PRICE MANIPULATION WITH DARK POOLS AND MULTI-PRODUCT SEPARATION IN INVENTORY HEDGING

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Yuemeng Sun

August 2011

© 2011 Yuemeng Sun

ALL RIGHTS RESERVED

PRICE MANIPULATION WITH DARK POOLS AND MULTI-PRODUCT SEPARATION IN INVENTORY HEDGING

Yuemeng Sun, Ph.D.
Cornell University 2011

This dissertation addresses two different problems within mathematical finance: an optimal execution problem with dark pools using a market impact model, and multi-product separation with financial hedging for inventory management.

In the first part of the dissertation we consider an optimal liquidation problem in which a large investor can sell on a traditional exchange or in a so-called dark pool. Dark pools differ from traditional exchanges in that the orders placed in it generate little to no price impact on the market price of the asset. Within the framework of the Almgren-Chriss market impact model, we study an extended model which includes the cross-impact between the two venues. By analyzing the optimal execution strategy, we identify those model specifications for which the corresponding order execution problem is stable in the sense that are no price manipulation strategies which can be beneficial.

In the second part of the dissertation, we propose financial hedging tools for inventory management. Based on a framework for hedging against the correlation of operational returns with financial market returns, we consider the general problem of optimizing simultaneously over both the operational policy and the hedging policy of the corporation. Our main goal is to achieve a separation result such that for a corporation with multiple products and inventory departments, the inventory decisions of each department can be made independently of the other departments' decisions. We focus initially on a single-period, multi-product hedging problem for inventory management, and model an economy experiencing monetary inflation. We use the Heath-Jarrow-

Morton model to represent the financial market. We then extend the model to consider multiple periods and more general market models. In both cases, we prove a separation result for inventory management that allows each inventory department to make decisions independently. In particular, the separation result for the multi-period problem is a global separation in the sense that no interaction needs to be considered among products in intermediate time periods. In addition, we propose a dynamic programming simplification of the multi-period single-item inventory problem which further simplifies the computation by reducing the dimension of the state space.

BIOGRAPHICAL SKETCH

Yuemeng "Sunny" Sun was born in Beijing, China on October 6, 1982. She did her undergraduate studies in the Department of Mathematics at Nanjing University. After earning a Bachelor of Science degree in June 2004, she majored in Applied Mathematics completing the Computational Finance track in the Department of Mathematics, Statistics and Computer Science at University of Illinois at Chicago. She obtained the Master of Science degree in May 2006. That same year, she was admitted to the Ph.D. program in the field of Operations Research and Information Engineering at Cornell University. She has a great interest in the field of mathematical finance and inventory management. From 2008 to 2009, she did her doctoral research under the guidance of Professor Alexander Schied, focused on the optimal execution problem with market impact. Since Professor Schied left Cornell University in 2009, she has been doing research under the guidance of Professor Peter Jackson, in the area of financial hedging in inventory management.

Upon completion of her Ph.D. she will join the Global Quant Group of Bank of America, Merrill Lynch.

To my parents:

Guoming Sun and Xiurong Wang

For their unconditional love and support

ACKNOWLEDGEMENTS

First of all, I would like to thank my advisor, Professor Peter Jackson for his invaluable guidance and encouragement. He has been an excellent advisor, and a great mentor both in work and life. Meetings with him have been delightful and inspiring. Each one was filled with not only useful insights into my work, but also valuable lessons for life outside of research. During the moments of frustration and anxiety in my research, Peter was always the person who encouraged me. I would also like to thank my former advisor Professor Alexander Schied, it was my great honor to work with him during his stay at Cornell University. Research experiences with Professor Schied lead me to the world of mathematical finance, and initiated my passion for this area. I am really thankful for the advice and suggestions he continues to offer me, despite now being in Germany. I also wish to thank the School of Operations Research and Information Engineering for the generous support and the great research environment.

Next, I want to thank Professor Johannes Wissel for working with me. I have been very fortunate to be able to work with such a brilliant scholar and a generous friend. The discussions with him were always illuminating to me. He has been providing me glowing insights and ideas in research and being so generous with his time. I am also very grateful to Professor Huseyin Topaloglu and Professor Michael Nussbaum for their valuable inputs while serving as members of my special committee. It was Professor Topaloglu's encouragement and advice that helped me get through the ups and downs.

In addition, I want to thank Professor Robert Bland for being such a heart-warming mentor. I want to thank Professor Kathryn Caggiano; it was a pleasure serving as a teaching assistant for her. My extended gratitude goes to all faculty and staff members in the School of Operations Research and Information Engineering. In particular, Kathryn King, Cindy Jay, and Eric Johnson for their patient, efficient administrative and technical support.

Special thanks to all my Ph.D. fellows and friends for their companionship and help. You made my life here enjoyable! To name a few, Fan Zhu, Yi Shen, Xiaoqing Xie, Hui Qu, Jie Chen, Juan Li, Collin Chan, Chao Ding, Mathew McLean, Jiawei Qian, Shanshan Zhang, Tuohua Wu, Sophia Liu, Gwen Spencer.

Last but not least, nothing would have been possible without the love and encouragement of my parents. I dedicate this thesis to you!

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Since the publication of The Theory of Speculation by Louis Bachelier in 1900, mathematical finance has been evolving as a rich and separate field of applied mathematics. There are a variety of different problems in the area of mathematical finance. For example, the portfolio optimization problem and the financial asset pricing problem are both classical topics. While its theoretical side is being enriched, mathematical finance tools are being extensively applied both in academia and industry. In this dissertation we address two different problems within mathematical finance: an optimal execution problem with dark pools and multi-product separation with financial hedging for inventory management. The first problem is a type of optimal execution problem, that is, a problem to find an optimal asset liquidation strategy in a market with limited liquidity. The particular problem addressed is the use of dark pools as an alternative to traditional exchanges. Transactions in dark pools are invisible to all but the parties directly engaged in the transaction. The second problem considers the integrated problem of managing retail inventories when the operational risk is correlated with financial instruments. We investigate conditions under which multi-product problems can be separated into single item problems.

This dissertation is structured into four chapters beyond this introduction. Chapter 2 presents the research on dark pools; Chapters 3 and 4 present the research on multi-product separation; and chapter 5 presents concluding remarks. A more detailed overview follows.

In chapter 2, we consider an optimal liquidation problem with two trading venues: a traditional exchange and a dark pool. A dark pool is an alternative trading platform whose use has been mushrooming in recent years. The first dark pool, Instinet's *After*

Hours Cross, was started in the fall of 1986. The Cross allowed clients to enter orders into a blind book which would then run a match at 6:30 pm Eastern time using that day's closing price for all traders. The match allowed large buyers and sellers to transact without pre-trade transparency and potential information leakage. Instinet soon had many competitors, both in the U.S. and around the world. As of 2010, there were over 40 different dark pools in the U.S. and they accounted for 12.1% of the U.S. equities market.

The main feature of a dark pool is that it provides dark liquidity; that is, orders placed in a dark pool cannot be seen by any potential market participant. As a result, dark pool orders do not influence the quoted price of an asset. Due to its 'dark' property, dark pools have been popular among traders who wish to move large numbers of shares without revealing themselves to the open market. Neither the price nor the identity of the trading company is displayed. Dark pools are popular among institutional investors. For them, dark pools provide many of the efficiencies associated with trading on the traditional exchanges' public limit order book but without showing their hand to others.

Dark pools vary greatly in their characteristics and makeup. It is common to divide them into the following five general categories:

- Public Crossing Networks. These are the most traditional dark pools. Most were started by agency-only brokerage firms with the single economic purpose of generating commissions. One of the distinguishing properties of public crossing networks is that the dark pool operator is barred from engaging in proprietary trades.
- Internalization Pools. These are designed primarily to internalize the operator's trade flow. They differ from public crossing networks in that they can include the operator's proprietary trades as well as the flow from their retail and institutional customers.

- Ping Destinations. The operators of these pools accept only Immediate or Cancel (IOC) orders, and their customers' flow interacts solely with the operator's own flow. The main operators of Ping destinations are large hedge funds or electronic market-makers.
- Exchange-Based Pools. There are two types of dark pools in this category: dark pools that are actually registered as Alternative Trading Systems (ATSs) by exchanges, and pools of liquidity created as a result of hidden order types supported by Electronic Communication Networks (ECNs) and exchanges. A distinguishing characteristic is that the hidden orders usually interact with regular displayed orders.
- Consortium-Based Pools. These are pools operated by numerous partnering brokers. These dark pools behave like a hybrid of public crossing networks and internalization pools. Unlike crossing network pools, the partners may engage in proprietary trades. However, unlike internalization pools, they are not typically owned by agency-only firms. They therefore provide somewhat more transparency.

In our work, we consider a model for order execution in two possible venues: a dark pool and an open exchange. Facing a liquidation deadline, the trader needs to execute a strategy using the two trading venues to maximize the expected revenue by reducing the market price impact. There are two tasks to accomplish: liquidation must be completed by the deadline and the impact of the strategy on market price must be minimized. While the dark pool promises a reduction of market impact and of liquidation costs, the risk is that the order placed in the dark pool cannot meet a matching order, and, as a result, cannot be fulfilled by the deadline. While using a traditional exchange guarantees execution of the trades, the risk is that the trades will generate a significant price impact. The optimal strategy, therefore, is likely to be a hybrid strategy exploiting both venues.

We propose a continuous-time stochastic model which extends the standard Almgren-Chriss market impact model to include exchange prices in a dark pool. We derive an optimal trading strategy that exploits both venues.

One distinguishing property of dark pools is that they do not have an intrinsic price-finding mechanism. Instead, the price at which orders are executed is derived from the publicly quoted price at an exchange. Thus, trades in a dark pool might be manipulated through placing large buy or sell orders in the corresponding exchange coincident with offsetting orders within the pool. Because of the importance of this issue for regulation and for market efficiency, we use our model to establish conditions under which the manipulations are not beneficial.

Chapters 3 and 4 of the dissertation address the financial hedging problem for a large corporation's inventory management. Traditional inventory management models focus on characterizing inventory policies so as to minimize the expected total cost over a planning horizon. This kind of objective is appropriate for risk-neutral decision makers. Corporate planners increasingly recognize that inventory investments and retail operations expose the company to significant financial risk; so, introducing a degree of risk aversion into inventory planning is appropriate.

Our model differs from much of the existing inventory literature in that we consider a non-financial corporation (a retail sales organization) doing financial hedging simultaneously with inventory management. We therefore consider both financial risk and non-financial risk. The financial risk comes from the financial market and hence can be hedged, to some extent, using financial instruments. The non-financial risk is assumed to be independent of the financial market, and hence cannot be hedged through financial trading. We assume that both financial and non-financial risk is observable. In the case of non-financial risk, this could be captured in macroeconomic indicators such as the rate of unemployment and market conditions. Because the non-financial risk cannot be hedged we must pose our model as an incomplete market. However, because the risk is observable we are able to pose the problem in terms of financial hedging.

This problem of hedging contingent claims by means of dynamic trading strategies in an incomplete market is a central problem in financial mathematics. There are abundant results in this framework. A classical approach to this problem is to control the hedging error by a quadratic criterion. Mathematically, this is equivalent to solving an optimal investment problem for a mean-variance type objective function. Due to the high degree of tractability, this approach is attractive to operations management. Hence, we consider a mean-variance type objective function for inventory management.

There is correlation in demand for products arising from common factors such as the business cycle and interest rates. Consequently, a risk-averse strategy for planning inventories should consider all of the products together as a large-scale portfolio problem. However, a typical retail company manages thousands of part numbers and employs dozens of inventory managers. The task of coordinating these inventory decisions as part of a portfolio optimization seems impractical with current technologies. Consequently, we explore conditions under which this optimization problem can be optimally decomposed into single-item inventory planning problems. In our view, for a practical implementation, the ideal solution for inventory managers is to solve a separate operational planning problem for each item and communicate their results to a finance department which would hedge the residual financial risk.

In chapter 3, we consider a single-period model in which the inventory decisions need only to be made at the beginning of the period. The financial risk we consider specifically is inflation risk. That is, we consider a period of rapidly inflating prices arising, for example, from a currency devaluation. We assume that high inflation affects retail operations in two ways: it leads to both higher sales prices but also to lower demand. In a period of rapid monetary inflation, an inventory manager may be tempted to convert as much cash as possible into hard assets such as retail inventory in order to preserve wealth. However, inventory is an inferior asset for the purpose of wealth preservation: it deteriorates with time and it can be hard to liquidate, especially if high prices are discouraging demand. Consequently, this strategy would make sense only if there is no alternative financial asset available in which to preserve wealth. We call such inventory investment a 'malinvestment.' If a reliable financial asset exists, the optimal strategy will be to restrict inventory investment to optimize operational tradeoffs and to hedge financial risk using the financial asset. The main achievement of this research is a separation result whereby the inventory decision of each product can be optimized separately and an optimal hedging strategy is developed subsequently.

In chapter 4, we extend the model to consider multiple time periods and more general market models. The challenge is to prove that a separation result is still valid in this case. Two types of separation might be possible within a multi-period problem. A so-called local separation might be possible in which inventory decisions within a given time period might be made independently but coordination and joint evaluation is required when considering the impact on future time periods. However, a global separation might also be possible in which the optimization problem decomposes into separate multi-period single-item problems with no joint evaluation required. Our work establishes a global separation result. For a multi-period problem, we also present a dynamic programming algorithm which reduces the dimension of the state space and admits a practical computation of the relevant inventory and hedging strategies. Concluding comments and suggestions for future research can be found in chapter 5.

CHAPTER 2

OPTIMAL EXECUTION WITH DARK POOLS AND THE ABSENCE OF PRICE MANIPULATION

2.1 Introduction

Recent years have seen a mushrooming of alternative trading platforms called dark pools. Orders placed in a dark pool are not visible to other market participants (hence the name) and thus do not influence the publicly quoted price of the asset. Thus, when dark-pool orders are executed against a matching order, no direct price impact is generated, although there may be certain indirect effects. Dark pools therefore promise a reduction of market impact and of liquidation costs. They are, hence, a popular platform for the execution of large orders.

Dark pools differ from standard limit order books in that they do not have an intrinsic price-finding mechanism. Instead, the price at which orders are executed is derived from the publicly quoted prices on an exchange. Thus, by manipulating the price at the exchange through placing buy or sell orders, the value of a possibly large amount of ”dark liquidity” in the dark pool can be altered. We refer to Mittal (2008) for a practical overview on dark pools and some related issues of market manipulation.

In this paper, we consider a stochastic model for order execution in two simultaneous possible venues: a dark pool and an open exchange. This model is a continuous-time variant of the one proposed by Kratz & Schöneborn (2010). It is a natural model because it extends the standard Almgren-Chriss market impact model for exchange prices to include a dark pool. We refer to Almgren (2003) for details on the Almgren-Chriss model and also to Bertsimas & Lo (1998) for a discrete-time precursor. Alternative approaches to modeling and analyzing dark pools have been proposed, e.g., by Degryse et al. (2009), Foucault & Menkveld (2008), Laruelle & Lehalle (2009), and Ye (2010).

Kratz & Schöneborn (2010) mainly investigate optimal order execution strategies for an investor who can trade in the exchange and in the dark pool. But, they are also interested in price manipulation strategies in the sense of Huberman & Stanzl (2004). Their Propositions 7.1 and 7.2 provide some first results on the existence and the absence of such strategies. One important reason for considering price manipulation strategies is that their existence leads to instabilities in the market impact model and often precludes the solvability of the optimal order execution problem. We refer to Huberman & Stanzl (2004), Gatheral (2010), Almgren (2003) for discussions.

Our main goal in this paper is to carry out an in-depth study of transaction-triggered price manipulation in this dark pool model. The observation is that the transaction-triggered price manipulation exists in such a model. Transaction-triggered price manipulation looks similar to the usual price manipulation strategies, but occurs only when triggered by a given transaction. More precisely, it involves strategies which decrease the expected execution costs of a sell (buy) program by intermediate buy (sell) trades. In section 4, a transaction-triggered price manipulation is identified, and it turns out that generation of such a phenomenon hinges in a subtle way on the interplay of all model parameters and of the liquidation time constraint. With further exploration of the optimal execution model, we discover only two cases in which transaction-triggered price manipulation exists. Finally, we tie these conditions to an constraint on model parameters which guarantees the absence of price manipulation.

The paper is organized as follows. In section 2 we introduce the model. The optimal execution strategy is stated in section 3. In section 4 we discuss the existence of transaction-triggered price manipulation and present sufficient conditions for the absence of price manipulation.

2.2 Formulation of the problem

Consider a seller who wants to liquidate an asset position of size x by time T. He/she has the choice of investing in the dark pool or in the exchange. Although the dark pool has the mechanics that the orders placed will not affect the market price of the asset, there exists a tradeoff in that the sellers or the buyers may never be able to find a counter-party for their trade. On the other hand, trading in the exchange guarantees that the seller can liquidate the asset at a certain price, but the transaction will have a price impact on the market, and as a result, effectively incur a transaction cost (the loss due to price impact).

Let $\bar{\mathcal{F}}_t$ be the filtration generated by the Brownian Motion B with constant volatility σ and $B_0 = 0$. The stock price dynamics in the market are given by

$$\begin{aligned} P_t &= P_0 + \sigma B_t + \gamma(X_t - X_0) + \eta \dot{X}_t \\ &= P_t^0 + \gamma(X_t - X_0) + \eta \dot{X}_t \end{aligned}$$

where γ is the parameter for permanent impact, η is the parameter for temporary impact, P_0 is the initial stock price in the exchange, P_t^0 is the unaffected stock price process, X_0 is the number of shares which need to be liquidated, and X_t is the number of shares yet to be liquidated at time t.

We model the arrival time of a matching buy order in the dark pool as an exponentially distributed random variable τ with parameter $\theta > 0$, i.e.

$$P(\tau < t) = \int_0^t \theta e^{-\theta s} ds.$$

The stopping time, τ, is independent of the Brownian Motion B. Define $\mathcal{F}_t$ as the τ-progressive enlargement of $\bar{\mathcal{F}}_t$; $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ is the corresponding probability space.

Consider the following trading strategy. The seller divides the total order into two parts at $t = 0$. $\hat{X}$ shares are placed in the dark pool in the hope of finding a counter-party to liquidate this portion, while $x - \hat{X}$ shares are to be executed in the exchange. If the transaction does not occur in the dark pool by some time point $\rho \in [0, T]$, the seller exits the dark pool and enters the exchange in order to meet the deadline of execution.

Let X_t be the number of shares held in the exchange at time t, satisfying boundary conditions $X_0 = x - \hat{X}$ and $X_T = 0$. If the seller cannot find a counter-party and decides to exit the dark pool at time ρ, then $X_{\rho_+} = X_\rho + \hat{X}$; otherwise, the matching buy order arrives at τ, $\tau \leq \rho$. Let $\dot{X}_t$ denote the derivative of X_t, and $\xi_t = -\dot{X}_t$ be the rate of liquidation at time t for a selling activity in the exchange. Assume ξ_t is progressively measurable and

$$\int_0^t \xi_s^2 ds < \infty, \quad \text{for all } t \leq T \ . \quad P - \text{a.s.}$$

We also assume that the strategies are admissible in the sense that the position in shares $X_t(\omega)$ is bounded uniformly in t and ω. Denote all admissible strategies by $\mathcal{X}(x, r, \rho, T)$.

An admissible strategy $(\hat{X}, \xi, \rho)$ will be called a *single-update strategy* if ρ is a deterministic time in $[0, T)$ and ξ is predictable with respect to the filtration generated by the stochastic process $\mathbb{1}_{\{\tau \leq t\}}$, $t \geq 0$.

Note that the process ξ of a single-update strategy evolves deterministically until there is an execution in the dark pool, i.e., until time τ. At that time, ξ can be updated. But, by assumption, the update can depend only on the time τ and not on any other random quantities. In particular, ξ can be written as

$$\xi_t = \begin{cases} \xi_t^0, & \text{if } t \leq \tau \text{ or } \tau > \rho, \\ \xi_t^1, & \text{if } t > \tau \text{ and } \tau \leq \rho, \end{cases} \tag{2.1}$$

where ξ^0 is deterministic, and ξ^1 depends on τ.

Our perspective is to maximize the expected revenue to the seller over $[0, T]$ by finding the optimal single-update strategy for the execution. Notice that if there is no execution in the dark pool by time ρ, the seller will withdraw $\hat{X}$ from the dark pool and place it in the exchange.

The following lemma characterizes the revenue from the execution.

Lemma 2.2.1. *With the price dynamics and the order arrival time in the dark pool defined as above, the seller's expected revenue by adapting a single-update strategy* $(\hat{X}, \xi, \rho)$ *is:*

$$\mathcal{R}_{[0,T]} = X_0 P_0^0 + \hat{X} P^0_{\{\tau\wedge\rho\}} + \int_0^T X_t \, dP_t^0 - \frac{\gamma}{2}(x - \mathbb{1}_{\{\tau<\rho\}}\hat{X})^2 - \eta \int_0^T \xi_t^2 \, dt$$
$$+ \mathbb{1}_{\{\tau<\rho\}}\hat{X}\gamma(X_\tau - X_0).$$

Proof. There are two scenarios to be considered: either the order is fulfilled by a counter-party before ρ in the dark pool or it is not.

In the first case, the revenue in $[0, T]$ is composed of two parts: the revenue in the exchange in $[0, T]$ and the revenue in the dark pool at the time point of fulfilling the order. That is, on the set $\{\omega : \tau(\omega) \le \rho\}$,

$$\begin{aligned}
\mathcal{R}_{[0,\tau]}|_\tau &= \int_0^\tau \xi_t P_t dt + \hat{X} P_\tau \\
&= X_0 P_0^0 - P_\tau^0 X_\tau + \int_0^\tau X_t dP_t^0 - \frac{\gamma}{2}(X_\tau - X_0)^2 - \eta \int_0^\tau \xi_t^2 dt + \gamma \hat{X}(X_\tau - X_0) \\
\mathcal{R}_{[\tau,T]}|_\tau &= \int_\tau^T \xi_t P_t dt \\
&= X_\tau P_\tau^0 - P_T^0 X_T + \int_\tau^T X_t dP_t^0 - \frac{\gamma}{2} X_0^2 + \frac{\gamma}{2}(X_\tau - X_0)^2 - \eta \int_\tau^T \xi_t^2 dt \\
\mathcal{R}_{[0,T]}|_\tau &= \mathcal{R}_{[0,\tau]}|_\tau + \mathcal{R}_{[\tau,T]}|_\tau \\
&= X_0 P_0^0 + \hat{X} P_\tau^0 + \int_0^T X_t dP_t^0 - \eta \int_0^T \xi_t^2 dt - \frac{\gamma}{2} X_0^2 + \gamma \hat{X}(X_\tau - X_0).
\end{aligned}$$

If there is no activity in the dark pool before ρ, then on the set $\{\omega : \tau(\omega) > \rho\}$,

$$
\begin{aligned}
\mathcal{R}_{[0,\tau]}|_\tau &= \int_0^\rho \xi_t P_t dt + \int_\rho^\tau \xi_t P_t dt \\
&= X_0 P_0^0 + P_{T_0}^0 \hat{X} - P_\tau^0 X_\tau + \int_0^\tau X_t dP_t^0 - \frac{\gamma}{2}(X_\tau - x)^2 - \eta \int_0^\tau \xi_t^2 dt \\
\mathcal{R}_{[\tau,T]}|_\tau &= X_\tau P_\tau^0 - P_T^0 X_T + \int_\tau^T X_t dP_t^0 - \frac{\gamma}{2}x^2 + \frac{\gamma}{2}(X_\tau - x)^2 - \eta \int_\tau^T \xi_t^2 dt \\
\mathcal{R}_{[0,T]}|_\tau &= X_0 P_0^0 + \hat{X} P_{T_0}^0 + \int_0^T X_t dP_t^0 - \eta \int_0^T \xi_t^2 dt - \frac{\gamma}{2}x^2.
\end{aligned}
$$

Combining the two scenarios above, the revenue during $[0, T]$ is

$$
\begin{aligned}
\mathcal{R}_{[0,T]} = X_0 P_0^0 + \hat{X} P_{\{\tau\wedge\rho\}}^0 + \int_0^T X_t \, dP_t^0 - \frac{\gamma}{2}(x - \mathbb{1}_{\{\tau<\rho\}}\hat{X})^2 - \eta \int_0^T \xi_t^2 \, dt \\
+ \mathbb{1}_{\{\tau<\rho\}} \hat{X} \gamma (X_\tau - X_0).
\end{aligned}
$$

□

2.3 The Optimal Execution Strategy

Single-update policies, as described in the previous section, may seem overly restrictive. The following proposition reveals that they are, in fact, optimal.

Proposition 2.3.1. *For any $X_0 \in \mathbb{R}$ and $T > 0$ there exists a single-update strategy that maximizes the expected revenues $\mathbb{E}[\mathcal{R}_T]$ in the class of all admissible strategies.*

Proof. Recall that

$$
\begin{aligned}
\mathcal{R}_{[0,T]} = X_0 P_0^0 + \hat{X} P_{\{\tau\wedge\rho\}}^0 + \int_0^T X_t \, dP_t^0 - \frac{\gamma}{2}(x - \mathbb{1}_{\{\tau<\rho\}}\hat{X})^2 - \eta \int_0^T \xi_t^2 \, dt \\
+ \mathbb{1}_{\{\tau<\rho\}} \hat{X} \gamma (X_\tau - X_0).
\end{aligned}
$$

Taking the conditional expectation with respect to $\mathcal{F}_{\tau\wedge\rho}$ yields

$$\mathbb{E}[\mathcal{R}_{[0,T]}\,|\,\mathcal{F}_{\tau\wedge\rho}] = X_0P_0^0 + \hat{X}P^0_{\{\tau\wedge\rho\}} + \int_0^{\tau\wedge\rho} X_t\,dP_t^0 - \frac{\gamma}{2}(x-\mathbb{1}_{\{\tau<\rho\}}\hat{X})^2 - \eta\int_0^{\tau\wedge\rho}\xi_t^2\,dt$$
$$+\,\mathbb{1}_{\{\tau<\rho\}}\hat{X}\gamma(X_\tau - X_0) - \mathbb{E}\Big[\eta\int_{\tau\wedge\rho}^T \xi_t^2\,dt\,\Big|\,\mathcal{F}_{\tau\wedge\rho}\Big].$$

Due to the liquidation constraint, we must have $\int_{\tau\wedge\rho}^T \xi_t\,dt = X_{\tau\wedge\rho} + \mathbb{1}_{\{\tau>\rho\}}\hat{X}$, and so

$$\int_{\tau\wedge\rho}^T \xi_t^2\,dt \geq \frac{(X_{\tau\wedge\rho}+\mathbb{1}_{\{\tau>\rho\}}\hat{X})^2}{T-\tau\wedge\rho}$$

with equality if, for $\tau\wedge\rho\leq t\leq T$,

$$\xi_t = \begin{cases} \dfrac{X_\tau}{T-\tau} & \text{on } \{\tau<\rho\} \\[2em] \dfrac{X_\rho+\hat{X}}{T-\rho} & \text{on } \{\rho\leq\tau\}. \end{cases} \tag{2.2}$$

These two possibilities will correspond to the single update of $\bar{\xi}$ at τ.

Note next that, due to the predictability of ξ and ρ, $(\xi_s)_{s\leq t}$ and $\rho\wedge t$ are independent of τ, conditional on $\{t\leq\tau\}$. It follows that

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_T] &= \mathbb{E}[\,\mathbb{E}[\mathcal{R}_T\,|\,\mathcal{F}_{\tau\wedge\rho}]\,] \\
&\leq xP_0^0 + \mathbb{E}\Big[-\frac{\gamma}{2}(X_0-\mathbb{1}_{\{\tau<\rho\}}\hat{X})^2 - \eta\int_0^{\tau\wedge\rho}\xi_t^2\,dt + \mathbb{1}_{\{\tau<\rho\}}\hat{X}\gamma(X_\tau-X_0) \\
&\quad -\eta\frac{(X_{\tau\wedge\rho}+\mathbb{1}_{\{\tau>\rho\}}\hat{X})^2}{T-\tau\wedge\rho}\Big] \\
&= xP_0^0 + \mathbb{E}\Big[\int_0^\infty du\,\theta e^{-\theta u}\Big\{-\frac{\gamma}{2}(X_0-\mathbb{1}_{\{u<\rho\}}\hat{X})^2 - \eta\int_0^{u\wedge\rho}\xi_t^2\,dt \\
&\quad + \mathbb{1}_{\{u<\rho\}}\hat{X}\gamma\int_0^u \xi_t\,dt - \eta\frac{(X_0+\int_0^{u\wedge\rho}\xi_t\,dt+\mathbb{1}_{\{u>\rho\}}\hat{X})^2}{T-u\wedge\rho}\Big\}\Big].
\end{aligned}$$

Consider the functional that maps $r\in[0,T]$ and $\xi\in L^p[0,T]$ to

$$\begin{aligned}
F(r,\xi) := \int_0^\infty du\,\theta e^{-\theta u}\Big\{&\frac{\gamma}{2}(X_0-\mathbb{1}_{\{u<r\}}\hat{X})^2 + \eta\int_0^{u\wedge r}\xi_t^2\,dt \\
&-\mathbb{1}_{\{u<r\}}\hat{X}\gamma\int_0^u\xi_t\,dt + \eta\frac{(X_0+\int_0^{u\wedge r}\xi_t\,dt+\mathbb{1}_{\{u>r\}}\hat{X})^2}{T-u\wedge r}\Big\}.
\end{aligned}$$

When F admits a minimizer (r^*, ξ^*), then concatenating ξ^* with (2.2) in $r^* \wedge \tau$ yields an optimal strategy that is a single-update strategy.

To show the existence of a minimizer of F, take any pair $(\tilde{r}, \tilde{\xi})$ for which $C := F(\tilde{r}, \tilde{\xi}) < \infty$. We then only need to look into those pairs (r, ξ) for which $F(r, \xi) \leq C$. Then the component ξ must be contained in the set

$$K_C := \Big\{ \xi \in L^1[0, T] \,\Big|\, \int_0^T \xi_t^2 \, dt \leq \widetilde{C} \Big\},$$

where $\widetilde{C}$ is a suitable constant.

The set K_C is a closed convex subset of $L^1[0, T]$. Hence it is also weakly closed in $L^1[0, T]$. It is also uniformly integrable according to the criterion of de la Vallée Poussin and our assumption that f has superlinear growth. Hence, the Dunford–Pettis theorem (Dunford & Schwartz 1988, Corollary IV.8.11) implies that K_C is weakly sequentially compact in $L^1[0, T]$. From now on we will endow K_C with the weak topology.

Next,

$$[0, T] \times K_C \ni (r, \xi) \longrightarrow \int_0^r \xi_t \, dt = \int_0^T \xi_t \mathbb{1}_{[0,r]}(t) \, dt$$

is a continuous map. Moreover,

$$[0, T] \times K_C \ni (r, \xi) \longmapsto \frac{1}{2} \int_0^r \xi_t^2 \, dt = \sup_{\varphi \in L^\infty} \left[\int_0^T \mathbb{1}_{[0,r]}(t) \xi_t \varphi_t \, dt - \frac{1}{2} \int_0^r \varphi_t^2 \, dt \right];$$

see, e.g., Rockafellar (1968). It follows that this map is lower semicontinuous.

Altogether, it follows that F is lower semicontinuous on the sequentially compact set $[0, T] \times K_C$ and so admits a minimizer. □

As we proved in proposition 2.3.1, for $\tau \wedge \rho \leq t \leq T$, the optimal single-update strategy is

$$\xi_t = \begin{cases} \dfrac{X_\tau}{T-\tau} & \text{on } \{\tau < \rho\} \\ \\ \dfrac{X_\rho + \hat{X}}{T-\rho} & \text{on } \{\rho \le \tau\}. \end{cases}$$

After plugging the single update of ξ at τ back into the revenue formula, the problem is reduced to an optimal execution problem in $[0, \rho]$, but with only one boundary condition: $X_0 = (1-r)x$. The righthand side boundary condition for $X(\rho)$ is free.

That is:

$$\max_{\xi} \int_0^\rho \eta \int_0^\tau \xi_t^2 dt de^{-\theta\tau} - \eta e^{-\theta\rho} \int_0^\rho \xi_t^2 dt - \eta e^{-\theta\rho} \frac{(x - \int_0^\rho \xi_t dt)^2}{T-\rho} - (1-e^{-\theta\rho})\frac{\gamma}{2}(x^2 - \hat{X}^2)$$
$$- \frac{\gamma}{2} x^2 e^{-\theta\rho} + \int_0^\rho \frac{\eta[(x-\hat{X}) - \int_0^\tau \xi_t dt]^2}{T-\tau} de^{-\theta\tau} - \int_0^\rho \gamma \hat{X} \left[x - \hat{X} - \int_0^\tau \xi dt \right] de^{-\theta\tau}$$

subject to

$$X_0 = x - \hat{X}, \quad \int_0^\rho \xi_t dt \le X_0 \tag{2.3}$$

where

$$\begin{aligned} f(\xi) \\ = \int_0^\rho \eta \xi_t^2 \int_t^\rho de^{-\theta\tau} dt + 2\eta(x-\hat{X})\theta \int_0^\rho \xi_t \int_t^\rho \frac{e^{-\theta\tau}}{T-\tau} d\tau dt - \eta\theta \int_0^\rho \frac{e^{-\theta\tau}}{T-\tau} \left(\int_0^\tau \xi_t dt \right)^2 d\tau \\ + \gamma \hat{X} \int_0^\rho \xi_t \int_t^\rho de^{-\theta\tau} dt - \eta e^{-\theta\rho} \int_0^\rho \xi_t^2 dt + 2\eta x \frac{e^{-\theta\rho}}{T-\rho} \int_0^\rho \xi_t dt - \frac{\eta e^{-\theta\rho}}{T-\rho} \left(\int_0^\rho \xi_t dt \right)^2. \end{aligned}$$

Assured of the existence of an optimal solution, we now solve for the optimal ξ^*.

Lemma 2.3.2. *The optimization problem* (2.3) *is equivalent to finding the solutions of an ordinary differential equation*

$$X''(t) - \theta X'(t) - \frac{\theta}{T-t} X(t) = A \tag{2.4}$$

subject to

$$X(0) = X_0 = x - \hat{X}, \quad X(\rho) \geq 0$$

where

$$A = -\frac{\gamma \hat{X} \theta}{2\eta}.$$

Proof. We prove the result using the calculus of variations.

