2.12 and Proposition 2.13 in the same reference) show that the local boundedness is never used. □

2.1.6 An Economy

We consider a pure exchange economy on a finite horizon $[0, T]$. An economy consists of a market $(\mathbb{F}, S)$ and a finite number of investors $(k = 1, ..., K)$ characterized by their beliefs, information, preferences, and endowments.

We let α^i denote the aggregate net supply of the i^{th} security. It is assumed that each α^i is non-random and constant over time, with $\alpha^0 = 0$ and $\alpha^i > 0$ for $i = 1, \ldots, d$.

There is a single consumption good that is perishable. The price of the consumption good, in units of the money market account, is denoted $\psi = \{\psi(t) : 0 \leq t \leq T\}$. We assume that $\psi(t)$ is strictly positive.

Each investor solves an optimization problem where he seeks to maximize utility from consumption. In [51], the optimizing agent receives endowments and consumes his wealth continuously through time, using a general incomplete semimartingale financial market to finance his consumption. The utility structure is very general, allowing among other things for state dependent utility functions. We adopt a similar setup. Let μ be the probability measure on $[0, T]$ such that $\mu(\{T\}) > 0$. Two canonical examples are $\mu([0, T)) = 0$, $\mu(\{T\}) = 1$, which corresponds to utility from terminal consumption only, and

$$\mu(dt) = \frac{1}{2T}dt + \frac{1}{2}\delta_{\{T\}}(dt),$$

which is diffuse on $[0, T)$ and has an atom $\{T\}$. This corresponds to utility from continuous consumption over $[0, T)$ and a bulk consumption at T. The use of the measure μ simplifies the notation by allowing us to treat utility from intermediate and final consumption within a single framework.

The k^{th} investor is characterized by the following quantities.

- *Beliefs and information* $(P_k, \mathbb{F})$. We assume that investor's beliefs P_k are equivalent to P. All investors have the same information set $\mathbb{F}$.
- A time dependent *utility function* $U_k : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ such that for each t in the support of μ, the function $U_k(t, \cdot)$ is concave and strictly increasing.

We also assume that $\lim_{x\to\infty} U_k(T, x) = \infty$. The utility that agent k derives from consuming $c(t)\mu(dt)$ at each time $t \leq T$ is

$$\mathcal{U}_k(c) = E_k\left[\int_0^T U_k(t, c(t))\mu(dt)\right],$$

where $E_k[\cdot]$ is expectation with respect to P_k. Since $\mu(\{T\}) > 0$, the utility is strictly increasing in the final consumption $c(T)$.

- *Initial wealth* x_k. Given a trading strategy $H = (H^1, \ldots, H^d)$, the investor will be required to choose his initial holding $H^0(0)$ in the money market account such that

$$x_k = H^0(0) + \sum_{i=1}^{d} H^i(0)S^i(0). \tag{2.2}$$

- A stochastic *endowment stream* $\epsilon_k(t)$, $t < T$ of the commodity. This means that the investors receive $\epsilon_k(t)\mu(dt)$ units of the commodity at time $t \leq T$. The cumulative endowment of the k^{th} investor, in units of the money market account, is given by

$$\mathcal{E}_k(t) = \int_0^t \psi(s)\epsilon_k(s)\mu(ds).$$

The setup is quite general and includes most formulations studied in the utility maximization literature. In [53], utility from terminal wealth in incomplete markets is considered, in which case $\psi \equiv 1$, $\mu(\{T\}) = 1$, and $\epsilon_k \equiv 0$. These results are extended in [12] to the case of random endowments, relaxing the condition $\epsilon_k \equiv 0$. In [51], the optimizing agent receives endowments and consumes his wealth continuously through time, so $\mu([0, T))$ is no longer zero. In fact, $\mu([0, t]) > 0$ is assumed for each $t < T$. All the above papers make additional assumptions on the utility function $U_k(t, \cdot)$ for some or all of their results. In particular, it is assumed that for each t in the support of μ, the function $U_k(t, \cdot)$ is strictly concave, strictly increasing, continuously differentiable, and satisfies the Inada conditions: $\partial_2 U_k(t, 0+) = \infty$ and $\partial_2 U_k(t, \infty) = 0$. Moreover, a condition that figures prominently is *reasonably asymptotic elasticity* condition. In [53] and [12] it takes the form

$$\limsup_{x \to \infty} \frac{x U_k'(x)}{U_k(x)} < 1,$$

where $U_k(x) = U_k(T, x)$. In [51], a uniform in time version of this condition is used, together with additional regularity conditions. It is also possible to relax other aspects of the utility structure. Moreover, they allow the utility function to evolve stochastically in a progressively measurable way. This would require boundedness assumptions on $\psi(t)$, see Example 3.4 in [51]. Finally, we mention [4], where utilities defined on $\mathbb{R}$ are considered.

Each investor chooses a consumption plan $\{c_k(t) : 0 \le t \le T\}$ with $c_k(t) \ge 0$, and a trading strategy in the money market account, H_k^0, and the risky securities, $H_k = (H_k^1, \ldots, H_k^d)$. The investor's wealth $W_k(t)$ at time t is

$$W_k(t) = H_k^0(t) + \sum_{i=1}^{d} H_k^i(t) S^i(t),$$

and the holdings $H_k^0(t)$ of the money market account must be chosen so that the strategy is *self-financing*, i.e.,

$$W_k(t) = x_k + \mathcal{E}_k(t) + (H_k \cdot S)_t - C_k(t)$$

where

$$C_k(t) = \int_0^t \psi(s) c_k(s) \mu(ds)$$

is the value of cumulative consumption. Note that the self-financing condition guarantees that (2.2) holds.

At time T, the investors' financial holdings are transformed into units of the consumption good, which can be consumed. That is, at time T the k^{th} investor receives a liquidating dividend of

$$\frac{H_k^0(T) + \sum_{i=1}^d H_k^i(T) S^i(T)}{\psi(T)},$$

in units of the consumption good.

A pair (c_k, H_k) is called *admissible* if c_k is progressively measurable, H_k is admissible in the usual sense, and it generates a wealth process W_k with nonnegative terminal wealth, $W_k(T) \geq 0$. The consumption rate process c_k is called admissible if there exists H_k such that (c_k, H_k) is admissible. We emphasize that admissibility of H_k means that $H_k \cdot S$ is uniformly bounded from below by some constant which is *independent of the initial capital* x_k. In particular, we do not require that $W_k(t)$ always be nonnegative. This is in contrast to some other work on utility maximization, for instance [53] and [67].

Investor k solves the following optimization problem:

The Investor's Problem: *To maximize* $U_k(c)$ *over all admissible consumption plans* $c = \{c(t) : 0 \leq t \leq T\}$. *For fixed endowments we write*

$$u_k(x) = \sup\{\mathcal{U}_k(c) : \; c \text{ is admissible}, \; x_k = x\}$$

In the utility maximization literature the existence of an optimal solution has been established under a wide range of assumptions. One common condition is to require $u_k(x) < \infty$ for some $x > 0$, together with the existence of an equivalent local martingale measure. In our setting, we directly assume the existence of an optimal solution to the investor's problem. This is a powerful assumption with several important consequences.

Lemma 2.1.3 *Assume that for some $x > 0$, the investor's problem has an optimal solution with a finite optimal value. Let $(\widehat{c}, \widehat{H})$ be an admissible pair such that $\widehat{c}$ achieves the optimum. Then $\widehat{H}$ is a maximal strategy.*

Proof. If $\widehat{H}$ is not maximal, there is an admissible strategy J such that $(J \cdot S)_T \geq (\widehat{H} \cdot S)_T$, with strict inequality with positive probability. Hence this strategy supports the same consumption $\widehat{c}(t)$ for $t < T$, as well as the final consumption

$$c'(T) = \widehat{c}(T) + \frac{(J \cdot S)_T - (H \cdot S)_T}{\mu(\{T\})}.$$

Since $U_k(T, \infty) = \infty$ and $\mu(\{T\}) > 0$, and the optimal solution has finite value by assumption, we must have $\widehat{c}(T) < \infty$. Hence $c'(T) \geq \widehat{c}(T)$, with positive probability that the inequality is strict. This strictly improves the utility of the investor, contradicting the optimality of $(\widehat{c}, \widehat{H})$. □

We note that as in [51] we may restrict the investors' portfolio choices to strategies $H_t \in \mathcal{K}$ a.s. for all $t \in [0, T]$ where $\mathcal{K}$ is a convex cone describing trading restrictions, such as a short sales prohibition. The proof of Lemma 2.1.3 still goes through, but maximality now refers to the restricted set of admissible strategies.

Lemma 2.1.4 *Assume that for some $x > 0$, the investor's problem has an optimal solution with finite optimal value. Then S satisfies NFLVR. Consequently, $\mathcal{M}_{loc}$ is nonempty.*

Proof. By a well-known characterization of NFLVR, it suffices to show that: (a) NA is satisfied, and (b) the set $\mathcal{K} = \{(H \cdot S)_T : H \text{ is } 1\text{−admissible}\}$ is bounded in L^0, see [17], Corollary 3.9.

Let $(\widehat{c}, \widehat{H})$ be an optimal consumption-investment plan. Suppose first NA fails, and let J be an arbitrage strategy. The strategy $\widetilde{H} = \widehat{H} + J$ is then admissible, and with $\widetilde{X}_T = (\widetilde{H} \cdot S)_T$ and $\widehat{X}_T = (\widehat{H} \cdot S)_T$, we have $\widetilde{X}_T \geq \widehat{X}_T$ and $P(\widetilde{X}_T > \widehat{X}_T) > 0$. Hence $\widehat{H}$ is not maximal, which is impossible by Lemma 2.1.3.

Next, the fact that the set $\mathcal{K}$ is bounded in L^0 follows from a straightforward adaptation of the proof of Proposition 4.19 in [49]. The argument goes through almost unchanged as soon as we have established that $u_k(\cdot)$ is concave. For this, choose arbitrary $x^i > 0$ for $i = 1, 2$ and $\lambda \in [0, 1]$, and set $x^0 = \lambda x^1 + (1 - \lambda)x^2$. There are sequences $\{c_i^n\}_{n \in \mathbb{N}}$, $i = 1, 2$, of consumption plans such that c_n^i is admissible given initial capital x^i, and

$$u_k(x^i) = \lim_{n \to \infty} E_k \left[\int_0^T U_k(t, c_n^i(t)) \mu(dt) \right].$$

Now, $c_n^0 = \lambda c_n^1 + (1 - \lambda)c_n^2$ is admissible with initial capital x^0. Hence, due to the concavity of $U_k(t, \cdot)$ for $t \in [0, T]$, we get

$$u_k(x^0) \geq \limsup_{n \to \infty} E_k \left[\int_0^T U_k(t, c_n^0(t)) \mu(dt) \right] \geq \lambda u_k(x^1) + (1 - \lambda)u_k(x^2).$$

Thus $u(\cdot)$ is concave, as claimed. □

This lemma is the formalization of the well-known result that the existence of an investor's optimal consumption choice implies that there are no arbitrage opportunities.

An *economy* is defined by the collection $(\{P_k\}_{k=1}^K, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$.

2.1.7 An Equilibrium

This section defines a market equilibrium and explores its implications. Given an economy $(\{P_k\}_{k=1}^K, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$, an economic equilibrium determines the price processes (ψ, S) by equating aggregate supply equal to aggregate demand. This is formalized in the following definition.

Definition 2.1.4 (Equilibrium) *Given an economy* $(\{P_k\}_{k=1}^K, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$*, a consumption good price index* ψ*, financial asset prices* $S = (S^0, S^1, \ldots, S^d)$*, and investor consumption-investment plans* $(\widehat{c}_k, \widehat{H}_k)$ *for* $k = 1, \ldots, K$*, the pair* (ψ, S) *is an equilibrium price process if for all* $0 \leq t \leq T$ *a. e.* P*,*

(i) securities markets clear:

$$\sum_{k=1}^K \widehat{H}_k^i(t) = \alpha^i, \qquad i = 0, \ldots, d;$$

(ii) commodity markets clear:

$$\sum_{k=1}^K \widehat{c}_k(t) = \sum_{k=1}^K \epsilon_k(t);$$

(iii) investors' choices are optimal: $(\widehat{c}_k, \widehat{H}_k)$ *solves the* k^{th} *investor's utility maximization problem and the optimal value is finite.*

Such an equilibrium is sometimes called an Arrow-Radner equilibrium. Sufficient conditions for the existence of such an equilibrium can be found in [24], [50], [16], [15, 14], and [66].

We now establish some properties that must hold in an economic equilibrium. Notice that NFLVR always holds in equilibrium as a consequence of Lemma 2.1.4.

Lemma 2.1.5 *Suppose an equilibrium is given. Then holding the market portfolio is a maximal strategy, i.e.* $H = (H^1, \ldots, H^d)$ *given by*

$$H^i(t) \equiv \alpha^i, \qquad i = 1, \ldots, d$$

is maximal.

Proof. By Lemma 2.1.4, $\mathcal{M}_{loc} \neq \emptyset$. Furthermore, Lemma 2.1.3 implies that each $\widehat{H}_k$ is maximal. By Lemma 2.1.2, their sum $H = \widehat{H}_0 + \cdots + \widehat{H}_K$ is also maximal. But the clearing condition for the securities markets implies that $H^i \equiv \alpha^i$ for each $i = 1, \ldots, d$. □

The next result shows that buying and holding assets in positive net supply is also a maximal strategy.

Lemma 2.1.6 *Suppose an equilibrium is given. Then, for each fixed $i \in \{0, 1, \ldots, d\}$, the strategy $H = (H^0, \ldots, H^d)$ given by*

$$\begin{cases} H^i \equiv 1 \\ \\ H^j \equiv 0, \quad j \neq i \end{cases}$$

is maximal, i.e. ND holds.

Proof. By Lemma 2.1.4, NFLVR and hence NA holds, so the claim is true for $i = 0$. Suppose $i \in \{1, \ldots, d\}$ and let $\widetilde{H}$ be the market portfolio from Lemma 2.1.5, multiplied by a factor $1/\alpha^i$. This is well-defined since $\alpha^i > 0$, and $\widetilde{H}$ is still maximal because maximality is not affected by positive scaling. By Lemma 2.1.5 and Lemma 2.1.1, there is a probability $Q \in \mathcal{M}_{loc}$ under which $\widetilde{H} \cdot S$ becomes a martingale. Due to the nonnegativity of asset prices,

$$\sum_{i=1}^{d} S^i(0) + \widetilde{H} \cdot S = S^i + \sum_{i \neq j} \frac{\alpha^j}{\alpha^i} S^j \geq S^i.$$

Hence under Q, S^i is a nonnegative local martingale dominated by a true martingale, and therefore itself a true martingale. Another application of Lemma 2.1.1 gives the maximality of H. □

As presented, our equilibrium is for an economy with symmetric information. An interesting extension is the asymmetric information case, where all traders share the same beliefs P but have different information sets represented by the filtrations $\mathbb{F}^k$. Furthermore, the market filtration $\mathbb{F} = \bigcap_k \mathbb{F}^k$ consists of the information that is available to all traders. In the investor's optimization problem, $\mathbb{F}^k$ replaces $\mathbb{F}$. Hence, the k^{th} investor's consumption and portfolio choices (c_k, H_k) are admissible with respect to $\mathbb{F}^k$. His optimal strategy $\widehat{H}_k$ will be $\mathbb{F}^k$-maximal, and since $\mathbb{F} \subset \mathbb{F}^k$ it is clear, at least on an informal level, that no $\mathbb{F}$-admissible strategy can dominate $\widehat{H}_k$. All else remains the same, with a market still being the pair $(\mathbb{F}, S)$. The definition of an equilibrium is unchanged with equilibrium prices reflecting the market clearing conditions (i) and (ii), and investors' decisions being optimal (iii), with the changed measurability requirements. When discussing NFLVR and ND, the market information set $\mathbb{F}$ is the relevant one. This asymmetric information extension relates our equilibrium notion to that of a Rational Expectations Equilibrium (REE), see [48] and [1] for reviews. Since $\mathbb{F}^S \subset \mathbb{F} \subset \mathbb{F}^k$, an investor's decisions are conditioned on the information revealed by prices. An equilibrium price process (ψ, S), therefore, confirms the investors' beliefs conditioned on $\mathbb{F}^S$.

2.2 Market Efficiency

This section defines an efficient market and provides two equivalent characterizations that are useful for empirical testing and theorem proving.

2.2.1 Definition

As discussed above, it is commonly believed that to test market efficiency, one needs to assume a particular equilibrium model in order to investigate its implications relating to the properties of the price process or the existence of abnormal trading profits. Both of these implications are derived from the martingale properties of the equilibrium price processes and they were first discovered by Samuelson [63]. If these implications are violated in the empirical study, then efficiency is rejected. In fact, Jensen [46], p. 96, in his review of the empirical literature uses these necessary conditions as the definition of an efficient market:

> *A market is efficient with respect to information set θ_t if it is impossible to make economic profits by trading on the basis of information set θ_t. By economic profits, we mean the risk adjusted returns net of all costs. Application of the zero profit condition to speculative markets under the assumption of zero storage costs and zero transactions costs gives us the result that asset prices (after the adjustment for required returns) will behave as a martingale with respect to the information set θ_t.*

Consistent with the intent of these definitions, we provide a model independent and rigorous definition of an efficient market.

Definition 2.2.1 *A market* $(\mathbb{F}, S)$ *is called* efficient on $[0, T]$ *with respect to* $\mathbb{F}$ *if there exists a consumption good price index* ψ *and an economy* $(\{P_k\}_{k=1}^K, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$ *for which* (ψ, S) *is an equilibrium price process* S *on* $[0, T]$*. If this holds for every* $T < \infty$*, the market is called* efficient *with respect to* $\mathbb{F}$*.*

This definition says that a market $(\mathbb{F}, S)$ is efficient with respect to $\mathbb{F}$ if there exists an economy whose equilibrium price process is consistent with S.[4]

2.2.2 Characterization Theorems

This section characterizes market efficiency. Our first characterization relates efficiency on $[0, T]$ to the economic notions of ND and NFLVR. The second gives a description in terms of equivalent martingale measures. The following theorem is the main result of this section.

Theorem 2.2.1 (Characterization of efficiency) *Let $(\mathbb{F}, S)$ be a market. The following statements are equivalent.*

(i) *$(\mathbb{F}, S)$ is efficient on $[0, T]$.*

(ii) *$(\mathbb{F}, S)$ satisfies both NFLVR and ND on $[0, T]$.*

(iii) *There exists a probability Q, equivalent to P, such that S is an $(\mathbb{F}, Q)$ martingale on $[0, T]$. That is, $\mathcal{M}(\mathbb{F}, S, T) \neq \emptyset$.*

Proof. $(i) \Longrightarrow (ii)$: If $(\mathbb{F}, S)$ is efficient on $[0, T]$, there is a consumption good price index ψ and an economy $(\{P_k\}_{k=1}^K, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$ such that (ψ, S) is an equilibrium price process. Hence by Lemma 2.1.4 and Lemma 2.1.6, both NFLVR and ND hold.

[4]In the context of an asymmetric information economy, a fully revealing REE is an equilibrium price process (ψ, S) such that $\mathbb{F}^S = \bigvee_{k=1}^K \mathbb{F}^k$, i.e. all private information is reflected in the market price process. Since also $\mathbb{F}^S \subset \mathbb{F}^k$, it follows that $\mathbb{F}^S = \mathbb{F}^k$ for each k. That is, all investors share the same information set, namely the information contained in the prices. A partially revealing REE is an equilibrium price process where this is not the case. A fully revealing REE corresponds to strong-form market efficiency, while a partially revealing REE corresponds to weak-form efficiency.

$(ii) \Longrightarrow (iii)$: If $(\mathbb{F}, S)$ satisfies ND and NFLVR, then all the strategies H^i in (2.1) are maximal. By Lemma 2.1.2, $H = H^1 + \ldots + H^n = (1, \ldots, 1)$ is then also maximal. Lemma 2.1.1 thus implies that there is $Q \in \mathcal{M}(\mathbb{F})$ turning

$$H \cdot S = (S^1 - S^1(0)) + \ldots + (S^n - S^n(0))$$

into a martingale. Using the nonnegativity of S, we see that each nonnegative Q local martingale S^i is dominated by a martingale, and is therefore itself a martingale.

$(iii) \Longrightarrow (i)$: Assume that there exists an equivalent martingale measure Q. We need to construct an equilibrium supporting the price process S. Let all investors have power utilities with parameter $0 < \gamma < 1$,

$$U_k(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & x > 0 \\ -\infty, & x \leq 0 \end{cases}$$

for each k, and suppose they only derive utility from terminal consumption, i.e. $\mu(\{T\}) = 1$. Set $\psi(t) \equiv 1$ and assume that the endowment streams ϵ_k are identically zero—then the investors only receive utility from the liquidating dividend.

