ANALYSIS OF FOUR PARTICLE SYSTEMS

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This thesis deals with four models of stochastic dynamics on relevant large finite systems.

The first one is the contact process on random graphs on n vertices with power law degree distributions. If the infection rate is λ, then nonrigorous mean field calculations suggest that the critical value λ_c of the infection rate is positive when the power α is larger than 3. Physicists seem to regard this as an established fact, since the result has recently been generalized to bipartite graphs in [25]. Here, we show that the critical value λ_c is zero for any value of α larger than 3, and the contact process starting from all vertices infected, with a probability tending to 1 as n increases to infinity, maintains a positive density of infected vertices for time at least $\exp(n^{1-\delta})$ for any positive δ. We also establish the existence of a quasi-stationary distribution in which a randomly chosen vertex is infected with probability $\rho(\lambda)$. It is expected that $\rho(\lambda)$ is asymptotically $C\lambda^\beta$ as λ decreases to zero for some positive constants C and β. Here we show that β lies between $\alpha - 1$ and $2\alpha - 3$, and so β is larger than 2 for any α larger than 3. Thus even though the graph is locally tree-like, β does not take the mean field critical value which equals 1.

The second one is a model for gene regulatory networks that is a modification of Kauffmann's [30] random Boolean networks. There are three parameters:

n = the number of nodes, r = the number of inputs to each node, and p = the expected fraction of 1's in the Boolean functions at each node. Following a standard practice in the physics literature, we use an appropriate threshold contact process on a random graph on n nodes, in which each node has in degree r, to approximate its dynamics. We show that if r is larger than 2 and $r \cdot 2p(1-p)$ is larger than 1, then the threshold contact process persists for a long time, which corresponds to chaotic behavior of the Boolean network. We prove that the persistence time is at least $\exp\left(cn^{b(p)}\right)$ with $b(p) > 0$ when $r \cdot 2p(1-p) > 1$, and $b(p) = 1$ when $(r-1) \cdot 2p(1-p) > 1$.

The third one is related to a gossip process defined by Aldous [3]. In this process, space is a discrete $N \times N$ torus, and the state of the process at time t is the set of individuals who know the information. Information spreads from a vertex to its nearest neighbors at rate $1/4$ each and at rate $N^{-\alpha}$ to a vertex chosen at random from the torus. We will be interested in the case in which α is smaller than 3, where the long range transmissions significantly accelerate the time at which everyone knows the information. We prove three results that precisely describe the spread of information in a slightly simplified model on the real torus. The time until everyone knows the information is asymptotically $(2 - 2\alpha/3)N^{\alpha/3}\log N$. After an appropriate random centering and scaling by $N^{\alpha/3}$, the fraction of informed population is almost a deterministic function which satisfies an integro-differential equation.

The fourth and the final one is about the discrete time threshold-two contact process on a random r-regular graph on n vertices. In this process, a vertex with at least two occupied neighbors at time t will be occupied at time $t+1$ with probability p, and vacant otherwise. We use a suitable isoperimetric inequality to show that if r is larger than 3 and p is close enough to 1, then starting from all vertices occupied, there is a positive density of occupied vertices up to time $\exp(c(p)n)$ for some positive constant $c(p)$. In the other direction, another appropriate isoperimetric inequality allows us to show that there is a decreasing function $\epsilon_2(p)$ so that if the number of occupied vertices in the initial configuration is smaller than $\epsilon_2(p)n$, then with high probability all vertices are vacant at time $2\log n/\log(2/(1+p))$. These two conclusions imply that the density of occupied vertices in the quasi-stationary distribution (defined in Chapter 5) is discontinuous at the critical probability $p_c \in (0,1)$.

BIOGRAPHICAL SKETCH

Shirshendu Chatterjee was born in November, 1982 in Kolkata, India. After completing schooling from Rahara Ramakrishna Mission Boys' Home, he joined the Bachelor of Statistics (B.Stat) program at the Indian Statistical Institute (ISI) in July, 2001. Upon graduating in 2004 with a Bachelor's degree, he decided to continue studying at the same institute for a Master of Statistics (M.Stat) degree.

After spending five wonderful years at ISI, Shirshendu joined the School of Operations Research and Information Engineering (ORIE) of Cornell University in August, 2006 to pursue his Ph.D. with concentration in applied probability and statistics.

Shirshendu has accepted an instructor position in the Courant Institute of Mathematical Sciences, New York University and he will join there soon after completing his Ph.D.

To my family

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Chapter 1

Overview

In recent years it has become increasingly clear that to effectively understand complex stochastic systems, it is crucial to analyze the interplay between the underlying spatial structure and the stochastic dynamics of the system. It has been unanimously established that many social, biological and technological systems are complex networks. However after the structures have been estimated and the geometric properties of the graphs such as their ”small world” nature have been studied, there remains the question: how does the structure of the networks affect the behavior of processes taking place on the networks? This thesis considers several processes in the context of epidemiology, biology and percolation of information that take place on large finite networks.

1.1 Contact process on power-law random graphs

There is empirical evidence that many real-world communication networks, such as the Internet network [22], social networks [39], human sexual contact networks [36] etc., have degree distributions with *power-law* tails, i.e. the degree of a typical vertex is k with probability

$$p_k \sim Ck^{-\alpha} \text{ as } k \to \infty \text{ for some constants } C, \alpha > 0. \tag{1.1.1}$$

However, in addition to estimating the degree distributions, one must consider the implications for the behavior of processes that take place on these networks.

One of the standard models used in the study of viral infections is the *contact process*, also called the *susceptible-infected-susceptible (SIS) model*, which has been studied extensively for bounded degree homogeneous graphs [35, Part I]. In this model, every vertex of the underlying graph is either infected or healthy (but susceptible). An infected vertex becomes healthy at rate 1 independent of the status of other vertices, and a healthy vertex becomes infected at a rate equal to the infection rate, λ, times the number of infected neighbors. In order to study epidemics on real-world networks, it is natural to consider the contact process on the networks with power-law degree distributions.

Motivated by this, we have studied the behavior of the contact process with infection rate λ on a random graph G_n on n vertices with power-law degree distributions, i.e. the degree d_i of any vertex i satisfies (1.1.1).

Nonrigorous mean field predictions [43, 44, 45] suggest that if the power $\alpha > 3$ (which is equivalent to the finite second moment condition for the degree distribution), then the critical value λ_c of the infection rate is positive, i.e. for small enough infection rate, everyone heals quickly. Also the critical value increases to 1 as α increases. Physicists seem to regard this as an established fact, since the result has recently been generalized to bipartite graphs [25]. We show that the critical value λ_c for the contact process on G_n is zero for any $\alpha > 3$. So there is always a chance of an epidemic even if the infection rate is small.

Ours was not the first result in this direction. The contact process on a generalization of the *preferential attachment graph* was considered by Berger, Borgs, Chayes, and Saberi [6]. They argue that the critical value $\lambda_c = 0$ for their graph model. Their arguments also suggest that the infection on such a graph model may persist for a time longer than $\exp\left(cn^{1/(\alpha-1)}\right)$ for some constant $c > 0$, when the associated degree distribution satisfies (1.1.1).

Based on the behavior of the contact process on $(\mathbb{Z} \mod n)$ [19, 21] and on $(\mathbb{Z} \mod n)^d$ [37], it is natural to conjecture that the right lower bound for the maximum possible survival of the infection for the contact process on G_n is $\exp(cn)$ for some constant $c > 0$. Here we almost prove the right lower bound for the persistence time: with high probability the contact process on G_n starting from all infected vertices maintains a positive density of infected vertices till time $\exp(n^{1-\delta})$ for any $\delta > 0$.

We also establish the existence of a quasi-stationary distribution in which a randomly chosen vertex is infected with probability $\rho(\lambda)$. It is expected that for some critical exponent $\beta > 0$, $\rho(\lambda) \sim C(\lambda - \lambda_c)^\beta$ as λ decreases to λ_c. We prove $\alpha - 1 \leq \beta \leq 2\alpha - 3$. Our bounds for β disproves the nonrigorous mean field predictions about the critical exponent as well.

In the physics literature the mean field arguments are widely used and believed to give correct results specially in case of locally tree-like graphs. But they lead to erroneous conclusions for the contact process on G_n, even though the random graph G_n is locally tree-like.

1.2 Random Boolean networks

Experimental evidence [1] suggests that the complex kinetics involved in different steps of a transcriptional pathway in real biological systems are, in many cases, reasonably well approximated by much simplified Boolean network models. In these models, each gene is represented by a node of a directed network and each node has one of two states: 'on' (i.e. expressing its target protein) or 'off'. The state of every node is simultaneously updated according to some function of its inputs, which approximates the action of activators (or inhibitors), i.e. proteins that act to increase (or decrease) expression of a given gene.

Random Boolean networks were originally developed by Kauffman [30] an abstraction of genetic regulatory networks. Recently similar approaches have been used in [29] and [33] to model the cell-cycle and transcriptional networks for yeast respectively. We consider a modification of Kauffman's model. There are three parameters: n = the number of nodes, r = the number of inputs to each node, and p = the expected fraction of 1's in the Boolean functions at each node. The state of a node $x \in V_n \equiv \{1, 2, \ldots, n\}$ at time $t = 0, 1, 2, \ldots$ is $\eta_t(x) \in \{0, 1\}$, and each node x receives input from r uniformly chosen distinct nodes $y_1(x), \ldots, y_r(x) \in V_n \setminus \{x\}$, which are called input nodes for x. We put oriented edges to each node from its input nodes to get a random graph G_n having uniform distribution over the collection of all directed graphs on the vertex set V_n in which each vertex has in-degree r. Once chosen the graph remains fixed through time. The updating rule for node x is $\eta_{t+1}(x) = f_x(\eta_t(y_1(x)), \ldots, \eta_t(y_r(x)))$, where the values $f_x(v)$, $x \in V_n$, $v \in \{0, 1\}^r$, chosen at the beginning and then fixed for all time, are independent and $= 1$ with probability p.

An important question for these Boolean network models is: *when is the network 'chaotic' (i.e. the values $(\eta_t(x), x \in V_n)$ fluctuate for a long time), and when is* *it 'ordered' (i.e. those values stabilize quickly)?* Real biological systems avoid the chaotic phase as expected, see e.g. [31, 46, 42]. A number of simulation studies have investigated the behavior of these Boolean network models, see e.g. [2] for a survey. The degenerate case of $r = 1$ has been studied [24] in detail.

Derrida and Pomeau [15] have argued that a network is *'chaotic'*, if $r \cdot 2p(1-p) > 1$, and *'ordered'*, if $r \cdot 2p(1-p) < 1$. To explain their conclusion, we have considered another process $\boldsymbol{\zeta} \equiv \{\zeta_t(x) \in \{0,1\} : t \geq 1, x \in V_n\}$, which they have called the *annealed approximation*, where $\zeta_t(x) = 1$ if and only if $\eta_t(x) \neq \eta_{t-1}(x)$. Following a standard practice in the physics literature, we have used a threshold contact process to approximate $\boldsymbol{\zeta}$.

$$P\left(\zeta_{t+1}(x) = 1 \middle|\, \zeta_t(y_1(x)) + \cdots + \zeta_t(y_r(x)) > 0\right) = 2p(1-p) \equiv q.$$

It is widely accepted that the condition for prolonged persistence of the threshold contact process is $qr > 1$. As in Section 1.1, the maximum possible persistence time is $\exp(\gamma n)$ for some constant $\gamma > 0$. We prove that if $q(r-1) > 1$, then the threshold contact process on G_n, starting from the all-one configuration, persists for time $\geq \exp(\gamma n)$ for some constant $\gamma > 0$.

The '$r-1$' in the condition occurs because we use an "isoperimetric inequality" to bound a worst-case scenario. We have also shown that if $qr > 1$, then the threshold contact process on G_n, starting from the all-one configuration, persists for time $\geq \exp\left(\gamma n^{B(q)}\right)$, where $B(q) \approx (1/8)\log(qr)/\log(r)$.

The quasi-stationary density of 1s' is given by the survival probability of an appropriate branching process.

1.3 Aldous' Gossip Process

In the last few years there has been a lot of interest in studying many real-world networks including social and professional networks. In these systems information sometimes reaches one part of the network, and then gradually circulates in the entire network. Exchange of information during *insider trading* in the financial market and *gossip* percolation in a society are two such examples. In order to study the percolation of information through networks one of the main technical tools is the *first-passage percolation process* associated with the communication strategy of the network agents.

In this context, Aldous [3] considered a first-passage percolation process, which he called short-long FPP, on the $N \times N$ torus. In this process, the state of any vertex is either 1 (informed) or 0 (uninformed). Once a vertex gets the information, it never loses it. If x is informed, each of its uninformed neighbors gets the information at rate $1/4$. In addition, at rate $N^{-\alpha}$ it sends the information to a vertex uniformly chosen from the torus.

The most important question in percolation of information is: *how quickly does the information spread and if T_N is the cover time, i.e. the time when everyone has got the information, then how does it grow with the size of the network?* In order to have a deeper understanding of the percolation process and to analyze its consequences, one also needs to know: *what are the appropriate centering and scaling factors for the size of the set of informed individuals and after the right centering and scaling how does the proportion of informed individuals increase from 0 to 1?*

Here we answer these questions for the short-long FPP process, but for a slightly simplified model on the (real) torus $(\mathbb{R} \mod N)^2$, which we call "balloon process" $\mathcal{C}_t$. The balloon process starts with one "center" chosen uniformly from the torus at time 0. When a center is born at x, a disk with radius 0 is put there, and its radius grows deterministically so that the area of the disk after time s is $s^2/2$. New centers are born at rate $N^{-\alpha}|\mathcal{C}_t|$. The location of each new center is chosen uniformly from the torus. A new center landing on $\mathcal{C}_t$ has no contribution. For the balloon process we have:

- if $\alpha \geq 3$ and T_N is the cover time, then T_N/N converges in distribution to a limit, which is a point mass at $\sqrt{\pi}$ if $\alpha > 3$,
- if $\alpha < 3$, then there is a random variable M so that for $\psi(t) := N^{\alpha/3}[(2 - 2\alpha/3)\log N - \log M + t]$, $N^{-2}|\mathcal{C}_{\psi(\cdot)}|$ converges in probability to a deterministic limit $h(\cdot)$ satisfying
$$h(t) = 1 - \exp\left(-\int_{-\infty}^{t} h(s)\frac{(t-s)^2}{2}\, ds\right)$$
uniformly on compact time sets.
- if $\alpha < 3$, then $T_N/N^{\alpha/3}\log N$ converges to $2 - 2\alpha/3$.

So, the long range transmission significantly accelerates the cover time only when $\alpha < 3$. In that case, there is a cutoff phenomenon, as the time that the fraction of covered area takes to reach a small level ϵ is $O(N^{\alpha/3}\log N)$, whereas the time that it takes to increases from ϵ to 1 is $O(N^{\alpha/3})$, which is much smaller than the previous time.

1.4 Threshold-two contact process on random regular graphs

In many situations, e.g. social networks, random graphs are better models than regular lattices for the spatial structure of the underlying system. Because of this, particle systems on random graphs need to be studied, as they often behave much differently from their Euclidean analogues.

Inspired by our study of random Boolean networks, and the fact that the sexual reproduction model has been studied extensively on regular lattices, see [20, 38], we study the behavior of the *threshold-two contact process* on random undirected r-regular graphs on n vertices. In this discrete time system, the state of a site $x \in V_n := \{1, 2, \ldots, n\}$ at time $t = 0, 1, \ldots$, $\zeta_t(x)$, is either 0 (vacant) or 1 (occupied). $\zeta_{t+1}(x) = 1$ with probability p, if at least two of the neighbors of x are occupied at time t, and $\zeta_{t+1}(x) = 0$ otherwise.

Like many other particle systems, the first question is whether there is any phase transition in the behavior of the system. The next concern is whether there is any quasi-stationary distribution as in the case of the contact process on power-law random graphs, and if yes, what are the properties of the corresponding density? Here we address these questions.

Using appropriate isoperimetric inequalities we prove that that the critical probability p_c, which defines the boundary between rapid convergence to all-zero configuration within logarithmically small time and exponentially prolonged persistence of changes, lies strictly between 0 and 1. We also show that for $p > p_c$ there is a quasi-stationary distribution with density $u(p) > 0$. Note that $u(p)$ is an analogue of the density of occupied vertices in the upper invariant measure for the contact process with sexual reproduction on regular lattices, which is conjectured to have a continuous phase transition (see Conjecture 1 and heuristic argument following that in [20]). Here we show

$$\inf_{p>p_c} u(p) > 0.$$

So, unlike the predicted behavior of its Euclidean analogue, the quasi-stationary density of the threshold-two contact process on a random regular graph is discontinuous at the critical value p_c.

Chapter 2

Contact process on power-law random graphs

2.1 Introduction

In this chapter we will study the contact process on random graphs with a power-law degree distribution, i.e., for some constant α, the degree of a typical vertex is k with probability $p_k \sim Ck^{-\alpha}$ as $k \to \infty$. Following Newman, Strogatz and Watts [40, 41], we construct the random graph G_n on the vertex set $\{1, 2, \ldots, n\}$ having degree distribution $\mathbf{p} = \{p_k : k \geq 0\}$ as follows. Let $d_1, \ldots, d_n$ be independent and have the distribution $P(d_i = k) = p_k$. We condition on the event $E_n = \{d_1 + \cdots + d_n \text{ is even}\}$ to have a valid degree sequence. As $P(E_n) \to 1/2$ as $n \to \infty$, the conditioning will have a little effect on the distribution of d_i's. Having chosen the degree sequence $(d_1, d_2, \ldots, d_n)$, we allocate d_i many half-edges to the vertex i, and then pair those half-edges at random. We also condition on the event that the graph is simple, i.e., it neither contains any self-loop at some vertex, nor contains multiple edges between two vertices. It can be shown (see e.g. [16, Theorem 3.1.2]) that if the degree distribution $\mathbf{p}$ has finite second moment, i.e., if $\alpha > 3$, the probability of the event that G_n is simple has a positive limit as $n \to \infty$, and hence the conditioning on this event will not have much effect on the distribution of d_i's.

We will be concerned with epidemics that take place on these random graphs. First consider the SIR (susceptible-infected-removed) model, in which sites begin as susceptible, and after being infected they get removed, i.e., become immune to further infection. In the simplest discrete-time formulation, an infected site x at time n will always be removed at time $n+1$ and for each susceptible neighbor y at time n x will cause y to become infected at time $n+1$ with probability p, with all of the infection events being independent.

In this case the spreading of the epidemic is equivalent to percolation. To compute the threshold p_c for a large, i.e., $O(n)$, epidemic to occur with positive probability, one notes that for a randomly chosen vertex x, the number of vertices at distance m from x, Z_m, is approximately a two-phase branching process in which the number of first generation children has distribution $\mathbf{p}$, but in the second and subsequent generations the offspring distribution is the size biased distribution $\mathbf{q} = \{q_k : k \geq 0\}$ satisfying

$$q_{k-1} = \frac{kp_k}{\mu} \quad \text{where } \mu = \textstyle\sum_k kp_k. \tag{2.1.1}$$

This occurs because vertices with degree k are k times as likely to be chosen for connections, and the edge that brings us to the new vertex uses up one of its degrees. For more details on this and the facts that we will quote in the next paragraph, see [16, Chapter 3].

With the above observation in hand, it is easy to compute the critical threshold for the SIR model. Let ν be the mean of the size biased distribution,

$$\nu = \sum_k kq_k. \tag{2.1.2}$$

Suppose we start the infection at a randomly chosen vertex x. Now if Y_m is the number of sites at distance m from x that become infected, then $EY_m =$

$p\mu(p\nu)^{m-1}$. So the epidemic is supercritical if and only if $p > 1/\nu$. In particular, if $p_k \sim Ck^{-\alpha}$ as $k \to \infty$ and $\alpha \le 3$, then $\nu = \infty$ and $p_c = 0$. Conversely if $\alpha > 3$ then $\nu < \infty$ and $p_c = 1/\nu > 0$. Hence for the SIR epidemic model on the random graph G_n with power-law degree distribution, there is a positive threshold for the infection to survive if and only if the power $\alpha > 3$.

We will study the continuous-time SIS (susceptible-infected-susceptible) model and show that its behavior differs from that of the SIR model. In the SIS model, at any time t each site x is either infected or healthy (but susceptible). We often refer to the infected sites as occupied, and the healthy sites as vacant. We define the functions $\{\zeta_t : t \ge 0\}$ on the vertex set so that $\zeta_t(x)$ equals 0 or 1 depending on whether the site x is healthy or infected at time t. An infected site becomes healthy at rate 1 independent of other sites and is again susceptible to the disease, while a susceptible site becomes infected at a rate λ times the number of its infected neighbors. Harris [27] introduces this model on the d-dimensional integer lattice and named it the *contact process*. See [35] for an account of most of the known results. We will make extensive use of the *self-duality property* property of this process. If we let $\xi_t \equiv \{x : \zeta_t(x) = 1\}$ to be the set of infected sites at time t, we obtain a set-valued process. If we write ξ_t^A to denote the process with $\xi_0^A = A$, then the self-duality property says that

$$P(\xi_t^A \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset) \tag{2.1.3}$$

for any two subsets A and B of vertices.

Pastor-Satorras and Vespignani [44, 43, 45] have made an extensive study of this model using mean-field methods. Their nonrigorous computations suggest the following conjectures about λ_c the threshold for "prolonged persistence" of the contact process.

- If $\alpha \leq 3$, then $\lambda_c = 0$.
- If $3 < \alpha \leq 4$, then $\lambda_c > 0$ but the critical exponent β, which controls the rate at which the equilibrium density of infected sites goes to 0, satisfies $\beta > 1$.
- If $\alpha > 4$, then $\lambda_c > 0$ and the equilibrium density $\sim C(\lambda - \lambda_c)$ as $\lambda \downarrow \lambda_c$, i.e. the critical exponent $\beta = 1$.

Notice that the conjectured behavior of λ_c for the SIS model parallels the results for p_c in the SIR model quoted above.

Gómez-Gardeñes et al. [25] have recently extended this calculation to the bipartite case, which they think of as a social network of sexual contacts between men and women. They define the polynomial decay rates for degrees in the two sexes to be γ_M and γ_F, and argue that the epidemic is supercritical when the transmission rates for the two sexes satisfy

$$\sqrt{\lambda_M \lambda_F} > \lambda_c = \sqrt{\frac{\langle k \rangle_M \langle k \rangle_F}{\langle k^2 \rangle_F \langle k^2 \rangle_M}}$$

where the angle brackets indicate expected value and k is shorthand for the degree distribution. Here λ_c is positive when $\gamma_M, \gamma_F > 3$.

Our first goal is to show that $\lambda_c = 0$ for all $\alpha > 3$. Our proof starts with the following observation due to Berger, Borgs, Chayes, and Saberi [6]. Here, we follow the formulation in [16, Lemma 4.8.2].

Lemma 2.1.1. *Suppose G is a star graph with center 0 and leaves $1, 2, \ldots, k$. Let A_t be the set of vertices infected in the contact process at time t when $A_0 = \{0\}$. If $k\lambda^2 \to \infty$, then $P(A_{\exp(k\lambda^2/10)} \neq \emptyset) \to 1$.*

Based on results for the contact process on $(\mathbf{Z} \bmod n)$ [19, 21], and on $(\mathbf{Z} \bmod n)^d$ [37], it is natural to conjecture that in the contact process on G_n, with probability tending to 1 as $n \to \infty$, the infection survives for time $\geq \exp(cn)$ for some constant c. It certainly cannot last longer, because the total number of edges is $O(n)$, and so even if all sites are occupied at time 0, there is a constant c so that with probability $\geq \exp(-cn)$ all sites will be vacant at time 1. Our next result falls a little short of that goal.

Theorem 2.1.2. *Consider a Newman, Strogatz and Watts random graphs G_n on the vertex set $\{1, 2, \ldots, n\}$, where the degrees d_i satisfy $P(d_i = k) \sim Ck^{-\alpha}$ as $k \to \infty$ for some constant C and some $\alpha > 3$, and $P(d_i \leq 2) = 0$. Let $\{\xi_t^1 : t \geq 0\}$ denote the contact process on the random graph G_n starting from all sites occupied, i.e., $\xi_0^1 = \{1, 2, \ldots, n\}$. Then for any value of the infection rate $\lambda > 0$, there is a positive constant $p(\lambda)$ so that for any $\delta > 0$*

$$\inf_{t \leq \exp\left(n^{1-\delta}\right)} P\left(\frac{|\xi_t^1|}{n} \geq p(\lambda)\right) \to 1 \quad \text{as } n \to \infty.$$

One could assume that $\nu > 1$ and look at the process on the giant component, but we would rather avoid this complication. The assumption $P(d_i \leq 2) = 0$ is convenient, because it implies the following.

Lemma 2.1.3. *Consider a Newman, Strogatz and Watts graphs, G_n, on n vertices, where the degrees of the vertices, d_i, satisfy $P(d_i \leq 2) = 0$, and the mean of the size* *biased degree distribution $\nu < \infty$. Then*

$$P(G_n \text{ is connected }) \to 1 \quad \text{as } n \to \infty,$$

and if D_n is the diameter of G_n,

$$P(D_n > (1+\epsilon)\log n/\log \nu) \to 0 \quad \text{for any } \epsilon > 0.$$

The size of the giant component in the graph is given by the nonextinction probability of the two-phase branching process, so $P(d_i \le 2) = 0$ is needed to have the size $\sim n$. Intuitively, Lemma 2.1.3 is obvious because the worst case is the random 3-regular graph, and in this case, the graph is not only connected and has diameter $\sim (\log n)/(\log 2)$, see [8, Sections 7.6 and 10.3], but the probability of a Hamiltonian cycle tends to 1, see [28, Section 9.3]. We have not been able to find a proof of Lemma 2.1.3 in the literature, so we give one in Section 5. By comparing the growth of the cluster with a branching process it is easy to show $P(D_n < (1-\epsilon)\log n/\log \nu) \to 0$ for any $\epsilon > 0$.

In a sense the main consequence of Theorem 1 is not new. Berger, Borgs, Chayes, and Saberi [6], see also [7], show that $\lambda_c = 0$ for a generalization of the Bárabasi-Albert model in which each new point has m edges which are with probability β connected to a vertex chosen uniformly at random and with probability $1-\beta$ to a vertex chosen with probability proportional to its degree. It has been shown [13, Theorem 2] that such graphs have power law degree distributions with $\alpha = 1 + 2/(1-\beta)$, so these examples have $\alpha \in [3,\infty)$ and $\lambda_c = 0$.

Having acknowledged the previous work of BBCS, it should be noted that (i) our result applies to a large class of power law graphs that have a different structure; and (ii) the BBCS proof yields a lower bound on the presistence time of $\exp(cn^{1/(\alpha-1)})$ compared to our $\exp(n^{1-\delta})$. Our improved bound on the survival times relies only on the power law degree distribution and the fact that the diameter is bounded by $C \log n$, so it also applies to graphs BBCS consider.

Theorem 2.1.2 shows that the fraction of infected sites in the graph G_n is bounded away from zero for a time longer than $\exp(n^{1/2})$. So using self-duality we can now define a quasi-stationary measure ξ^1_∞ on the subsets of $\{1, 2, \ldots, n\}$ as follows. For any subset of vertices A, $P(\xi^1_\infty \cap A \neq \emptyset) \equiv P(\xi^A_{\exp(n^{1/2})} \neq \emptyset)$. Let X_n be uniformly distributed on $\{1, 2, \ldots, n\}$ and let $\rho_n(\lambda) = P(X_n \in \xi^1_\infty)$. Berger, Borgs, Chayes and Saberi [6] show that for the contact process on their preferential attachment graphs, there are positive, finite constants so that

$$b\lambda^C \leq \rho_n(\lambda) \leq B\lambda^c.$$

In contrast, we get reasonably good numerical bounds on the critical exponent.

Theorem 2.1.4. *Suppose $\alpha > 3$. There is a $\lambda_0 > 0$ so that if $0 < \lambda < \lambda_0$ and $0 < \delta < 1$, then there exists two constants $c(\alpha, \delta)$ and $C(\alpha, \delta)$ so that as $n \to \infty$*

$$P(c\lambda^{1+(\alpha-2)(2+\delta)} \leq \rho_n(\lambda) \leq C\lambda^{1+(\alpha-2)(1-\delta)}) \to 1.$$

When α is close to 3 and δ is small, the powers in the lower and upper bounds are close to 3 and 2. The ratio of the two powers is $\leq (2 + \delta)/(1 - \delta) \approx 2$ when δ is small.

The intuition behind the lower bound is that if the infection starts from a vertex of degree $d(x) \geq (10/\lambda)^{2+\delta}$, then it survives for a long time with a probability bounded away from 0. The density of such points is $C\lambda^{(2+\delta)(\alpha-1)}$, but we can improve the bound to the one given by looking at neighbors of these vertices, which have density $C\lambda^{(2+\delta)(\alpha-2)}$ and will infect their large degree neighbor with probability $\geq c\lambda$.

For the upper bound we show that if $m(\alpha, \delta)$ is large enough and the infection starts from a vertex x such that there is no vertex of degree $\geq \lambda^{-(1-\delta)}$ within distance m from x, then its survival is very unlikely. To get the extra factor of λ we note that the first event must be a birth. Based on the proof of Lemma 2.1.1, we expect that survival is unlikely if there is no nearby vertex of degree $\geq \lambda^{-2}$ and hence the lower bound gives the critical exponent.