Note that $\xi(t) = -X'(t)$, let

$$\begin{aligned} & F(t, X(t), X'(t)) \\ = & \eta X'(t)^2 \int_t^\rho de^{-\theta\tau} + 2\eta(x - \hat{X})\theta X'(t) \int_t^\rho \frac{e^{-\theta\tau}}{T-\tau} d\tau + \eta\theta \frac{e^{-\theta t}}{T-t} \left(\int_0^t X'(\tau) d\tau \right)^2 \\ & + \gamma \hat{X} X'(t) \int_t^\rho de^{-\theta\tau} + \eta e^{-\theta\rho} X'^2(t) + 2\eta x \frac{e^{-\theta\rho}}{T-\rho} X'(t) + \frac{\eta e^{-\theta\rho}}{T-\rho} X'(t) \int_0^\rho X'(t) dt. \end{aligned}$$

Then the objective function has the form:

$$f = \int_0^\rho F(t, X(t), X'(t)) dt.$$

Let $g_\epsilon(t) = X(t) + \epsilon h(t)$ be a perturbation of $X(t)$, where h is a differentiable function satisfying $h(0) = h(\rho) = 0$. The perturbed objective function has the form

$$\begin{aligned} f(\epsilon) = & \int_0^\rho F(t, g_\epsilon(t), g'_\epsilon(t)) dt \\ = & \int_0^\rho \eta(X'(t) + \epsilon h'(t))^2 (e^{-\theta t} - e^{-\theta\rho}) dt + 2\eta(x - \hat{X})\theta \int_0^\rho (X'(t) + \epsilon h'(t)) \int_t^\rho \frac{e^{-\theta\tau}}{T-\tau} d\tau dt \\ & + \eta\theta \int_0^\rho \frac{e^{-\theta\tau}}{T-\tau} (\int_0^\tau (X'(t) + \epsilon h'(t)) dt)^2 d\tau + \gamma \hat{X} \int_0^\rho (X'(t) + \epsilon h'(t))(e^{-\theta\rho} - e^{-\theta t}) dt \\ & + \eta e^{-\theta\rho} \int_0^\rho (X'(t) + \epsilon h'(t))^2 dt + 2\eta x \frac{e^{-\theta\rho}}{T-\rho} \int_0^\rho (X'(t) + \epsilon h'(t)) dt \\ & + \frac{\eta e^{-\theta\rho}}{T-\rho} (\int_0^\rho (X'(t) + \epsilon h'(t)) dt)^2. \end{aligned}$$

Calculating the total derivative of $f(\epsilon)$ with respect to ϵ, we have

$$\begin{aligned}
\frac{df(\epsilon)}{d\epsilon} &= \int_0^\rho 2\eta(X'(t)+\epsilon h'(t))h'(t)(e^{-\theta t}-e^{-\theta\rho})dt + 2\eta(x-\hat{X})\theta\int_0^\rho \frac{e^{-\theta t}}{T-t}h(t)dt \\
&+ 2\eta\theta\int_0^\rho \frac{e^{-\theta\tau}}{T-\tau}\int_0^\tau (X'(t)+\epsilon h'(t))dt\int_0^\tau h'(t)dtd\tau + \gamma\hat{X}\int_0^\rho h'(t)(e^{-\theta\rho}-e^{-\theta t})dt \\
&+ 2\eta e^{-\theta\rho}\int_0^\rho (X'(t)+\epsilon h'(t))h'(t)dt + 2\eta x\frac{e^{-\theta\rho}}{T-\rho}\int_0^\rho h'(t)dt \\
&+ \frac{2\eta e^{-\theta\rho}}{T-\rho}\int_0^\rho (X'(t)+\epsilon h'(t))dt\int_0^\rho h'(t)dt.
\end{aligned}$$

Since the extreme value is obtained at $\epsilon = 0$, and hence $\frac{df(\epsilon)}{d\epsilon}|_{\epsilon=0} = 0$, we have

$$\begin{aligned}
\frac{df(\epsilon)}{d\epsilon}|_{\epsilon=0} &= \int_0^\rho 2\eta X'(t)h'(t)(e^{-\theta t}-e^{-\theta\rho})dt + 2\eta(x-\hat{X})\theta\int_0^\rho \frac{e^{-\theta t}}{T-t}h(t)dt \\
&+ 2\eta\theta\int_0^\rho \frac{e^{-\theta\tau}}{T-\tau}\int_0^\tau X'(t)dt\int_0^\tau h'(t)dtd\tau + \gamma\hat{X}\int_0^\rho h'(t)(e^{-\theta\rho}-e^{-\theta t})dt \\
&+ 2\eta e^{-\theta\rho}\int_0^\rho X'(t)h'(t)dt + 2\eta x\frac{e^{-\theta\rho}}{T-\rho}\int_0^\rho h'(t)dt \\
&+ \frac{2\eta e^{-\theta\rho}}{T-\rho}\int_0^\rho X'(t)dt\int_0^\rho h'(t)dt \\
&= -2\eta\int_0^\rho [X''(t)(e^{-\theta t}-e^{-\theta\rho}) - \theta X'(t)e^{-\theta t}]h(t)dt + 2\eta(x-\hat{X})\theta\int_0^\rho \frac{e^{-\theta t}}{T-t}h(t)dt \\
&+ 2\eta\theta\int_0^\rho \frac{e^{-\theta t}}{T-t}(X(t)-X(0))h(t)dt - \gamma\hat{X}\int_0^\rho \theta e^{-\theta t}h(t)dt \\
&- 2\eta e^{-\theta\rho}\int_0^\rho X''(t)h(t)dt.
\end{aligned}$$

According to the fundamental lemma of the calculus of variations, this yields

$$-2\eta X''(t)e^{-\theta t} + 2\eta\theta X'(t)e^{-\theta t} + 2\eta\theta\frac{e^{-\theta t}}{T-t}X(t) - \gamma\hat{X}\theta e^{-\theta t} = 0$$

which is equivalent to

$$X''(t) - \theta X'(t) - \frac{\theta}{T-t}X(t) = -\frac{\gamma\hat{X}\theta}{2\eta}.$$

□

Lemma 2.3.3. *The general solution of the differential equation (2.4) is:*

$$\begin{aligned}
X(t) &= C_1[e^{\theta t} - \theta(T-t)e^{\theta T}Ei(\theta(T-t))] + C_2(T-t) \\
&+ [Ei(\theta T) - Ei(\theta(T-t))]e^{\theta T}(T-t)A(T-\frac{1}{\theta}) \\
&- \frac{A}{\theta}(T-t) + \frac{A}{\theta}Te^{\theta t} + \frac{A}{\theta^2}(1-e^{\theta t}) - \frac{A(T-t)}{\theta}\ln\frac{T}{T-t}
\end{aligned}$$

where $Ei(t) = \int_t^\infty \frac{e^{-s}}{s}ds$ *is the exponential integral.*

Proof. First solve the homogeneous differential equation:

$$-2\eta X''(t)e^{-\theta t} + 2\eta\theta X'(t)e^{-\theta t} + 2\eta\theta\frac{e^{-\theta t}}{T-t}X(t) = 0.$$

The general solution is

$$X(t) = C_1[e^{\theta t} - \theta(T-t)e^{\theta T}Ei(\theta(T-t))] + C_2(T-t)$$

where the two basic solutions are:

$$\begin{aligned}
X_1(t) &= e^{\theta t} - \theta(T-t)e^{\theta T}Ei(\theta(T-t)) \\
X_2(t) &= T-t.
\end{aligned}$$

The particular solution can be obtained from the formula:

$$\begin{aligned}
X^*(t) &= \int_{t_0}^t \frac{X_1(s)X_2(t) - X_1(t)X_2(s)}{X_1(s)X_2'(s) - X_2(s)X_1'(s)}Ads \\
&= A\int_{t_0}^t [\theta(T-s)e^{\theta(T-s)}Ei(\theta(T-s)) - 1](T-t)ds \\
&+ A\int_{t_0}^t [e^{\theta(t-s)}(T-s) - \theta(T-t)(T-s)e^{\theta(T-s)}Ei(\theta(T-t))]ds \\
&= -\frac{A}{\theta}(T-t) + \frac{A}{\theta}Te^{\theta t} + \frac{A}{\theta^2}(1-e^{\theta t}) - \frac{A(T-t)}{\theta}\ln\frac{T}{T-t} \\
&+ [Ei(\theta T) - Ei(\theta(T-t))]e^{\theta T}(T-t)A(T-\frac{1}{\theta}).
\end{aligned}$$

Hence the general solution for the nonhomogeneous differential equation is:

$$\begin{aligned}
X(t) &= C_1X_1(t) + C_2X_2(t) + X^*(t) \\
&= C_1[e^{\theta t} - \theta(T-t)e^{\theta T}Ei(\theta(T-t))] + C_2(T-t) \\
&- \frac{A}{\theta}(T-t) + \frac{A}{\theta}Te^{\theta t} + \frac{A}{\theta^2}(1-e^{\theta t}) - \frac{A(T-t)}{\theta}\ln\frac{T}{T-t} \\
&+ [Ei(\theta T) - Ei(\theta(T-t))]e^{\theta T}(T-t)A(T-\frac{1}{\theta}).
\end{aligned}$$

□

The constant coefficients in the general solutions to (2.4) can be solved by boundary conditions. From the initial boundary condition $X(0) = X_0 = x - \hat{X}$, we obtain

$$C_2 = \frac{X_0}{T} - \frac{C_1}{T}[1 - \theta Te^{\theta T}Ei(\theta T)].$$

Hence

$$\begin{aligned}
X(t) &= C_1[e^{\theta t} - \theta(T-t)e^{\theta T}Ei(\theta(T-t)) - \frac{T-t}{T} + \theta(T-t)e^{\theta T}Ei(\theta T)] \\
&+ (T-t)[\frac{X_0}{T} - \frac{A}{\theta} - \frac{A}{\theta}\ln\frac{T}{T-t}] + \frac{A}{\theta}Te^{\theta t} \\
&+ \frac{A}{\theta^2}(1-e^{\theta t}) + [Ei(\theta T) - Ei(\theta(T-t))]e^{\theta T}(T-t)A(T-\frac{1}{\theta}).
\end{aligned}$$

2.4 The Existence of Price Manipulation

In the optimal execution strategy, it is possible to have $X(\rho) < 0$, which means the seller sells more than $x - \hat{X}$ shares by ρ. The incentive for doing this would be to reduce the cost from temporary impact due to what may be a high speed liquidation during $[\rho, T]$. For example, if the seller realizes that the probability of liquidating in the dark pool is small, then he or she will expect a significant infusion into the exchange from the dark pool at ρ to be liquidated in time interval $[\rho, T]$. If ρ is close to T then the selling has to be done in a relatively short period; this will require a higher liquidation speed followed by a correspondingly larger cost due to the temporary impact. In order to balance the transaction pressure between the two periods, such a seller will tend to short the asset over $[0,\rho]$, and pay it back at ρ when the asset is withdrawn from the dark pool. Such short selling is a conscious manipulation of the exchange price to facilitate a large trade.

In this section, we give a necessary and sufficient condition to exclude price manipulation.

Lemma 2.4.1. *A necessary and sufficient condition for the absence of an incentive for short selling in a market with a dark pool and an exchange is:*

$$C_1 < \tilde{C}_1$$

where

$$\tilde{C}_1 = \frac{G_2}{G_1}$$

and

$$\begin{aligned} G_1 &= e^{\theta\rho} - \theta(T-\rho)e^{\theta T}(Ei(\theta(T-\rho)) - Ei(\theta T)) - \frac{T-\rho}{T} \\ G_2 &= -\frac{X_0}{T}(T-\rho) + \frac{A}{\theta}(T-\rho) - \frac{A}{\theta}(T-\rho)\ln\frac{T-\rho}{T} - \frac{A}{\theta}Te^{\theta\rho} \\ &\quad - \frac{A}{\theta^2}(1-e^{\theta\rho}) + A(T-\rho)e^{\theta T}(T-\frac{1}{\theta})[Ei(\theta(T-\rho)) - Ei(\theta T)]. \end{aligned}$$

Proof. The existence of price manipulation is equivalent to the boundary condition

$$X(\rho) < 0.$$

Notice that

$$X(\rho) < 0 \Leftrightarrow C_1G_1 < G_2.$$

Let $\tilde{C}_1 = \frac{G_2}{G_1}$, since

$$\begin{aligned} G_1 &= e^{\theta\rho} - \theta(T-\rho)e^{\theta T}[Ei(\theta(T-\rho)) - Ei(\theta T)] - \frac{T-\rho}{T} \\ &\geq e^{\theta\rho} - \frac{T-\rho}{T} \\ &= e^{\theta\rho} - 1 + \frac{\rho}{T} > 0. \end{aligned}$$

Hence

$$\begin{aligned} X(\rho) < 0 \Leftrightarrow C_1G_1 &< G_2 \\ \Leftrightarrow C_1 &< \frac{G_2}{G_1} = \tilde{C}_1. \end{aligned}$$

□

To exclude the case of short selling described in the introduction to this section, we impose the boundary condition $X(\rho) \geq 0$, which is equivalent to:

$$C_1 \geq \tilde{C}_1. \tag{2.5}$$

Now we have reduced the objective functional to be a function of C_1, which can be optimized by varying C_1. Notice that the rate of liquidation is:

$$\begin{aligned} \xi(t) &= -X'(t) \\ &= C_1[\theta e^{\theta T}Ei(\theta T) - \theta e^{\theta T}Ei(\theta(T-t)) - \frac{1}{T}] \\ &+ \frac{X_0}{T} - \frac{A}{\theta}\ln\frac{T}{T-t} + Ae^{\theta T}(T - \frac{1}{\theta})[Ei(\theta T) - Ei(\theta(T-t))] \\ &= C_1P(t) + Q(t) \end{aligned}$$

where

$$\begin{aligned} P(t) &= \theta e^{\theta T}Ei(\theta T) - \theta e^{\theta T}Ei(\theta(T-t)) - \frac{1}{T} \\ Q(t) &= \frac{X_0}{T} - \frac{A}{\theta}\ln\frac{T}{T-t} + Ae^{\theta T}(T - \frac{1}{\theta})[Ei(\theta T) - Ei(\theta(T-t))]. \end{aligned}$$

Taking the partial derivative of the objective function $f(\xi)$ with respect to C_1, we have

$$\frac{\partial f(\xi)}{\partial C_1} = 2C_1P_1 + Q_1$$

where

$$\begin{aligned}
P_1 &= \int_0^{\rho} \eta P^2(t)e^{-\theta t}dt + \int_0^{\rho} \eta\theta \frac{e^{-\theta\tau}}{T-\tau}\left(\int_0^{\tau} P(t)dt\right)^2 d\tau + \frac{\eta e{-}\theta\rho}{T-\rho}\left(\int_0^{\rho} P(t)dt\right)^2 \\
Q_1 &= \int_0^{\rho} 2\eta P(t)Q(t)e^{-\theta t}dt + 2\eta\theta\int_0^{\rho}\frac{e^{-\theta\tau}}{T-\tau}\int_0^{\tau}P(t)dt\int_0^{\tau}Q(t)dtd\tau \\
&+ 2\eta\frac{e^{-\theta\rho}}{T-\rho}\int_0^{\rho}P(t)dt\int_0^{\rho}Q(t)dt - 2\eta\theta X_0\int_0^{\rho}\frac{e^{-\theta\tau}}{T-\tau}\int_0^{\tau}P(t)dtd\tau \\
&- \gamma\hat{X}\int_0^{\rho}(e^{-\theta\rho} - e^{-\theta t})P(t)dt - 2\eta x\frac{e^{-\theta\rho}}{T-\rho}\int_0^{\rho}P(t)dt.
\end{aligned}$$

Now setting the partial derivative equal to 0 we obtain the optimal solution for C_1:

$$\tilde{\tilde{C}}_1 = -\frac{Q_1}{2P_1}.$$

If $\tilde{\tilde{C}}_1$ does not satisfy inequality (2.5), we know from the monotonicity of the number of shares to be liquidated, the optimal solution of the objective functional can be obtained by taking $C_1^* = \tilde{C}_1$.

The optimal liquidation strategy for $t \in [0, \rho)$ is therefore:

$$\xi^*(t) = C_1^*P(t) + Q(t)$$

where

$$C_1^* = \begin{cases} \tilde{\tilde{C}}_1 & \text{if } \tilde{\tilde{C}}_1G_1 \geq G_2 \\ \tilde{C}_1 & \text{otherwise.} \end{cases}$$

To this point, we have explored the behavior of an optimal position and liquidation strategy in the exchange before ρ. This strategy is determined at time 0. Once the seller is notified that there is a match in the dark pool at time $t_0 < \rho$, he/she updates the strategy to $\bar{\xi}^*(t)$; otherwise, strategy $\bar{\bar{\xi}}^*(t)$ is adopted at time ρ.

In fact, $C_1^* = \tilde{C}_1$ is just a special case of $C_1^* = \tilde{\tilde{C}}_1$ due to the following lemma.

Lemma 2.4.2. *$\tilde{C}_1 = \tilde{\tilde{C}}_1$ if the optimal asset position $X^*(\rho) = 0$.*

Proof. From the expression of $X(t)$, we know if $X^*(\rho) > 0$, $\tilde{\tilde{C}}_1$ satisfies

$$\begin{aligned} X^*(\rho) = \tilde{\tilde{C}}_1 & \left[e^{\theta\rho} - \theta(T-\rho)e^{\theta T} Ei(\theta(T-\rho)) - \frac{T-\rho}{T} + \theta(T-\rho)e^{\theta T} Ei(\theta T) \right] \\ & + (T-\rho)\left[\frac{X_0}{T} - \frac{A}{\theta} - \frac{A}{\theta}\ln\frac{T}{T-\rho} \right] + \frac{A}{\theta} T e^{\theta\rho} \\ & + \frac{A}{\theta^2}(1-e^{\theta\rho}) + \left[Ei(\theta T) - Ei(\theta(T-\rho)) \right] e^{\theta T}(T-\rho)A(T-\frac{1}{\theta}). \end{aligned}$$

Hence from the expression of $\tilde{C}_1$,

$$\lim_{X^*(\rho)\to 0} \tilde{\tilde{C}}_1 = \tilde{C}_1.$$

$\square$

Based on the arguments above, we can narrow the following analysis to deal with the solution in the case that

$$\xi^*(t) = \tilde{\tilde{C}}_1 P(t) + Q(t), \qquad t \in [0, \rho].$$

Figures 2.1-2.5 demonstrate asset positions and liquidation rates over time $[0, \rho]$ for optimal trading strategies with different parameter values. It is noticed that it might be optimal sometimes to purchase assets in the exchange in order to push the price level up in the hope of selling at a better price in the dark pool.

In other words, it is possible that an agency enters the market with the intention of selling, however, due to the existence of the dark pool, it's beneficial for them to buy the asset for the purpose of a "pump and dump". By buying the asset in $[0, \rho]$, the market price of the asset increases due to the purchasing orders, which enables the seller to liquidate the rest of asset at a higher price. This observation shows that the existence of the dark pool creates a motivation for price manipulation. Therefore, it is important that the regulatory agency takes action to prevent this type of activity.

We observe that this manipulation strategy dominates if one of the following condition is satisfied:

- the probability of finding a counter party is sufficiently, large
- the temporary impact is sufficiently small,
- the permanent impact is sufficiently large,
- the proportion of order in the dark pool is sufficiently large, or
- the exiting time from dark pool ρ is sufficiently large.

We give a sufficient condition for the absence of manipulation in theorem 2.4.8 below . Several lemmas are required before we state the theorem.

The following lemmas show that the strategy before update, $\xi(t)$, is a nondecreasing function of t for any $t \leq \tau$; hence, it suffices to examine $\xi(0)$ to detect the manipulation.

Lemma 2.4.3. *For the optimal strategy $\xi^*(t)$, and the corresponding asset position $X^*(t)$, $t \in [0, \rho]$, we have*

$$\xi^*(\rho) = \frac{X^*(\rho) + \hat{X}}{T - \rho}.$$

Proof. The idea of the proof is to consider the expected revenue $\mathbb{E}[\mathcal{R}_{[t,T]}]$ on the set $\{\omega : \tau(\omega) \geq t\}$. Then we set $t \to \rho_-$, since we know that under the optimal updated strategy, $\xi(t)$, $t \in [\tau \wedge \rho, T]$, the expected revenue can be reduced to a functional of $\xi(\rho)$, and the optimal $\xi^*(\rho)$ is the one that maximizes the expression.

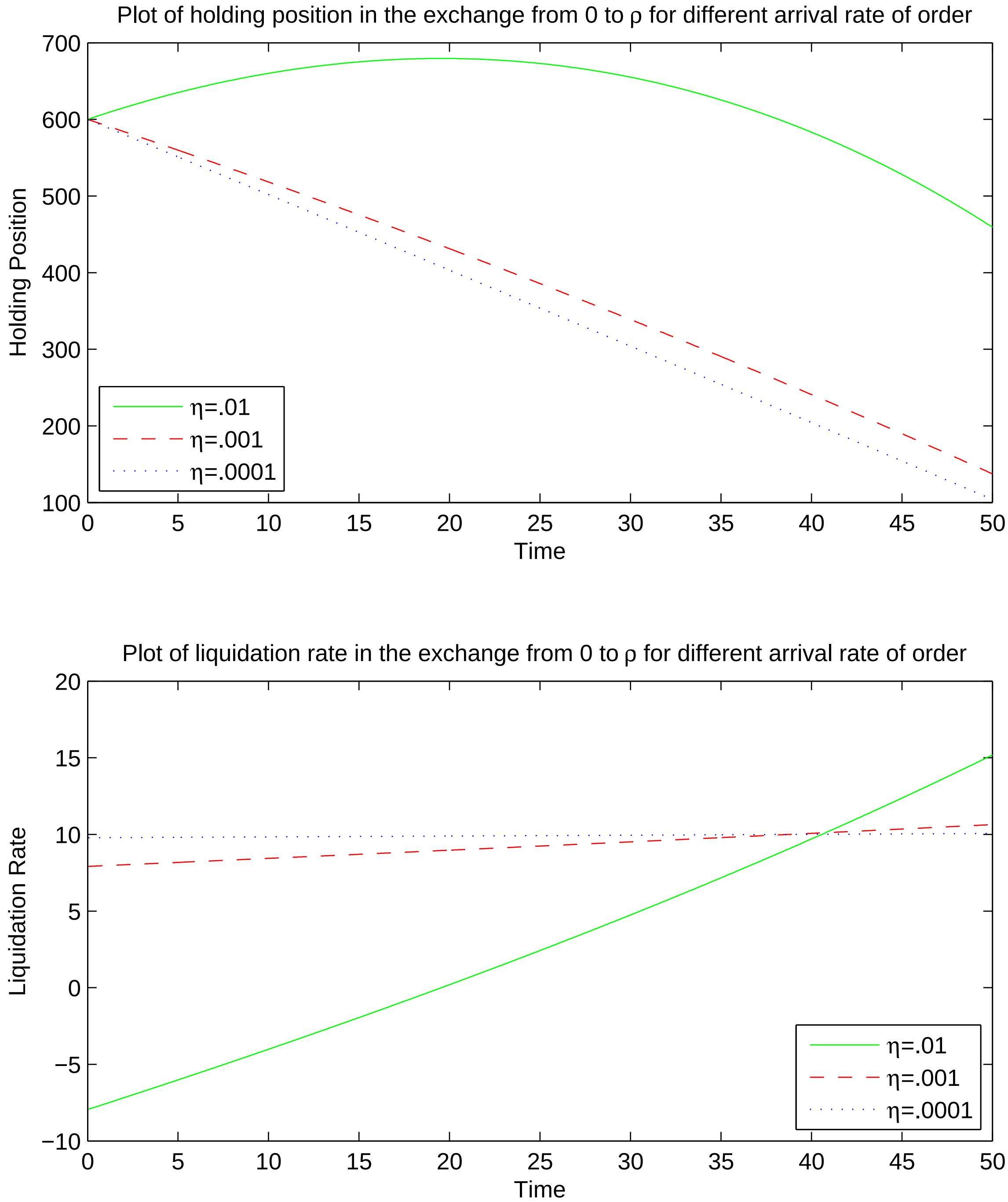


Figure 2.1: Asset position X_t and liquidation rate ξ_t over time $[0, \rho]$ for optimal trading strategies with $X = 1,000$, $T = 100$, $\rho = 50$, $r = 0.4$, $\gamma = 0.01$, $\eta = 0.04$. The solid line corresponds to $\theta = 0.01$, the dashed line to $\theta = 0.001$, and the dotted line to $\theta = .0001$.

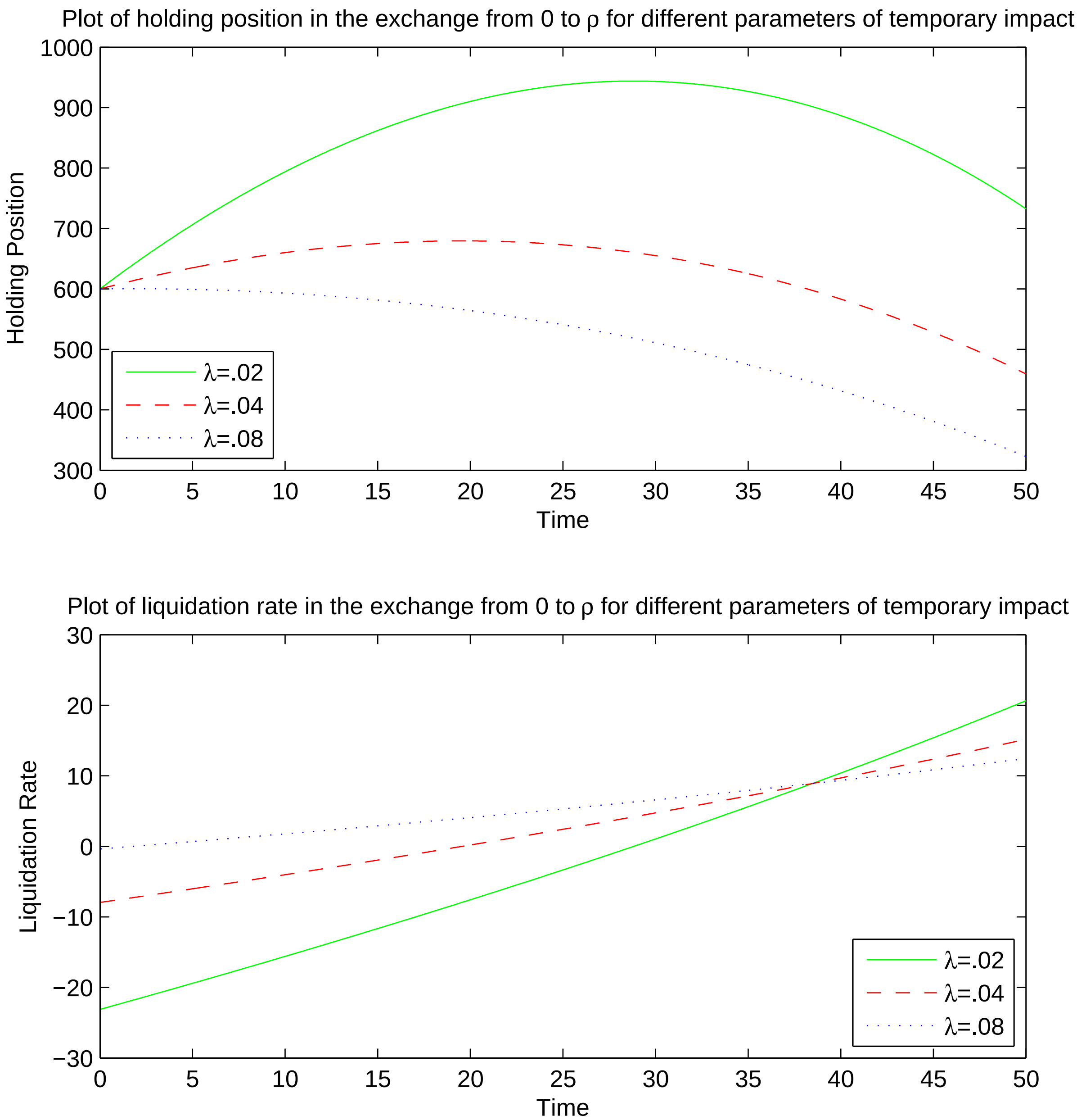
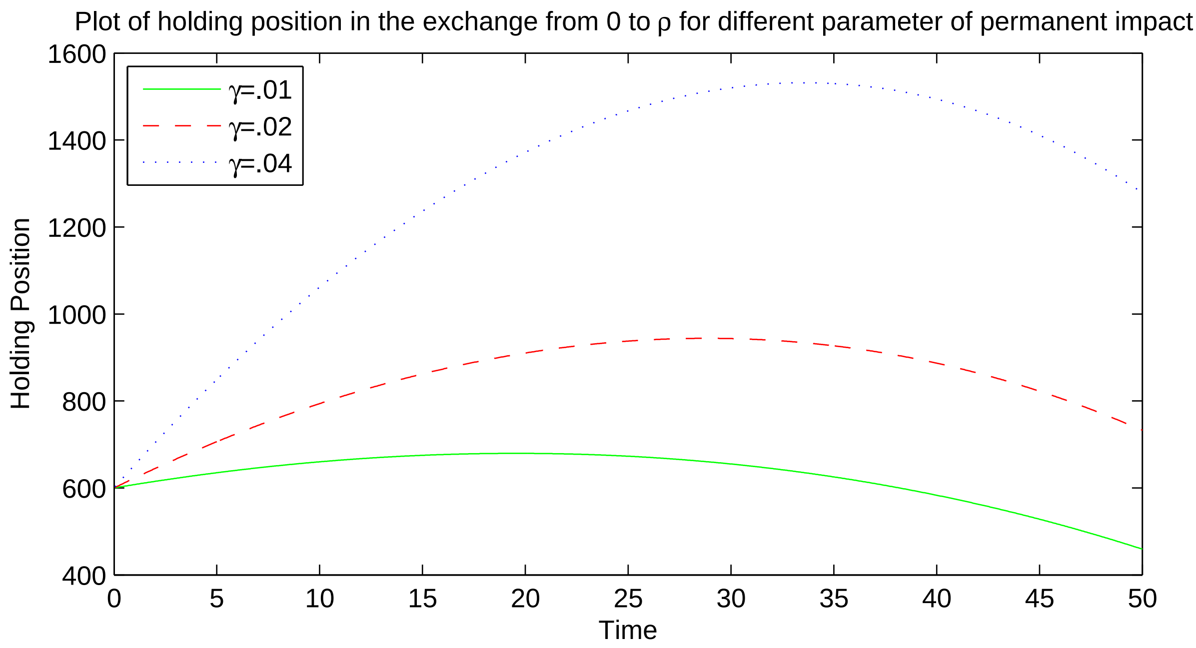


Figure 2.2: Asset position X_t and liquidation rate ξ_t over time $[0, \rho]$ for optimal trading strategies with $X = 1,000$, $T = 100$, $\rho = 50$, $\theta = 0.01$, $r = 0.4$, $\gamma = 0.01$. The solid line corresponds to $\eta = 0.02$, the dashed line to $\eta = 0.04$, and the dotted line to $\eta = 0.08$.



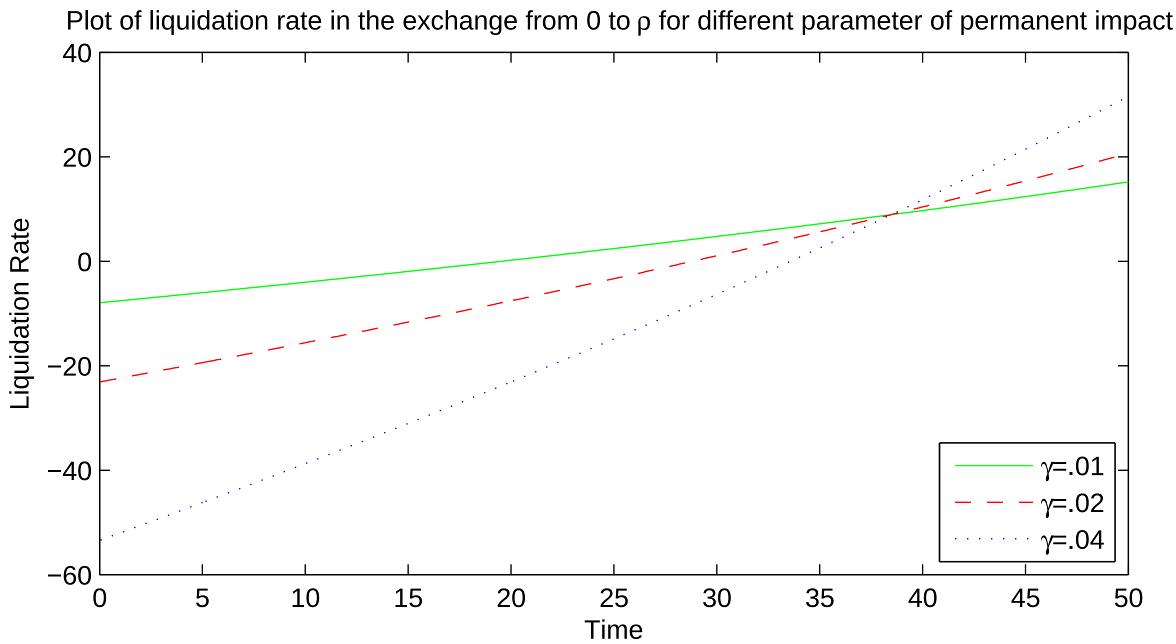


Figure 2.3: Asset position X_t and liquidation rate ξ_t over time $[0, \rho]$ for optimal trading strategies with $X = 1,000$, $T = 100$, $\rho = 50$, $\theta = 0.01$, $r = 0.4$, $\eta = 0.04$. The solid line corresponds to $\gamma = 0.1$, the dashed line to $\gamma = 0.2$, and the dotted line to $\gamma = 0.4$.