Next, suppose that the investor beliefs are given by an equivalent probability P^*, which we define via

$$\frac{dP^*}{dQ} = \frac{Z(T)^\gamma}{E^Q[Z(T)^\gamma]},$$

where

$$Z(t) = \frac{\alpha^1 S^1(t) + \cdots + \alpha^d S^d(t)}{\alpha^1 S^1(0) + \cdots + \alpha^d S^d(0)},$$

which is a strictly positive Q-martingale by hypothesis, with $E^Q[Z(T)] = 1$. Note that since $\gamma < 1$, $E^Q[Z(T)^\gamma] < \infty$, so P^* is well-defined. The k^{th} investor's optimization problem is then

$$\sup\left\{E^{P^*}[U_k(X(T))] \;:\; X(T) = x_k + \int_0^T H(s)dS(s),\ H \text{ admissible}\right\}.$$

Since $U_k(x) = -\infty$ for $x \leq 0$, we may restrict attention to strategies for which $X(T) > 0$. Then, due to the supermartingale property of $X = x_k + \int H(t)dS(t)$ under Q, $X(t) \geq E^Q(X(T) \mid \mathcal{F}_t) \geq 0$ for all $t \leq T$. Hence, in fact, we only need to consider x_k-admissible strategies.

We now show that, with the preferences and beliefs described above, the optimal strategy for each investor is to invest his initial wealth in the market portfolio until the time horizon T. As a consequence, there is an equilibrium supporting the market $(\mathbb{F}, S)$. To prove this, first note that, by the definition of P^* and U_k,

$$E^{P^*}[U_k(x_k X(T))] = \frac{x_k^{1-\gamma}}{1-\gamma}\frac{1}{E^Q[Z(T)^\gamma]}E^Q\Big[Z(T)^\gamma Z(T)^{1-\gamma}\Big] = \frac{x_k^{1-\gamma}}{1-\gamma}\frac{1}{E^Q[Z(T)^\gamma]},$$

since Z is a Q martingale with expectation one. Thus the candidate optimal utility is finite. Next, let H be any 1-admissible strategy, and set $X = 1 + \int HdS$. The concavity of U_k, the definition of P^*, and the supermartingale (resp. martingale) property of X (resp. Z) under Q yield

$$\begin{aligned}
E^{P^*}[U_k(x_k X(T)) - U_k(x_k Z(T))] &\leq E^{P^*}[U_k'(x_k Z(T))(x_k X(T) - x_k Z(T))] \\
&= \frac{x_k^{1-\gamma}}{E^Q[Z(T)^\gamma]}E^{P^*}\left[\frac{dQ}{dP^*}(X(T) - Z(T))\right] \\
&= \frac{x_k^{1-\gamma}}{E^Q[Z(T)^\gamma]}\Big(E^Q[X(T)] - E^Q[Z(T)]\Big) \\
&\leq 0.
\end{aligned}$$

Hence

$$E^{P^*}[U_k(x_k X(T))] \leq E^{P^*}[U_k(x_k Z(T))],$$

and since the final payoff from any x_k-admissible strategy is of the form $x_k X(T)$ with X as above, this proves the optimality of $x_k Z(T)$.

It is now straightforward to verify that we have an equilibrium. With preferences as described above, the k^{th} investor's holdings in the i^{th} asset at time t is given by

$$\widehat{H}_k^i(t) = x_k \frac{\alpha^i}{\alpha^1 S^1(0) + \cdots + \alpha^d S^d(0)}.$$

Summing over k and using that $\sum_{k=1}^K x_k = \alpha^1 S^1(0) + \cdots + \alpha^d S^d(0)$ shows that the securities markets clear. The commodity markets also clear, since there is no intermediate consumption or endowments. This concludes the proof. □

Condition (iii) formalizes the connection between martingales and efficiency as first noted in [63] and [27], and it is equivalent to the definition of efficiency used in [61]. As pointed out previously, by the Fundamental Theorem of Asset Pricing, NFLVR on $[0, T]$ implies that $\mathcal{M}_{loc}(\mathbb{F}, S, T) \neq \emptyset$. The efficiency condition is stronger. It requires that $\mathcal{M}(\mathbb{F}, S, T) \neq \emptyset$ where

$$\mathcal{M}(\mathbb{F}, S, T) = \{Q \sim P : S \text{ is an } (\mathbb{F}, Q) \text{ martingale on } [0, T]\}.$$

The set $\mathcal{M}(\mathbb{F}, S, T)$ can equivalently be described as consisting of the equivalent measures that turn S into a uniformly integrable martingale on $[0, T]$. When there is no risk of confusion we write $\mathcal{M}$, $\mathcal{M}(\mathbb{F})$, etc.

Consistent with this observation, there exist markets that satisfy NFLVR but are not efficient. An example is any complete market with a price bubble, see [43]. To see this, consider the following simple economy consisting of only two traded assets, the money market account and S^1. Let S^1 be an inverse Bessel process[5]. Then $\mathcal{M}_{loc}$ consists of a single element under which S is a strict local martingale (i.e. a local martingale that is not a martingale), and hence $\mathcal{M} = \emptyset$. Theorem 2.2.1 then shows that this market, where we can take $\mathbb{F} = \mathbb{F}^S$, is not efficient. This example is discussed in more detail in [18], as well as in Chapter 4 below.

[5]The inverse Bessel process can be defined as $1/\|B\|$, where B is a three-dimensional Brownian motion starting from $(1, 0, 0)$. See [10] for details.

The alternative characterization of efficiency in terms of ND and NFLVR makes precise the meaning of "no economic profits" in the definition of an efficient market as given in [46] and quoted above. "No economic profits" means NFLVR and ND. As stated, it is self-evident that the notions of NFLVR and ND are independent of any particular equilibrium model; they must be satisfied by all such equilibrium models. It is this characterization that facilitates empirical tests of market efficiency that are independent of the joint model hypothesis.

Indeed, given any market $(\mathbb{F}, S)$, to disprove efficiency one just needs to identify an arbitrage opportunity (FLVR) or a dominating trading strategy. Conversely, if one can show that no such strategies exist, then the market is efficient. To show that no such strategies exist, one can use Theorem 2.2.1, and show that a martingale probability measure Q exists. Given a specification for the stochastic process S, an empirical investigation of the process's parameters could confirm or reject this possibility. In contrast to the classical joint hypothesis test of an efficient market, this alternative provides a test of market efficiency where the additional hypothesis can be independently validated (see Chapter 3 below).

This theorem also helps us to understand the relationship between an efficient market and asset price bubbles. As shown in [43, 44], a complete market that is efficient (satisfies both NFLVR and ND) has no price bubbles. However, the authors provide numerous examples of efficient but incomplete markets that contain price bubbles. Hence, there is a weak relationship between market efficiency and the non-existence of asset price bubbles, the link is the notion of a complete market.

Our second theorem deals with the case where $(\mathbb{F}, S)$ is *efficient with respect to* $\mathbb{F}$, i.e. where efficiency on $[0, T]$ holds for every finite T (see Definition 2.2.1).

Theorem 2.2.2 *The market $(\mathbb{F}, S)$ is efficient if and only if there is a family of probabilities $\{Q_t\}_{t\geq 0}$, where Q_t is defined on $\mathcal{F}_t$, such that*

(*i*) $Q_t = Q_s$ *on* $\mathcal{F}_s$ *for all* $s < t$,

(*ii*) $Q_t \sim P$ *on* $\mathcal{F}_t$ *and* S *is a* $(\mathbb{F}, Q_t)$ *martingale on* $[0, t]$.

Proof. Sufficiency follows by considering Q_T and applying Theorem 2.2.1 to $(\mathbb{F}, S)$ restricted to $[0, T]$. For necessity, it suffices to construct measures Q^n, $n \in \mathbb{N}$, such that $Q^n \sim P$, $Q^{n+1} = Q^n$ on $\mathcal{F}_n$, and S is a Q^n martingale on $[0, n]$. We construct the Q^n inductively. Let $Q^0 = P$. Suppose Q^{n-1} has been constructed, and choose $\widetilde{Q}^n$, equivalent to P, such that S becomes a uniformly integrable martingale on $[0, n]$. Such a measure exists due to the hypothesis and Theorem 2.2.1. Let $Z^{n-1}_t = E^P\left[\frac{dQ^{n-1}}{dP} \,\middle|\, \mathcal{F}_t\right]$ and $\widetilde{Z}^n_t = E^P\left[\frac{d\widetilde{Q}^n}{dP} \,\middle|\, \mathcal{F}_t\right]$, and define

$$Z^n_t = \begin{cases} Z^{n-1}_t & t < n-1, \\ Z^{n-1}_{n-1}\frac{\widetilde{Z}^n_t}{\widetilde{Z}^n_{n-1}} & t \geq n-1. \end{cases}$$

The measure Q^n given by $\frac{dQ^n}{dP} = Z^n_n$ has density process Z^n, which coincides with Z^{n-1} on $[0, n-1]$ implying that $Q^n = Q^{n-1}$ on $\mathcal{F}_{n-1}$.

It remains to check that S is a Q^n martingale on $[0, n]$, so pick $0 \leq s < t \leq n$ and $A \in \mathcal{F}_s$. First, if $t \leq n-1$, then $E^{Q^n}\left[\mathbf{1}_A(S^i_t - S^i_s)\right] = E^{Q^{n-1}}\left[\mathbf{1}_A(S^i_t - S^i_s)\right] = 0$ for each i. If instead $s \geq n-1$, then Bayes' rule yields

$$E^{Q^n}\left[S^i_t \mid \mathcal{F}_s\right] = \frac{1}{Z^n_s}E^P\left[Z^n_t S^i_t \mid \mathcal{F}_s\right] = \frac{1}{\widetilde{Z}^n_s}E^P\left[\widetilde{Z}^n_t S^i_t \mid \mathcal{F}_s\right] = E_{\widetilde{Q}^n}\left[S^i_t \mid \mathcal{F}_s\right] = S^i_s.$$

Finally, if $s \leq n-1 \leq t$, then

$$E^{Q^n}\left[\mathbf{1}_A(S^i_t - S^i_s)\right] = E^{Q^n}\left[\mathbf{1}_A(S^i_t - S^i_{n-1})\right] + E^{Q^n}\left[\mathbf{1}_A(S^i_{n-1} - S^i_s)\right] = 0,$$

by the two previous cases. The proof is complete. □

Most of the empirical literature testing for market efficiency utilizes discrete time markets (see [27, 28, 29] and [46] for reviews). Hence it is important to understand the characterization of market efficiency in a discrete time model. Specifically, let $(\mathbb{F}, S)$ be a market in discrete time, $t \in \{0, 1, \ldots\}$. Then $(\mathbb{F}, S)$ is efficient on $\{0, \ldots, T\}$ with respect to $\mathbb{F}$ if and only if it satisfies NFLVR on $\{0, \ldots, T\}$. The proof of this claim is straightforward and therefore omitted. In fact, NFLVR implies (in our setting) that a true martingale measure exists, so the Dalang-Morton-Willinger (DMW) Theorem, see [13], lets us conclude that in discrete time, NFLVR excludes arbitrage using strategies that are not necessarily admissible. Conversely, if no such arbitrage opportunities exist, the DMW Theorem gives an equivalent martingale measure, thus showing that the market is efficient. This connection is relevant, because in discrete time the setting of the DMW Theorem is arguably more suitable than that of NFLVR.

2.3 Different Information Sets

In this section we study how market efficiency is affected by changes in the information sets, both information reductions and expansions. More formally we consider nested filtrations $\mathbb{F} \subset \mathbb{G}$, and study conditions under which efficiency with respect to $\mathbb{F}$ carries over to $\mathbb{G}$, and vice-versa. The results in this section relies crucially on the characterization of efficiency in terms of equivalent martingale measures. The corresponding analysis in the context of an equilibrium model would be much more complicated.

2.3.1 Filtration Shrinkage

If $(\mathbb{G}, S)$ is known to be efficient and we want to deduce the efficiency of $(\mathbb{F}, S)$, the analysis is particularly simple. We therefore start by treating this case. The following result is classical, see e.g. [59], Theorem I.21:

Lemma 2.3.1 *Let a filtered probability space with time set* $[0, T]$ *be given. A càdlàg, adapted process* M *such that*

$$E[|M_\tau|] < \infty \qquad \text{and} \qquad E[M_\tau] = E[M_0]$$

for every stopping time τ *is a uniformly integrable martingale.*

Theorem 2.3.1 *Let* S *be an n-dimensional* $\mathbb{G}$ *semimartingale with nonnegative components and suppose that the market* $(\mathbb{G}, S)$ *is efficient on* $[0, T]$*. If* $\mathbb{F} \subset \mathbb{G}$ *is a filtration to which* S *is adapted, then* S *is an* $\mathbb{F}$ *semimartingale, and* $(\mathbb{F}, S)$ *is efficient.*

Proof. By Theorem 2.2.1 there is $Q \sim P$ such that S is a $(\mathbb{G}, Q)$ uniformly integrable martingale. Let τ be any $\mathbb{F}$ stopping time. It is then also a $\mathbb{G}$ stopping time, so $E^Q[|S^i_\tau|] < \infty$ and $E^Q[S^i_\tau] = E^Q[S^i_0]$ for each i by the optional stopping theorem. But then S is a uniformly integrable $(\mathbb{F}, Q)$ martingale by Lemma 2.3.1, and we may conclude by Theorem 2.2.1. □

With respect to the model described in Section 2.1 and the information sets discussed in the finance literature, efficiency of $(\mathbb{F}, S)$ is called semi-strong form efficiency, since in our economy $\mathbb{F}$ corresponds to publicly available information. Theorem 2.3.1 then proves that semi-strong form efficiency implies weak-form efficiency. Weak-form efficiency corresponds to the information set generated by past security prices $(\mathbb{F}^S, S)$, and in our economy $\mathbb{F}^S \subset \mathbb{F}$. In contrast, strong-form efficiency, inside information, corresponds to an information set expansion. This is discussed in the next section.

2.3.2 Filtration Expansion

For market efficiency under information expansion, we start with an efficient market $(\mathbb{F}, S)$ and consider a larger filtration $\mathbb{G} \supset \mathbb{F}$. In general, it is well known in the finance literature (e.g. [27], p. 388, or [46], p. 97) that when the information set is expanded to include inside information, market efficiency need not be preserved. Using our characterization theorems, we can easily confirm these insights with a simple example. In this example, the additional information is knowing the risky security's price at a later date. Given this information, an arbitrage strategy is easily constructed.

Consider a market consisting of only two assets, the money market account and a single risky security. Let the risky security's price process be $S^1_t = \exp(B_t - \frac{1}{2}t)$ where B is a Brownian motion on $[0, 1]$ with the natural augmented filtration $\mathbb{F}$. We know that the market $(\mathbb{F}, S)$ is efficient since there exists a martingale probability measure. Indeed, S is already a martingale under P.

Next, consider the inside information set $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le 1}$ where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(S^1_1)$ represents all information, including the future realizations of the risky security's time 1 value. This information is known at time 0. Then, one can show (see [38]) that S^1 is a $\mathbb{G}$ semimartingale. The market $(\mathbb{G}, S)$ is not efficient. Indeed, consider the admissible strategy $H_t = \mathbf{1}_{\{S^1_1 \ge 2\}} \mathbf{1}_{(0,1]}(t)$ whose final payoff is $(S^1_1 - 1)\mathbf{1}_{\{S^1_1 \ge 2\}}$. If $P\{S^1_1 \ge 2\} > 0$, then this admissible strategy is an arbitrage opportunity. Hence, NA is violated, thus also ND and NFLVR. Therefore, by Theorem 2.2.1, the market based on the augmented information set $(\mathbb{G}, S)$ is not efficient.

A different and perhaps more important question in this context is the following: if $(\mathbb{F}, S)$ is efficient and $(\mathbb{G}, S)$ satisfies NFLVR, when is $(\mathbb{G}, S)$ efficient? We know, via Theorem 2.2.1, that a necessary and sufficient condition is that ND holds also for $(\mathbb{G}, S)$. The next section gives an explicit example where passing from $(\mathbb{F}, S)$ to $(\mathbb{G}, S)$ can yield an inefficient market, which however still satisfies NFLVR.

Example (An NFLVR but Inefficient Market)

We now give an example of a market $(\mathbb{F}, S)$ that is efficient, and where under information expansion $\mathbb{G} \supset \mathbb{F}$, the market $(\mathbb{G}, S)$ satisfies NFLVR but not ND. The example is based on a construction in [21], which we repeat here for clarity of the presentation. For simplicity we let the time set be $[0, \infty]$; an example in the finite horizon case can be achieved through a suitable time change. The values at infinity of all involved processes are determined by their limits as $t \to \infty$, which always exist.

Let the filtration $\mathbb{F}$ be the natural augmented filtration generated by two independent Brownian motions W and B. In this example we take $\mathcal{F} = \mathcal{F}_\infty$. Define the stopping times

$$\tau = \inf\{t \geq 0 : \mathcal{E}(W)_t = 2\} \qquad \text{and} \qquad \rho = \inf\{t \geq 0 : \mathcal{E}(B)_t = 1/2\}$$

where $\mathcal{E}(B)_t = \exp(B_t - \frac{1}{2}t)$ is the stochastic exponential of the Brownian motion

B, and similarly for $\mathcal{E}(W)$. Define processes S and Z by

$$S = \mathcal{E}(B)^{\tau\wedge\rho} \qquad \text{and} \qquad Z = \mathcal{E}(W)^{\tau\wedge\rho}.$$

Lemma 2.3.2 (Delbaen, Schachermayer [21]) *The following statements hold:*

(i) *S is a non-uniformly integrable P local martingale.*

(ii) *Z is a uniformly integrable P martingale with $Z_\infty > 0$ a.s. and $E[Z_\infty] = 1$.*

(iii) *SZ is a uniformly integrable P martingale, implying that S is a uniformly integrable martingale under the measure $Q \sim P$ given by $dQ = Z_\infty dP$.*

(iv) $P(\tau < \infty) = \frac{1}{2}$.

The next step is to construct a filtration $\mathbb{G} \supset \mathbb{F}$ such that the price process S still satisfies NFLVR ($\mathcal{M}_{loc}(\mathbb{G}) \neq \emptyset$), but no $R \in \mathcal{M}_{loc}(\mathbb{G})$ exists under which S becomes uniformly integrable. We let $\mathbb{G}$ be the *initial expansion* of $\mathbb{F}$ with the stopping time τ, i.e. the right-continuous completion of

$$\mathbb{F} \vee \sigma(\tau) = (\mathcal{F}_t \vee \sigma(\tau))_{t\geq 0}.$$

(Note that $\mathcal{G}_\infty = \mathcal{F}_\infty = \mathcal{F}$.) Initial expansions of filtrations have been studied extensively by several authors, see e.g. [40] and the book [47]. However, our example is sufficiently simple that we do not need the general theory of initial expansions.

Lemma 2.3.3 *The process B is Brownian motion with respect to $(\mathbb{G}, P)$.*

Proof. Fix $0 \leq s < t < \infty$. The distribution under P of $B_t - B_s$ does not depend on the filtration, so it remains normally distributed with zero mean and variance

$t - s$. Moreover, B is certainly $\mathbb{G}$ adapted. It remains to prove that $B_t - B_s$ is independent of $\mathcal{G}_s$ under P. The filtration $\mathbb{G}$ is the right-continuous completion of

$$(\mathcal{G}_t^0)_{t\geq 0} = (\mathcal{F}_t^B \vee \mathcal{F}_t^W \vee \sigma(\tau))_{t\geq 0},$$

where $(\mathcal{F}_t^B)_{t\geq 0}$ and $(\mathcal{F}_t^W)_{t\geq 0}$ denote the natural augmented filtrations of B and W, respectively. Pick any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, and define $F = f(B_t - B_s)$. Let X, Y, and Z be bounded random variables measurable with respect to $\mathcal{F}_s^B$, $\mathcal{F}_s^W$, and $\sigma(\tau)$, respectively. Since FX is $\mathcal{F}_\infty^B$-measurable, YZ is $\mathcal{F}_\infty^W$-measurable, and B and W are independent under P, it follows that FX and YZ are independent under P. Similarly, X and YZ are independent. Moreover, since B is Brownian motion, F is independent of $\mathcal{F}_s^B$, and thus of X. This yields

$$E^P[FXYZ] = E^P[FX]E^P[YZ] = E^P[F]E^P[X]E^P[YZ] = E^P[F]E^P[XYZ].$$

By the Monotone Class Theorem, we get $E^P[Fg] = E^P[F]E^P[g]$ for every bounded, $\mathcal{G}_s^0$-measurable g. Now let $F^\varepsilon = f(B_t - B_{s+\varepsilon})$ for $\varepsilon > 0$ small, and pick any bounded, $\mathcal{G}_s$-measurable g. Then g is $\mathcal{G}_{s+\varepsilon}^0$-measurable, so by the above, $E^P[F^\varepsilon g] = E^P[F^\varepsilon]E^P[g]$. Letting $\varepsilon \downarrow 0$ and using continuity and boundedness of f, we obtain $E^P[Fg] = E^P[F]E^P[g]$. This suffices to conclude that $B_t - B_s$ and $\mathcal{G}_s$ are independent. □

As a consequence of Lemma 2.3.3 and the fact that $\tau \wedge \rho$ is a $\mathbb{G}$ stopping time, $S = \mathcal{E}(B)^{\tau\wedge\rho}$ remains a $(\mathbb{G}, P)$ local martingale. In particular, S satisfies NFLVR with respect to $\mathbb{G}$. However, the following result shows that ND fails, which completes our example.