It is natural to speculate that the density of the quasi-stationary measure $\rho_n(\lambda) \to \rho(\lambda)$ as $n \to \infty$. By the heuristics for the computation of λ_c in the SIR model, it is natural to guess that, when $\alpha > 2$, $\rho(\lambda)$ is the expected probability of weak survival for the contact process on a tree generated by the two-phase branching process, starting with the origin occupied.

Here the phrase 'weak survival' refers to set of infected sites being not empty for all times, in contrast to 'strong survival' where the origin is reinfected infinitely often. As in the case of the contact process on the Bollobás-Chung small world studied in [18], it is the weak survival critical value that is the threshold for prolonged persistence on the finite graph.

Sketch of the proof of Theorem 1.

The remainder of the chapter is devoted to proofs. Let V_n^ϵ be the set of vertices in the graph G_n with degree at least n^ϵ. We call the points in V_n^ϵ *stars*. We say that a star of degree k is *hot* if at least $\lambda k/4$ of its neighbors are infected and is *lit* if at least $\lambda k/10$ of its neighbors are infected. Our first step, taken in Lemma 2.2.2, is to improve the proof of Lemma 2.1.1 to show that a hot star will remain lit for time $\exp(cn^{\epsilon})$ with high probability.

To keep the system going for a long time, we cannot rely on just one star. There are $O(n^{1-\epsilon(\alpha-1)})$ stars in this graph which has diameter $O(\log n)$. If one star goes out, presence of a lit star can make it hot again within a time $2n^{\epsilon/3}$ with probability at least n^{-b}. See Lemmas 2.2.3 and 2.2.4 for this. Lemma 2.2.6 shows that a lit star gets hot within $2\exp(n^{\epsilon/3})$ units of time with probability

$$\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16),$$

and Lemma 2.2.5 shows that a hot star eventually succeeds to make a non-lit star hot within $\exp(n^{\epsilon/2})$ units of time with probability

$$\geq 1 - 8e^{-\lambda^2 n^{\epsilon}/80}.$$

Using these estimates, we can show that the number of lit stars dominates a random walk with a strong positive drift, and hence more than 3/4's of the collection will stay lit for a time $O(\exp(n^{1-\alpha\epsilon}))$. See Proposition 2.2.7 at the end of Section 2 for the argument.

To get a lower bound on the density of infected sites, first we bound the probability of the event that the dual process, starting from a vertex of degree $(10/\lambda)^{2+\delta}$, reaches more than 3/4's of the stars. We do this in two steps. In the first step (see Lemma 2.3.2) we get a lower bound for the probability of the dual process reaching one of the stars. To do this, we consider a chain of events in which we reach vertices with degree $(10/\lambda)^{k+\delta}$ for $k \geq 2$ sequentially. In the second step (see Lemma 2.3.4) we again use a comparison with random walk to show that, with probability tending to 1, the dual process, starting from any lit star, will light up more than 3/4's of the stars. Then we show that the above events are asymptotically uncorrelated, and use a second moment argument to complete the proof of Theorem 2.1.2 and the lower bound for the density in Theorem 2.1.4.

Open Problem. *Improve the bounds in Theorem 2.1.4 and extend the result to $\alpha > 1$.*

When $2 < \alpha < 3$ the size biased distribution has infinite mean. Chung and Lu [11, 12] obtained bounds on the diameter in this case, and later it has been shown [48] that if H_n is the distance between 1 and 2 then

$$H_n \sim \frac{2 \log \log n}{-\log(\alpha - 2)}$$

When $1 < \alpha < 2$ the size-biased distribution has infinite mass. It has been shown [47] in this case

$$\lim_{n\to\infty} P(H_n = 2) = \lim_{n\to\infty} 1 - P(H_n = 3) = p \in (0, 1)$$

so the graph is very small.

All of the results about the persistence of infection at stars in Section 2 are valid for any α, since they only rely on properties of the contact process on a star graph and an upper bound on the diameter. The results in Section 3, rely on the existence of the size biased distribution and hence are restricted to $\alpha > 2$. The proof of the lower bound should be extendible to that case, but the proof of the upper bound given in Section 4 relies heavily on the size-biased distribution having finite mean. When $1 < \alpha < 2$, the size-biased distribution does not exist and the situation changes drastically. We guess that in this case $\rho_n(\lambda) = O(\lambda)$.

2.2 Persistence of infection at stars

Let $\epsilon > 0$ and let V_n^ϵ be the set of vertices in our graph G_n with degree at least n^ϵ. We call these vertices *stars*. We say that a vertex of degree k is *hot* if it has at least $L = \lambda k/4$ infected neighbors and we call it *lit* if it has at least $0.4L = \lambda k/10$ infected neighbors. We will show that if ϵ is small, then in the contact process starting from all vertices occupied, most of the stars in V_n^ϵ will remain lit for time $O(\exp(n^{1-\alpha\epsilon})$.

We begin with a slight improvement of Lemma 2.1.1 which gives a numerical estimate of the failure probability, but before that we need two simple estimates.

Lemma 2.2.1. *If $0 \le x \le a \le 1$ then $e^x \le 1 + (1+a)x$ and $e^{-x} \le 1 - (1 - 2a/3)x$.*

Proof. Using the series expansion for e^x

$$\begin{aligned} e^x &\le 1 + x + \frac{ax}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right) \\ e^{-x} &\le 1 - x + \frac{ax}{2}\left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\right) \end{aligned}$$

and summing the geometric series gives the result. □

Lemma 2.2.2. *Let G be a star graph with center 0 and leaves $1, 2, \ldots, k$. Let A_t be the set of vertices infected in the contact process at time t. Suppose $\lambda \le 1$ and $\lambda^2 k \ge 50$. Let $L = \lambda k/4$ and let $T = \exp(k\lambda^2/80)/4L$. Let $P_{L,i}$ denote the probability when at* *time 0 the center is at state i and L leaves are infected. Then*

$$P_{L,i}\left(\inf_{t\le T}|A_t|\le 0.4L\right)\le 7e^{-\lambda^2 k/80}\quad \textit{for } i=0,1.$$

Proof. Write the state of the system as (m,n) where m is the number of infected leaves and $n=1$ if the center is infected and 0 otherwise. To reduce to a one dimensional chain, we will concentrate on the first coordinate. When the state is $(m,0)$ with $m>0$, the next event will occur after exponential time with mean $1/(m\lambda+m)$, and the probability that it will be the reinfection of the center is $\lambda/(\lambda+1)$. So the number of leaf infections N that will die while the center is 0 has a shifted geometric distribution with success probability $\lambda/(\lambda+1)$, i.e.,

$$P(N=j)=\left(\frac{1}{\lambda+1}\right)^j\cdot\frac{\lambda}{\lambda+1}\quad \text{for } j\ge 0.$$

Let N_L be the realization of N when the state of the system is $(L,0)$. Then N_L will be more than $0.1L$ with probability

$$P_{L,0}(N_L>0.1L)\le(1+\lambda)^{-0.1L}\le e^{-\lambda L/20}=e^{-\lambda^2 k/80}. \tag{2.2.1}$$

Here we use the inequality $1+\lambda\ge e^{\lambda/2}$. If $N_L\le 0.1L$, then there will be at least $0.9L$ infected leaves when the center is infected.

The next step is to modify the chain so that the infection rate is 0 when the number of infected leaves is $L=\lambda k/4$ or greater. In this case the number of infected leaves $\ge Y_t$ where

	at rate	
$Y_t\to Y_t-1$	$\lambda k/4$	
$Y_t\to Y_t+1$	$3\lambda k/4$	for $Y_t<L$.

$$Y_t \to Y_t - N \qquad 1$$

To bound the survival time of this chain, we will estimate the probability that starting from $0.8L$ it will return to $0.4L$ before hitting L. During this time Y_t is a random walk that jumps at rate $\lambda k + 1$. Let X be the change in the random walk in one step. Then

$$X = \begin{cases} -1 & \text{with probability } (\lambda k/4)/(\lambda k + 1) \\ +1 & \text{with probability } (3\lambda k/4)/(\lambda k + 1) \\ -N & \text{with probability } 1/(\lambda k + 1), \end{cases}$$

and so

$$\begin{aligned} Ee^{\theta X} &= e^{\theta} \cdot \frac{3}{4} \cdot \frac{\lambda k}{\lambda k + 1} + e^{-\theta} \cdot \frac{1}{4} \cdot \frac{\lambda k}{\lambda k + 1} \\ &\quad + \frac{1}{\lambda k + 1} \sum_{j=0}^{\infty} e^{-\theta j} \left(\frac{1}{\lambda + 1} \right)^{j} \cdot \frac{\lambda}{\lambda + 1}. \end{aligned}$$

If $e^{-\theta}/(\lambda + 1) < 1$, the third term on the right is

$$\frac{\lambda}{\lambda k + 1} \cdot \frac{1}{1 + \lambda - e^{-\theta}}.$$

If we pick $\theta < 0$ so that $e^{-\theta} = 1 + \lambda/2$, then

$$Ee^{\theta X} = \frac{\lambda k}{\lambda k + 1} \left(\frac{1}{1 + \lambda/2} \cdot \frac{3}{4} + (1 + \lambda/2) \cdot \frac{1}{4} + \frac{2}{\lambda k} \right).$$

Since $1/(1 + x) < 1 - x + x^2$ for $0 < x < 1$,

$$\begin{aligned} &\frac{1}{1 + \lambda/2} \cdot \frac{3}{4} + (1 + \lambda/2) \cdot \frac{1}{4} + \frac{2}{\lambda k} - 1 \\ &< \left(-\frac{\lambda}{2} + \frac{\lambda^2}{4} \right) \frac{3}{4} + \frac{\lambda}{8} + \frac{2}{\lambda k} \\ &< -\frac{3\lambda}{16} + \frac{\lambda}{8} + \frac{2}{\lambda k}, \end{aligned}$$

where in the last inequality, we have used $\lambda < 1$. Since we have assumed $\lambda^2 k \geq 50$, the right-hand side is < 0.

To estimate the hitting probability we note that if $\phi(x) = \exp(\theta x)$ and $Y_0 \geq 0.6L$, then $\phi(Y_t)$ is a supermartingale until it hits L. Let q be the probability that Y_t hits the interval $(-\infty, 0.4L]$ before returning to L. Since $\theta < 0$, we have $\phi(x) \geq \phi(0.4L)$ for $x \leq 0.4L$. So using the optional stopping theorem we have

$$q\phi(0.4L) + (1-q)\phi(L) \leq \phi(0.8L),$$

which implies that

$$q \leq \phi(0.8L)/\phi(0.4L) = \exp(0.4\theta L) \leq e^{-\lambda^2 k/40},$$

as $e^{-\theta} = 1 + \lambda/2 \geq e^{\lambda/4}$ when $\lambda/4 < 1/2$ (sum the series for e^x).

At this point we have estimated the probability that the chain started at a point $\geq 0.8L$ will go to L before going below $0.4L$. When the chain is at L, the time until the next jump is exponential with mean $1/(L+1) \geq 1/2L$. The probability that the jump takes us below $0.8L$ is (since $1 + \lambda \geq e^{\lambda/2}$)

$$\leq (1+\lambda)^{-0.2L} \leq e^{-\lambda L/10} = e^{-\lambda^2 k/40}.$$

Thus the probability that the chain fails to return to L, $M = e^{\lambda^2 k/80}$ times before going below $0.4L$ is

$$\leq 2e^{-\lambda^2 k/80}.$$

Using Chebyshev's inequality on the sum, S_M of M exponentials with mean 1 (and hence variance 1),

$$P(S_M < M/2) \leq 4/M.$$

Multiplying by $1/2L$ we see that the total time, T_M of the first M excursions satisfies

$$P(T_M < M/4L) \leq 4e^{-\lambda^2 k/80}.$$

Combining this with the previous estimate on the probability of having fewer than M returns and the error probability in (2.2.1) proves the desired result. □

Thus Lemma 2.2.2 shows that a hot star will remain lit for a long time with probability very close to 1. Our next step is to investigate the process of transferring the infection from one star to another. The first step in doing that is to estimate what happens when only the center of the star infected.

Lemma 2.2.3. *Let G be a star graph with center 0 and leaves $1, 2, \ldots, k$. Let $0 < \lambda < 1$, $\delta > 0$ and suppose $\lambda^{2+\delta} k \geq 10$. Again let $P_{l,i}$ denote the probability when at time 0 the center is in state i and l leaves are infected. Let τ_0 be the first time 0 becomes healthy, and let T_j be the first time the number of infected leaves equals j. If $L = \lambda k/4$, $\gamma = \delta/(4+2\delta)$, and $K = \lambda k^{1-\gamma}/4$, then for $k \geq k_0(\delta)$*

$$\begin{aligned}
P_{0,1}(T_K > \tau_0) &\leq 2/k^\gamma, \\
P_{K,1}(T_0 < T_L) &\leq \exp(-\lambda^2 k^{1-\gamma}/16) \leq 1/k^\gamma, \\
E_{0,1}(T_L | T_L < \infty) &\leq 2.
\end{aligned}$$

Combining the first two inequalities $P_{0,1}(T_L < \infty) \geq 1 - 2/k^\gamma$, and using Markov's inequality, if we can infect a vertex of degree at least k such that $k \geq k_0(\delta)$ and $\lambda^{2+\delta} k > 10$, then with probability $\geq 1 - 5/k^\gamma$ the vertex gets hot within the next k^γ units of time.

Proof. Note that $\tau_0 \sim \exp(1)$, and for any $t \leq \tau_0$, the leaves independently becomes healthy at rate 1 and infected at rate λ. Let $p_0(t)$ is the probability that leaf j is infected at time t when the central vertex of the star has remained infected for all times $s \leq t$. $p_0(0) = 0$ and

$$\frac{dp_0(t)}{dt} = -p_0(t) + (1 - p_0(t))\lambda = \lambda - (\lambda + 1)p_0(t).$$

So solving gives $p_0(t) = \int_0^t \lambda e^{-(\lambda+1)(t-s)}\, ds = \frac{\lambda}{\lambda+1}\left(1 - e^{-(\lambda+1)t}\right)$. From this it follows that

$$P_{0,1}(T_K < \tau_0) \geq P(\text{Binomial}(k, p_0(k^{-\gamma})) > K)P(\tau_0 > k^{-\gamma}). \tag{2.2.2}$$

Now if $k^\gamma > 8/3$, $(\lambda + 1)k^{-\gamma} \leq 3/4$ and it follows from Lemma 2.2.1 that

$$p_0(k^{-\gamma}) \geq \lambda k^{-\gamma}/2.$$

Writing $p = p_0(k^{-\gamma})$ to simplify formulas, if $\theta > 0$

$$P(\text{Binomial}(k, p) \leq K) \leq e^{\theta K}\left(1 - p + pe^{-\theta}\right)^k.$$

Since $\log(1 + x) \leq x$ the right-hand side is

$$\leq \exp\left(\frac{\theta\lambda k^{1-\gamma}}{4} + (e^{-\theta} - 1)\frac{\lambda k^{1-\gamma}}{2}\right).$$

Taking $\theta = 1/2$ and using Lemma 2.2.1 to conclude $e^{-1/2} - 1 \leq -1/3$, the above is

$$\leq \exp(-\lambda k^{1-\gamma}/24) \leq \exp(-k^{1/2-\gamma}/8),$$

since $\lambda^2 k \geq 9$. Using this in (2.2.2), the right-hand side is

$$\geq (1 - \exp(-k^{1/2-\gamma}/8))(1 - k^{-\gamma}) \geq 1 - 2/k^\gamma,$$

if $k^{1/2-\gamma} \geq 8\gamma \log k$.

Using the supermartingale from the proof of Lemma 2.2.2, if $q = P_{K,1}(T_0 < T_L)$, then we have

$$q \cdot 1 + (1-q)e^{\theta L} \leq e^{\theta K},$$

and so $q \leq e^{\theta K} \leq e^{-\lambda K/4}$. In the last step we have used $e^{\theta} = 1/(1+\lambda/2) \leq e^{-\lambda/4}$, which comes from Lemma 2.2.1. Filling in the value of K, $e^{-\lambda K/4} = e^{-\lambda^2 k^{1-\gamma}/16}$. Now

$$\lambda^2 k^{1-\gamma} = (\lambda^{2+\delta} k)^{2/(2+\delta)} k^{1-\gamma-2/(2+\delta)} \geq 10^{2/(2+\delta)} k^{\delta/(4+2\delta)}.$$

So if $k^{\delta/(4+2\delta)} > 16 \cdot 10^{-2/(2+\delta)} \gamma \log k$, then $e^{-\lambda K/4} \leq 1/k^{\gamma}$.

To bound the time we use the lower bound random walk Y_t from Lemma 2.2.2. $EN = 1/\lambda$, so

$$EY_t = \left(\frac{\lambda k}{2} - \frac{1}{\lambda}\right) t = \left(\frac{\lambda^2 k - 2}{2\lambda}\right) t.$$

Let T_L^Y be the hitting time of L for the random walk Y_t. Using the optional stopping theorem for the martingale $Y_t - (\lambda^2 k - 2)t/2\lambda$ and the bounded stopping time $T_L^Y \wedge t$ we get

$$EY_{T_L^Y \wedge t} - \left(\frac{\lambda^2 k - 2}{2\lambda}\right) E\left(T_L^Y \wedge t\right) = EY_0 = 0.$$

Since $EY_{T_L^Y \wedge t} \leq L = \lambda k/4$, it follows that

$$E(T_L^Y \wedge t) \leq \left(\frac{2\lambda}{\lambda^2 k - 2}\right) L = \frac{\lambda^2 k/2}{\lambda^2 k - 2} = \frac{1}{2 - 4/\lambda^2 k} \leq 1,$$

as by our assumption $\lambda^2 k \geq 4$. Letting $t \to \infty$ we have $ET_L^Y \leq 1$. Since Y_t is a lower bound for the number of infected leaves, $T_L \mathbf{1}_{[T_L < \infty]} \leq T_L^Y$. Hence

$$\begin{aligned} E_{0,1}(T_L | T_L < \infty) &= \frac{E_{0,1}\left(T_L \mathbf{1}_{[T_L<\infty]}\right)}{P_{0,1}(T_L < \infty)} \\ &\leq \frac{E_{0,1} T_L^Y}{P_{0,1}(T_K < \tau_0) P_{K,1}(T_L < T_0)} \leq \frac{1}{1/2} = 2 \end{aligned}$$

for large k. $\square$

To transfer infection from one vertex to another we use the following Lemma.

Lemma 2.2.4. *Let $v_0, v_1, \dots v_m$ be a path in the graph and suppose that v_0 is infected at time 0. Then the probability that v_m will become infected by time m is $\geq (e^{-1}(1-e^{-\lambda})e^{-1})^m$.*

Proof. The first factor is the probability that the infection at v_0 lasts for time 1, the second the probability that v_0 infects v_1 by time 1, and the third the probability that the infection at v_1 remains until time 1. Iterating this m times brings the infection from 0 to m. $\square$

When the diameter of the graph is $\leq 2\log n$, the probability in Lemma 2.2.4 is $\geq n^{-b}$ for some $b \in (1/2, \infty)$, and the time required is $\leq 2\log n$. Combining this with Lemma 2.2.3 (with $k = n^\epsilon$ and $\gamma = 1/3$) shows that if n is large, then with probability $\geq Cn^{-b}$ we can use one hot star to make another star hot within time $2n^{\epsilon/3}$. Using Lemma 2.2.2 and trying repeatedly gives the following Lemma.

Lemma 2.2.5. *Let s_1 and s_2 be two stars in V_n^ϵ and suppose that s_1 is hot at time 0. Then, for large n, s_2 will be hot by time $T = \exp(n^{\epsilon/2})$ with probability*

$$\geq 1 - 8e^{-\lambda^2 n^\epsilon/80}.$$

Proof. If n is large, Lemma 2.2.2 shows that s_1 remains lit for T units of time with probability $\geq 1 - 7e^{-\lambda^2 n^\epsilon/80}$. Let $t_n = 2n^{\epsilon/3}$ and consider the discrete time points $t_n, 2t_n, \dots$. At all of these time points we can think of a path starting from an infected neighbor of s_1 up to s_2. Using one such path the infection gets transmitted to s_2 and it gets hot in $2n^{\epsilon/3}$ units of time with probability $\geq Cn^{-b}$ for some constant C. So s_1 fails to make s_2 hot by time T with probability

$$\leq (1 - Cn^{-b})^{T/t_n} \leq \exp(-Cn^{-b}T/t_n) \leq \exp(-\lambda^2 n^\epsilon/80)$$

for large n. For the first inequality we use $1 - x \leq e^{-x}$. Combining with the first error probability in this proof, we get the result. $\square$

Next we show that a lit star becomes hot with a high probability, and then helps to make other non-lit stars lit.

Lemma 2.2.6. *Let s be a star of V_n^ϵ and suppose that s is lit at time 0. Then s will be hot by time $2\exp(n^{\epsilon/3})$ with probability*

$$\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16), \quad \textit{if } n \textit{ is large.}$$

Proof. Since s is lit, it has at least $\lambda n^\epsilon/10$ infected neighbors at time 0. If s itself is not infected at time 0, let N be the number of leaf infections that die out before s gets infected. Using similar argument as in the beginning of the proof of Lemma 2.2.2,

$$P(N = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0,$$

which implies

$$P(N > \lambda n^\epsilon/20) \leq (1+\lambda)^{-\lambda n^\epsilon/20} \leq e^{-\lambda^2 n^\epsilon/40},$$

as $1 + \lambda > e^{\lambda/2}$ by Lemma 2.2.1. Also the time T_M taken for $M = \lambda n^\epsilon/20$ leaf infections to die out is a sum of M exponentials with mean at most $1/(\lambda+1)M \leq 1/M$. Now if $n^{2\epsilon/3} > 40/16$, the above error probability is $\leq e^{-\lambda^2 n^{\epsilon/3}/16}$.

Using Chebyshev's inequality on the sum, S_M of M exponentials with mean 1 (and hence variance 1), we see that if $\exp(n^{\epsilon/3}) \geq 2$, i.e., $n^{\epsilon/3} > \log 2$

$$\begin{aligned} P(S_M > M\exp(n^{\epsilon/3})) &\leq \frac{1}{M(\exp(n^{\epsilon/3})-1)^2} \\ &\leq \frac{4}{M\exp(2n^{\epsilon/3})} \leq \exp(-\lambda^2 n^{\epsilon/3}/16). \end{aligned}$$

where in the final inequality we have used $M > 4$, i.e., $n^\epsilon > 80/\lambda$, and $\lambda^2/16 < 2$.

Multiplying by $1/M$ we see that the total time, T_M, satisfies

$$P(T_M > \exp(n^{\epsilon/3})) \leq \exp(-\lambda^2 n^{\epsilon/3}/16).$$

Combining these two error probabilities gives that s will be infected along with at least $\lambda n^\epsilon/20$ infected neighbors within $\exp(n^{\epsilon/3})$ units of time with error probability

$$\leq 2\exp(-\lambda^2 n^{\epsilon/3}/16). \tag{2.2.3}$$

Now $\lambda n^\epsilon/20 \geq \lambda n^{\epsilon/3}/4$, when $n^{2\epsilon/3} > 5$. So if s is infected and has at least $\lambda n^\epsilon/20$ infected neighbors, then using the second inequality of Lemma 2.2.3 (with $\gamma = 2/3$ and $k = n^\epsilon$), s becomes hot with error probability

$$\leq \exp(-\lambda^2 n^{\epsilon/3}/16).$$

Finally using Markov's inequality and the third inequality of Lemma 2.2.3, the time T_s taken by s to get hot, after it became infected, is more than $T = \exp(n^{\epsilon/3})$ with probability

$$\leq 2\exp(-n^{\epsilon/3}) \leq 2\exp(-\lambda^2 n^{\epsilon/3}/16),$$

as $\lambda < 1$. Combining all these error probabilities proves the Lemma. $\square$

We now use Lemmas 2.2.5, 2.2.6 and 2.2.2 to prove that if the contact process starts from all sites infected, then for a long time at least 3/4's of the stars will be lit.

Proposition 2.2.7. *Let $I_{n,t}^{\epsilon}$ be the set of stars in V_n^{ϵ} which are lit at time t in the contact process $\{\xi_t^1 : t \geq 0\}$ on G_n. Let $t_n = 2\exp(n^{\epsilon/2})$ and $M_n = \exp(n^{1-\alpha\epsilon})$. Then there is a stopping time T_n such that $T_n \geq M_n \cdot t_n$ and*

$$P\left(\left|I_{n,T_n}^{\epsilon}\right| \leq (3/4)\left|V_n^{\epsilon}\right|\right) \leq \exp(-Cn^{\epsilon}).$$

Proof. Let $\alpha_n = |V_n^{\epsilon}|$. Clearly $|I_{n,0}^{\epsilon}| = \alpha_n$. We will estimate the probability that starting from $(7/8)\alpha_n$ lit stars, the number goes below $(3/4)\alpha_n$ before reaching α_n. Define the stopping times τ_is' and σ_is' as follows. Let $\tau_0 = \sigma_0 = 0$ and for $i \geq 0$ let

$$\begin{aligned}
\tau_{i+1} &\equiv \inf\left\{t > \tau_i + \sigma_i t_n : \left|I_{n,t}^{\epsilon}\right| = (7/8)\alpha_n\right\}, \\
\sigma_{i+1} &\equiv \min\left\{s \in \mathbb{N} : \left|I_{n,\tau_{i+1}+s\cdot t_n}^{\epsilon}\right| \notin ((3/4)\alpha_n, \alpha_n)\right\}.
\end{aligned}$$

We need to look at time lags that are multiples of t_n in the definition of σ_i because in our worst nightmare (which is undoubtedly a paranoid delusion) all the lit stars of degree $k \geq n^{\epsilon}$ at time τ_{i+1} have exactly $0.1k$ infected neighbors .

Lemma 2.2.6 implies that a lit star of V_n^{ϵ} gets hot within time $2\exp(n^{\epsilon/3}) \leq \exp(n^{\epsilon/2})$ (for large n) with probability $\geq 1 - 5\exp(-\lambda^2 n^{\epsilon/3}/16)$. Combining with Lemma 2.2.2 gives that a lit star at time 0 gets hot by time $t_n/2$ and remains lit at time t_n with probability $\geq 1 - 6\exp(-\lambda^2 n^{\epsilon/3}/16)$ for large n. Now if $|I_{n,t}^{\epsilon}| < \alpha_n$ for some t, then the number of lit stars will increase at time $t + t_n$ with probability $\geq P(A \cap B)$, where

- A: All the lit stars will get hot by $t_n/2$ units of time, and be lit for time t_n.
- B: A non-lit star will become hot by time $t_n/2$ in presence of another hot star, and remain lit for another $t_n/2$ units of time.

Now using the above argument $P(A) \geq 1 - 6n\exp(-\lambda^2 n^{\epsilon/3}/16)$, as there are at most n stars. Combining Lemma 2.2.5 and 2.2.2 gives $P(B) \geq 1 - 9\exp(-\lambda^2 n^{\epsilon}/80)$. So $P(A \cap B) \geq 1 - \exp(-n^{\epsilon/4})$ for large n. Using the stopping times $\left|I^{\epsilon}_{n,\tau_i + r\cdot t_n}\right| \geq W_r$ for $r \leq \sigma_i$, where $\{W_r : r \geq 0\}$ is a discrete time random walk satisfying

$$\begin{aligned} &W_r \to W_r - 1 \text{ with probability } \exp\left(-n^{\epsilon/4}\right), \\ &W_r \to W_r + 1 \text{ with probability } 1 - \exp\left(-n^{\epsilon/4}\right), \end{aligned} \tag{2.2.4}$$

and $W_0 = (7/8)\alpha_n$. Now θ^{W_r} is a martingale where

$$\theta = \frac{\exp(-n^{\epsilon/4})}{1 - \exp(-n^{\epsilon/4})} < \exp(-n^{\epsilon/4}/2). \tag{2.2.5}$$

If q is the probability that W_r goes below $(3/4)\alpha_n$ before hitting α_n, then applying the optional stopping theorem

$$q \cdot \theta^{(3/4)\alpha_n} + (1-q)\cdot \theta^{\alpha_n} \leq \theta^{(7/8)\alpha_n},$$

which implies

$$q \leq \theta^{(\alpha_n/8)} \leq \exp\left(-Cn^{1-(\alpha-1)\epsilon}\right),$$

as $\alpha_n \sim Cn^{1-(\alpha-1)\epsilon}$ for some constant C. So the probability that the random walk fails to return to α_n at least $M_n = \exp(n^{1-\alpha\epsilon})$ times before going below $(3/4)\alpha_n$ is $\leq \exp(-Cn^{\epsilon})$. Now if

$$K = \min\left\{i \geq 1 : \left|I^{\epsilon}_{n,\tau_i+\sigma_i\cdot t_n}\right| \leq (3/4)\alpha_n\right\},$$

the coupling with the random walk will imply $P(K \leq M_n) \leq \exp(-Cn^{\epsilon})$, and hence for $T_n \equiv \tau_{M_n} + \sigma_{M_n} \cdot t_n$

$$P\left(\left|I^{\epsilon}_{n,T_n}\right| \leq (3/4)\left|V^{\epsilon}_n\right|\right) \leq \exp(-Cn^{\epsilon}).$$

As $\sigma_i \geq 1$ for all i, by our construction $T_n \geq M_n \cdot t_n$, and we get the result. □

So the infection persists for time longer than $\exp\left(n^{1-\alpha\epsilon}\right)$ in the stars of V^{ϵ}_n.