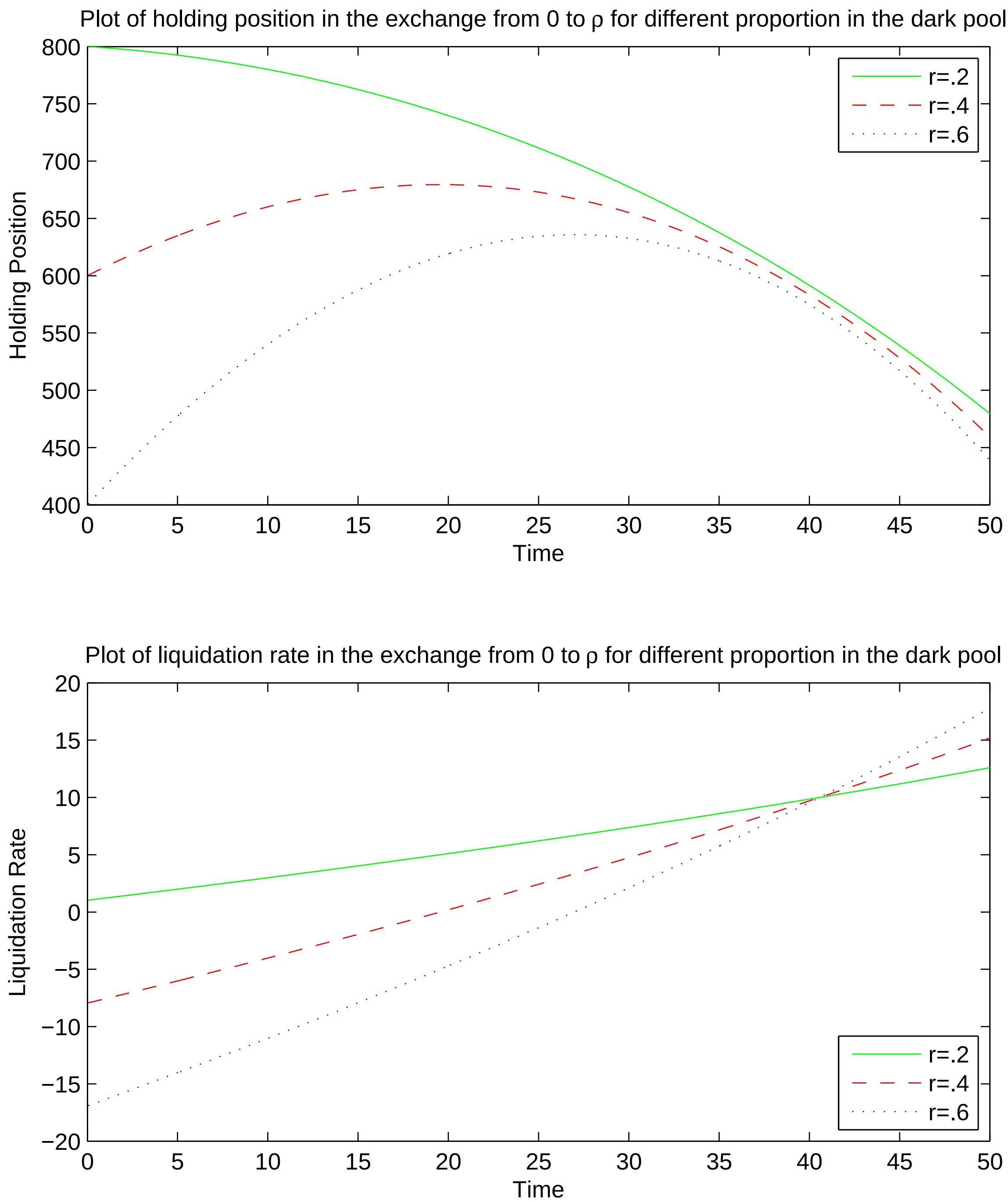


Figure 2.4: Asset position X_t and liquidation rate ξ_t over time $[0, \rho]$ for optimal trading strategies with $X = 1,000$, $T = 100$, $\rho = 50$, $\theta = 0.01$, $\eta = 0.04$, $\gamma = 0.01$. The solid line corresponds to $r = 0.2$, the dashed line to $r = 0.4$, and the dotted line to $r = 0.6$.

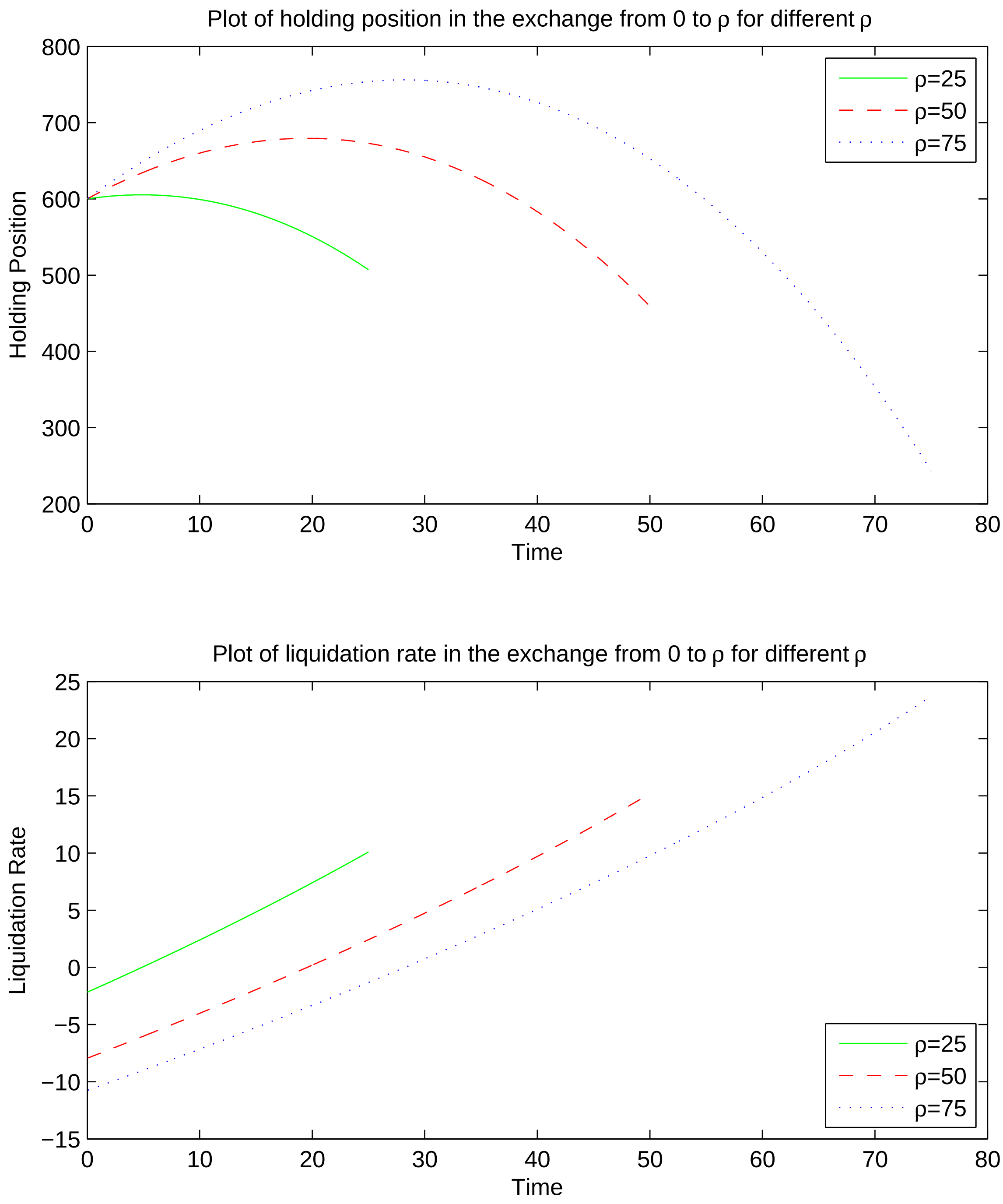


Figure 2.5: Asset position X_t and liquidation rate ξ_t over time $[0, \rho]$ for optimal trading strategies with $X = 1,000$, $T = 100$, $\theta = 0.01$, $r = 0.4$, $\gamma = 0.01$, $\eta = 0.04$. The solid line corresponds to $\rho = 25$, the dashed line to $\rho = 50$, and the dotted line to $\rho = 75$.

For all $t \in [0,\rho)$, if there is no transaction in the dark pool in $[0,t)$, the expected revenue from t to T is:

$$
\begin{aligned}
&\mathbb{E}[\mathcal{R}_{[t,T]}; \tau \geq t] \\
&= \mathbb{E}[\mathcal{R}_{[t,T]}; t \leq \tau \leq \rho] + \mathbb{E}[\mathcal{R}_{[t,T]}; \tau > \rho] \\
&= \mathbb{E}[\mathbb{E}[\mathcal{R}_{[t,T]}|\tau]; t \leq \tau \leq \rho] + \mathbb{E}[\mathbb{E}[\mathcal{R}_{[t,T]}|\tau]; \tau > \rho] \\
&= \int_t^\rho \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2\right) d\tau \\
&= \int_t^\rho \theta e^{-\theta\tau} \left(-\eta \int_t^\tau \xi_s^2 ds + \gamma\hat{X}(X_\tau - X_0) - \frac{\gamma}{2}X_0^2 - \eta \int_\tau^T \left(\frac{X_\tau}{T-\tau}\right)^2 ds\right) d\tau \\
&+ \int_\rho^\infty \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2 - \eta \int_t^\rho \xi_s^2 ds - \frac{\gamma}{2}x^2 - \eta \int_\rho^T \left(\frac{X_\rho + \hat{X}}{T-\rho}\right)^2 ds\right) d\tau
\end{aligned}
$$

which is a functional of $\xi(t)$, and

$$
\begin{aligned}
g(\xi) = &\int_t^\rho \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2 - \eta \int_t^\tau \xi_s^2 ds + \gamma\hat{X}(X_\tau - X_0) - \frac{\gamma}{2}X_0^2 - \eta\frac{X_\tau^2}{T-\tau}\right) d\tau \\
&+ \int_\rho^\infty \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2 - \eta \int_t^\rho \xi_s^2 ds - \frac{\gamma}{2}x^2 - \eta\frac{(X_\rho + \hat{X})^2}{T-\rho}\right) d\tau \\
= &\int_t^\rho \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2 - \eta \int_t^\tau \xi_s^2 ds + \gamma\hat{X}(X_\tau - X_0) - \frac{\gamma}{2}X_0^2 - \eta\frac{X_\tau^2}{T-\tau}\right) d\tau \\
&+ \int_\rho^\infty \theta e^{-\theta\tau} \left(P_0^0(X_t + \hat{X}) + \frac{\gamma}{2}(X_t - X_0)^2 - \eta \int_t^\rho \xi_s^2 ds - \frac{\gamma}{2}x^2 - \eta\frac{(X_t - \int_t^\rho \xi_s ds + \hat{X})^2}{T-\rho}\right) d\tau.
\end{aligned}
$$

By L'Hospital's rule:

$$
\begin{aligned}
&\lim_{t\to\rho^-} \frac{-\int_t^\rho \theta e^{-\theta\tau}\eta \int_t^\tau \xi_s^2 ds d\tau}{\rho - t} = \lim_{t\to\rho^-} \left(-\theta e^{-\theta t}\eta \int_t^t \xi_s^2 ds - \int_t^\rho \theta e^{-\theta\tau}\eta\xi_t^2 d\tau\right) = 0 \\
&\lim_{t\to\rho^-} \frac{\int_t^\rho \theta e^{-\theta\tau}\frac{X_\tau^2}{T-\tau}d\tau}{\rho - t} = \lim_{t\to\rho^-} \frac{\int_t^\rho \theta e^{-\theta\tau}\frac{(X_t - \int_t^\tau \xi_s ds)^2}{T-\tau}d\tau}{\rho - t} = 0 \\
&\lim_{t\to\rho^-} \frac{-\int_\rho^\infty \theta e^{-\theta\tau}\eta \int_t^\rho \xi_s^2 ds d\tau}{\rho - t} = \lim_{t\to\rho^-} -e^{-\theta\rho}\eta\xi_t^2 = -e^{-\theta\rho}\eta\xi_\rho^2 \\
&\lim_{t\to\rho^-} \frac{-\int_\rho^\infty \theta e^{-\theta\tau}\frac{\eta}{T-\rho}(\int_t^\rho \xi_s ds)^2 d\tau}{\rho - t} = \lim_{t\to\rho^-} \frac{-2e^{-\theta\rho}\eta\int_t^\rho \xi_s ds \xi_t}{T-\rho} = 0 \\
&\lim_{t\to\rho^-} \frac{\int_\rho^\infty \theta e^{-\theta\tau}2\eta\frac{(X_t + \hat{X})}{T-\rho}\int_t^\rho \xi_s ds d\tau}{\rho - t} \\
&= \lim_{t\to\rho^-} \frac{1}{T-\rho}\left(2\eta e^{-\theta\rho}(X_t + \hat{X})\xi_t\right) = \frac{2\eta(X_\rho + \hat{X})\xi_\rho}{T-\rho}.
\end{aligned}
$$

Hence

$$\lim_{t\to\rho^-}\frac{g(\xi)}{\rho-t}=-e^{-\theta\rho}\eta\xi_\rho^2+2e^{-\theta\rho}\frac{\eta(X_\rho+\hat{X})\xi_\rho}{T-\rho}+\bar{G}$$

where $\bar{G}$ is a function which does not depend on ξ_ρ.

It is obvious that the optimal solution of this functional will be obtained at

$$\xi^*(\rho)=\frac{X^*(\rho)+\hat{X}}{T-\rho}.$$

□

Lemma 2.4.4. *The optimal constant $\tilde{\tilde{C}}_1$ always satisfies*

$$\tilde{\tilde{C}}_1=-\hat{X}e^{-\theta\rho}+\frac{A(T-\rho)}{\theta}e^{-\theta\rho}-\frac{A}{\theta}T-\frac{A}{\theta^2}(e^{-\theta\rho}-1).$$

Proof. Notice that

$$\begin{aligned}X^*(\rho)&=\tilde{\tilde{C}}_1\left[e^{\theta\rho}-\theta(T-\rho)e^{\theta T}Ei(\theta(T-\rho))-\frac{T-\rho}{T}+\theta(T-\rho)e^{\theta T}Ei(\theta T)\right]\\&+(T-\rho)\left[\frac{X_0}{T}-\frac{A}{\theta}-\frac{A}{\theta}\ln\frac{T}{T-\rho}\right]+\frac{A}{\theta}Te^{\theta\rho}\\&+\frac{A}{\theta^2}(1-e^{\theta\rho})+\left[Ei(\theta T)-Ei(\theta(T-\rho))\right]e^{\theta T}(T-\rho)A(T-\frac{1}{\theta})\end{aligned}$$

and

$$\begin{aligned}\xi^*(\rho)&=\tilde{\tilde{C}}_1[\theta e^{\theta T}Ei(\theta T)-\theta e^{\theta T}Ei(\theta(T-\rho))-\frac{1}{T}]\\&+\frac{X_0}{T}-\frac{A}{\theta}\ln\frac{T}{T-\rho}+Ae^{\theta T}(T-\frac{1}{\theta})[Ei(\theta T)-Ei(\theta(T-\rho))].\end{aligned}$$

This lemma is then a direct result from the previous lemma. □

The following theorem is one of the main results of the paper. It implies that there are only two possible cases of transaction -triggered price manipulation. The first case is characterized by $\xi(0)<0$, which means that instead of selling, the trader starts with buying at time 0. The second case is characterized by $X(\rho)<0$; that is, the exchange venue ends up with the seller shorting at time ρ.

Theorem 2.4.5. *The optimal strategy $\xi(t)$ is a nondecreasing function of t, for $t \in [0, \rho]$.*

Proof. It suffices to prove for all $t \in [0, \rho]$

$$\xi'(t) = -\frac{1}{T-t}[\tilde{\tilde{C}}_1 \theta e^{\theta t} + Ae^{\theta t}(T - \frac{1}{\theta}) + \frac{A}{\theta}] \geq 0.$$

Note from the previous lemma, we have

$$\begin{aligned}
&\tilde{\tilde{C}}_1 \theta e^{\theta t} + Ae^{\theta t}(T - \frac{1}{\theta}) + \frac{A}{\theta} \\
&= \frac{\theta^2}{2}\hat{X}(T-\rho)\theta e^{\theta(t-\rho)} - \hat{X}\theta e^{\theta(t-\rho)} + A(T-\rho)e^{\theta(t-\rho)} - ATe^{\theta t} \\
&+ \frac{A}{\theta}(e^{\theta t} - e^{\theta(t-\rho)}) + Ae^{\theta t}T - \frac{A}{\theta}e^{\theta t} + \frac{A}{\theta} \\
&= -\frac{\gamma \hat{X}}{2\eta}\theta(T-\rho)e^{\theta(t-\rho)} + \frac{A}{\theta}(1 - e^{\theta(t-\rho)}) \\
&\leq 0
\end{aligned}$$

by noticing $A \leq 0$. Hence

$$\xi'(t) \geq 0 \qquad \text{for } t \in [0, \rho].$$

□

Since $\xi(t)$ is a nondecreasing function of t, $t \in [0, \rho]$, we know that if $\xi(0) > 0$, there will not be price manipulation in $[0, \rho]$.

Lemma 2.4.6. *The manipulation will not exist if $\tilde{\tilde{C}}_1 \leq X_0$.*

Proof. Note

$$\xi^*(0) = -\frac{C_1}{T} + \frac{X_0}{T}$$

so if $\tilde{\tilde{C}}_1 \leq X_0$, from the previous lemma we know there will not be purchasing activities in $[0, \rho]$. □

Lemma 2.4.7. *$\tilde{\tilde{C}}_1$ is a decreasing function of the temporary impact parameter η, and an increasing function of the permanent impact parameter γ.*

Proof. From lemma 2.4.4, we have

$$\begin{aligned}\frac{\partial \tilde{\tilde{C}}_1}{\partial A} &= (T - \frac{1}{\theta})(e^{-\theta\rho} - 1) - \rho e^{-\theta\rho} \\ &\leq (\rho - \frac{1}{\theta})(e^{-\theta\rho} - 1) - \rho e^{-\theta\rho} \\ &\leq -\rho + \frac{1}{\theta}\theta\rho \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial A}{\partial \eta} &= \frac{\gamma \hat{X}\theta}{2\eta^2} \geq 0 \\ \frac{\partial A}{\partial \gamma} &= -\frac{\hat{X}\theta}{2\eta} \leq 0.\end{aligned}$$

Hence

$$\frac{\partial \tilde{\tilde{C}}_1}{\partial \eta} = \frac{\partial \tilde{\tilde{C}}_1}{\partial A}\frac{\partial A}{\partial \eta} \leq 0$$

and

$$\frac{\partial \tilde{\tilde{C}}_1}{\partial \gamma} = \frac{\partial \tilde{\tilde{C}}_1}{\partial A}\frac{\partial A}{\partial \gamma} \geq 0.$$

□

Theorem 2.4.8. *Let*

$$H = \frac{\frac{\hat{X}}{2}(1 - e^{-\theta\rho}) - x}{\frac{\hat{X}}{2}\left(\left[(e^{-\theta\rho} - 1)(T - \frac{1}{\theta}) - \rho e^{-\theta\rho}\right]\right)}.$$

Then $\xi(0) \leq 0$ if $\frac{\eta}{\gamma} \leq H$, and $\xi(0) > 0$ if $\frac{\eta}{\gamma} > H$.

Proof. First notice that

$$\begin{aligned}
\tilde{\tilde{C}}_1 - X_0 &= \frac{\gamma}{\eta}\frac{\hat{X}}{2}\left[(e^{-\theta\rho}-1)(T-\frac{1}{\theta}) - \rho e^{-\theta\rho}\right] - \frac{\hat{X}}{2}(1-e^{-\theta\rho}) + x \\
&= \frac{\gamma}{\eta}H_1 - H_2
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= \frac{\hat{X}}{2}\left[(e^{-\theta\rho}-1)(T-\frac{1}{\theta}) - \rho e^{-\theta\rho}\right] \\
&= \frac{\hat{X}}{2}\frac{\partial \tilde{\tilde{C}}_1}{\partial A} \leq 0
\end{aligned}$$

and

$$\begin{aligned}
H_2 &= \frac{\hat{X}}{2}(1-e^{-\theta\rho}) - x \\
&\leq \frac{\hat{X}}{2} - x \\
&\leq 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\xi(0) \geq 0 &\Leftrightarrow \tilde{\tilde{C}}_1 - X_0 \leq 0 \\
&\Leftrightarrow \frac{\gamma}{\eta}H_1 - H_2 \leq 0 \\
&\Leftrightarrow \frac{\gamma}{\eta} \geq \frac{H_2}{H_1} = H.
\end{aligned}$$

Similarly, we can reverse the inequality and get

$$\begin{aligned}
\xi(0) < 0 &\Leftrightarrow \tilde{\tilde{C}}_1 - X_0 > 0 \\
&\Leftrightarrow \frac{\gamma}{\eta}H_1 - H_2 > 0 \\
&\Leftrightarrow \frac{\gamma}{\eta} < \frac{H_2}{H_1} = H.
\end{aligned}$$

□

2.5 Conclusion

In this paper, we focus on the optimal liquidation problem with two trading venues: a traditional exchange and a dark pool. We extend the market price impact model to include the cross impact between exchange and dark pool, and analyze the optimal execution strategy. We observe that price manipulation strategies could be beneficial to traders under certain conditions, and we identify those model specifications for which the corresponding order execution problem is stable in the sense that there are no price manipulation strategies which can be beneficial.

CHAPTER 3

MULTI-PRODUCT SEPARATION RESULT FOR INVENTORY MANAGEMENT UNDER INFLATION RISK

3.1 Introduction

For a risk-averse corporation, an important decision problem to consider is how to manage the tradeoff between the risks and the expected return of inventory activities. This question is often formulated as a mean-variance type of decision problem; several papers such as Choi et al. (2008), and Wu et al. (2009) discuss the optimal operational decision for this problem. A non-financial corporation can be exposed to various sources of risk, which can be subsumed into two types: financial risk and non-financial risk. The financial risk comes from the financial market and hence can be hedged, to some extent, using financial instruments. The non-financial risk is assumed to be independent of the financial market, and hence cannot be hedged through financial trading. This can be characterized as a financial hedging problem in an incomplete market. A recent line of research addresses incorporating hedging in operations management. In particular, the financial department of a non-financial corporation can trade in financial markets to hedge risks arising from operational activities. This kind of problem leads to making financial and operational decisions simultaneously. Different inventory models with hedging have been proposed (see, for instance, Caldentey & Haugh (2006), Caldentey & Haugh (2009) and Gaur & Seshadri (2005)).

Financial hedging in incomplete markets is a widely studied field in mathematical finance. A classical approach to this problem is to control the hedging error by a quadratic criterion. This is mathematically equivalent to solving an optimal investment problem for a mean-variance type of objective function. From an operations management point of view, an attractive feature of this approach is its high degree of tractability. We refer to Schweizer et al. (1999) for a thorough overview of the quadratic hedging literature.

We use the phrase inventory hedging to refer to the general problem of using financial instruments to hedge away the financial risk associated with inventory management activities. One of the challenges that will need to be faced for any practical implementation of inventory hedging is in scaling the techniques to cope with the cardinality of the problem. Retail organizations, for example, manage tens, if not hundreds, of thousands of inventory items. Optimizing the risk-return tradeoff among these items is potentially a very large scale portfolio optimization problem. In our approach, we limit the interaction between inventory items to correlations with common market factors and we consider only those market factors which can be hedged with financial instruments. With this restriction, we show that the overall inventory hedging problem can be decomposed into separate problems, one for each inventory item, and the optimal financial hedging policy can be determined after the inventory policies are determined. Hence, our major contribution is the achievement of a multi-product separation result for inventory hedging.

In the special case of a single product operational decision problem, our work closely follows Caldentey & Haugh (2006) who propose a dynamic hedging strategy for the profits of a risk-averse corporation when these profits are correlated with returns in the financial markets. We depart from their framework by considering a slightly different mean-variance type objective function. This change allows us to extend their results to a multi-product problem which admits a separation theorem.

One example of financial risk that can affect a corporation's profits is monetary inflation, defined as the general increase in prices caused by a debasement of the underlying currency. We use the model in Jarrow & Yildirim (2003) to describe a market of inflation-related financial securities. This setting enables us to characterize an inflationary economy. We discuss a classical newsvendor problem in which demand for the product is negatively correlated with inflation. As will be proved in section 3.3, in the absence of financial hedging instruments monetary inflation leads to a malinvestment in inventory. Our motivation is to provide a correction to the malinvestment in inventory that may exist under rapid monetary inflation. However, as in Caldentey & Haugh (2006), our main results in section 3 are formulated for a generic financial asset, and thus can be applied in the context of other sources of financial risk.

The paper is organized as follows. In section 2 we introduce the financial market model and the inventory model, and formulate the hedging problem for multiple product inventory management. Our main results are stated in section 3, where we solve the problem via separation. We conclude the paper with numerical examples in section 4. Proofs are collected in section 5, also referred to as the Appendix.

3.2 Model and Problem Formulation

Fix a time horizon $T^* \in (0, \infty)$. Our set of states is given by the product probability space $(\Omega, \mathcal{F}, P) = (\Omega^W \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$, where $(\Omega^W, \mathcal{F}^W, \mathcal{F}_t^W, P^W)$ and $(E, \mathcal{E}, \mathcal{E}_t, P^E)$ are two complete filtered probability spaces. In particular, $(\Omega^W, \mathcal{F}^W, \mathcal{F}_t^W, P^W)$ is a probability space endowed with Brownian motions $(W_n(t), W_r(t), W_I(t) : t \in [0, T^*])$ with correlations given by

$$dW_n(t)dW_r(t) = \rho_{nr}dt$$

$$dW_n(t)dW_I(t) = \rho_{nI}dt$$

$$dW_r(t)dW_I(t) = \rho_{rI}dt.$$

The subscripts n, r, and I are used to suggest that the corresponding processes are sources of randomness for nominal; real; and inflation related instruments, respectively.

The space $\mathcal{E}$ represents an additional source of randomness which affects the market, where $\{\mathcal{E}_t : t \in [0, T^*]\}$ is the standard filtration generated by the $N-$dimensional Brownian motion $\mathcal{B}(t) = (\mathcal{B}_1(t), \ldots, \mathcal{B}_N(t))$, $t \in [0, T^*]$, independent of $\mathcal{F}_t^W$.

3.2.1 Financial market model

To analyze the impact of inflation risk on inventory management, we start by describing a market for inflation-related financial securities. We use the Heath-Jarrow-Morton type term structure model as applied by Jarrow & Yildirim (2003) where the tradable assets in the market are a bank account, nominal zero-coupon treasury bonds, and the Treasury Inflation-Protected Securities (TIPS) zero-coupon bonds. The following notation for financial markets is used in this paper:

- 'r' for real, 'n' for nominal.
- $P_n(t, T)$: time t price of a nominal zero-coupon bond maturing at time T in dollars.
- $I(t)$: time t CPI inflation index, i.e. dollars per CPI unit.
- $P_r(t, T)$: time t price of a real zero-coupon bond maturing at time T in CPI units.
- $f_k(t, T)$: time t nominal ($k = n$), respectively real ($k = r$), forward rates for date T, i.e.

$$P_k(t, T) = \exp\left\{\int_t^T f_k(t, u)du\right\}.$$

- $r_k(t) = f_k(t, t)$: the time t nominal ($k = n$), respectively real ($k = r$), spot rate.

- $B_n(t)$: time t money market account value, i.e.

$$B_n(t) = \exp\left\{\int_0^t r_n(v)dv\right\}.$$

- $P_{TIPS}(t,T)$: time t TIPS zero-coupon bond maturing at time T. i.e.

$$P_{TIPS}(t,T) = I(t)P_r(t,T).$$

Given the initial forward rate curve $f_k(0,T)$ with $T \in [0,T^*]$, $k \in \{r,n\}$, we assume that the nominal and real T-maturity forward rate evolves as:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t) \tag{3.1}$$

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t) \tag{3.2}$$

for $0 \leq t \leq T \leq T^*$, where $\alpha_k(t,T)$ and $\sigma_k(t,T)$ are stochastic processes satisfying certain technical measurability and integrability conditions.[1]

The inflation index's evolution is given by

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I(t) \tag{3.3}$$

for $t \in [0,T^*]$, where $\mu_I(t)$ and $\sigma_I(t)$ are stochastic processes satisfying certain technical measurability and integrability conditions.[2]

As stated in Shreve (2004), the financial market $B_n(t)$, $P_n(t,T)$, $P_{TIPS}(t,T)$, $0 \leq t \leq T \leq T^*$, is arbitrage-free if there exists a probability measure Q equivalent to P^W on $(\Omega, \mathcal{F}^W)$ such that:

$$\frac{P_n(t,T)}{B_n(t)}, \frac{P_{TIPS}(t,T)}{B_n(t)} \text{ are } Q-\text{local martingales for all } T \in [0,T^*].$$

[1] The process $\alpha_k(t,T)$ is $\mathcal{F}_t$-adapted and jointly measurable with $\int_0^T |\alpha_k(t,T)|dt < \infty$ P-a.s. and $\sigma_k(t,T)$ satisfies $\int_0^T \sigma_k^2(t,T)dt < \infty$ P-a.s.

[2] The process $\mu_I(t)$ is $\mathcal{F}_t$-adapted with $E[\int_0^\tau |\mu_I(t)|^2 dt] < \infty$ and $\sigma_I(t)$ is a deterministic function of time with $\int_0^\tau \sigma_I^2(v)dv < \infty$ P-a.s.

By Girsanov's theorem, given that $(W_n(t), W_r(t), W_I(t) : t \in [0,T])$ is a P-Brownian motion and that Q is a probability measure equivalent to P, then there exist market prices of risk $(\lambda_n(t), \lambda_r(t), \lambda_I(t) : t \in [0,T])$ such that

$$\widetilde{W}_k(t) = W_k(t) - \int_0^t \lambda_k(s)ds \quad \text{for } k \in \{n, r, I\} \tag{3.4}$$

are $Q-$ Brownian motions.

The following proposition characterizes the necessary and sufficient conditions for the economy to be arbitrage-free.

Proposition 3.2.1. $\frac{P_n(t,T)}{B_n(t)}, \frac{P_{TIPS}(t,T)}{B_n(t)}$ *are $Q-$local martingales for all $T \in [0,T^*]$ if and only if there exists functions $(\lambda_n(t), \lambda_r(t), \lambda_I(t) : t \in [0,T])$ satisfying (3.4) such that:*

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right) \tag{3.5}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \tag{3.6}$$

$$\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t). \tag{3.7}$$

The proof can be found in the Appendix.

We further restrict the model parameters to satisfy:

$$\sigma_I(t) = \sigma_I$$

$$\sigma_k(t,T) = \sigma_k \exp(-a_k(T-t)), \quad k \in \{n, r\}$$

where $\sigma_I, \sigma_n, \sigma_r$, a_n and a_r are constants. Under these assumptions, the bond prices and inflation index follow a lognormal model under the risk-neutral measure Q. The processes $\frac{P_n(t,T)}{B_n(t)}, \frac{P_{TIPS}(t,T)}{B_n(t)}$ are martingales under Q.

Proposition 3.2.2. *Under the risk neutral measure Q, the dynamics are:*

$$df_n(t,T) = -\frac{\sigma_n^2}{a_n} e^{-a_n(T-t)} \left(e^{-a_n(T-t)} - 1\right) dt + \sigma_n e^{-a_n(T-t)} d\widetilde{W}_n(t) \tag{3.8}$$

$$df_r(t,T) = -\sigma_r e^{-a_r(T-t)} \left(\frac{\sigma_r}{a_r}(e^{-a_r(T-t)} - 1) - \sigma_I \rho_{rI}\right) dt + \sigma_r e^{-a_r(T-t)} d\widetilde{W}_r(t) \tag{3.9}$$

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]dt + \sigma_I d\widetilde{W}_I(t) \tag{3.10}$$

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt + \frac{\sigma_n}{a_n}(e^{-a_n(T-t)} - 1)d\widetilde{W}_n(t) \tag{3.11}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) + \rho_{rI}\sigma_I \frac{\sigma_r}{a_r}(e^{-a_r(T-t)} - 1)\right] dt + \frac{\sigma_r}{a_r}\left[e^{-a_r(T-t)} - 1\right] d\widetilde{W}_r(t) \tag{3.12}$$

$$\frac{dP_{TIPS}(t,T)}{P_{TIPS}(t,T)} = r_n(t)dt + \sigma_I d\widetilde{W}_I(t) + \frac{\sigma_r}{a_r}(e^{-a_r(T-t)} - 1)d\widetilde{W}_r(t). \tag{3.13}$$

The proof can be found in Jarrow & Yildirim (2003) Proposition 2.

To simplify the problem, we fix the time horizon T of our inventory management problem, use $P_{TIPS}(\cdot, T)$ as numeraire, and immediately pass to quantities discounted with $P_{TIPS}(\cdot, T)$. This means that $P_{TIPS}(\cdot, T)$ has (discounted) price 1 at all times and the discounted nominal bond price is $X(\cdot) := P_n(\cdot, T)/P_{TIPS}(\cdot, T)$. The following proposition characterizes the dynamics of the discounted nominal bond:

Proposition 3.2.3. *Let $X(t) = P_n(t,T)/P_{TIPS}(t,T)$ be the discounted nominal bond process using the same maturity TIPS as numeraire. Its price process under the measure P^W is*

$$\frac{dX(t)}{X(t)} = \mu(t)dt + \sigma(t)dW(t)$$

where $W(t)$ is a P^W-Brownian motion defined as

$$W(t) = \int_0^t \frac{1}{\sigma(s)} \left(\sum_{k=n,r} \frac{\sigma_k}{a_k}\left(e^{-a_k(T-s)} - 1\right) dW_k(s) - \sigma_I dW_I(s)\right) \tag{3.14}$$

and $\mu(t)$ and $\sigma(t)$ are defined as

$$\begin{aligned}
\mu(t) = &-\lambda_n(t)\frac{\sigma_n}{a_n}(e^{-a_n(T-t)} - 1) + \lambda_I(t)\sigma_I + \lambda_r(t)\frac{\sigma_r}{a_r}(e^{-a_r(T-t)} - 1) - \rho_{nI}\sigma_I\frac{\sigma_n}{a_n}(e^{-a_n(T-t)} - 1) \\
&- \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}(e^{-a_r(T-t)} - 1)(e^{-a_n(T-t)} - 1) + \frac{\sigma_r^2}{a_r^2}(e^{-a_r(T-t)} - 1)^2 + \sigma_I^2 \qquad (3.15)\\
\sigma(t)^2 = &\frac{\sigma_n^2}{a_n^2}(e^{-a_n(T-t)} - 1)^2 + \sigma_I^2 + \frac{\sigma_r^2}{a_r^2}(e^{-a_r(T-t)} - 1)^2 - 2\rho_{nI}\frac{\sigma_n\sigma_I}{a_n}(e^{-a_n(T-t)} - 1) \\
&+ 2\rho_{rI}\frac{\sigma_r\sigma_I}{a_r}(e^{-a_r(T-t)} - 1) - 2\rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}(e^{-a_n(T-t)} - 1)(e^{-a_r(T-t)} - 1). \qquad (3.16)
\end{aligned}$$

The proof can be found in the Appendix.

In Jarrow & Yildirim (2003), the authors described in detail the procedure to estimate parameters a_k, $k \in \{n, r\}$, σ_k, $k \in \{n, r, I\}$ and correlations $\rho_{rI}, \rho_{nI}, \rho_{rn}$ from three different data sets: Treasury bond data, TIPS prices, and CPI-U data. For our application, we also need to know the parameters λ_k, $k \in \{n, r\}$, or equivalently, we need to estimate the drifts of the financial assets. This is a difficult problem in econometrics. We leave this practical issue as an open question for now.