Theorem 2.3.2 *The market $(\mathbb{G}, S)$ constructed above does not satisfy ND.*

Proof. We will prove that $\mathcal{M}(\mathbb{G}) = \emptyset$. Define the $\mathbb{G}$ adapted process $\widetilde{S} = \mathbf{1}_{\{\tau=\infty\}}S$.

We claim that if S is a $(\mathbb{G}, R)$ uniformly integrable martingale for some $R \sim P$, then so is $\widetilde{S}$. Indeed, in this case

$$\widetilde{S}_t = \mathbf{1}_{\{\tau=\infty\}} S_t = \mathbf{1}_{\{\tau=\infty\}} E^R[S_\infty \mid \mathcal{G}_t] = E^R[\mathbf{1}_{\{\tau=\infty\}} S_\infty \mid \mathcal{G}_t],$$

so that $\widetilde{S}$ is closed by $\mathbf{1}_{\{\tau=\infty\}} S_\infty$. Suppose for contradiction that such an R exists. Then

$$E^R[\widetilde{S}_\infty] = E^R[\widetilde{S}_0] = R(\tau = \infty).$$

On the other hand,

$$E^R[\widetilde{S}_\infty] = E^R[\mathbf{1}_{\{\tau=\infty\}} \mathcal{E}(B)_\rho] = \frac{1}{2} R(\tau = \infty).$$

Since $R \sim P$ and $P(\tau = \infty) = \frac{1}{2} > 0$, this is a contradiction. It follows that $\widetilde{S}$ cannot be a $(\mathbb{G}, R)$-uniformly integrable martingale for any $R \sim P$, so neither can S. □

The remainder of this section looks for alternative conditions that imply efficiency (or equivalently ND) under an information set expansion. We discover three sufficient conditions; if the market is either: (i) discrete time, (ii) complete, or (iii) the H-hypothesis holds.

Discrete Time Markets

In a discrete time market, if $(\mathbb{F}, S)$ is efficient and $(\mathbb{G}, S)$ satisfies NFLVR, then $(\mathbb{G}, S)$ is efficient. This follows directly from our earlier observation that under this hypothesis NFLVR is a sufficient condition for the efficiency of $(\mathbb{G}, S)$.

Complete Markets

If $(\mathbb{F}, S)$ is a complete and efficient market and $(\mathbb{G}, S)$ satisfies NFLVR, then $(\mathbb{G}, S)$ is efficient. This follows because in a complete market, strategies which are maximal in the smaller filtration also remain maximal in the larger filtration (subject to certain regularity conditions). Hence, information expansion introduces no new profitable trading strategies. To prove this claim, we start with the definition of a complete market.

We will use the following definition of completeness; it says that there is only one risk-neutral measure on $\mathcal{F}_\infty$.

Definition 2.3.1 (Completeness) *A market* $(\mathbb{F}, S)$ *is called* complete *if it satisfies NFLVR and all* $Q \in \mathcal{M}(\mathbb{F})$ *coincide on* $\mathcal{F}_\infty$.

For the rest of this section, *we restrict attention to the case where the security process S is strictly positive and $\mathbb{F}$ locally bounded*. This guarantees that S is a special semimartingale, which is needed for the proof of the following lemma.

Lemma 2.3.4 *Let S be an n-dimensional, locally bounded $\mathbb{F}$ semimartingale with positive components, satisfying NFLVR with respect to $\mathbb{F}$. If $\mathbb{G} \supset \mathbb{F}$ is a larger filtration, then* $\mathcal{M}_{loc}(\mathbb{G}) \subset \mathcal{M}_{loc}(\mathbb{F})$.

Proof. A theorem by Stricker [65] says that if M is a positive $\mathbb{G}$ local martingale, then it is an $\mathbb{F}$ supermartingale, and if in addition M is $\mathbb{F}$ special, then it is an $\mathbb{F}$ local martingale. Each S^i satisfies these conditions under any $Q \in \mathcal{M}_{loc}(\mathbb{G})$, taking into account that S is locally bounded with respect to $\mathbb{F}$ and hence special.
□

Theorem 2.3.3 *Let $(\mathbb{F}, S)$ be a complete market, and suppose that S locally bounded with strictly positive components. If $\mathbb{G} \supset \mathbb{F}$ is a larger filtration such that $(\mathbb{G}, S)$ satisfies*

NFLVR, then every locally bounded $\mathbb{F}$-maximal strategy is $\mathbb{G}$-maximal. In particular, if $(\mathbb{F}, S)$ is efficient, then so is $(\mathbb{G}, S)$.

Proof. Since S satisfies NFLVR with respect to $\mathbb{G}$, it is a $\mathbb{G}$ semimartingale. By Theorem IV.33 in [59], the stochastic integral $H \cdot S$ does not depend on the filtration ($\mathbb{F}$ or $\mathbb{G}$) as long as H is $\mathbb{F}$ predictable and locally bounded. Now, let H be a locally bounded, $\mathbb{F}$-maximal strategy. Then $E^Q[H \cdot S]_\infty = 0$ for some $Q \in \mathcal{M}_{loc}(\mathbb{F})$ by Lemma 2.1.1. However, $(\mathbb{G}, S)$ satisfies NFLVR, so with Lemma 2.3.4 and the completeness assumption we get that

$$\emptyset \neq \mathcal{M}_{loc}(\mathbb{G}) \subset \mathcal{M}_{loc}(\mathbb{F}) = \{Q\}.$$

Therefore $Q \in \mathcal{M}_{loc}(\mathbb{G})$, so another application of Lemma 2.1.1 shows that H is $\mathbb{G}$-maximal. Finally, ND and hence completeness of $(\mathbb{G}, S)$ now follows from the fact that the strategies $H^i = (0, \ldots, 0, 1, 0, \ldots, 0)$, which are $\mathbb{F}$-maximal by assumption, are also $\mathbb{G}$-maximal. □

An interpretation of Theorem 2.3.3 is that given a complete and efficient market $(\mathbb{F}, S)$, any additional information that introduces inefficiencies in $(\mathbb{G}, S)$ will in fact introduce arbitrage opportunities as well, in the sense of NFLVR.

Hypothesis H

This section shows that if $(\mathbb{F}, S)$ is an efficient market, $(\mathbb{G}, S)$ satisfies NFLVR, and $\mathbb{G} \supset \mathbb{F}$ is such that the Hypothesis H holds, then $(\mathbb{G}, S)$ is efficient. Hypothesis H refers to the property that given two nested filtrations $\mathbb{F} \subset \mathbb{G}$ and a probability P, any $(\mathbb{F}, P)$ martingale is again a $(\mathbb{G}, P)$ martingale. An alternative terminology is that $\mathbb{F}$ *is immersed in* $\mathbb{G}$ *under* P.

In modeling credit risk, information expansion and reduction are important considerations. First, differential information characterizes the relationship between structural and reduced form credit risk models. A reduced form model can be obtained via information reduction in a structural model (see [42] for a review). Second, within a reduced form credit risk model, an economy is often characterized by the evolution of a set of state variables yielding the information set $\mathbb{F}$. And, default information is usually included via an expansion of this filtration to include the information generated by a set of default times, yielding the larger information set $\mathbb{G}$. One then studies the conditions under which the martingale pricing technology extends from $\mathbb{F}$ to $\mathbb{G}$. The H-hypothesis guarantees this martingale pricing extension, see [26] and [5]. It is not surprising, therefore, that the H-hypothesis also plays an important role in understanding information expansion with respect to market efficiency. Similar questions have been studied by [34] and [2], among others.

The following characterization of Hypothesis H is due to [6].

Theorem 2.3.4 (Brémaud-Yor) *The following are equivalent:*

(i) Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under the measure P.

(ii) $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. That is, for every $\mathcal{F}_\infty$-measurable nonnegative F and $\mathcal{G}_t$-measurable nonnegative G_t,

$$E^P[FG_t \mid \mathcal{F}_t] = E^P[F \mid \mathcal{F}_t]E^P[G_t \mid \mathcal{F}_t].$$

The next result was proved by [11] in the special case of progressive expansions with random times. A modification of their argument leads to the following result, where now the expanded filtration $\mathbb{G} \supset \mathbb{F}$ is completely general.

Lemma 2.3.5 *Suppose that $Q \in \mathcal{M}_{loc}(\mathbb{F})$ and that Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under some equivalent measure $R \sim Q$. Then there is $Q^* \in \mathcal{M}_{loc}(\mathbb{F})$ such that $\mathbb{F}$ is immersed in $\mathbb{G}$ under Q^*, and $Q^* = Q$ on $\mathcal{F}_\infty$.*

Proof. Let $Z = E^R\left[\frac{dQ}{dR} \,\middle|\, \mathcal{F}_\infty\right]$ and define Q^* via $dQ^* = ZdR$. Then for $A \in \mathcal{F}_\infty$,

$$E^{Q^*}(\mathbf{1}_A) = E^R[Z\mathbf{1}_A] = E^R\left[E^R\left[\frac{dQ}{dR}\mathbf{1}_A \,\middle|\, \mathcal{F}_\infty\right]\right] = E^Q[\mathbf{1}_A],$$

so $Q = Q^*$ on $\mathcal{F}_\infty$. In particular, then, $Q^* \in \mathcal{M}_{loc}(\mathbb{F})$. Now, choose any $\mathcal{F}_\infty$-measurable $F \geq 0$ and $\mathcal{G}_t$-measurable $G_t \geq 0$. Bayes' rule, immersion under R, and the fact that Z is $\mathcal{F}_\infty$-measurable and nonnegative yield

$$\begin{aligned} E^{Q^*}[FG_t \mid \mathcal{F}_t] &= \frac{E^R[ZFG_t \mid \mathcal{F}_t]}{E^R[Z \mid \mathcal{F}_t]} \\ &= \frac{E^R[ZF \mid \mathcal{F}_t]}{E^R[Z \mid \mathcal{F}_t]} E^R[G_t \mid \mathcal{F}_t] \\ &= E^{Q^*}[F \mid \mathcal{F}_t] E^R[G_t \mid \mathcal{F}_t]. \end{aligned}$$

Similarly,

$$E^{Q^*}[G_t \mid \mathcal{F}_t] = \frac{E^R[ZG_t \mid \mathcal{F}_t]}{E^R[Z \mid \mathcal{F}_t]} = E^R[G_t \mid \mathcal{F}_t].$$

Hence $E^{Q^*}[FG_t \mid \mathcal{F}_t] = E^{Q^*}[F \mid \mathcal{F}_t] E^{Q^*}[G_t \mid \mathcal{F}_t]$, so immersion holds under Q^*, as desired. □

We now give the key theorem of this section. We note that Hypothesis H only has to hold under some arbitrary equivalent measure, not necessarily P or some $Q \in \mathcal{M}_{loc}(\mathbb{F})$.

Theorem 2.3.5 *Let $(\mathbb{F}, S)$ be a market that satisfies NFLVR. Suppose that $\mathbb{G} \supset \mathbb{F}$ is a larger filtration such that Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under some equivalent measure. Then $(\mathbb{G}, S)$ satisfies NFLVR, and every locally bounded $\mathbb{F}$-maximal strategy is $\mathbb{G}$-maximal. In particular, if $(\mathbb{F}, S)$ is efficient, then so is $(\mathbb{G}, S)$.*

Proof. By Lemma 2.3.5, the intersection $\mathcal{M}_{loc}(\mathbb{F}) \cap \mathcal{M}_{loc}(\mathbb{G})$ is non-empty, so $(\mathbb{G}, S)$ satisfies NFLVR. Let H be locally bounded and $\mathbb{F}$-maximal, so that $E^Q(H \cdot S)_T = 0$ for some $Q \in \mathcal{M}_{loc}(\mathbb{F})$. By Lemma 2.3.5 there is $Q^* \in \mathcal{M}_{loc}(\mathbb{G})$ coinciding with Q on $\mathcal{F}_T$, so $E^{Q^*}(H \cdot S)_T = 0$ and H is $\mathbb{G}$-maximal. As in the proof of Theorem 2.3.3, the local boundedness of H ensures that $H \cdot S$ does not depend on the filtration. Also as in the proof of Theorem 2.3.3, the efficiency of $(\mathbb{G}, S)$ follows from the fact that the strategies $H^i = (0, \ldots, 0, 1, 0, \ldots, 0)$ remain maximal in $\mathbb{G}$. □

CHAPTER 3

EFFICIENCY IN HIGH DIMENSIONAL STOCHASTIC VOLATILITY MODELS

In this chapter we consider some models for price processes useful for pricing options on equities and equity indices. We investigate when these price processes are consistent with market efficiency. We take the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ to be generated by an n-dimensional Brownian motion $W = (W^1, \dots, W^n)$. As before, there are d risky assets with positive price processes $S = (S_t^1, \dots, S_t^d)_{0\le t\le T}$, $S_t^i \ge 0$ for all i, as well as a risk free asset whose price is set to $S_t^0 \equiv 1$. Trading strategies are modeled by admissible integrands. We will assume throughout that NFLVR is satisfied, and that $P \in \mathcal{M}_{loc}$ (see Section 2.1.3). Due to the characterization theorem in Chapter 2, Theorem 2.2.1, the question of whether efficiency holds is reduced to determining whether ND holds, or, equivalently, whether $\mathcal{M}$ is nonempty.

We first consider quite general local volatility models, where we observe that a certain dichotomy is present: if NFLVR holds, then either $\mathcal{M}_{loc} = \mathcal{M}$ or $\mathcal{M} = \emptyset$. In the first case, by Theorem 2.2.1, the market $(\mathbb{F}, S)$ is efficient, while in the second case it is not. We then proceed with a detailed analysis of a class of stochastic volatility models, giving sufficient and (in a certain sense to be described later) necessary conditions for efficiency. Our goal is to show that there are large classes of efficient models, many of them with price processes that are strict local martingales with respect to the measure under which their dynamics would typically be specified. Results in this vein are well known in the one-dimensional case, see especially Sin [64]. In contrast, our results are established in the multi-dimensional case, which is the appropriate setting since

$(\mathbb{F}, S)$ should be thought of as a model for an entire market.

These results have two uses. First, they provide an alternative method for testing market efficiency based on a joint hypothesis. Here the joint hypothesis is the specification of a particular stochastic process for asset prices. This additional hypothesis is testable independently of market efficiency. Once this is done, efficiency can be accepted or rejected depending on the estimated parameters. In contrast, the classical joint hypothesis—specifying a particular equilibrium model—is not independently testable. The equilibrium model and efficiency are both accepted or rejected in unison.

Second, these results are useful for pricing securities in positive net supply when one wants to impose more structure on the price process than just NFLVR. In particular, one may only want to consider price processes that are consistent with some economic equilibrium, or alternatively stated, are consistent with an efficient market. Our characterization theorems enable one to understand the additional structure required. Such restrictions have already proven useful in the context of asset price bubbles, see [43, 44].

3.1 Local Volatility

Assume that the price process $S = (S^1, \dots, S^d)$ is governed by the following system of stochastic differential equations.

$$dS^i_t = \sigma^i(t, S_t)^\top dW_t + b^i(t, S_t)dt \qquad (i = 1, \dots, d), \tag{3.1}$$

where $\sigma^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$ and $b^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ are such that a weak solution, unique in law, exists with $S^i_t > 0$ for all $t \in [0, T]$. We also assume that the σ^i are sufficiently regular to guarantee weak existence and uniqueness if the b^i are set to be identically zero.

Assume now that NFLVR holds, so that $\mathcal{M}_{loc} \neq \emptyset$. By the martingale representation theorem, the density process $Z_t = E^P\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]$ associated with some $Q \in \mathcal{M}_{loc}$ can be expressed as $dZ_t = Z_t\theta_t^\top dW_t$ for some adapted, $\mathbb{R}^n$-valued process θ that depends on Q. Defining $W^Q = W - \int_0^\cdot \theta_s ds$, Girsanov's theorem implies that

$$dS_t^i = \sigma^i(t, S_t)^\top dW_t^Q + \left(\sigma^i(t, S_t)^\top \theta_t + b^i(t, S_t)\right) dt \qquad (i = 1, \ldots, d).$$

Since S^i is a local martingale under Q, the drift term is identically zero, so that

$$dS_t^i = \sigma^i(t, S_t)^\top dW_t^Q \qquad (i = 1, \ldots, d).$$

Since W^Q is Brownian motion under Q uniqueness in law implies that S has the same law under every $Q \in \mathcal{M}_{loc}$. This immediately yields the following theorem.

Theorem 3.1.1 *If the local volatility model described in (3.1) satisfies NFLVR, then it is either a true martingale under every $Q \in \mathcal{M}_{loc}$ and $(\mathbb{F}, S)$ is efficient, or it is a strict local martingale under every $Q \in \mathcal{M}_{loc}$ and $(\mathbb{F}, S)$ is inefficient.*

Which of the two possibilities actually holds is determined entirely by the properties of σ. Necessary and sufficient conditions under various regularity assumptions on σ have been investigated by several authors, see for example [9] and [56]. For example, in the case where $d = 1$ and $\sigma^1(t, x) = \sigma(x)$ for some measurable function $\sigma(\cdot)$ satisfying weak regularity conditions, the price process is a true martingale under Q if and only if for some $c > 0$,

$$\int_c^\infty \frac{x}{\sigma(x)^2} dx = \infty.$$

We remark that the question of whether the local volatility model described above satisfies NFLVR or not is less interesting; this is almost always assumed, and the risk-neutral dynamics are then specified directly (i.e., one does not model the b^i.)

3.2 Stochastic Volatility, Constant Correlation Structure

Let us now describe a class of stochastic volatility models where the correlation structure between the different processes does not change with time. We expand upon earlier work of Sin [64], who considers a similar model in the one-dimensional case. See also Hobson [36], who investigates related problems in the one-dimensional case.

The price process is given by the following system of stochastic differential equations.

$$\begin{aligned} dS_t^i &= S_t^i f^i(t, v_t)\sigma_i^\top dW_t \qquad (i = 1, \ldots, d) \\ dv_t^j &= a_j^\top dW_t + b^j(t, v_t^j)dt \qquad (j = 1, \ldots, m). \end{aligned}$$

Here $\sigma_i, a_j \in \mathbb{R}^n$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$. Moreover, each $b^j : [0, T]\times\mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz. This guarantees that the SDE for $v_t = (v_t^1, \ldots, v_t^m)$ has a strong (non-explosive) solution on $[0, T]$. If, for instance, $f^i : [0, T] \times \mathbb{R}^m \to \mathbb{R}_+$ is locally bounded, the local martingales

$$S_t^i = S_0^i \exp\left(\int_0^t f^i(s, v_s)\sigma_i^\top dW^s - \frac{1}{2}\|\sigma_i\|^2 \int_0^t f^i(s, v_s)^2 ds\right), \qquad i = 1, \ldots, d,$$

stay strictly positive (we assume that $S_0^i > 0$ for all i.) This will be the case under the conditions we will impose on the f^i. Notice that NFLVR is automatically satisfied since each S^i is a local martingale under the original measure. Specifying the model in this way is typical in applications, and allows us to focus on the question of whether ND holds.

The following condition will be imposed on the model.

Condition 3.2.1 *The functions f^i are Lipschitz on $(-\infty, C]^m$ for every $C > 0$. More precisely, there exist constants K_C such that for $i = 1, \ldots, d$,*

$$|f^i(y,t) - f^i(z,t)| \leq K_C |y - z|$$

for every $t \in [0, T]$ and $y, z \in \mathbb{R}^m$ with $y^j \leq C$, $z^j \leq C$, $j = 1, \ldots, m$.

At first glance, this condition may seem somewhat restrictive. However, note that $f^i(t, y)$ is always nonnegative and should be thought of as being increasing in each volatility component y^j. Moreover, the condition imposes no restrictions on the growth rate of $f^i(t, y)$ as the components of y become large.

An important special case where (i) holds is when $b^j(t, v_t) = \rho_j(\kappa_j - v_t^j)$ for some positive constants ρ_j and κ_j, i.e. where the volatilities are mean-reverting. In this case the part of (ii) pertaining to b^j is also satisfied. This is similar to the situation considered by Sin [64].

3.2.1 A First Sufficient Condition

In this section we provide a sufficient condition for ND within the stochastic volatility model with constant correlation structure described above. The result will be strengthened later (see Section 3.2.2), but it is nonetheless useful to study this simpler condition first. This is because it allows one to explicitly construct the density of a martingale measure. The stronger result in Section 3.2.2 is then obtained by combining the "dual" techniques used here with "primal" arguments, reasoning about the existence of arbitrage strategies.

Theorem 3.2.1 *Consider the stochastic volatility model with constant correlations described above, and assume that Condition 3.2.1 is satisfied. If there is a vector* $\theta \in \mathbb{R}^d$ *such that for all* i *and* j,

$$\theta^\top \sigma_i = 0, \qquad \theta^\top a_j \geq \sigma_i^\top a_j, \qquad \theta^\top a_j \geq 0,$$

then $\mathcal{M} \neq \emptyset$. *If* $\sigma_i^\top a_j \leq 0$ *for all* i *and all* j, *then* S *is already a martingale under* P.