2.3 Density of infected stars

Proposition 2.2.7 implies that if the contact process starts with all vertices infected, most of the stars remain lit even after $\exp(n^{1-\alpha\epsilon})$ units of time. In this section we will show that the density of infected stars is bounded away from 0 and we will find a lower bound for the density. We start with the following Lemma about the growth of clusters in the random graph G_n, when we expose the neighbors of a vertex one at a time. For more details on this procedure see [16, Section 3.2].

Lemma 2.3.1. *Suppose $0 < \delta \leq 1/8$. Let A be the event that the two clusters, starting from 1 and 2 respectively, intersect before their sizes grow to n^{δ}. Then*

$$P(A) \leq Cn^{-(\frac{1}{4}-\delta)}.$$

Proof. If $d_1, \ldots, d_n$ are the degrees of the vertices, then

$$P\left(\max_{1\leq i\leq n} d_i > n^{3/(2\alpha-2)}\right) \leq n \cdot P(d_1 > n^{3/(2\alpha-2)}) \leq c/\sqrt{n} \tag{2.3.1}$$

for some constant c. Suppose all the degrees are at most $n^{3/(2\alpha-2)}$. Suppose R_1 and R_2 are the clusters starting from 1 and 2 up to size n^δ. Let B be the event that R_1 contains a vertex of degree $\geq n^{1/(2\alpha-2)}$. Let e_n be the sum of degrees of all those vertices with degree $\geq n^{1/(2\alpha-2)}$. While growing R_1 the probability that a vertex of degree $\geq n^{1/(2\alpha-2)}$ will be included on any step is

$$\leq \frac{e_n}{\sum_{i=1}^n d_i - n^{\delta+3/(2\alpha-2)}} \equiv \beta_n.$$

Since the size biased distribution is $q_k \sim Ck^{-(\alpha-1)}$ as $k \to \infty$, $\sum_{s\geq k} q_s \sim Ck^{-(\alpha-2)}$ as $k \to \infty$, and we have $e_n \sim Cn^{1-(\alpha-2)/(2\alpha-2)}$ and hence $\beta_n \sim Cn^{-(\alpha-2)/(2\alpha-2)}$ as $n \to \infty$. So for large n $\beta_n \leq c_1 n^{-1/4}$ for some constant c_1, when $\alpha > 3$. Thus

$$P(B^c) \geq 1 - c_1/n^{1/4-\delta}.$$

If B^c occurs, all the degrees of the vertices of R_1 are at most $n^{1/(2\alpha-2)}$. In that case, while growing R_2 the probability of choosing one vertex from R_1 is

$$\leq \frac{n^{\delta+1/(2\alpha-2)}}{\sum_{i=1}^n d_i - n^{\delta+3/(2\alpha-2)}} \leq c_2/n^{1-\delta-1/(2\alpha-2)}.$$

So the conditional probability

$$P(A^c|B^c) \geq \left(1 - c_2 n^{-(1-\delta-1/(2\alpha-2))}\right)^{n^\delta} \geq 1 - c_2/n^{1-2\delta-1/(2\alpha-2)}.$$

Hence combining these two

$$P(A^c) \geq (1 - c_1/n^{1/4-\delta})(1 - c_2/n^{1-2\delta-1/(2\alpha-2)}) \geq 1 - C_1/n^{1/4-\delta},$$

and that completes the proof. □

Lemma 2.3.1 will help us to show that in the dual contact process, staring from any vertex of degree $\geq (10/\lambda)^{2+\delta}$ for some $\delta > 0$, the infection reaches a star of V_n^ϵ, with probability bounded away from 0.

Lemma 2.3.2. *Let ξ_t^A be the contact process on G_n starting from $\xi_0^A = A$. Suppose $0 < \epsilon < 1/20(\alpha-1)$. Then there are constants $\lambda_0 > 0$, $n_0 < \infty$, $c_0 = c_0(\lambda, \epsilon)$ and $p_i > 0$ independent of $\lambda < \lambda_0$, $n \geq n_0$ and ϵ such that if $T = n^{c_0}$, v_2 is a vertex with degree $d(v_2) \geq (10/\lambda)^{2+\delta}$ for some $0 < \delta < 1$ and v_1 is a neighbor of v_2,*

$$P\left(\xi_T^{\{v_2\}} \cap V_n^\epsilon\right) \geq p_2, \qquad P\left(\xi_{T+1}^{\{v_1\}} \cap V_n^\epsilon\right) \geq p_1\lambda.$$

Proof. The second conclusion follows immediately from the first, since the probability that v_1 will infect v_2 before time 1, and that v_2 will stay infected until time 1 is

$$\geq \frac{\lambda}{\lambda+1}(1-e^{-(\lambda+1)})e^{-1} \geq c\lambda.$$

Let Λ_m be the set of vertices in G_n of degree $\geq (10/\lambda)^{m+\delta}$ for $m \geq 2$. Define $\gamma = \frac{\delta}{2(2+\delta)}$ and

$$B = 2(\alpha-1)\log(10/\lambda), \qquad u = \left(e^{-1}(1-e^{-\lambda})e^{-1}\right)^{-(B+1)},$$

$$w_n \equiv \log(n^\epsilon)/\log(10/\lambda) - \delta \qquad T_m = T_m^1 + T_m^2$$

where $T_m^1 = (10/\lambda)^{(m+\delta)\gamma}$, $T_m^2 = u^m$, and we let $n^{c_0} = \sum_{m=2}^{w_n} T_m$.

Define $E_2 = \left\{\xi_{T_2}^{\{v_2\}} \cap \Lambda_3 \neq \emptyset\right\}$ and for $m \geq 3$, having defined $E_2, \ldots, E_{m-1}$, we let

$$E_m = \left\{\xi_{T_m}^{\{v_m\}} \cap \Lambda_{m+1} \neq \emptyset\right\}, \quad \text{and} \quad v_m \in \xi_{T_{m-1}}^{\{v_{m-1}\}} \cap \Lambda_m.$$

Let A_m be the event that the clusters of size $(10/\lambda)^{(m+\delta+1)(\alpha-2)}$ starting from two neighbors of v_m do not intersect and

$$F = \cap_{m=2}^{w_n} A_m.$$

Since $\epsilon < 1/20(\alpha - 1)$, the cluster size $(10/\lambda)^{(m+\delta+1)(\alpha-2)}$ is at most $n^{1/10}$ for $m \le w_n$. So using Lemma 2.3.1 and the fact $\binom{k}{2} < k^2$,

$$P(F^c) \le \left(\sum_{m=2}^{w_n} (10/\lambda)^{2m+2\delta} \right) cn^{-(1/4-1/10)}$$
$$\le n^{2\epsilon} cn^{-(1/4-1/10)} < cn^{-(1/4-3/20)} < 1/6$$

for large n.

Since each vertex has degree at least 3, if F occurs then by the choice of B the neighborhood of radius Bm around v_m will contain more than $(10/\lambda)^{(m+\delta+1)(\alpha-2)+m}$ vertices. Let G_m be the event that the neighborhood of radius Bm around v_m intersects Λ_{m+1}. In the neighborhood of v_m probability of having a vertex of Λ_{m+1} is at least $c(\lambda/10)^{(m+\delta+1)(\alpha-2)}$. Hence

$$P(G_m^c \cap F) \le \left(1 - c(\lambda/10)^{(m+\delta+1)(\alpha-2)}\right)^{(10/\lambda)^{m+(m+\delta+1)(\alpha-2)}}$$
$$\le \exp(-(10/\lambda)^m).$$

If λ is small, $\sum_{m=2}^{\infty} \exp(-(10/\lambda)^m) \le 1/6$.

On the intersection of F and G_m we have a vertex of Λ_{m+1} within radius Bm of v_m. Using Lemma 2.2.2 and Lemma 2.2.3, in the contact process $\left\{\xi_t^{\{v_m\}} : t \ge 0\right\}$, v_m gets hot at time T_m^1 and remains lit till time T_m with error probability $\le c\lambda^{(m+\delta)\gamma}$ for small λ. If v_m is lit, then Lemma 2.2.4 shows that v_m fails to transfer the infection to some vertex in Λ_{m+1} within time T_m^2 with probability

$$\le \left[1 - (e^{-1}(1-e^{-\lambda})e^{-1})^{Bm}\right]^{T_m^2/(Bm)}$$
$$\le \exp\left[-(e^{-1}(1-e^{-\lambda})e^{-1})^{-m}/(Bm)\right] \equiv \eta_m.$$

where $\equiv$ indicates we are making a definition, and hence $P(E_m^c G_m F) \leq c\lambda^{(m+\delta)\gamma} + \eta_m$. If λ is small $\sum_{m=2}^{w_m}[c\lambda^{(m+\delta)\gamma} + \eta_m] \leq 1/6$, we can take $p_2 = 1/2$ and the proof is complete. $\square$

Lemma 2.3.2 gives a lower bound on the probability that an infection starting from a neighbor of a vertex of degree $\geq (10/\lambda)^{2+\delta}$ reaches a star. Lemma 2.2.3 shows that if the infection reaches a star, then with probability tending to 1 the star gets hot within $n^{\epsilon/3}$ units of time. Combining these two we get the following.

Proposition 2.3.3. *Suppose* $0 < \epsilon < 1/20(\alpha-1)$. *There are constants* $\lambda_0 > 0$, $n_0 < \infty$ $c_1 = c_1(\lambda, \epsilon)$ *and* $p_1 > 0$, *which does not depend on* $\lambda < \lambda_0$, $n \geq n_0$ *and* ϵ, *such that for any vertex* v_1 *with a neighbor* v_2 *of degree* $d(v_2) \geq (10/\lambda)^{2+\delta}$ *for some* $\delta \in (0, 1)$, *and* $T = n^{c_1}$ *the probability that* $\xi_T^{\{v_1\}}$ *contains a hot star is bounded below by* $p_1\lambda$.

Next we will show that if we start with one lit star, then after time $\exp(n^{\epsilon/2})$ at least 3/4's of the stars will be lit.

Lemma 2.3.4. *Let* $I_{n,t}^{\epsilon}$ *be the set of stars which are lit at time* t *in the contact process on* G_n *such that* $|I_{n,0}^{\epsilon}| = 1$. *Then for* $T' = \exp(n^{\epsilon/2})$

$$P(|I_{n,T'}^{\epsilon}| < (3/4)|V_n^{\epsilon}|) \leq 7\exp(-\lambda^2 n^{\epsilon/3}/16).$$

Proof. Let s_1 be the lit star at time 0. As seen in Proposition 2.2.7, s_1 remains lit at time $T' = \exp(n^{\epsilon/2})$ with probability $\geq 1 - 6\exp(-\lambda^2 n^{\epsilon/3}/16)$ for large n. With probability $\geq Cn^{-b}$ another star gets hot within time $t_n = 2n^{\epsilon/3}$ and remains lit at time T'. Using similar argument as in Lemma 2.2.5, the process fails to make

$(3/4)|V_n^\epsilon|$ many stars lit by time T' with probability

$$\leq (3/4)|V_n^\epsilon|(1 - Cn^{-b})^{T'/t_n}$$

$$\leq (3/4)|V_n^\epsilon| \exp(-Cn^{-b}T'/t_n) \leq \exp(-\lambda^2 n^{\epsilon/3}/16),$$

as $|V_n^\epsilon| = Cn^{1-(\alpha-1)\epsilon}$ and $1 - x \leq e^{-x}$. So combining with the earlier error probability we get the result. □

Now we are almost ready to prove our main result. However, we need one more Lemma that we will use in the proof of the theorem.

Lemma 2.3.5. *Let F and G be two events which involve exposing n^δ many vertices starting at 1 and 2 respectively for some $0 < \delta \leq 1/8$. Then*

$$|P(F \cap G) - P(F)P(G)| \leq Cn^{-(1/4-\delta)}.$$

Proof. Let R_1 and R_2 be the clusters for exposing n^δ many vertices starting from 1 and 2 respectively, and let A be the event that they intersect. Clearly

$$\begin{aligned} P(F \cap G) &\leq P(A) + P(F \cap G \cap A^c) \\ &= P(A) + P(F \cap A^c)\, P(G \cap A^c) \\ &\leq P(A) + P(F)P(G). \end{aligned}$$

Using similar argument for F^c and G we get

$$|P(F \cap G) - P(F)P(G)| \leq P(A).$$

We estimate $P(A)$ using Lemma 2.3.1. □

Lemma 2.3.5 shows that two events which involve exposing at most $n^{1/8}$ vertices starting from two different vertices are asymptotically uncorrelated. Now we give the proof of the main theorem.

Proof of Theorem 1. Given $\delta > 0$, choose $\epsilon = \min\{\delta/\alpha, 1/20(\alpha-1)\}$. Let A_n be the set of vertices in G_n with a neighbor of degree at least $(10/\lambda)^{2+\delta}$. Clearly $|A_n|/n \to c_0(\lambda/10)^{(2+\delta)(\alpha-2)}$ as $n \to \infty$ for some constant c_0. Define the random variables $Y_x, x \in A_n$ as $Y_x = 1$ if the dual contact process starting from x can light up a star of V_n^ϵ and 0 otherwise. By Proposition 2.3.3, $EY_x \geq p_1\lambda$ for some constant $p_1 > 0$ and for any $x \in A_n$.

If we grow the cluster starting from $x \in A_n$ and exposing one vertex at a time, we can find a star on any step with probability at least $cn^{-(\alpha-2)\epsilon}$. So with probability $1-\exp(-cn^\epsilon)$, we can find a star of V_n^ϵ within the exposure of at most $n^{\alpha\epsilon}$ vertices. So, with high probability, lighting a star up is an event involving at most $n^{(\alpha+1)\epsilon}$ many vertices. As $(\alpha+1)\epsilon < 1/8$, using Lemma 2.3.5, we can say

$$\begin{aligned} &P(Y_x = 1, Y_z = 1) - P(Y_x = 1)\, P(Y_z = 1) \\ &\leq (1 - \exp(-cn^\epsilon))\, Cn^{-(1/4-(\alpha+1)\epsilon)} + \exp(-cn^\epsilon) \equiv \theta_n. \end{aligned}$$

Using our bound on the covariances,

$$\text{var}\left(\sum_{x\in A_n} Y_x\right) \leq n + \binom{n}{2}\theta_n,$$

and Chebyshev's inequality gives

$$P\left(\left|\sum_{x\in A_n}(Y_x - EY_x)\right| \geq n\gamma\right) \leq \frac{n + \binom{n}{2}\theta_n}{n^2\gamma^2} \to 0 \quad \text{as } n \to \infty,$$

for any $\gamma > 0$, since $\theta_n \to 0$ as $n \to \infty$. Since $EY_x \geq p_1\lambda$ and $|A_n|/n \to c_0(\lambda/10)^{(2+\delta)(\alpha-2)}$, if we take $p_l \equiv p_1\lambda \cdot c_0(\lambda/10)^{(2+\delta)(\alpha-2)}/2$ then

$$\lim_{n\to\infty} P\left(\sum_{x\in A_n} Y_x \geq np_l\right) = 1. \tag{2.3.2}$$

Now if $Y_x = 1$, Proposition 2.3.3 says that the dual process starting from x makes a star hot after $T_1 = n^{c_1}$ units of time. Then by Lemma 2.3.4 within next

$T_2 = \exp(n^{\epsilon/2})$ units of time the dual process lights up 75% of all the stars with probability $1 - 7\exp(-\lambda^2 n^{\epsilon/3}/16)$.

Let $I^{\epsilon}_{n,t}$ be the set of stars which are lit at time t in the contact process $\{\xi^1_t : t \geq 0\}$ and

$$T_3 = \inf\left\{t > \exp(n^{1-\alpha\epsilon}) : \left|I^{\epsilon}_{n,t}\right| \geq (3/4)\left|V^{\epsilon}_n\right|\right\}.$$

By Proposition 2.2.7, $P(T_3 < \infty) \geq 1 - \exp(-cn^{\epsilon})$. Let

$$\mathcal{S} = \left\{S \subset \{1, 2, \ldots, n\} : \xi^1_t = S \Rightarrow |I^{\epsilon}_{n,t}| \geq (3/4)|V^{\epsilon}_n|\right\}.$$

Using the Markov property and self-duality of the contact process we get the following inequality. For any subset B of the vertex set, and for the event $F_n \equiv [T_3 < \infty]$ we have

$$\begin{aligned}
&P\left[\left(\xi^{\mathbf{1}}_{T_1+T_2+T_3} \supset B\right) \cap F_n\right] \\
&= \sum_{S \in \mathcal{S}} P\left(\xi^{S}_{T_1+T_2} \supset B\right) P\left(\xi^1_{T_3} = S | F_n\right) P(F_n) \\
&= \sum_{S \in \mathcal{S}} P\left(\xi^{\{x\}}_{T_1+T_2} \cap S \neq \emptyset \; \forall x \in B\right) P\left(\xi^1_{T_3} = S | F_n\right) P(F_n) \\
&\geq \sum_{S \in \mathcal{S}} P\left(|\xi^{\{x\}}_{T_1+T_2} \cap I^{\epsilon}_{n,T_3}| > (3/4)|V^{\epsilon}_n| \; \forall x \in B\right) P\left(\xi^1_{T_3} = S | F_n\right) P(F_n) \\
&\geq P(Y_x = 1 \; \forall x \in B)\left(1 - 7|B| \exp\left(-\lambda^2 n^{\epsilon/3}/16\right)\right) P(F_n) \\
&\geq P(Y_x = 1 \; \forall x \in B)(1 - 2\exp\left(-cn^{\epsilon/4}\right)),
\end{aligned}$$

as $|B| \leq n$ and $P(F_n) \geq 1 - \exp(-cn^{\epsilon})$. Hence for $T = T_1 + T_2 + T_3$, combining with (2.3.2) and using the attractiveness property of the contact process we conclude that as $n \to \infty$

$$\inf_{t \leq T} P\left(\frac{|\xi^{\mathbf{1}}_t|}{n} > p_l\right) = P\left(\frac{|\xi^{\mathbf{1}}_T|}{n} > p_l\right) \tag{2.3.3}$$

$$\geq P\left(\xi_T^{\mathbf{1}} \supseteq \{x : Y_x = 1\}, \sum_{x\in A_n} Y_x \geq np_l\right) \to 1,$$

which completes the proof of Theorem 1, and proves the lower bound in Theorem 2.

2.4 Upper bound in Theorem 2

For the upper bound, we will show that if the infection starts from a vertex x with no vertex of degree $> 1/\lambda^{1-\delta}$ nearby, it has a very small chance to survive. To get the 1 in upper bound we need to use the fact that first event in the contact process starting at x has to be a birth so we begin with that calculation.

Let Λ_δ be the set of vertices of degree $> \lambda^{\delta-1}$. Define $Z_x, x \in \{1, 2, \dots, n\}$ as $Z_x = 1$ if the dual contact process $\{\xi_t^{\{x\}} : t \geq 0\}$ starting from x survives for $T' = 1/\lambda^{\alpha-1}$ units of time, and 0 otherwise. We will show $EZ_x \leq C\lambda^{1+(\alpha-2)(1-\delta)}$ for some constant C. If T_1 is the time for the first event in the dual process, then $ET_1 \leq 1$ and using Markov's inequality $P(T_1 > 1/\lambda^{\alpha-1}) < \lambda^{\alpha-1}$. So if $T_1 < 1/\lambda^{\alpha-1}$, the first event must be a birth for Z_x to be 1. So for $x \in \Lambda_\delta$,

$$\begin{aligned} P(Z_x = 1) &\leq P(T_1 > 1/\lambda^{\alpha-1}) + \sum_{i>\lambda^{\delta-1}} p_i \frac{\lambda i}{\lambda i + 1} \\ &\leq \lambda^{\alpha-1} + C\lambda \sum_{i>\lambda^{\delta-1}} i^{-(\alpha-1)} \\ &\leq \lambda^{\alpha-1} + C\lambda \cdot \lambda^{(\alpha-2)(1-\delta)}. \end{aligned}$$

For $x \in \Lambda_\delta^c$, let $w(\lambda) \leq C\lambda^{(\alpha-2)(1-\delta)}$ be the size-biased probability of having a vertex of Λ_δ in its neighborhood. If $d(x) = i$, the expected number of vertices in a radius m around x is at most $i \cdot EZ_m$, where Z_m is the total progeny up to m^{th} generation of the branching process with offspring distribution $q_k = (k+1)p_{k+1}/\mu \sim ck^{\alpha-1}$. So the expected number of vertices, which are within a distance $m = \lceil(\alpha-1)/\delta\rceil$, the smallest integer larger than $(\alpha-1)/\delta$, from x and belong to Λ_δ, is

$$\leq \sum_{i=2}^{(1/\lambda)^{1-\delta}} p_i \cdot i \cdot EZ_m \cdot C\lambda^{(\alpha-2)(1-\delta)} \leq C\lambda^{(\alpha-2)(1-\delta)}.$$

Using Markov's inequality the probability of having at least one vertex of Λ_δ within a distance m from x has the same upper bound as above.

Until we reach Λ_δ, $|\xi_t^{\{x\}}| \leq Y_t$ where

$$\begin{aligned} Y_t &\to Y_1 - 1 \quad \text{at rate} \quad Y_t \\ Y_t &\to Y_t + 1 \quad \text{at rate} \quad Y_t\lambda \cdot (1/\lambda)^{1-\delta} = Y_t\lambda^\delta \end{aligned}$$

So Y_t jumps at rate $Y_t(1+\lambda^\delta)$ and it jumps to Y_t+1 with probability $\lambda^\delta/(1+\lambda^\delta) < \lambda^\delta$. If $T_1 < 1/\lambda^{\alpha-1}$, the first event in the dual process $\xi_t^{\{x\}}$ must be a birth for Z_x to be 1. Let T_{2m} is the time of the $2m^{th}$ event after the first event. Then $ET_{2m} \leq 2m/(1+\lambda^\delta)$ and using Markov's inequality

$$P(T_{2m} > 1/\lambda^{\alpha-1}) \leq C\lambda^{\alpha-1}.$$

Now if $T_{2m} < 1/\lambda^{\alpha-1}$ and there is no vertex of Λ_δ within a distance m of x, the infection starting at x survives for time T' only if Y_t has at least m up jumps before hitting 0. If there are $\leq m-1$ up jumps in the first $2m$ then Y_t will hit 0 by T_{2m}, as $Y_0 = 2$. The probability of this event is

$$\begin{aligned} &\leq P(B \geq m) \quad \text{where} \quad B \sim \text{ Binomial } (2m, \lambda^\delta) \\ &\leq 2^{2m}\lambda^{m\delta} \leq 2^{2m}\lambda^{\alpha-1}. \end{aligned}$$

Combining all three error probabilities, for any $x \in \Lambda_\delta^c$,

$$\begin{aligned} P(Z_x = 1) \le P(T_1 > 1/\lambda^{\alpha-1}) + P(T_{2m} > 1/\lambda^{\alpha-1}) \\ + \sum_{i \le \lambda^{\delta-1}} p_i \frac{\lambda i}{\lambda i + 1} \cdot C\lambda^{(\alpha-2)(1-\delta)} \\ \le C\lambda^{1+(\alpha-2)(1-\delta)}. \end{aligned}$$

Using an argument similar to one at the end of the proof of Theorem 2.1.2

$$P\left(\left|\sum_x (Z_x - EZ_x)\right| > n\gamma\right) \to 0 \quad \text{as } n \to \infty$$

for any $\gamma > 0$. Since $EZ_x \le C\lambda^{1+(\alpha-2)(1-\delta)}$ for all $x \in \{1, 2, \ldots, n\}$, if we take $p_u = 3C\lambda^{1+(\alpha-2)(1-\delta)}$, then

$$P\left(\sum_x Z_x \ge np_u\right) \to 0 \quad \text{as } n \to \infty.$$

So by making C larger in the definition of p_u and using the attractiveness of the contact process

$$\inf_{t \ge T'} P(|\xi_t^1| \le p_u n) \to 1$$

as $n \to \infty$. $\square$

2.5 Proof of connectivity and diameter

We conclude the chapter with the proof of Lemma 2.1.3. We begin with a large deviations result. The fact is well-known, but the proof is short so we give it for completeness.

Lemma 2.5.1. *Let $X_1, X_2, \dots$ be i.i.d., nonnegative with mean μ. If $\rho < \mu$, then there is a constant $\gamma > 0$ so that*

$$P(X_1 + \cdots + X_k \le \rho k) \le e^{-\gamma k}$$

Proof. Let $\phi(\theta) = Ee^{-\theta X}$. If $\theta > 0$ then

$$e^{-\theta\rho k} P(X_1 + \cdots + X_k \le \rho k) \le \phi(\theta)^k.$$

So we have

$$P(X_1 + \cdots + X_k \le \rho k) \le \exp(k\{\theta\rho + \log\phi(\theta)\}).$$

$\log(\phi(0)) = 0$ and as $\theta \to 0$

$$\frac{d}{d\theta}\log(\phi(\theta)) = \frac{\phi'(\theta)}{\phi(\theta)} \to -\mu.$$

So $\log\phi(\theta) \sim -\mu\theta$ as $\theta \to 0$, and the result follows by taking θ small. $\square$

Proof of Lemma 2.1.3. We will prove the result in the following steps.

Step 1: Let $k_n = (\log n)^2$. The size of the cluster C_x, starting from $x \in \{1, 2, \dots n\}$, reaches size k_n with probability $1 - o(n^{-1})$.

Step 2: There is a $B < \infty$ so that if the size of C_x reaches size $B \log n$, it will reach $n^{2/3}$ with probability $1 - O(n^{-2})$.

Step 3: Let $\zeta > 0$. Two clusters C_x and C_y, starting from x and y respectively, of size $n^{(1/2)+\zeta}$ will intersect with probability $1 - o(n^{-2})$.

Steps 2 and 3 follow from the proof of Theorem 3.2.2 of [16], so it is enough to do Step 1. Before doing this, note that if $d_1, \dots, d_n$ are the degrees of the vertices, and $\eta > 0$ then as $n \to \infty$,

$$P\left(\max_{1\le i\le n} d_i > n^{(1+\eta)/(\alpha-1)}\right) \le n \cdot P(d_1 > n^{(1+\eta)/(\alpha-1)}) \sim Cn^{-\eta}.$$

Given $\alpha > 3$, we choose $\eta > 0$ small enough so that $(1+\eta)/(\alpha-1) < 1/2$.

To prove step 1, we will expose one vertex at a time. Following the notation of [16], suppose A_t, U_t and R_t are the sets of active, unexplored and removed sites respectively at time t in the process of growing the cluster starting from 1, with $R_0 = \{1\}$, $A_0 = \{z : 1 \sim z\}$ and $U_0 = \{1, 2, \ldots, n\} - A_0 \cup R_0$. At time $\tau = \inf\{t : A_t = \emptyset\}$ the process stops. If $A_t \neq \emptyset$, pick i_t from A_t in some way measurable with respect to the process up to that time and let

$$\begin{aligned}
R_{t+1} &= R_t \cup \{i_t\} \\
A_{t+1} &= A_t \cup \{z \in U_t : i_t \sim z\} - \{i_t\} \\
U_{t+1} &= U_t - \{z \in U_t : i_t \sim z\}.
\end{aligned}$$

Here $|R_t| = t+1$ for $t \le \tau$ and so $C_1 = \tau + 1$. If there were no collisions, then $|A_{t+1}| = |A_t| - 1 + Z$ where Z has the size biased degree distribution q. Let q^η be the distribution of $(Z|Z \le n^{(1+\eta)/(\alpha-1)})$. Then on the event $\{\max_i d_i \le n^{(1+\eta)/(\alpha-1)}\}$, $|A_t|$ is dominated by a random walk $S_t = S_0 + Z_1 + \cdots + Z_t$, where $S_0 = A_0$ and $Z_i \sim q^\eta$. Since $q_{k-1} = kp_k/\mu$, we have $q_0 = q_1 = 0$ and hence $q_0^\eta = q_1^\eta = 0$. Then S_t increases monotonically.

If we let $T = \inf\{m : S_m \ge k_n\}$ then

$$P(|C_1| \le k_n) \le P(S_t - |A_t| \ge 4 \text{ for some } t \le T). \tag{2.5.1}$$

As observed above, if n is large, all of the vertices have degree $\le n^\beta$ where $\beta = (1+\eta)/(\alpha-1) < 1/2$. As long as $S_t \le 2k_n$, each time we add a new vertex and the probability that it is in the active set is at most

$$\gamma_n = \frac{2k_n n^\beta}{\sum_{i=1}^n d_i - 2k_n n^\beta} \le C k_n n^{\beta-1}$$

for large n. Thus the probability of two or more collisions while $S_t \le 2k_n$ is $\le (2k_n)^2 \gamma_n^2 = o(n^{-1})$.

If $S_T - S_{T-1} \le k_n$, then the previous argument suffices, but $S_T - S_{T-1}$ might be as large as n^β. Letting $m > 1/(1-2\beta)$, we see that the probability of m or more collisions is at most

$$(n^\beta)^m (C n^{\beta-1})^m = o(n^{-1}).$$

To grow the cluster we will use a breadth first search: we will expose all the vertices at distance 1 from the starting point, then those at distance 2, etc. When a collision occurs, we do not add a vertex, and we delete the one with which a collision has occurred, so two are lost. There is at most one collision while $S_t \le 2k_n$. Since $S_0 \ge 3$, it is easy to see that the worst thing that can happen in terms of the growth of the cluster is for the collision to occur on the first step, reducing S_0 to 1. After this the number of vertices doubles at each step so size k_n is reached before we have gone a distance $\log_2 k_n$ from the starting point.