3.2.2 Inventory model

We consider a classical single-period, multi-product newsvendor model for inventory management. There are N different products. At time $t = 0$, the operations manager makes the product purchase decisions $\gamma = (\gamma_1, \ldots, \gamma_N)$, which is a vector control, to satisfy a future stochastic demand $D(T) = (D_1(T), \ldots, D_N(T))$. At time $t = T$, the demand is realized. For any product j, $j = 1, \ldots, N$, the net profit at time T will be

$$
\begin{aligned}
& H_T(\gamma_j) \\
&= R_j(T)\min\{D_j(T), \gamma_j\} + s_j(T)(\gamma_j - D_j(T))^+ - q_j(T)(D_j(T) - \gamma_j)^+ - p_j(0)\frac{P(T)}{P(0)}\gamma_j \\
&= (R_j(T) - s_j(T))D_j(T) + s_j(T)\gamma_j - (R_j(T) + q_j(T) - s_j(T))(D_j(T) - \gamma_j)^+ \\
&\quad - p_j(0)\frac{P(T)}{P(0)}\gamma_j
\end{aligned}
$$

with γ_j the corresponding operational decision for product j, R_j is the unit retail price, s_j is the salvage value of unsold units, q_j is an additional lost sales penalty per unit of unsatisfied demand, p_j is the unit purchase price, and $P(t)$ is the price of the financial asset used as numeraire (or accounting). Notice that the purchasing occurs at time 0 and the retail activities are realized at time T.

In an economy with monetary inflation, we can expect that both price and demand will be affected by the inflation index. For example, wages may not keep pace with cost of living increases. In particular, we consider products whose demand depends on the level of the inflation index, and the nominal prices of these products increase with the index. The model we have is that for any time t, a price equals a fundamental price multiplied by the inflation index. That is, for $j = 1, \ldots, N$:

$$
\begin{aligned}
R_j(t) &= R_j(0)I(t) \\
p_j(t) &= p_j(0)I(t) \\
s_j(t) &= s_j(0)I(t) \\
q_j(t) &= q_j(0)I(t)
\end{aligned}
\tag{3.17}
$$

where $R_j(0), p_j(0), s_j(0)$ and $q_j(0)$ are constants satisfying $R_j(0) > \frac{p(0)}{P_{TIPS}(0,T)} > s_j(0)$. We further assume that the demand is a power function of the inflation-linked price:

$$D_j(t) = a_j e^{-b_j \log R_j(t) + c_j \mathcal{B}_j(t)}$$

with constants $a_j > 0$ and $b_j, c_j \in \mathbb{R}$, and $\mathcal{B}_j(t)$ is the jth element of $\mathcal{B}(t) = (\mathcal{B}_1(t), \ldots, \mathcal{B}_N(t))$, an N-dimensional Brownian motion independent of $W(t)$. $\mathcal{B}(t)$ denotes the non-financial noise. Thus, in this model, there are two sources of randomness in the demand process for a product: a risky financial variable (the CPI index) and a non-financial noise. As stated in the model setup, the filtration $\mathcal{F}_t^W \otimes \mathcal{E}_t$, $t \in [0, T^*]$ represents the evolution of observable information in the model. We consider the non-financial noise $\mathcal{B}_j(t)$, $j = 1, \ldots, N$ to be observable. For example, $\mathcal{B}_j(t)$ could represent the relative appeal of product j to a typical consumer. This power function model of demand is more realistic than the linear model of demand considered by Caldentey & Haugh (2006) and others.

The total payoff function of the corporation is the sum of net profits over all products:

$$H_T(\gamma) = \sum_{j=1}^{N} H_T(\gamma_j).$$

3.2.3 Hedging in the financial market

Consider a financial market consisting of a riskless and a risky asset with prices $P(t)$ and $S(t)$, respectively. We express all value and price processes in terms of the riskless asset P as numeraire. In particular, in numeraire P, the price of the riskless asset P itself is equal to 1, and the price of the risky asset S is given by $X(t) = \frac{S(t)}{P(t)}$. We assume that $X(t)$ satisfies the stochastic differential equation (SDE):

$$\frac{dX(t)}{X(t)} = \mu(t)dt + \sigma(t)dW(t)$$

where $\mu(t)$ and $\sigma(t)$ are given in proposition 3.2.3.

We further assume that the so-called mean-variance trade-off $\eta(t) := \mu(t)/\sigma(t)$ is a bounded and deterministic function. In our application, we take $P(t) = P_{TIPS}(t, T)$ and

$S(t) = P_n(t,T)$, so inflation-adjusted time T-dollars are interpreted as the riskless asset and nominal time T-dollars are the risky asset.

With everything expressed in inflation-adjusted dollars, the corresponding payoff in discounted units is given by:

$$\begin{aligned} H_T^D(\gamma_j) &= \frac{H_T(\gamma_j)}{P_{TIPS}(T,T)} \\ &= (R_j(0) - s_j(0))D_j(T) - (R_j(0) + q_j(0) - s_j(0))(D_j(T) - \gamma_j)^+ \\ &+ s_j(0)\gamma_j - p_j(0)\frac{\gamma_j}{P_{TIPS}(0,T)} \end{aligned} \tag{3.18}$$

where we have used $P_{TIPS}(T,T) = I(T)$ and the parameters defined in (3.17).

The total discounted payoff is

$$H_T^D(\gamma) = \sum_{j=1}^{n} H_T^D(\gamma_j).$$

Define the set of self-financing trading strategies Θ to be the collection of $\mathcal{F}^W \otimes \mathcal{E}$-predictable processes $(\theta_t)_{0\le t\le T}$ such that

$$E\left[\int_0^T \theta_t^2 X(t)^2 dt\right] < \infty. \tag{3.19}$$

The strategy variable, θ_t, denotes the number of shares in the risky asset $X(t)$ held at time t. The (discounted) gain process $G_t(\theta)$ associated with trading strategy $\theta \in \Theta$ is defined by

$$G_t(\theta) := \int_0^t \theta_s dX(s), \qquad \text{for all } t \in [0,T].$$

Consider a risk-averse non-financial corporation that operates during $[0,T]$. It earns a discounted profit H_T^D which depends on an operating strategy $\gamma \in \Gamma$, and it gains $G_T(\theta)$ from trading in the financial market. We let Γ be the set of $\mathcal{F}_0^W \otimes \mathcal{E}_0$-predictable policies $\gamma = (\gamma_1, \ldots, \gamma_N)$ with N components. H_T^D is an $\mathcal{F}_T^W \otimes \mathcal{E}_T$-measurable random variable.

Since $\sigma(X(t) \,|\, 0 \leq t \leq T) \subsetneq \mathcal{F}_T^W \otimes \mathcal{E}_T$, the market is now incomplete. In other words, there is risk in the stochastic demands $D_j(t)$ (modeled by $\mathcal{B}(t)$) which cannot be hedged by trading in the financial market using asset $X(t)$.

Starting with an initial wealth W_0, the corporation makes an operational decision and implements a self-financing hedging strategy. As the result of operational and financial activities, the final discounted wealth at time T will be

$$Y_T^{(\gamma,\theta)} := \hat{W}_0 + H_T^D(\gamma) + G_T(\theta)$$

where $\hat{W}_0 = \frac{W_0}{P_{TIPS}(0,T)}$ is the discounted initial wealth. For a fixed risk-aversion parameter $\kappa > 0$, we are interested in the optimal solution to the problem

$$U = \max_{\gamma\in\Gamma,\theta\in\Theta} \left(E\left[Y_T^{(\gamma,\theta)} \right] - \kappa Var\left[Y_T^{(\gamma,\theta)} \right] \right). \tag{3.20}$$

This completes our presentation of the multi-product inventory hedging problem in an economy with monetary inflation. In the literature, the most closely related model to this is that proposed and analyzed by Caldentey & Haugh (2006). Our model is derived from theirs in that we consider a single-period newsvendor-style payoff function for the inventory problem. We also assume that demand for the product is correlated with a financial asset and that a self-financing hedging strategy can be implemented based on this asset. There are four important differences in our approach, as compared to Caldentey & Haugh (2006). First of all, we consider a multi-product problem whereas Caldentey & Haugh (2006) explore a single product model. Secondly, we use a different demand model. Caldentey & Haugh (2006) use a linear model relating demand to price. We use a nonlinear demand model, and we characterize the impact of inflation. The advantage of the power function for demand is that it ensures that demand will not be negative or zero. Thirdly, in what follows we use a different dual criterion as the basis for the hedging strategy solution. Finally, Caldentey & Haugh (2006) develop solutions for both complete and incomplete information models, according to whether the non-financial noise is observable or not. We assume that the non-financial noise is observable, which corresponds to the complete information scenario in Caldentey & Haugh (2006). These differences enable us to focus on a realistic model for multi-product separation.

3.3 Hedging of multiple products

In the previous section, the optimization problem (3.20) has been defined. It involves optimizing over operational and financial decisions. Instead of finding the optimal controls simultaneously, we first the fix operational control $\gamma \in \Gamma$ and consider the restricted hedging problem

$$U^{\gamma} = \sup_{\theta \in \Theta} \left(E\left[Y_T^{(\gamma,\theta)} \right] - \kappa Var\left[Y_T^{(\gamma,\theta)} \right] \right). \tag{3.21}$$

This problem can be reformulated as follows. Let $B^{\gamma}(m)$ denote a variance minimizing problem

$$B^{\gamma}(m) = \inf_{\theta \in \Theta} \left\{ Var\left[Y_T^{(\gamma,\theta)} \right] \mid E\left[Y_T^{(\gamma,\theta)} \right] = m \right\}, \quad \text{for each } m \in \mathbb{R}. \tag{3.22}$$

Then

$$U^{\gamma} = \sup_{m \in \mathbb{R}} \left(m - \kappa B^{\gamma}(m) \right). \tag{3.23}$$

On the other hand, define the auxiliary problem

$$A_T^{\gamma}(\lambda) = \inf_{\theta \in \Theta} E\left[(Y_T^{(\gamma,\theta)} - \lambda)^2 \right], \quad \text{for each } \lambda \in \mathbb{R}. \tag{3.24}$$

The following theorem states that the auxiliary problem $A_T^{\gamma}(\lambda)$ is conjugate to the variance minimization problem.

Theorem 3.3.1. *With $A_T^\gamma(\lambda)$ and $B^\gamma(m)$ defined as in (3.24) and (3.22), we have*

$$B^\gamma(m) = \sup_{\lambda\in\mathbb{R}} \left(A_T^\gamma(\lambda) - (m-\lambda)^2\right) \tag{3.25}$$

and with λ_m the optimizer in (3.25), the optimal control in $B^\gamma(m)$ is equal to the optimal control in $A_T^\gamma(\lambda)$ with $\lambda = \lambda_m$.

The proof can be found in the Appendix.

By theorem 3.3.1, to solve the optimization problem (3.21), it suffices to find the optimal solution of the dual problem $A_T^\gamma(\lambda)$. It turns out that $A_T^\gamma(\lambda)$ is the auxiliary problem of a quadratic hedging problem. Quadratic hedging is a classical mathematical finance topic. We introduce the quadratic hedging problem in the following section, and use it to solve the restricted hedging problem (3.21).

3.3.1 Quadratic hedging problem and Föllmer-Schweizer decomposition

In this section, we start with considering the auxiliary problem, and show that the optimization over the hedging strategy can be eliminated. We further find the optimizer in the restricted hedging problem and the duality problem, and finally derive the separation result of the restricted hedging problem using duality and the auxiliary problem.

Instead of considering the optimization problem (3.20), the quadratic hedging problem (3.24) can be seen to arise from maximizing the expected quadratic utility of terminal wealth, where the utility function is defined as $u(w) = w - lw^2$. Indeed, the quadratic utility problem

$$\max_{(\gamma,\theta)\in\Gamma\times\Theta} E[u(\hat{W}_0 + H_T^D(\gamma) + G_T(\theta))] \tag{3.26}$$

is equivalent to

$$\min_{(\gamma,\theta)\in\Gamma\times\Theta} E\left[\left(\hat{W}_0 + H_T^D(\gamma) + G_T(\theta) - \lambda\right)^2\right]$$

with $\lambda = \frac{1}{2l}$. To solve this, we first fix $\gamma \in \Gamma$ and consider the optimization problem

$$\min_{\theta\in\Theta} E\left[\left(\hat{W}_0 + H_T^D(\gamma) + G_T(\theta) - \lambda\right)^2\right]. \tag{3.27}$$

This leads to the auxiliary problem (3.24).

Given the assumption that the market price of risk, $\eta(t)$, is a bounded and deterministic function, the solution of (3.27) can be found using the minimal equivalent martingale measure (MEMM, see Föllmer & Schweizer (1991)) defined by

$$\frac{d\hat{P}}{dP} := \exp\left\{\int_0^T \eta(t)dW(t) - \frac{1}{2}\int_0^T \eta^2(t)dt\right\}. \tag{3.28}$$

By Girsanov's theorem, both the financial asset X and non-financial noise $\mathcal{B}$ are square-integrable martingales under $\hat{P}$. We use $\hat{E}[\cdot]$ to denote the expectation under $\hat{P}$. The following theorem is the key result in quadratic hedging. It has been established in a number of modeling setups by different authors; we refer to Černỳ & Kallsen (2007) for a treatment of the quadratic hedging problem in a general semimartingale model and for a discussion of the literature on this problem. The version we are using here is due to Schweizer (1992).

Theorem 3.3.2. *For any $\mathcal{F}_T$-measurable claim $H_T^D(\gamma_j) \in \mathfrak{L}^p(P)$, $j = 1, \ldots, N$ for some $p > 2$, there is a hedging strategy, $\vartheta^{(\gamma_j)}$, and a process $\delta^{(\gamma_j)} \in \mathfrak{L}^2(P)$, such that $H_T^D(\gamma_j)$ admits the decomposition*

$$H_T^D(\gamma_j) = V_0^{(\gamma_j)} + \int_0^T \vartheta_t^{(\gamma_j)} dX(t) + \int_0^T \delta_t^{(\gamma_j)} d\mathcal{B}_j(t) \tag{3.29}$$

where $V_0^{(\gamma_j)} := \hat{E}[H_T^D(\gamma_j)]$. As a result, $H_T^D(\gamma) = \sum_{j=1}^N H_T^D(\gamma_j)$ admits the decomposition

$$H_T^D(\gamma) = V_0^{(\gamma)} + \int_0^T \vartheta_t^{(\gamma)} dX(t) + \int_0^T \delta_t^{(\gamma)} d\overline{\mathcal{B}}(t) \tag{3.30}$$

with

$$V_0^{(\gamma)} = \sum_{j=1}^{N} V_0^{(\gamma_j)} \tag{3.31}$$

$$\vartheta_t^{(\gamma)} = \sum_{j=1}^{N} \vartheta_t^{(\gamma_j)} \tag{3.32}$$

$$\delta_t^{(\gamma)} = \sqrt{\sum_{j=1}^{N} \left(\delta_t^{(\gamma_j)}\right)^2} \tag{3.33}$$

$$\overline{\mathcal{B}}(t) = \int_0^t \frac{1}{\delta_s^{(\gamma)}} \sum_{j=1}^{N} \delta_s^{(\gamma_j)} d\mathcal{B}_j(s) \tag{3.34}$$

with $\overline{\mathcal{B}}(t)$ *a Brownian motion under* P *and* $\hat{P}$.

In addition, the optimal strategy, θ^**, that solves (3.27) is given by* $\theta^* = \Phi(G_t^*)$ *where*

$$\Phi(G_t^*) = \vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda), \tag{3.35}$$

where G_t^* *solves the stochastic differential equation (SDE)*

$$dG_t^* = -\Phi(G_t^*)dX(t) \tag{3.36}$$

$$G_0^* = 0 \tag{3.37}$$

and $V_t^{(\gamma)}$ *is the intrinsic value process defined by*

$$V_t^{(\gamma)} := \hat{E}[H_T^D(\gamma)|\mathcal{F}_t] = V_0^{(\gamma)} + \int_0^t \vartheta_s^{(\gamma)} dX(s) + \int_0^t \delta_s^{(\gamma)} d\overline{\mathcal{B}}(s). \tag{3.38}$$

The decomposition (3.38) is known as the Galtchouk-Kunita-Watanabe (GKW) decomposition of $V_t^{(\gamma)}$ *under* $\hat{P}$ *with respect to* X.

Remark: The decomposition in the theorem is also known as the Föllmer-Schweizer decomposition of $H_T^D(\gamma)$ with respect to the semimartingale X. In particular, when the price of the discounted risky asset X is a martingale, as in our model, the Föllmer-Schweizer decomposition coincides with the Galtchouk-Kunita-Watanabe(GKW) decomposition under P.

This is a direct result from the result of Schweizer (1992).

The importance of the next theorem is that it establishes the value of the auxiliary process in terms that do not involve the optimal hedging strategy.

Theorem 3.3.3. *Define the auxiliary process*

$$A_T^{(\gamma)} := E[(V_T^{(\gamma)} + G_T^* + \hat{W}_0 - \lambda)^2] \tag{3.39}$$

and $K_t := \int_0^t \eta(s)^2 ds$, then $A_T^{(\gamma)}$ is given by

$$\begin{aligned} A_T^{(\gamma)}(\lambda) &= e^{-K_T}\left(\left(\hat{W}_0 + V_0^{(\gamma)} - \lambda\right)^2 + \int_0^T e^{K_s} E\left[(\delta_s^{(\gamma)})^2\right] ds\right) \\ &= e^{-K_T}\left(\left(\hat{W}_0 + V_0^{(\gamma)} - \lambda\right)^2 + \int_0^T e^{K_s} \sum_{j=1}^N E\left[(\delta_s^{(\gamma_j)})^2\right] ds\right). \end{aligned} \tag{3.40}$$

Observe that (3.40) involves the intrinsic value $V_0^{(\gamma)}$ and the non-financial noise term $\delta_t^{(\gamma)}$ from the decomposition (3.38). The optimization over θ in (3.24) has been eliminated.

Since the right-hand-side of (3.40) exists, we can replace the *inf* and *sup* with *min* and *max*, respectively, in (3.24), (3.21), (3.22) and (3.25). Let λ_m and m_{opt} denote the optimizers of (3.25) and (3.23), respectively. Theorem (3.3.4) permits us to solve for λ_m and m_{opt}, and these explicit solutions enable us to separate the multi-product problem (3.20) by product. Theorem (3.3.4) can be considered the main result of this paper.

Theorem 3.3.4. *The optimizer and the corresponding optimal value of problem (3.25) is*

$$\lambda_m = \frac{m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)})}{1 - e^{-K_T}} \tag{3.41}$$

$$B^\gamma(m) = \frac{e^{-K_T}}{1 - e^{-K_T}}(\hat{W}_0 + V_0^{(\gamma)} - m)^2 + e^{-K_T}\int_0^T e^{K_u} \sum_{j=1}^N E\left[(\delta_s^{\gamma_j})^2\right] ds. \tag{3.42}$$

The optimizer m_{opt} of problem (3.23) is given by

$$m_{opt} = \frac{1}{2\kappa}\frac{1 - e^{-K_T}}{e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)} \tag{3.43}$$

and the optimal value in problem (3.21) is

$$\begin{aligned} U^\gamma &= \hat{W}_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + V_0^{(\gamma)} - \kappa e^{-K_T} \int_0^T e^{K_s} \sum_{j=1}^N E\left[(\delta_s^{(\gamma_j)})^2\right] ds \\ &= \hat{W}_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + \sum_{j=1}^N V_0^{(\gamma_j)} - \kappa e^{-K_T} \int_0^T e^{K_s} \sum_{j=1}^N E\left[(\delta_s^{(\gamma_j)})^2\right] ds. \end{aligned} \tag{3.44}$$

Finally, the optimal control γ in (3.20) can be found by maximizing (3.44) over γ.

With this theorem, we achieve separation for the multi-product problem as stated in the following corollary.

Corollary 3.3.5. *With U^γ defined as in (3.21), the problem*

$$\max_\gamma \; U^\gamma \tag{3.45}$$

is equivalent to solving

$$\max_{\gamma_j} \left(V_0^{(\gamma_j)} - \kappa e^{-K_T} \int_0^T e^{K_s} E\left[(\delta_s^{(\gamma_j)})^2\right] ds \right) \tag{3.46}$$

for each $j = 1, \ldots, N$.

The following theorem proves that the problem above is well-defined.

Theorem 3.3.6. *There exists an optimal solution γ_j for problem (3.46).*

The proof can be found in the Appendix.

Armed with the existence of the optimal operation strategy γ_j, problem (3.46) can be solved numerically after we obtain $V_t^{(\gamma_j)}$ and $\delta_t^{(\gamma_j)}$ via the F-S decomposition. The following theorem provides this decomposition in explicit form.

Theorem 3.3.7. *The intrinsic value of discounted profit* $V_t^{(\gamma_j)} = \hat{E}[H_T^D(\gamma_j)|\mathcal{F}_t]$ *from product* j *is given by*

$$V_t^{(\gamma_j)} = (R_j(0) - s_j(0))N_t^{(j)} + s_j(0)\gamma_j - (R_j(0) + q_j(0) - s_j(0))M_t^{(\gamma_j)} - \frac{p_j(0)\gamma_j}{P_{TIPS}(0,T)} \tag{3.47}$$

for all $t \in [0, T]$*, and it has the Galtchouk-Kunita-Watanabe decomposition*

$$\begin{aligned} V_t^{(\gamma_j)} &= \hat{E}[H_T^D(\gamma_j)|\mathcal{F}_t] \\ &= V_0^{(\gamma_j)} + \int_0^t \vartheta_s^{(\gamma_j)} dX(s) + \int_0^t \delta_s^{(\gamma_j)} d\mathcal{B}_j(s) \end{aligned}$$

where

$$\begin{aligned} V_0^{(\gamma_j)} &= (R_j(0) - s_j(0))D_j(0) + s_j(0)\gamma_j - (R_j(0) + q_j(0) - s_j(0))(D_j(0) - \gamma_j)^+ \\ &\quad - \frac{p_j(0)\gamma_j}{P_{TIPS}(0,T)} \end{aligned} \tag{3.48}$$

$$\vartheta_t^{(\gamma_j)} = \frac{b_j}{X(t)} J_j(t) L^{(\gamma_j)}(t) \tag{3.49}$$

$$\delta_t^{(\gamma_j)} = c_j J_j(t) L^{(\gamma_j)}(t) \tag{3.50}$$

$$J_j(t) = a_j \exp\left(-b_j \log R_j(0) + b_j \log X(t) + c_j \mathcal{B}_j(t)\right) \tag{3.51}$$

$$\begin{aligned} L^{(\gamma_j)}(t) &= -(R_j(0) + q_j(0) - s_j(0))F_j(t)\Phi\left(\frac{\mu_z^j(t) + \log\frac{J_j(t)}{\gamma_j}}{\sigma_z^j(t)} + \sigma_z^j(t)\right) \\ &\quad + (R_j(0) - s_j(0))F_j(t) \end{aligned} \tag{3.52}$$

$$F_j(t) = e^{\mu_z^j(t) + \frac{1}{2}\sigma_z^j(t)^2} \tag{3.53}$$

$$M^{(\gamma_j)}(t) = J_j(t)F_j(t)\Phi\left(\frac{\mu_z^j(t) + \log\frac{J_j(t)}{\gamma_j}}{\sigma_z^j(t)} + \sigma_z^j(t)\right) - \gamma_j\Phi\left(\frac{\mu_z^j(t) + \log\frac{J_j(t)}{\gamma_j}}{\sigma_z^j(t)}\right) \tag{3.54}$$

$$N_t^{(j)} = J_j(t)F_j(t) \tag{3.55}$$

$$\mu_z^j(t) = b_j \mu_Y(t) \tag{3.56}$$

$$\sigma_z^j(t)^2 = b_j^2 \sigma_Y^2(t) + c_j^2(T - t) \tag{3.57}$$

$$\mu_Y(t) = -\frac{1}{2}\sigma_Y^2(t) \tag{3.58}$$

and

$$\sigma_Y^2(t) = \sum_{k=n,r} \frac{\sigma_k^2}{a_k^2}\left(\frac{1}{2a_k}\left(1-e^{-2a_k(T-t)}\right) - \frac{2}{a_k}\left(1-e^{-a_k(T-t)}\right) + T - t\right) + \sigma_I^2(T-t)$$
$$- 2\rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}\left[\frac{1}{a_n+a_r}\left(1-e^{-(a_n+a_r)(T-t)}\right) - \sum_{k=n,r}\frac{1}{a_k}\left(1-e^{-a_k(T-t)}\right) + (T-t)\right]$$
$$- \sum_{k=n,r} 2\rho_{nI}\frac{\sigma_k\sigma_I}{a_k}\left[\frac{1}{a_k}\left(1-e^{-a_k(T-t)}\right) - (T-t)\right]$$
$$= \int_t^T \sigma(s)^2 ds \tag{3.59}$$

and where $\Phi(\cdot)$, $\phi(\cdot)$ *are the CDF and pdf of the standard normal random variable respectively.*

The proof can be found in the Appendix.

Remark: Notice that in (3.46) the drifts of the financial assets enter only via the risk aversion parameters κe^{-K_T} and e^{K_s}. As stated before, the evaluation of drifts of financial assets is non-trivial. In problem (3.46) it corresponds to evaluating the market risk aversion, but this is also a difficult issue in econometrics.

The optimal hedging strategy $\theta^* = \theta^*(\gamma)$ in (3.20) can now be computed by solving (3.21) with the optimal $\gamma = (\gamma_1, \ldots, \gamma_N)$. This is achieved by using the duality in theorem 1 and the optimal control given in theorem 3.3.2. Combining these results, we find that $\theta^*(\gamma)$ is given in feedback form by

$$\theta_t^*(\gamma) = -\left(\vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})\right) \tag{3.60}$$

where G_t^* is the solution of the stochastic differential equation (SDE):

$$dG_t^* = -\left[\vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))\left(V_t^{(\gamma)} + G_t^* - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)}\right)\right] dX(t) \tag{3.61}$$

where $\vartheta^{(\gamma)} = \sum_{j=1}^N \vartheta^{(\gamma_j)}$. The solution of this SDE can be expressed in terms of a stochastic integral with respect to X. We discuss how to solve it numerically in the next section.

3.3.2 Approximation of quadratic hedging strategy

According to theorem 3.3.2, we need to solve a stochastic differential equation in order to obtain the optimal hedging strategy. In general, this requires numerical techniques for stochastic differential equations. In practice, a strategy which can be quickly and easily calculated is desirable. Hence we introduce an approximation hedging strategy.

We suppress the product index j from this point, as the argument will apply to any product. First, recall that the optimal quadratic hedging gain process for the discounted problem satisfies the stochastic differential equation:

$$dG_t^* = -\left[\vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})\right] dX(t) \tag{3.62}$$

with $G_0^* = 0$. The optimal hedging strategy is then given by

$$\theta_t^*(\gamma) = -\left(\vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})\right). \tag{3.63}$$

To avoid solving an SDE for each step, we propose an approximation hedging strategy. The following theorem gives the approximation and evaluates the quality of the approximation by considering the expected squared difference of the gain processes.

Theorem 3.3.8. *Consider the approximation strategy*

$$\widetilde{\theta}_t(\gamma) = -\left(\vartheta_t^{(\gamma)} + \mu(t)/(\sigma(t)^2 X(t))\left(V_t^{(\gamma)} + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)}\right)\right) \tag{3.64}$$

and the gain process under the approximation strategy

$$\widetilde{G}_t = \int_0^t \widetilde{\theta}_s(\gamma) dX(s). \tag{3.65}$$

If $|\eta(t)| \le \epsilon_1$, $|\sigma(t)| \le \epsilon_2$, *we have*

$$E[(G_t^* - \widetilde{G}_t)^2] < \epsilon_1^2(1 + t\epsilon_1^2)t^2\Upsilon_t^* \tag{3.66}$$

where

$$\upsilon_t = \vartheta_t^{(\gamma)}\sigma(t)X(t) + \eta(t)\left(V_t + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)}\right)$$

and

$$\Upsilon_t^* = \sup_{u\in[0,t]} E[\upsilon_u^2].$$

The proof can be found in the Appendix.

In practice, our conjecture is that the optimal hedging strategy can be approximated with smaller error via a forward finite difference method. Consider m discrete time points in $[0, T]$. For any $i = 1, \ldots, m$ we are interested in solving

$$\begin{aligned} &G_{t_i}^* - G_{t_{i-1}}^* \\ &= -\left[\vartheta_{t_i}^{(\gamma)} + \mu(t_i)/(\sigma(t_i)^2 X(t_i))(V_{t_i}^{(\gamma)} + G_{t_{i-1}}^* + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})\right](X(t_i) - X(t_{i-1})). \end{aligned} \tag{3.67}$$

The difference between this approach and (3.62) is that the observed gain process of the optimal hedging strategy is used to replace G_t^* on the right hand side of the SDE. This reduces the difficulties of solving the nonhomogeneous linear SDE (3.62). It also worth mentioning that the numerical approach is a two-dimensional procedure which yields the optimal hedging strategy θ_t^* and gain process G_t^* simultaneously. In fact, we can obtain both values as in (3.63) for each time step.

3.3.3 Comparison of optimal inventory decisions of hedging and non-hedging

In this section, we compare the optimal inventory decision in the case when the financial instrument for hedging is not available with the case in which hedging is available. We show that if there is no inflation-protected financial instrument, in a high inflation economy, the investor tends to purchase as much inventory as possible to preserve wealth. In other words, inflation distorts the inventory decision and causes a malinvestment. On the other hand, optimal hedging enables the operations department to make the correct inventory decision while the financial department takes care of inflation.

In the following theorem, we consider a single-product case without loss of generality.

Theorem 3.3.9. *There exist a critical value μ_I^* such that for all $\mu_I > \mu_I^*$, the optimal inventory decision with hedging is less than the optimal inventory decision without hedging, that is, $\gamma_H^* < \gamma_{NH}^*$.*

Proof. If there is no inflation-protected financial instrument, at time 0, the corporation purchases inventory with unit price $p(0)$, and the riskless asset in this case is bank account. So, at T, the present value of purchase cost is $p(0)\gamma B_n(0)$. In contrast, with a hedging opportunity, the riskless asset we consider in (3.18) is the TIPS. As a result, the non-hedging discounted payoff is

$$\overline{H}_T^D(\gamma) = (R(0) - s(0))D(T) - (R(0) + q(0) - s(0))(D(T) - \gamma)^+ + s(0)\gamma - p(0)\frac{\gamma B_n(0)}{P_{TIPS}(T,T)}$$

and the objective function under this case is

$$\max_\gamma \left(E[\overline{H}_T^D] - \kappa Var[\overline{H}_T^D] \right).$$

As proved in Wu et al. (2009), theorem 2.4, the variance function is bounded in $\gamma \in [0, +\infty)$. Also notice that

$$\lim_{\mu_I \to +\infty} E[\frac{B_n(0)}{P_{TIPS}(T,T)}] = 0, \tag{3.68}$$

and hence $E[\overline{H}_T^D]$ is an increasing function of γ for μ_I sufficiently large. That is, the optimal inventory decision without hedging $\gamma_{NH}^* \to \infty$ as $\mu_I \to \infty$.

On the other hand, we have proved in theorem 3.3.6 that the optimizer γ_H^* of problem (3.46) exists and is finite. As a result, with all other parameters the same, as μ_I increases, the optimal inventory decision γ_{NH}^* increases to $+\infty$ while γ_H^* remains unchanged, hence there is a critical value μ_I^* such that for all $\mu_I > \mu_I^*$, $\gamma_H^* < \gamma_{NH}^*$. □

3.4 Numerical example

In this section we demonstrate the multi-product separation result via a numerical example. In particular, we are interested in demonstrating the impact of hedging on products whose demands are correlated with the inflation index to different extents.

The following inventory parameter values are used for the example.

$$R0 = \$600,\ p0 = \$500,\ s0 = \$200,\ b0 = \$300,\ \kappa = 0.2,\ T = 2 \text{ years}.$$

We use the calibration result in Jarrow & Yildirim (2003) for the following financial market parameter values.

$$a_n = 0.013398,\ a_r = 0.014339,\ \sigma_n = 0.0566,\ \sigma_r = 0.0299$$
$$\rho_{nI} = 0.01482,\ \rho_{nr} = 0.06084,\ \rho_{rI} = -0.032127.$$

Furthermore, we assume

$$\alpha_n = 0.1,\ \alpha_r = 0.02,\ r_n(0) = 0.2,\ r_r(0) = 0, I_0 = 1.$$

Finally, we use the CPI parameter $\sigma_I = 0.1874$.

We consider two products exhibiting different correlations with the CPI: products 1 and 2 with CPI correlations $b_1 = 0.2$ and $b_2 = 0.9$. Furthermore, to demonstrate the hedging effect, we require that at time T, the realized demands have the same distribution, which leads to the same optimal inventory decision in the absence of hedging.

Practically, this can be done by fixing a_1, b_1, b_2 and c_1, and calculate a_2, c_2 via

$$a_2 = \exp(\log a_1 - (b_1 - b_2)\log R(0) + (b_1 - b_2)\mu_Y(0))$$

and

$$c_2 = \sqrt{(b_1^2 - b_2^2)\sigma_Y^2(0)/T + c_1^2}.$$

λ_I	Product	Non Hedging		Hedging	
		γ^*	Objective Function	γ^*	Objective Function
1.5	1	8.1574×10^4	1.1536×10^{14}	1.9436×10^5	4.2672×10^{13}
	2			2.1430×10^5	4.4424×10^{13}
1	1	8.1218×10^4	7.1696×10^{13}	1.0805×10^5	4.4948×10^{13}
	2			1.0737×10^5	3.6380×10^{13}
0.5	1	8.8479×10^4	8.6252×10^{13}	7.7591×10^4	6.8909×10^{13}
	2			6.6116×10^4	4.5046×10^{13}
0.1	1	9.5204×10^4	9.9335×10^{13}	6.7280×10^4	7.8694×10^{13}
	2			4.9799×10^4	4.0859×10^{13}
0.05	1	1.0041×10^5	1.5937×10^{14}	6.6303×10^4	9.0930×10^{13}
	2			4.8791×10^4	5.1474×10^{13}

Table 3.1: Optimal inventory decision and objective function value for different products.

We vary the drift of CPI by changing λ_I. To mimic a high-inflation economy, a small λ_I value is required.

Table 3.4 displays the results of the experiment. The observations from the experiment are that

- For both products, and for sufficiently high inflation, the optimal inventory decision with hedging becomes smaller than the one without hedging.
- The impact on the optimal inventory decision as inflation increases is weaker in product 1 compared to product 2.

The first observation illustrates theorem 3.3.9, showing that the malivestment will occur under a high-inflation economy while the application of hedging avoids it. The second observation shows that as the inflation level changes, the optimal inventory decision with hedging changes. Moreover, the product with the higher dependence on inflation has the more significant change in the optimal inventory decision.

3.5 Appendix to chapter 3

Proof of proposition 3.2.1.

Proof. According to the fundamental theorem of asset pricing, any finite subfamiliy of the market is arbitrage-free if there exists $Q \approx P$ such that all $\frac{P_n(t,T)}{B_n(t)}$ and all $\frac{P_{TIPS}(t,T)}{B_n(t)}$ are Q-local martingales.