Remark. One noteworthy special case where the last part of Theorem 3.2.1 applies is when each of the vectors a_j is orthogonal to all the σ_i. In this case there are, after a change of coordinates, two independent sets of Brownian motions, one driving the S^i and the other driving the v^j.

The following corollary gives a simple geometric condition that guarantees the existence of the vector θ required in Theorem 3.2.1. For a set of vectors $y_1, \ldots, y_n$, let $\mathrm{conv}(y_1, \ldots, y_n)$ denote their convex hull, and $\mathrm{span}(y_1, \ldots, y_n)$ their linear span.

Corollary 3.2.1 *Consider the stochastic volatility model with constant correlations described above, and assume that Condition 3.2.1 is satisfied. If*

$$\mathrm{conv}(a_1, \ldots, a_m) \cap \mathrm{span}(\sigma_1, \ldots, \sigma_d) = \emptyset,$$

then $\mathcal{M} \neq \emptyset$.

Proof. Since $\mathrm{conv}(a_1, \ldots, a_m)$ is compact and convex, and $\mathrm{span}(\sigma_1, \ldots, \sigma_d)$ is closed and convex they can be strictly separated by a hyperplane. In particular, there exists $\theta \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\theta^\top a_j > \alpha$ for all j and $\theta^\top(\lambda\sigma_i) \leq \alpha$ for all i and all $\lambda \in \mathbb{R}$. Take $\lambda = \pm 1$ to see that $\alpha = 0$ and $\theta^\top \sigma_i = 0$ for all i. By positive scaling we may assume that $\theta^\top a_j \geq \sigma_i^\top a_j$ for all i and j. Apply Theorem 3.2.1 with this θ. □

Remark. The larger $n-m$, the "easier" it is for the condition in Corollary 3.2.1 to be satisfied. In particular, it holds if $m = 1$ and a_1 is not in the span of $\sigma_1, \ldots, \sigma_d$. On the other hand, if $\text{span}(\sigma_1, \ldots, \sigma_d) = \mathbb{R}^n$, then Corollary 3.2.1 cannot be applied. This is the case of a complete market. In this situation, going back to Theorem 3.2.1, the only candidate for θ is the zero vector, in which case one would need $\sigma_i^\top a_j \leq 0$ for all i and j in order to deduce efficiency. In fact, having $\sigma_i^\top a_j \leq 0$ for all i and j is *necessary* when the σ_i span $\mathbb{R}^n$, in a sense that will be discussed in Section 3.2.3.

The proof of Theorem 3.2.1 requires two lemmas, both of which are similar to results that are well-known in the literature. For later use we state them in a more general form than needed for Theorem 3.2.1. The first lemma is a slight modification of a comparison theorem due to Ikeda and Watanabe, see [37], Theorem 1.1.

Lemma 3.2.1 *Suppose that for $j = 1, 2$ and some continuous $a : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^n$, we have*

$$Y_t^j = Y_0^j + \int_0^t a(s, Y_s^j)^\top dW_s + \int_0^t \beta_s^j ds,$$

where W is n-dimensional Brownian motion and β^j are adapted processes. Suppose the following conditions are satisfied:

(i) $\beta_t^1 \geq b^1(t, Y_\cdot^1)$ and $b^2(t, Y_\cdot^2) \geq \beta_t^2$ for some predictable path functionals[1] b^1, b^2 with $b^1(t, y_\cdot) \geq b^2(t, y_\cdot)$ for all y and t.

[1]See Rogers and Williams [60], Chapter V.2, for the definition of predictable path functionals, corresponding notions of Lipschitz continuity, etc.

(ii) *There is an increasing* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\rho(0) = 0$, $\int_{0+} \rho(u)^{-2} du = \infty$ *such that for all* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$, *a satisfies*

$$\|a(t, x) - a(t, y)\| \leq \rho(\|x - y\|).$$

(iii) $Y_0^1 \geq Y_0^2$.

(iv) *Pathwise uniqueness holds for one of* $dY_t = a(t, Y_t)^\top dW_t + b^j(t, Y_\cdot)dt$, $j = 1, 2$.

Then $Y_t^1 \geq Y_t^2$ *for all* t.

Proof. Theorem 1.1 in [37] contains the above statement, but for the case $n = 1$ and b^1, b^2 being defined on $\mathbb{R} \times \mathbb{R}_+$, rather than path space. However, their proof remains valid for our setup. □

The second lemma uses the same techniques as the proof of Lemma 4.2 in Sin [64]. See also [8], [9], [55]. For completeness and since the proof is quite short, we provide the details. Thanks are due to Younes Kchia, who pointed out an error in an earlier version of this lemma.

Lemma 3.2.2 *Let* Y *be an* $\mathbb{R}^d$*-valued diffusion on* $[0, T]$ *satisfying a stochastic differential equation*

$$dY_t = A(t, Y_\cdot)dW_t + b(t, Y_\cdot)dt,$$

where W *is* n*-dimensional Brownian motion and* A *and* b *are predictable path functionals with values in* $\mathbb{R}^{d \times n}$ *and* $\mathbb{R}^d$, *respectively. Assume that a non-explosive solution exists and is pathwise unique on* $[0, T]$. *If* f *is an* $\mathbb{R}^n$*-valued predictable path functional, locally Lipschitz, such that the auxiliary SDE*

$$d\widehat{Y}_t = A(t, \widehat{Y}_\cdot)dW_t + [b(t, \widehat{Y}_\cdot) + A(t, \widehat{Y}_\cdot)f(t, \widehat{Y}_\cdot)]dt, \qquad \widehat{Y}_0 = Y_0 \tag{3.2}$$

has a non-explosive and pathwise unique solution on $[0, T]$*, then the positive local martingale* X *given by*

$$X_t = \exp\left(\int_0^t f(s, Y_\cdot)^\top dW_s - \frac{1}{2}\int_0^t |f(s, Y_\cdot)|^2 ds\right)$$

is a true martingale on $[0, T]$*.*

Proof. Define stopping times

$$\tau_k = \inf\left\{t \geq 0 : \int_0^t \|f(s, Y_\cdot)\|^2 ds \geq k\right\} \wedge T$$

and processes $X^k = X^{\tau_k} = \exp\{M^k - \langle M^k, M^k\rangle\}$, where $M_t^k = \int_0^{t\wedge\tau_k} f(s, Y_\cdot)^\top dWs$. By Novikov's criterion, each X^k is a true martingale. It stays strictly positive, so we define equivalent measures Q^k by $dQ^k = X_T^k dP$. By Girsanov's theorem,

$$dY_t = A(t, Y_\cdot)dW_t^k + \left[b(t, Y_\cdot) + \mathbf{1}_{\{\mathbf{t}\leq\tau_\mathbf{k}\}}A(t, Y_\cdot)f(t, Y_\cdot)\right] dt,$$

where $dW_t^k = dW_t - \mathbf{1}_{\{\mathbf{t}\leq\tau_\mathbf{k}\}}f(t, Y_\cdot)dt$ is Brownian motion under Q^k. Next, define stopping times

$$\widehat{\tau}_k = \inf\left\{t \geq 0 : \int_0^t \|f(s, \widehat{Y}_\cdot)\|^2 ds \geq k\right\} \wedge T$$

By the non-explosion of Y and $\widehat{Y}$, the stopping times τ_k and $\widehat{\tau}_k$ are equal to T for all sufficiently large k, almost surely. Moreover, by pathwise uniqueness, the law of $\widehat{\tau}_k$ under P is the same as the law of τ_k under Q^k. These facts yield

$$\begin{aligned}
E^P[X_T] &= \lim_{k\to\infty} E^P[X_T \mathbf{1}_{\{\tau_\mathbf{k}=\mathbf{T}\}}] \\
&= \lim_{k\to\infty} E^P[X_{T\wedge\tau_k} \mathbf{1}_{\{\tau_\mathbf{k}=\mathbf{T}\}}] \\
&= \lim_{k\to\infty} Q^k(\tau_k = T) \\
&= \lim_{k\to\infty} P(\widehat{\tau}_k = T) = 1.
\end{aligned}$$

This shows that X has constant expectation and hence is a martingale. □

We are now ready to give the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. The goal is to find a measure $Q \sim P$ under which each S^i becomes a martingale. We split the proof into a number of steps.

Step 1. As a candidate density process for a measure change, let Z be the stochastic exponential of $-\int_0^\cdot h(t, v_t)\theta^\top dW_t$, where we define $h : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ by $h(t, y) = \max_{i=1,\dots,n} f^i(t, y)$. Then Z is the unique solution of

$$dZ_t = -Z_t h(t, v_t)\theta^\top dW_t, \qquad Z_0 = 1. \tag{3.3}$$

Since v_t is non-explosive, Z is a strictly positive local martingale. Lemma 3.2.2 implies that it is a true martingale if $\widehat{v}_t$ is non-explosive and pathwise unique, where

$$d\widehat{v}^j_t = a_j^\top dW_t + \left[b^j(t, \widehat{v}^j_t) - h(t, \widehat{v}_t)a_j^\top \theta\right] dt, \quad \widehat{v}^j_0 = v^j_0 \qquad (j = 1, \dots, m).$$

Step 2. Due to Condition 3.2.1, $\widehat{v}_t$ is non-explosive and pathwise unique at least up to τ_k, where

$$\tau_k = \inf\left\{t \geq 0 : \max_{j=1,\dots,m} \widehat{v}^j_t \geq k\right\}.$$

We need to show that, almost surely, $\tau_k \geq T$ for large enough k. Since $a_j^\top \theta \geq 0$, the drift coefficient of $\widehat{v}^j_t$ is bounded above by $b^j(t, \widehat{v}^j_t)$. Lemma 3.2.1 then shows that $\widehat{v}^j_t \leq w^j_t$ up to time τ_k, where w^j is the solution of

$$dw^j_t = a_j^\top dW_t + b^j(t, w^j_t)dt, \quad w_0 = v^j_0,$$

which is pathwise unique. Note that the condition on the volatility coefficient in Lemma 3.2.1 is satisfied since a_j is constant. Since b^j is Lipschitz, each w^j is non-explosive and we deduce that no $\widehat{v}^j$ can explode to $+\infty$. This shows that $\tau_k \geq T$ for large enough k.

Step 3. From Steps 1–2 it follows that Z is a true martingale on $[0, T]$, so it is the density process of the measure Q given by $dQ = Z_T dP$. Then $dB_t = dW_t + h(t, v_t)\theta dt$ is Brownian motion under Q by Girsanov's theorem, and the dynamics of S and v can be written

$$\begin{aligned} dS_t^i &= S_t^i f^i(t, v_t)\sigma_i^\top dB_t \qquad (i = 1, \ldots, d) \\ dv_t^j &= a_j^\top dB_t + \left[b^j(t, v_t^j) - h(t, v_t)a_j^\top \theta\right] dt \qquad (j = 1, \ldots, m), \end{aligned}$$

taking into account that $\theta^\top \sigma_i = 0$ for all i. The auxiliary SDE associated with S^i is

$$d\widehat{v}_t^j = a_j^\top dB_t + \left[b^j(t, \widehat{v}_t^j) + f^i(t, \widehat{v}_t)\sigma_i^\top a_j - h(t, \widehat{v}_t)\theta^\top a_j\right] dt, \quad \widehat{v}_0^j = v_0^j \qquad (j = 1, \ldots, m).$$

Since $\theta^\top a_j \geq \sigma_i^\top a_j$ and $h(t, \widehat{v}_t) \geq f^i(t, \widehat{v}_t)$, the drift coefficient is bounded above by $b^j(t, \widehat{v}_t^j) + f^i(t, \widehat{v}_t)[\sigma_i^\top a_j - \theta^\top a_j] \leq b^j(t, \widehat{v}_t^j)$. The same argument as in Step 2 shows that $\widehat{v}_t$ does not explode on $[0, T]$. This proves that S^i is a martingale under Q for each i and finishes the proof of part (i) of the theorem.

To prove the last assertion, notice that if $\sigma_i^\top a_j \leq 0$ for all i and j, then $\theta = 0$ works. Therefore S is already a martingale under the original measure. □

3.2.2 A strengthened sufficiency result

Our goal is now to obtain a sufficient condition for efficiency, which is applicable in certain situations where Theorem 3.2.1 is not. This condition will in fact also turn out to be necessary in a certain sense, see Section 3.2.3. The following result is an intermediate step.

Proposition 3.2.1 *Consider the stochastic volatility model with constant correlations, assume Condition 3.2.1 holds, and fix* $k \in \{1, \ldots, d\}$. *If there exists* $\theta \in \mathbb{R}^n$ *such that for* *all i and j, $\theta^\top \sigma_i = 0$ and $\theta^\top a_j \geq \sigma_k^\top a_j$, then there is a strictly positive local martingale Z such that ZS^i is a local martingale for each i, and a true martingale for $i = k$.*

Proof. We use the same notation as in the proof of Theorem 3.2.1. Define Z as in (3.3), that is, $dZ_t = -Z_t h(t, v_t)\theta^\top dW_t$ and $Z_0 = 1$, and write $X^i = ZS^i$. Since $\theta^\top \sigma_i = 0$, Itô's formula yields

$$dX_t^i = X_t^i (f^i(t, v_t)\sigma_i - h(t, v_t)\theta)^\top dW_t,$$

showing that ZS^i is a local martingale for each i. By Lemma 3.2.2, X^k is a martingale if $\widehat{v}$ is non-explosive, where

$$d\widehat{v}_t^j = a_j^\top dW_t + \left[b^j(t, \widehat{v}_t^j) + f^k(t, \widehat{v}_t)\sigma_k^\top a_j - h(t, \widehat{v}_t)\theta^\top a_j \right] dt.$$

Just as in Step 4 of the proof of Theorem 3.2.1, the assumption $\theta^\top a_j \geq \sigma_k^\top a_j$, implies that $\widehat{v}$ is non-explosive. □

Notice the difference between the hypothesis of Proposition 3.2.1 and that of Theorem 3.2.1: in the former, we have dropped the requirement that $\theta^\top a_j \geq 0$ (and allow θ to depend on k.) The price to pay is that the process Z is not necessarily a martingale, and thus may not be the density process of any equivalent measure. Such a process could therefore exist even if ND is violated, i.e. if $\mathcal{M} = \emptyset$. However, the asset S^k is nonetheless undominated, as the following result shows. We now only assume that S is a continuous local martingale, not necessarily following the stochastic volatility model described above.

Theorem 3.2.2 *Let S be a d-dimensional continuous local martingale with nonnegative components, and fix $k \in \{1, \ldots, d\}$. If there is a strictly positive local martingale Z such that ZS^i is a local martingale for each i and a true martingale for $i = k$, then S^k is undominated.*

Proof. If Z is a martingale, then under Q given by $dQ = Z_T dP$, S^k becomes a martingale, so the statement holds by Lemma 2.1.1. So assume that Z is a strict local martingale. Suppose for contradiction that S^k is dominated. Then there is, for some $a > 0$, an a-admissible strategy H such that $(H \cdot S)_T \geq S_T^k - S_0^k$, with positive probability of having strict inequality. Together with the martingale property of ZS^k this yields

$$E\left[Z_T(H \cdot S)_T\right] > E[Z_T S_T^k] - S_0^k E[Z_T] = S_0^k(1 - E[Z_T]).$$

Since each ZS^i is a local martingale, $\langle Z, H \cdot S\rangle = H \cdot \langle Z, S\rangle = 0$, implying that $Z(H \cdot S)$ is a local martingale. By a-admissibility, $Z(a + H \cdot S)$ is a nonnegative local martingale, hence a supermartingale. Therefore,

$$E\left[Z_T(H \cdot S)_T\right] = E\left[Z_T(a + (H \cdot S)_T)\right] - aE[Z_T] \leq a(1 - E[Z_T]).$$

Combining the two previous displays, $a(1 - E[Z_T]) > S_0^k(1 - E[Z_T])$, and hence $a > S_0^k$. This holds for every a such that H is still a-admissible. In particular, H is not S_0^k-admissible, and it follows that

$$P\Big((H \cdot S)_t < -S_0^k \text{ for some } t \in (0, T)\Big) > 0.$$

Therefore, there is an $\varepsilon > 0$ such that $P(\tau < T) > 0$, where

$$\tau = \inf\left\{t \geq 0 : (H \cdot S)_t \leq -S_0^k - \varepsilon\right\}.$$

Consider now the strategy $\widetilde{H} = H\mathbf{1}_{[\tau,T]}$. This is predictable and S-integrable, and letting $a \geq S_0^k + \varepsilon$ be such that H is a-admissible, we have

$$(\widetilde{H} \cdot S)_t = (H \cdot S)_t - (H \cdot S)_{t\wedge\tau} \geq -a + S_0^k + \varepsilon.$$

Hence $\widetilde{H}$ is admissible. Finally, $(\widetilde{H} \cdot S)_T = 0$ on $\{\tau \geq T\}$, and on $\{\tau < T\}$,

$$(\widetilde{H} \cdot S)_T = (H \cdot S)_T - (H \cdot S)_\tau \geq S_T^k - S_0^k + S_0^k + \varepsilon \geq \varepsilon,$$

by the choice of H and nonnegativity of S^k. In other words, $\widetilde{H}$ is an arbitrage strategy. This contradicts NFLVR and shows that S^k cannot be a dominated asset. □

Remark. Note the "primal" nature of the proof of Theorem 3.2.2: instead of constructing a martingale measure for S^k, a dual quantity, the core of the proof is the construction of a certain (arbitrage) strategy, which is a primal quantity.

Our strengthened sufficiency result for ND now follows immediately as a corollary, but we state it as a theorem to emphasize its importance.

Theorem 3.2.3 *Consider the stochastic volatility model with constant correlations, and assume Condition 3.2.1 holds. If for each $k \in \{1, \dots, d\}$ there exists $\theta \in \mathbb{R}^n$ (possibly depending on k) such that for all i and j,*

$$\theta^\top \sigma_i = 0 \qquad \text{and} \qquad \theta^\top a_j \geq \sigma_k^\top a_j,$$

then ND holds.

Proof. For each k, we first apply Proposition 3.2.1 and then Theorem 3.2.2 to deduce that S^k is undominated. □

3.2.3 Necessary Conditions

We cannot in general expect the sufficient conditions of Theorem 3.2.1 and/or Theorem 3.2.3 to also be necessary for ND. This is because they are independent of the choice of f^i and b^j. By choosing appropriate f^i, for instance by making them bounded, we can always guarantee that ND holds, independently of

$a_1, \ldots, a_m$ and $\sigma_1, \ldots, \sigma_d$. Instead we will identify conditions on the correlation structure under which one can find functions f^i and b^j such that ND fails. (Of course, the f^i and b^j we consider should always satisfy the assumptions of Section 3.2, in particular Condition 3.2.1.)

Theorem 3.2.4 *Consider the stochastic volatility model with constant correlations, and assume there is a vector $\eta \in \mathrm{conv}(a_1, \ldots, a_m) \cap \mathrm{span}(\sigma_1, \ldots, \sigma_d)$ with $\eta^\top \sigma_k > 0$ for some k. Then there exist functions f^i and b^j that satisfy the assumptions of Section 3.2, in particular Condition 3.2.1, such that S^k is a strict local martingale under every $Q \in \mathcal{M}_{loc}$.*

Proof. Assume for notational simplicity that $\|\eta\| = \|\sigma_k\| = 1$. Write $\eta = \lambda^1 a_1 + \cdots + \lambda^m a_m$ for convex weights λ^j, and define

$$f^k(t, y) = \exp\left(\sum_{j=1}^{m} \lambda^j y^j - \frac{1}{2} t\right), \qquad f^i(t, y) \equiv 1 \quad (i \neq k),$$

and

$$b^j(t, y^j) \equiv 0 \quad (j = 1, \ldots, m).$$

Define also $B^1_t = \eta^\top W_t$ and $B^2_t = \sigma_k^\top W_t$, which are one-dimensional Brownian motions with $d\langle B^1, B^2 \rangle_t = \eta^\top \sigma_k dt$, where $\eta^\top \sigma_k > 0$. With $u_t = \exp(B_t - \frac{1}{2} t)$, we then have

$$\begin{aligned} dS^k_t &= S^k_t u_t dB^2_t \\ du_t &= u_t dB^1_t. \end{aligned}$$

From Lemma 4.2 and Lemma 4.3 in [64], we deduce that S^k is a strict local martingale. Now, pick an arbitrary $Q \in \mathcal{M}_{loc}$ and let Z be the corresponding density process. By martingale representation, $dZ_t = Z_t \theta_t^\top dW_t$ for some process θ. Since every S^i remains a local martingale under Q, it follows that $\langle Z, S^i \rangle = 0$. But

$$\langle Z, S^i \rangle_t = \int_0^t S^i_s f^i(s, v_s) Z_s \sigma_i^\top \theta_t dt,$$

so because $S^i_s f^i(s, v_s) Z_s > 0$, we have $\sigma_i^\top \theta_t = 0$. Since $\eta \in \text{span}(\sigma_1, \ldots, \sigma_d)$, we also have $\eta^\top \theta_t = 0$. Thus B^1 and B^2 are still Brownian motions under Q, so the law of (S^k, u) is unchanged and we deduce that S^k is a strict local martingale under Q. This completes the proof. □

Remark. In the case where $\sigma_1, \ldots \sigma_d$ span $\mathbb{R}^n$, we immediately see that the sufficient condition of Theorem 3.2.1, $\sigma_i^\top a_j \leq 0$ for all i and j, is indeed also necessary for efficiency, as claimed in the remark after the proof of Corollary 3.2.1.