In the final step we might have a jump $S_\tau - S_{\tau-1} \ge k_n$ and m collisions, but as long as $k_n = (\log n)^2 > 2m$ we do not lose any ground. In the growth before time T, each vertex, except for possibly one collision, has added two new vertices to the active set. From this it is easy to see that the number of vertices in the active set is at least $k_n/2 - 2m$.

To grow the graph now, we will expose all of the vertices in the current active set, then expose all of the neighbors of these vertices, etc. Let $\epsilon > 0$. The proof of Theorem 3.2.2 of [16, page 78] shows that if δ is small then until $n\delta$ vertices have been exposed, the cluster growth dominates a random walk with mean $\nu - \epsilon$. Let $J_1, J_2, \ldots$ be the successive sizes of the active set when these phases are complete. The large deviations result, Lemma 2.5.1, implies that there is a $\gamma > 0$ so that

$$P(J_{i+1} \leq (\nu - 2\epsilon)J_i | J_i = j_i) \leq \exp(-\gamma j_i)$$

Since $J_1 \geq (\log n)^2/2 - 8$, it follows from this result that with probability $\geq 1 - o(n^{-1})$, in at most

$$\left(\frac{1}{2} + \zeta\right) \frac{\log n}{\log(\nu - \epsilon)}$$

steps, the active set will grow to size $n^{(1/2)+\zeta}$. Using the result from Step 3 and noting that the initial phase of the growth has diameter $\leq \log_2 k_n = O(\log \log n)$ the desired result follows. $\square$

Chapter 3

Random Boolean networks

3.1 Introduction

Random Boolean networks were originally developed by Kauffman [30] as an abstraction of genetic regulatory networks. The idea is to identify generic properties and patterns of behavior for the model, then compare them with the behavior of real systems. Protein and RNA concentrations in networks are often modeled by systems of differential equations. However, in large networks the number of parameters such as decay rates, production rates and interaction strengths can become huge. Recent work in [1] on the segment polarity network in *Drosophila melanogaster*, see also [10], has shown that Boolean networks can in some cases outperform differential equation models. Random Boolean networks have been used in [29]to model the yeast transcriptional network, and this approach have been used tin [33] to model the yeast cell-cycle network.

In our version of his model, the state of each node $x \in V_n \equiv \{1, 2, \ldots, n\}$ at time $t = 0, 1, 2, \ldots$ is $\eta_t(x) \in \{0, 1\}$, and each node x receives input from r distinct nodes $y_1(x), \ldots, y_r(x)$, which are chosen randomly from $V_n \setminus \{x\}$.

We construct our random directed graph G_n on the vertex set $V_n = \{1, 2, \ldots, n\}$ by putting oriented edges to each node from its input nodes. To be precise, we define the graph by creating a random mapping $\phi : V_n \times \{1, 2, \ldots, r\} \to V_n$, where $\phi(x, i) = y_i(x)$, such that $y_i(x) \neq x$ for $1 \leq i \leq r$

and $y_i(x) \neq y_j(x)$ when $i \neq j$, and taking the edge set $E_n \equiv \{(y_i(x), x) : 1 \leq i \leq r, x \in V_n\}$. So each vertex has in-degree r in our random graph G_n. The total number of choices for ϕ is $[(n-1)(n-2)\cdots(n-r)]^n$. However, the resulting graph G_n will remain the same under any permutation of the vector $\mathbf{y}_x \equiv (y_1(x), \ldots, y_r(x))$ for any $x \in V_n$. So if $e_{zx} \in \{0,1\}$ is the number of directed edges from node z to node x in G_n, then $\sum_{z=1}^n e_{z,x} = r$, and the total number of permutations of the vectors $\mathbf{y}_x, 1 \leq x \leq n$, that correspond to the same graph is $(r!)^n$. So if $\mathbb{P}$ denotes the distribution of G_n, then

$$\mathbb{P}(e_{zx}, 1 \leq z, x \leq n) = \frac{(r!)^n}{[(n-1)(n-2)\cdots(n-r)]^n} = \frac{1}{\left[\binom{n-1}{r}\right]^n},$$

if $e_{z,x} \in \{0,1\}, e_{x,x} = 0$ and $\sum_{z=1}^n e_{zx} = r$ for all $x \in V_n$, and $\mathbb{P}(e_{zx}, 1 \leq x, z \leq n) = 0$ otherwise. So our random graph G_n has uniform distribution over the collection of all directed graphs on the vertex set V_n in which each vertex has in-degree r. Once chosen the network remains fixed through time. The rule for updating node x is

$$\eta_{t+1}(x) = f_x(\eta_t(y_1(x)), \ldots, \eta_t(y_r(x))),$$

where the values $f_x(v)$, $x \in V_n$, $v \in \{0,1\}^r$, chosen at the beginning and then fixed for all time, are independent and $= 1$ with probability p.

A number of simulation studies have investigated the behavior of this model. See [2] for survey. Flyvberg and Kjaer [24] have studied the degenerate case of $r = 1$ in detail. Derrida and Pommeau [15] have argued that for $r \geq 3$ there is a phase transition in the behavior of these networks between rapid convergence to a fixed point and exponentially long persistence of changes, and identified the phase transition curve to be given by the equation $r \cdot 2p(1-p) = 1$.

The networks with parameters below the curve have behavior that is 'ordered', and those with parameters above the curve have 'chaotic' behavior. Since chaos is not healthy for a biological network, it should not be surprising that real biological networks avoid this phase. See [31], [46] and [42].

To explain the intuition behind the conclusion of [15], we define another process $\{\zeta_t(x) : t \geq 1\}$ for $x \in V_n$, which they called the *annealed approximation*. The idea is that $\zeta_t(x) = 1$ if and only if $\eta_t(x) \neq \eta_{t-1}(x)$, and $\zeta_t(x) = 0$ otherwise. Now if the state of at least one of the inputs $y_1(x), \dots, y_r(x)$ into node x has changed at time t, then the state of node x at time $t+1$ will be computed by looking at a different value of f_x. If we ignore the fact that we may have used this entry before, we get the dynamics of the threshold contact process

$$\begin{aligned} P\left(\zeta_{t+1}(x) = 1 \middle| \, \zeta_t(y_1(x)) + \cdots + \zeta_t(y_r(x)) > 0\right) &= 2p(1-p), \text{ and} \\ P\left(\zeta_{t+1}(x) = 0 \middle| \, \zeta_t(y_1(x)) + \cdots + \zeta_t(y_r(x)) = 0\right) &= 1. \end{aligned}$$

Conditional on the state at time t, the decisions on the values of $\zeta_{t+1}(x)$, $x \in V_n$, are made independently.

We content ourselves to work with the threshold contact process, since it gives an approximate sense of the original model, and we can prove rigorous results about its behavior. To simplify notation and explore the full range of threshold contact processes we let $q \equiv 2p(1-p)$, and suppose $0 \leq q \leq 1$. As mentioned above, it is widely accepted that the condition for prolonged persistence of the threshold contact process is $qr > 1$. To explain this, we note that vertices in the graph G_n have average out-degree r, so a value of 1 at a vertex will, on the average, produce qr 1's in the next generation.

We will also write the threshold contact process as a set valued process. Let $\xi_t \equiv \{x : \zeta_t(x) = 1\}$. We will refer to the vertices $x \in \xi_t$ as occupied at time t. So if P_{G_n} is the distribution of the threshold contact process $\boldsymbol{\xi} \equiv \{\xi_t : t \geq 0\}$ conditioned on the graph G_n, then

$$\begin{aligned} P_{G_n}\left(x \in \xi_{t+1} | \{y_1(x), \ldots, y_r(x)\} \cap \xi_t \neq \emptyset\right) &= q, \text{ and} \\ P_{G_n}\left(x \in \xi_{t+1} | \{y_1(x), \ldots, y_r(x)\} \cap \xi_t = \emptyset\right) &= 0, \end{aligned}$$

and if $\mathbf{P}$ denotes the distribution of the threshold contact process on the random graph G_n, which has distribution $\mathbb{P}$, then

$$\mathbf{P}(\cdot) = \mathbb{E}P_{G_n}(\cdot), \tag{3.1.1}$$

where $\mathbb{E}$ is the expectation corresponding to the probability distribution $\mathbb{P}$.

Let $\boldsymbol{\xi}^A \equiv \left\{\xi_t^A : t \geq 0\right\}$ denote the threshold contact process starting from $\xi_0^A = A \subset V_n$, and $\boldsymbol{\xi}^1 \equiv \{\xi_t^1 : t \geq 0\}$ denote the special case when $A = V_n$. Let ρ be the survival probability of a branching process with offspring distribution $p_r = q$ and $p_0 = 1 - q$. By branching process theory

$$\rho = 1 - \theta, \text{ where } \theta \in (0,1) \text{ satisfies } \theta = 1 - q + q\theta^r. \tag{3.1.2}$$

Using all the ingredients above we now present our first result.

Theorem 3.1.1. *Suppose $q(r-1) > 1$ and let $\delta > 0$. Let $\mathbf{P}$ be the probability distribution in* (3.1.1). *Then there is a positive constant $C(\delta)$ so that as $n \to \infty$*

$$\inf_{t \leq \exp(C(\delta)n)} \mathbf{P}\left(\frac{|\xi_t^1|}{n} \geq \rho - 2\delta\right) \to 1.$$

The threshold contact process will eventually die out on any finite graph. But it certainly cannot last longer than $\exp(O(n))$ units of time, because the number of vertices is n, and so even if all vertices are occupied at time 0, there is a probability $\geq (1-q)^n$ that all of them will be vacant at time 1.

To prove Theorem 3.1.1, we will consider the dual coalescing branching process $\hat{\boldsymbol{\xi}} \equiv \{\hat{\xi}_t : t \geq 0\}$. In this process if x is occupied at time t, then with probability q all of the sites $y_1(x), \ldots, y_r(x)$ will be occupied at time $t+1$, and with probability $1-q$ none of them will be occupied at time $t+1$. Birth events from different sites are independent. Let $\hat{\boldsymbol{\xi}}^A \equiv \{\hat{\xi}_t^A : t \geq 0\}$ be the dual process starting from $\hat{\xi}_0^A = A \subset V_n$. The two processes can be constructed on the same sample space so that for any choices of A and B for the initial sets of occupied sites, $\boldsymbol{\xi}^A$ and $\hat{\boldsymbol{\xi}}^B$ satisfies the following duality relationship, see [26].

$$\left\{\xi_t^A \cap B \neq \emptyset\right\} = \left\{\hat{\xi}_t^B \cap A \neq \emptyset\right\}, \quad t = 0, 1, 2, \ldots. \tag{3.1.3}$$

Taking $A = \{1, 2, \ldots, n\}$ and $B = \{x\}$ this says

$$\left\{x \in \xi_t^1\right\} = \left\{\hat{\xi}_t^{\{x\}} \neq \emptyset\right\}, \tag{3.1.4}$$

or, taking probabilities of both the events above, the density of occupied sites in $\boldsymbol{\xi}^1$ at time t is equal to the probability that $\hat{\boldsymbol{\xi}}^{\{x\}}$ survives until time t. Since over small distances our graph looks like a tree in which each vertex has r descendants, the last quantity $\approx \rho$.

From (3.1.3) it should be clear that we can prove Theorem 3.1.1 by studying the coalescing branching process. The key to this is an "isoperimetric inequality". Let $\hat{G}_n$ be the graph obtained from our original graph $G_n = (V_n, E_n)$ by reversing the edges. That is, $\hat{G}_n = (V_n, \hat{E}_n)$, where $\hat{E}_n = \{(x, y) : (y, x) \in E_n\}$. Given a set $U \subset V_n$, let

$$U^* = \{y \in V_n : x \to y \text{ for some } x \in U\}, \tag{3.1.5}$$

where $x \to y$ means $(x, y) \in \hat{E}_n$. Note that U^* can contain vertices of U. The idea behind this definition is that if U is occupied at time t in the coalescing branching process, then the vertices in U^* may be occupied at time $t+1$.

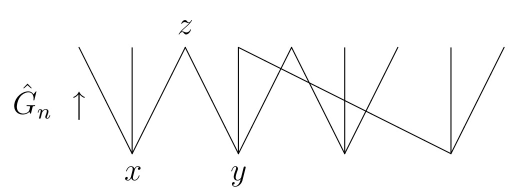
Theorem 3.1.2. *Let $E(m, k)$ be the event that there is a subset $U \subset V_n$ with size $|U| = m$ so that $|U^*| \leq k$. Given $\eta > 0$, there is an $\epsilon_0(\eta) > 0$ so that for $m \leq \epsilon_0 n$*

$$\mathbb{P}\left[E(m, (r-1-\eta)m)\right] \leq \exp(-\eta m \log(n/m)/2).$$

In words, the isoperimetric constant for small sets is $r-1$. It is this result that forces us to assume $q(r-1) > 1$ in Theorem 3.1.1.

Claim. There is a $c > 0$ so that if n is large, then, with high probability, for each $m \leq cn$ there is a set U_m with $|U_m| = m$ and $|U_m^*| \leq 1 + (r-1)m$.

Sketch of Proof. Define an undirected graph H_n on the vertex set V_n so that x and y are adjacent in H_n if and only if there is a z so that $x \to z$ and $y \to z$ in $\hat{G}_n$. The drawing illustrates the case $r = 3$.



The mean number of neighbors of a vertex in H_n is $r^2 \geq 9$, so standard arguments show that there is a $c > 0$ so that, with probability tending to 1 as $n \to \infty$, there is a connected component K_n of H_n with $|K_n| \geq cn$. If U is a connected subset of K_n with $|U| = \lfloor cn \rfloor$, then by building up U one vertex at a time and keeping it connected we get a sequence of sets $\{U_m, m = 1, 2, \ldots, \lfloor cn \rfloor\}$ with $|U_m| = m$ and $|U_m^*| \le 1 + (r-1)m$. □

Since the isoperimetric constant is $\le r-1$, it follows that when $q(r-1) < 1$, then for any $\epsilon > 0$ there are bad sets A with $|A| \le n\epsilon$, so that $E\left|\hat{\xi}_1^A\right| \le |A|$. Computations from the proof of Theorem 5.1.5 suggest that there are a large number of bad sets. We have no idea how to bound the amount of time spent in bad sets, so we have to take a different approach to show persistence when $1/r < q \le 1/(r-1)$.

Theorem 3.1.3. *Suppose $qr > 1$. If δ_0 is small enough, then for any $0 < \delta < \delta_0$, there are constants $C(\delta) > 0$ and $B(\delta) = (1/8 - 2\delta)\log(qr - \delta)/\log r$ so that as $n \to \infty$*

$$\inf_{t \le \exp\left(C(\delta)\cdot n^{B(\delta)}\right)} \mathbf{P}\left(\frac{|\xi_t^1|}{n} \ge \rho - 2\delta\right) \to 1.$$

Based on results for the basic contact process on $(\mathbf{Z} \mod n)$ [19, 21] and on $(\mathbf{Z} \mod n)^d$ [37], it is natural to believe that the conclusion of Theorem 3.1.1 holds in all situations with $qr > 1$. But here we content ourselves with the rather weak result.

To prove Theorem 3.1.3, we will again investigate persistence of the dual. Let

$$\begin{aligned} d_0(x,y) &\equiv \text{length of a shortest oriented path from } x \text{ to } y \text{ in } \hat{G}_n, \\ d(x,y) &\equiv \min_{z \in V_n}[d_0(x,z) + d_0(y,z)], \end{aligned} \tag{3.1.6}$$

and for any subset A of vertices let

$$m(A,K) = \max_{S \subseteq A}\{|S| : d(x,y) \ge K \text{ for all } x, y \in S, x \ne y\}. \tag{3.1.7}$$

Let $R \equiv \log n/\log r$ be the average value of $d_0(1,2)$. So R is an average distance between any two distinct vertices of the graph. Also let $a = 1/8 - \delta$ and $B = (a-\delta)\log(qr-\delta)/\log r$. We will show that if $m\left(\hat{\xi}^A_s, 2\lceil aR\rceil\right) < \lfloor n^B \rfloor$ at some time s, then with high probability, we will later have $m\left(\hat{\xi}^A_t, 2\lceil aR\rceil\right) \geq \lfloor n^B \rfloor$ for some $t > s$. To do this we explore the vertices in $\hat{G}_n$ one at a time using a breadth-first search algorithm based on the distance function d_0. We say that a collision has occurred if we encounter a vertex more than once in the exploration process. First we show in Lemma 3.3.1 that, with probability tending to 1 as $n \to \infty$, there can be at most one collision in the set $\{u : d_0(x,u) \leq 2\lceil aR\rceil\}$ for any $x \in V_n$. Then we argue in Lemma 3.3.2 that when we first have $m\left(\hat{\xi}^A_s, 2\lceil aR\rceil\right) < \lfloor n^B \rfloor$, there is a subset N of occupied sites so that $|N| \geq (q-\delta)\lfloor n^B \rfloor$, and $d(z,w) \geq 2\lceil aR\rceil - 2$ for any two distinct vertices $z, w \in N$, and $\{u : d_0(z,u) \leq 2\lceil aR\rceil - 1\}$ has no collision. We run the dual process starting from the vertices of N until time $\lceil aR\rceil - 1$, so they are independent. With high probability there will be at least one vertex $w \in N$ for which $\left|\hat{\xi}^{\{w\}}_{\lceil aR\rceil -1}\right| \geq \lceil n^B \rceil$. By the choice of N, for any two distinct vertices $x, z \in \hat{\xi}^{\{w\}}_{\lceil aR\rceil -1}$, $d(x,z) \geq 2\lceil aR\rceil$. It seems foolish to pick only one vertex w, but we do not know how to guarantee that the vertices are suitably separated if we pick more.

3.2 Proof of Theorem 3.1.1

We begin with the proof of the isoperimetric inequality, Theorem 5.1.5.

Proof of Theorem 2. Let $p(m,k)$ be the probability that there is a set U with $|U| =$

m and $|U^*| = k$. First we will estimate $p(m,\ell)$ where $\ell = \lfloor (r-1-\eta)m \rfloor$.

$$p(m,\ell) \le \sum_{\{(U,U'):|U|=m,|U'|=\ell\}} \mathbb{P}(U^* = U') \le \sum_{\{(U,U'):|U|=m,|U'|=\ell\}} \mathbb{P}(U^* \subset U').$$

According to the construction of G_n, for any $x \in U$ the other ends of the r edges coming out of it are distinct and they are chosen at random from $V_n \setminus \{x\}$. So

$$\mathbb{P}(U^* \subset U') = \left[\frac{\binom{|U'|}{r}}{\binom{n-1}{r}} \right]^{|U|} \le \left(\frac{|U'|}{n-1} \right)^{r|U|},$$

and hence

$$p(m,\ell) \le \binom{n}{m}\binom{n}{\ell}\left(\frac{\ell}{n-1}\right)^{rm}. \tag{3.2.1}$$

To bound the right-hand side, we use the trivial bound

$$\binom{n}{m} \le \frac{n^m}{m!} \le \left(\frac{ne}{m}\right)^m, \tag{3.2.2}$$

where the second inequality follows from $e^m > m^m/m!$. Using (3.2.2) in (3.2.1)

$$p(m,\ell) \le (ne/m)^m (ne/\ell)^\ell \left(\frac{\ell}{n}\right)^{rm} \left(\frac{n}{n-1}\right)^{rm}.$$

Recalling $\ell \le (r-1-\eta)m$, and accumulating the terms involving $(m/n), r-1-\eta$ and e the last expression becomes

$$\begin{aligned} &\le \quad e^{m(r-\eta)}(m/n)^{m[-1-(r-1-\eta)+r]}(r-1-\eta)^{-(r-1-\eta)m+rm}[n/(n-1)]^{rm} \\ &= \quad e^{m(r-\eta)}(m/n)^{m\eta}(r-1-\eta)^{m(1+\eta)}[n/(n-1)]^{rm}. \end{aligned}$$

Letting $c(\eta) = r-\eta+r\log(n/(n-1))+(1+\eta)\log(r-1-\eta) \le C$ for $\eta \in (0, r-1)$, we have

$$p\left(m, \lfloor (r-1-\eta)m \rfloor\right) \le \exp\left(-\eta m \log(n/m) + Cm\right).$$

Summing over integers $k = (r-1-\eta')m$ with $\eta' \ge \eta$, and noting that there are fewer than rm terms in the sum, we have

$$\mathbb{P}\left[E(m,(r-1-\eta)m)\right] \le \exp(-\eta m \log(n/m) + C'm).$$

To clean up the result to the one given in Theorem 5.1.5, choose ϵ_0 such that $\eta \log(1/\epsilon_0)/2 > C'$. Hence for any $m \leq \epsilon_0 n$,

$$\eta \log(n/m)/2 \geq \eta \log(1/\epsilon_0)/2 > C',$$

which gives the desired result. □

Our next goal is to show that the graph $\hat{G}_n$ locally looks like a tree with high probability. For that we explore all the vertices in V_n one at a time, starting from a vertex x, and using a breadth-first search algorithm based on the distance function d_0 of (3.1.6). More precisely, for each $x \in V_n$, we define the sets A_x^k, which we call the active set at the k^{th} step, and R_x^k, which we call the removed set at k^{th} step, for $k = 0, 1, \ldots, \beta_x$, where $\beta_x \equiv \min\{l : A_x^l = \emptyset\}$, sequentially as follows. $R_x^0 \equiv \emptyset$ and $A_x^0 \equiv \{x\}$. Let $D(x, l) = \{y : d_0(x, y) \leq l\}$. For $0 \leq k < \beta_x$, we get $k_0 = \min\{l : 0 \leq l \leq k, A_x^k \cap D(x, l) \neq \emptyset\}$, and choose $x_k \in A_x^k \cap D(x, k_0)$ with the minimum index.

If $x_k \in R_x^k$, then $A_x^{k+1} \equiv A_x^k \setminus \{x_k\}, R_x^{k+1} \equiv R_x^k$ and

if $x_k \notin R_x^k$, then $A_x^{k+1} \equiv A_x^k \cup \{y_1(x_k), \ldots, y_r(x_k)\} \setminus \{x_k\}, R_x^{k+1} \equiv R_x^k \cup \{x_k\}$.

If $x_k \in R_x^k$, we say that a collision has occurred while exploring $\hat{G}_n$ starting from x. The choice of x_k ensures that while exploring the graph starting from x, for any $j \geq 1$, we consider the vertices, which are at d_0 distance j from x, prior to those, which are at d_0 distance $j+1$ from x.

The next Lemma shows that with high probability R_x^k will have k vertices, and for $x \neq z$, R_x^k and R_z^k do not intersect each other, when $k \leq n^{1/2-\delta}$. For the lemma we need the following stopping times.

$$\pi_x^1 \equiv \min\left\{l \geq 1 : |R_x^l| < l\right\},$$

$$\pi_{x,z} \equiv \min\left\{l \geq 1 : R_x^l \cap R_z^l \neq \emptyset\right\}, x \neq z,$$
$$\alpha_x^{n,\delta} \equiv \min\left\{l \geq 1 : |R_x^l| \geq \lceil n^{1/2-\delta}\rceil\right\}, \delta < 1/2, \tag{3.2.3}$$
$$\beta_x = \min\left\{l \geq 1 : A_x^l = \emptyset\right\}$$

So π_x^1 is the time of first collision while exploring $\hat{G}_n$ starting from x, and $\pi_{x,z}$ is the time of first collision while exploring $\hat{G}_n$ simultaneously from x and z.

Lemma 3.2.1. *Suppose* $0 < \delta < 1/2$. *Let* I_x^1*,* $x \in V_n$*, and* $I_{x,z}$*,* $x, z \in V_n, x \neq z$*, be the events*

$$I_x^1 \equiv \left\{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,\delta}\right\}, \quad I_{x,z} \equiv I_x^1 \cap I_z^1 \cap \left\{\pi_{x,z} \geq \alpha_x^{n,\delta} \vee \alpha_z^{n,\delta}\right\},$$

where $\pi_x^1, \pi_{x,z}, \alpha_x^{n,\delta}$ *and* β_x *are the stopping times defined in* (3.2.3)*. Then*

$$\mathbb{P}\left[\left(I_x^1\right)^c\right] \leq n^{-2\delta}, \quad \mathbb{P}(I_{x,z}^c) \leq 5n^{-2\delta} \tag{3.2.4}$$

for large enough n*.*

Note that the randomness, which determines whether the events I_x^1 and $I_{x,z}$ occur or not, arises only from the construction of the random graph G_n, and does not involve the threshold contact process $\boldsymbol{\xi}^1$ on G_n.

Proof. Let $\delta' = 1/2 - \delta$. Since in the construction of the random graph G_n the input nodes $y_i(z), 1 \leq i \leq r$, for any vertex z are distinct and different from z, there are at least $n - r$ choices for each $y_i(z)$. Also $\left|R_x^l\right| \leq l$ for any l. So

$$\mathbb{P}(|R_x^k| = |R_x^{k-1}|) \leq (k-1)/(n-r). \tag{3.2.5}$$

It is easy to check that $\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,\delta}$ if $|R_x^k| \neq |R_x^{k-1}|$ for $k = 1, 2, \ldots, \lceil n^{\delta'}\rceil$. So

$$\mathbb{P}\left[\left(I_x^1\right)^c\right] \leq \mathbb{P}\left[\cup_{k=1}^{\lceil n^{\delta'}\rceil}\left(\left|R_x^k\right| = \left|R_x^{k-1}\right|\right)\right] \leq \sum_{k=1}^{\lceil n^{\delta'}\rceil} \mathbb{P}\left(|R_x^k| = |R_x^{k-1}|\right)$$

$$\le \sum_{k=1}^{\lceil n^{\delta'} \rceil} (k-1)/(n-r) \le n^{2\delta'}/n = n^{-2\delta}$$

for large enough n. For the other assertion, note that $I_{x,z}$ occurs if $|R_x^k| \neq |R_x^{k-1}|, |R_z^k| \neq |R_z^{k-1}|$ and $R_x^k \cap R_z^k = \emptyset$ for $k = 1, 2, \ldots, \lceil n^{\delta'} \rceil$. Also if for some $k \ge 1$ $R_x^k \cap R_z^k \neq \emptyset$ and $R_x^l \cap R_z^l = \emptyset$ for all $1 \le l < k$, then either $R_x^k = R_x^{k-1} \cup \{x_{k-1}\}$ and $x_{k-1} \in R_z^{k-1}$, or $R_z^k = R_z^{k-1} \cup \{z_{k-1}\}$ and $z_{k-1} \in R_x^k$. Now since each of the input nodes in the construction of G_n has at least $n-r$ choices, and $|R_x^l|, |R_z^l| \le l$ for any l,

$$\mathbb{P}\left(R_x^k \cap R_z^k \neq \emptyset, R_x^l \cap R_z^l = \emptyset, 1 \le l < k\right) \le \mathbb{P}\left(x_{k-1} \in R_z^{k-1}\right) + \mathbb{P}\left(z_{k-1} \in R_x^k\right) \le (2k-1)/(n-r). \tag{3.2.6}$$

Combining the error probabilities of (3.2.5) and (4.4.6)

$$\mathbb{P}\left(I_{x,z}^c\right) \le \mathbb{P}\left[\cup_{k=1}^{\lceil n^{\delta'} \rceil} \left(\left|R_x^k\right| = \left|R_x^{k-1}\right|\right) \cup_{k=1}^{\lceil n^{\delta'} \rceil} \left(\left|R_z^k\right| = \left|R_z^{k-1}\right|\right) \cup_{k=1}^{\lceil n^{\delta'} \rceil} \left(R_x^k \cap R_z^k \neq \emptyset\right)\right]$$
$$\le \sum_{k=1}^{\lceil n^{\delta'} \rceil} \left[\mathbb{P}\left(\left|R_x^k\right| = \left|R_x^{k-1}\right|\right) + \mathbb{P}\left(\left|R_z^k\right| = \left|R_z^{k-1}\right|\right) + \mathbb{P}\left(R_x^k \cap R_z^k \neq \emptyset, R_x^l \cap R_z^l = \emptyset, 1 \le l < k\right)\right]$$
$$\le \sum_{k=1}^{\lceil n^{\delta'} \rceil} (4k-3)/(n-r) \le 5n^{2\delta'-1} = 5n^{-2\delta}$$

for large n. □

Lemma 3.2.1 shows that $\hat{G}_n$ is locally tree-like. The number of vertices in the induced subgraph $\hat{G}_{x,M}$ with vertex set $G_n \cap \{u : d_0(x,u) \le M\}$ is at most $1+r+\cdots+r^M \le 2r^M$. So if I_x^1 occurs, then, for any M satisfying $2r^M \le n^{1/2-\delta}$, the subgraph $\hat{G}_{x,M}$ is an oriented finite $r-$tree, where each vertex except the leaves has out-degree r. Similarly if $I_{x,z}$ occurs, then for any such M, $\hat{G}_{x,M} \cap \hat{G}_{z,M} = \emptyset$.

In the next lemma, we will use this to get a bound on the survival of the dual process for small times. Let ρ be the branching process survival probability defined in (3.1.2).