Suppose there exists such a Q as above. By Itô's representation theorem and Girsanov's theorem, there exist predictable processes $\lambda_k(t)$, $k \in \{n, r, I\}$ such that

$$d\widetilde{W}_k(t) = dW_k(t) - \lambda_k(t)dt, \quad k \in \{n, r, I\}$$

are Q-Brownian motions. Itô's lemma yields

$$\begin{aligned}
& d\left(\frac{P_n(t,T)}{B_n(t)}\right) \\
&= \frac{1}{B_n(t)}\left(dP_n(t,T) - P_n(t,T)r_n(t)dt\right) \\
&= \frac{P_n(t,T)}{B_n(t)}\left(\left[\int_t^T \alpha_n(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma_n(t,s)ds\right)^2 + f_n(t,t) - r_n(t)\right]dt\right. \\
&\left. - \left[\int_t^T \sigma_n(t,s)ds\right]\left(d\widetilde{W}_n(t) + \lambda_n(t)dt\right)\right).
\end{aligned}$$

The processes are Q-local martingale for all maturities $T \geq t$ if and only if the drifts vanish, i.e.

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right),$$

noticing that $r_n(t) = f_n(t,t)$.

Similarly, Itô's lemma also yields

$$\begin{aligned}
& d\left(\frac{P_{TIPS}(t,T)}{B_n(t)}\right) \\
&= \frac{P_{TIPS}(t,T)}{B_n(t)}\left(\left[-\int_t^T \alpha_r(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma_r(t,s)ds\right)^2 + f_r(t,t) - r_n(t)\right]dt\right. \\
&- \left[\int_t^T \sigma_r(t,s)ds\right]\left(d\widetilde{W}_r(t) + \lambda_r(t)dt\right) + \sigma_I(t)\left(d\widetilde{W}_I(t) + \lambda_I(t)dt\right) \\
&\left. + \mu_I(t)dt - \int_t^T \sigma_r(t,s)ds \cdot \sigma_I(t)\rho_{rI}dt\right).
\end{aligned}$$

The processes are Q-local martingales for all $T \geq t$ if and only if the drifts vanish, i.e.

$$\begin{aligned}
\alpha_r(t,T) &= \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \\
\mu_I(t) &= r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t).
\end{aligned}$$

□

Proof of proposition 3.2.3.

Proof. We discount the nominal bond by TIPS. By Itô's lemma and proposition 3.2.2, the discounted nominal bond price is given by

$$
\begin{aligned}
\frac{dX(t)}{X(t)} &= \frac{d\left(\frac{P_n(t,T)}{P_{TIPS}(t,T)}\right)}{\frac{P_n(t,T)}{P_{TIPS}(t,T)}} \\
&= \left[\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right)d\widetilde{W}_n(t) - \sigma_I d\widetilde{W}_I(t) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)}-1\right)d\widetilde{W}_r(t)\right] \\
&- \left[\rho_{nI}\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right) + \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}\left(e^{-a_r(T-t)}-1\right)\left(e^{-a_n(T-t)}-1\right) - \sigma_I^2\right. \\
&\left.-\frac{\sigma_r^2}{a_r^2}\left(e^{-a_r(T-t)}-1\right)^2\right]dt \\
&= \left[\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right)dW_n(t) - \sigma_I dW_I(t) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)}-1\right)dW_r(t)\right] \\
&- \left[\lambda_n(t)\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right) - \lambda_I(t)\sigma_I - \lambda_r(t)\frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)}-1\right)\right]dt \\
&- \left[\rho_{nI}\sigma_I\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right) + \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}\left(e^{-a_r(T-t)}-1\right)\left(e^{-a_n(T-t)}-1\right)\right. \\
&\left.-\sigma_I^2 - \frac{\sigma_r^2}{a_r^2}\left(e^{-a_r(T-t)}-1\right)^2\right]dt.
\end{aligned}
$$

Let

$$
\begin{aligned}
\mu(t) &= -\lambda_n(t)\frac{\sigma_n}{a_n}(e^{-a_n(T-t)}-1) + \sigma_I\lambda_I(t) + \frac{\sigma_r}{a_r}(e^{-a_r(T-t)}-1)\lambda_r(t) + \frac{\sigma_r^2}{a_r^2}(e^{-a_r(T-t)}-1)^2 \\
&\quad - \rho_{nI}\sigma_I\frac{\sigma_n}{a_n}(e^{-a_n(T-t)}-1) - \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}(e^{-a_r(T-t)}-1)(e^{-a_n(T-t)}-1) + \sigma_I^2 \\
\sigma(t)^2 &= \frac{\sigma_n^2}{a_n^2}(e^{-a_n(T-t)}-1)^2 + \sigma_I^2 + \frac{\sigma_r^2}{a_r^2}(e^{-a_r(T-t)}-1)^2 - 2\rho_{nI}\frac{\sigma_n\sigma_I}{a_n}(e^{-a_n(T-t)}-1) \\
&\quad + 2\rho_{rI}\frac{\sigma_r\sigma_I}{a_r}(e^{-a_r(T-t)}-1) - 2\rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}(e^{-a_n(T-t)}-1)(e^{-a_r(T-t)}-1).
\end{aligned}
$$

and

$$
W(t) = \int_0^t \frac{1}{\sigma(s)}\left(\frac{\sigma_n}{a_n}(e^{-a_n(T-s)}-1)dW_n(s) - \sigma_I dW_I(s) - \frac{\sigma_r}{a_r}(e^{-a_r(T-s)}-1)dW_r(s)\right)
$$

Then $W(t)$ is a P-Brownian motion. Hence we have rewritten the discounted process $X(t)$ with respect to a one-dimension P-Brownian motion. This finishes the proof of proposition 3.2.3. □

Proof of theorem 3.3.1.

Proof. Recall that

$$A^{(\gamma)}(\lambda) = \inf_{\theta\in\Theta} E\left[(Y_T^{(\gamma,\theta)} - \lambda)^2\right] \tag{3.69}$$

$$B^{\gamma}(m) = \inf_{\theta\in\Theta}\left\{Var\left[Y_T^{(\gamma,\theta)}\right]\middle| E\left[Y_T^{(\gamma,\theta)}\right] = m\right\},\quad m\in\mathbb{R} \tag{3.70}$$

We want to prove that

$$A^{\gamma}(\lambda) = \inf_{m}\left[B^{\gamma}(m) + (m-\lambda)^2\right] \quad \lambda\in\mathbb{R}, \tag{3.71}$$

$$B^{\gamma}(m) = \sup_{\lambda}\left[A^{\gamma}(\lambda) - (m-\lambda)^2\right],\quad m\in\mathbb{R} \tag{3.72}$$

and $\forall\ m\in\mathbb{R}$, the optimal control of $B^{\gamma}(m)$ is equal to the optimal control in (3.72). Notice

$$E[(Y_T^{\lambda,\theta} - \lambda)^2] = Var[Y_T^{(\gamma,\theta)}] + (E[Y_T^{(\gamma,\theta)}] - \lambda)^2 \tag{3.73}$$

By definition of $B^{\gamma}(m)$, for each $\epsilon > 0$ we can find $\theta^{\epsilon}\in\Theta$ with controlled diffusion $Y^{\gamma,\theta^{\epsilon}}$, such that $E[Y_T^{\gamma,\theta^{\epsilon}}] = m$ and $Var[Y_T^{\gamma,\theta^{\epsilon}}] \le B^{\gamma}(m) + \epsilon$. i.e.

$$E[(Y_T^{\gamma,\theta^{\epsilon}} - \lambda)^2] \le B^{\gamma}(m) + (m-\lambda)^2 + \epsilon \tag{3.74}$$

and hence

$$A^{\gamma}(\lambda) = \inf_{\theta\in\Theta} E[(Y_T^{\gamma,\theta^{\epsilon}} - \lambda)^2] \le B^{\gamma}(m) + (m-\lambda)^2 \quad \forall m,\lambda\in\mathbb{R}. \tag{3.75}$$

On the other hand, for $\lambda\in\mathbb{R}$, let $\hat{\theta}_{\lambda}\in\Theta$ with controlled diffusion $\hat{Y}_T^{\gamma,\theta,\lambda}$, and optimal control for $A^{\gamma}(\lambda)$. Set $m_{\lambda} = E[\hat{Y}_T^{\gamma,\theta,\lambda}]$.

$$\begin{aligned} A^{\gamma}(\lambda) &= Var[\hat{Y}_T^{\gamma,\theta,\lambda}] + (m_{\lambda} - \lambda)^2 \\ &\ge B^{\gamma}(m_{\lambda}) + (m_{\lambda} - \lambda)^2. \end{aligned} \tag{3.76}$$

Combining (3.75) and (3.76),

$$\begin{aligned} A^\gamma(\lambda) &= \inf_m [B^\gamma(m) + (m-\lambda)^2] \\ &= B^\gamma(m_\lambda) + (m_\lambda - \lambda)^2 \end{aligned} \tag{3.77}$$

and $\hat{\theta}_\lambda$ is the solution to $B^\gamma(m_\lambda)$.

Also, since $X \mapsto Var[X]$ is convex in X, the function $B^\gamma(m)$ is convex in m, and since

$$A^\gamma(\lambda) = \inf_m [B^\gamma(m) + (m-\lambda)^2] \tag{3.78}$$

$$\frac{(\lambda^2 - A^\gamma(\lambda))}{2} = \sup_m \left[m\lambda - \frac{B^\gamma(m) + m^2}{2} \right] \tag{3.79}$$

the function $\lambda \mapsto \frac{\lambda^2 - A^\gamma(\lambda)}{2}$ is the Fenchel-Legendre transform of the convex function $m \mapsto \frac{(B^\gamma(m)+m^2)}{2}$. We then have the duality relation

$$\frac{(B^\gamma(m) + m^2)}{2} = \sup_\lambda \left[m\lambda - \frac{(\lambda^2 - A^\gamma(\lambda))}{2} \right] \tag{3.80}$$

and hence (3.72):

$$B^\gamma(m) = \sup_\lambda [A^\gamma(\lambda) - (m-\lambda)^2]. \tag{3.81}$$

Finally, $\forall m \in \mathbb{R}$, let $\lambda_m \in \mathbb{R}$ be the argument maximum of $B^\gamma(m)$ in (3.72). Then m is an argument minimum of $A^\gamma(\lambda)$ in (3.72). Since

$$m \mapsto B^\gamma(m) + (m-\lambda)^2 \qquad \text{is strictly convex,}$$

this argument minimum is unique, so $m = m_{\lambda_m} = E[\hat{Y}_T^{\gamma,\theta,\lambda_m}]$. Hence,

$$\begin{aligned} B^\gamma(m) &= A^\gamma(\lambda_m) + (m - \lambda_m)^2 \\ &= E[\hat{Y}_T^{\gamma,\theta,\lambda_m}]^2 + \left(E[\hat{Y}_T^{\gamma,\theta,\lambda_m}] - \lambda_m \right)^2 \\ &= Var[\hat{Y}_T^{\gamma,\theta,\lambda_m}] \end{aligned}$$

i.e. $\hat{\theta}_{\lambda_m}$ is a solution to $B^\gamma(m)$. □

Proof of theorem 3.3.3.

Proof. Define the process $N_t^{(\gamma)} := (V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda)^2$. Using Itô's lemma:

$$dN_t^{(\gamma)} = 2(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda)(dV_t^{(\gamma)} + dG_t^*) + d < V^{(\gamma)} + G^*, V^{(\gamma)} + G^* >_t .$$

Using the definition of $V_t^{(\gamma)}$ and G_t^*, we obtain

$$\begin{aligned} N_t^{(\gamma)} &= N_0^{(\gamma)} + 2\int_0^t (V_s^{(\gamma)} + G_s^* + \hat{W}_0 - \lambda)(\delta_s^{(\gamma)} d\overline{\mathcal{B}}(s) - \eta_s(V_s^{(\gamma)} + G_s^* + \hat{W}_0 - \lambda)dW(s)) \\ &+ \int_0^t (\delta_s^{(\gamma)^2} - \eta_s^2(V_s^{(\gamma)} + G_s^* + \hat{W}_0 - \lambda)^2)ds. \end{aligned}$$

Taking expectations, canceling all martingale terms, and using Fubini's theorem with the deterministic mean-variance assumption, we obtain

$$A_t^{(\gamma)} = E[N_t^{(\gamma)}] = E[N_0^{(\gamma)}] + \int_0^t (E[\delta_s^{(\gamma)^2}] - \eta_s^2 A_s^{(\gamma)})ds.$$

This implies the ODE:

$$\frac{d}{dt}A_t^{(\gamma)} + \eta_t^2 A_t^{(\gamma)} = E[\delta_t^{(\gamma)^2}].$$

Finally, use the integrating factor $K_t = \exp\left(\int_0^t \eta_s^2 ds\right)$ and the boundary condition $A_0^{(\gamma)} = (V_0^{(\gamma)} + \hat{W}_0 - \lambda)^2$ to obtain the desired result. □

Proof of theorem 3.3.4.

Proof. By theorem 3.3.1 we have

$$\begin{aligned} B^\gamma(m) &= \min_\theta \left\{ Var[\hat{W}_0 + H_T^\gamma + G_T(\theta)] \,|\, E[\hat{W}_0 + H_T^\gamma + G_T(\theta)] = m \right\} \\ &= \max_\lambda (A^\gamma(\lambda) - (m - \lambda)^2). \end{aligned}$$

The maximum is achieved for

$$0 = \frac{\partial}{\partial \lambda} A^\gamma(\lambda) + 2(m - \lambda)$$

where

$$A^\gamma(\lambda) = e^{-K_T}\left((\hat{W}_0 + V_0^{(\gamma)} - \lambda)^2 + \int_0^T e^{K_u}\sum_{j=1}^n E[(\delta_u^{\gamma_j})^2]\right).$$

So the optimal condition is

$$\begin{aligned}
&-2e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)} - \lambda) + 2(m-\lambda) = 0 \\
&\Leftrightarrow (1-e^{-K_T})\lambda = m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)}) \\
&\Leftrightarrow \lambda_m^* = \frac{m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)})}{1-e^{-K_T}}.
\end{aligned}$$

Plugging the result into the problem, we obtain

$$\begin{aligned}
B^{\gamma^*}(m) &= A^\gamma(\lambda_m^*) - (m-\lambda_m^*)^2 \\
&= e^{-K_T}\left(\frac{W_0 + V_0^{(\gamma)} - m}{1-e^{-K_T}}\right)^2 - \left(\frac{e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)} - m)}{1-e^{-K_T}}\right)^2 \\
&\quad + e^{-K_T}\int_0^T e^{K_u}\sum_{j=1}^n E[(\delta_u^{\gamma_j})^2]du.
\end{aligned}$$

This proves the first part of the theorem. To solve the problem

$$U^\gamma = \max_m(m - \kappa B^\gamma(m)),$$

note that the first order condition is

$$\begin{aligned}
&1 - \kappa\frac{\partial}{\partial m}B^\gamma(m) = 0 \\
&\Leftrightarrow 2(m^* - \hat{W}_0 - V_0^{(\gamma)})\frac{e^{-K_T}}{1-e^{-K_T}} = \frac{1}{\kappa} \\
&\Leftrightarrow m^* = \frac{1}{2\kappa}\frac{e^{-K_T}}{1-e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)} \\
&\Leftrightarrow (m^* - \hat{W}_0 - V_0^{(\gamma)})^2 = \frac{1}{4\kappa^2}\left(\frac{e^{-K_T}}{1-e^{-K_T}}\right)^2.
\end{aligned}$$

That is,

$$
\begin{aligned}
U^{\gamma} &= m^* - \kappa B^{\gamma}(m^*) \\
&= \frac{1}{2\kappa}\frac{e^{-K_T}}{1-e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)} - \frac{1}{4\kappa}\frac{e^{-K_T}}{1-e^{-K_T}} - \kappa e^{-K_T}\int_0^T e^{K_u}\sum_{j=1}^{N} E[(\delta_u^{\gamma_j})^2]du \\
&= \hat{W}_0 + \sum_{j=1}^{N} V_0^{(\gamma_j)} + \frac{1}{4\kappa}(e^{K_T}-1) - \kappa e^{-K_T}\int_0^T e^{K_u}\sum_{j=1}^{N} E[(\delta_u^{\gamma_j})^2]du.
\end{aligned}
$$

This finishes the second part of the theorem. □

Proof of theorem 3.3.6.

Proof. For simplicity, we suppress the dependence of product j for the proof. The problem we consider is

$$
\max_{\gamma}\left(V_0^{(\gamma)} - \kappa e^{-K_T}\int_0^T e^{K_s}E[(\delta_s^{(\gamma)})^2]ds\right) \tag{3.82}
$$

where $V_0^{(\gamma)}$ and $\delta_s^{(\gamma)}$ are given in theorem 3.3.7. Both amounts are independent of μ_I.

The objective function above, in general, is not a concave function of γ. We want to prove the existence by proving the concavity of $V_0^{(\gamma)}$ and showing the variance part is bounded with respect to γ.

First observe that

$$
\begin{aligned}
\frac{dV_0^{(\gamma)}}{d\gamma} &= s - \frac{p(0)}{P_{TIPS}(0,T)} + (R(0)+q(0)-s(0))\Phi\left(\frac{\mu_z(t)+\log\frac{J(0)}{\gamma}}{\sigma_z(0)}\right) \\
\frac{d^2V_0^{(\gamma)}}{d\gamma^2} &= -(R(0)+q(0)-s(0))\frac{1}{\gamma}\phi\left(\frac{\mu_z(0)+\log\frac{J(0)}{\gamma}}{\sigma_z(0)}\right) < 0
\end{aligned}
$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and CDF of the standard normal distribution. This proves that $V_0^{(\gamma)}$ is a concave function of γ.

Furthermore,

$$\lim_{\gamma\to 0}\delta_t^{(\gamma)} = cJ(t)(-(R(0)+q(0)-s(0))F(t)+(R(0)-s(0))F(t)) = -cJ(t)q(0)F(t)$$
$$\lim_{\gamma\to +\infty}\delta_t^{(\gamma)} = cJ(t)\left(-(R(0)+q(0)-s(0))F(t)\Phi\left(\frac{\mu_z(t)}{\sigma_z(t)}+\sigma_z(t)\right)+(R(0)-s(0))F(t)\right).$$

Notice that $E[J(t)^2] < \infty$, which implies that $E[(\delta_s^{(\gamma)})^2]$ is bounded for any time s as $\gamma \in [0, +\infty)$.

We also need

$$\lim_{\gamma\to +\infty}\frac{dV_0^{(\gamma)}}{d\gamma} = s - \frac{p(0)}{P_{TIPS}(0,T)} < 0$$

where the last inequality is due to the assumption.

Up to this point, we have proved that problem (3.46) is composed of a concave function and a bounded function of γ, hence we have proved that problem (3.46) is well-defined, the optimizer γ^* exists and is finite. □

The following lemmas will be used in proving theorem 3.3.7. In particular, lemma 3.5.1 is contributed to proof of lemma 3.5.2, which will be the building blocks to the proof of theorem 3.3.7.

Lemma 3.5.1. *Under the MEMM $\hat{P}$, the discounted nominal bond $X(t)$ is a $\hat{P}$-local martingale with dynamics*

$$\frac{dX(t)}{X(t)} = \frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)}-1\right)d\hat{W}_n(t) - \sigma_I d\hat{W}_I(t) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)}-1\right)d\hat{W}_r(t)$$

where $\hat{W}_k(t)$, $k \in \{n, I, r\}$ are $\hat{P}$-Brownian motions defined as

$$\hat{W}_I(t) = \widetilde{W}_I(t) - \int_0^t \left(\sigma_I + \rho_{rI}\frac{\sigma_r}{a_r}\left(e^{-a_r(T-s)-1}\right)\right)ds$$
$$\hat{W}_n(t) = \widetilde{W}_n(t) - \int_0^t \left(\rho_{nI}\sigma_I + \rho_{nr}\frac{\sigma_r}{a_r}\left(e^{-a_r(T-s)}-1\right)\right)ds$$
$$\hat{W}_r(t) = \widetilde{W}_r(t) - \int_0^t \left(\sigma_I\rho_{rI} + \frac{\sigma_r}{a_r}\left(e^{-a_r(T-s)}-1\right)\right)ds.$$

Proof of lemma 3.5.1.

Proof. First notice the dynamics of $X(t)$ under risk-neutral measure Q are:

$$\frac{dX(t)}{X(t)} = \left[\frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)} - 1\right) d\widetilde{W}_n(t) - \sigma_I d\widetilde{W}_I(t) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)} - 1\right) d\widetilde{W}_r(t)\right]$$
$$- \left[\rho_{nI}\sigma_I \frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)} - 1\right) + \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}\left(e^{-a_r(T-t)} - 1\right)\left(e^{-a_n(T-t)} - 1\right)\right.$$
$$\left. -\sigma_I^2 + \frac{\sigma_r^2}{a_r^2}\left(e^{-a_r(T-t)} - 1\right)^2\right] dt$$

According to Lévy's theorem, $\hat{W}_k(t)$, $k \in \{n, I, r\}$ defined as in the lemma will be $\hat{P}$ Brownian motion, and $X(t)$ is a $\hat{P}$-local martingale. □

Lemma 3.5.2. *Under the MEMM measure $\hat{P}$, $Y(t) = \log X(t)$ has dynamics*

$$dY(t) = d\log X(t)$$
$$= \left[\rho_{nI}\frac{\sigma_n}{a_n}\sigma_I\left(e^{-a_n(T-t)} - 1\right) + \rho_{nr}\frac{\sigma_n\sigma_r}{a_n a_r}\left(e^{-a_n(T-t)} - 1\right)\left(e^{-a_r(T-t)} - 1\right) - \frac{1}{2}\sigma_I^2\right.$$
$$\left. -\rho_{rI}\frac{\sigma_r\sigma_I}{a_r}\left(e^{-a_r(T-t)} - 1\right) - \frac{1}{2}\frac{\sigma_n^2}{a_n^2}\left(e^{-a_n(T-t)} - 1\right)^2 - \frac{1}{2}\frac{\sigma_r^2}{a_r^2}\left(e^{-a_r(T-t)} - 1\right)^2\right] dt$$
$$+ \frac{\sigma_n}{a_n}\left(e^{-a_n(T-t)} - 1\right) d\hat{W}_n(t) - \sigma_I d\hat{W}_I(t) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-t)} - 1\right) d\hat{W}_r(t).$$

So given $\mathcal{F}_t$, $Y(T) - Y(t)$ is a normally distributed random variable with mean $\mu_Y(t)$ and variance $\sigma_Y^2(t)$ defined in theorem 3.3.7. Moreover, we have

$$\sigma_Y^2(t) = \int_t^T \sigma(s)^2 ds. \tag{3.83}$$

Proof of lemma 3.5.2.

Proof. The dynamics of $Y(t)$ are a direct consequence of Itô's lemma. By lemma 3.5.1:

$$
\begin{aligned}
&Y(T) - Y(t)\\
&= \log X(T) - \log X(t)\\
&= \int_t^T \left[\rho_{nI}\frac{\sigma_n}{a_n}\sigma_I\left(e^{-a_n(T-s)} - 1\right) + \rho_{nr}\frac{\sigma_n\sigma_r}{a_na_r}\left(e^{-a_n(T-s)} - 1\right)\left(e^{-a_r(T-s)} - 1\right)\right.\\
&\left.-\rho_{rI}\frac{\sigma_r\sigma_I}{a_r}\left(e^{-a_r(T-s)} - 1\right) - \frac{1}{2}\sigma_I^2 - \frac{1}{2}\frac{\sigma_n^2}{a_n^2}\left(e^{-a_n(T-s)} - 1\right)^2 - \frac{1}{2}\frac{\sigma_r^2}{a_r^2}\left(e^{-a_r(T-s)} - 1\right)^2\right] ds\\
&+ \int_t^T \left[\frac{\sigma_n}{a_n}\left(e^{-a_n(T-s)} - 1\right) d\hat{W}_n(s) - \sigma_I d\hat{W}_I(s) - \frac{\sigma_r}{a_r}\left(e^{-a_r(T-s)} - 1\right) d\hat{W}_r(s)\right]
\end{aligned}
$$

which is a normally distributed random variable, given $\mathcal{F}_t$, with mean $\mu_Y(t)$ and variance $\sigma_Y^2(t)$.

To prove (3.83), recall that by assumption

$$\frac{dX(t)}{X(t)} = \mu(t)dt + \sigma(t)dW(t)$$

so

$$
\begin{aligned}
dY(t) &= d\log X(t)\\
&= (\mu(t) - \frac{1}{2}\sigma(t)^2)dt + \sigma(t)dW(t).
\end{aligned}
$$

Girsanov theorem then implies (3.83). □

Proof of theorem 3.3.7.

Proof. For each product j, $j = 1, \ldots, N$, the intrinsic value of the discounted payoff is

$$
\begin{aligned}
&V_t^{(\gamma_j)} = \hat{E}[H^D(\gamma_j)|\mathcal{F}_t]\\
&= \hat{E}\left[(R_j(0) - s_j(0))D_j(T) + s_j(0)\gamma_j - (R_j(0) + q_j(0) - s_j(0))(D_j(T) - \gamma_j)^+ - \frac{p_j(0)\gamma_j}{P_{TIPS}(0,T)}|\mathcal{F}_t\right].
\end{aligned}
$$

Let

$$M_t^{(\gamma_j)} = \hat{E}[(D_j(T) - \gamma_j)^+ | \mathcal{F}_t]$$

and

$$N_t^{(j)} = \hat{E}[D_j(T) | \mathcal{F}_t].$$

Notice that the demand is given by

$$\begin{aligned}
D_j(T) &= a_j e^{-b_j \log R_j(T) + c_j \mathcal{B}_j(T)} \\
&= a_j e^{-b_j \log R_j(0) - b_j \log I(T) + c_j \mathcal{B}_j(T)} \\
&= a_j \exp\left(-b_j \log R_j(0) - b_j \log \frac{P_{TIPS}(T,T)}{P_n(T,T)} + c_j \mathcal{B}_j(T)\right) \\
&= a_j \exp\left(-b_j \log R_j(0) + b_j \log X(T) + c_j \mathcal{B}_j(T)\right).
\end{aligned}$$

Let $J_j(t) = a_j e^{-b_j \log R_j(0) + b_j \log X(t) + c_j \mathcal{B}_j(t)}$. Conditioning on $\mathcal{F}_t$, we have

$$\begin{aligned}
M_t^{(\gamma_j)} &= \hat{E}[(D_j(T) - \gamma_j)^+ | \mathcal{F}_t] \\
&= \hat{E}\left[\left(a_j e^{-b_j \log R_j(0) + b_j \log X(t) + c_j \mathcal{B}_j(t)} e^{b_j(\log X(T) - \log X(t)) + c_j(\mathcal{B}_j(T) - \mathcal{B}_j(t))} - \gamma_j\right)^+ | \mathcal{F}_t\right] \\
&= \hat{E}\left[\left(J_j(t) e^{b_j(\log X(T) - \log X(t)) + c_j(\mathcal{B}_j(T) - \mathcal{B}_j(t))} - \gamma_j\right)^+ | \mathcal{F}_t\right].
\end{aligned}$$

Let

$$\begin{aligned}
Z_j(t,T) &= b_j(\log X(T) - \log X(t)) + c_j(\mathcal{B}_j(T) - \mathcal{B}_j(t)) \\
&= b_j(Y(T) - Y(t)) + c_j(\mathcal{B}_j(T) - \mathcal{B}_j(t))
\end{aligned}$$

which is a normal random variable with mean $\mu_z^j(t)$ and variance $\sigma_z^j(t)^2$. By lemma 3.5.2, we have $\mu_z^j(t) = b_j \mu_Y^j(t)$ and $\sigma_z^j(t)^2 = b_j^2 \sigma_Y^j(t)^2 + c_j^2(T-t) = b_j^2 \int_0^t \sigma(s)^2 ds + c_j^2(T-t)$. We can calculate the conditional expectation

$$M^{(\gamma_j)}(t) = J_j(t) F_j(t) \Phi\left(\frac{\mu_z^j(t) + \log \frac{J_j(t)}{\gamma_j}}{\sigma_z^j(t)} + \sigma_z^j(t)\right) - \gamma_j \Phi\left(\frac{\mu_z^j(t) + \log \frac{J_j(t)}{\gamma_j}}{\sigma_z^j(t)}\right) \tag{3.84}$$

with $\Phi(\cdot)$ being the CDF of the standard normal distribution and $F_j(t) = e^{\mu_z^j(t)+\frac{1}{2}\sigma_z^j(t)^2}$.

Similarly,

$$N_t^{(j)} = \hat{E}[D_j(T)|\mathcal{F}_t] = J_j(t)\hat{E}[e^{Z_j(t,T)}|\mathcal{F}_t] = J_j(t)F_j(t). \tag{3.85}$$

Hence the intrinsic value of discounted profit for product j is

$$V_t^{(\gamma_j)} = (R_j(0) - s_j(0))N_t^{(\gamma_j)} + s_j(0)\gamma_j - (R_j(0) + q_j(0) - s_j(0))M_t^{(\gamma_j)} - p_j(0)\frac{\gamma_j}{P_{TIPS}(0,T)}. \tag{3.86}$$

So the decomposition with respect to $X(t)$ can be obtained by Itô's formula and finally we have the desired result. □

The following proposition is dedicated to the proof of theorem 3.3.7.

Proposition 3.5.3. *Let* $\upsilon_t = \vartheta_t^{(\gamma)}\sigma(t)X(t) + \eta(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})$. *Assuming* $|\eta(t)| \le \epsilon_1$, $|\sigma(t)| \le \epsilon_2$, *there exists*

$$\Upsilon_t^* = \sup_{u\in[0,t]} E[\upsilon_u^2]$$

Proof of proposition 3.5.3.

Proof. First recall from the proof of theorem 3.3.6 that

$$\vartheta_t^{(\gamma)} = \frac{b}{X(t)}J(t)L(t).$$

Hence

$$\upsilon_t = b\sigma(t)J(t)L(t) + \eta(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})$$

and

$$\begin{aligned}\upsilon_t^2 &= b^2\sigma(t)^2J^2(t)L^2(t) + \eta^2(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})^2 \\ &\quad + 2b\sigma(t)J(t)L(t)\eta(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa}e^{K_T} - V_0^{(\gamma)})\end{aligned}$$

We want to find an upper bound for $|L(t)|$ and $|V_t|$. Notice we have by assumption $R(0) > p(0) > s(0)$; hence, for any $u \in [0, t]$,

$$\begin{aligned}
|L(u)| &\leq |(R(0) + q(0) - s(0))F(u)\Phi_{z,u}(\log \frac{J(u)}{\gamma})| + |(R(0) - s(0))F(u)| \\
&\leq (R(0) + q(0) - s(0))F(u)1 + (R(0) - s(0))F(u) \\
&=: \overline{L}(u)
\end{aligned}$$

and

$$\begin{aligned}
|V_u| &\leq |(R(0) - s(0))J(u)F(u)| + |s(0)\gamma| + |(R(0) + q(0) - s(0))J(u)F(u)\Phi_{z,u}(\log \frac{J(u)}{\gamma} - \sigma_z(u)^2)| \\
&+ |(R(0) + q(0) - s(0))\gamma\Phi_{z,u}(\log \frac{J(u)}{\gamma})| + |p(0)\frac{\gamma}{P_{TIPS}(0, T)}| \\
&\leq (R(0) - s(0))J(u)F(u) + s(0)\gamma + (R(0) + q(0) - s(0))(J(u)F(u) + \gamma) + p(0)\frac{\gamma}{P_{TIPS}(0, T)} \\
&=: \overline{V}(t).
\end{aligned}$$

Let

$$\begin{aligned}
\Delta_1(u) &= (R(0) + q(0) - s(0))F(u) + (R(0) - s(0))F(u), \\
\Delta_2(u) &= (2R(0) + q(0) - 2s(0))F(u),
\end{aligned}$$

and

$$\Delta_3(u) = s(0)\gamma + (R(0) + q(0) - s(0))\gamma + p(0)\frac{\gamma}{P_{TIPS}(0, T)} + \hat{W}_0 + \frac{1}{2\kappa}e^{K_u} + V_0^{(\gamma)}.$$

Then

$$\begin{aligned}
E[v_u^2] &\leq E[b^2\sigma(u)^2 J^2(u)\overline{L}(u)^2 + \eta^2(u)\overline{V}(u)^2 + 2|b|\sigma(u)\eta(u)J(u)\overline{L}(u)\overline{V}(u)] \\
&= b^2\sigma(u)^2 E[J^2(u)\Delta_1(u)^2] + \eta^2(u)E[(\Delta_2(u)J(u) + \Delta_3(u))^2] \\
&+ 2|b|\sigma(u)\eta(u)E[J(u)\Delta_1(u)(\Delta_2(u)J(u) + \Delta_3(u))] \\
&= (b^2\sigma(u)^2\Delta_1(u)^2 + \eta(u)^2\Delta_2(u)^2 + 2b\eta(u)\sigma(u)\Delta_1(u)\Delta_2(u))E[J^2(u)] \\
&+ 2(\eta(u)^2\Delta_2(u)\Delta_3(u) + |b|\eta(u)\sigma(u)\Delta_1(u)\Delta_3(u))E[J(u)] + \eta^2(u)\Delta_3(u)^2.
\end{aligned}$$

To calculate the expectation, it suffices to calculate $E[J(u)]$ and $E[J^2(u)]$:

$$E[J(u)] = ae^{-b\log R(0)+\mu_z(u)+\frac{1}{2}\sigma_z(u)^2}$$

and

$$E[J^2(u)] = a^2e^{-2b\log R(0)+2\mu_z(u)+2\sigma_z(u)^2}.$$

Also notice that, for any $u \in [0,t]$,

$$\begin{aligned}
\sigma_z(u)^2 &= b^2\sigma_Y(u)^2 + c^2(T-u) = b^2\int_0^u \sigma(s)^2 ds + c^2(T-u) \le b^2 t\epsilon_2^2 + c^2T =: \sigma_z^*(t)^2, \\
\mu_z(u) &= b\mu_Y(u) = -\frac{b}{2}\sigma_Y(u)^2 \le \frac{|b|}{2}\int_0^u \sigma(s)^2 ds \le \frac{|b|}{2}t\epsilon_2^2 =: \mu_z^*(t), \\
F(u) &= e^{\mu_z(u)+\frac{1}{2}\sigma_z(u)^2} \le e^{\mu_z^*(t)+\frac{1}{2}\sigma_z^*(t)^2} =: F^*(t), \\
\Delta_1(u) &\le (R(0)+q(0)-s(0))F^*(t) + (R(0)-s(0))F^*(t) =: \Delta_1^*(t), \\
\Delta_2(u) &\le (2R(0)+q(0)-2s(0))F^*(t) =: \Delta_2^*(t), \\
\Delta_3(u) &\le s(0)\gamma + (R(0)+q(0)-s(0))\gamma + p(0)\frac{\gamma}{P_{TIPS}(0,T)} + \hat{W}_0 + \frac{1}{2\kappa}e^{\epsilon_1^2 t} + V_0^{(\gamma)} =: \Delta_3^*(t).
\end{aligned}$$

Hence

$$\begin{aligned}
E[\upsilon_u^2] \le\ & (b^2\epsilon_2^2\Delta_1^*(t)^2 + \epsilon_1^2\Delta_2^*(t)^2 + 2|b|\epsilon_1\epsilon_2\Delta_1^*(t)\Delta_2^*(t))a^2e^{2(-b\log R(0)+\mu_z^*(t)+\sigma_z^*(t)^2)} \\
&+ 2(\epsilon_1^2\Delta_2^*(t)\Delta_3^*(t) + |b|\epsilon_1\epsilon_2\Delta_1^*(t)\Delta_3^*(t))ae^{-b\log R(0)}F^*(t) + \epsilon_1^2\Delta_3^*(t)^2 \\
&=: \Upsilon_t.
\end{aligned}$$

Now, Υ_t is bounded; hence, $\Upsilon_t^* \le \Upsilon_t$ is also bounded. This finishes the proof. □

Proof of theorem 3.3.8.