The following simple results use convex duality to clarify the precise relationship between the conditions in Theorem 3.2.1, Theorem 3.2.3, and Theorem 3.2.4.

Proposition 3.2.2 *Fix $k \in \{1, \ldots, k\}$. The following two conditions are equivalent:*

(i) There exists $\theta \in \mathbb{R}^d$ such that for all i and j, $\theta^\top \sigma_i = 0$ and $\theta^\top a_j \geq \sigma_k^\top a_j$.

(ii) Every vector $\eta \in \text{conv}(a_1, \ldots, a_m) \cap \text{span}(\sigma_1, \ldots, \sigma_d)$ satisfies $\eta^\top \sigma_k \leq 0$.

Proof. Fix $k \in \{1, \ldots, d\}$ and consider the following linear program with variables $(\theta, t) \in \mathbb{R}^n \times \mathbb{R}$.

$$(P) \quad \begin{cases} \text{minimize} & & & t & & \\ \quad \text{s.t.} & A\theta & + & et & \geq & A\sigma_k \\ & \Sigma\theta & & & = & 0 \\ & & & t & \geq & 0, \end{cases}$$

where $A = [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m\times n}$, $\Sigma = [\sigma_1, \ldots, \sigma_d]^\top \in \mathbb{R}^{d\times n}$, and $e = (1, \ldots, 1)^\top \in \mathbb{R}^m$ is the vector of ones. Note that (P) is always feasible; take for instance $\theta = 0$ and t large enough. Also, denoting its optimal value by V_P, we have $V_P \geq 0$. It is easily verified that $V_P = 0$ if and only if there is a θ such that $\theta^\top \sigma_i = 0$ for all i, and $\theta^\top a_j \geq \sigma_k^\top a_j$ for all j, that is, if condition (i) holds.

Consider now the linear programming dual of (P). It has variables $(\lambda, z) \in \mathbb{R}^m \times \mathbb{R}^d$ and is given by

$$(D) \quad \begin{cases} \text{maximize} & \sigma_k^\top A^\top \lambda \\ \quad \text{s.t.} & A^\top \lambda + \Sigma^\top z = 0 \\ & e^\top \lambda \leq 1 \\ & \lambda \geq 0. \end{cases}$$

Note that it is always feasible, and let V_D denote its optimal value. The strong duality theorem from linear programming implies that $V_P = V_D$, so it remains to verify that $V_D = 0$ if and only if (ii) holds. Suppose first (ii) fails. Then there is a vector $\lambda \in \mathbb{R}^m$ of convex weights, as well as a vector $z \in \mathbb{R}^d$ such that $\eta = A^\top \lambda = -\Sigma^\top z$ and $\eta^\top \sigma_k > 0$. Hence (λ, z) is feasible for (D), with objective value strictly greater than zero. So $V_D > 0$. Conversely, assume $V_D > 0$. Then there is $\lambda \in \mathbb{R}^m$ with $\lambda \geq 0$ and $e^\top \lambda \leq 1$ such that

$$A^\top \lambda \in \operatorname{span}(\sigma_1, \ldots, \sigma_d) \qquad \text{and} \qquad (A^\top \lambda)^\top \sigma_k > 0.$$

In particular, $e^\top \lambda > 0$, so we can define $\widetilde{\lambda} = \frac{1}{e^\top \lambda} \lambda$. Then $A^\top \widetilde{\lambda} \in \operatorname{conv}(a_1, \ldots, a_m) \cap \operatorname{span}(\sigma_1, \ldots, \sigma_d)$ and $(A^\top \widetilde{\lambda})^\top \sigma_k > 0$. Hence (ii) fails. □

It follows directly from Proposition 3.2.2 that the conditions of Theorem 3.2.3 and Theorem 3.2.4 are complementary in the sense that one holds if and only if the other fails. We may formulate this observation as follows.

Corollary 3.2.2 *Consider the stochastic volatility model with constant correlations. The condition of Theorem 3.2.3 is satisfied if and only if ND holds for every choice of f^i and b^j that satisfy the assumptions of Section 3.2, in particular Condition 3.2.1.*

Proof. This is immediate from Theorem 3.2.3, Theorem 3.2.4, and Proposition 3.2.2. □

For completeness, we give a result similar to Proposition 3.2.2, that clarifies the relationship between Theorem 3.2.1 (our initial sufficient condition) and the necessary condition of Theorem 3.2.4.

Proposition 3.2.3 *The following two conditions are equivalent:*

(i) *There exists $\theta \in \mathbb{R}^d$ such that for all i and j, $\theta^\top \sigma_i = 0$, $\theta^\top a_j \geq 0$, and $\theta^\top a_j \geq \sigma_i^\top a_j$.*

(ii) *Whenever $\lambda^1 a_1 + \cdots + \lambda^m a_m \in \operatorname{span}(\sigma_1, \ldots, \sigma_d)$ for convex weights $\lambda^1, \ldots, \lambda^m$, we have*

$$\lambda^j > 0 \quad \Longrightarrow \quad \sigma_i^\top a_j \leq 0 \quad \text{for all } i.$$

Proof. The proof is similar to that of Proposition 3.2.2, so we omit the details. The main difference is that we now consider the linear programs

$$(P) \quad \begin{cases} \text{minimize} & & & t & & \\ \quad \text{s.t.} & A\theta & + & et & \geq & \gamma \\ & \Sigma\theta & & & = & 0 \\ & & & t & \geq & 0, \end{cases}$$

where the vector $\gamma \in \mathbb{R}^m$ has components $\gamma^j = \max\{0,\ \sigma_i^\top a_j;\ i = 1, \ldots, d\}$, and its dual,

$$(D) \qquad \begin{cases} \text{maximize} & \eta^\top \lambda \\ \text{s.t.} & A^\top \lambda + \Sigma^\top z = 0 \\ & e^\top \lambda \le 1 \\ & \lambda \ge 0. \end{cases}$$

□

We end this section with an example illustrating the different situations we have encountered.

Example 3.2.1 *Let $m = 2$, $d = 1$, and set $\sigma_1 = (1, 0)$. Consider the following three cases, illustrated in Figure 3.1:*

(i) $a_1 = (1, 1)$, $a_2 = (-2, 0)$. *Taking* $\theta = (0, 1)$, *Theorem 3.2.1 shows that ND holds.*

(ii) $a_1 = (1, 1)$, $a_2 = (-1/2, -1)$. *With* $\eta = (1/4, 0)$, *Theorem 3.2.4 shows that ND fails for some* f^i, b^j.

(iii) $a_1 = (1, 1)$, $a_2 = (-3, -1)$. *Then* $\frac{1}{2}a_1 + \frac{1}{2}a_2 = (-1, 0)^\top$, *which equals* $-\sigma_1$. *But* $a_1^\top \sigma_1 = 1$, *so the condition in Theorem 3.2.1 is violated (this follows from Proposition 3.2.3.) On the other hand,* $\eta = (-1, 0)^\top$ *is the only point in* $\operatorname{conv}(a_1, \ldots, a_m) \cap \operatorname{span}(\sigma_1, \ldots, \sigma_d)$, *and since* $\eta^\top \sigma_1 = -1 < 0$, *the condition of Theorem 3.2.4 is also violated. However, Theorem 3.2.3 can be used to deduce that ND in fact holds: take* $\theta = (0, 1)$.

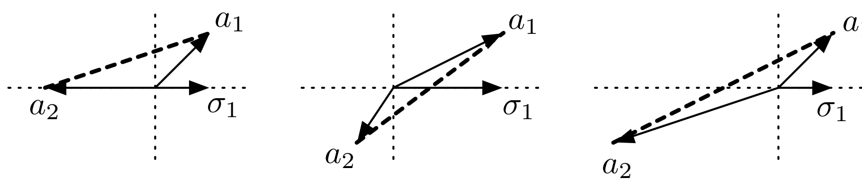

Figure 3.1: *Left:* The equivalent conditions of Proposition 3.2.3 are satisfied, so $\mathcal{M} \neq \emptyset$. *Middle:* The condition of Theorem 3.2.4 is satisfied, so $\mathcal{M} = \emptyset$. *Right:* None of the previous conditions are satisfied. However, Theorem 3.2.3 still lets us conclude that ND holds. The dashed lines indicate $\text{conv}(a_1, \ldots, a_m)$.

3.3 Stochastic Volatility, Variable Correlation Structure

The stochastic volatility model with constant correlation structure described in Section 3.2 can be generalized to one where the vectors $\sigma_1, \ldots, \sigma_d$ and $a_1, \ldots, a_m$ depend on v_t and t. Under the appropriate non-degeneracy assumptions, this generalized model can again be shown to be efficient. Specifically, consider the model

$$\begin{aligned} dS_t^i &= S_t^i f^i(t, v_\cdot)\sigma_i(t, v_t)^\top dW_t \qquad (i = 1, \ldots, d) \\ dv_t^j &= a_j(t, v_t^j)^\top dW_t + b^j(t, v_t^j)dt \qquad (j = 1, \ldots, m), \end{aligned}$$

where f^i is a predictable path functional with values in $\mathbb{R}_+$, and the σ_i and a_j are measurable with values in $\mathbb{R}^n$. As a normalization, assume that

$$\|\sigma_i(t, y)\| \equiv 1.$$

Allowing f^i to depend on the whole path of v is convenient because it allows us to use stochastic exponentials of the v^j. In order to get pathwise uniqueness of solutions for the volatility processes, as well as various auxiliary processes, we assume that the σ_i, a_j and b^j are all Lipschitz continuous, uniformly in t.

We need to impose a modification of Condition 3.2.1:

Condition 3.3.1 *The functionals f^i are Lipschitz on $(-\infty, C]^m$ for every $C > 0$. More precisely, there exist constants K_C such that for $i = 1, \ldots, d$ and $t \leq T$,*

$$|f^i(t, y_\cdot) - f^i(t, z_\cdot)| \leq K_C \sup_{0 \leq s \leq t} \|y_s - z_s\|$$

for all paths y, z with $\sup_{0 \leq s \leq t} y_s^j$ and $\sup_{0 \leq s \leq t} z_s^j$ dominated by C for $j = 1, \ldots, m$.

Remark. It would be desirable to extend the specification of the volatility process to allow each a_j to depend on the entire vector $v_t = (v_t^1, \ldots, v_t^m)$, and not just v_t^j itself. Unfortunately this leads to complications due to the lack of tractable non-explosion criteria for multidimensional diffusions.

Define the sets

$$C(t, y) = \text{conv}(a_1(t, y^j), \ldots, a_m(t, y^m))$$

and

$$\mathcal{S}(t, y) = \text{span}(\sigma_1(t, y), \ldots, \sigma_d(t, y)),$$

as well as the distance between them,

$$\begin{aligned} D(t, y) &= \inf \{\|u - v\| : u \in C(t, y),\ v \in \mathcal{S}(t, y)\} \\ &= \inf \{\|x\| : x \in C(t, y) - \mathcal{S}(t, y)\}. \end{aligned}$$

The set $C(t, y) - \mathcal{S}(t, y)$ is closed, convex and non-empty, which implies that $D(t, y)$ is finite and attained by a *unique* element $x^* = x^*(t, y)$ of $C(t, y) - \mathcal{S}(t, y)$. This x^* is characterized by the property that $x^* \in C(t, y) - \mathcal{S}(t, y)$ and $x^{*\top}(x - x^*) \geq 0$ for every $x \in C(t, y) - \mathcal{S}(t, y)$. These standard facts from convex analysis can be found in, for instance, Bertsekas et al. [3].

The following lemma establishes some properties of $x^*(t, y)$ that will be needed to prove Theorem 3.3.1 below. Its proof is technical but straightforward, and is given at the end of this section.

Lemma 3.3.1 *The following properties hold:*

(i) $x^*(t, y)$ *is orthogonal to* $\sigma_i(t, y)$ *for* $i = 1, \dots, n$*;*

(ii) The mapping $(t, y) \mapsto x^*(t, y)$ *is measurable.*

Theorem 3.3.1 *Consider the stochastic volatility model with variable correlation structure, and assume that Condition 3.3.1 holds. If there is a constant* $0 \leq c < \infty$ *such that for all* $(t, y) \in [0, T] \times \mathbb{R}^m$*,* $i = 1, \dots, d$ *and* $j = 1, \dots, m$*,*

$$\sigma_i(t, y)^\top a_j(t, y) \leq cD(t, y),$$

then $\mathcal{M} \neq \emptyset$*.*

As a particular case, notice that if $a_j(t, y)$ is bounded, it suffices that $D(y)$ be bounded away from zero for Theorem 3.3.1 to apply. (Recall that $\|\sigma_i(t, y)\| \equiv 1$.)

Proof. For notational simplicity we drop the argument (t, y) and write $a_j = a_j(t, y)$, $\sigma_i = \sigma_i(t, y)$, etc. Define the function $\theta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$\theta = \left(0 \vee \max_{i,j} \sigma_i^\top a_j\right) \frac{x^*}{\|x^*\|^2}.$$

By Lemma 3.3.1, $\theta = \theta(t, y)$ is measurable in (t, y), and $\theta^\top \sigma_i \equiv 0$ for all i. Moreover,

$$|\theta| = 0 \vee \frac{\max_{i,j} \sigma_i^\top a_j}{\|x^*\|} = 0 \vee \frac{\max_{i,j} \sigma_i^\top a_j}{D} \leq c < \infty$$

for all (t, y). We have, for $k = 1, \dots, m$,

$$\theta^\top a_k = \left(0 \vee \max_{i,j} \sigma_i^\top a_j\right) \frac{a_k^\top x^*}{\|x^*\|^2} \geq 0 \vee \max_{i,j} \sigma_i^\top a_j,$$

using that $a^k \in \mathcal{C} - \mathcal{S}$ and $x^{*\top}(x - x^*) \geq 0$, and hence $x^{*\top} x \geq \|x^*\|^2$, for every $x \in \mathcal{C} - \mathcal{S}$. Having established these properties of θ, it is straightforward to check that if we replace the process Z in (3.3) by

$$dZ_t = -Z_t h(t, v_\cdot)\theta(t, v_t)^\top dW_t, \qquad Z_0 = 1,$$

where $h(t, y_\cdot) = \max_{i=1,\ldots,n} f^i(t, y_\cdot)$, the proof of Theorem 3.2.1 goes through with minor modifications. The only point that needs special verification is that $a_j(t, y)$ satisfies condition (*ii*) of Lemma 3.2.1. But this follows from the fact that all a_j are Lipschitz. □

Similarly to the constant correlation case, it is possible to weaken the condition given in Theorem 3.3.1.

Theorem 3.3.2 *Consider the stochastic volatility model with variable correlation structure, and assume that Condition 3.3.1 holds. Assume that for each $k \in \{1, \ldots, d\}$ there is a measurable function $\theta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^n$ such that*

$$\begin{cases} \theta^\top \sigma_i(t, y) &= 0 & (i = 1, \ldots, d) \\ \theta^\top a_j(t, y) &\geq \sigma_k(t, y)^\top a_j(t, y) & (j = 1, \ldots, m) \end{cases}$$

and

$$\sup\left\{\|\theta(t, y)\| : (t, y) \in \mathbb{R}^m \times [0, T]\right\} < \infty.$$

Then ND holds.

Proof. As in the constant correlation case (Proposition 3.2.1), we construct for each k a process Z such that ZS^i is a local martingale for each i, and a true martingale for $i = k$. As in the proof of Theorem 3.3.1, the construction still goes through as long as we note that, under our hypotheses, Lemma 3.2.1 and

Lemma 3.2.2 are still valid in the path-dependent case. We may now apply Theorem 3.2.2 to get the result. □

It only remains to prove Lemma 3.3.1. To this end we establish a more general result on the measurability of the optimal solution to certain parameterized families of optimization problems. It uses the above-mentioned fact that for a non-empty, closed, convex subset $\mathcal{K}$ of some Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, the point $x^* = \arg\min_{x\in\mathcal{K}}\|x\|$ exists, is unique, and can be characterized as the only $x^* \in \mathcal{K}$ such that $\langle x^*, x - x^*\rangle \geq 0$ for every $x \in \mathcal{K}$.

Lemma 3.3.2 *Let $F : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$, $(y,z) \mapsto F(y,z)$ be Borel measurable in y and linear in z. Let Z be a convex subset of $\mathbb{R}^p$, and let $\mathcal{K}(y)$ be the convex set*

$$\mathcal{K}(y) = F(y, Z) = \{F(y,z) : z \in Z\}.$$

Assume $\mathcal{K}(y)$ is closed and let $x^*(y)$ be the unique minimizer of $\inf_{x\in\mathcal{K}(y)}\|x\|$. Then the mapping $y \mapsto x^*(y)$ is Borel measurable.

Proof. Let C be any closed subset of $\mathbb{R}^n$. It suffices to show that $\{y \in \mathbb{R}^m : x^*(y) \in C\}$ is a Borel set. Let C_0 and Z_0 be countable, dense subsets of C and Z, respectively. The claim follows once we show that

$$\{y : x^*(y) \in C\} = \bigcap_{k\geq 1}\bigcup_{x^k\in C_0}\bigcup_{z^k\in Z_0}\bigcap_{z\in Z_0}\left\{y : G(y; x^k, z^k) < \frac{1}{k} \text{ and } H(y; k, x^k, z) \geq -\frac{4}{\sqrt{k}}\right\},$$

where

$$G(y; x^k, z^k) = \|F(y, z^k) - x^k\| \quad \text{and} \quad H(y; k, x^k, z) = \langle x^k, F(y,z) - x^k\rangle \mathbf{1}_{\{\|\mathbf{F}(\mathbf{y},\mathbf{z})\|\leq\sqrt{k}\}}.$$

To prove "$\subset$", fix $y \in \mathbb{R}^m$ and suppose $x^* = x^*(y) \in C$. Then for each $k \geq 1$, there is $x^k \in C_0$ with $\|x^* - x^k\| < 1/k$ since C_0 is dense. Since $x^* \in \mathcal{K}(y)$, $x^* = F(y, z^*)$ for some $z^* \in Z$. Hence for any $z^k \in Z_0$,

$$\|F(y,z^k) - x^k\| \le \|x^* - x^k\| + \|F(y,z^k) - F(y,z^*)\| < \frac{1}{k} + \|F(y,z^k) - F(y,z^*)\|,$$

so taking z^k sufficiently close to z^* makes the right side less than $1/k$. That is, there exists $z^k \in Z$ such that $G(y; x^k, z^k) = \|F(y,z^k) - x^k\| < 1/k$.

Next, for any $x \in \mathcal{K}(y)$ we have that $\langle x^*, x - x^* \rangle \ge 0$, so together with the Cauchy-Schwartz inequality,

$$\begin{aligned} \langle x^k, x - x^k \rangle &= \langle x^k - x^*, x - x^k - x^* \rangle + \langle x^*, x - x^* \rangle \\ &\ge \langle x^k - x^*, x - x^k - x^* \rangle \\ &\ge -\|x^k - x^*\| \, \|x - x^k - x^*\|. \end{aligned}$$

We know that $\|x^k - x^*\| < 1/k$. Moreover, since $\|x^*\| \le \|x\|$ by definition of x^*, we get

$$\|x - x^k - x^*\| = \|x - 2x^* - (x^k - x^*)\| \le \|x\| + 2\|x^*\| + \|x^k - x^*\| \le 3\|x\| + \frac{1}{k}.$$

Combining the two displays yields $\langle x^k, x - x^k \rangle \ge -\frac{3}{k}\|x\| - \frac{1}{k^2}$, implying that $\langle x^k, x - x^k \rangle \ge -4/\sqrt{k}$ for all $x \in K(y)$ with $\|x\| \le \sqrt{k}$. This statement is equivalent to having $H(y; k, x^k, z) \ge -4/\sqrt{k}$ for every $z \in Z$, or equivalently, for every $z \in Z_0$.

What we have shown so far is the following: $x^*(y) \in C$ implies that for every $k \ge 1$, there exist $x^k \in C_0$ and $z^k \in Z_0$ such that for every $z \in Z_0$ we have $G(y; x^k, z^k) < 1/k$ and $H(y; k, x^k, z) \ge -4/\sqrt{k}$. This is the desired inclusion.