Lemma 3.2.2. *If $q > 1/r$, $\delta \in (0, qr-1)$, $\gamma = (20\log r)^{-1}$, and $b = \gamma\log(qr-\delta)$ then for any $x \in V_n$, if n is large,*

$$\mathbf{P}\left(\left|\hat{\xi}^{\{x\}}_{\lceil 2\gamma\log n\rceil}\right| \geq \lceil n^b \rceil\right) \geq \rho - \delta.$$

Proof. Let I_x^1 be the event

$$I_x^1 = \left\{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,1/4}\right\},$$

where $\pi_x^1, \beta_x, \alpha_x^{n,1/4}$ are as in (3.2.3). Let P_{Z^x} be the distribution of a branching process $\mathbf{Z}^x \equiv \{Z_t^x : t = 0,1,2,\ldots\}$ with $Z_0^x = 1$ and offspring distribution $p_0 = 1-q$ and $p_r = q$. Since $q > 1/r$, this is a supercritical branching process. Let B_x be the event that the branching process survives. Then

$$P_{Z^x}(B_x) = \rho,$$

where ρ is as in (3.1.2). If we condition on B_x, then, using a large deviation result for branching processes from [4],

$$P_{Z^x}\left(\left|\frac{Z_{t+1}^x}{Z_t^x} - qr\right| > \delta \,\middle|\, B_x\right) \leq e^{-c(\delta)t} \tag{3.2.7}$$

for some constant $c(\delta) > 0$ and for large enough t. So if $F_x = \{Z_{t+1}^x \geq (qr-\delta)Z_t^x$ for $\lfloor\gamma\log n\rfloor \leq t < \lceil 2\gamma\log n\rceil\}$, then

$$P_{Z^x}(F_x^c|B_x) \leq \sum_{t=\lfloor\gamma\log n\rfloor}^{(\lceil 2\gamma\log n\rceil)-1} e^{-c(\delta)t} \leq C_\delta n^{-c(\delta)\gamma/2} \tag{3.2.8}$$

for some constant $C_\delta > 0$ and for large enough n. On the event $B_x \cap F_x$,

$$Z^x_{\lceil 2\gamma\log n\rceil} \geq (qr-\delta)^{\lceil 2\gamma\log n\rceil - \lfloor\gamma\log n\rfloor} \geq (qr-\delta)^{\gamma\log n} = n^{\gamma\log(qr-\delta)},$$

since $Z^x_{\lfloor \gamma \log n \rfloor} \geq 1$ on B_x.

Now coming back to the dual process $\hat{\boldsymbol{\xi}}^{\{x\}}$, let $P_{I_x^1}$ denote the conditional distribution of $\hat{\boldsymbol{\xi}}^{\{x\}}$ given I_x^1. This does not specify the entire graph but we will only use the conditional law for events that involve the process on the subtree whose existence is guaranteed by I_x^1. By the choice of γ, the number of vertices in the subgraph induced by $\{u : d_0(x,u) \leq \lceil 2\gamma \log n \rceil\}$ is at most $2r^{\lceil 2\gamma \log n \rceil} < n^{1/4}$. Then it is easy to see that we can couple $P_{I_x^1}$ with P_{Z^x} so that

$$P_{I_x^1}\left[\left(\left|\hat{\xi}_t^{\{x\}}\right|, 0 \leq t \leq \lceil 2\gamma \log n \rceil\right) \in \cdot\right] = P_{Z^x}\left[(Z_t^x, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot\right].$$

Combining the error probabilities of (3.2.4) and (3.2.8)

$$\begin{aligned}
\mathbf{P}\left(\left|\hat{\xi}^{\{x\}}_{\lceil 2\gamma \log n \rceil}\right| \geq \lceil n^b \rceil\right) &\geq P_{I_x^1}\left(\left|\hat{\xi}^{\{x\}}_{\lceil 2\gamma \log n \rceil}\right| \geq \lceil n^b \rceil\right) \mathbb{P}(I_x^1) \\
&= P_{Z^x}\left(Z^x_{\lceil 2\gamma \log n \rceil} \geq \lceil n^b \rceil\right) \mathbb{P}(I_x^1) \\
&\geq P_{Z^x}(B_x \cap F_x) \mathbb{P}(I_x^1) \\
&= P_{Z^x}(B_x) P_{Z^x}(F_x | B_x) \mathbb{P}(I_x^1) \\
&\geq \rho\left(1 - C_\delta n^{-c(\delta)\gamma/2}\right)\left(1 - n^{-1/2}\right) \geq \rho - \delta
\end{aligned}$$

for large enough n. □

Lemma 3.2.2 shows that the dual process starting from one vertex will with probability $\geq \rho - \delta$ survive until there are $\lceil n^b \rceil$ many occupied sites. The next lemma will show that if the dual starts with $\lceil n^b \rceil$ many occupied sites, then for some $\epsilon > 0$ it will have $\lceil \epsilon n \rceil$ many occupied sites within time $\lceil \epsilon n \rceil$ with high probability.

Lemma 3.2.3. *If $q(r-1) > 1$, then there exists $\epsilon_1 > 0$ such that for any A with*

$|A| \geq \lceil n^b \rceil$ the dual process $\hat{\boldsymbol{\xi}}^A$ satisfies

$$\mathbf{P}\left(\max_{t \leq \lceil \epsilon_1 n - n^b \rceil} \left| \hat{\xi}_t^A \right| < \epsilon_1 n \right) \leq \exp\left(-n^{b/4}\right).$$

Proof. Choose $\eta > 0$ such that $(q-\eta)(r-1-\eta) > 1$, and let $\epsilon_0(\eta)$ be the constant in Theorem 5.1.5. Take $\epsilon_1 \equiv \epsilon_0(\eta)$. Let $\nu \equiv \min\left\{t : \left|\hat{\xi}_t^A\right| \geq \lceil \epsilon_1 n \rceil\right\}$. Let $F_t \equiv \left\{\left|\hat{\xi}_t^A\right| \geq \left|\hat{\xi}_{t-1}^A\right| + 1\right\}$, and

$$B_t \equiv \left\{\text{at least } (q-\eta)\left|\hat{\xi}_t^A\right| \text{ occupied sites of } \hat{\xi}_t^A \text{ give birth}\right\},$$
$$C_t \equiv \left\{|U_t^*| \geq (r-1-\eta)|U_t|\right\}, \text{ where } U_t = \left\{x \in \hat{\xi}_t^A : x \text{ gives birth}\right\}.$$

Now if B_t and C_t occur, then

$$\left|\hat{\xi}_{t+1}^A\right| = |U_t^*| \geq (r-1-\eta)|U_t| \geq (r-1-\eta)(q-\eta)\left|\hat{\xi}_t^A\right| > \left|\hat{\xi}_t^A\right|, \qquad (3.2.9)$$

i.e. F_{t+1} occurs. So $F_{t+1} \supseteq B_t \cap C_t$ for all $t \geq 0$. Using the binomial large deviations, see [16, Lemma 2.3.3, page 40],

$$P_{G_n}\left(B_t | \, \hat{\xi}_t^A\right) \geq 1 - \exp\left(-\Gamma((q-\eta)/q)q\left|\hat{\xi}_t^A\right|\right), \qquad (3.2.10)$$

where $\Gamma(x) = x\log x - x + 1 > 0$ for $x \neq 1$. If we take $H_0 \equiv \left\{\left|\hat{\xi}_0^A\right| \geq \lceil n^b \rceil\right\}$ and $H_t \equiv \cap_{s=1}^t F_s$, then $\left|\hat{\xi}_t^A\right| \geq \lceil n^b \rceil$ on the event H_t for all $t \geq 0$. Keeping that in mind we can replace $\left|\hat{\xi}_t^A\right|$ in the right side of (3.2.10) by n^b to have

$$P_{G_n}(B_t^c \cap H_t) \leq P_{G_n}\left(B_t^c \cap \left\{\left|\hat{\xi}_t^A\right| \geq \lceil n^b \rceil\right\}\right) \leq \exp\left(-\Gamma((q-\eta)/q)qn^b\right) \quad \forall t \geq 0. \qquad (3.2.11)$$

The same bound also works for the unconditional probability distribution $\mathbf{P}$. Next we see that $P_{G_n}(C_t|U_t) \geq \mathbf{1}_{E^c}$, where $E = E(|U_t|, (r-1-\eta)|U_t|)$, as defined in Theorem 5.1.5. Taking expectation with respect to the distribution of G_n,

$\mathbf{P}(C_t|U_t) \geq \mathbb{P}(E^c)$. Since for $t < \nu$, $|U_t| < \epsilon_0(\eta)n$, and $|U_t| \geq (q-\eta)n^b \geq n^b/(r-1)$ on $H_t \cap B_t$, using Theorem 5.1.5

$$\begin{aligned}\mathbf{P}(C_t^c \cap B_t \cap H_t \cap \{t < \nu\}) &\leq \mathbf{P}[C_t^c \cap \{(n^b/(r-1)) \leq |U_t| < \epsilon_1 n\}] \\ &\leq \exp\left(-\frac{\eta}{2}\frac{n^b}{r-1}\log\frac{n(r-1)}{n^b}\right). \qquad (3.2.12)\end{aligned}$$

Combining these two bounds of (3.2.11) and (3.2.12) we get

$$\begin{aligned}\mathbf{P}(F_{t+1}^c \cap H_t \cap \{t < \nu\}) &\leq \mathbf{P}((B_t \cap C_t)^c \cap H_t \cap \{t < \nu\}) \\ &\leq \mathbf{P}(B_t^c \cap H_t) + \mathbf{P}(C_t^c \cap B_t \cap H_t \cap \{t < \nu\}) \leq \exp\left(-n^{b/2}\right)\end{aligned}$$

for large n. Since $\nu \leq \lceil \epsilon_1 n - n^b \rceil$ on $H_{\lceil \epsilon_1 n - n^b \rceil}$,

$$\begin{aligned}\mathbf{P}\left(\nu > \lceil \epsilon_1 n - n^b \rceil\right) &\leq \mathbf{P}\left[\left(\nu > \lceil \epsilon_1 n - n^b \rceil\right) \cap \left(\cup_{t=1}^{\lceil \epsilon_1 n - n^b \rceil} F_t^c\right)\right] \\ &\leq \sum_{t=1}^{\lceil \epsilon_1 n - n^b \rceil} \mathbf{P}(F_t^c \cap H_{t-1} \cap \{\nu > t-1\}) \\ &\leq (\lceil \epsilon_1 n - n^b \rceil)\exp\left(-n^{b/2}\right) \leq \exp\left(-n^{b/4}\right)\end{aligned}$$

for large n and we get the result. □

The next result shows that if there are $\lceil \epsilon n \rceil$ many occupied sites at some time for some $\epsilon > 0$, then the dual process survives for at least $\exp(cn)$ units of time for some constant c.

Lemma 3.2.4. *If $q(r-1) > 1$, then there exist constants $c > 0$ and $\epsilon_1 > 0$ as in Lemma 3.2.3 such that for $T = \exp(cn)$ and any A with $|A| \geq \lceil \epsilon_1 n \rceil$,*

$$\mathbf{P}\left(\inf_{t \leq T}\left|\hat{\xi}_t^A\right| < \epsilon_1 n\right) \leq 2\exp(-cn).$$

Proof. Choose $\eta > 0$ so that $(q-\eta)(r-1-\eta) > 1$, and then choose $\epsilon_0(\eta) > 0$ as in Theorem 5.1.5. Take $\epsilon_1 = \epsilon_0(\eta)$. For any A with $|A| \geq \lceil \epsilon_1 n \rceil$, let $U_t' =$

$\left\{x \in \hat{\xi}_t^A : x \text{ gives birth}\right\}$, $t = 0, 1, \ldots$. If $|U_t'| \le \lfloor \epsilon_1 n \rfloor$, then take $U_t = U_t'$. If $|U_t'| > \epsilon_1 n$, we have too many vertices to use Theorem 5.1.5, so we let U_t be the subset of U_t' consisting of the $\lfloor \epsilon_1 n \rfloor$ vertices with smallest indices. Let

$$F_t = \left\{\left|\hat{\xi}_t^A\right| \ge \lceil \epsilon_1 n \rceil\right\}, \qquad H_t = \cap_{s=0}^t F_s,$$
$$B_t = \left\{\text{at least } (q-\eta)\left|\hat{\xi}_t^A\right| \text{ many occupied sites of } \hat{\xi}_t^A \text{ give birth}\right\},$$
$$C_t = \{|U_t^*| \ge (r-1-\eta)|U_t|\}.$$

Now using an argument similar for the one for (3.2.9), $F_{t+1} \cap H_t \supset B_t \cap C_t \cap H_t$ for any $t \ge 0$. Using our binomial large deviations result (3.2.10) again, $P_{G_n}\left(B_t | \, \hat{\xi}_t^A\right) \ge 1 - \exp\left(-\Gamma((q-\eta)/q)q\left|\hat{\xi}_t^A\right|\right)$. On the event F_t, $\left|\hat{\xi}_t^A\right| \ge \lceil \epsilon_1 n \rceil$, and so

$$P_{G_n}(B_t^c \cap H_t) \le P_{G_n}\left(B_t^c \cap \left\{\left|\hat{\xi}_t^A\right| \ge \lceil \epsilon_1 n \rceil\right\}\right) \le \exp\left(-\Gamma((q-\eta)/q)q\epsilon_1 n\right).$$

The same bound works for the unconditional probability distribution $\mathbf{P}$.

Since $|U_t| \le \epsilon_1 n$, and on the event $H_t \cap B_t$ $|U_t| \ge (q-\eta)\epsilon_1 n \ge \epsilon_1 n/(r-1)$, using Theorem 5.1.5 and similar argument which leads to (3.2.12) we have

$$\mathbf{P}(C_t^c \cap H_t \cap B_t) \le \exp\left(-\frac{\eta}{2}\frac{\epsilon_1 n}{r-1}\log\frac{r-1}{\epsilon_1}\right).$$

Combining these two bounds

$$\begin{aligned}\mathbf{P}(F_{t+1}^c \cap H_t) &\le \mathbf{P}[(B_t \cap C_t)^c \cap H_t] \\ &\le \mathbf{P}(B_t^c \cap H_t) + \mathbf{P}(C_t^c \cap B_t \cap H_t) \le 2\exp(-2c(\eta)n),\end{aligned}$$

where

$$c(\eta) = \frac{1}{2}\min\left\{\Gamma\left(\frac{q-\eta}{q}\right)q\epsilon_1, \frac{\eta}{2}\frac{\epsilon_1}{r-1}\log\frac{r-1}{\epsilon_1}\right\}.$$

Hence for $T \equiv \exp(c(\eta)n)$

$$\begin{aligned}\mathbf{P}\left(\inf_{t\le T}\left|\hat{\xi}_t^A\right| < \epsilon_1 n\right) &\le \mathbf{P}\left(\cup_{t=1}^{\lfloor T\rfloor} F_t^c\right)\\ &\le \sum_{t=0}^{\lfloor T\rfloor -1} \mathbf{P}(F_{t+1}^c \cap G_t) \le 2T\exp(-2c(\eta)n) = 2\exp(-c(\eta)n).\end{aligned}$$

which completes the proof. □

Lemma 3.2.4 confirms prolonged persistence for the dual. We will now give the

Proof of Theorem 3.1.1. Choose $\delta \in (0, qr-1)$ and $\gamma = (20\log r)^{-1}$. Define the random variables $Y_x, 1 \le x \le n$, so that $Y_x = 1$ if the dual process $\hat{\boldsymbol{\xi}}^{\{x\}}$ starting at x satisfies $\left|\hat{\xi}^{\{x\}}_{\lceil 2\gamma\log n\rceil}\right| \ge \lceil n^b \rceil$ for $b = \gamma\log(qr-\delta)$, and $Y_x = 0$ otherwise. By Lemma 3.2.2, if n is large, then

$$\mathbf{E}Y_x \ge \rho - \delta \quad \text{for any } x.$$

Let π_x^1, $\pi_{x,z}$ and $\alpha_x^{n,3/10}$ be the stopping times as in (3.2.3), and $I_x^1, I_{x,z}$ be the corresponding events as in Lemma 3.2.1. Recall that $\hat{G}_{x,M}$ is the subgraph with vertex set $V_n \cap \{u : d_0(x,u) \le M\}$. On the event $I_{x,z}$, $\hat{G}_{x,\lceil 2\gamma\log n\rceil}$ and $\hat{G}_{z,\lceil 2\gamma\log n\rceil}$ are oriented finite $r-$trees consisting of disjoint sets of vertices, since $2r^{\lceil 2\gamma\log n\rceil} \le n^{1/5}$ by the choice of γ. Hence if $P_{I_{x,z}}$ is the conditional distribution of $\left(\hat{\boldsymbol{\xi}}^{\{x\}}, \hat{\boldsymbol{\xi}}^{\{z\}}\right)$ given $I_{x,z}$, then

$$\begin{aligned}&P_{I_{x,z}}\left[\left(\hat{\xi}_t^{\{x\}}, 0\le t\le \lceil 2\gamma\log n\rceil\right)\in\cdot, \left(\hat{\xi}_t^{\{z\}}, 0\le t\le \lceil 2\gamma\log n\rceil\right)\in\cdot\right]\\ =\ &P_{I_{x,z}}\left[\left(\hat{\xi}_t^{\{x\}}, 0\le t\le \lceil 2\gamma\log n\rceil\right)\in\cdot\right] P_{I_{x,z}}\left[\left(\hat{\xi}_t^{\{z\}}, 0\le t\le \lceil 2\gamma\log n\rceil\right)\in\cdot\right].\end{aligned}$$

Having all the ingredients ready we will now estimate the covariance between the events $\{Y_x = 1\}$ and $\{Y_z = 1\}$ for $x \ne z$. Standard probability arguments give the inequalities

$$\begin{aligned}
\mathbf{P}(Y_x=1,Y_z=1) &\leq \mathbf{P}[(Y_x=1,Y_z=1)\cap I_{x,z}]+\mathbb{P}(I^c_{x,z}) \\
&= P_{I_{x,z}}(Y_x=1,Y_z=1)\mathbb{P}(I_{x,z})+\mathbb{P}(I^c_{x,z}) \\
&= P_{I_{x,z}}(Y_x=1)P_{I_{x,z}}(Y_z=1)\mathbb{P}(I_{x,z})+\mathbb{P}(I^c_{x,z}) \\
&= \mathbf{P}[(Y_x=1)\cap I_{x,z}]\mathbf{P}[(Y_z=1)\cap I_{x,z}]/\mathbb{P}(I_{x,z})+\mathbb{P}(I^c_{x,z}) \\
&\leq \mathbf{P}(Y_x=1)\mathbf{P}(Y_z=1)/\mathbb{P}(I_{x,z})+\mathbb{P}(I^c_{x,z}).
\end{aligned}$$

Subtracting $\mathbf{P}(Y_x=1)\mathbf{P}(Y_z=1)$ from both sides gives

$$\begin{aligned}
&\mathbf{P}(Y_x=1,Y_z=1)-\mathbf{P}(Y_x=1)\mathbf{P}(Y_z=1) \\
\leq\ &\mathbf{P}(Y_x=1)\mathbf{P}(Y_z=1)\left(\frac{1}{\mathbb{P}(I_{x,z})}-1\right)+\mathbb{P}(I^c_{x,z}) \\
\leq\ &\mathbb{P}(I^c_{x,z})[1+1/\mathbb{P}(I_{x,z})], \qquad (3.2.13)
\end{aligned}$$

where in the last inequality we replaced the two probabilities by 1. Now from Lemma 3.2.1 $\mathbb{P}(I^c_{x,z})\leq 5n^{-3/5}$, and so

$$\mathbf{P}(Y_x=1,Y_z=1)-\mathbf{P}(Y_x=1)\mathbf{P}(Y_z=1)\leq 5n^{-3/5}\left(1+1/\left(1-5n^{-3/5}\right)\right)\leq 15n^{-3/5}$$

for large enough n. Using this bound,

$$\mathbf{var}\left(\sum_{x=1}^{n}Y_x\right)\leq n+15n(n-1)n^{-3/5},$$

and Chebyshev's inequality shows that as $n\to\infty$

$$\mathbf{P}\left(\left|\sum_{x=1}^{n}(Y_x-\mathbf{E}Y_x)\right|\geq n\delta\right)\leq\frac{n+15n(n-1)n^{-3/5}}{n^2\delta^2}\to 0.$$

Since $\mathbf{E}Y_x\geq\rho-\delta$, this implies

$$\lim_{n\to\infty}\mathbf{P}\left(\sum_{x=1}^{n}Y_x\geq n(\rho-2\delta)\right)=1. \qquad (3.2.14)$$

Our next goal is to show that ξ_T^1 contains the random set $D \equiv \{x : Y_x = 1\}$ at $T = T_1 + T_2$, a time that grows exponentially fast in n. We choose $\eta > 0$ so that $(q-\eta)(r-1-\eta) > 1$. Let ϵ_1 and $c(\eta)$ be the constants in Lemma 3.2.4. If $Y_x = 1$, then $\left|\hat{\xi}_{T_1}^{\{x\}}\right| \geq \lceil n^b \rceil$ for $T_1 = \lceil 2\gamma \log n \rceil$. Combining the error probabilities of Lemmas 3.2.3 and 3.2.4 shows that for $T_2 = \lfloor \exp(c(\eta)n) \rfloor + \lceil \epsilon_1 n - n^b \rceil$, and for any subset A of vertices with $|A| \geq \lceil n^b \rceil$

$$\mathbf{P}\left(\left|\hat{\xi}_{T_2}^{A}\right| \geq \lceil \epsilon_1 n \rceil\right) \geq 1 - 3\exp\left(-n^{b/4}\right) \tag{3.2.15}$$

for large n.

Let $\mathcal{C}$ be the set of all subsets of V_n of size at least $\lceil n^b \rceil$, and denote $C_x \equiv \hat{\xi}_{T_1}^{\{x\}}$. Using the duality relationship of (3.1.4) for the conditional probability distribution

$$\mathcal{P}(\cdot) = \mathbf{P}\left(\cdot \left| \hat{\xi}_t^{\{x\}}, 0 \leq t \leq T_1, x \in V_n \right.\right),$$

we see that

$$\begin{aligned}
\mathcal{P}\left(\xi_{T_1+T_2}^1 \supseteq D\right) &= \mathcal{P}\left[\cap_{x\in D}\left\{x \in \xi_{T_1+T_2}^1\right\}\right] \\
&= \mathcal{P}\left[\cap_{x\in D}\left\{\hat{\xi}_{T_1+T_2}^{\{x\}} \neq \emptyset\right\}\right].
\end{aligned}$$

Since $D = \{x : Y_x = 1\}$, it follows from the definition of Y_x that $C_x \in \mathcal{C}$ for all $x \in D$. So by the Markov property of the dual process the above is

$$\begin{aligned}
&= \sum_{C_x \in \mathcal{C}, x\in D} \mathcal{P}\left[\cap_{x\in D}\left(\hat{\xi}_{T_1+T_2}^{\{x\}} \neq \emptyset, \hat{\xi}_{T_1}^{\{x\}} = C_x\right)\right] \\
&= \sum_{C_x \in \mathcal{C}, x\in D} \mathbf{P}\left[\cap_{x\in D}\left(\hat{\xi}_{T_2}^{C_x} \neq \emptyset\right)\right] \mathcal{P}\left[\cap_{x\in D}\left(\hat{\xi}_{T_1}^{\{x\}} = C_x\right)\right].
\end{aligned}$$

Using (5.8.1) $\mathbf{P}\left(\hat{\xi}_{T_2}^{C_x} \neq \emptyset\right) \geq \mathbf{P}\left(\left|\hat{\xi}_{T_2}^{C_x}\right| \geq \lceil \epsilon_1 n \rceil\right) \geq 1 - 3\exp\left(-n^{b/4}\right)$. So the above is

$$\geq \left(1 - 3|D|\exp\left(-n^{b/4}\right)\right) \sum_{C_x \in \mathcal{C}, x\in D} \mathcal{P}\left[\cap_{x\in D}\left(\hat{\xi}_{T_1}^{\{x\}} = C_x\right)\right]$$

$$\geq \quad 1 - 3n \exp\left(-n^{b/4}\right).$$

For the last inequality we use $|D| \leq n$ and $\mathcal{P}(Y_x = 1 \forall x \in D) = 1$. Since the lower bound only depends on n, the unconditional probability

$$\mathbf{P}\left(\xi^1_{T_1+T_2} \supseteq \{x : Y_x = 1\}\right) \geq 1 - 3n \exp\left(-n^{b/4}\right).$$

Hence for $T = T_1 + T_2$ using the attractiveness property of the threshold contact process, and combining the last calculation with (3.2.14) we conclude that as $n \to \infty$

$$\begin{aligned}\inf_{t \leq T} \mathbf{P}\left(\frac{|\xi^1_t|}{n} > \rho - 2\delta\right) &= \mathbf{P}\left(\frac{|\xi^1_T|}{n} > \rho - 2\delta\right) \\ &\geq \mathbf{P}\left(\xi^1_T \supseteq \{x : Y_x = 1\}, \sum_{x=1}^{n} Y_x \geq n(\rho - 2\delta)\right) \to 1.\end{aligned}$$

This completes the proof of Theorem 3.1.1. □

3.3 Proof of Theorem 3.1.3

Recall the definition of the active sets $A^k_x, k = 0, 1, \ldots, \beta_x$, and the removed sets $R^k_x, k = 0, 1, \ldots, \beta_x$, introduced before Lemma 3.2.1. Also recall the stopping times π^1_x and $\alpha^{n,\delta}_x$ in (3.2.3) and define

$$\pi^2_x \equiv \min\left\{l > \pi^1_x : \left|R^l_x\right| < l - 1\right\}.$$

This is the time of second collision while exploring $\hat{G}_n$ starting from x. First we show that with high probability for every vertex $x \in V_n$ the second collision occurs after $\lceil n^{1/4-\delta} \rceil$ many steps for any $\delta \in (0, 1/4)$.

Lemma 3.3.1. *Let* $\delta \in (0, 1/4)$ *and* I_x^2 *be the event*

$$I_x^2 \equiv \left\{\pi_x^2 \wedge \beta_x \geq \alpha_x^{n,1/4+\delta}\right\}.$$

Then for $I \equiv \cap_{x \in V_n} I_x^2$*,* $\mathbb{P}(I^c) \leq 2n^{-4\delta}$ *for large enough* n*.*

Proof. Let $\delta' = (1/4) - \delta$. Since in the construction of the random graph G_n the input nodes $y_i(z), 1 \leq i \leq r$, for any vertex z are distinct and different from z, there are at least $n - r$ choices for each $y_i(z)$. Also $\left|R_x^l\right| \leq l$ for any l. So $\mathbb{P}(|R_x^k| = |R_x^{k-1}|) \leq (k-1)/(n-r)$. Now if I_x^2 fails to occur, then there will be k_1 and k_2 such that $1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil$ and $|R_x^{k_i}| = |R_x^{k_i - 1}|$ for $i = 1, 2$. So

$$\begin{aligned}
\mathbb{P}\left[\left(I_x^2\right)^c\right] &\leq \sum_{1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil} \mathbb{P}\left(\left|R_x^{k_1}\right| = \left|R_x^{k_1-1}\right|, \left|R_x^{k_2}\right| = \left|R_x^{k_2-1}\right|\right) \\
&\leq \sum_{1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil} \frac{(k_1-1)(k_2-1)}{(n-r)^2} \leq \sum_{1 \leq k_1, k_2 \leq \lceil n^{\delta'} \rceil} 2\frac{(k_1-1)(k_2-1)}{n^2} \leq 2n^{4\delta'-2}
\end{aligned}$$

for large enough n. The second inequality holds because the choices of the input nodes are independent. Hence $\mathbb{P}(I^c) \leq \sum_{x \in V_n} \mathbb{P}\left[\left(I_x^2\right)^c\right] \leq 2n^{4\delta'-1} = 2n^{-4\delta}$. □

Lemma 3.3.1 shows that with high probability for all vertices there will be at most one collision until we have explored $\lceil n^{1/4-\delta} \rceil$ many vertices starting from any vertex of $\hat{G}_n$. Now recall the definition of the distance functions d_0 and d from (3.1.6), and $m(A, K)$ given in (3.1.7). Let $R = \log n / \log r$, $a = (1/8 - \delta)$ and let ρ be the branching process survival probability defined in (3.1.2).

Lemma 3.3.2. *Let* P_I *denote the conditional distribution of* $\hat{\boldsymbol{\xi}}^{\{x\}}, x \in V_n$ *given* I*, where* I *is the event defined in Lemma 3.3.1. If* $qr > 1$ *and* δ_0 *is small enough, then for any* $0 < \delta < \delta_0$ *there are constants* $C(\delta) > 0$*,* $B(\delta) = (1/8 - 2\delta)\log(qr - \delta)/\log r$ *and a stopping time* T *satisfying*

$$P_I\left(T < 2\exp\left(C(\delta)n^{B(\delta)}\right)\right) \leq 2\exp\left[-C(\delta)n^{B(\delta)}\right],$$

such that for any A *with* $m(A, 2\lceil aR \rceil) \geq \lfloor n^{B(\delta)} \rfloor$, $\left| \hat{\xi}_T^A \right| \geq \lfloor n^{B(\delta)} \rfloor$.