Proof. We compute

$$
\begin{aligned}
&E[(G_t^* - \widetilde{G}_t)^2] \\
&= E\left[\left(\int_0^t \frac{\mu(s)}{\sigma(s)^2 X(s)} G_s^* dX(s)\right)^2\right] \\
&= E\left[\left(\int_0^t \frac{\mu(s)}{\sigma(s)^2 X(s)} G_s^* (\mu(s)X(s)ds + \sigma(s)X(s)dW(s))\right)^2\right] \\
&\leq 2E\left[\left(\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^* ds\right)^2\right] + 2E\left[\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^{*2} ds\right] \\
&\leq 2E\left[\int_0^t t\frac{\mu(s)^4}{\sigma(s)^4} G_s^{*2} ds\right] + 2E\left[\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^{*2} ds\right] \\
&\leq 2\epsilon_1^2(1 + t\epsilon_1^2)\int_0^t E\left[G_s^{*2}\right] ds.
\end{aligned}
$$

Here we used the inequality $(a+b)^2 \leq 2(a^2+b^2)$, Itô isometry, and Jensen's inequality. Notice that G_s^* is the solution of a linear stochastic differential equation, which is given by

$$
G_t^* = -Z_t \int_0^t \frac{\upsilon_u}{Z_u}\big(2\eta(u)du + dW(u)\big)
$$

where

$$
Z_t = \exp\left(-\frac{3}{2}\int_0^t \eta(u)^2 du - \int_0^t \eta(u)dW(u)\right)
$$

and

$$
\upsilon_t = \vartheta_t^{(\gamma)}\sigma(t)X(t) + \eta(t)(V_t + \hat{W}_0 - \lambda).
$$

Let $\frac{d\bar{P}}{dP} = e^{-\int_0^t 2\eta(u)dW(u) - \int_0^t 2\eta(u)^2 du}$. Then $d\bar{W}(t) = 2\eta(t)dt + dW(t)$ is a $\bar{P}$-Brownian motion by Girsanov's theorem. Hence

$$
\begin{aligned}
E\left[G_t^{*2}\right] &= E\left[e^{-\int_0^t 2\eta(u)dW(u)-\int_0^t 3\eta(u)^2du}\left(\int_0^t \frac{\upsilon_u}{Z_u}\big(2\eta(u)du + dW(u)\big)\right)^2\right] \\
&= e^{-\int_0^t \eta(u)^2du}\bar{E}\left[\left(\int_0^t \frac{\upsilon_u}{Z_u}d\bar{W}(u)\right)^2\right] \\
&= e^{-\int_0^t \eta(u)^2du}\bar{E}\left[\int_0^t \frac{\upsilon_u^2}{Z_u^2}du\right] \\
&= e^{-\int_0^t \eta(v)^2dv}\int_0^t e^{\int_0^u \eta(v)^2dv}\bar{E}\left[e^{\int_0^t 2\eta(u)d\bar{W}(u)-\int_0^u 2\eta(v)^2dv}\upsilon_u^2\right]du \\
&= \int_0^t e^{-\int_u^t \eta(v)^2dv}E\left[\upsilon_u^2\right]du.
\end{aligned}
$$

Let $\Upsilon_t^* = \sup_{u\in[0,t]} E[\upsilon_u^2]$ as proved in proposition 3.5.3. In combination with the last estimate we obtain

$$
\begin{aligned}
E[(G_t^* - \widetilde{G}_t)^2] &\le 2\epsilon_1^2(1+t\epsilon_1^2)\int_0^t\int_0^s e^{-\int_u^s \eta(v)^2dv}E\left[\upsilon_u^2\right]duds \\
&\le \epsilon_1^2(1+t\epsilon_1^2)t^2\Upsilon_t^*.
\end{aligned}
$$

□

CHAPTER 4

MULTI-PERIOD, MULTI-PRODUCT SEPARATION RESULT FOR INVENTORY MANAGEMENT UNDER FINANCIAL RISK

4.1 Introduction

Decision problems under risk aversion have been widely studied in operations management. While there are various criteria for risk aversion, one of the most common approaches is to optimize the tradeoff between the variance of return and the expected return; for example, Choi et al. (2008) and Wu et al. (2009) discuss the mean-variance type of operational decision problem. A recent research interest is to implement financial hedging in operations management. In our paper, we consider a non-financial corporation that dynamically hedges its profits when these profits are correlated with financial markets. In this framework, there are two types of risk for the non-financial corporation: financial risk and non-financial risk. The financial risk comes from the financial market and can be hedged using financial instruments. The non-financial risk is assumed to be observable but cannot be hedged through financial trading. This framework allows us to apply tools from the theory of hedging in an incomplete market.

In this paper, we consider a corporation which aims to simultaneously solve an operational and a financial decision problem. Inventory management provides an important example of such a problem. The classical inventory management problem optimizes an inventory decision for a stochastic future demand variable. If the demand is affected by a financial market risk, this risk can be partially hedged via trading in the market. The corporation is then faced with a combined optimal inventory decision and optimal hedging problem. We assume that there is an inventory department and a financial department in the non-financial corporation. The financial department implements dynamic hedging in the financial markets, while the inventory department makes inventory decisions periodically. We refer to Caldentey & Haugh (2006), Caldentey & Haugh (2009), Gaur & Seshadri (2005) and Sun et al. (2011) for similar hedging models in inventory management.

We employ the tool of financial hedging in an incomplete market from the literature in mathematical finance. Although there is recent progress in solving hedging problems with general utility function objectives, we focus on the mean-variance objective for tractability of hedging problems. Schweizer et al. (1999) provides a thorough overview of the quadratic hedging results.

For a large corporation carrying different type of products, multiple inventory decisions have to be made by different departments. The decisions are naturally interconnected if the demands for different products are exposed to the same financial risk to some extent. Optimizing the risk-return tradeoff across a portfolio of products is a daunting task, especially considering that the number of products can easily run into the tens of thousands and more. Consequently, it is our goal to develop models or approximations which allow the optimization problem to be separated and solved one product at a time. For a single-period decision model, Sun et al. (2011) achieve such a separation result. In this paper, we extend the model to a multi-period case. A sequence of inventory decisions needs to be made at the beginning of each period by inventory managers, while the financial department executes a global dynamic hedging strategy throughout time. The main contribution of this paper is that we achieve a global separation result for the multiple-product, multiple-period operational decision problem. More specifically, the inventory decisions for a particular product are independent of other products decision processes. We refer to Karmarkar (1987) for a discussion of the multi-period inventory model.

As mentioned, our work is a multi-period extension of Sun et al. (2011). Sun et al. (2011) concentrate on the case in which prices and demand are affected by inflation, and use a financial market model for inflation-related securities. That approach makes critical use of the Föllmer-Schweizer (F-S) decomposition of hedging strategies. We use a general financial market model in our paper to allow for various economic applications. We also consider a general demand model in our paper as opposed to the exponential demand model in Sun et al. (2011).

The separation result reduces the global optimization problem to a dynamic programming problem for each product. This dynamic program can be difficult to solve. We propose a Fast Fourier Transformation approach such that, provided a density function of the demand for each period is available, the problem can be solved in an efficient way. In addition, the approach we suggest enables us to find the F-S decomposition for a discretized value function as opposed to an analytical value function as in Sun et al. (2011).

The paper is organized as follows. In section 2 we review the separation result for single-period, multiple-product inventory management problem. Formulation of the multi-period extension of the problem can be found in section 3. Our main separation result is presented in section 4. In section 5 we give a tractable numerical scheme for implementation. The proofs can be found in section 6, also referred to as the Appendix.

4.2 Separation result for hedging in inventory management with multi-product and single period

In this section we introduce the previously established model and results for inventory management hedging with multiple products and a single period. This is the building block for the multi-period model.

4.2.1 Model setup

Let $T^* \in (0, \infty)$ be the fixed time horizon, consider the probability space $(\Omega, \mathcal{F}, P) = (\Omega^W \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$ endowed with two independent Brownian motions: a 1-dimensional Brownian motion $W(t)$ and a N-dimensional Brownian motion $\mathcal{B}(t) = (\mathcal{B}^1(t), \ldots, \mathcal{B}^N(t))$, $t \in [0, T^*]$. The space Ω^W represents the randomness of the financial instrument which will be used for hedging, and E represents the non-financial noise which affects the market.

The financial market that we consider consists of a riskless and a risky asset with prices $P(t)$ and $S(t)$, respectively. The riskless asset will be used as numeraire to discount all value and price processes. In other words, the price of riskless asset P is equal to 1 with numeraire P, and the price of risky asset S is $X(t) = \frac{S(t)}{P(t)}$, which is assumed to satisfy the stochastic differential equation:

$$\frac{dX(t)}{X(t)} = \mu_t dt + \sigma_t dW(t) \tag{4.1}$$

where μ_t and σ_t are assumed to be bounded adapted processes. Furthermore, we assume that the so-called mean-variance trade-off $\eta_t := \mu_t / \sigma_t$ is a bounded and deterministic function.

Remark: The model we propose here for the financial market is general. In Sun et al. (2011), the riskless asset P is taken to be a TIPS bond, and the risky asset S to be a nominal dollar bond.

Let the set Θ be the family of self-financing trading strategies, with $\theta \in \Theta$ being $\mathcal{F}^W \otimes \mathcal{E}$-predictable processes such that for all $T \leq T^*$

$$E\left[\int_0^T \theta_t^2 X(t)^2 dt\right] < \infty \quad P-a.s. \tag{4.2}$$

The trading variable, θ_t, denotes the number of shares in the risky asset $X(t)$ held at time t. The corresponding (discounted) gain process is defined as

$$G_t(\theta) := \int_0^t \theta_s dX(s) \quad \text{ for all } t \in [0, T].$$

For a fixed $T \leq T^*$, let $D(t) = (D^1(t), \ldots, D^N(t))$, $t \in [0, T]$ be the stochastic demand of N different products with

$$\boldsymbol{D}^j(t) = f^j(X(t), \mathcal{B}^j(t)), t \in [0, T], j = 1, \ldots, N \tag{4.3}$$

with initial demand $D^0 = (D^{0,1}, \ldots, D^{0,N}) = f(X(0), \mathcal{B}(0))$ at time 0 and a realized demand $\boldsymbol{D}_T = (D_T^1, \ldots, D_T^N) = (D^1(T), \ldots, D^N(T))$ at the end of period T.

There are two sources of randomness in the demand process: a risky financial asset X and a non-financial noise $\mathcal{B}$. Furthermore, the non-financial noise $\mathcal{B}(t) = (\mathcal{B}^1(t), \ldots, \mathcal{B}^N(t))$ is considered to be observable.

Remark: The assumption (4.3) for the demand together with the independence of X and $\mathcal{B}$ and the mutual independence of $\mathcal{B}^j$ is crucial to our result. It enables us to use a tradable financial asset to hedge the financial risk, in a way that allows separate solutions for individual inventory decision problems. See corollary 3.3.5 below.

We consider a risk-averse non-financial corporation which plans over the period $[0, T]$. At time $t = 0$ the operations manager makes the inventory decision

$\boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^N)$, which is $\mathcal{F}_0^W \otimes \mathcal{E}_0$-predictable. With an initial inventory level $\boldsymbol{x}_0 = (x_0^1, \ldots, x_0^N)$, a discounted payoff for all products $H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T)$ will be realized at T

$$H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T) = \sum_{j=1}^{N} H_T(\gamma^j; x_0^j, D_T^j). \tag{4.4}$$

During the period $[0, T]$, the financial department of the corporation implements a dynamic hedging strategy with risky asset $X(t)$. Notice that $H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T)$ is a $\mathcal{F}_T^W \otimes \mathcal{E}_T$-measurable random variable, and $\sigma(X(t) \,|\, 0 \leq t \leq T) \subsetneq \mathcal{F}_T^W \otimes \mathcal{E}_T$, hence we are dealing with an incomplete market. Intuitively, this is because the non-financial risk in the stochastic demand $\boldsymbol{D}(t)$ cannot be hedged by trading $X(t)$.

Starting with an initial discounted wealth ω_0, the payoff from the operational and financial activities of the corporation at time T is

$$Y_T(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T) := \omega_0 + H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T) + G_T(\theta).$$

The optimization problem we are interested in is

$$U = \max_{\gamma, \theta} \left(E\left[Y_T(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T) \right] - \kappa Var\left[Y_T(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T) \right] \right). \tag{4.5}$$

4.2.2 Separation result for multi-product single period model

One difficulty in problem (4.5) is that the operational decisions of different products are inter-connected. This is due to the fact that all demand processes depend on the risky financial asset, possibly to different extent. The main contribution of Sun et al. (2011) is to provide a separation result of the multi-product problem such that the inventory decision of each product can be made separately. In this section, we review the main results.

Our objective is to solve the global optimization problem (4.5), which involves optimizing over operational and financial decisions simultaneously. As a first step, let us consider the following problem for the fixed operational decision $\gamma \in \Gamma$

$$U^{\gamma} = \sup_{\theta \in \Theta} \left(E\left[Y_T(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T) \right] - \kappa Var\left[Y_T(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T) \right] \right) \tag{4.6}$$

which is equivalent to

$$U^{\gamma} = \sup_{m \in \mathbb{R}} \left(m - \kappa B^{\gamma}(m) \right), \tag{4.7}$$

with

$$B^{\gamma}(m) = \inf_{\theta \in \Theta} \left\{ Var\left[Y_T^{(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T)} \right] \mid E\left[Y_T^{(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{x}_0, \boldsymbol{D}_T)} \right] = m \right\}, \quad \text{for each } m \in \mathbb{R}. \tag{4.8}$$

We also define the auxiliary problem as

$$A_T^{\gamma}(\lambda) = \inf_{\theta \in \Theta} E\left[(Y_T^{(\boldsymbol{\gamma}, \theta; \omega_0, \boldsymbol{D}_0, \boldsymbol{x}_0)} - \lambda)^2 \right] \quad \text{for each } \lambda \in \mathbb{R}. \tag{4.9}$$

The following theorem states a duality result between (4.9) and (4.8).

Theorem 4.2.1 ((Sun et al. 2011, Theorem 1))**.** *With $A_T^{\gamma}(\lambda)$ and $B^{\gamma}(m)$ defined as in (4.9) and (4.8), we have*

$$B^{\gamma}(m) = \sup_{\lambda \in \mathbb{R}} \left(A^{\gamma}(\lambda) - (m - \lambda)^2 \right) \tag{4.10}$$

and with λ_m the optimizer in (4.10), the optimal control in $B^{\gamma}(m)$ is equal to the optimal control in $A^{\gamma}(\lambda)$ with $\lambda = \lambda_m$.

According to theorem 3.3.1, for fixed operational decision γ, solving (4.6) is equivalent to finding the optimal control $\theta \in \Theta$ for (4.9) and then optimizing over λ. Fortunately, there are well-established results for problem (4.9) in the quadratic hedging literature (see Černỳ & Kallsen (2007) and Schweizer et al. (1999) for an overview). The following subsection formulates the main results from this literature in the context of our model.

Quadratic hedging problem for multi-product

The objective function of a quadratic hedging problem is

$$\max_{(\gamma,\theta)\in\Gamma\times\Theta} E[u(\omega_0 + H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T) + G_T(\theta))] \tag{4.11}$$

with $u(w) = w - lw^2$, or equivalently

$$\min_{(\gamma,\theta)\in\Gamma\times\Theta} E\left[(\omega_0 + H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T) + G_T(\theta) - \lambda)^2\right]$$

with $\lambda = \frac{1}{2l}$. We first fix the operational decision $\gamma \in \Gamma$ and find the optimal hedging strategy for

$$\min_{\theta\in\Theta} E\left[(\omega_0 + H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T) + G_T(\theta) - \lambda)^2\right]. \tag{4.12}$$

Under the assumption that the market price of risk, η_t, is a bounded and deterministic function, (4.12) can be solved using the minimal equivalent martingale measure(MEMM) which is defined by

$$\frac{d\hat{P}}{dP} := \exp\left\{\int_0^T \eta_t dW(t) - \frac{1}{2}\int_0^T \eta^2(t)dt\right\}. \tag{4.13}$$

It can be proved by Girsanov's theorem, as in Shreve (2004), that both X and $\mathcal{B}$ are square-integrable martingales under $\hat{P}$. Denote the expectation under measure $\hat{P}$ as $\hat{E}[\cdot]$. We have the following theorem summarizing the key result on the quadratic hedging problem.

Theorem 4.2.2 ((Sun et al. 2011, Theorem 2)). *For any $\mathcal{F}_T$-measurable claim $H_T^D(\gamma^j; x_0^j, D_T^j) \in \mathfrak{L}^p(P)$, $j = 1, \ldots, N$ for some $p > 2$, there is a hedging strategy, $\vartheta^{(\gamma^j;x_0^j,D_T^j)}$, and a process $\delta^{(\gamma^j;x_0^j,D_T^j)} \in \mathfrak{L}^2(P)$, such that $H_T(\gamma^j; x_0^j, D_T^j)$ admits the decomposition*

$$H_T(\gamma^j; x_0^j, D_T^j) = V_0^{(\gamma^j;x_0^j,D^{0,j})} + \int_0^T \vartheta_t^{(\gamma^j;x_0^j,D_T^j)} dX(t) + \int_0^T \delta_t^{(\gamma^j;x_0^j,D_T^j,)} d\mathcal{B}_j(t) \tag{4.14}$$

where $V_0^{(\gamma^j;x_0^j,D^{0,j})} := \hat{E}[H_T(\gamma^j;x_0^j,D_T^j)]$.

As a result, $H_T(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T) = \sum_{j=1}^N H_T(\gamma^j;x_0^j,D_T^j)$ *admits the decomposition*

$$H_T(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T) = V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}^0)} + \int_0^T \vartheta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} dX(t) + \int_0^T \delta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} d\overline{\mathcal{B}}(t) \tag{4.15}$$

with

$$V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}^0)} = \sum_{j=1}^N V_0^{(\gamma^j;x_0^j,D^{0,j})} \tag{4.16}$$

$$\vartheta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} = \sum_{j=1}^N \vartheta_t^{(\gamma^j;x_0^j,D_T^j)} \tag{4.17}$$

$$\delta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} = \sqrt{\sum_{j=1}^N \left(\delta_t^{(\gamma^j;x_0^j,D_T^j)}\right)^2} \tag{4.18}$$

$$\overline{\mathcal{B}}(t) = \int_0^t \frac{1}{\delta_s^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}} \sum_{j=1}^N \delta_s^{(\gamma^j;x_0^j,D_T^j,)} d\mathcal{B}_j(s) \tag{4.19}$$

with $\overline{\mathcal{B}}(t)$ *a Brownian motion under* P *and* $\hat{P}$.

In addition, the optimal strategy θ^* *that solves (4.12) is given by* $\theta^* = \Phi(G_t^*)$, *where* G_t^* *is the solution to the SDE*

$$dG_t^* = -\Phi(G_t^*)dX(t) \tag{4.20}$$

with $G_0^* = 0$ *and* $\Phi(G_t^*) = \vartheta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} + \mu_t/(\sigma_t^2 X(t))(V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} + G_t^* + \omega_0 - \lambda)$, *and* $V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}$ *is the intrinsic value process defined by*

$$V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} := \hat{E}[H_T^D(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)|\mathcal{F}_t] = V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} + \int_0^t \vartheta_s^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} dX(s) + \int_0^t \delta_s^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{B}_T)} d\overline{\mathcal{B}}(s). \tag{4.21}$$

The decomposition (4.21) is known as the Galtchouk-Kunita-Watanabe(GKW) decomposition of $V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}$ *under* $\hat{P}$ *with respect to* X.

The Galtchouk-Kunita-Watanabe(GKW) decomposition under $\hat{P}$ in the theorem is also known as the Föllmer-Schweizer(F-S) decomposition of $H_T(\boldsymbol{\gamma}; \boldsymbol{x}_0, \boldsymbol{D}_T)$ with respect to the semimartingale X under P.

The following theorem gives an explicit expression for the optimal value in (4.12) by solving the optimal hedging strategy.

Theorem 4.2.3 ((Sun et al. 2011, Theorem 3)). *Define the auxiliary process*

$$A_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} := E[(V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)} + G_t^* + \omega_0 - \lambda)^2] \tag{4.22}$$

and $K_t = \int_0^t \eta(s)^2 ds$, *then* $A_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}$ *is given by*

$$\begin{aligned} A_T^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}(\lambda) &= e^{-K_T}\left(\left(\omega_0 + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)} - \lambda\right)^2 + \int_0^T e^{K_s} E\left[(\delta_s^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)})^2\right] ds\right) \\ &= e^{-K_T}\left(\left(\omega_0 + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)} - \lambda\right)^2 + \int_0^T e^{K_s} \sum_{j=1}^N E\left[(\delta_s^{(\gamma^j;x_0^j,D_T^j)})^2\right] ds\right). \end{aligned}$$

The first application of theorem 4.2.3 is to note that if we can find the intrinsic value $V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)}$ and the F-S decomposition term $\delta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}$, the auxiliary process can be obtained in a form that does not involve the optimal hedging strategy. Secondly, notice that the value in problem (4.9) equals $A_T^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}$. Since the optimizer of (4.9) exists, we can replace the *inf* and *sup* with *min* and *max* in (4.9), (4.6), (4.8) and (4.10) from now on. Let m_{opt} denote the optimizer of 4.7.

The following theorem is the main result of the multi-product optimal hedging inventory management problem. It solves the optimal control to the multi-product problem. It also achieves the separation of the original problem (4.5) by product.

Theorem 4.2.4 ((Sun et al. 2011, Theorem 4)). *The optimizer and the corresponding* *optimal value of problem (4.10) is*

$$\lambda_m = \frac{m - e^{-K_T}(\omega_0 + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)})}{1 - e^{-K_T}} \tag{4.23}$$

$$B^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}(m) = \frac{e^{-K_T}}{1 - e^{-K_T}}(\omega_0 + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)} - m)^2 \tag{4.24}$$

$$+ e^{-K_T}\int_0^T e^{K_u}\sum_{j=1}^N E\left[(\delta_s^{(\gamma^j;x_0^j,D_T^j)})^2\right] ds.$$

The optimizer m_{opt} of problem (4.7) is given by

$$m_{opt} = \frac{1}{2\kappa}\frac{1 - e^{-K_T}}{e^{-K_T}} + \omega_0 + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)} \tag{4.25}$$

and the optimal value in problem (4.6) is

$$\begin{aligned} U^\gamma &= \omega_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)} - \kappa e^{-K_T}\int_0^T e^{K_s}\sum_{j=1}^N E\left[(\delta_s^{(\gamma^j;x_0^j,D_T^j)})^2\right] ds \\ &= \omega_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + \sum_{j=1}^N V_0^{(\gamma^j;x_0^j,D_0^j)} - \kappa e^{-K_T}\int_0^T e^{K_s}\sum_{j=1}^N E\left[(\delta_s^{(\gamma^j;x_0^j,D_T^j)})^2\right] ds. \end{aligned} \tag{4.26}$$

Finally the optimal control $\boldsymbol{\gamma}$ in (4.5) can be found by maximizing (4.26) over $\boldsymbol{\gamma}$.

Finally, we state the corollary which gives the separated optimization objective function.

Corollary 4.2.5 ((Sun et al. 2011, Corollary 1)). *With U^γ defined as in (4.6), the problem*

$$\max_{\gamma} \ U^{\gamma} \tag{4.27}$$

is equivalent to solving

$$\max_{\gamma_j}\left(V_0^{(\gamma^j;x_0^j,D_0^j)} - \kappa e^{-K_T}\int_0^T e^{K_s} E\left[(\delta_s^{(\gamma^j;x_0^j,D_T^j)})^2\right] ds\right) \tag{4.28}$$

for each $j = 1, \ldots, N$.

We refer to Sun et al. (2011) for the proof of the existence of a solution to problem (4.28). In practice, the optimal inventory decision can be solved numerically for each γ^j

once the intrinsic value $V_t^{(\gamma^j;x_0^j,D_T^j)}$ and the F-S decomposition $\delta_t^{(\gamma^j;x_0^j,D_T^j)}$ are obtained. The optimal hedging strategy $\theta^*(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)$ can be computed via solving the feedback form SDE:

$$\begin{aligned}&\theta_t^*(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)\\ &=-\left(\vartheta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}+\mu_t/(\sigma_t^2X(t))(V_t(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)+G_t^*+\omega_0-\frac{1}{2\kappa}e^{K_T}-V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)})\right)\end{aligned}\tag{4.29}$$

where G_t^* is the solution of the stochastic differential equation (SDE):

$$dG_t^*=-\left[\vartheta_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}+\mu_t/(\sigma_t^2X(t))\left(V_t^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}+G_t^*-\frac{1}{2\kappa}e^{K_T}-V_0^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_0)}\right)\right]dX(t)\tag{4.30}$$

with $\vartheta^{(\boldsymbol{\gamma};\boldsymbol{x}_0,\boldsymbol{D}_T)}=\sum_{j=1}^N\vartheta^{(\gamma^j;x_0^j,D_T^j)}$.

4.3 Optimal hedging inventory management problem for multi-period, multi-product

In this section, we use the results of single-period problem as building blocks to solve a multi-period, multi-product optimal hedging problem in inventory management. The setting we use is similar to the one in the single period, and we aim at achieving a separation result by product. In other words, instead of solving the dynamic programming problem of a product portfolio, we prove that the global optimization problem is equivalent to solving N independent dynamic programming problems corresponding to each product.

Consider T periods with the i-th period $[t_i,t_{i+1})$, with $t_0=0$ and $t_T=T^*$. Let γ_i^j be the inventory purchase of product j at the time t_i, $i=0,\dots,T-1$, $j=1,\dots,N$.

The operational decision matrix is defined as

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_0^1 & \cdots & \gamma_0^N \\ \vdots & \ddots & \vdots \\ \gamma_{T-1}^1 & \cdots & \gamma_{T-1}^N \end{pmatrix}_{T\times N}$$

where $\boldsymbol{\gamma}_i = (\gamma_i^1, \ldots, \gamma_i^N)$ is $\mathcal{F}_{t_i}^W \otimes \mathcal{E}_{t_i}$-measurable, with γ_i^j being the inventory purchase for product j at time t_i.

Let $\theta(t)$, $t \in [0, T^*]$ be the $\mathcal{F}^W \otimes \mathcal{E}$-predictable continuous hedging strategy process. For simplicity of notation, denote by $\theta_i(t)$ the restriction of the process $\theta(t)$ to the interval $(t_i, t_{i+1}]$.

Denote the gains from trading during $[0, t]$ as $G_t(\theta)$; the gain during period $[t_i, t_{i+1}]$ is

$$G_{t_{i+1}}(\theta) - G_{t_i}(\theta) = \int_{t_i}^{t_{i+1}} \theta_i(s) dX(s). \tag{4.31}$$

Let $\boldsymbol{D}_i = \boldsymbol{D}_i(t_i) = (D_i^1(t_i), \ldots, D_i^N(t_i)) = (D_i^1, \ldots, D_i^N)$ be total demand during time interval $[t_{i-1}, t_i]$. Define a process $\boldsymbol{D}_i(t) = (D_i^1(t), \ldots, D_i^N(t))$, $t \in [t_{i-1}, t_i)$ as the $\mathcal{F}^W \otimes \mathcal{E}$-adapted demand process. $D_i^j(t)$ is related to the time-t-projection of $D_i^j(t_i)$ via

$$E[D_i^j(t_i)|\mathcal{F}_t] = \bar{f}(t, D_i^j(t)) \tag{4.32}$$

and we assume $\bar{f}$ is a deterministic function which depends on the distribution of $D_i^j(t_i)$. In (4.59), below, we provide an example for the structure of $D_i^j(t_i)$, and an explicit formula for $\bar{f}$.

When solving the multi-period problem via a dynamic programming approach, we shall need the quantity $D_i^{0,j}$ which is defined as

$$D_i^{0,j} := D_i^j(t_{i-1}).$$

That is, $D_i^{0,j}$ is the projection (or forecast) made at time t_{i-1} for the total demand for product j to be realized during time interval $[t_{i-1}, t_i]$. Notice that by definition, we have

$$D_i^j := D_i^j(t_i).$$

During the period $[t_i, t_{i+1})$, $i = 0, \ldots, T-1$, the state variables are

- ω_i: wealth at time t_i.
- $D_{i+1}^{0,j} = D_{i+1}^j(t_i)$: is the demand of product j at time t_i.
- x_i^j: inventory position at time t_i of product j before purchase, but after observation of D_i^j.

Similar to the decision variables notation, denote the demand vector and inventory position vector of the i-th period for all products as $\boldsymbol{D}_{i+1}^0$ and $\boldsymbol{x}_i$ respectively.

The gain from inventory activities during $[t_i, t_{i+1}]$ for product j is of the form $H_{i+1}^j(\gamma_i^j; x_i^j, D_{i+1}^j)$ with inventory decision γ_i^j at time t_i. The total gain for the corporation during $[t_i, t_{i+1}]$ is $H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = \sum_{j=1}^N H_{i+1}^j(\gamma_i^j; x_i^j, D_{i+1}^j)$.

The state variables have dynamics:

$$\omega_i = \omega_{i-1} + H_i\left(\boldsymbol{\gamma}_{i-1}; \boldsymbol{x}_{i-1}, \boldsymbol{D}_i\right) + \int_{t_{i-1}}^{t_i} \theta_{i-1}(s)dX(s) \tag{4.33}$$

and

$$x_i^j = (x_{i-1}^j + \gamma_{i-1}^j - D_i^j)^+ \tag{4.34}$$

for $i = 1, \ldots, T-1$ with initial wealth ω_0, demand process initial value $D_1^{0,j}$, and the initial inventory position x_0^j, $j = 1, \ldots, N$. Note in (4.34) we have assumed a lost sales model; that is, demand in excess of available inventory is lost. We also assume that the stock acquisition lead time is less than one period in duration.

Assume that all inventory at the end of time T will be sold at a salvage price. Let $\boldsymbol{s}_T = (s_T^1, \ldots, s_T^N)$ be the vector indicating the (discounted) unit salvage value at time T. The boundary condition is then

$$\omega_T = \omega_{T-1} + H_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) + \int_{t_{T-1}}^{t_T} \theta_{T-1}(s)dX(s) + \boldsymbol{s}_T \boldsymbol{x}_T^{\intercal}. \tag{4.35}$$

Finally, fix a risk-aversion coefficient, $\kappa > 0$. The corporation's goal is to solve the mean-variance optimization problem

$$U(\kappa) = \max_{\boldsymbol{\Gamma},\theta}\left(E[\omega_T] - \kappa Var[\omega_T]\right). \tag{4.36}$$

Note that we are considering a global optimization over financial hedging strategies θ and inventory decisions $\boldsymbol{\Gamma}$ on the time interval $[0, T^*]$. The inventory strategy is defined over discrete points in time, $t \in [t_0, t_1, \ldots, t_{T-1}]$ whereas the hedging strategy is continuous over $[0, T^*]$.

4.4 Separation result for multi-period, multi-product model

In this section, we state a separation result for the multi-period mean-variance optimization problem.

Let $\boldsymbol{\Gamma}_i = (\boldsymbol{\gamma}_i, \ldots, \boldsymbol{\gamma}_{T-1})$ be a $\sigma(\omega_i, x_i, D_{i+1}^0, (X_t, \mathcal{B}_t)_{t \geq t_i})$-measurable inventory decision vector for the time interval $[t_i, T^*]$, $i = 0, \ldots, T-1$. For fixed inventory decision $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0$, consider the auxiliary problem:

$$A^{(\lambda,\boldsymbol{\Gamma})} = \min_{\theta} E[(\omega_T - \lambda)^2], \tag{4.37}$$

and denote by $\hat{E}_i[\cdot]$ the conditional expectation $\hat{E}[\cdot|\mathcal{F}_{t_i}]$.

For fixed $\mathbf{\Gamma}$, define $A_T^{(\lambda)} := (\omega_T - \lambda)^2$ and for any period i, $i = T-1, \ldots, 0$, define recursively

$$A_i^{(\lambda,\mathbf{\Gamma}_i)}\left(\omega_i, \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0\right) = \min_{\boldsymbol{\theta}_i} E_i\left[A_{i+1}^{(\lambda,\mathbf{\Gamma}_{i+1})}(\omega_{i+1}, \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+1})\right] \tag{4.38}$$

$$H_i^0(\boldsymbol{\gamma}_{i-1}; \boldsymbol{x}_{i-1}, \boldsymbol{D}_i^0) = \hat{E}_{i-1}[H_i(\boldsymbol{\gamma}_{i-1}; \boldsymbol{x}_{i-1}, \boldsymbol{D}_i)] \tag{4.39}$$

$$F_l^i(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) = \hat{E}_i[H_{l+1}^0(\boldsymbol{\gamma}_l; \boldsymbol{x}_l, \boldsymbol{D}_{l+1}^0)], \quad l \geq i+1 \tag{4.40}$$

$$\bar{F}_l^{i+1}(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = \hat{E}_{i+1}[H_{l+1}^0(\boldsymbol{\gamma}_l; \boldsymbol{x}_l, \boldsymbol{D}_{l+1}^0)], \quad l \geq i+1 \tag{4.41}$$

$$\Delta_l^i(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0)^2(s) = E_i[\delta_l^{(\boldsymbol{\gamma}_l;\boldsymbol{x}_l,\boldsymbol{D}_{l+1})}(s)^2] \tag{4.42}$$

According to Schweizer (1992), there exists the F-S decomposition for

$$V_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) \tag{4.43}$$

conditional on $\mathcal{F}_{t_i}$. Let $\delta_i^{(\mathbf{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)$ be the corresponding integrand in the orthogonal component of the F-S decomposition for the non-financial noise.