For the reverse inclusion "$\supset$", let y be an element of the right side, and choose, for each $k \ge 1$, the corresponding $x^k \in C_0$ and $z^k \in Z_0$. If we let $\bar{x}^k = F(y, z^k)$, the hypothesis says that $\|\bar{x}^k - x^k\| < 1/k$ and $\langle x^k, x - x^k \rangle \ge -4/\sqrt{k}$ for every $x \in F(y, Z_0)$ with $\|x\| \le \sqrt{k}$. Since $F(y, Z_0)$ is dense in $F(y, Z) = \mathcal{K}(y)$, this actually holds for every $x \in \mathcal{K}(y)$ with $\|x\| \leq \sqrt{k}$. In particular, for all k so large that $\|x^*\| \leq k$,

$$\langle x^k, x^k - x^* \rangle \leq 4/\sqrt{k}. \tag{3.4}$$

Moreover, if $\|x^*\| \leq k$,

$$\begin{aligned} \langle x^*, x^k - x^* \rangle - \langle x^*, \bar{x}^k - x^* \rangle &= \langle x^*, x^k - \bar{x}^k \rangle \\ &\geq -\|x^*\| \, \|x^k - \bar{x}^k\| \\ &\geq -\sqrt{k}\frac{1}{k}. \end{aligned}$$

Therefore, since $\langle x^*, \bar{x}^k - x^* \rangle \geq 0$ (recall that $\bar{x}^k \in \mathcal{K}(y)$), we get that $\langle x^*, x^k - x^* \rangle \geq -1/\sqrt{k}$ for all sufficiently large k. Subtracting from (3.4) yields

$$\|x^k - x^*\| = \langle x^k - x^*, x^k - x^* \rangle \leq \frac{5}{\sqrt{k}}.$$

This implies that $\lim_{k\to\infty} x^k = x^*$. But x^k lies in C_0, whose closure is C, so $x^* \in C$. This yields the second inclusion and finishes the proof. □

Remark. The statement and proof of Lemma 3.3.2 remain valid if $\mathbb{R}^m$ is replaced by a measurable space $\mathbb{Y}$, $\mathbb{R}^p$ by a separable normed vector space $\mathbb{Z}$, and $\mathbb{R}^p$ by a finite-dimensional Euclidean space $\mathbb{X}$. Of course, the statements relating to Borel measurability must be modified to match the measurable structure of the space $\mathbb{Y}$.

Proof of Lemma 3.3.1. For (*i*), the statement is clearly true if $x^*(t, y) = 0$, so assume that $x^*(t, y) \neq 0$. In this case a standard calculation shows that $x^*(t, y)$ separates $\text{span}(\sigma_1, \ldots, \sigma_d)$ and $\text{conv}(a_1, \ldots, a_m)$. The same argument as in the beginning of the proof of Theorem 3.2.1 then gives the result.

For (*ii*), define $F : [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ as the map

$$(t, y; \lambda, z) \mapsto A(t, y)^\top \lambda - \Sigma(t, y)^\top z,$$

where $A(t, y)^\top$ is the $n \times m$-matrix with columns $a_1(t, y), \ldots, a_m(t, y)$, and $\Sigma(t, y)^\top$ is the $n \times d$-matrix with columns $\sigma_1(t, y), \ldots, \sigma_n(t, y)$. Let Δ^m be the unit simplex in $\mathbb{R}^m$, and observe that

$$C(t, y) - \mathcal{S}(t, y) = F(t, y; \mathbb{R}^d \times \Delta^m).$$

Since $C(t, y) - \mathcal{S}(t, y)$ is closed, Lemma 3.3.2 yields the result. □

CHAPTER 4

FILTRATION SHRINKAGE IN THE ABSENCE OF EFFICIENCY

In the previous chapters we have seen that the distinction between martingales and strict local martingales is crucial in the context of efficiency: the market is efficient if and only if the discounted price process becomes a martingale under some equivalent measure. We also showed in Section 2.3 that a reduction of the information set preserves efficiency, as long as the smaller filtration is still large enough that the prices process remains adapted. In this chapter we investigate a situation that differs from the previous setting in two ways. Firstly, the process under consideration is a strict local martingale (and in the key example studied in Section 4.5, the inverse Bessel process, no equivalent martingale measure exists); and secondly, this process will be projected onto a filtration to which it is not adapted.

4.1 Background and Notation

It is a simple fact that the optional projection of a martingale onto a subfiltration is again a martingale. However, for local martingales the situation is different, as was observed by Föllmer and Protter in [33]. They consider, among other things, three-dimensional Brownian motion $B = (B^1, B^2, B^3)$ starting from $(1, 0, 0)$, defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, Q)$ where the filtration $\mathbb{G}$ is generated by B. In this setting they study optional projections of the process $N = 1/\|B\|$ onto subfiltrations $\mathbb{F}^1$ and $\mathbb{F}^{1,2}$ generated by B^1 and (B^1, B^2), respectively. It is well-known that N, the reciprocal of a BES^3 process, is a local martingale in $\mathbb{G}$. The same turns out to be true for the optional projection onto $\mathbb{F}^{1,2}$. However, the optional projection onto $\mathbb{F}^1$ *is not* a local martingale. Indeed, it was shown in [33], Theorem 5.1, that

$$E^{Q}\left[N_t \mid \mathcal{F}_t^1\right] = 1 + \int_0^t u_x(s, B_s^1)dB_s^1 - \int_0^t \frac{1}{s}dL_s^0,$$

where the function u is given by

$$u(t,x) = \sqrt{\frac{2\pi}{t}}\exp\left(\frac{x^2}{2t}\right)\left(1 - \Phi(|x|/\sqrt{t})\right), \tag{4.1}$$

and L^0 is the local time of B^1 at level zero. Here $\Phi(\cdot)$ is the standard Normal cumulative distribution function. A superficial reason for the appearance of the local time is the non-differentiability of u at $x = 0$, but that is of course highly specific to this particular example. A major goal of this chapter is to shed further light on when optional projections of local martingales fail to be local martingales, and, when this is the case, what can be said about the behavior of their finite variation parts. This type of results have bearing on incomplete information models, currently appearing primarily in a credit risk context (cf. [42] and the references therein.) See also the discussion in Section 4.6 below.

A crucial tool in the analysis is a variant of the *Föllmer measure*, whose construction we briefly review in Section 4.2. A non-uniqueness property of (this variant of) the Föllmer measure leads us to formulate a measure extension problem (Problem 1), which, when a solution exists, allows us to interpret the finite variation part of the optional projection as the compensator of a certain stopping time (Theorem 4.3.1). This is done in Section 4.3. After these general considerations we turn in Section 4.4 to a specific model system in a diffusion setting. The additional structure allows us obtain more detailed results (in particular Theorem 4.4.1), and lets us elaborate on the inverse Bessel example described above. The details of this specific example are given in Section 4.5.

Let us fix some notation that will be in force throughout this chapter. $(\Omega, \mathcal{G}, \mathbb{G})$ is a filtered measurable space, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the right-continuous modification of a *standard system*. That is, $\mathcal{G}_t = \cap_{u>t}\mathcal{G}^o_u$, where each $\mathcal{G}^o_t$ is Standard Borel (see Parthasarathy [58], Definition V.2.2) and any decreasing sequence of atoms has a non-empty intersection[1]. A key example of a standard system is when $(\mathcal{G}^o_t)_{t\geq 0}$ is generated by the coordinate process on the space of right-continuous paths that can explode in finite time and be absorbed at infinity. See the Appendix in Föllmer [32] for details. We assume for simplicity that $\mathcal{G} = \mathcal{G}_\infty = \bigvee_{t\geq 0}\mathcal{G}_t$. The space $(\Omega, \mathcal{G}, \mathbb{G})$ will eventually be endowed with various probability measures. However, *we do not then automatically assume the usual hypotheses*—but this is does not cause any serious complications, due to the following result.

Lemma 4.1.1 *Let P be a probability measure on $\mathcal{G}$, and denote by $(\overline{\mathcal{G}}, \overline{\mathbb{G}})$ the augmentation of $(\mathcal{G}, \mathbb{G})$ with respect to P. Then*

(i) Every $\overline{\mathbb{G}}$ optional (predictable) process is P-indistinguishable from a $\mathbb{G}$ optional (predictable) process.

(ii) Every right-continuous $(\mathbb{G}, P)$ martingale is a $(\overline{\mathbb{G}}, P)$ martingale.

Proof. Part (i) is Lemma 7 in Appendix 1 of [23]. Part (ii) follows from Theorem IV.3 in the same reference. □

Next, let M be a càdlàg adapted process on $(\Omega, \mathcal{G}, \mathbb{G})$ such that $M_0 = 1$, $0 \leq M_t < \infty$ for all $t \geq 0$. Define stopping times

$$\tau_n = n \wedge \inf\left\{t \geq 0 : M_t \leq \frac{1}{n}\right\}, \qquad \tau_0 = \lim_{n\to\infty}\tau_n,$$

[1]This means that if $(t_n)_{n\geq 0}$ is a nonnegative increasing sequence, $A_n \in \mathcal{G}^o_{t_n}$ is an atom for each $n \geq 1$, and $A_n \supset A_{n+1}$, then $\cap_n A_n \neq \emptyset$.

and assume that $M_t = 0$ for all $t \geq \tau_0$. Note that $\mathcal{G}_{\tau_0-} = \bigvee_{n\geq 1} \mathcal{G}_{\tau_n}$, see for instance [22], Theorem IV.56(d). Define also a $(0, \infty]$-valued càdlàg and adapted process N by

$$N_t = \begin{cases} \frac{1}{M_t}, & M_t > 0 \\ \infty, & M_t = 0. \end{cases}$$

Notice that no probability measure has been specified so far. The preceding relations are therefore assumed to hold *for all* $\omega \in \Omega$.

Finally, let Q be a probability on $\mathcal{G}$ such that N becomes a local martingale. In particular, this implies that N_t is finite valued for all $t \geq 0$, Q-a.s.

4.2 The Föllmer Measure

Following similar ideas as in Delbaen and Schachermayer [18] and Pal and Protter [57], which originated with the paper by Föllmer [32], we can construct a new probability P_0 on $\mathcal{G}_{\tau_0-}$ as follows. For each $n \geq 1$, the stopped process $(N_{t\wedge\tau_n})_{t\geq 0}$ is a strictly positive $(\mathbb{G}, Q)$ uniformly integrable martingale[2], so we may define a probability $P_n \sim Q$ on $\mathcal{G}_{\tau_n}$ by $dP_n = N_{\tau_n} dQ$. The optional stopping theorem and uniform integrability yield

$$N_{t\wedge\tau_n} = \lim_{u\to\infty} E^Q\left[N_{u\wedge\tau_{n+1}} \mid \mathcal{G}_{t\wedge\tau_n}\right] = E^Q\left[N_{\tau_{n+1}} \mid \mathcal{G}_{t\wedge\tau_n}\right],$$

and sending t to infinity gives $N_{\tau_n} = E^Q[N_{\tau_{n+1}} \mid \mathcal{G}_{\tau_n}]$. The measures $(P_n)_{n\geq 1}$ thus form a consistent family. Next, by Remark 6.1 in the Appendix of [32], $(\mathcal{G}_{\tau_n-})_{n\geq 1}$ is a standard system, so Parthasarathy's extension theorem (Theorem V.4.2 in [58]) applies: there exists a probability measure P_0 on $\mathcal{G}_{\tau_0-}$ that coincides with P_n on $\mathcal{G}_{\tau_n-}$, for each n.

[2]Indeed, the positivity of N gives $N_{t\wedge\tau_n} \leq n + \Delta N_{\tau_n} \leq n + N_{\tau_n}$. Also, $E^Q[N_{\tau_n}] \leq N_0 = 1$, since every nonnegative local martingale is a supermartingale by Fatou's lemma.

Here is the key point: P_0 is only defined on $\mathcal{G}_{\tau_0-}$, not on all of $\mathcal{G}$. There are typically many ways in which P_0 can be extended to a measure P on $\mathcal{G}$, and we will see that the choice of extension is crucial in the context of filtration shrinkage. In particular, the existence of an extension P with certain properties is intimately connected with the behavior of the optional projection of N (under Q) onto smaller filtrations $\mathbb{F} \subset \mathbb{G}$.

The following lemma shows that no matter which extension P one chooses, M is always the density process relative to Q. In particular it is a (true) P martingale.

Lemma 4.2.1 *Suppose P is an extension of P_0 to all of $\mathcal{G}$. Then $Q|_{\mathcal{G}_t} \ll P|_{\mathcal{G}_t}$ for every $t \geq 0$, and*

$$M_t = \frac{dQ}{dP}\Big|_{\mathcal{G}_t}.$$

Proof. The argument uses well-known ideas, see for instance [32]. Fix $t \geq 0$ and pick $A \in \mathcal{G}_t$. Using that $M_t = 0$ for $t \geq \tau_0$, monotone convergence, and the fact that $M_{t\wedge\tau_n} = \frac{dQ}{dP}|_{\mathcal{G}_{t\wedge\tau_n}}$, we obtain

$$\begin{aligned}
E^P[M_t \mathbf{1}_A] &= E^P[M_t \mathbf{1}_{A\cap\{\tau_0>t\}}] \\
&= \lim_{n\to\infty} E^P[M_t \mathbf{1}_{A\cap\{\tau_n>t\}}] \\
&= \lim_{n\to\infty} E^P[M_{t\wedge\tau_n} \mathbf{1}_{A\cap\{\tau_n>t\}}] \\
&= \lim_{n\to\infty} Q(A\cap\{\tau_n>t\}) \\
&= Q(A\cap\{\tau_0>t\}) = Q(A).
\end{aligned}$$

This is the desired statement. □

It is well-known that if N is a strict local martingale under Q, then $P(\tau_0 < \infty) > 0$, and our focus will be on this case. In particular, this means that P and

Q cannot be equivalent. In fact, they may even be singular, which is the case if $P(\tau_0 < \infty) = 1$. On the other hand, Lemma 4.2.1 guarantees that we always have *local absolute continuity*: for each t, $Q|_{\mathcal{G}_t} \ll P|_{\mathcal{G}_t}$. "Global" absolute continuity, $Q \ll P$, holds when $(M_t)_{t\geq 0}$ is uniformly integrable under P.

Note that the converse construction is straightforward: starting with a a measure P on $\mathcal{G}$ under which M is a martingale, the measures Q_n on $\mathcal{G}_n$ given by $dQ_n = M_n dP$ form a consistent family, extendable to a measure Q on $\mathcal{G}$ using Parthasarathy's theorem. Local absolute continuity is immediate, and "global" absolute continuity holds when M is uniformly integrable.

Let us finally comment on how the question of uniqueness has been treated previously in the literature. In Föllmer's original paper [32], a measure is constructed on the product space $(0, \infty] \times \Omega$, specifically on the predictable σ-field. This measure assigns zero mass to the stochastic interval $]\!]\tau_0, \infty]\!]$, which is key to obtaining uniqueness. On the other hand, neither [18] nor [57] consider the product space, but work directly on Ω. However, N is now taken to be the coordinate process, with $+\infty$ as an absorbing state. Hence there is "no more randomness" contained in the probability space after τ_0, which gives uniqueness of P. In the recent paper [52], Kardaras et al. consider more general probability spaces, and in particular discuss the question of non-uniqueness.

4.3 Predictable Compensators and a Measure Extension Problem

Consider now a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$, assumed to be the right-continuous modification of a standard system. Again, completeness is not assumed. We suppose that P is an extension of P_0 as discussed in Section 4.2. By Theorem 6 in Appendix 1 of [23], optional projections of M and N exist under P and Q, respectively. When we write $E^P[M_t \mid \mathcal{F}_t]$ and $E^Q[N_t \mid \mathcal{F}_t]$ we always refer to these optional projections. Moreover, the projections have càdlàg paths. This follows from the càdlàg property of the optional projections onto the augmentation of $\mathbb{F}$ (under P respectively Q), together with Lemma 4.1.1 and the uniqueness of the projection. A subtlety arises here: the optional projection of N under Q is unique up to a Q-evanescent set. However, this set need not be P-evanescent. We will return to this issue in the remark after Problem 1 below.

The following lemma relates the optional projection of N under Q to certain conditional expectations with respect to P, which are typically better behaved.

Lemma 4.3.1 *Let $t \geq 0$. Then $P(\tau_0 > t \mid \mathcal{F}_t) > 0$ P-a.s. if and only if $E^P[M_t \mid \mathcal{F}_t] > 0$ P-a.s. In this case the restrictions of P and Q to $\mathcal{F}_t$ are equivalent, and we have*

$$P(\tau_0 > t \mid \mathcal{F}_t) = E^P[M_t \mid \mathcal{F}_t]\, E^Q[N_t \mid \mathcal{F}_t], \qquad P- \text{ and } Q-\text{a.s.} \tag{4.2}$$

Proof. In the following, we emphasize that inclusions and equalities are up to P-nullsets. Let $A = \{E^P[M_t \mid \mathcal{F}_t] = 0\} \in \mathcal{F}_t$. Then

$$E^P[\mathbf{1}_A M_t] = E^P\left[\mathbf{1}_A E^P[M_t \mid \mathcal{F}_t]\right] = 0,$$

so $M_t = 0$ on A. Hence $\tau_0 \le t$ on A, so

$$P\left(\mathbf{1}_A P(\tau_0 > t \mid \mathcal{F}_t)\right) = P\left(A \cap \{\tau_0 > t\}\right) = 0,$$

and we deduce that $P(\tau_0 > t \mid \mathcal{F}_t) = 0$ on A. The reverse inclusion, $\{P(\tau_0 > t \mid \mathcal{F}_t) = 0\} \subset A$, is proved similarly. Next, the restrictions of P and Q to $\mathcal{F}_t$ are equivalent since $E^P[M_t \mid \mathcal{F}_t]$ is the density of $Q|_{\mathcal{F}_t}$ with respect to $P|_{\mathcal{F}_t}$. To prove formula (4.2), we use that $Q(\tau_0 > t) = 1$, Bayes' rule, and the fact that $\frac{dQ}{dP}|_{\mathcal{G}_t} = M_t$ to get

$$E^Q\left[N_t \mid \mathcal{F}_t\right] = E^Q\left[\frac{1}{M_{t\wedge\tau_0}}\mathbf{1}_{\{\tau_0 > t\}} \,\middle|\, \mathcal{F}_t\right] = \frac{E^P\left[\frac{dQ}{dP}|_{\mathcal{G}_t}\frac{1}{M_t}\mathbf{1}_{\{\tau_0 > t\}} \mid \mathcal{F}_t\right]}{E^P\left[M_t \mid \mathcal{F}_t\right]} = \frac{P(\tau_0 > t \mid \mathcal{F}_t)}{E^P\left[M_t \mid \mathcal{F}_t\right]}.$$

This gives the desired conclusion. □

The fact that the restrictions of P and Q to $\mathcal{F}_t$ are equivalent if and only if $P(\tau_0 > t \mid \mathcal{F}_t) > 0$ P-almost surely suggests the following *measure extension problem*:

Problem 1 *Given the probability measure P_0 constructed in Section 4.2, and the sub-filtration $\mathbb{F} \subset \mathbb{G}$, find a probability P on $(\Omega, \mathcal{G})$ such that*

(i) $P = P_0$ *on* $\mathcal{G}_{\tau_0 -}$,

(ii) $P(\tau_0 > t \mid \mathcal{F}_t) > 0$ *for all* $t \ge 0$, *P-almost surely.*

Remark. The issue of P-non-uniqueness of the optional projection of N under Q is resolved if P solves the measure extension problem. Because, if N' and N'' are two modifications of $E^Q[N_t \mid \mathcal{F}_t]$, then for every $T \ge 0$, $(N'_t)_{t\le T}$ and $(N''_t)_{t\le T}$ coincide on a set A_T with $Q(A_T) = 1$. But $A_T \in \mathcal{F}_T$, so $P(A_T) = 1$ as well. It follows that $N' = N''$ P-a.s.

A solution P to the measure extension problem, when it exists, leads to an interpretation of the finite variation part of the Q optional projection onto $\mathbb{F}$ of the local martingale N. To see how, let us define

$$Z_t = P(\tau_0 > t \mid \mathcal{F}_t).$$

This is an $(\mathbb{F}, P)$ supermartingale, therefore it has a càdlàg modification since $\mathbb{F}$ is right-continous. If in addition it is strictly positive, it has a multiplicative Doob-Meyer decomposition

$$Z_t = e^{-\Lambda_t} K_t, \tag{4.3}$$

where Λ is nondecreasing, predictable, of finite variation with $\Lambda_0 = 0$, and K is an $(\mathbb{F}, P)$ martingale with $K_0 = 1$.

Proposition 4.3.1 *Suppose P is a solution to the measure extension problem (Problem 1). Then $E^Q[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, Q)$ supermartingale, with multiplicative decomposition*

$$E^Q[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t,$$

where Λ is as in (4.3) and U is an $(\mathbb{F}, Q)$ martingale. As a consequence, $E^Q[N_t \mid \mathcal{F}_t]$ is of Class (DL).[3]

Proof. If P solves the measure extension problem, Lemma 4.3.1 implies that

$$E^P[M_t \mid \mathcal{F}_t] e^{\Lambda_t} E^Q[N_t \mid \mathcal{F}_t] = K_t,$$

an $(\mathbb{F}, P)$ martingale. Since $E^P[M_t \mid \mathcal{F}_t] = \frac{dQ}{dP}|_{\mathcal{F}_t}$, it follows that $e^{\Lambda_t} E^Q[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, Q)$ martingale. Denoting this process by U yields the claimed decomposition. Since $0 \leq E^Q[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t \leq U_t$, the Class (DL) property follows from the

[3] An $\mathbb{F}$-optional process X is of Class (DL) if the family $\{X_\tau : \tau \text{ a stopping time} \leq T\}$ is uniformly integrable, for each $T \geq 0$.

optional stopping theorem and the fact that a nonnegative process dominated by a process of Class (DL) is itself of Class (DL). □

Remark. The fact that $E^Q[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, Q)$ supermartingale also follows from Theorem 2.3 in [33].