Proof. Let $m_t \equiv m\left(\hat{\xi}_t^A, 2\lceil aR \rceil\right)$. We define the stopping times σ_i and τ_i as follows. $\sigma_0 \equiv 0$, and for $i \geq 0$

$$\tau_{i+1} \equiv \min\left\{t > \sigma_i : m_t < \lfloor n^B \rfloor\right\},$$
$$\sigma_{i+1} \equiv \min\left\{t > \tau_{i+1} : m_t \geq \lfloor n^B \rfloor\right\}.$$

Since $\tau_i > \sigma_{i-1}$ for $i \geq 1$, $m_{\tau_i - 1} \geq \lfloor n^B \rfloor$, and hence there is a set $X_i \subset \hat{\xi}_{\tau_i - 1}^A$ of size at least $\lfloor n^B \rfloor$ such that $d(u, v) \geq 2\lceil aR \rceil$ for any two distinct vertices $u, v \in X_i$. Let E_i be the event that at least $(q - \delta)|X_i|$ many vertices of X_i give birth at time τ_i. Using the binomial large deviation estimate (3.2.10)

$$P_{G_n}(E_i) \geq 1 - \exp\left(-\Gamma((q - \delta)/q)q\lfloor n^B \rfloor\right), \tag{3.3.1}$$

where $\Gamma(x) = x \log x - x + 1$.

Now let I be the event defined in Lemma 3.3.1. Since $|\{z : d_0(x, z) \leq 2\lceil aR \rceil\}|$ is at most $2r^{2\lceil aR \rceil} \leq 2rn^{2a} \leq n^{1/4-\delta}$, so if I occurs, then for any vertex $x \in V_n$ there is at most one collision in $\{z : d_0(x, z) \leq 2\lceil aR \rceil\}$, and hence there are at least $r - 1$ input nodes $u_1(x), \ldots, u_{r-1}(x)$ of x such that $\{z : d_0(u_i(x), z) \leq 2\lceil aR \rceil - 1\}$ is a finite oriented $r-$tree for each $1 \leq i \leq r - 1$. Since the right hand side of (4.3.8) depends only on n,

$$P_I(I \cap E_i) = P_I(E_i) \geq 1 - \exp\left(-c_1(\delta)n^B\right),$$

where $c_1(\delta) = \Gamma((q - \delta)/q)q/2$. If $I \cap E_i$ occurs, then we can choose one suitable offspring of each of the vertices in X_i, which give birth, to form a subset $N_i \subset \hat{\xi}_{\tau_i}^A$ such that $|N_i| \geq (q - \delta)\lfloor n^B \rfloor$, $d(u, v) \geq 2\lceil aR \rceil - 2$ for any two distinct vertices

$u, v \in N_i$, and $\{z : d_0(u,z) \leq 2\lceil aR \rceil - 1\}$ is a finite oriented $r-$tree for each $u \in N_i$.

By the definition of N_i it is easy to see that for each $x \in N_i$

$$P_I\left[\left(\left|\hat{\xi}_t^{\{x\}}\right|, 0 \leq t \leq 2\lceil aR \rceil - 1\right) \in \cdot\right] = P_{Z^x}\left[(Z_t^x, 0 \leq t \leq 2\lceil aR \rceil - 1) \in \cdot\right],$$

where $\mathbf{Z}^x$ is a supercritical branching process, as introduced in Lemma 3.2.2, with distribution P_{Z^x} and mean offspring number qr. Let B_x be the event of survival for $\mathbf{Z}^x$, and $F_x = \cap_{t=\lfloor \delta R \rfloor - 1}^{\lceil aR \rceil - 2}\left\{Z_{t+1}^x \geq (qr-\delta)Z_t^x\right\}$. So $P_{Z^x}(B_x) = \rho > 0$ as in (3.1.2). Using the error probability of (3.2.7)

$$P_{Z^x}(F_x^c | B_x) \leq \sum_{t=\lfloor \delta R \rfloor - 1}^{\lceil aR \rceil - 2} e^{-c'(\delta)t} \leq C_\delta e^{-c'(\delta)\delta \log n/(2\log r)} = C_\delta n^{-c'(\delta)\delta/(2\log r)} \qquad (3.3.2)$$

for some constants $C_\delta, c'(\delta) > 0$. On the event $B_x \cap F_x$,

$$Z_{\lceil aR \rceil - 1}^x \geq (qr-\delta)^{(\lceil aR \rceil - 1) - (\lfloor \delta R \rfloor - 1)} \geq (qr-\delta)^{(a-\delta)R} = n^{(a-\delta)\log(qr-\delta)/\log r} = n^B.$$

Hence for $Q_x \equiv \left\{\left|\hat{\xi}_{\lceil aR \rceil - 1}^{\{x\}}\right| \geq \lceil n^B \rceil\right\}$ for $x \in N_i$, we use standard probability arguments and (4.4.26) to have

$$\begin{aligned} P_I(Q_x) &= P_I\left(\left|\hat{\xi}_{\lceil aR \rceil - 1}^{\{x\}}\right| \geq \lceil n^B \rceil\right) = P_{Z^x}\left(Z_{\lceil aR \rceil - 1}^x \geq \lceil n^B \rceil\right) \\ &\geq P_{Z^x}(B_x \cap F_x) \geq P_{Z^x}(B_x) P_{Z^x}(F_x | B_x) \geq \rho - \delta \qquad (3.3.3) \end{aligned}$$

for large enough n.

Since $d(u,v) \geq 2\lceil aR \rceil - 2$ for any two distinct vertices $u, v \in N_i$, $\hat{\xi}_t^{N_i}$ is a disjoint union of $\hat{\xi}_t^{\{x\}}$ over $x \in N_i$ for $t \leq \lceil aR \rceil - 1$. Let H_i be the event that there is at least one $x \in N_i$ for which Q_x occurs. Then recalling that $|N_i| \geq (q-\delta)\lfloor n^B \rfloor$ on E_i,

$$P_I(H_i^c | E_i) \leq (1-\rho+\delta)^{(q-\delta)\lfloor n^B \rfloor} = \exp\left(-c_2(\delta) n^B\right), \qquad (3.3.4)$$

where $c_2(\delta) = (q-\delta)\log(1/(1-\rho+\delta))/2$.

If $H_i \cap E_i$ occurs, choose any vertex $w_i \in N_i$ such that Q_{w_i} occurs and let $S_i \equiv \hat{\xi}^{\{w_i\}}_{\lceil aR \rceil - 1}$. By the choice of w_i, $|S_i| \geq \lfloor n^B \rfloor$. Since $(\lceil aR \rceil - 1) + \lceil aR \rceil = 2\lceil aR \rceil - 1$, for any two distinct vertices $x, z \in S_i$ the subgraphs induced by $\{u : d_0(x,u) \leq \lceil aR \rceil\}$ and $\{u : d_0(z,u) \leq \lceil aR \rceil\}$ are finite $r-$trees consisting of disjoint sets of vertices, and hence $d(x,z) \geq 2\lceil aR \rceil$. Hence using monotonicity of the dual process $\sigma_i \leq \tau_i + \lceil aR \rceil - 1$ on this event $H_i \cap E_i$. So

$$P_I(\sigma_i > \tau_i + \lceil aR \rceil - 1) \leq P_I(E_i^c) + P_I(H_i^c | E_i) \leq 2\exp(-2C(\delta)n^B),$$

where $C(\delta) \equiv \min\{c_1(\delta), c_2(\delta)\}/2$. Let $L = \inf\{i \geq 1 : \sigma_i > \tau_i + \lceil aR \rceil - 1\}$. Then

$$\begin{aligned} P_I\left[L > \exp\left(C(\delta)n^B\right)\right] &\geq \left[1 - 2\exp(-2C(\delta)n^B)\right]^{\exp\left(C(\delta)n^B\right)} \\ &\geq 1 - 2\exp\left(-C(\delta)n^B\right). \end{aligned}$$

Since $\sigma_i > \tau_i > \sigma_{i-1}$, $\sigma_{L-1} \geq 2(L-1)$. As $\left|\hat{\xi}^A_{\sigma_{L-1}}\right| \geq \lfloor n^B \rfloor$, we get our result if we take $T = \sigma_{L-1}$. $\square$

As in the proof of Theorem 3.1.1, survival of the dual process gives persistence of the threshold contact process.

Proof of Theorem 3.1.3. Let $0 < \delta < \delta_0, \rho$, $a = (1/8 - \delta)$ and $B = (1/8 - 2\delta)\log(qr - \delta)/\log r$ be the constants from the previous proof. Define the random variables $Y_x, 1 \leq x \leq n$, as $Y_x = 1$ if the dual process $\hat{\boldsymbol{\xi}}^{\{x\}}$ starting at x satisfies $\left|\hat{\xi}^{\{x\}}_{\lceil aR \rceil - 1}\right| > \lfloor n^B \rfloor$ and $Y_x = 0$ otherwise.

Consider the event $I_x^1 = \left\{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,1/4+\delta}\right\}$, where π_x^1, β_x and $\alpha_x^{n,1/4+\delta}$ are stopping times defined as in (3.2.3). Using Lemma 3.2.1 and 3.3.1

$$P_I\left[\left(I_x^1\right)^c\right] \le \frac{\mathbb{P}\left[\left(I_x^1\right)^c\right]}{\mathbb{P}(I)} \le \frac{n^{-2(1/4+\delta)}}{1-2n^{-4\delta}} \le 2n^{-(1/2+2\delta)}. \tag{3.3.5}$$

Let $J_x \equiv I \cap I_x^1$ and P_{J_x} be the conditional distribution of $\hat{\boldsymbol{\xi}}^{\{x\}}$ given J_x. Since the number of vertices in the set $\{u : d_0(x,u) \le \lceil aR \rceil - 1\}$ is at most $2r^{\lceil aR \rceil - 1} \le 2r^{aR} < n^{1/4-\delta}$ by the choice of a,

$$P_{J_x}\left[\left(\left|\hat{\xi}_t^{\{x\}}\right|, 0 \le t \le \lceil aR \rceil - 1\right) \in \cdot\right] = P_{Z^x}\left[(Z_t^x, 0 \le t \le \lceil aR \rceil - 1) \in \cdot\right],$$

where $\mathbf{Z}^x$ is a supercritical branching process, as introduced in Lemma 3.2.2, with distribution P_{Z^x} and mean offspring number qr. Let B_x and $F_x = \cap_{t=\lfloor \delta R \rfloor - 2}^{\lceil aR \rceil - 2}\left\{Z_{t+1}^x \ge (qr-\delta)Z_t^x\right\}$. So $P_{Z^x}(B_x) = \rho > 0$ as in (3.1.2), and similar to (4.4.26)

$$P_{Z^x}(F_x^c | B_x) \le \sum_{t=\lfloor \delta R \rfloor - 2}^{\lceil aR \rceil - 2} e^{-c'(\delta)t} \le C_\delta n^{-c'(\delta)\delta/(2\log r)}$$

for some constants $C_\delta, c'(\delta) > 0$. On the event $B_x \cap F_x$, $Z_{\lceil aR \rceil - 1}^x \ge (qr - \delta)^{(\lceil aR \rceil - 1) - (\lfloor \delta R \rfloor - 2)} > (qr-\delta)^{(a-\delta)R} \ge \lfloor n^B \rfloor$. Hence using (4.3.7)

$$\begin{aligned}
P_I(Y_x = 1) &\ge P_I\left(I_x^1 \cap \left\{\left|\hat{\xi}_{\lceil aR \rceil - 1}^{\{x\}}\right| > \lfloor n^B \rfloor\right\}\right) \\
&= P_{J_x}\left(\left|\hat{\xi}_{\lceil aR \rceil - 1}^{\{x\}}\right| > \lfloor n^B \rfloor\right) P_I(I_x^1) \\
&= P_{Z^x}\left(Z_{\lceil aR \rceil - 1}^x > \lfloor n^B \rfloor\right) P_I(I_x^1) \\
&\ge P_{Z^x}(B_x \cap F_x) P_I(I_x^1) = P_{Z^x}(B_x) P_{Z^x}(F_x | B_x) P_I(I_x^1) \ge \rho - \delta
\end{aligned}$$

for large enough n.

Next we estimate the covariance between the events $\{Y_x = 1\}$ and $\{Y_z = 1\}$. We consider the stopping times $\pi_x^1, \beta_x, \pi_{x,z}, \alpha_x^{n,1/4+\delta}$ as in (3.2.3) and the corresponding event $I_{x,z}$ as in Lemma 3.2.1. We can use similar argument, which leads to (3.2.13), to conclude

$$P_I(Y_x = 1, Y_z = 1) - P_I(Y_x = 1)P_I(Y_z = 1) \le P_I(I^c_{x,z})(1 + 1/P_I(I_{x,z})).$$

From Lemma 3.2.1 and 3.3.1,

$$P_I(I^c_{x,z}) \le \frac{\mathbb{P}(I^c_{x,z})}{\mathbb{P}(I)} \le \frac{5n^{-2(1/4+\delta)}}{1 - 2N^{-4\delta}} \le 10n^{-(1/2+2\delta)}$$

for large enough n, and so

$$P_I(Y_x = 1, Y_z = 1) - P_I(Y_x = 1)P_I(Y_z = 1) \le 30n^{-(1/2+2\delta)}$$

for large n. Using the bound on the covariances,

$$\mathrm{var}_I\left(\sum_{x=1}^{n} Y_x\right) \le n + 30n(n-1)n^{-2\delta},$$

and Chebyshev's inequality gives that as $n \to \infty$

$$P_I\left(\left|\sum_{x=1}^{n}(Y_x - \mathbf{E}Y_x)\right| \ge n\delta\right) \le \frac{n + 30n(n-1)n^{-2\delta}}{n^2\delta^2} \to 0.$$

Since $\mathbf{E}Y_x \ge \rho - \delta$ for all $x \in V_n$, this implies

$$\lim_{n\to\infty} P_I\left(\sum_{x=1}^{n} Y_x \ge n(\rho - 2\delta)\right) = 1. \tag{3.3.6}$$

Our next goal is to show that ξ^1_T contains the random set $D \equiv \{x : Y_x = 1\}$ with high probability for a suitable choice of T. If $Y_x = 1$, then $\left|\hat{\xi}^{\{x\}}_{T_1}\right| > \lfloor n^B \rfloor$, where $T_1 = \lceil aR \rceil - 1$. Note that $\lceil aR \rceil - 1 + \lceil aR \rceil \le 2\lceil aR \rceil$, and on the event I there can be at most one collision in $\{u : d_0(x,u) \le 2\lceil aR \rceil\}$. Even though the first collision occurs between descendants of two vertices in $\hat{\xi}^{\{x\}}_{T_1}$, still we can exclude one vertex from $\hat{\xi}^{\{x\}}_{T_1}$ to have a set $W_x \subset \hat{\xi}^{\{x\}}_{T_1}$ of size at least $\lfloor n^B \rfloor$ such that for any two distinct vertices $z, w \in W_x$, the subgraphs induced by $\{u : d_0(z,u) \le \lceil aR \rceil\}$ and $\{v : d_0(w,v) \le \lceil aR \rceil\}$ are finite oriented $r-$trees consisting of disjoint sets of vertices, i.e. $d(z,w) \geq 2\lceil aR \rceil$. So if $Y_x = 1$, then $m\left(\hat{\xi}^{\{x\}}_{T_1}, 2\lceil aR \rceil\right) \geq \lfloor n^B \rfloor$ on the event I. Using Lemma 3.3.2, after an additional $T_2 \geq 2\exp\left(C(\delta)n^B\right)$ units of time, the dual process contains at least $\lfloor n^B \rfloor$ many occupied sites with P_I probability $\geq 1 - 2\exp\left(-C(\delta)n^B\right)$.

Let $\mathcal{F}$ be the set of all subsets of V_n of size $> \lfloor n^B \rfloor$, and denote $F_x \equiv \hat{\xi}^{\{x\}}_{T_1}$. Using the duality relationship of (3.1.4) for the conditional probability $\mathcal{P}_I(\cdot) \equiv \mathcal{P}(\cdot|I)$, where

$$\mathcal{P}(\cdot) = \mathbf{P}\left(\cdot \middle| \hat{\xi}^{\{x\}}_t, 0 \leq t \leq T_1, x \in V_n\right),$$

we see that

$$\begin{aligned}
\mathcal{P}_I\left(\xi^1_{T_1+T_2} \supseteq D\right) &= \mathcal{P}_I\left[\cap_{x\in D}\left\{x \in \xi^1_{T_1+T_2}\right\}\right] \\
&= \mathcal{P}_I\left[\cap_{x\in D}\left\{\hat{\xi}^{\{x\}}_{T_1+T_2} \neq \emptyset\right\}\right].
\end{aligned}$$

Since $D = \{x : Y_x = 1\}$, $F_x \in \mathcal{F}$ for all $x \in D$. So by the Markov property of the dual process the above is

$$\begin{aligned}
&= \sum_{F_x\in\mathcal{F}, x\in D} \mathcal{P}_I\left[\cap_{x\in D}\left(\hat{\xi}^{\{x\}}_{T_1+T_2} \neq \emptyset, \hat{\xi}^{\{x\}}_{T_1} = F_x\right)\right] \\
&= \sum_{F_x\in\mathcal{F}, x\in D} P_I\left[\cap_{x\in D}\left(\hat{\xi}^{F_x}_{T_2} \neq \emptyset\right)\right] \mathcal{P}_I\left[\cap_{x\in D}\left(\hat{\xi}^{\{x\}}_{T_1} = F_x\right)\right].
\end{aligned}$$

Now since $W_x \subset F_x$, using monotonicity of the dual process, $P_I\left(\hat{\xi}^{F_x}_{T_2} \neq \emptyset\right) \geq P_I\left(\hat{\xi}^{W_x}_{T_2} \neq \emptyset\right)$. Also using Lemma 3.3.2, $P_I\left(\left|\hat{\xi}^{W_x}_{T_2}\right| \geq \lfloor n^B \rfloor\right) \geq 1 - 2\exp\left(-C(\delta)n^B\right)$ for any $F_x \in \mathcal{F}$. So the above is

$$\begin{aligned}
&\geq \left(1 - 2|D|\exp\left(-C(\delta)n^B\right)\right) \sum_{F_x\in\mathcal{F}, x\in D} \mathcal{P}_I\left[\cap_{x\in D}\left(\hat{\xi}^{\{x\}}_{T_1} = F_x\right)\right] \\
&\geq 1 - 2n\exp\left(-C(\delta)n^B\right).
\end{aligned}$$

For the last inequality we use $|D| \leq n$ and $\mathcal{P}_I(Y_x = 1 \forall x \in D) = 1$. Since the lower bound only depends on n,

$$\begin{aligned}
P_I\left(\xi^1_{T_1+T_2} \supseteq \{x : Y_x = 1\}\right) &\geq 1 - 2n\exp\left(-C(\delta)n^B\right) \\
\Rightarrow \quad \mathbf{P}\left(\xi^1_{T_1+T_2} \supseteq \{x : Y_x = 1\}\right) &\geq \mathbb{P}(I)\left[1 - 3n\exp\left(-C(\delta)n^B\right)\right] \to 1,
\end{aligned}$$

as $n \to \infty$, since $\mathbb{P}(I) \geq 1 - 2n^{-4\delta}$ by Lemma 3.3.1.

Hence for $T = T_1 + T_2$ using the attractiveness property of the threshold contact process, and combining the last calculation with (3.3.6) we conclude that as $n \to \infty$

$$\begin{aligned}
\inf_{t\leq T}\mathbf{P}\left(\frac{|\xi^1_t|}{n} > \rho - 2\delta\right) &= \mathbf{P}\left(\frac{|\xi^1_T|}{n} > \rho - 2\delta\right) \\
&\geq \mathbf{P}\left(\xi^1_T \supseteq \{x : Y_x = 1\}, \sum_{x=1}^{n} Y_x \geq n(\rho - 2\delta)\right) \to 1,
\end{aligned}$$

which completes the proof of Theorem 3.1.3. $\square$

Chapter 4

Aldous' Gossip Process

4.1 Introduction

We study a model introduced by Aldous [3] for the spread of gossip and other more economically useful information. His paper considers various game theoretic aspects of random percolation of information through networks. Here we concentrate on one small part, a first passage percolation model with nearest neighbor and long-range jumps introduced in his Section 6.2. The work presented here is also related to work in [23] and [9], where the impact of long-range dispersal on the spread of epidemics and invading species have been considered.

Space is the discrete torus $\Lambda(N) = (\mathbb{Z} \bmod N)^2$. The state of the process at time t is $\xi_t \subset \Lambda(N)$, the set of individuals who know the information at time t. Information spreads from i to j at rate

$$\nu_{ij} = \begin{cases} 1/4 & \text{if } j \text{ is a (nearest) neighbor of } i \\ \lambda_N/N^2 & \text{if not.} \end{cases}$$

If $\lambda_N = 0$, this is ordinary first passage percolation on the torus. If we start with $\xi_0 = \{(0,0)\}$, then the shape theorem for nearest-neighbor first passage percolation, see [14] or [32], implies that until the process exits $(-N/2, N/2)^2$, the radius of the set ξ_t grows linearly and ξ_t has an asymptotic shape. From this we see that if $\lambda_N = 0$, then there is a constant c_0 so that the time T_N, until everyone knows the information, satisfies

$$\frac{T_N}{N} \xrightarrow{P} c_0,$$

where $\xrightarrow{P}$ denotes convergence in probability.

To simplify things, we will remove the randomness from the nearest neighbor part of the process, and formulate it on the (real) torus $\Gamma(N) = (\mathbb{R} \bmod N)^2$. One should be able to prove a similar result for the first passage percolation model but there are two difficulties. The first and easier to handle is that the limiting shape is not round. The second and more difficult issue is that the growth is not deterministic but has fluctuations. One should be able to handle both of these problems, but the proof is already long enough.

We consider what we call the "balloon process", in which the state of the process at time t is $\mathcal{C}_t \subset \Gamma(N)$. It starts with one "center" chosen uniformly from the torus at time 0. When a center is born at x, a disk with radius 0 is put there, and its radius grows deterministically as $r(s) = s/\sqrt{2\pi}$, so that the area of the disk at time s after its birth is $s^2/2$. If the area covered at time t is C_t, then births of new centers occur at rate $\lambda_N C_t$. The location of each new center is chosen uniformly from the torus. If the new point lands at $x \in \mathcal{C}_t$, it will never contribute anything to the growth of the set, but we will count it in the total number of centers, which we denote by $\tilde{X}_t$.

Before turning to the details of our analysis we would like to point out that a related balloon process was used by Barbour and Reinert [5] in their study of distances on the small world graph. Consider a circle of radius L and introduce a Poisson mean $\rho L/2$ number of chords with length 0 connecting randomly chosen points on the circle. To study the distance between a fixed point O and a point chosen at random one wants to examine $S(t) = \{x : \text{dist}(O, x) \leq t\}$. If we ignore overlaps and let $M(t)$ be the number of intervals in $S(t)$ then $S'(t) = 2M(t)$ and $M(t)$ is a Yule process with births at rate $2\rho M(t)$ due to the interval ends encountering points in the Poisson process of chords. This is a balloon process in which the new births come from the boundaries. As in our case, one first studies the growth of the balloon process and then estimates the difference from the real process to prove the desired results. There are interesting parallels and differences between the two proofs, see [16, Section 5.2] for a proof.

Here we will be concerned with $\lambda_N = N^{-\alpha}$. To begin we will get rid of trivial cases. If the diameter of $\mathcal{C}_t$ grows linearly, then $\int_0^{c_0 N} C_t \, dt = O(N^3)$. So if $\alpha > 3$, with probability tending to 1 as N goes to ∞, there is no long range jump before the initial disk covers the entire torus, and the time T_N until the entire torus is covered satisfies

$$\frac{T_N}{N} \xrightarrow{P} c_1, \qquad \text{where} \quad c_1 = \sqrt{\pi}.$$

If $\alpha = 3$, then with probabilities bounded away from 0, (i) there is no long range jump and $T_N \approx c_1 N$, and (ii) there is one that lands close enough to $(N/2, N/2)$ to make $T_N \leq (1 - \delta) N c_1$. Using $\Rightarrow$ for weak convergence, this suggests that

Theorem 0. *When $\alpha = 3$, $T_N/N \Rightarrow$ a random limit concentrated on $[0, c_1]$ and with an atom at c_1.*

Proof. Suppose without loss of generality that the initial center is at 0, and view the torus as $(-N/2, N/2]^2$. The key observation is that the set-valued process

$\{\mathcal{C}_{Nt}/N, t \geq 0\}$ converges to a limit $\mathcal{D}_t$. Before the first long-range dispersal, the state of $\mathcal{D}_t$ is the intersection of the disk of radius $t/\sqrt{2\pi}$ with $(-1/2, 1/2]^2$. Long range births occur at rate equal to the area of $\mathcal{D}_t$ and are dispersed uniformly. Since the distance from (0,0) to (1/2,1/2) is $1/\sqrt{2}$, if there are no long range births before time $c_1 = \sqrt{\pi}$ or if all long range births land inside $\mathcal{D}_t$ then the torus is covered at time c_1. Computing the distribution of the cover time when it is $< c_1$ is complicated, but the answer is a continuous functional of the limit process, and standard weak convergence results give the result. $\square$

For the remainder of the paper we suppose $\lambda_N = N^{-\alpha}$ with $\alpha < 3$. The overlaps between disks in $\mathcal{C}_t$ pose a difficulty in analyzing the process, so we begin by studying a simpler "balloon branching process" $\mathcal{A}_t$, in which A_t is the sum of the areas of all of the disks at time t, births of new centers occur at rate $\lambda_N A_t$, and the location of each new center is chosen uniformly from the torus. Let X_t be the number of centers at time t in $\mathcal{A}_t$.

Suppose we start $\mathcal{C}_0$ and $\mathcal{A}_0$ from the same randomly chosen point. The areas $C_t = A_t$ until the time of the first birth, which can be made to be the same in the two processes. If we couple the location of the new centers at that time, and continue in the obvious way letting $\mathcal{C}_t$ and $\mathcal{A}_t$ give birth at the same time with the maximum rate possible, to the same place when they give birth simultaneously, and letting $\mathcal{A}_t$ give birth by itself otherwise, then we will have

$$\mathcal{C}_t \subset \mathcal{A}_t, \quad C_t \leq A_t, \quad \tilde{X}_t \leq X_t \quad \text{for all } t \geq 0. \tag{4.1.1}$$

X_t is a Crump-Mode-Jagers branching process, but saying these words does not magically solve our problems. Define the length process L_t to be $\sqrt{2\pi}$ times the sum of the radii of all the disks at time t.

$$L_t = \int_0^t (t-s)dX_s = \int_0^t X_s\,ds, \tag{4.1.2}$$
$$A_t = \int_0^t \frac{(t-s)^2}{2}dX_s = \int_0^t L_s\,ds.$$

Here and later we use $\int_0^t$ for integration over the closed interval $[0,t]$, i.e., we include the contribution from the atom in dX_s at 0. ($X_0 = 1$ while $X_s = 0$ for $s < 0$.) For the second equality on each line integrate by parts or note that $L_t' = X_t$ and $A_t' = L_t$. Since X_t increases by 1 at rate $\lambda_N A_t$, (X_t, L_t, A_t) is a Markov process.

To simplify formulas, we will often drop the subscript N from λ_N. For comparison with C_t, the parameter λ is important, but in the analysis of A_t it is not. If we let

$$X_t^1 = X(t\lambda^{-1/3}), \quad L_t^1 = \lambda^{1/3}L(t\lambda^{-1/3}), \quad A_t^1 = \lambda^{2/3}A(t\lambda^{-1/3}), \tag{4.1.3}$$

then (X_t^1, L_t^1, A_t^1) is the process with $\lambda = 1$.

To study the growth of A_t, first we will compute the means of X_t, L_t, and A_t. Let $F(t) = \lambda t^3/3!$. Using the independent and identical behavior of all the disks in $\mathcal{A}_t$ it is easy to show that (see the proof of Lemma 4.2.4)

$$EX_t = 1 + \int_0^t EX_{t-s}\,dF(s).$$

Solving the above renewal equation and using (4.1.2), we can show

$$EX_t = \sum_{k=0}^{\infty} F^{*k}(t) = V(t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!},$$
$$EL_t = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+1}}{(3k+1)!}, \tag{4.1.4}$$

$$EA_t = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+2}}{(3k+2)!}.$$

To evaluate $V(t)$ we note that $V'''(t) = \lambda V(t)$ with $V(0) = 1$, $V'(0) = V''(0) = 0$, so

$$V(t) = \frac{1}{3}\left[\exp(\lambda^{1/3}t) + \exp(\lambda^{1/3}\omega t) + \exp(\lambda^{1/3}\omega^2 t)\right]. \tag{4.1.5}$$

Here $\omega = \left(-1 + i\sqrt{3}\right)/2$ is one of the complex cube roots of 1 and $\omega^2 = \left(-1 - i\sqrt{3}\right)/2$ is the other. Note that each of ω and ω^2 has real part $-1/2$. So the second and third terms in (4.1.5) go to 0 exponentially fast.

If $\mathcal{F}_s = \sigma\{X_r, L_r, A_r : r \leq s\}$, then

$$\frac{d}{dt} E\left[\begin{array}{c|c} X_t & \\ L_t & \mathcal{F}_s \\ A_t & \end{array}\right]\Bigg|_{t=s} = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} X_s \\ L_s \\ A_s \end{bmatrix}. \tag{4.1.6}$$

Let Q be the matrix in (4.1.6). By computing the determinant of $Q - \eta I$ it is easy to see that Q has eigenvalues $\eta = \lambda^{1/3}, \omega\lambda^{1/3}, \omega^2\lambda^{1/3}$, and $e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t)$ is a (complex) martingale. To treat the three martingales separately, let

$$\begin{aligned} I_t &= X_t + \lambda^{1/3} L_t + \lambda^{2/3} A_t, & M_t &= \exp\left(-\lambda^{1/3}t\right) I_t, \\ J_t &= X_t + (\omega\lambda^{1/3}) L_t + (\omega\lambda^{1/3})^2 A_t, & \tilde{J}_t &= \exp\left(-\omega\lambda^{1/3}t\right) J_t, \\ K_t &= X_t + (\omega^2\lambda^{1/3}) L_t + (\omega^2\lambda^{1/3})^2 A_t, & \tilde{K}_t &= \exp\left(-\omega^2\lambda^{1/3}t\right) K_t, \end{aligned}$$

so that M_t is the real martingale, and $\tilde{J}_t$ and $\tilde{K}_t$ are the complex ones.