Let $\delta_i^{\left(\Gamma_i^j; x_i^j, D_{i+1}^j\right)}(s)$ be the orthogonal component of the F-S decomposition of

$$V_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) = H_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}\left(\Gamma_i^j; x_i^j, D_{i+1}^j\right) \tag{4.44}$$

conditional on $\mathcal{F}_{t_i}$ corresponding to product j. We have the following theorem:

Theorem 4.4.1. *(a) For fixed* Γ *and for any period* $i \in \{0, 1, \ldots, T-1\}$*, we have*

$$\begin{aligned} &A_i^{(\lambda,\Gamma_i)}\left(\omega_i, \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0\right) \\ &= e^{-\int_{t_i}^{t_T} \eta_t^2 dt}\left(\omega_i + H_{i+1}^0(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) + \sum_{l=i+1}^{T-1} F_l^i(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) - \lambda\right)^2 \\ &+ e^{-\int_{t_i}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} E_i\left[\delta_i^{(\mathbf{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)^2\right] ds \\ &+ \sum_{l=i+1}^{T-1} e^{-\int_{t_i}^{t_{l+1}} \eta_t^2 dt} \int_{t_l}^{t_{l+1}} e^{\int_{t_l}^{s} \eta_v^2 dv} \Delta_l^i(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0)^2(s) ds; \end{aligned} \tag{4.45}$$

and

(b)

$$E_i[\delta_i^{(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)^2] = \sum_{j=1}^{N} E_i[\delta_i^{(\Gamma_i^j;\boldsymbol{x}_i^j,D_{i+1}^j)}(s)^2]$$

with

$$H_{i+1}(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}) = \sum_{j=1}^{N} H_{i+1}(\gamma_i^j;x_i^j,D_{i+1}^j).$$

The proof can be found in the Appendix.

Remark: In the theorem above, we need to obtain the integrand $\delta_i^{\left(\gamma_i^j;x_i^j,D_{i+1}^j\right)}(s)$ in the orthogonal component of the F-S decomposition (we refer to the integrand as the orthogonal component of the F-S decomposition) for (4.43). Since a backward induction method is used for dynamic programming, the value (4.43) has to be computed numerically. In section 5, we discuss a technique to handle the decomposition for a discretized value function as opposed to an analytical value function as in Sun et al. (2011).

Letting $i = 0$ in theorem 4.4.1, we can rewrite the global optimization problem as in the following corollary.

Corollary 4.4.2. *The global minimization problem (4.37) has optimal value*

$$A^{(\lambda,\Gamma)} = e^{-\int_{t_0}^{t_T}\eta_t^2 dt}\left(\omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0\left(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0\right)] - \lambda\right)^2 \tag{4.46}$$
$$+\sum_{i=0}^{T-1} e^{-\int_{t_0}^{t_{i+1}}\eta_t^2 dt}\int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s}\eta_v^2 dv} E[\delta_i^{(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)^2]ds$$

where $\delta_i^{(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)$ *is the orthogonal component of the F-S decomposition of*

$$V_{i+1}(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}) = H_{i+1}(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}) + \sum_{l=i+1}^{T-1}\bar{F}_l^{i+1}(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}) \tag{4.47}$$

with

$$\bar{F}_l^{i+1}(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}) = \hat{E}_{i+1}[H_{l+1}^0(\boldsymbol{\Gamma}_l;\boldsymbol{x}_l,\boldsymbol{D}_{l+1}^0)]. \tag{4.48}$$

The following theorem is the main result of this paper.

Theorem 4.4.3. *The optimal value and optimal* Γ *in (4.36) is given by*

$$U(\kappa) = \max_{\mathbf{\Gamma}} \left\{ \omega_0 + \sum_{i=0}^{T-1} \hat{E}[H^0_{i+1}\left(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}\right)] \right.$$
$$\left. -\kappa \sum_{i=0}^{T-1} e^{-\int_{t_0}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} E\left[\boldsymbol{\delta}_i^{(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}^0_{i+1})}(s)^2\right] ds \right\} \tag{4.49}$$

where $\delta_i^{(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1})}(s)$ *is the orthogonal component of the F-S decomposition conditional on* $\mathcal{F}_{t_i}$ *of*

$$V_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) \tag{4.50}$$

with

$$\bar{F}_l^{i+1}(\mathbf{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = \hat{E}_{i+1}[H^0_{l+1}(\boldsymbol{\gamma}_l; \boldsymbol{x}_l, \boldsymbol{D}^0_{l+1})]. \tag{4.51}$$

Product-wise, we have

$$H^0_{i+1}\left(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}^0_{i+1}\right) = \sum_{j=1}^{N} H^{0,j}_{i+1}\left(\gamma_i^j; x_i^j, D^{0,j}_{i+1}\right) \tag{4.52}$$

$$E[\boldsymbol{\delta}_i^{(\mathbf{\Gamma}_i; \boldsymbol{x}_i £ \boldsymbol{D}_{i+1})}(s)^2] = \sum_{j=1}^{N} E[\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}(s)^2] \tag{4.53}$$

with $\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}(s)$ *the orthogonal component conditional on* $\mathcal{F}_{t_i}$ *of the F-S decomposition of product* j *with respect to*

$$V_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) = H_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\Gamma_i^j; x_i^j, D_{i+1}^j). \tag{4.54}$$

The proof is based on corollary 4.2.5 and 4.4.2. See the Appendix.

The implication of theorem 4.4.3 is that the inventory optimization can be performed product by product. Let $\Gamma_i^j = (\gamma_i^j, \dots, \gamma_{T-1}^j)$, $i = 0, \dots, T-1$ be the inventory decision vector starting from time t_i for product j. The following corollary states the main result in a product-wise formulation.

Corollary 4.4.4. *The optimal inventory decision* $\Gamma^j = (\gamma_0^j, \ldots, \gamma_{T-1}^j)$ *is given by*

$$\max_{\Gamma^j} \left\{ \omega_0 + \sum_{i=0}^{T-1} \hat{E}[H_{i+1}^0 \left(\gamma_i^j; x_i^j, D_{i+1}^{0,j} \right)] \right.$$
$$\left. -\kappa \sum_{i=0}^{T-1} e^{-\int_{t_0}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} E\left[\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}(s)^2 \right] ds \right\} \tag{4.55}$$

where $\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}(s)$ *is the orthogonal component conditional on* $\mathcal{F}_{t_i}$ *of the F-S decomposition of*

$$V_{i+1}(\Gamma_i^j; x_i^j, D_{i+1}^j) = H_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\Gamma_i^j; x_i^j, D_{i+1}^j) \tag{4.56}$$

with

$$\bar{F}_l^{i+1}(\Gamma_i^j; x_i^j, D_{i+1}^j) = \hat{E}_{i+1}[H_{l+1}^0(\gamma_l^j; x_l^j, D_{l+1}^{0,j})]. \tag{4.57}$$

Remark: Notice that there are two different measures involved in problem (4.55): the real world measure P and the risk-neutral MEMM $\hat{P}$.

4.5 Solution via dynamic programming

In this section, we describe a dynamic programming approach to solve problem (4.55). In light of the separation result in corollary 4.4.4, we can focus on the single product case for simplicity of notation.

We assume a power function formulation for (4.3) and we assume that the non-financial noise in demand is memoryless. That is, for period i

$$D_{i+1}(t) = a_{i+1} e^{b_{i+1} \log X(t) + c_{i+1}(\mathcal{B}(t) - \mathcal{B}(t_i))}. \tag{4.58}$$

With this assumption, the $\bar{f}(\cdot,\cdot)$ function in (4.32) can be calculated as follows:

$$\begin{aligned}
&E\left[D_i(t_i)|\mathcal{F}_t\right]\\
&= E\left[a_i e^{b_i \log X(t)+c_i(\mathcal{B}(t)-\mathcal{B}(t_{i-1}))}|\mathcal{F}_t\right]\\
&= a_i e^{b_i(\log X(t_i)-\log X(t))+c_i(\mathcal{B}(t)-\mathcal{B}(t_{i-1}))} E[e^{b_i(\log X(t_i)-\log X(t))+c_i(\mathcal{B}(t_i)-\mathcal{B}(t))}]\\
&= D_i(t) e^{b_i \int_t^{t_i}\mu(s)ds+\frac{1}{2}\left(b_i^2\int_t^{t_i}\sigma^2(s)ds+c_i^2(t_i-t)\right)}\\
&=: \bar{f}(t, D_i(t))
\end{aligned} \tag{4.59}$$

since $b_i(\log X(t_i) - \log X(t)) + c_i(\mathcal{B}(t_i) - \mathcal{B}(t))$ is a normally distributed random variable with mean $b_i \int_t^{t_i}\mu(s)ds$ and variance $b_i^2 \int_t^{t_i}\sigma^2(s)ds + c_i(t_i - t)$.

We also assume that the financial asset follows a Black-Scholes model

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(t).$$

Notice that there are two different measures involved in problem (4.55): the real world measure P and the risk-neutral MEMM $\hat{P}$. To initialize the dynamic programming, we rewrite the objective function in terms of the risk-neutral MEMM $\hat{P}$. Recall the Radon-Nikodým derivative

$$Z = \frac{d\hat{P}}{dP} = e^{\eta W_{T^*}-\frac{1}{2}\eta^2 T^*}$$

with $\eta = \frac{\mu}{\sigma}$ and the Radon-Nikodým process

$$Z_t = e^{\eta W_t - \frac{1}{2}\eta^2 t}.$$

We also define

$$Z_{\tau,t} = e^{\eta(W_t - W_\tau)-\frac{1}{2}\eta^2(t-\tau)}.$$

Notice that

$$E\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2\right] = \hat{E}\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2 \frac{1}{Z_{t_T}}\right].$$

Under the MEMM $\hat{P}$, problem (4.55) then reads

$$\max_{\Gamma}\left\{\omega_0+\sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0\left(\gamma_i;x_i,D_{i+1}^0\right)]\right.$$
$$\left.-\kappa\sum_{i=0}^{T-1}e^{-\int_{t_0}^{t_{i+1}}\eta_t^2dt}\int_{t_i}^{t_{i+1}}e^{\int_{t_i}^{s}\eta_v^2dv}\hat{E}\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2\frac{1}{Z_{t_T}}\right]ds\right\}. \tag{4.60}$$

Further notice that

$$\begin{aligned}\hat{E}_i\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2\frac{1}{Z_{t_T}}\right] &= \hat{E}_i\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2\frac{1}{Z_{t_i}Z_{t_i,t_T}}\right]\\ &= \frac{1}{Z_{t_i}}\hat{E}_i\left[\delta_i^{(\Gamma_i;x_i,D_{i+1})}(s)^2\frac{1}{Z_{t_i,t_T}}\right].\end{aligned}$$

With the demand model in (4.58), we have

$$Z_{t_i}=e^{\frac{\mu}{b_{i+1}\sigma^2}\left[\log D_{i+1}^0-\log a_{i+1}-b_{i+1}(\log X_0+\frac{1}{2}(\mu-\sigma^2)t_i)\right]}.$$

This enables us to characterize the problem using the state variables $(\gamma_i;x_i,D_{i+1}^0)$. Iterated conditioning in (4.60) then yields the following proposition for the dynamic programming algorithm.

Proposition 4.5.1. *Define the terminal conditions*

$$\Phi_T=0,$$

and

$$V_T(\gamma_{T-1};x_{T-1},D_T)=H_T(\gamma_{T-1};x_{T-1},D_T)+s_T(\gamma_{T-1}+x_{T-1}-D_T)^+.$$

Then problem (4.55) can be solved via the dynamic programming recursion: For $i=0,\ldots,T-1$*,*

$$\Phi_i(x_i,D_{i+1}^0) \tag{4.61}$$
$$=\sup_{\gamma_i}\left\{H_{i+1}^0(\gamma_i;x_i,D_{i+1}^0)-\kappa e^{-\int_{t_0}^{t_{i+1}}\eta_t^2dt}\int_{t_i}^{t_{i+1}}e^{\int_{t_i}^{s}\eta_v^2dv}\frac{1}{Z_{t_i}}\hat{E}_i\left[\frac{\delta_i^{*(\gamma_i;x_i,D_{i+1})}(s)^2}{Z_{t_i,t_T}}\right]ds\right.$$
$$\left.+\hat{E}_i[\Phi_{i+1}((x_i+\gamma_i-D_{i+1})^+,D_{i+2}^0)]\right\}, \tag{4.62}$$

where $\delta_i^{*(\gamma_i;x_i,D_{i+1})}(s)$ *is the orthogonal component of the F-S decomposition conditional on* $\mathcal{F}_{t_i}$ *of* $V_{i+1}(\gamma_i;x_i,D_{i+1})$ *with*

$$V_i(\gamma_{i-1};x_{i-1},D_i) = H_i(\gamma_{i-1};x_{i-1},D_i) + \hat{E}_i[V_{i+1}(\gamma_i^*;(x_{i-1}+\gamma_{i-1}-D_i)^+,D_{i+1})],$$

and $\gamma_i^* = \gamma_i^*(x_i)$ *is the optimal inventory decision for period i, that is, the optimizer in (4.61).*

We can further simplify the problem with the demand assumption (4.58) and a newsvendor inventory model such that the discounted payoff of each period i is

$$\begin{aligned} &H_{i+1}(\gamma_i;x_i,D_{i+1}) \\ &= R_{i+1}D_{i+1} - (R_{i+1}+q_{i+1})(D_{i+1}-\gamma_i-x_i)^+ - p_{i+1}\gamma_i + \mathbb{1}_{\{i=T-1\}}s_T(y_{T-1}-D_T)^+ \end{aligned} \tag{4.63}$$

where R_{i+1} is the unit retail price, q_{i+1} is the penalty cost for unsatisfied demand and p_{i+1} is the unit purchase price.

Before we state the simplified dynamic programming recursion, we need the following lemma.

Lemma 4.5.2. *Let* $y_i = \gamma_i + x_i$ *be the inventory level after the inventory decision of period i is made,* $y_i = (y_i^1,\ldots,y_i^N)$*. The inventory payoff of any period i,* $i = 0,\ldots,T-1$ *can be rewritten as*

$$H_{i+1}(\gamma_i;x_i,D_{i+1}) = h_{i+1}(x_i+\gamma_i,D_{i+1}) + R_{i+1}x_i \tag{4.64}$$

$$= h_{i+1}(y_i,D_{i+1}) + R_{i+1}x_i \tag{4.65}$$

where

$$h_{i+1}(y_i,D_{i+1}) = R_{i+1}(D_{i+1}-y_i)^- - q_{i+1}(D_{i+1}-y_i)^+ + (R_{i+1}-p_{i+1})y_i + \mathbb{1}_{\{i=T-1\}}s_T(y_{T-1}-D_T)^+. \tag{4.66}$$

Proof: This follows directly from rewriting (4.63).

We finally have the following corollary for the simplified dynamic programming algorithm.

Corollary 4.5.3. *Let $h_{i+1}(y_i, D_{i+1})$ for $i = 0, ..., T-1$ be the functions defined in (4.66). The dynamic programming recursion for problem (4.36) is given by the terminal condition*

$$\Psi_T = 0, \tag{4.67}$$

$$M_T(y_{T-1}, D_T) = h_T(y_{T-1}, D_T) \tag{4.68}$$

and the recursion which for each $i = T-1, ..., 0$ computes functions $\Psi_i(x, D)$ and $M_i(y, D)$ from functions $\Psi_{i+1}(x, D)$ and $M_{i+1}(y, D)$ as follows. First, compute the orthogonal component $\delta_i^{*(y_i;D_{i+1})}(s)$ of the F-S decomposition of $M_{i+1}(y_i, D_{i+1})$. Then, compute the function

$$\begin{aligned}\Psi_i(x_i, D^0_{i+1}) = \sup_{y_i \geq x_i} \Bigg\{ & \hat{E}_i[h_{i+1}(y_i; D_{i+1})] + R_{i+1}x_i - \kappa e^{-\int_{t_0}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} \frac{1}{Z_{t_i}} \hat{E}_i \left[\frac{\delta_i^{*(y_i;D_{i+1})}(s)^2}{Z_{t_i,t_T}} \right] ds \\ & + \hat{E}_i[\Psi_{i+1}((y_i - D_{i+1})^+, D^0_{i+2})] \Bigg\}\end{aligned} \tag{4.69}$$

and let $y_i^*(x_i, D^0_{i+1})$ denote the maximizer in (4.69) if it exists. Finally, compute

$$\begin{aligned}M_i(y_{i-1}, D_i) = {} & h_i(y_{i-1}, D_i) + R_{i+1}(y_{i-1} - D_i)^+ \\ & + \hat{E}_i\Big[M_{i+1}\Big(y_i^*\big((y_{i-1} - D_i)^+, D^0_{i+1}\big), D_{i+1}\Big)\Big].\end{aligned} \tag{4.70}$$

We refer to $M_i(\cdot, \cdot)$ as the intermediate value function.

Remark: Observe that (4.69) resembles a classical formulation of the risk-neutral multi-period Newsvendor problem. The difference is captured in the third term on the right hand side. It is not surprising, therefore, to see in this term the risk-aversion factors and an integration over time of a hedging process, δ, which are features unique to this risk-averse model.

The following theorem states that the optimizer of the multi-period problem exists, so that we can replace the *sup* in (4.69) with *max*.

Theorem 4.5.4. *The optimizer of problem (4.69) exists.*

The proof can be found in the Appendix.

4.6 Numerical implementation and the F-S decomposition of intermediate value function

In this section we describe an algorithm for solving the dynamic programming problem in corollary 4.5.3. There are two major difficulties in implementation: how to obtain the orthogonal component δ_i of the F-S decomposition for M_{i+1}; and how to store the numerical value of M_i in recursion formula (4.70).

In light of the separation result (corollary 4.4.4), we can restrict our discussion to the single-product case as in section 4.5. In particular, we seek the orthogonal component, $\delta_i^{(y_i;D_{i+1})}(s)$, in the F-S decomposition of $M_{i+1}(y_i; D_{i+1})$ which is defined by the recursion as in (4.70). For fixed state variables y_i and D^0_{i+1} , the value of (4.70) is based on realizations of D_{i+1}. The difficulty is that the value function in (4.70) is no longer presented in an analytical form such that we can apply Itô's lemma to obtain the F-S decomposition. However, by applying a Fast Fourier Transformation, we can achieve the F-S decomposition in numerical form.

Let $N_{i+1} = \log D_{i+1}$. We suppose we have a numerical representation of the function

$$g(N_{i+1}) = M_{i+1}(y_i; N_{i+1}) \tag{4.71}$$

given in (4.70).

The Fourier transformation of $g(N_{i+1})$ is

$$\alpha(\xi) = \int_{-\infty}^{+\infty} g(N_{i+1}) e^{-2\pi \mathbb{i} N_{i+1}\xi} dN_{i+1} \tag{4.72}$$

where $\mathbb{i}^2 = -1$, and the inverse Fourier transformation is

$$g(N_{i+1}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \alpha(\xi) e^{2\pi \mathbb{i} N_{i+1}\xi} d\xi. \tag{4.73}$$

By theorem 4.2.2, we can obtain the F-S decomposition of the analytical function $e^{2\pi \mathbb{i} N_{i+1}\xi}$ as

$$e^{2\pi \boldsymbol{i} N_{i+1}\xi} = \hat{E}\left[e^{2\pi \mathbb{i} N_{i+1}\xi} | \mathcal{F}_{t_i}\right] + \int_{t_i}^{t_{i+1}} \bar{\delta}(v,\xi) d\mathcal{B}(v) + \int_{t_i}^{t_{i+1}} \bar{\vartheta}(v,\xi) dX(v).$$

for suitable choices of $\bar{\delta}$ and $\bar{\vartheta}$. In particular, assume that the demand has the exponential form in (4.58)

$$D_{i+1}(t) = a_{i+1} e^{b_{i+1} \log X(t) + c_{i+1}(\mathcal{B}(t) - \mathcal{B}(t_i))}$$

where $X_{i+1}(t)$ and $\mathcal{B}_{i+1}(t)$ for $t \in [t_i, t_{i+1}]$ are risky asset price and nonfinancial noise at time t, respectively. Thus,

$$N_{i+1}(t) = \log a_{i+1} + b_{i+1} \log X_{i+1}(t) + c_{i+1}(\mathcal{B}_{i+1}(t) - \mathcal{B}(t_i)).$$

The following lemma gives an explicit formula for the F-S decomposition of $e^{2\pi \mathbb{i} N_{i+1}\xi}$.

Lemma 4.6.1. *The F-S decomposition of* $e^{2\pi \mathbb{i} N_{i+1}\xi}$ *can be written explicitly in terms of*

$$\hat{E}\left[e^{2\pi \mathbb{i} N_{i+1}\xi} | \mathcal{F}_{t_i}\right] = e^{2\pi \mathbb{i} N_{i+1}\xi + 2\pi\xi \mathbb{i} b_{i+1}\mu_z^i - 2\pi^2\xi^2 (b_{i+1}^2 (\sigma_z^i)^2 + c_{i+1}^2 (t_{i+1} - t_i))},$$

$$\bar{\delta}(v,\xi) = 2\pi \mathbb{i} \xi c_{i+1} e^{2\pi \mathbb{i} \xi N_{i+1}(v)},$$

and

$$\bar{\vartheta}(v,\xi) = 2\pi\mathrm{i}\xi\frac{b_{i+1}}{X_{i+1}(v)}e^{2\pi\mathrm{i}\xi N_{i+1}(v)}$$

with $\mu_z^i = -\frac{1}{2}(\sigma_z^i)^2$ *and* $(\sigma_z^i)^2 = \int_{t_i}^{t_{i+1}}\sigma(s)^2 ds$.

This implies that the inverse Fourier transformation also has a decomposition of the following form:

$$\begin{aligned}
g(N_{i+1}) &= \int_{-\infty}^{+\infty}\alpha(\xi)\hat{E}\left[e^{2\pi\mathrm{i}N_{i+1}\xi}|\mathcal{F}_{t_i}\right]d\xi + \int_{-\infty}^{+\infty}\int_{t_i}^{t_{i+1}}\alpha(\xi)\bar{\delta}(v,\xi)d\mathcal{B}(v)d\xi \\
&\quad + \int_{-\infty}^{+\infty}\int_{t_i}^{t_{i+1}}\alpha(\xi)\bar{\vartheta}(v,\xi)dX(v)d\xi \\
&= \int_{-\infty}^{+\infty}\alpha(\xi)\hat{E}\left[e^{2\pi\mathrm{i}N_{i+1}\xi}|\mathcal{F}_{t_i}\right]d\xi + \int_{t_i}^{t_{i+1}}\int_{-\infty}^{+\infty}\alpha(\xi)\bar{\delta}(v,\xi)d\xi d\mathcal{B}(v) \\
&\quad + \int_{t_i}^{t_{i+1}}\int_{-\infty}^{+\infty}\alpha(\xi)\bar{\vartheta}(v,\xi)d\xi dX(v).
\end{aligned}$$

The orthogonal component of the F-S decomposition we need is essentially the term

$$\delta_i^{(\gamma_i;x_i,D_{i+1})} = \int_{-\infty}^{+\infty}\alpha(\xi)\bar{\delta}(v,\xi)d\xi. \tag{4.74}$$

(4.74) can be computed numerically, once we have an approximation of $\alpha(\xi)$ in (4.72). To apply the Fast Fourier Transformation, we employ regular spacing of size ϖ and ϱ for N_{i+1} and ξ respectively. In particular, fix a large n and m, define

$$N_{i+1}^k = (k-n-1)\varpi \quad \text{for } k = 1,\ldots,2n \tag{4.75}$$

$$\xi_u = -b + \varrho(u-1) \quad \text{for } u = 1,\ldots,m \tag{4.76}$$

with

$$b = \frac{1}{2}m\varrho \tag{4.77}$$

for $u = 1,\ldots,m$. Then an approximation for (4.72) is

$$\alpha(\xi_u) = \sum_{k=1}^{2n} e^{-2\pi\mathrm{i}N_{i+1}^k\xi_u}g(N_{i+1}^k)\varpi \tag{4.78}$$

$$= \sum_{k=1}^{2n} e^{2\pi\mathrm{i}bN_{i+1}^k}e^{-2\pi\mathrm{i}\varrho\varpi(u-1)(k-n-1)}g(N_{i+1}^k)\varpi. \tag{4.79}$$

This approximation can be efficiently computed using the Fast Fourier Transform method.

Note that as (4.71) also depends on y_i, we suppress the dependence on y_i for a compact notation. In any period i, we have to compute the expectation under the MEMM $\hat{E}_i[\cdot]$ in (4.69). Since $\bar{\delta}(v;\xi)$ is an $\mathcal{F}_v$-measurable random variable, we can compute the expectation $\hat{E}_i[\cdot]$ under ds and $d\xi$ integrals explicitly. Finally, a discrete approximation to the double integral on ds and $d\xi$ is required. The following theorem states the formula of the third term in (4.69).

Theorem 4.6.2. *For $i = 0, \ldots, T-1$, we have*

$$\int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} \frac{1}{Z_{t_i}} \hat{E}_i \left[\frac{\delta_i^{*(y_i;D_{i+1})}(s)^2}{Z_{t_i,t_T}} \right] ds \tag{4.80}$$

$$= -\int_{t_i}^{t_{i+1}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\int_{t_i}^{s} \eta_v^2 dv} \frac{1}{Z_{t_i}} \alpha(\xi_1)\alpha(\xi_2)\xi_1\xi_2 c_{i+1}^2 e^{\varsigma(\xi_i,\xi_2)} d\xi_1 d\xi_2 ds \tag{4.81}$$

where

$$\begin{aligned}
\varsigma(\xi_1,\xi_2) &= 2\pi \mathbb{i} b_{i+1}(\xi_1+\xi_2)\left(-\frac{1}{2}\sigma^2(t_s-t_i) + \log X(t_i)\right) + 2\pi\mathbb{i}(\xi_1+\xi_2)\log a_{i+1} \\
&\quad - \eta^2(t_T - t_i) + \frac{1}{2}(t_s-t_i)\iota_1^2 + \frac{1}{2}(t_s-t_i)\iota_2^2 \\
\iota_1 &= 2\pi\mathbb{i} b_{i+1}\sigma(\xi_1+\xi_2) \\
\iota_2 &= 2\pi\mathbb{i} c_{i+1}(\xi_1+\xi_2).
\end{aligned}$$

The proof can be found in the Appendix.

4.7 Appendix to chapter 4

Proof of theorem 4.4.1:

Proof. We prove *a)* and *b)* simultaneously by backward induction on $i = T-1, \dots, 0$.

For fixed inventory decision $\mathbf{\Gamma}$, recall that $\boldsymbol{s}_T = (s_T^1, \dots, s_T^N)$ is the vector indicating the (discounted) unit salvage value at time T, where s_T^j is the corresponding (discounted) salvage price for product j, $j = 1, \dots, N$. The final wealth is

$$\begin{aligned}\omega_T &= \omega_{T-1} + H_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) + \int_{t_{T-1}}^{t_T} \theta_{T-1}(s)dX(s) + \boldsymbol{s}_T\boldsymbol{x}_T^\intercal \\ &= \omega_{T-1} + \tilde{H}_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) + \int_{t_{T-1}}^{t_T} \theta_{T-1}(s)dX(s)\end{aligned}$$

where

$$\tilde{H}_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) = H_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) + \boldsymbol{s}_T[\boldsymbol{x}_{T-1} + \boldsymbol{\gamma}_{T-1} - \boldsymbol{D}_T]^\intercal. \tag{4.82}$$

For $i = T-1$ we obtain from theorem 4.2.3 applied to the time period $[t_{T-1}, t_T]$

$$\begin{aligned}&A_{T-1}^{(\lambda,\mathbf{\Gamma}_{T-1})}(\omega_{T-1}, \boldsymbol{x}_{T-1}, \boldsymbol{D}_T^0) \\ &= \min_{\boldsymbol{\theta}_{T-1}} E_{T-1}[(\omega_T - \lambda)^2] \\ &= \min_{\boldsymbol{\theta}_{T-1}} E_{T-1}\left[\left(\omega_{T-1} + \tilde{H}_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right) + \int_{t_{T-1}}^{t_T} \theta_{T-1}(s)dX(s) - \lambda\right)^2\right] \\ &= e^{-\int_{t_{T-1}}^{t_T}\eta_t^2 dt}\left(\omega_{T-1} + H_T^0\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T^0\right) - \lambda\right)^2 \\ &\quad + e^{-\int_{t_{T-1}}^{t_T}\eta_t^2 dt}\int_{t_{T-1}}^{t_T} e^{\int_{t_{T-1}}^{s}\eta_v^2 dv} E_{T-1}\left[\delta_{T-1}^{(\mathbf{\Gamma}_{T-1};\boldsymbol{x}_{T-1},\boldsymbol{D}_T)}(s)^2\right] ds\end{aligned}$$

where the intrinsic value of payoff $\tilde{H}_T$ incurred during $[T-1, T)$ at T is

$$\begin{aligned}&H_T^0\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T^0\right) \\ &= \hat{E}_{T-1}[\tilde{H}_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T^0\right)] \\ &= \hat{E}_{T-1}[H_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T^0\right) + \boldsymbol{s}_T[\boldsymbol{x}_{T-1} + \boldsymbol{\gamma}_{T-1} - \boldsymbol{D}_T^0]^\intercal]\end{aligned}$$

and $\delta_{T-1}^{(\mathbf{\Gamma}_{T-1};\boldsymbol{x}_{T-1},\boldsymbol{D}_T)}$ is the orthogonal component of the F-S decomposition of $\tilde{H}_T\left(\boldsymbol{\gamma}_{T-1}; \boldsymbol{x}_{T-1}, \boldsymbol{D}_T\right)$.

Let $\delta_{T-1}^{(\Gamma_{T-1}^{j};x_{T-1}^{j},D_{T}^{j})}$ be the orthogonal component of the F-S decomposition of $\tilde{H}_T\left(\gamma_{T-1}^{j};x_{T-1}^{j},D_T^{j}\right)$ corresponding to product j. Then by theorem 4.2.2

$$E_{T-1}\left[\delta_{T-1}^{(\boldsymbol{\Gamma}_{T-1};\boldsymbol{x}_{T-1},\boldsymbol{D}_T)}(s)^2\right] = \sum_{j=1}^{N} E_{T-1}\left[\delta_{T-1}^{(\gamma_{T-1}^{j};x_{T-1}^{j},D_T^{j})}(s)^2\right]$$

and

$$\tilde{H}_T^0\left(\boldsymbol{\gamma}_{T-1};\boldsymbol{x}_{T-1},\boldsymbol{D}_T^0\right) = \sum_{j=1}^{N}\tilde{H}_T^0\left(\gamma_{T-1}^{j};x_{T-1}^{j},D_T^{0,j}\right).$$

This finishes the proof of the $i = T-1$ period.

For the induction step, assume that for any period $\tilde{k} = i+1,\ldots,T-1$, we have

(4.61). Then for period i,

$$
\begin{aligned}
&A_i^{(\lambda,\Gamma_i)}(\omega_i, \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) \\
&= \min_{\boldsymbol{\theta}_i} E_i[A_{i+1}^{(\lambda,\Gamma_{i+1})}(\omega_{i+1}, \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0)] \\
&= \min_{\boldsymbol{\theta}_i} E_i \left[A_{i+1}^{(\lambda,\Gamma_{i+1})} \left(\underbrace{\omega_i + H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + \int_{t_i}^{t_{i+1}} \theta_i(s) dX(s)}_{\omega_{i+1}}, \underbrace{(\boldsymbol{x}_i + \boldsymbol{\gamma}_i - \boldsymbol{D}_{i+1})^+}_{\boldsymbol{x}_{i+1}}, \boldsymbol{D}_{i+2}^0 \right) \right] \\
&= \min_{\boldsymbol{\theta}_i} E_i \Bigg[e^{-\int_{t_{i+1}}^{t_T} \eta_t^2 dt} \Bigg(\omega_i + H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + \int_{t_i}^{t_{i+1}} \theta_i(s) dX(s) + H_{i+2}^0(\boldsymbol{\gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0) \\
&+ \sum_{l=i+2}^{T-1} F_l^{i+2}(\boldsymbol{\Gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0,) - \lambda \Bigg)^2 + e^{-\int_{t_{i+1}}^{t_T} \eta_t^2 dt} \int_{t_{i+1}}^{t_{i+2}} e^{\int_{t_{i+1}}^{s} \eta_v^2 dv} E_{i+1} \left[\delta_{i+1}^{(\boldsymbol{\Gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2})}(s)^2 \right] ds \\
&+ \sum_{l=i+2}^{T-1} e^{-\int_{t_{i+1}}^{t_{l+1}} \eta_t^2 dt} \int_{t_l}^{t_{l+1}} e^{\int_{t_l}^{s} \eta_v^2 dv} \Delta_l^{i+1}(\boldsymbol{\Gamma}_{i+1}\ \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0)(s)^2 ds \Bigg] \qquad (4.83) \\
&= \min_{\boldsymbol{\theta}_i} E_i \Bigg[e^{-\int_{t_{i+1}}^{t_T} \eta_t^2 dt} \Bigg(\omega_i + H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + H_{i+2}^0(\boldsymbol{\gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0) + \int_{t_i}^{t_{i+1}} \theta_i(s) dX(s) \\
&+ \sum_{l=i+2}^{T-1} F_l^{i+2}(\boldsymbol{\Gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0) - \lambda \Bigg)^2 \Bigg] + e^{-\int_{t_{i+1}}^{t_T} \eta_t^2 dt} \int_{t_{i+1}}^{t_{i+2}} e^{\int_{t_{i+1}}^{s} \eta_v^2 dv} E_i \left[E_{i+1} \left[\delta_{i+1}^{(\boldsymbol{\gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2})}(s)^2 \right] ds \right] \\
&+ \sum_{l=i+2}^{T-1} e^{-\int_{t_{i+1}}^{t_{l+1}} \eta_t^2 dt} \int_{t_l}^{t_{l+1}} e^{\int_{t_l}^{s} \eta_v^2 dv} E_i \left[\Delta_l^{i+1}(\boldsymbol{\Gamma}_{i+1}; \boldsymbol{x}_{i+1}, \boldsymbol{D}_{i+2}^0)(s)^2 ds \right] \\
&= e^{-\int_{t_i}^{t_T} \eta_t^2 dt} \left(\omega_i + H_{i+1}^0(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) + \sum_{l=i+1}^{T-1} F_l^i(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0) - \lambda \right)^2 \\
&+ e^{-\int_{t_i}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} E_i[\delta_i^{(\boldsymbol{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1})}(s)^2] ds \\
&+ \sum_{l=i+1}^{T-1} e^{-\int_{t_i}^{t_{l+1}} \eta_t^2 dt} \int_{t_l}^{t_{l+1}} e^{\int_{t_l}^{s} \eta_v^2 dv} \Delta_l^i(\boldsymbol{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0)(s)^2 ds \qquad (4.84)
\end{aligned}
$$

where $\delta_i^{(\boldsymbol{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1})}$ is the orthogonal component of the F-S decomposition of

$$H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\boldsymbol{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1})$$

The last equation in (4.83) follows from theorem 4.2.3 applied to the time period $[t_i, t_{i+1}]$.

Let $\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}$ be the orthogonal component of the F-S decomposition of

$$H_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j) \tag{4.85}$$

corresponding to product j. Then

$$E_i[\delta_i^{(\boldsymbol{\Gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1})}(s)^2] = \sum_{j=1}^{N} E_i[\delta_i^{(\Gamma_i^j; x_i^j, D_{i+1}^j)}(s)^2]$$

and

$$H_{i+1}(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}) = \sum_{j=1}^{N} H_{i+1}(\gamma_i^j; x_i^j, D_{i+1}^j)$$

This proofs for the i-th period, (4.61) holds.