Corollary 4.3.1 *Suppose N is a strict $(\mathbb{G}, Q)$ local martingale. If $E^Q[N_t \mid \mathcal{F}_t]$ is also an $(\mathbb{F}, Q)$ local martingale, then the measure extension problem has no solution.*

Proof. Suppose a solution exists. Then, since $E^Q[N_t \mid \mathcal{F}_t]$ is a local martingale, the process Λ in Proposition 4.3.1 is identically zero, and $E^Q[N_t \mid \mathcal{F}_t] = U_t$ is a martingale. Hence $E^Q[N_t] = E^Q[E^Q[N_t \mid \mathcal{F}_t]] = 1$ for all $t \geq 0$, contradicting the assumption that N is a strict local martingale. □

The finite variation part Λ appearing when N is projected onto the smaller filtration can now be interpreted as the *predictable compensator of* τ_0, viewed in the appropriate filtration. The key step is an application of the Jeulin-Yor theorem from the theory of filtration expansions.

Theorem 4.3.1 *Let $\mathbb{F}^{\tau_0}$ be the progressive expansion of $\mathbb{F}$ with τ_0, that is, the smallest filtration that contains $\mathbb{F}$, satisfies the usual hypotheses (with respect to P), and makes τ_0 a stopping time. If P solves the measure extension problem, then*

(i) the process

$$\mathbf{1}_{\{\tau_0 \leq t\}} - \int_0^{t \wedge \tau_0} d\Lambda_s$$

is an $(\mathbb{F}^{\tau_0}, P)$ martingale,

(ii) τ_0 is not predictable, provided $P(\tau_0 < \infty) > 0$.

Proof. The proof uses stochastic integration, which assumes the usual hypotheses. This causes no complications: by Lemma 4.1.1, we may first pass to the P-completion $\overline{\mathbb{F}}$ of $\mathbb{F}$ without losing the semimartingale property of any of the processes involved, carry out the computations there, and then go back to $\mathbb{F}$ at the cost of changing things on a P-nullset.

The integration by parts formula yields

$$Z_t = 1 + \int_0^t e^{-\Lambda_{s-}} dK_s + [e^{-\Lambda}, K]_t - \int_0^t e^{-\Lambda_{s-}} K_{s-} d\Lambda_s.$$

By Yoeurp's lemma, $[e^{-\Lambda}, K]$ is a local martingale, so we have the additive Doob-Meyer decomposition $Z_t = \mu_t - a_t$, where

$$\mu_t = 1 + \int_0^t e^{-\Lambda_{s-}} dK_s + [e^{-\Lambda}, K]_t \qquad \text{and} \qquad a_t = \int_0^t Z_{s-} d\Lambda_s.$$

By the Jeulin-Yor Theorem (see Theorem 1.1 in [35]), the process

$$\mathbf{1}_{\{\tau_0 \leq t\}} - \int_0^{t \wedge \tau_0} \frac{1}{Z_{s-}} da_s$$

is an $(\mathbb{F}^{\tau_0}, P)$ martingale. Substituting for da_s yields (i).

To prove (ii), assume for contradiction that there is a strictly increasing sequence of $\mathbb{F}^{\tau_0}$ stopping times ρ_n such that $\lim_n \rho_n = \tau_0$. By the Lemma on page 370 in [59], there are $\mathbb{F}$ stopping times σ_n such that $\sigma_n \wedge \tau_0 = \rho_n \wedge \tau_0$. But since $\rho_n < \tau_0$ P-a.s., we have $\sigma_n = \rho_n$ P-a.s. It follows that τ_0 is P-a.s. equal to an $\mathbb{F}$ stopping time, implying that

$$P(\tau_0 > t \mid \mathcal{F}_t) = \mathbf{1}_{\{\tau_0 > t\}} \quad P\text{-a.s.}$$

This contradicts the assumption that P solves the measure extension problem, since by hypothesis $P(\tau_0 < \infty) > 0$. □

The significance of Theorem 4.3.1 is that it shows *when* the $(\mathbb{F}, Q)$ supermartingale $E^Q[N_t \mid \mathcal{F}_t]$ loses mass: it happens exactly when the compensator of τ_0 increases, i.e., when there is an increased probability, conditionally on $\mathbb{F}$, that τ_0 has already happened. This corresponds to a kind of smoothing over time of the sets $\{\tau_0 \leq t\}$ when we pass to the smaller filtration $\mathbb{F}$. This smoothing is necessary to make the restrictions of P and Q equivalent, since $\{\tau_0 \leq t\}$ is Q-null but not necessarily P-null.

4.4 A Diffusion Model

We now examine a specific model system in a diffusion setting. Let

$$f : \mathbb{R}^d \to \mathbb{R}$$

be a continuous function that is C^2 on the set $\{f > 0\} = \{x \in \mathbb{R}^d : f(x) > 0\}$, which is assumed nonempty. We also consider functions

$$\mu : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \qquad \text{and} \qquad \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d},$$

where $\mathbb{R}^{d \times d}$ denotes the space of $d \times d$ matrices. We assume that

$$\begin{gathered} \mu \text{ and } \sigma \text{ are locally Lipschitz on } [0, \infty) \times \{f > 0\}, \\ \sigma(t, \cdot) \text{ is invertible on } \{f > 0\} \text{ for all } t \in [0, \infty), \end{gathered} \tag{4.4}$$

and

$$\nabla f(y)^\top \mu(t, y) + \frac{1}{2} \mathrm{Tr}\Big(\sigma(t, y)\sigma(t, y)^\top \nabla^2 f(y)\Big) = 0 \qquad \text{on} \qquad [0, \infty) \times \{f > 0\}. \tag{4.5}$$

Consider the SDE

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t, \qquad Y_0 \in \{f > 0\},$$

where $W = (W^1, \ldots, W^d)$ is d-dimensional Brownian motion. Due to the local Lipschitz property of μ and σ, the SDE has a unique strong solution on $[0, \tau_0 \wedge \zeta)$, where

$$\tau_0 = \lim_{n\to\infty} \tau_n, \qquad \tau_n = n \wedge \inf\left\{t \geq 0 : f(Y_t) \leq \frac{1}{n}\right\}$$

and

$$\zeta = \lim_{n\to\infty} \zeta_n, \qquad \zeta_n = n \wedge \inf\{t \geq 0 : \|Y_t\| \geq n\}.$$

Moreover, we assume that this solution is non-explosive in the sense that

$$\tau_0 < \zeta \text{ on the set } \{\tau_0 < \infty\}. \tag{4.6}$$

We may then omit all instances of the stopping time ζ. Now, let $(\Omega, \mathcal{G}, \mathbb{G})$ be the space of $\mathbb{R}^d$-valued paths that are continuous up to their (possibly finite) explosion time, and then absorbed at infinity. $\mathbb{G}$ is the right-continuous modification of the filtration generated by the coordinate process $Y = (Y^1, \ldots, Y^d)$. This is a standard system. Furthermore, let P be a probability measure such that the law of the coordinate process Y on $[0, \tau_0)$ is described by the above SDE. We then define W on $[0, \tau_0)$ via

$$W_t = \int_0^{t\wedge\tau_0} \sigma(s, Y_s)^{-1} dY_s - \int_0^{t\wedge\tau_0} \sigma(s, Y_s)^{-1} \mu(s, Y_s) ds,$$

which is Brownian motion stopped at τ_0. Note that the stochastic integral is computed in the P-augmentation of $\mathbb{G}$, so that the right side need not be $\mathbb{G}$-adapted. However, we can always choose W to be a $\mathbb{G}$-adapted process indistinguishable from it.

The measure extension problem (Problem 1) now consists in specifying (the law of) Y appropriately on all of $[0, \zeta)$.

Remark. We emphasize that a key reason for being explicit about the structure of the sample space (as opposed to just considering the law of the processes involved) is that this structure is crucial when we project onto smaller filtrations.

The process M we have worked with previously is given by

$$M_t = \begin{cases} f(Y_t) & t < \tau_0 \\ \\ 0 & t \geq \tau_0. \end{cases}$$

This makes the notation τ_n and τ_0 consistent with previous sections. We can also define $N = 1/M$ as before. It is clear that M is a local martingale on $[0, \tau_0)$ in the sense that there are stopping times ρ_n, increasing to τ_0, such that $(M_{t\wedge\rho_n})_{t\geq 0}$ is a local martingale for each n. We will assume something stronger, namely:

$$M \text{ is a } (\mathbb{G}, P) \text{ martingale.} \tag{4.7}$$

Note that this does not depend on the behavior of Y after τ_0. That is, it does not depend on the specific choice of P.

A probability measure Q on $\mathcal{G}$ such that $\frac{dQ}{dP}|_{\mathcal{G}_t} = M_t$ can now be constructed as in Section 4.2. Again, the behavior of Y after τ_0 does not matter for Q. It follows that Y can be prescribed there at will, without invalidating any of what we have done so far. In particular, in this setting, *solving the measure extension problem* simply means to specify Y on $[\tau_0, \zeta)$ in such a way that part (*ii*) of Problem 1 is satisfied (part (*i*) holds by construction). Of course, for this to have a meaning we need to describe the smaller filtration $\mathbb{F}$, and this is our next step.

We construct $\mathbb{F}$ as follows. Let $\mathcal{X} \subset \mathbb{R}^d$ be measurable, and suppose

$$g : \mathbb{R}^d \to \mathcal{X}$$

is continuous. One important example that we will revisit later is where $\mathcal{X} = \mathbb{R}^k$, $k < d$, and g is a coordinate projection of $\mathbb{R}^d$ onto $\mathbb{R}^k$. Given Y, we can define a new process $X = (X_t)_{t\geq 0}$ by setting

$$X_t = g(Y_t), \qquad t \geq 0.$$

Then X is càdlàg since g is continuous and Y càdlàg. Moreover, it is clear that X is $\mathbb{G}$ adapted, so the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ given by

$$\mathcal{F}_t = \bigcap_{u>t} \sigma(X_s : s \leq u),$$

which is right-continuous but not P-augmented, is a subfiltration of $\mathbb{G}$. Finally, we introduce the following set:

$$\mathcal{X}_0 = g(\{f = 0\}) = \{x \in \mathcal{X} : x = g(y) \text{ for some } y \text{ with } f(y) = 0\}. \tag{4.8}$$

The importance of the set $\mathcal{X}_0$ comes from the fact that if $X_t \notin \mathcal{X}_0$, then Y_t cannot lie in $\{f = 0\}$. Therefore the probability of τ_0 occurring at any of those times is zero. In order to makes this statement precise we first need a preliminary result.

Lemma 4.4.1 *Assume that P solves the measure extension problem. Then X is continuous, P-a.s.*

Proof. Suppose for contradiction that $P(\Delta X_\sigma \neq 0) > 0$ for some $\mathbb{F}$ stopping time σ. Then also $P(\Delta Y_\sigma \neq 0) > 0$, since $\Delta Y_\sigma = 0$ implies that $\Delta X_\sigma = g(Y_\sigma) - g(Y_{\sigma-}) = 0$ and thus $\{\Delta X_\sigma \neq 0\} \subset \{\Delta Y_\sigma \neq 0\}$. But Y is continuous on $[0, \tau_0)$, so we must have $\tau_0 \leq \sigma$ on $\{\Delta Y_\sigma \neq 0\}$. Hence

$$Z_\sigma = P(\tau_0 > t \mid \mathcal{F}_t)|_{t=\sigma} = 0 \qquad \text{on} \quad \{\Delta X_\sigma \neq 0\},$$

which has positive probability by assumption. However, this is impossible since P solves the measure extension problem. □

Recall the multiplicative decomposition $E^Q[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t$ of the positive $(\mathbb{F}, Q)$ supermartingale $E^Q[N_t \mid \mathcal{F}_t]$, which is of Class (DL) by Proposition 4.3.1, provided the measure extension problem has a solution. We can say the following about the points of increase of Λ:

Theorem 4.4.1 *Assume that P solves the measure extension problem and let Λ be as in Proposition 4.3.1. Then the random measure $d\Lambda_t$ is supported on the set $\{t : X_t \in \mathcal{X}_0\}$.*

Proof. Consider the filtration $\mathbb{F}^{\tau_0}$ described in Theorem 4.3.1. By that theorem,

$$\mathbf{1}_{\{\tau_0 \leq t\}} - \int_0^{t\wedge\tau_0} d\Lambda_s$$

is an $(\mathbb{F}^{\tau_0}, P)$ martingale. Pick two bounded $\mathbb{F}$ stopping times ρ and σ such that $X_t \notin \mathcal{X}_0$ for $\rho < t \leq \sigma$. They are also $\mathbb{F}^{\tau_0}$ stopping times, so the martingale property and the optional sampling theorem yield

$$E^P\left[\mathbf{1}_{\{\rho<\tau_0\leq\sigma\}}\right] = E^P\left[\int_{\rho\wedge\tau_0}^{\sigma\wedge\tau_0} d\Lambda_s\right]. \tag{4.9}$$

We claim that $P(\rho < \tau_0 \leq \sigma) = 0$. Indeed, if $\rho < \tau_0 \leq \sigma$, the choice of ρ and σ implies that $g(Y_{\tau_0-}) = X_{\tau_0-} = X_{\tau_0} \notin \mathcal{X}_0$, using also that X is continuous by Lemma 4.4.1. But this is a contradiction, because $f(Y_{\tau_0-}) = \lim_{n\to\infty} f(Y_{\tau_n}) = 0$, so that $g(Y_{\tau_0-}) \in \mathcal{X}_0$ by definition of $\mathcal{X}_0$. It follows that $\rho < \tau_0 \leq \sigma$ is impossible, proving the claim.

An immediate consequence of this claim is that $(\Lambda_\sigma - \Lambda_\rho)\mathbf{1}_{\{\rho<\tau_0\leq\sigma\}} = 0$. Moreover, the left, and hence right, side of (4.9) is zero, implying that $\Lambda_{\sigma\wedge\tau_0} - \Lambda_{\rho\wedge\tau_0} = 0$. Thus

$$(\Lambda_\sigma - \Lambda_\rho)\mathbf{1}_{\{\rho<\tau_0\leq\sigma\}} = (\Lambda_{\sigma\wedge\tau_0} - \Lambda_{\rho\wedge\tau_0})\mathbf{1}_{\{\rho<\tau_0\leq\sigma\}} = 0.$$

We deduce that $\Lambda_\sigma - \Lambda_\rho = (\Lambda_\sigma - \Lambda_\rho)\mathbf{1}_{\{\tau_0\leq\rho\}}$, so that on $\{\Lambda_\sigma - \Lambda_\rho > 0\}$, which lies in $\mathcal{F}_\sigma$, we necessarily have $\tau_0 \leq \rho$. It follows that on this set,

$$Z_\sigma = P(\tau_0 > t \mid \mathcal{F}_t)|_{t=\sigma} = 0.$$

To avoid a contradiction with the assumption that P solves the measure extension problem, we must have $P(\Lambda_\sigma - \Lambda_\rho > 0) = 0$. From this we deduce the key fact that

$$\Lambda_\sigma - \Lambda_\rho = 0 \qquad P\text{−a.s.},$$

for any bounded $\mathbb{F}$ stopping times ρ and σ such that $X_t \notin \mathcal{X}_0$ when $\rho < t \leq \sigma$. Since X is continuous (Lemma 4.4.1), we can cover $\{X \notin \mathcal{X}_0\}$ with countably many intervals of the form $(\rho, \sigma]$, where ρ and σ have these properties. This completes the proof. □

As a corollary, we can give a simple sufficient condition for Λ to have singular paths, as in the example studied by Föllmer and Protter [33] that was mentioned in the Introduction.

Corollary 4.4.1 *Assume the law of X_t under P admits a density for each $t \geq 0$. Then, if $\mathcal{X}_0$ is a nullset in $\mathcal{X}$, the paths of Λ are singular.*

Proof. By Fubini's theorem,

$$E^P\left[\int_0^t \mathbf{1}_{\{\mathbf{X_s}\in\mathcal{X}_\mathbf{0}\}} ds\right] = \int_0^t P(X_s \in \mathcal{X}_0) ds = 0,$$

since X_s has a density and $\mathcal{X}_0$ is a nullset. Since $\int_0^t \mathbf{1}_{\{\mathbf{X_s}\in\mathcal{X}_\mathbf{0}\}} ds$ is nonnegative it is in fact zero. The set $\{t : X_t \in \mathcal{X}_0\}$ is therefore a nullset P-a.s., and it contains the support of $d\Lambda_t$ by Theorem 4.4.1. This proves the claim. □

4.5 The Inverse Bessel Process

So far we have *assumed* that the measure extension problem has a solution. Let us now consider a particular example where such a solution can indeed be found. The example falls into the diffusion framework described in Section 4.4.

We take $d = 3$ and let $f(y) = \|y\|$ be the Euclidean norm (which of course is C^2 on the non-empty set $\{f > 0\}$.) Next, we take $\sigma(t, y) \equiv I$, the identity matrix, and set $\mu(t, y) = \mu(y) = -y/\|y\|^2$ for $y \neq 0$. For $y = 0$, we can set $\mu(y) = 0$, for example. Note that both σ and μ certainly are locally Lipschitz on $\{f > 0\} = \mathbb{R}^3 \backslash \{0\}$. Moreover, a calculation shows that (4.5) is satisfied. We thus choose P so that the law of the coordinate process Y, restricted to $[0, \tau_0 \wedge \zeta)$, is given by the SDE

$$dY_t^i = dW_t^i - \frac{Y_t^i}{\|Y_t\|^2} dt, \qquad i = 1, 2, 3,$$

where we set $Y_0 = (1, 0, 0)$. Furthermore, M is given by

$$M_t = \begin{cases} \|Y_t\|, & t < \tau_0 \wedge \zeta \\ \\ 0, & t \geq \tau_0 \wedge \zeta. \end{cases}$$

The following lemma shows that M has a simple structure. In particular, it immediately implies that $\zeta = \infty$ P-a.s. and that M is a martingale. The basic assumptions (4.4), (4.5), (4.6) and (4.7) on the diffusion model in Section 4.4 are therefore satisfied.

Lemma 4.5.1 *The process M is Brownian motion starting from one and absorbed at zero. It can be written*

$$M_t = 1 + \sum_{i=1}^{3} \int_0^{t \wedge \tau_0} \frac{Y_s^i dW_s^i}{\|Y_s\|}.$$

Proof. Define stopping times $\rho_n = \tau_n \wedge \zeta_n$, $n \geq 1$. An application of Itô's formula yields

$$M_{t \wedge \rho_n} = 1 + \sum_{i=1}^{3} \int_0^{t \wedge \rho_n} \frac{Y_s^i dW_s^i}{\|Y_s\|}$$

for $n \geq 1$. Consequently, $\langle M, M \rangle_{t \wedge \rho_n} = t \wedge \rho_n$, so $(M_{t \wedge \rho_n})_{t \geq 0}$ is Brownian motion stopped at ρ_n. Thus $\lim_{n \to \infty} \rho_n < \zeta$ on the set where the limit is finite, since Brownian motion does not explode in finite time. We deduce that $\lim_{n \to \infty} \rho_n = \tau_0$

and that $\tau_0 < \zeta$ on $\{\tau_0 < \infty\}$. It also follows that $M_{t\wedge\tau_0}$, which is equal to M_t, is Brownian motion stopped at τ_0. □

It remains to specify Y after τ_0, and this can be done in different ways. One possibility that makes the computations relatively simple is to kill the drift at time τ_0, and let Y continue from there as a standard three-dimensional Brownian motion. That is, Y satisfies

$$Y_t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + W_t - \int_0^{t\wedge\tau_0} \frac{Y_s}{\|Y_s\|^2} ds.$$

We will show later that this indeed corresponds to a solution to the measure extension problem when the filtration $\mathbb{F}$ is generated by Y^1.

Recall that the measure Q is given by $dQ|_{\mathcal{G}_t} = M_t dP|_{\mathcal{G}_t}$. It is well-known, and easy to check by an application of the Girsanov theorem (see [59], Theorem III.41), that M becomes a BES^3 process under Q, so that $N = 1/M$ is an inverse Bessel process. In particular, $Q(\tau_0 = \infty) = 1$. But a more precise statement is true:

Lemma 4.5.2 *Under Q, the process Y is three-dimensional Brownian motion starting from* $(1, 0, 0)$.