Theorem 4.1.1. $\{M_t : t \geq 0\}$ *is a positive square integrable martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$. $EM_t = M_0 = 1$.

$$EM_t^2 = \frac{8}{7} - \frac{1}{3}\exp(-\lambda^{1/3}t) + O\left(\exp(-5\lambda^{1/3}t/2)\right),$$

$$E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 = \frac{1}{6}\exp(2\lambda^{1/3}t) + O\left(\exp(\lambda^{1/3}t/2)\right).$$

If we let $M = \lim_{t\to\infty} M_t$, then $P(M > 0) = 1$ and

$$\exp(-\lambda^{1/3}t)X_t,\ \lambda^{1/3}\exp(-\lambda^{1/3}t)L_t,\ \lambda^{2/3}\exp(-\lambda^{1/3}t)A_t \to M/3$$

a.s. and in L^2. The distribution of M does not depend on λ.

The last result follows from (4.1.3), which with (4.1.2) explains why the three quantities converge to the same limit. The key to the proof of the convergence results is to note that $1 + \omega + \omega^2 = 0$ implies

$$3X_t = I_t + J_t + K_t,$$

$$3\lambda^{1/3}L_t = I_t + \omega^2 J_t + \omega K_t,$$

$$3\lambda^{2/3}A_t = I_t + \omega J_t + \omega^2 K_t.$$

The real parts of ω and ω^2 are $-1/2$. Although the results for $E|\tilde{J}_t|^2$ and $E|\tilde{K}_t|^2$ show that the martingales $\tilde{J}_t$ and $\tilde{K}_t$ are not L^2 bounded, it is easy to show that $\exp\left(-\lambda^{1/3}t\right) J_t$ and $\exp\left(-\lambda^{1/3}t\right) K_t \to 0$ a.s. and in L^2, and Theorem 5.1.2 then follows from $M_t = \exp\left(-\lambda^{1/3}t\right) I_t \to M$.

Recall that $\lambda_N = N^{-\alpha}$ and let

$$a(t) = (1/3)N^{2\alpha/3}\exp(N^{-\alpha/3}t), \quad l(t) = N^{-\alpha/3}a(t), \quad x(t) = N^{-2\alpha/3}a(t), \tag{4.1.7}$$

so that $A_t/a(t), L_t/l(t), X_t/x(t) \to M$a.s.. Let

$$S(\epsilon) = N^{\alpha/3}[(2 - 2\alpha/3)\log N + \log(3\epsilon)], \tag{4.1.8}$$

so $a(S(\epsilon)) = \epsilon N^2$. Let

$$\sigma(\epsilon) = \inf\{t : A_t \geq \epsilon N^2\} \quad \text{and} \quad \tau(\epsilon) = \inf\{t : C_t \geq \epsilon N^2\}. \tag{4.1.9}$$

The first of these is easy to study.

Theorem 4.1.2. *If* $0 < \epsilon < 1$, *then as* $N \to \infty$

$$N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) \xrightarrow{P} -\log(M).$$

The coupling in (4.1.1) *implies* $\tau(\epsilon) \geq \sigma(\epsilon)$. *In the other direction, for any* $\gamma > 0$

$$\limsup_{N\to\infty} P[\tau(\epsilon) > \sigma((1+\gamma)\epsilon)] \leq P\left(M \leq (1+\gamma)\epsilon^{1/3}\right) + 11\frac{\epsilon^{1/3}}{\gamma}.$$

The last result implies that for $\epsilon < 1$

$$\tau(\epsilon) \sim (2 - 2\alpha/3)N^{\alpha/3}\log N. \tag{4.1.10}$$

Our next goal is to obtain more precise information about $\tau(\epsilon)$ and about how $|C_t|/N^2$ increases from a small positive level to reach 1.

The first result in Theorem 4.1.2 shows that $(\sigma(\epsilon) - S(\epsilon))/N^{\alpha/3}$ is determined by the random variable M from Theorem 5.1.2, which in turn is determined by what happens early in the growth of the branching balloon process. Let

$$R = N^{\alpha/3}[(2 - 2\alpha/3)\log N - \log(M)], \tag{4.1.11}$$

R is defined so that $a(R) = (1/3)N^2/M$, and hence $A_R/N^2 \xrightarrow{P} 1/3$. Define

$$\psi(t) \equiv R + N^{\alpha/3}t, \quad W \equiv \psi(\log(3\epsilon)), \text{ and } I_{\epsilon,t} = [\log(3\epsilon), t] \tag{4.1.12}$$

for $\log(3\epsilon) \leq t$. W is defined so that $a(W) = \epsilon N^2/M$ and hence $A_W/N^2 \xrightarrow{P} \epsilon$. The arguments that led to Theorem 4.1.2 will show that if ϵ is small then C_W/A_W is close to 1 with high probability.

To get a lower bound on the growth of C_t after time W we declare that the centers in $\mathcal{C}_W$ and $\mathcal{A}_W$ to be generation 0 in $\mathcal{C}_t$ and $\mathcal{A}_t$ respectively, and we number the succeeding generations in the obvious way, a center born from an area of generation k is in generation $k+1$. For $t \geq \log(3\epsilon)$, let $C^k_{W,\psi(t)}$ and $A^k_{W,\psi(t)}$ denote the areas covered at time $\psi(t)$ by respective centers of generations $j \in \{0, 1, \dots, k\}$ and let

$$g_0(t) = \epsilon \left[1 + (t - \log(3\epsilon)) + \frac{(t - \log(3\epsilon))^2}{2} \right], \quad f_0(t) = g_0(t) - \epsilon^{7/6}. \tag{4.1.13}$$

To explain these definitions, we note that Lemma 4.4.3 will show that for any t, there is an $\epsilon_0 = \epsilon_0(t)$ so that for any $0 < \epsilon < \epsilon_0$

$$\lim_{N\to\infty} P\left(\sup_{s \in I_{\epsilon,t}} \left| N^{-2} A^0_{W,\psi(s)} - g_0(s) \right| > \eta \right) = 0 \quad \text{for any } \eta > 0,$$
$$P\left(\inf_{s \in I_{\epsilon,t}} N^{-2} (C^0_{W,\psi(s)} - A^0_{W,\psi(s)}) < -\epsilon^{7/6} \right) \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Since $C^0_{W,\psi(t)} \leq A^0_{W,\psi(t)}$, if ϵ is small, with high probability $g_0(t)$ and $f_0(t)$ provide upper and lower bounds respectively for $C^0_{W,\psi(t)}$.

To begin to improve these bounds we let

$$f_1(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2} f_0(s)\, ds \right),$$

and define g_1 similarly. To explain this equation note that an $x \notin C^0_{W,\psi(t)}$ will not be in $C^1_{W,\psi(t)}$ if and only if no generation 1 center is born in the space-time cone

$$K^\epsilon_{x,t} \equiv \left\{ (y,s) \in \Gamma(N) \times [W, \psi(t)] : |y - x| \leq (\psi(t) - s)/\sqrt{2\pi} \right\}.$$

Lemma 4.4.4 shows that for $0 < \epsilon < \epsilon_0$ and $\delta > 0$,

$$\limsup_{N\to\infty} P\left(\inf_{s \in I_{\epsilon,t}} N^{-2} C^1_{W,\psi(s)} - f_1(s) < -\delta \right) \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

To iterate this we will let

$$f_{k+1}(t) = 1 - (1 - f_k(t)) \exp\left(- \int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2} (f_k(s) - f_{k-1}(s))\, ds \right)$$

for $k \geq 1$. The difference $f_k(s) - f_{k-1}(s)$ in the integral comes from the fact that a new point in generation $k+1$ must come from a point that is in generation k but not in generation $k-1$. Combining these equations we have

$$\begin{aligned}
1 - f_{k+1}(t) &= (1 - f_k(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} (f_k(s) - f_{k-1}(s))\, ds \right) \\
&= (1 - f_{k-1}(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \sum_{l=k-1}^{k} (f_l(s) - f_{l-1}(s))\, ds \right) \\
\cdots &= (1 - f_0(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \sum_{l=1}^{k} (f_l(s) - f_{l-1}(s)) + f_0(s)\, ds \right)
\end{aligned}$$

so that

$$f_{k+1}(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} f_k(s)\, ds \right). \tag{4.1.14}$$

Since $f_1(t) \geq f_0(t)$, letting $k \to \infty$, $f_k(t) \uparrow f_\epsilon(t)$, where f_ϵ is the unique solution of

$$f_\epsilon(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} f_\epsilon(s)\, ds \right) \tag{4.1.15}$$

with $f_\epsilon(\log(3\epsilon)) = \epsilon - \epsilon^{7/6}$. $g_k(t)$ and $g_\epsilon(t)$ are defined similarly.

$g_\epsilon(t)$ and $f_\epsilon(t)$ provide upper and lower bounds on the growth of $C_{\psi(t)}$ for $t \geq \log(3\epsilon)$. To close the gap between these bounds we let $\epsilon \to 0$.

Lemma 4.1.3. *For any $t < \infty$, if $I_{\epsilon,t} = [\log(3\epsilon), t]$, then as $\epsilon \to 0$,*

$$\sup_{s \in I_{\epsilon,t}} |f_\epsilon(s) - h(s)|, \ \sup_{s \in I_{\epsilon,t}} |g_\epsilon(s) - h(s)| \to 0$$

for some nondecreasing h with (a) $\lim_{t \to -\infty} h(t) = 0$, (b) $\lim_{t \to \infty} h(t) = 1$,

$$(c) \qquad h(t) = 1 - \exp\left(- \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds \right),$$

and (d) $0 < h(t) < 1$ for all t.

If one removes the 2 from inside the exponential, this is equation (36) in [3]. Since there is no initial condition, the solution is only unique up to time translation.

Theorem 4.1.4. *Let h be the function in Lemma 5.8.5. For any $t < \infty$ and $\delta > 0$,*

$$\lim_{N\to\infty} P\left(\sup_{s\leq t} |N^{-2}C_{\psi(s)} - h(s)| \leq \delta\right) = 1.$$

This result shows that the displacement of $\tau(\epsilon)$ from $(2-2\alpha/3)N^{\alpha/3}\log N$ on the scale $N^{\alpha/3}$ is dictated by the random variable M that gives the rate of growth of the branching balloon process, and that once C_t reaches ϵN^2, the growth is deterministic.

The solution $h(t)$ never reaches 1, so we need a little more work to show that

Theorem 4.1.5. *Let T_N be the first time the torus is covered. As $N \to \infty$*

$$T_N/(N^{\alpha/3}\log N) \xrightarrow{P} 2 - 2\alpha/3.$$

The remainder of the paper is organized as follows. In section 2, we prove the properties of $\mathcal{A}_t$ presented in Theorem 5.1.2. In section 3, we prove the properties of the hitting times s $\sigma(\epsilon)$ and $\tau(\epsilon)$ stated in Theorem 4.1.2. In section 4, we prove the limiting behavior of $\mathcal{C}_t$ mentioned in Theorem 4.1.4. Finally in section 5, we prove Theorem 4.1.5.

4.2 Properties of the balloon branching process $\mathcal{A}_t$

Lemma 4.2.1. $\int_0^t s^m(t-s)^n ds = \frac{m!n!}{(m+n+1)!}t^{m+n+1}$.

Proof. If you can remember the definition of the beta distribution, this is trivial. If you can't then integrate by parts and use induction. $\square$

Let $F(t) = \lambda t^3/3!$ for $t \geq 0$, and $F(t) = 0$ for $t < 0$. Let $V(t) = \sum_{k=0}^{\infty} F^{*k}(t)$, where $*k$ indicates the k-fold convolution.

Lemma 4.2.2. *If* $\omega = \left(-1 + i\sqrt{3}\right)/2$, *then*

$$V(t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!} = \frac{1}{3}\left[\exp\left(\lambda^{1/3} t\right) + \exp\left(\lambda^{1/3}\omega t\right) + \exp\left(\lambda^{1/3}\omega^2 t\right)\right].$$

Proof. We first use induction to show that

$$F^{*k}(t) = \begin{cases} \lambda^k t^{3k}/(3k)! & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{4.2.1}$$

This holds for $k = 0, 1$ by our assumption. If the equality holds for $k = n$, then using Lemma 4.2.1 we have for $t \geq 0$

$$F^{*(n+1)}(t) = \int_0^t F^{*n}(t-s)\, dF(s) = \int_0^t \frac{\lambda^n (t-s)^{3n}}{(3n)!} \frac{\lambda s^2}{2}\, ds = \frac{\lambda^{n+1} t^{3n+3}}{(3n+3)!}.$$

It follows by induction that $V(t) = \sum_{k=0}^{\infty} \lambda^k t^{3k}/(3k)!$. To evaluate the sum we note that setting $\lambda = 1$, $U(t) = \sum_{k=0}^{\infty} t^{3k}/(3k)!$ solves

$$U'''(t) = U(t) \quad \text{with } U(0) = 1 \text{ and } U'(0) = U''(0) = 0.$$

This differential equation has solutions of the from $e^{\gamma t}$, where $\gamma^3 = 1$, i.e. $\gamma = 1, \omega$ and ω^2. This leads to the general solution

$$U(t) = Ae^t + Be^{\omega t} + Ce^{\omega^2 t}$$

for some constants A, B, C. Using the initial conditions for $U(t)$ we have

$$A + B + C = 1, \quad A + B\omega + C\omega^2 = 0, \quad A + B\omega^2 + C\omega = 0.$$

Since $1+\omega+\omega^2=0$, we have $A=B=C=1/3$. Since $V(t)=U(\lambda^{1/3}t)$, we have proved the desired result. $\square$

Our next step is to compute the first two moments of X_t, L_t and A_t. For that we need the following lemma in addition to the previous one.

Lemma 4.2.3. *Let $\{N_t : t \geq 0\}$ be a Poisson process on $[0,\infty)$ with intensity $\lambda(\cdot)$ and let Π_t be the set of points at time t. If $\{Y_t, Z_t : t \geq 0\}$ are two complex valued stochastic processes satisfying*

$$Y_t = y(t) + \sum_{s_i \in \Pi_t} Y^i_{t-s_i}, \qquad Z_t = z(t) + \sum_{s_i \in \Pi_t} Z^i_{t-s_i},$$

where (Y^i, Z^i), $i = 1, 2, \ldots$ are i.i.d. copies of (Y, Z), and independent of N, then

$$EY_t = y(t) + \int_0^t EY_{t-s}\lambda(s)\,ds,$$
$$E(Y_tZ_t) = (EY_t)(EZ_t) + \int_0^t E(Y_{t-s}Z_{t-s})\lambda(s)\,ds.$$

Proof. N_t has Poisson distribution with mean $\Lambda_t = \int_0^t \lambda(s)ds$. Given $N_t = n$, the conditional distribution of Π_t is same as the distribution of $\{t_1, \ldots, t_n\}$, where $t_1, \ldots, t_n$ are i.i.d. from $[0,t]$ with density $\beta(\cdot) = \lambda(\cdot)/\Lambda_t$. Hence

$$E(Y_t|N_t) = y(t) + \sum_{i=1}^{N_t} EY^i_{t-t_i} = y(t) + N_t \int_0^t EY_{t-s}\,\beta(s)\,ds,$$

and taking expected values $EY_t = y(t) + \int_0^t EY_{t-s}\lambda(s)\,ds$.

Similarly $EZ_t = z(t) + \int_0^t EZ_{t-s}\lambda(s)ds$. Using the conditional distribution of Π_t given N_t, $E(Y_tZ_t|N_t)$ is

$$= y(t)z(t) + y(t)E\sum_{i=1}^{N_t} Z^i_{t-t_i} + z(t)E\sum_{i=1}^{N_t} Y^i_{t-t_i} + E\left[\sum_{i=1}^{N_t} Y^i_{t-t_i}Z^i_{t-t_i} + \sum_{i\neq j} Y^i_{t-t_i}Z^j_{t-t_j}\right]$$

$$= y(t)z(t) + y(t)N_t \int_0^t EZ_{t-s}\,\beta(s)\,ds + z(t)N_t \int_0^t EY_{t-s}\,\beta(s)\,ds$$
$$+ N_t \int_0^t E(Y_{t-s}Z_{t-s})\,\beta(s)\,ds + N_t(N_t-1)\int_0^t EY_{t-s}\,\beta(s)\,ds \int_0^t EZ_{t-s}\beta(s)ds.$$

Taking expectation on both sides and using $EN_t(N_t-1) = \Lambda_t^2$, we get

$$E(Y_tZ_t) = (EY_t)(EZ_t) + \int_0^t E(Y_{t-s}Z_{t-s})\lambda(s)ds,$$

which completes the proof. □

Now we use Lemma 4.2.2 and 4.2.3 to have the first moments.

Lemma 4.2.4. $E(X_t, L_t, A_t) = (V(t), V''(t)/\lambda, V'(t)/\lambda)$.

Proof. Recall that $F(t) = \lambda t^3/3!$. In the balloon branching process, the initial center x gives birth to new centers at rate $F'(t) = \lambda t^2/2$, and all the centers behave independently and with the same distribution as the one at x. So

$$X_t = 1 + \sum_{s_i \in \Pi_t} X^i_{t-s_i}, \tag{4.2.2}$$

where $\Pi_t \subset [0,t]$ is the set of times when new centers are born in $\mathcal{A}_t$ and $X^i, i = 1, 2, \ldots,$ are i.i.d. copies of X, and using Lemma 4.2.3,

$$EX_t = 1 + \int_0^t EX_{t-s}\,dF(s).$$

Using (4.5) from Chapter 3 of [17] and then (4.1.2):

$$\begin{aligned} EX_t = V(t) &= \sum_{k=0}^\infty \frac{\lambda^k t^{3k}}{(3k)!}, \\ EL_t = \int_0^t EX_s\,ds &= \sum_{k=0}^\infty \frac{\lambda^k t^{3k+1}}{(3k+1)!}, \\ EA_t = \int_0^t EL_s\,ds &= \sum_{k=0}^\infty \frac{\lambda^k t^{3k+2}}{(3k+2)!}. \end{aligned} \tag{4.2.3}$$

Since $V(t) = 1 + \sum_{k=0}^{\infty} \lambda^{k+1} t^{3k+3}/(3k+3)!$, it is easy to see that $EA_t = V'(t)/\lambda$ and $EL_t = V''(t)/\lambda$. $\square$

Lemma 4.2.5. *If* $M_t = \exp(-\lambda^{1/3} t)[X_t + \lambda^{1/3} L_t + \lambda^{2/3} A_t]$, *then* $\{M_t : t \geq 0\}$ *is a square integrable martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$. $EM_t = 1$ *and*

$$EM_t^2 = \frac{8}{7} - \frac{1}{3} \exp\left(-\lambda^{1/3} t\right) + \theta_t \quad \textit{where} \quad |\theta_t| \leq \frac{4}{15} \exp(-5\lambda^{1/3} t/2).$$

and hence $(8/7) - EM_t^2 \leq \exp(-\lambda^{1/3} t)$.

Proof. Let $h(t, x, \ell, a) = \exp(-\lambda^{1/3} t)[x + \lambda^{1/3} \ell + \lambda^{2/3} a]$, and let $\mathcal{L}$ be the generator of the Markov process (t, X_t, L_t, A_t). (4.1.6) implies $\mathcal{L}h = 0$, so M_t is a martingale from Dynkin's formula. $EM_t = EM_0 = 1$.

To compute EM_t^2 we use Lemma 4.2.3 as follows. Let $Y_t = Z_t = X_t + \lambda^{1/3} L_t + \lambda^{2/3} A_t$ and $g(t) \equiv (EY_t)^2$. Since $EM_t = 1$, $g(t) = \exp\left(2\lambda^{1/3} t\right)$. Combining (4.1.2) and (4.2.2), letting $L_t^i = \int_0^t X_s^i \, ds$ and $A_t^i = \int_0^t L_s^i \, ds, i = 1, 2, \ldots,$ and changing the variables $u = s - s_i$, we see that

$$L_t = \int_0^t \left[1 + \sum_{s_i \in \Pi_s} X_{s-s_i}^i\right] \, ds = t + \sum_{s_i \in \Pi_t} \int_0^{t-s_i} X_u^i \, du = t + \sum_{s_i \in \Pi_t} L_{t-s_i}^i, \text{ and hence}$$

$$A_t = \int_0^t \left[t + \sum_{s_i \in \Pi_s} L_{s-s_i}^i\right] \, ds = t^2/2 + \sum_{s_i \in \Pi_t} \int_0^{t-s_i} L_u^i \, du = t^2/2 + \sum_{s_i \in \Pi_t} A_{t-s_i}^i.$$

Thus all of X_t, L_t and A_t satisfy the hypothesis of Lemma 4.2.3 and so do Y_t and Z_t, as they are linear combinations of X_t, L_t and A_t. So applying Lemma 4.2.3

$$EY_t^2 = g(t) + \int_0^t EY_{t-s}^2 \, dF(s).$$

Solving the renewal equation using (4.8) in Chapter 3 of [17],

$$EY_t^2 = g * V(t) = \exp\left(2\lambda^{1/3} t\right) + \int_0^t \exp(2\lambda^{1/3}(t-s)) V'(s) \, ds,$$

where $V = \sum_{k=0}^{\infty} F^{*k}$. To evaluate the integral we use Lemma 4.2.2 to conclude

$$\begin{aligned}
&\int_0^t \exp\left(-2\lambda^{1/3}s\right) V'(s)\,ds \\
&= \frac{1}{3}\int_0^t \exp\left(-2\lambda^{1/3}s\right)\cdot \lambda^{1/3}\left[\exp\left(\lambda^{1/3}s\right) + \omega\exp\left(\lambda^{1/3}\omega s\right) + \omega^2\exp\left(\lambda^{1/3}\omega^2 s\right)\right] ds \\
&= \frac{1}{3}\left[\frac{1}{1-2}\left\{\exp\left(-\lambda^{1/3}t\right) - 1\right\} + \frac{\omega}{\omega-2}\left\{\exp\left((\omega-2)\lambda^{1/3}t\right) - 1\right\}\right. \\
&\qquad\qquad\qquad\left. + \frac{\omega^2}{\omega^2-2}\left\{\exp\left((\omega^2-2)\lambda^{1/3}t\right) - 1\right\}\right].
\end{aligned}$$

Now using $1 = -\omega - \omega^2$ and $\omega^3 = 1$,

$$1 - \frac{\omega}{\omega-2} - \frac{\omega^2}{\omega^2-2} = 1 - \frac{\omega^3 - 2\omega + \omega^3 - 2\omega^2}{\omega^3 - 2\omega^2 - 2\omega^2 + 4} = 1 - \frac{4}{7} = \frac{3}{7}.$$

Since $\omega = \left(-1 + i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1 - i\sqrt{3}\right)/2$, the remaining error satisfies

$$\begin{aligned}
3|\theta_t| &= \left|\frac{\omega}{\omega-2}\exp\left((\omega-2)\lambda^{1/3}t\right)\right| + \left|\frac{\omega^2}{\omega^2-2}\exp\left((\omega^2-2)\lambda^{1/3}t\right)\right| \\
&= \left(\frac{1}{|\omega-2|} + \frac{1}{|\omega^2-2|}\right)\exp\left(-5\lambda^{1/3}t/2\right) \le 2\cdot\frac{2}{5}\exp\left(-5\lambda^{1/3}t/2\right),
\end{aligned}$$

since $\omega - 2$ and $\omega^2 - 2$ each have real part $-5/2$. Putting all together

$$\int_0^t \exp\left(-2\lambda^{1/3}s\right) V'(s)\,ds = \frac{1}{7} - \frac{1}{3}\exp\left(-\lambda^{1/3}t\right) + \theta_t, \tag{4.2.4}$$

Since $EM_t^2 = \exp\left(-2\lambda^{1/3}t\right) EY_t^2$, the desired result follows. □

We use the previous calculation to get bounds for EA_t^2, EL_t^2 and EX_t^2, which will be useful later.

Lemma 4.2.6. *Let $a(\cdot), l(\cdot)$ and $x(\cdot)$ be as in* (4.1.7). *Then*

$$EA_t^2 \le \frac{27}{2}a^2(t), \quad EL_t^2 \le \frac{27}{2}l^2(t), \quad EX_t^2 \le \frac{27}{2}x^2(t).$$

Proof. By (4.2.4) we have

$$\int_0^t \exp\left(-2\lambda^{1/3}s\right) V'(s)\, ds \leq \frac{1}{7} + \frac{4}{15} = \frac{43}{105} \leq \frac{1}{2}. \tag{4.2.5}$$

Now using Lemma 4.2.3

$$EA_t^2 = (EA_t)^2 + \int_0^t EA_{t-s}^2\, dF(s), \quad EL_t^2 = (EL_t)^2 + \int_0^t EL_{t-s}^2\, dF(s),$$
$$EX_t^2 = (EX_t)^2 + \int_0^t EX_{t-s}^2\, dF(s).$$

Solving the renewal equations $EA_t^2 = \phi_a * V(t), EL_t^2 = \phi_l * V(t)$ and $EX_t^2 = \phi_x * V(t)$, where $V(\cdot)$ is as in Lemma 4.2.2 and $\phi_a(t) = (EA_t)^2, \phi_l(t) = (EL_t)^2$ and $\phi_x(t) = (EX_t)^2$. A crude upper bound for $\phi_a(t)$ is $9a^2(t)$. Since $a(t-s) = a(t)\exp\left(-\lambda^{1/3}s\right)$,

$$a^2 * V(t) = a^2(t)\left[1 + \int_0^t \exp\left(-\lambda^{1/3}s\right) V'(s)\, ds\right] \leq \frac{3a^2(t)}{2} \tag{4.2.6}$$

by (4.2.5). Hence $EA_t^2 \leq 9a^2 * V(t) \leq (27/2)a^2(t)$.

Similarly using the bounds $9l^2(t)$ and $9x^2(t)$ for $\phi_l(t)$ and $\phi_x(t)$ respectively and noting that $l(t-s)/l(t) = x(t-s)/x(t) = \exp\left(-\lambda^{1/3}s\right)$, we get the desired bounds for EL_t^2 and EX_t^2. □

Lemma 4.2.7. *Let $\tilde{J}_t, \tilde{K}_t = e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t)$ with $\eta = \omega\lambda^{1/3}$, $\omega^2\lambda^{1/3}$ respectively. Then $\tilde{J}_t$ and $\tilde{K}_t$ are complex martingales with respect to the filtration $\mathcal{F}_t$, and*

$$E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 = \frac{1}{6}\exp(2\lambda^{1/3}t) + \frac{1}{2} + \theta_t, \quad \text{where } |\theta_t| \leq \frac{2}{3}\exp\left(\lambda^{1/3}t/2\right),$$

and hence $E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 \leq (4/3)\exp\left(2\lambda^{1/3}t\right)$.

Proof. Let $h(t,x,\ell,a) = e^{-\eta t}(x + \eta\ell + \eta^2 a)$, and let $\mathcal{L}$ be the generator of the Markov process (t, X_t, L_t, A_t). (4.1.6) implies $\mathcal{L}h = 0$ when $\eta = \lambda^{1/3}\omega, \lambda^{1/3}\omega^2$, so that $\tilde{J}_t$ and $\tilde{K}_t$ are complex martingales by Dynkin's formula.

First we compute $E|J_t|^2$, where $J_t = \exp\left(\lambda^{1/3}\omega t\right)\tilde{J}_t$. For that we use Lemma 4.2.3 with $Y_t = J_t$ and $Z_t = \bar{J}_t$, the complex conjugate. Since $\tilde{J}_t$ is a complex martingale with $\tilde{J}_0 = 1$ and $\omega = \left(-1 + i\sqrt{3}\right)/2$, $E\tilde{J}_t = 1$ and hence

$$|EJ_t|^2 = \exp(-\lambda^{1/3}t).$$

Using Lemma 4.2.3 $E|J_t|^2 = |EJ_t|^2 + \int_0^t E|J_{t-s}|^2\, dF(s)$. Solving the renewal equation as we have done twice before

$$E|J_t|^2 = \exp(-\lambda^{1/3}t) + \int_0^t \exp(-\lambda^{1/3}(t-s))V'(s)\, ds.$$

Repeating the first part of the proof for $K_t = \exp\left(\lambda^{1/3}\omega^2 t\right)\tilde{K}_t$, we see that $E|K_t|^2$ is also equal to the right-hand side above.