Hence for all period $i = 0, \ldots, T-1$, we have (4.61). □

Proof of theorem 4.4.3:

Proof. For fixed inventory decisions $\boldsymbol{\Gamma}$, let the variance minimization problem for the multi-period problem be

$$B^{\boldsymbol{\Gamma}}(m) = \inf_{\theta \in \Theta} \{Var[\omega_T] | E[\omega_T] = m\}.$$

In theorem 4.2.1 Sun et al. (2011) it is proved that for fixed inventory strategy $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_0, \ldots, \boldsymbol{\gamma}_{T-1})$

$$B^{\boldsymbol{\Gamma}}(m) = \sup_{\lambda} (A^{\boldsymbol{\Gamma}}(\lambda) - (m - \lambda)^2)$$

where the optimum is achieved for

$$0 = \frac{\partial}{\partial \lambda} A^{\boldsymbol{\Gamma}}(\lambda) + 2(m - \lambda).$$

This is equivalent to

$$-2e^{-\int_{t_0}^{t_T}\eta_t^2 dt}\left(\omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)] - \lambda\right) + 2(m-\lambda) = 0$$

which has solution

$$\lambda_m^* = \frac{m - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}(\omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)])}{1 - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}.$$

Plugging this back into the duality equation yields

$$\begin{aligned}
B^{\boldsymbol{\Gamma}^*} &= A^{\boldsymbol{\Gamma}}(\lambda_m^*) - (m-\lambda_m^*)^2 \\
&= e^{-\int_{t_0}^{t_T}\eta_t^2 dt}\left(\frac{\omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)] - m}{1 - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}\right)^2 \\
&- \left(\frac{e^{-\int_{t_0}^{t_T}\eta_t^2 dt}(\omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)] - m)^2}{1 - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}\right) \\
&+ \sum_{i=0}^{T-1} e^{-\int_{t_0}^{t_{i+1}}\eta_t^2 dt}\int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s}\eta_v^2 dv} E[\delta_i^{(\boldsymbol{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)^2]ds.
\end{aligned}$$

As a result, with $U^{\boldsymbol{\Gamma}}$ defined as

$$U^{\boldsymbol{\Gamma}} = \sup_{m\in\mathbb{R}}(m - \kappa B^{\boldsymbol{\Gamma}}(m))$$

we have

$$U(\kappa) = \max_{\boldsymbol{\Gamma}} U^{\boldsymbol{\Gamma}}.$$

and

$$\begin{aligned}
U^{\boldsymbol{\Gamma}} &= \max_m(m - \kappa B^{\boldsymbol{\Gamma}}(m)) \\
&\Leftrightarrow 2\left(m^* - \omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)]\right)\frac{e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}{1 - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}} = \frac{1}{\kappa} \\
&\Leftrightarrow \left(m^* - \omega_0 + \sum_{i=0}^{T-1}\hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1}^0)]\right)^2 = \frac{1}{4\kappa^2}\left(\frac{e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}{1 - e^{-\int_{t_0}^{t_T}\eta_t^2 dt}}\right)^2
\end{aligned}$$

which implies

$$U^{\mathbf{\Gamma}} = \omega_0 + \sum_{i=0}^{T-1} \hat{E}[H_{i+1}^0(\boldsymbol{\gamma}_i; \boldsymbol{x}_i, \boldsymbol{D}_{i+1}^0)] + \frac{1}{\kappa}\left(e^{-\int_{t_0}^{t_T}\eta_t^2 dt} - 1\right)$$
$$- \kappa \sum_{i=0}^{T-1} e^{-\int_{t_0}^{t_{i+1}}\eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s}\eta_v^2 dv} E[\delta_i^{(\mathbf{\Gamma}_i;\boldsymbol{x}_i,\boldsymbol{D}_{i+1})}(s)^2]ds.$$

□

The following lemmas and corollary are used to get corollary 4.5.3.

Lemma 4.7.1. *Let $y_i = \gamma_i + x_i$ be the inventory level after the inventory decision of period i is made, $y_i = (y_i^1, \ldots, y_i^N)$.*

(a) The random variable $\bar{F}_l^{*i+1}(\gamma_i; x_i, D_{i+1}) := \hat{E}_{i+1}[H_{l+1}^0(\gamma_l^*; x_l, D_{l+1}^0)]$ can be written as a function $\bar{F}_l^{*i+1}(y_i; D_{i+1})$, and the orthogonal component $\delta_i^{*(\gamma_i;x_i,D_{i+1})}$ of the F-S decomposition conditional on $\mathcal{F}_{t_i}$ of $V_{i+1}(\gamma_i, x_i, D_{i+1})$ in proposition 1 can be rewritten as a function of y_i and D_{i+1}:

$$\delta_i^{*(\gamma_i;x_i,D_{i+1})} = \delta_i^{*(x_i+\gamma_i;D_{i+1})} = \delta_i^{*(y_i;D_{i+1})}. \tag{4.86}$$

(b) We can decompose

$$V_{i+1}(\gamma_i; x_i, D_{i+1}) = \bar{V}_{i+1}(y_i; x_i, D_{i+1}) = M_{i+1}(y_i; D_{i+1}) + R_{i+1}x_i \tag{4.87}$$

with

$$M_{i+1}(y_i; D_{i+1}) = h_{i+1}(y_i; D_{i+1}) + (R_{i+1} - p_{i+1})D_{i+1} + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1^*}(y_i; D_{i+1}). \tag{4.88}$$

Moreover we have for $\bar{V}_i$ and M_i the terminal conditions

$$\bar{V}_T(y_{T-1}; x_{T-1}, D_T) = M_T(y_{T-1}; D_T) + R_T x_{T-1} \tag{4.89}$$

$$M_T(y_{T-1}; D_T) = h_T(y_{T-1}; D_T) + (R_T - p_T)D_T + s_T(y_{T-1} - D_T)^+ \tag{4.90}$$

with the iteration

$$\bar{V}_i(y_{i-1}; x_{i-1}, D_i) = M_i(y_{i-1}; D_i) + R_i x_{i-1} \tag{4.91}$$

$$M_i(y_{i-1}; D_i) = h_i(y_{i-1}; D_i) + (R_i - p_i)D_i + J_i(y_{i-1}; D_i), \tag{4.92}$$

where we define

$$J_i(y_{i-1}; D_i) = \hat{E}_i[\bar{V}_{i+1}\left(y_i^*((y_{i-1} - D_i)^+, D_{i+1}^0); (y_{i-1} - D_i)^+, D_{i+1}\right)] \tag{4.93}$$

and $y_i^*(x_i, D_{i+1}^0) = \gamma_i^*(x_i, D_{i+1}^0) + x_i$ *is the optimal inventory level after decision of period* i.

The following lemma is used to prove lemma 4.7.1.

Lemma 4.7.2. *For each period* i, $i = 0, \ldots, T-1$, *the orthogonal component* $\bar{\delta}_i$ *of F-S decomposition conditional on* $\mathcal{F}_{t_i}$ *for payoff function* $H_{i+1}(\gamma_i; x_i, D_{i+1})$ *is a function of* $y_i = x_i + \gamma_i$ *and* D_{i+1}, *that is,* $\bar{\delta}_i = \bar{\delta}_i^{(y_i, D_{i+1})}$.

Proof of lemma 4.7.2:

Proof. Recall for each period i, $i = 0, \ldots, T-1$, the payoff function from operations is

$$\begin{aligned}
&H_{i+1}(\gamma_i; x_i, D_{i+1}) \\
&= R_{i+1}\min(D_{i+1}, x_i + \gamma_i) - q_{i+1}(D_{i+1} - (x_i + \gamma_i))^+ - p_{i+1}\gamma_i \\
&= R_{i+1}D_{i+1} - (R_{i+1} + q_{i+1})(D_{i+1} - \gamma_i - x_i)^+ - p_{i+1}\gamma_i.
\end{aligned}$$

With model assumptions of (4.58) and (4.63), for $t \in [t_i, t_{i+1}]$, it is proved in Sun et al. (2011), theorem 3.3.7, that the orthogonal component of the F-S decomposition is given by:

$$\begin{aligned}
&\bar{\delta}_i^{(\gamma_i; x_i, D_{i+1})}(t) \\
&= c_{i+1}D_{i+1}(t)\left(-(R_{i+1} + q_{i+1})F_i(t)\Phi\left(\frac{\mu_z^i(t) + \log\frac{D_{i+1}(t)}{\gamma_i + x_i}}{\sigma_z^i(t)} + \sigma_z(t)\right) + R_{i+1}F_i(t)\right)
\end{aligned}$$

for $i = 0, \ldots, T-2$, and, for $i = T-1$,

$$\bar{\delta}_i^{(\gamma_i;x_i,D_{i+1})}(t)$$
$$= c_{i+1}D_{i+1}(t)\left(-(R_{i+1}+q_{i+1}-s_{i+1})F_i(t)\Phi\left(\frac{\mu_z^i(t)+\log\frac{D_{i+1}(t)}{\gamma_i+x_i}}{\sigma_z^i(t)}+\sigma_z(t)\right)+(R_{i+1}-s_{i+1})F_i(t)\right)$$

with

$$\begin{aligned}
F_i(t) &= e^{b_{i+1}\mu_z^i(t)+\frac{1}{2}(b_{i+1}^2\sigma_z^i(t)^2+c_{i+1}^2(t_{i+1}-t_i))}\\
\mu_z^i(t) &= -\frac{1}{2}\sigma_z^i(t)^2\\
\sigma_z^i(t) &= \int_{t_i}^{t_{i+1}}\sigma(s)^2 ds.
\end{aligned}$$

As a result, for $i = 0, \ldots, T-1$,

$$\bar{\delta}_i^{(\gamma_i;x_i,D_{i+1})} = \bar{\delta}_i^{(x_i+\gamma_i;D_{i+1})} = \bar{\delta}_i^{(y_i;D_{i+1})}.$$

This finishes the proof.

$\square$

Proof of lemma 4.7.1:

Proof. To prove (a), recall that for period $[t_i, t_{i+1}]$, $\delta_i^{*(\gamma_i;x_i,D_{i+1})}$ is the orthogonal component of F-S decomposition conditional on $\mathcal{F}_{t_i}$ of

$$V_{i+1}(\gamma_i;x_i,D_{i+1}) = H_{i+1}(\gamma_i;x_i,D_{i+1}) + \sum_{l=i+1}^{T-1}\bar{F}_l^{i+1^*}(\gamma_i;x_i,D_{i+1}).$$

By lemma 4.7.2 it suffices to look at the orthogonal component of F-S decomposition conditional on $\mathcal{F}_{t_i}$ of

$$\sum_{l=i+1}^{T-1}\bar{F}_l^{i+1^*}(\gamma_i;x_i,D_{i+1}).$$

Denote this by $\bar{\bar{\delta}}_i^{*(\gamma_i;x_i,D_{i+1})}$. We prove by backward induction on $i+1 \le k \le l$, for fixed $l = i+1, \ldots, T-1$ and $i = 0, \ldots, T-1$

$$\bar{F}_l^{k^*}(\gamma_{k-1}; x_{k-1}, D_k) = \bar{F}_l^{k^*}(y_{k-1}; D_k). \tag{4.94}$$

The desired result

$$\bar{\bar{\delta}}_i^{*(\gamma_i;x_i,D_{i+1})} = \bar{\bar{\delta}}_i^{*(y_i;D_{i+1})} \tag{4.95}$$

is then obvious once we have (4.94).

For fixed i and l, $l \ge i+1$, $i = 0, \ldots, T-1$, use backward induction on $i+1 \le k \le l$. For $k = l$

$$\begin{aligned}
\bar{F}_l^{k^*}(\gamma_{k-1}; x_{k-1}, D_k) &= \hat{E}_k[H_{l+1}^0(\gamma_l^*; x_l, D_{l+1}^0)] \\
&= \hat{E}_l[H_{l+1}^0(\gamma_l^*; x_l, D_{l+1}^0)] \\
&= \hat{E}_l[H_{l+1}^0(\gamma_l^*(x_l); x_l, D_{l+1}^0)] \\
&= \hat{E}_l[H_{l+1}^0(\gamma_l^*((y_{l-1} - D_l)^+); (y_{l-1} - D_l)^+, D_{l+1}^0)] \\
&= \bar{F}_l^{l^*}(y_{l-1}; D_l) \\
&= \bar{F}_l^{k^*}(y_{k-1}; D_k)
\end{aligned}$$

where we used that $x_l = (x_{l-1} + \gamma_{l-1} - D_l)^+ = (y_{l-1} - D_l)^+$.

This proves (4.94) for $k = l$.

Assuming that for any $k = l, \ldots, \bar{k}+1$, we have (4.94), i.e.

$$\bar{F}_l^{k^*}(\gamma_{k-1}; x_{k-1}, D_k) = \bar{F}_l^{k^*}(y_{k-1}; D_k),$$

then for $k = \bar{k}$,

$$\begin{aligned}
\bar{F}_l^{\bar{k}^*}(\gamma_{\bar{k}-1}; x_{\bar{k}-1}, D_{\bar{k}}) &= \hat{E}_{\bar{k}}[H_{l+1}^0(\gamma_l^*; x_l, D_{l+1}^0)] \\
&= \hat{E}_{\bar{k}}[\hat{E}_{\bar{k}+1}[H_{l+1}^0(\gamma_l^*; x_l, D_{l+1}^0)]] \\
&= \hat{E}_{\bar{k}}[\bar{F}_l^{\bar{k}+1^*}(\gamma_{\bar{k}}; x_{\bar{k}}, D_{\bar{k}+1})] \\
&= \hat{E}_{\bar{k}}[\bar{F}_l^{\bar{k}+1^*}(y_{\bar{k}}; D_{\bar{k}+1})] \\
&= \hat{E}_{\bar{k}}[\bar{F}_l^{\bar{k}+1^*}(x_{\bar{k}} + \gamma_{\bar{k}}; D_{\bar{k}+1})] \\
&= \hat{E}_{\bar{k}}[\bar{F}_l^{\bar{k}+1^*}((x_{\bar{k}-1} + \gamma_{\bar{k}-1} - D_{\bar{k}})^+ + \gamma_{\bar{k}}^*((x_{\bar{k}-1} + \gamma_{\bar{k}-1} - D_{\bar{k}})^+); D_{\bar{k}+1})] \\
&= \bar{F}_l^{\bar{k}^*}(y_{\bar{k}-1}; D_{\bar{k}})
\end{aligned}$$

since

$$x_{\bar{k}} = (x_{\bar{k}-1} + \gamma_{\bar{k}-1} - D_{\bar{k}})^+ = (y_{\bar{k}-1} - D_{\bar{k}})^+.$$

This finishes the proof of (4.94).

As a result,

$$\sum_{l=i+1}^{T-1} \bar{F}_l^{i+1^*}(\gamma_i; x_i, D_{i+1}) = \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1^*}(y_i; D_{i+1}), \tag{4.96}$$

so the orthogonal component of F-S decomposition of the function above depends only on $(y_i; D_{i+1})$, and, hence is of the form $\bar{\bar{\delta}}_i^{*(y_i;D_{i+1})}$.

Further notice that $\delta_i = \bar{\delta}_i + \bar{\bar{\delta}}_i$, and combine lemma 4.7.2 and the result above. This yields

$$\delta_i^{*(\gamma_i;x_i D_{i+1})} = \delta_i^{*(y_i;D_{i+1})}.$$

This proves (4.86) and concludes the proof of *(a)*.

For *(b)*, recall from Proposition 1 part (b) that the iteration formula for $V_{i+1}(\gamma_i; x_i, D_{i+1})$ is

$$V_i(\gamma_{i-1}; x_{i-1}, D_i) = H_i(\gamma_{i-1}; x_{i-1}, D_i) + \hat{E}_i[V_{i+1}(\gamma_i^*; (x_{i-1} + \gamma_{i-1} - D_i)^+, D_{i+1})]. \tag{4.97}$$

We write

$$\bar{V}_{i+1}(y_i; x_i, D_{i+1}) = M_{i+1}(y_i; D_{i+1}) + R_{i+1}x_i$$

where

$$\begin{aligned} M_{i+1}(y_i; D_{i+1}) &= h_{i+1}(y_i; D_{i+1}) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1^*}(\gamma_i; x_i, D_{i+1}) \\ &= h_{i+1}(y_i; D_{i+1}) + \sum_{l=i+1}^{T-1} \bar{F}_l^{i+1^*}(y_i; D_{i+1}) \end{aligned}$$

is of the form $J_i(y_i, D_{i+1})$ for some function J_i by (4.96).

Also notice

$$\hat{E}_i[\bar{V}^*_{i+1}(\gamma_i; x_i, D_{i+1})] = \sum_{l=i+1}^{T-1} \bar{F}_l^{i^*}(\gamma_i; x_i, D_{i+1}) \tag{4.98}$$

and, hence,

$$\hat{E}_i[\bar{V}^*_{i+1}(\gamma_i; x_i, D_{i+1})] = \hat{E}_i[\bar{V}_{i+1}\left(y_i^*((y_{i-1} - D_i)^+, D^0_{i+1}); (y_{i-1} - D_i)^+, D_{i+1}\right)] := J_i(y_i; D_{i+1})$$

by (4.96).

Combining (a) and (4.98) finishes the proof. □

Proof of corollary 4.5.3:

Proof. From lemmas 4.7.1 and 4.7.2, $\delta_i^{*(y_i;D_{i+1})}$ is the orthogonal component of F-S decomposition conditional on $\mathcal{F}_{t_i}$ for

$$\bar{V}_{i+1}(y_i; x_i, D_{i+1}) = M_{i+1}(y_i; D_{i+1}) + R_{i+1}x_i$$

since $R_{i+1}x_i$ is $\mathcal{F}_{t_i}$-measurable, $\delta_i^{*(y_i;D_{i+1})}$ is the orthogonal component of F-S decomposition conditional on $\mathcal{F}_{t_i}$ for $M_{i+1}(y_i; D_{i+1})$. Notice that

$$\begin{aligned} \bar{V}_{i+1}(y_i^*; x_i, D_{i+1}) &= M_{i+1}(y_i^*; D_{i+1}) + R_{i+1}x_i \\ &= M_{i+1}(y_i^*((y_{i-1} - D_i)^+, D^0_{i+1}); D_{i+1}) + R_{i+1}(y_{i-1} - D_i)^+. \end{aligned}$$

The recursion for M_i is

$$\begin{aligned}M_i(y_{i-1}; D_i) &= h_i(y_{i-1}; D_i) + \hat{E}_i[\bar{V}_{i+1}(y_i^*((y_{i-1} - D_i)^+, D_{i+1}^0); (y_{i-1} - D_i)^+, D_{i+1})] \\ &= h_i(y_{i-1}; D_i) + \hat{E}_i[M_{i+1}(y_i^*((y_{i-1} - D_i)^+, D_{i+1}^0); D_{i+1}) + R_{i+1}(y_{i-1} - D_i)^+] \\ &= h_i(y_{i-1}; D_i) + \hat{E}_i[M_{i+1}(y_i^*((y_{i-1} - D_i)^+, D_{i+1}^0); D_{i+1})] + R_{i+1}(y_{i-1} - D_i)^+.\end{aligned}$$

This proves the recursion formula for M_i as in (4.70).

Let Ψ be the value function such that

$$\Psi_T = 0.$$

Combining the definition above and proposition 4.5.1, we obtain (4.69):

$$\begin{aligned}\Psi_i(x_i, D_{i+1}^0) = \sup_{y_i \geq x_i} &\left\{ \hat{E}_i[h_{i+1}(y_i; D_{i+1})] + R_{i+1}x_i - \kappa e^{-\int_{t_0}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} \frac{1}{Z_{t_i}} \hat{E}_i \left[\frac{\delta_i^{*(y_i; D_{i+1})}(s)^2}{Z_{t_i, t_T}} \right] ds \right. \\ &\left. + \hat{E}_i[\Psi_{i+1}((y_i - D_{i+1})^+, D_{i+2}^0)] \right\}. \qquad (4.99)\end{aligned}$$

□

Proof of theorem 4.5.4:

Proof. We prove by backward induction that for any $i = 0, \ldots, T-1$, the optimizer for (4.69) $y_i^* < +\infty$.

For $i = T-1$,

$$\lim_{y_{T-1} \to +\infty} \hat{E}_i[h_{i+1}(y_i; D_{i+1})] = -\infty$$

and $\kappa e^{-\int_{t_0}^{t_{i+1}} \eta_t^2 dt} \int_{t_i}^{t_{i+1}} e^{\int_{t_i}^{s} \eta_v^2 dv} \frac{1}{Z_{t_i}} \hat{E}_i \left[\frac{\delta_i^{*(y_i; D_{i+1})}(s)^2}{Z_{t_i, t_T}} \right] ds > 0$. Hence,

$$\lim_{y_{T-1} \to +\infty} \left\{ \hat{E}_{T-1}[h_T(y_{T-1}; D_T)] - \kappa e^{-\int_{t_0}^{t_T} \eta_t^2 dt} \int_{t_{T-1}}^{t_T} e^{\int_{t_{T-1}}^{s} \eta_v^2 dv} \frac{1}{Z_{t_{T-1}}} \hat{E}_{T-1} \left[\frac{\delta_{T-1}^{*(y_{T-1}; D_T)}(s)^2}{Z_{t_{T-1}, t_T}} \right] ds \right\} = -\infty.$$

That is, $y^*_{T-1} < +\infty$.

Assume for $i = T-1, \dots, k+1$, $y^*_i < +\infty$, then for $i = k$,

$$\lim_{y_k \to +\infty} \hat{E}_k[\Psi_{k+1}((y_k - D_{k+1})^+, D^0_{k+2})] = \lim_{x_{k+1} \to +\infty} \hat{E}_k[\Psi_{k+1}(x_{k+1}, D^0_{k+2})] = -\infty$$

due to the induction and the fact that $y_{k+1} \geq x_{k+1}$.

As a result,

$$\lim_{y_k \to +\infty} \left\{ \hat{E}_k[h_{k+1}(y_k; D_{k+1})] - \kappa e^{-\int_{t_0}^{t_{k+1}} \eta_t^2 dt} \int_{t_k}^{t_{k+1}} e^{\int_{t_k}^{s} \eta_v^2 dv} \frac{1}{Z_{t_k}} \hat{E}_k \left[\frac{\delta_k^{*(y_k;D_{k+1})}(s)^2}{Z_{t_k,t_T}} \right] ds \right.$$
$$\left. + \hat{E}_k[\Psi_{k+1}((y_k - D_{k+1})^+, D^0_{k+2})] \right\} = -\infty.$$

This shows that $y_k = +\infty$ is not an optimizer for $i = k$. This finishes the proof. □

Proof of theorem 4.6.2:

Proof. We want to calculate $\hat{E}_i \left[\frac{\delta_i^{(y_i;D_{i+1})}(s)^2}{Z_{t_i,t_T}} \right]$.

First notice that

$$\hat{W}(t) = W(t) + \eta t$$

is a $\hat{P}$-Brownian motion, where $W(t)$ is a P-Brownian motion. hence

$$\begin{aligned}
D_{i+1}(t_s) &= a_{i+1} e^{b_{i+1} \log X(t_s) + c_{i+1}(\mathcal{B}(t_s) - \mathcal{B}(t_i))} \\
&= a_{i+1} e^{b_{i+1}(\log X(t_s) - \log X(t_i)) + b_{i+1} \log X(t_i) + c_{i+1}(\mathcal{B}(t_s) - \mathcal{B}(t_i))} \\
&= a_{i+1} e^{b_{i+1}\left((\mu - \frac{1}{2}\sigma^2)(t_s - t_i) + \sigma(W(t_s) - W(t_i))\right) + b_{i+1} \log X(t_i) + c_{i+1}(\mathcal{B}(t_s) - \mathcal{B}(t_i))} \\
&= a_{i+1} e^{b_{i+1}\left(-\frac{1}{2}\sigma^2(t_s - t_i) + \sigma(\hat{W}(t_s) - \hat{W}(t_i))\right) + b_{i+1} \log X(t_i) + c_{i+1}(\mathcal{B}(t_s) - \mathcal{B}(t_i))},
\end{aligned}$$

and

$$\begin{aligned}
Z_{t_i,t_T} &= e^{\eta(W(t_T) - W(t_i)) - \frac{1}{2}\eta^2(t_T - t_i)} \\
&= e^{\eta(W(t_T) - W(t_s)) + \eta(W(t_s) - W(t_i)) - \frac{1}{2}\eta^2(t_T - t_i)} \\
&= e^{\eta(\hat{W}(t_T) - \hat{W}(t_s)) + \eta(\hat{W}(t_s) - \hat{W}(t_i)) - \frac{3}{2}\eta^2(t_T - t_i)}.
\end{aligned}$$

Let

$$u_1 = \hat{W}(t_s) - \hat{W}(t_i)$$

$$u_2 = \hat{W}(t_T) - \hat{W}(t_s)$$

$$u_3 = \mathcal{B}(t_s) - \mathcal{B}(t_i)$$

where (u_1, u_2, u_3) is a 3-dimensional multivariate normal $\mathcal{N}(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} t_s - t_i & 0 & 0 \\ 0 & t_T - t_s & 0 \\ 0 & 0 & t_s - t_i \end{pmatrix}$$

and $\psi(u_1, u_2, u_3)$ is the corresponding probability density function. As a result, we have

$$\begin{aligned}
&\hat{E}_i\left[\frac{\delta_i^{(y_i;D_{i+1})}(s)^2}{Z_{t_i,t_T}}\right] \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\eta u_1 - \eta u_2 - \frac{3}{2}\eta^2(t_T - t_i)} \\
&\quad \cdot \left[\int_{-\infty}^{+\infty} \alpha(\xi)\mathbb{i}\xi a_{i+1}c_{i+1}e^{2\pi\mathbb{i}\xi\left[b_{i+1}(-\frac{1}{2}\sigma^2(t_s-t_i)+\log X(t_i)+\sigma u_1)+c_{i+1}u_3\right]}d\xi\right]^2 \psi(u_1, u_2, u_3)du_1du_2du_3 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\eta u_1 - \eta u_2 - \frac{3}{2}\eta^2(t_T - t_i)} \\
&\quad \cdot \int_{-\infty}^{+\infty} \alpha(\xi_1)\mathbb{i}\xi_1 a_{i+1}c_{i+1}e^{2\pi\mathbb{i}\xi_1\left[b_{i+1}(-\frac{1}{2}\sigma^2(t_s-t_i)+\log X(t_i)+\sigma u_1)+c_{i+1}u_3\right]}d\xi_1 \\
&\quad \cdot \int_{-\infty}^{+\infty} \alpha(\xi_2)\mathbb{i}\xi_2 a_{i+1}c_{i+1}e^{2\pi\mathbb{i}\xi_2\left[b_{i+1}(-\frac{1}{2}\sigma^2(t_s-t_i)+\log X(t_i)+\sigma u_1)+c_{i+1}u_3\right]}d\xi_2\psi(u_1, u_2, u_3)du_1du_2du_3 \\
&= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \alpha(\xi_1)\alpha(\xi_2)\xi_1\xi_2 a_{i+1}^2 c_{i+1}^2 e^{2\pi\mathbb{i}(\xi_1+\xi_2)b_{i+1}\left(-\frac{1}{2}\sigma^2(t_s-t_i)+\log X(t_i)\right)-\frac{3}{2}\eta^2(t_T-t_i)} \\
&\quad \cdot e^{-\eta(u_1+u_2)+2\pi\mathbb{i}b_{i+1}\sigma u_1(\xi_1+\xi_2)+2\pi\mathbb{i}c_{i+1}u_3(\xi_1+\xi_2)}\psi(u_1, u_2, u_3)du_1du_2du_3d\xi_1d\xi_2 \\
&= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \alpha(\xi_1)\alpha(\xi_2)\xi_1\xi_2 a_{i+1}^2 c_{i+1}^2 e^{2\pi\mathbb{i}(\xi_1+\xi_2)b_{i+1}\left(-\frac{1}{2}\sigma^2(t_s-t_i)+\log X(t_i)\right)-\frac{3}{2}\eta^2(t_T-t_i)} \\
&\quad \cdot e^{\frac{1}{2}(t_s-t_i)\iota_1^2+\frac{1}{2}(t_s-t_i)\iota_2^2+\frac{1}{2}\eta^2(t_T-t_i)}d\xi_1d\xi_2
\end{aligned}$$

with

$$\iota_1 = 2\pi\mathbb{i}b_{i+1}\sigma(\xi_1 + \xi_2)$$

$$\iota_2 = 2\pi\mathbb{i}c_{i+1}(\xi_1 + \xi_2).$$

(4.80) follows immediately by considering the integration on time interval $[t_i, t_{i+1}]$. $\square$

CHAPTER 5

CONCLUSION

The first part of the dissertation focuses on an optimal liquidation problem with dark pools using a market impact model. In chapter 2, we propose a market impact model which includes the cross-impact between two venues, and we derive the optimal execution strategy. Observing that there exists the possibility for transaction-triggered price manipulation, we use this model to identify a market condition such that price manipulation is not beneficial.

There is much more research that could be conducted on dark pools. As an alternative trading venue which is relatively new to the public, dark pools have not been thoroughly studied. To our knowledge, there is no existing model which characterizes the mechanisms of dark pools in general. For example, our model assumes that there is no partial fulfillment of dark pool orders. This should be extended to accommodate partial orders. Furthermore, we have considered only a single order type in the dark pool. In practice, different dark pools are experimenting with a variety of order types (Limit, market, peg-to-national-best-bid, peg-to-midpoint, national-best-offer, minimum-quantity, day and IOC, etc.). There are no models available now to allow consideration of these different order types. Another challenge is to propose a proper model for the execution price in dark pools. As we discussed in our work, price manipulation can exist in a market impact extended model. Proposing a price model for dark pools which guarantees the absence of price manipulation would be meaningful for both regulation and market efficiency.

The second part of the dissertation solves a multi-product inventory hedging problem. We consider both the single- and multiple-period problems, and prove, in both cases, a separation result for inventory management. This allows each inventory department to make decisions independently. In particular, the separation result for the multi-period problem is a global separation in the sense that no interaction needs to be considered among products in intermediate time periods. In addition, we propose a dynamic programming algorithm of the multi-period single-item inventory problem which further simplifies the computation by reducing the dimensionality of the state space. In the literature, the Föllmer-Schweizer decomposition is used for analytical representations. We extend this result with a Fast Fourier Transformation scheme to apply the Föllmer-Schweizer decomposition numerically.

The separation results for inventory hedging introduced in this dissertation are among the first in the literature to deal with multi-product inventory hedging issues. Our work completes the separation results in the sense that it solves both single and multiple period problems. Despite that, there are still some extensions to be considered in future research. For example, instead of considering a mean-variance type objective function, alternative risk-averse objectives should be analyzed, such as the exponential utility function. Alternatively, one can replace our assumption that the retail prices are exogenous, and consider a pricing problem instead, which leads to an equilibrium model.

Thus, we have successfully extended the ideas of Louis Bachelier into the new world of dark pool trading and into the old world of inventory management.

BIBLIOGRAPHY

Almgren, R. (2003), 'Optimal execution with nonlinear impact functions and trading-enhanced risk', *Applied Mathematical Finance* **10**(1), 1–18.

Bertsimas, D. & Lo, A. (1998), 'Optimal control of execution costs', *Journal of Financial Markets* **1**(1), 1–50.

Caldentey, R. & Haugh, M. (2006), 'Optimal control and hedging of operations in the presence of financial markets', *Mathematics of Operations Research* **31**(2), 285–304.

Caldentey, R. & Haugh, M. (2009), 'Supply contracts with financial hedging', *Operations Research* **57**(1), 47–65.

Černỳ, A. & Kallsen, J. (2007), 'On the structure of general mean-variance hedging strategies', *The Annals of Probability* **35**(4), 1479–1531.

Chen, X., Sim, M., Simchi-Levi, D. & Sun, P. (2007), 'Risk aversion in inventory management', *Operations Research* **55**(5), 828.

Choi, T., Li, D. & Yan, H. (2008), 'Mean–Variance Analysis for the Newsvendor Problem', *Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on* **38**(5), 1169–1180.

Degryse, H., Van Achter, M. & Wuyts, G. (2009), 'Dynamic order submission strategies with competition between a dealer market and a crossing network', *Journal of Financial Economics* **91**(3), 319–338.

Dunford, N. & Schwartz, J. (1988), 'Linear operators. Part I. General theory. With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1958 original'.

Föllmer, H. & Schweizer, M. (1991), *Hedging of contingent claims under incomplete information*, Vol. 5, Gordon and Breach, London/New York.

Foucault, T. & Menkveld, A. (2008), ‘Competition for order flow and smart order routing systems’, *The Journal of Finance* **63**(1), 119–158.

Gatheral, J. (2010), ‘No-dynamic-arbitrage and market impact’, *Quantitative Finance* **10**(7), 749–759.

Gaur, V. & Seshadri, S. (2005), ‘Hedging inventory risk through market instruments’, *Manufacturing & Service Operations Management* **7**(2), 103–120.

Huberman, G. & Stanzl, W. (2004), ‘Price Manipulation and Quasi-Arbitrage’, *Econometrica* **72**(4), 1247–1275.

Jarrow, R. & Yildirim, Y. (2003), ‘Pricing treasury inflation protected securities and related derivatives using an HJM model’, *Journal of Financial and Quantitative Analysis* **38**(02), 337–358.

Karmarkar, U. (1987), ‘The multilocation multiperiod inventory problem: Bounds and approximations’, *Management Science* **33**(1), 86–94.

Kratz, P. & Schöneborn, T. (2010), ‘Optimal liquidation in dark pools’, *Preprint* .

Laruelle, S. & Lehalle, C. (2009), ‘Optimal split of orders across liquidity pools: a stochastic algorithm approach’, *Arxiv preprint arXiv:0910.1166* .

Mittal, H. (2008), ‘Are You Playing in a Toxic Dark Pool?’, *The Journal of Trading* **3**(3), 20–33.

Pham, H. (2009), *Continuous-time stochastic control and optimization with financial applications*, Vol. 61, Springer Verlag.

Rockafellar, R. (1968), ‘Integrals which are convex functionals.’, *Pacific journal of mathematics* **24**(3), 525–539.

Schied, A. & Schöneborn, T. (2007), 'Optimal portfolio liquidation for CARA investors', *preprint* .

Schweizer, M. (1992), 'Mean-variance hedging for general claims', *The Annals of Applied Probability* **2**(1), 171–179.

Schweizer, M. (1995), 'On the minimal martingale measure and the Föllmer-schweizer decomposition', *Stochastic Analysis and Applications* **13**(5), 573–599.

Schweizer, M. et al. (1999), *A guided tour through quadratic hedging approaches*, Sonderforschungsbereich 373.

Shreve, S. (2004), *Stochastic Calculus for Finance II (Continuous-Time Models)*, Springer.

Sun, Y., Wissel, J. & Jackson, P. (2011), 'Multi-product separation result for inventory management under inflation risk'.

Wu, J., Li, J., Wang, S. & Cheng, T. (2009), 'Mean-variance analysis of the newsvendor model with stockout cost', *Omega* **37**(3), 724–730.

Ye, M. (2010), 'A glimpse into the dark: Price formation, transaction costs, and market share in the crossing newwork.', *Preprint* .