Proof. By Lemma 4.5.1, $\langle M, W^i \rangle_t = \int_0^{t\wedge\tau_0} \frac{Y_s^i}{\|Y_s\|} ds$. Hence on $[0, \tau_0)$, which is the same as $[0, \infty)$ Q-a.s., we have

$$Y_0^i + W_t^i - \int_0^t \frac{1}{M_s} d\langle M, W^i \rangle_s = Y_t^i + \int_0^t \frac{Y_s^i}{\|Y_s\|^2} ds - \int_0^t \frac{Y_s^i}{\|Y_s\|^2} ds = Y_t^i.$$

This is a $(\mathbb{G}, Q)$ local martingale by the the Girsanov theorem. Since $\langle Y^i, Y^i \rangle_t = t$ Q-a.s., and $\langle Y^i, Y^j \rangle = 0$ for $i \neq j$, Lévy's theorem gives the result. □

We are now exactly in the situation of Föllmer and Protter [33], as described in the Introduction: Under Q, $N = 1/\|Y\|$ is the reciprocal of the norm of a three-dimensional Brownian motion starting from $(1,0,0)$. We can therefore apply their results on the behavior of its optional projection. More precisely, let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be given by

$$\mathcal{F}_t = \bigcap_{u>t} \sigma(Y_s^1 : s \leq u),$$

which corresponds to g being a projection map from $\mathbb{R}^3$ onto $\mathbb{R}$:

$$\mathcal{X} = \mathbb{R}, \qquad g(y) = y^1, \qquad \mathcal{X}_0 = \{0\}.$$

Then

$$E^Q\left[N_t \mid \mathcal{F}_t\right] = 1 + \int_0^t u_x(s, Y_s^1) dY_s^1 - \int_0^t \frac{1}{s} dL_s^0,$$

where $u(t,x)$ is given by (4.1) and L^0 is the local time of Y^1 at level zero.

It remains to verify that P indeed solves the measure extension problem. In particular, this will require techniques from filtering theory. We start with two lemmas.

Lemma 4.5.3 *We have*

$$E^P\left[\frac{1}{\|Y_t\|}\mathbf{1}_{\{\tau_0>t\}}\right] = E^Q\left[N_t^2\right].$$

Proof. Since $\frac{dQ}{dP}|_{\mathcal{G}_t} = M_t = \|Y_t\|$ on $\{\tau_0 > t\}$, and $Q(\tau_0 > t) = 1$, we get

$$E^P\left[\frac{1}{\|Y_t\|}\mathbf{1}_{\{\tau_0>t\}}\right] = E^Q\left[\frac{1}{M_t^2}\mathbf{1}_{\{\tau_0>t\}}\right] = E^Q\left[N_t^2\right],$$

as desired. □

Lemma 4.5.4 *Let $\xi = (\xi_t)_{t\geq 0}$ be a measurable process with $E^P[\int_0^t |\xi_s| ds] < \infty$ for all $t \geq 0$. Then*

$$E^P\left[\int_0^t \xi_s ds \,\middle|\, \mathcal{F}_t\right] - \int_0^t E^P[\xi_s \mid \mathcal{F}_s] ds$$

is a martingale. Here $E^P[\xi_t \mid \mathcal{F}_t]$ refers to the optional projection.

Proof. This is a well-known result from filtering theory. □

We can now give the first result on the structure of Y^1 in the shrunken filtration $\mathbb{F}$, under the measure P.

Theorem 4.5.1 *The process Y^1 can be decomposed as*

$$Y_t^1 = 1 + B_t - \int_0^t \theta_s ds,$$

where B is $(\mathbb{F}, P)$ Brownian motion and θ satisfies, for every $t \geq 0$,

$$\theta_t = E^P\left[\frac{Y_t^1}{||Y_t||^2}\mathbf{1}_{\{\tau_0>\mathbf{t}\}} \,\middle|\, \mathcal{F}_t\right] \qquad \text{and} \qquad E^P\left[\int_0^t |\theta_s| ds\right] < \infty.$$

Proof. For any $y = (y^1, \ldots, y^d) \in \mathbb{R}^d$ we have

$$\frac{|y^i|}{||y||} \leq \frac{||y||_1}{||y||} \leq \sqrt{d},$$

where $||y||_1 = |y^1| + \ldots + |y^d|$ is the ℓ^1-norm on $\mathbb{R}^d$. This inequality and Lemma 4.5.3 imply that

$$E^P\left[\frac{|Y_t^1|}{||Y_t||^2}\mathbf{1}_{\{\tau_0>\mathbf{t}\}}\right] \leq \sqrt{3}E^P\left[\frac{1}{||Y_t||}\mathbf{1}_{\{\tau_0>\mathbf{t}\}}\right] = \sqrt{3}E^Q\left[N_t^2\right]. \tag{4.10}$$

This allows us to define θ as the optional projection of $\frac{Y^1}{||Y||^2}\mathbf{1}_{\{\tau_0>\mathbf{t}\}}$ onto $\mathbb{F}$. In particular,

$$\theta_t = E^P\left[\frac{Y_t^1}{||Y_t||^2}\mathbf{1}_{\{\tau_0>\mathbf{t}\}} \,\middle|\, \mathcal{F}_t\right]$$

P-a.s. for each $t \geq 0$. Moreover, θ is a measurable process, and from Jensen's inequality and (4.10) we get

$$\int_0^t E^P[|\theta_s|]ds \leq \int_0^t E^P\left[\frac{|Y_s^1|}{||\mathbf{Y}_s||^2}\mathbf{1}_{\{\tau_0>\mathbf{t}\}}\right]ds \leq \sqrt{3}\int_0^t E^Q\left[N_s^2\right]ds.$$

The right side is finite due to the well-known fact that $t \mapsto E^Q[N_t^2]$ is locally bounded on $[0,\infty)$, see for instance Chapter 1.10 in [10].

We now turn to the decomposition of Y^1. Using the $\mathbb{F}$ adaptedness of Y^1 we get

$$\begin{aligned} Y_t^1 = E^P\left[Y_t^1 \mid \mathcal{F}_t\right] &= 1 + E^P\left[W_t^1 \mid \mathcal{F}_t\right] - E^P\left[\int_0^t \frac{Y_s^1}{\|Y_s\|^2}\mathbf{1}_{\{\tau_0>s\}} ds \,\middle|\, \mathcal{F}_t\right] \\ &= 1 + B_t - \int_0^t \theta_s ds, \end{aligned}$$

where we define

$$B_t = E^P\left[W_t^1 \mid \mathcal{F}_t\right] + \int_0^t E^P\left[\frac{Y_s^1}{\|Y_s\|^2}\mathbf{1}_{\{\tau_0>s\}} \,\middle|\, \mathcal{F}_s\right] ds - E^P\left[\int_0^t \frac{Y_s^1}{\|Y_s\|^2}\mathbf{1}_{\{\tau_0>s\}} ds \,\middle|\, \mathcal{F}_t\right].$$

It remains to prove that B is Brownian motion. First, since W^1 is a $(\mathbb{G}, P)$ martingale, $E^P[W_t^1 \mid \mathcal{F}_t]$ is an $(\mathbb{F}, P)$ martingale. Second, an application of Lemma 4.5.4 with $H_t = (Y_t^1/\|Y_t\|)\mathbf{1}_{\{\tau_0>t\}}$ shows that $B_t - E^P\left[W_t^1 \mid \mathcal{F}_t\right]$ is an $(\mathbb{F}, P)$ martingale as well. Hence B is an $(\mathbb{F}, P)$ martingale, and its quadratic variation coincides with $\langle Y^1, Y^1\rangle_t = t$. By Lévy's theorem it is therefore a Brownian motion, and the proof is complete. □

To show that we indeed have a solution to the measure extension problem, we need to prove that the restrictions of P and Q to $\mathcal{F}_t$ are equivalent for each $t \geq 0$. We start with the following simple refinement of Bayes' rule.

Lemma 4.5.5 *Suppose $Q \ll P$ are two probability measures, and let X be a random variable in $L^1(Q)$. Let $\mathcal{F}$ be a sub-σ-field and suppose $A \subset \{E^P[\frac{dQ}{dP} \mid \mathcal{F}] > 0\}$. Then $E^Q[X \mid \mathcal{F}]$ is uniquely defined on A up to a P-nullset, and we have*

$$E^P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}\right] E^Q[X \mid \mathcal{F}]\,\mathbf{1}_A = E^P\left[\frac{dQ}{dP} X \mathbf{1}_A \,\middle|\, \mathcal{F}\right]$$

P-a.s. (and hence Q-a.s.)

Proof. To prove the first statement, let Y and Y' be two versions of $E^Q[X \mid \mathcal{F}]$. Then $Q(Y \neq Y') = 0$, and we get

$$0 = Q(\{Y \neq Y'\} \cap A) = E^P\left[E^P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}\right]\mathbf{1}_{\{Y \neq Y'\}\cap A}\right].$$

Since $E^P[\frac{dQ}{dP} \mid \mathcal{F}] > 0$ on A, we get $P(\{Y \neq Y'\} \cap A) = 0$, as desired. The second statement follows from the following calculation, where $B \in \mathcal{F}$ is arbitrary:

$$\begin{aligned} E^P\left[E^P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}\right]E^Q[X \mid \mathcal{F}]\mathbf{1}_{A\cap B}\right] &= E^P\left[\frac{dQ}{dP}E^Q[X \mid \mathcal{F}]\mathbf{1}_{A\cap B}\right] \\ &= E^Q[X\mathbf{1}_{A\cap B}] \\ &= E^P\left[\frac{dQ}{dP}X\mathbf{1}_{A\cap B}\right]. \end{aligned}$$

□

The next lemma is the key to proving equivalence.

Lemma 4.5.6 *For each $t \geq 0$,*

$$\int_0^t \theta_s^2 ds < \infty \qquad P\text{–a.s.}$$

Proof. Define $\sigma_0 = \inf\{t \geq 0 : P(\tau_0 > t \mid \mathcal{F}_t) = 0\}$. Then $\tau_0 \leq \sigma_0$, so by Theorem 4.5.1,

$$\theta_t \mathbf{1}_{\{\sigma_0 \leq t\}} = E^P\left[\frac{Y_t^1}{||Y_t||^2}\mathbf{1}_{\{\tau_0 > t\}\cap\{\sigma_0 \leq t\}} \,\middle|\, \mathcal{F}_t\right] = 0.$$

Hence $\theta_t = \theta_t \mathbf{1}_{\{\sigma_0 > t\}}$. Now, set $X = \frac{Y_t^1}{||Y_t||^3}\mathbf{1}_{\{\tau_0 > t\}}$. Then $E^Q[|X|] = E^P[|M_t X|] = E^P[|\theta_t|]$, which is finite by Theorem 4.5.1. Since also $E^P[M_t \mid \mathcal{F}_t] > 0$ on $\{\sigma_0 > t\}$ by Lemma 4.3.1, we may apply Lemma 4.5.5 to get

$$\theta_t = E^P\left[\frac{Y_t^1}{||Y_t||^2}\mathbf{1}_{\{\tau_0 > t\}} \,\middle|\, \mathcal{F}_t\right]\mathbf{1}_{\{\sigma_0 > t\}} = E^Q\left[\frac{Y_t^1}{||Y_t||^3} \,\middle|\, \mathcal{F}_t\right]E^P[M_t \mid \mathcal{F}_t]\mathbf{1}_{\{\sigma_0 > t\}}.$$

Since $E^P[M_t \mid \mathcal{F}_t]$ is a finite, càdlàg process, it is pathwise bounded on each $[0, t]$ (with the bound depending on ω and t in a possibly non-predictable way.) It thus suffices to prove that $\int_0^t \xi_s^2 ds < \infty$, where $\xi_s = E^Q[\frac{Y_s^1}{\|Y_s\|^3} \mid \mathcal{F}_s]$. To do this, first note that

$$\frac{|Y_s^1|}{\|Y_s\|^3} \leq \frac{1}{|Y_s^1|^2}.$$

Thus, since $Y_0^1 = 1$ and Y^1 is continuous, there is a non-empty time interval $[0, \varepsilon)$, depending on ω, on which ξ_s is bounded. Next, since Y^1 and (Y^2, Y^3) are independent under Q, we have for each $s > 0$,

$$\xi_s = E^Q\left[\frac{y}{[y^2 + ((Y_s^2)^2 + (Y_s^3)^2)]^{3/2}}\right]_{y=Y_s^1} = E^Q\left[\frac{y}{(y^2 + sZ)^{3/2}}\right]_{y=Y_s^1},$$

where $Z = (s^{-1/2}Y_s^2)^2 + (s^{-1/2}Y_s^3)^2$ is χ_2^2 distributed. Thus

$$E^Q\left[\frac{1}{(y^2 + sZ)^{3/2}}\right] = \frac{1}{2}\int_0^\infty (y^2 + sz)^{-3/2} e^{-z/2} dz \leq \frac{1}{2}\int_0^\infty (y^2 + sz)^{-3/2} dz = \frac{1}{2s|y|},$$

and hence $|\xi_s| \leq \frac{1}{2s}$, which of course is square integrable over $[\varepsilon, t]$. Together with the boundedness on $[0, \varepsilon)$, this proves the claim. □

We can now finally prove that P solves the measure extension problem.

Theorem 4.5.2 *We have $E^P[M_t \mid \mathcal{F}_t] > 0$ for all $t \geq 0$, P-a.s. Hence P solves the measure extension problem.*

Proof. By Lemma 4.3.1 it is indeed enough to prove $E^P[M_t \mid \mathcal{F}_t] > 0$ for all $t \geq 0$. Throughout the proof we will freely use Lemma 4.1.1 to pass between $\mathbb{F}$ and its P-completion, without explicit mentioning. We let B and θ be as in Theorem 4.5.1. Now, thanks to Lemma 4.5.6 above, Theorem 4.5.3 (see the end of this section) is applicable. In particular it follows that $E^P[M_t \mid \mathcal{F}_t]$ is continuous, hence strictly positive on $t \leq \sigma_n$ for each n, where

$$\sigma_n = \inf\left\{t \geq 0 : E^P[M_t \mid \mathcal{F}_t] \leq \frac{1}{n}\right\}.$$

The stochastic exponential representation for strictly positive martingales and Theorem 4.5.3 therefore imply that for each n there is a process ξ^n such that

$$E^P[M_t \mid \mathcal{F}_t] = \exp\left\{\int_0^t \xi_s^n dB_s - \frac{1}{2}\int_0^t (\xi_s^n)^2 ds\right\}, \qquad t \le \sigma_n.$$

Recalling that $E^P[M_t \mid \mathcal{F}_t] = \frac{dQ}{dP}|_{\mathcal{F}_t}$, we apply the Girsanov theorem to obtain that

$$B_{t\wedge\sigma_n} - \int_0^{t\wedge\sigma_n} \xi_s^n ds$$

is an $(\mathbb{F}, Q)$ local martingale. It is equal to

$$Y^1_{t\wedge\sigma_n} - 1 + \int_0^{t\wedge\sigma_n} (\theta_s - \xi_s^n) ds,$$

and since Y^1 is $(\mathbb{F}, Q)$ Brownian motion, the finite variation term must be zero. Hence for each n, $\xi_t^n = \theta_t \; dt \otimes dP$-a.e. on $[0, \sigma_n]$, and we deduce that

$$E^P[M_t \mid \mathcal{F}_t] = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \qquad t < \lim_{n\to\infty} \sigma_n.$$

In particular, $\lim_{n\to\infty} \sigma_n = \inf\{t \ge 0 : \int_0^t \theta_s^2 ds = \infty\}$, which is infinite by Lemma 4.5.6. This concludes the proof. □

Having established that P solves the measure extension problem, let us make a few comments on the general results established in Section 4.3 and Section 4.4 in the context of the above example. First, it was observed in [33] that $E^P[N_t \mid \mathcal{F}_t]$ is of Class (D). This is consistent with, albeit stronger than, the conclusion of Proposition 4.3.1. Next, the processes Λ and U in the multiplicative decomposition $E^Q[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t$ are easily computed:

$$\Lambda_t = \int_0^t \frac{1}{su(s, Y_s^1)} dL_s^0, \qquad U_t = \exp\left(\int_0^t \frac{u_x}{u}(s, Y_s^1) dY_s^1 - \frac{1}{2}\int_0^t \frac{u_x}{u}(s, Y_s^1)^2 ds\right).$$

In particular, Λ has singular paths. This is consistent with Corollary 4.4.1, since Y_t^1 admits a density and $\mathcal{X}_0 = \{0\}$. Moreover, Λ only increases on $\{Y_t^1 \in \mathcal{X}_0\} = \{Y_t^1 = 0\}$ since $d\Lambda_t \ll dL_t^0$. Finally, if we instead let $\mathbb{F}$ be generated by (Y^1, Y^2), the measure extension problem does not have a solution. This follows from Corollary 4.3.1, since Theorem 5.2 in [33] shows that the optional projection of N under Q is still a local martingale.

We end this section with the following result, which was needed in the proof of Theorem 4.5.2. It seems likely that the result is known; however, we have not been able to find it stated explicitly, and therefore provide full details. The setting is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Theorem 4.5.3 *Let X be a stochastic process and assume that $\mathbb{F}$ is its natural filtration, made right-continuous and augmented with the P-nullsets. Suppose*

$$X_t = X_0 + B_t - \int_0^t \theta_s ds$$

for a Brownian motion B and a predictable process θ such that $\int_0^t \theta_s^2 ds < \infty$ for all $t \geq 0$. Then B has the predictable representation property: for every local martingale U there is a predictable process ξ with $\int_0^t \xi_s^2 ds < \infty$ for all $t \geq 0$, such that

$$U_t = U_0 + \int_0^t \xi_s dB_s.$$

Proof. Define stopping times $\rho_n = \inf\{t \geq 0 : \int_0^t \theta_s^2 ds \geq n\}$. Due to the integrability of θ, $\lim_{n\to\infty} \rho_n = \infty$ a.s. Fix n and define

$$dZ_t^n = Z_t^n \theta_t \mathbf{1}_{\{\mathbf{t} \leq \rho_\mathbf{n}\}} dB_t, \qquad Z_0^n = 1,$$

which is a strictly positive local martingale, and a uniformly integrable martingale by Novikov's criterion. Then define $Q^n \sim P$ by $dQ^n = Z_\infty^n dP$, and introduce the filtration $\mathbb{F}^n = (\mathcal{F}_{t\wedge T_n})_{t\geq 0}$. Girsanov's theorem together with Lévy's characterization of Brownian motion shows that

$$W_t^n = X_{t\wedge\rho_n} - X_0 = B_{t\wedge\rho_n} - \int_0^{t\wedge\rho_n} \theta_s ds$$

is $(\mathbb{F}^n, Q^n)$ Brownian motion on $[0, T_n]$, and it clearly generates $\mathbb{F}^n$. Hence every $(\mathbb{F}^n, Q^n)$ local martingale can be represented as a stochastic integral with respect to W^n. Now, Girsanov's theorem applied to the $(\mathbb{F}^n, P)$ local martingale $U^n = (U_{t\wedge\rho_n})_{t\geq 0}$ shows that

$$U_t^n - \int_0^t \frac{1}{Z_s^n} d\langle U^n, Z^n\rangle_s$$

is an $(\mathbb{F}^n, Q)$ local martingale, and thus equal to $U_0 + \int_0^t \xi_s^n dW_s^n$ for some predictable ξ^n with $\int_0^t (\xi_s^n)^2 ds < \infty$ for all $t \geq 0$. Furthermore, the Kunita-Watanabe inequality implies that $d\langle U^n, Z^n\rangle_t \ll d\langle Z^n, Z^n\rangle_t \ll dt$, so there is some $\mathbb{F}^n$ adapted process η^n such that $d\langle U^n, Z^n\rangle_t = \eta_t^n dt$. Combining these facts yields

$$U_t^n = U_0 + \int_0^t \xi_s^n dW_s^n + \int_0^t \frac{\eta_s^n}{Z_s^n} ds = U_0 + \int_0^{t\wedge\rho_n} \xi_s^n dB_s + \int_0^{t\wedge\rho_n} \left(\frac{\eta_s^n}{Z_s^n} - \xi_s^n \theta_s\right) ds.$$

But U^n is an $(\mathbb{F}, P)$ local martingale, so the finite variation term is zero. We can now construct the desired process ξ by setting $\xi = \xi^n$ on $(\rho_{n-1}, \rho_n]$ for $n \geq 1$. □

4.6 Financial implications

The financial interpretation of the fact that a strict local martingale may lose the local martingale property when projected onto a filtration to which it is no longer adapted was discussed in [33]. The main conclusion there was that a price process which is arbitrage free in the sense of NFLVR, may be perceived as containing arbitrage opportunities by an investor who only has access to incomplete observations of the price process. We may now qualify this statement: if the market is *inefficient*, the partially observed price process may look as though arbitrage opportunities exist. In other words, the presence of dominated assets (but absence of arbitrage opportunities) translates into the appearance of arbitrage. On the other hand, in an efficient market this phenomenon does not occur: the optional projection of a martingale is always a martingale, as follows from the calculation,

$$E[E[M_t \mid \mathcal{F}_t] \mid \mathcal{F}_s] = E[E[M_t \mid \mathcal{G}_s] \mid \mathcal{F}_s] = E[M_s \mid \mathcal{F}_s].$$

Therefore, the efficiency property is stable with respect to *any* reductions of information set, while NFLVR is not.

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