The integral is $\exp(-\lambda^{1/3}t)$ times

$$\begin{aligned}
&\frac{1}{3}\int_0^t \exp\left(\lambda^{1/3}s\right)\cdot\lambda^{1/3}\left[\exp\left(\lambda^{1/3}s\right) + \omega\exp\left(\lambda^{1/3}\omega s\right) + \omega^2\exp\left(\lambda^{1/3}\omega^2 s\right)\right]\, ds \\
&= \frac{1}{3}\left[\frac{1}{1+1}\left\{\exp\left(2\lambda^{1/3}t\right) - 1\right\} + \frac{\omega}{\omega+1}\left\{\exp\left((\omega+1)\lambda^{1/3}t\right) - 1\right\}\right. \\
&\qquad\qquad\qquad \left. + \frac{\omega^2}{\omega^2+1}\left\{\exp\left((\omega^2+1)\lambda^{1/3}t\right) - 1\right\}\right].
\end{aligned}$$

Now using $1 = -\omega - \omega^2$ and $\omega^3 = 1$,

$$-\frac{1}{2} - \frac{\omega}{\omega+1} - \frac{\omega^2}{\omega^2+1} = -\frac{1}{2} - \frac{\omega^3+\omega+\omega^3+\omega^2}{\omega^3+\omega^2+\omega+1} = -\frac{3}{2}.$$

Since $\omega = \left(-1+i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1-i\sqrt{3}\right)/2$, if we take

$$\theta_t = \frac{1}{3}\left[\frac{\omega}{\omega+1}\exp\left((\omega+1)\lambda^{1/3}t\right) + \frac{\omega^2}{\omega^2+1}\exp\left((\omega^2+1)\lambda^{1/3}t\right)\right], \text{ then}$$

$$3|\theta_t| \leq \left(\frac{1}{|\omega+1|} + \frac{1}{|\omega^2+1|}\right)\exp\left(\lambda^{1/3}t/2\right) \leq 2\exp\left(\lambda^{1/3}t/2\right),$$

since each of $\omega+1$ and ω^2+1 has real part $1/2$. Putting all together

$$E|J_t|^2 \leq \frac{1}{6}\exp\left(\lambda^{1/3}t\right) + \frac{1}{2}\exp\left(-\lambda^{1/3}t\right) + \frac{2}{3}\exp\left(-\lambda^{1/3}t/2\right), \tag{4.2.7}$$

which completes the proof, since $E|\tilde{J}_t|^2/E|J_t|^2 = \exp(\lambda^{1/3}t) = E|\tilde{K}_t|^2/E|K_t|^2$.

□

Lemma 4.2.8. *If $M = \lim_{t\to\infty} M_t$, we have $P(M > 0) = 1$ and*

$$\exp(-\lambda^{1/3}t)X_t,\ \lambda^{1/3}\exp(-\lambda^{1/3}t)L_t,\ \lambda^{2/3}\exp(-\lambda^{1/3}t)A_t \to \frac{M}{3} \quad \textit{a.s. and in } L^2.$$

Proof. $M = \lim_{t\to\infty} M_t$ exists a.s. and in L^2, since M_t is an L^2 bounded martingale. Recall that

$$I_t = X_t + \lambda^{1/3}L_t + \lambda^{2/3}A_t,$$

$$J_t = X_t + \omega\lambda^{1/3}L_t + \omega^2\lambda^{2/3}A_t,$$

$$K_t = X_t + \omega^2\lambda^{1/3}L_t + \omega\lambda^{2/3}A_t.$$

Since $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$,

$$\begin{aligned} 3X_t &= I_t + J_t + K_t, \\ 3\lambda^{1/3}L_t &= I_t + \omega^2 J_t + \omega K_t, \\ 3\lambda^{2/3}A_t &= I_t + \omega J_t + \omega^2 K_t. \end{aligned} \tag{4.2.8}$$

Since $M_t = \exp(-\lambda^{1/3}t)I_t \to M$, it suffices to show that $\exp(-\lambda^{1/3}t)J_t$ and $\exp(-\lambda^{1/3}t)K_t$ go to 0 a.s. and in L^2. We will only prove this for J_t, since the argument for K_t is almost identical. $\tilde{J}_t$ is a complex martingale, so $|\tilde{J}_t|$ is a real submartingale. Using the L^2 maximal inequality, (4.3) in Chapter 4 of [17], and Lemma 4.2.7,

$$E\left(\max_{0\le s\le t}|\tilde{J}_s|^2\right) \le 4E|\tilde{J}_t|^2 \le \frac{16}{3}\exp(2\lambda^{1/3}t). \tag{4.2.9}$$

The real part of ω is $-1/2$. So writing $\tilde{J}_s = \exp(\lambda^{1/3}(1-\omega)s)\cdot\exp(-\lambda^{1/3}s)J_s$, we see that

$$E\left(\max_{u\le s\le t}|\tilde{J}_s|^2\right) \ge \exp(3\lambda^{1/3}u)E\left(\max_{u\le s\le t}\left|\exp(-\lambda^{1/3}s)J_s\right|^2\right). \tag{4.2.10}$$

Combining these bounds with Chebyshev inequality, and taking $t_n = 2\lambda^{-1/3}\log n$ for $n = 1, 2, \ldots$

$$P\left(\max_{t_n \le s \le t_{n+1}} \left|\exp\left(-\lambda^{1/3}s\right) J_s\right|^2 \ge \epsilon\right) \le \epsilon^{-2} E\left(\max_{t_n \le s \le t_{n+1}} \left|\exp(-\lambda^{1/3}s) J_s\right|^2\right)$$
$$\le \frac{16}{3}\epsilon^{-2}\exp\left(\lambda^{1/3}(2t_{n+1} - 3t_n)\right) = \frac{16}{3}\epsilon^{-2}\frac{(n+1)^4}{n^6} \tag{4.2.11}$$

for any $\epsilon > 0$. Summing over n, and using the Borel-Cantelli lemma

$$|\exp(-\lambda^{1/3}s)J_s| \to 0 \quad \text{a.s.}$$

To get convergence in L^2 we use (4.2.7).

$$E\left|\exp\left(-\lambda^{1/3}t\right) J_t\right|^2 \le \frac{4}{3}\exp\left(-\lambda^{1/3}t\right) \to 0 \text{ as } t \to \infty.$$

To prove that $P(M > 0) = 1$ we begin by noting that convergence in L^2 implies that $P(M > 0) > 0$. Every time a new balloon is born it has positive probability of starting a process with a positive limit, so this will happen eventually and $P(M > 0) = 1$. □

4.3 Hitting times for $\mathcal{A}_t$ and $\mathcal{C}_t$

Recall that $\sigma(\epsilon) = \inf\{t : A_t \ge \epsilon N^2\}$ and $\tau(\epsilon) = \inf\{t : C_t \ge \epsilon N^2\}$. Also recall the definitions of $a(\cdot), l(\cdot), x(\cdot)$ and $S(\cdot)$ from (4.1.7) and (4.1.8). Note that $a(S(\epsilon)) = \epsilon N^2$ and $A_t/a(t), L_t/l(t), X_t/x(t) \to M$ a.s. by Theorem 5.1.2. We begin by estimating the difference between M and each of $A_t/a(t), L_t/l(t)$ and $X_t/x(t)$.

Lemma 4.3.1. *For any* $\gamma, u > 0$

$$P\left(\sup_{t \ge u} |A_t/a(t) - M| \ge \gamma^2\right) \le C\gamma^{-4}\exp\left(-\lambda^{1/3}u\right)$$

for some constant C. The same bound holds for $P\left(\sup_{t\geq u}|L_t/l(t)-M|\geq\gamma^2\right)$ and $P\left(\sup_{t\geq u}|X_t/x(t)-M|\geq\gamma^2\right)$.

Proof. Using (4.2.8) $A_t/a(t) = M_t + \omega\exp\left(-\lambda^{1/3}t\right)J_t + \omega^2\exp\left(-\lambda^{1/3}t\right)K_t$. For $0<u\leq t$ the triangle inequality implies

$$|A_t/a(t)-M| \leq |M_t-M| + \left|\exp\left(-\lambda^{1/3}t\right)J_t\right| + \left|\exp\left(-\lambda^{1/3}t\right)K_t\right|. \tag{4.3.1}$$

Taking the supremum over t,

$$\begin{aligned} P\left(\sup_{t\geq u}|A_t/a(t)-M|\geq\gamma^2\right) &\leq P\left(\sup_{t\geq u}|M_t-M|\geq\gamma^2/3\right) \\ +P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right|\geq\gamma^2/3\right) &+ P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)K_t\right|\geq\gamma^2/3\right). \end{aligned} \tag{4.3.2}$$

To bound the first term in the right hand side of (4.3.2) we note that

$$E\left(\sup_{t\geq u}|M_t-M|^2\right) = \lim_{U\to\infty}E\left(\max_{u\leq t\leq U}|M_t-M|^2\right).$$

Using triangle inequality $|M_t-M|\leq|M_t-M_u|+|M_u-M|$. Taking supremum over $t\in[u,U]$ and using the inequality $(a+b)^2\leq 2(a^2+b^2)$,

$$E\left(\max_{u\leq t\leq U}|M_t-M|^2\right) \leq 2\left(E\left(\max_{u\leq t\leq U}|M_t-M_u|^2\right)+E|M_u-M|^2\right).$$

Using the L^2 maximal inequality, (4.3) in Chapter 4 of [17], and orthogonality of martingale increments,

$$E\left(\max_{u\leq t\leq U}|M_t-M_u|^2\right) \leq 4E(M_U-M_u)^2 = 4(EM_U^2-EM_u^2).$$

Since the martingale M_t converges to M in L^2, $EM^2=\lim_{t\to\infty}EM_t^2=8/7$. Then using orthogonality of martingale increments and Lemma 4.2.5,

$$E(M_u-M)^2 = EM^2-EM_u^2 \leq \exp\left(-\lambda^{1/3}u\right).$$

Combining the last four bounds with Lemma 4.2.5, and using Chebyshev inequality

$$P\left(\sup_{t\geq u}|M_t - M| \geq \gamma^2/3\right) \leq 9\gamma^{-4}\cdot 10\exp\left(-\lambda^{1/3}u\right). \tag{4.3.3}$$

To bound the second term in the right hand side of (4.3.2) we take $t_n = u + 2\lambda^{-1/3}\log n$ for $n = 1, 2, \ldots$ and use an argument similar to the one leading to (4.2.11) together with Chebyshev inequality to get

$$\begin{aligned}
P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right| \geq \gamma^2/3\right) &\leq \sum_{n=1}^{\infty} P\left(\max_{t_n\leq t\leq t_{n+1}}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right| \geq \gamma^2/3\right)\\
&\leq 9\gamma^{-4}\sum_{n=1}^{\infty} E\left(\max_{t_n\leq t\leq t_{n+1}}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right|\right)^2\\
&\leq 9\cdot\frac{16}{3}\gamma^{-4}\sum_{n=1}^{\infty}\exp(\lambda^{1/3}(2t_{n+1}-3t_n))\\
&= 48\gamma^{-4}\exp(-\lambda^{1/3}u)\sum_{n=1}^{\infty}\frac{(n+1)^4}{n^6}.
\end{aligned} \tag{4.3.4}$$

Repeating the previous argument for the third term in the right hand side of (4.3.2) we get the same upper bound as in (4.3.4). Combining (4.3.2), (4.3.3) and (4.3.4) we get the desired bound for $A_t/a(t)$.

The bound in (4.3.1) also works for both $L_t/l(t)$ and $X_t/x(t)$, since using (4.2.8)

$$L_t/l(t) = M_t + \omega^2\exp(-\lambda^{1/3}t)J_t + \omega\exp(-\lambda^{1/3}t)K_t,$$

$$X_t/x(t) = M_t + \exp(-\lambda^{1/3}t)J_t + \exp(-\lambda^{1/3}t)K_t,$$

and so the assertion of this lemma holds if $A_t/a(t)$ is replabed by $L_t/l(t)$ or $X_t/x(t)$. □

We now use Lemma 4.3.1 to study the limiting behavior of $\sigma(\epsilon)$.

Lemma 4.3.2. *Let $W_\epsilon = S(\epsilon/M)$, where $S(\cdot)$ is as in* (4.1.8) *and M is the limit random variable in Theorem 5.1.2. Then for any $\eta > 0$*

$$\lim_{N\to\infty} P(|A_{W_\epsilon} - \epsilon N^2| > \eta N^2) = \lim_{N\to\infty} P(|L_{W_\epsilon} - \epsilon N^{2-\alpha/3}| > \eta N^{2-\alpha/3})$$
$$= \lim_{N\to\infty} P(|X_{W_\epsilon} - \epsilon N^{2-2\alpha/3}| > \eta N^{2-2\alpha/3}) = 0.$$

Proof. Since $P(M > 0) = 1$, given $\theta > 0$, we can choose $\gamma = \gamma(\theta) > 0$ so that $\gamma < \eta/\epsilon$ and

$$P(M < \gamma) < \theta. \tag{4.3.5}$$

Using Lemma 4.3.1 we can choose a constant $b = b(\gamma, \theta)$ such that

$$P\left(\sup_{t\geq bN^{\alpha/3}} |A_t/a(t) - M| > \gamma^2\right) < \theta.$$

Combining with (4.3.5)

$$P\left(\sup_{t\geq bN^{\alpha/3}} |A_t/a(t) - M| > \gamma M\right) < 2\theta.$$

Since $a(W_\epsilon) = \epsilon N^2/M$, by the choices of γ and b,

$$\begin{aligned} P(|A_{W_\epsilon} - \epsilon N^2| \geq \eta N^2) &\leq P(|A_{W_\epsilon} - \epsilon N^2| \geq \epsilon\gamma N^2) \\ &= P(|A_{W_\epsilon}/a(W_\epsilon) - M| \geq \gamma M) < 2\theta + P\left(W_\epsilon < bN^{\alpha/3}\right). \end{aligned}$$

By the definition of $S(\cdot)$,

$$P\left(W_\epsilon < bN^{\alpha/3}\right) = P\left(M > \frac{3\epsilon}{b} N^{2-2\alpha/3}\right) \to 0$$

as $N \to \infty$, and so $\limsup_{N\to\infty} P\left(|A_{W_\epsilon} - \epsilon N^2| > \eta N^2\right) \leq 2\theta$. Since $\theta > 0$ is arbitrary, we have shown that

$$\lim_{N\to\infty} P\left(|A_{W_\epsilon} - \epsilon N^2| \geq \eta N^2\right) = 0.$$

Repeating the argument for L_{W_ϵ} and X_{W_ϵ}, and noting that $l(W_\epsilon) = \epsilon N^{2-\alpha/3}/M$ and $x(W_\epsilon) = \epsilon N^{2-2\alpha/3}/M$, we get the other two assertions. $\square$

As a corollary of Lemma 4.3.2 we get the first conclusion of Theorem 4.1.2.

Corollary 4.3.3. *As* $N \to \infty$, $N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) \xrightarrow{P} -\log(M)$.

Proof. For any $\eta > 0$ choose $\gamma > 0$ so that $\log(1+\gamma) < \eta$ and $\log(1-\gamma) > -\eta$. Let W_ϵ be as in Lemma 4.3.2. Clearly $W_{(1+\gamma)\epsilon} = S(\epsilon) + N^{\alpha/3}[\log(1+\gamma) - \log M]$ and $W_{(1-\gamma)\epsilon} = S(\epsilon) + N^{\alpha/3}[\log(1-\gamma) - \log M]$. Using Lemma 4.3.2

$$\begin{aligned} &P\left[N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) > -\log M + \eta\right] \\ &\qquad \le P\left(\sigma(\epsilon) > W_{(1+\gamma)\epsilon}\right) = P\left(A_{W_{(1+\gamma)\epsilon}} < \epsilon N^2\right) \to 0, \\ &P\left[N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) < -\log M - \eta\right] \\ &\qquad \le P\left(\sigma(\epsilon) < W_{(1-\gamma)\epsilon}\right) = P\left(A_{W_{(1-\gamma)\epsilon}} > \epsilon N^2\right) \to 0 \end{aligned}$$

as $N \to \infty$, and the proof is complete. □

The second conclusion in Theorem 2 follows from $C_t \le A_t$. To get the third we have to wait till Lemma 4.3.6. First we need to show that when A_t/N^2 is small, C_t/N^2 is not very much smaller. To prepare for that we need the following result.

Lemma 4.3.4. *Let* $F(t) = \lambda t^3/3!$. *If* $u(\cdot)$ *and* $\beta(\cdot)$ *are functions such that* $u(t) \le \beta(t) + \int_0^t u(t-s)dF(s)$ *for all* $t \ge 0$, *then*

$$u(t) \le \beta * V(t) = \beta(t) + \int_0^t \beta(t-s)dV(s),$$

where $V(\cdot)$ *is as in Lemma 4.2.2.*

Proof. Define $\tilde{\beta}(t) \equiv \beta(t) + \int_0^t u(t-s)dF(s) - u(t)$. So $\tilde{\beta}(t) \ge 0$ for all $t \ge 0$. If $\hat{\beta}(t) \equiv \beta(t) - \tilde{\beta}(t)$, then

$$u(t) = \hat{\beta}(t) + \int_0^t u(t-s)dF(s).$$

Solving the renewal equation we get $u(t) = \hat{\beta} * V(t)$, where $V(\cdot)$ is as in Lemma 4.2.2. Since $\hat{\beta}(t) \leq \beta(t)$ for all $t \geq 0$, we get the result. $\square$

We now apply Lemma 4.3.4 to estimate the difference between EA_t and EC_t.

Lemma 4.3.5. *For any $t \geq 0$ and $a(\cdot)$ as in* (4.1.7)*,*

$$EC_t \geq EA_t - \frac{11a^2(t)}{N^2}.$$

Proof. In either of our processes, if a center is born at time s, then the radius of the corresponding disk at time $t > s$ will be $(t-s)/\sqrt{2\pi}$. Thus x will be covered at time t if and only if there is a center in the space-time cone

$$K_{x,t} \equiv \left\{(y,s) \in \Gamma(N) \times [0,t] : |y-x| \leq (t-s)/\sqrt{2\pi}\right\}. \tag{4.3.6}$$

If $0 = s_0, s_1, s_2, \ldots$ are the birth times of new centers in $\mathcal{C}_t$, then

$$P(x \notin \mathcal{C}_t | s_0, s_1, s_2, \ldots) = \prod_{i: s_i \leq t} \left[1 - \frac{(t-s_i)^2}{2N^2}\right] \leq \exp\left[-\sum_{i: s_i \leq t} \frac{(t-s_i)^2}{2N^2}\right],$$

since $1 - x \leq e^{-x}$. Let $q(t) \equiv P(x \notin \mathcal{C}_t)$, which does not depend on x, since we have a random chosen starting point. Recall that $\tilde{X}_t$ is the number of centers born by time t in $\mathcal{C}_t$. Using the last inequality

$$q(t) \leq E\exp\left[-\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right],$$

and $EC_t = N^2(1-q(t))$. Integrating $e^{-y} \geq 1 - y$ gives $1 - e^{-x} \geq x - x^2/2$ for $x \geq 0$. So

$$\begin{aligned} EC_t &\geq N^2 E\left[1 - \exp\left(-\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right)\right] \qquad (4.3.7) \\ &\geq N^2 E\left[\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s - \frac{1}{2}\left(\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right)^2\right]. \end{aligned}$$

For the first term on the right we use $E\tilde{X}_t = 1 + \lambda \int_0^t EC_s ds$. For the second term on the right, we use the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$ described in the introduction, see (4.1.1), so that we have $\int_0^t (t-s)^2 d\tilde{X}_s \leq \int_0^t (t-s)^2 dX_s$. Combining these two facts

$$\begin{aligned} EC_t &\geq \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{1}{2N^2} E\left[\int_0^t \frac{(t-s)^2}{2} dX_s\right]^2 \\ &= \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{EA_t^2}{2N^2}. \end{aligned} \tag{4.3.8}$$

The last equality follows from (4.1.2), as does the next equation for EA_t:

$$EA_t = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} V'(s)\, ds = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EA_s ds. \tag{4.3.9}$$

Here $V(\cdot)$ is as in Lemma 4.2.2 and $EA_t = V'(t)/\lambda$ by Lemma 4.2.4. Combining (4.3.8) and (4.3.9), if $u(t) \equiv EA_t - EC_t$, and $F(s) = \lambda s^3/3!$, then

$$u(t) \leq \frac{EA_t^2}{2N^2} + \int_0^t \frac{(t-s)^2}{2} \lambda u(s)\, ds = \frac{EA_t^2}{2N^2} + \int_0^t u(t-r)\, dF(r),$$

where the last step is obtained by changing variables $s \mapsto t - r$. If $\beta(t) = EA_t^2/2N^2$, then by Lemma 4.2.6 $\beta(t) \leq 27a^2(t)/4N^2$, and using Lemma 4.3.4 and (4.2.6)

$$u(t) \leq \beta * V(t) \leq \frac{27}{4N^2}(a^2) * V(t) \leq \frac{27}{4N^2}\frac{3}{2}a^2(t),$$

which gives the result, since $81/8 \leq 11$. □

To complete the proof of Theorem 4.1.2 it remains to show the third conclusion of it, which we separate as the following lemma and prove it using Lemma 4.3.5.

Lemma 4.3.6. *For any $\gamma > 0$*

$$\limsup_{N\to\infty} P(\tau(\epsilon) > \sigma((1+\gamma)\epsilon)) \leq P\left(M \leq (1+\gamma)\epsilon^{1/3}\right) + 11\frac{\epsilon^{1/3}}{\gamma}.$$

Proof. Let $U = \sigma((1+\gamma)\epsilon)$ and $T = S(\epsilon^{2/3})$, where $S(\cdot)$ is as in (4.1.8). Now

$$S(\epsilon^{2/3}) - S((1+\gamma)\epsilon) = N^{\alpha/3}\left[-\frac{1}{3}\log(\epsilon) - \log(1+\gamma)\right].$$

It follows from Corollary 4.3.3 that $\limsup_{N\to\infty} P(U \geq T)$

$$\leq P\left(-\log(M) \geq -\frac{1}{3}\log(\epsilon) - \log(1+\gamma)\right) = P\left(M \leq (1+\gamma)\epsilon^{1/3}\right).$$

Using Markov's inequality, Lemma 4.3.5, and $a(T) = \epsilon^{2/3}N^2$,

$$P\left(|A_T - C_T| > \gamma\epsilon N^2\right) \leq \frac{E(A_T - C_T)}{\gamma\epsilon N^2} \leq \frac{6(a(T))^2}{\gamma\epsilon N^4} \leq 11 \cdot \frac{\epsilon^{1/3}}{\gamma}. \tag{4.3.10}$$

Using these two bounds and the fact that $|A_t - C_t|$ is nondecreasing in t, we get

$$\begin{aligned}
&\limsup_{N\to\infty} P[\tau(\epsilon) > \sigma((1+\gamma)\epsilon)] = \limsup_{N\to\infty} P\left[|A_U - C_U| > \gamma\epsilon N^2\right] \\
&\leq \limsup_{N\to\infty} P(U \geq T) + \limsup_{N\to\infty} P\left[|A_U - C_U| > \gamma\epsilon N^2, U < T\right] \\
&\leq \limsup_{N\to\infty} P(U \geq T) + P\left(|A_T - C_T| > \gamma\epsilon N^2\right),
\end{aligned}$$

which completes the proof. □

4.4 Limiting behavior of $\mathcal{C}_t$

Let $\mathcal{C}^0_{s,t}$ be the set of points covered in $\mathcal{C}_t$ at time t by the balloons born before time s. If we number the generations of centers in $\mathcal{C}_t$ starting with those existing at time s as $\mathcal{C}_t$-centers of generation 0, then $\mathcal{C}^0_{s,t}$ is the set of points covered at time t by the generation 0 centers of $\mathcal{C}_t$. Let $\mathcal{C}^1_{s,t}$ be the set of points, which are either in $\mathcal{C}^0_{s,t}$, or are covered at time t by a balloon born from this area. This is the set of points covered by $\mathcal{C}_t$-centers of generations ≤ 1 at time t , ignoring births from $\mathcal{C}^1_{s,t} \setminus \mathcal{C}^0_{s,t}$, which are second generation centers. Continuing by induction, we let $\mathcal{C}_{s,t}^k$ be the set of points and $C_{s,t}^k = \left|\mathcal{C}_{s,t}^k\right|$ be the total area covered by $\mathcal{C}_t$-centers of generations $0 \le j \le k$ at time t. Similarly $A_{s,t}^k$ denotes the total area of the balloons in $\mathcal{A}_t$ of generations $j \in \{0, 1, \dots, k\}$ at time t, where generation 0 centers are those existing at time s.

Recall the following definitions from (4.1.7), (4.1.8), (4.1.11) and (4.1.12).

$$a(t) = (1/3)N^{2\alpha/3} \exp\left(N^{-\alpha/3}t\right),$$
$$S(\epsilon) = N^{\alpha/3}[(2 - 2\alpha/3)\log N + \log(3\epsilon)],$$
$$R = N^{\alpha/3}[(2 - 2\alpha/3)\log N - \log(M)],$$

where M is the limit random variable in Theorem 5.1.2, and for $\log(3\epsilon) \le t$,

$$\psi(t) \equiv R + N^{\alpha/3}t, \quad W \equiv \psi(\log(3\epsilon)), \text{ and } I_{\epsilon,t} = [\log(3\epsilon), t].$$

Note that $\psi(t) \le 0$ only if $M \ge N^{2-2\alpha/3}t$.

Obviously $C_{s,t}^0 \le A_{s,t}^0$. For the other direction we have the following lemma.

Lemma 4.4.1. *For any* $0 < s < t$,

$$EC_{s,t}^0 \ge EA_{s,t}^0 - \frac{a^2(s)}{N^2} p\left((t-s)\lambda^{1/3}\right),$$

where for some positive constants c_1, c_2 *and* c_4,

$$p(x) = c_1 + c_2 x^2/2! + c_4 x^4/4!. \tag{4.4.1}$$

Proof. By the definition of $A_{s,t}^0$,

$$A_{s,t}^0 = \int_0^s \frac{(t-r)^2}{2}\, dX_r = \frac{(t-s)^2}{2} X_s + (t-s)L_s + A_s. \tag{4.4.2}$$

For the second equality we have written $(t-r)^2 = (t-s)^2 + 2(t-s)(s-r) + (s-r)^2$ and used (4.1.2). As in Lemma 4.3.5, a point x is not covered by time t by the balloons born before time s, if and only if no center is born in the truncated space-time cone

$$K_{x,s,t} \equiv \left\{(y,r) \in \Gamma(N) \times [0,s] : |y-x| \le (t-r)/\sqrt{2\pi}\right\}.$$

So using arguments similar to the ones for (4.3.7) and $1-e^{-x} \ge x - x^2/2$,

$$\begin{aligned} EC^0_{s,t} &\ge N^2 E\left[1 - \exp\left(-\int_0^s \frac{(t-r)^2}{2N^2}\, d\tilde{X}_r\right)\right] \\ &\ge N^2\left[E\int_0^s \frac{(t-r)^2}{2N^2}\, d\tilde{X}_r - \frac{1}{2}E\left(\int_0^s \frac{(t-r)^2}{2N^2}\, d\tilde{X}_r\right)^2\right]. \end{aligned}$$

For the first term on the right, we use $E\tilde{X}_t = 1 + \lambda\int_0^t EC_s ds$. For the second term on the right, we use the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$ described in the introduction, see (4.1.1), to conclude that

$$\int_0^s (t-r)^2\, d\tilde{X}_r \le \int_0^s (t-r)^2\, dX_r = 2A^0_{s,t}.$$

Combining these two facts, using the first equality in (4.4.2), $EX_t = 1 + \lambda\int_0^t EA_s\, ds$, and Lemma 4.3.5,

$$\begin{aligned} EC^0_{s,t} &\ge \frac{t^2}{2} + \int_0^s \frac{(t-r)^2}{2}\lambda EC_r\, dr - \frac{E(A^0_{s,t})^2}{2N^2} \\ &\ge \frac{t^2}{2} + \int_0^s \frac{(t-r)^2}{2}\lambda EA_r\, dr - 11\int_0^s \frac{(t-r)^2}{2}\frac{\lambda a^2(r)}{N^2}\, dr - \frac{E(A^0_{s,t})^2}{2N^2} \\ &= EA^0_{s,t} - 11\int_0^s \frac{(t-r)^2}{2}\frac{\lambda a^2(r)}{N^2} dr - \frac{E(A^0_{s,t})^2}{2N^2}. \end{aligned} \tag{4.4.3}$$

To estimate the second term in the right side of (5.8.1), we write

$$(t-r)^2/2 = (t-s)^2/2 + (t-s)(s-r) + (s-r)^2/2,$$

change variables $r = s-q$, and note $a(s-q) = a(s)\exp\left(-\lambda^{1/3}q\right)$, to get

$$\int_0^s \frac{(t-r)^2}{2}\lambda a^2(r)\, dr = a^2(s)\left[\frac{(t-s)^2}{2}\lambda^{2/3}\int_0^s \lambda^{1/3}\exp\left(-2\lambda^{1/3}q\right)\, dq\right.$$

$$+ (t-s)\lambda^{1/3}\int_0^s \lambda^{2/3} q \exp\left(-2\lambda^{1/3}q\right)\, dq + \int_0^s \lambda \frac{q^2}{2}\exp\left(-2\lambda^{1/3}q\right)\, dq\Bigg]$$

$$\le \frac{a^2(s)}{2}\left[\frac{(t-s)^2}{2}\lambda^{2/3} + (t-s)\lambda^{1/3} + 1\right]. \tag{4.4.4}$$

For the last inequality we have used

$$\int_0^s r^k \exp(-\mu r)\, dr \le \int_0^\infty r^k \exp(-\mu r)\, dr = \frac{k!}{\mu^{k+1}}.$$

To estimate the third term in the right side of (5.8.1) we use (4.4.2) to get

$$E\left[(A^0_{s,t})^2\right] \le 3[EX_s^2(t-s)^4/4 + EL_s^2(t-s)^2 + EA_s^2].$$

Applying Lemma 4.2.6 and using the fact that $a(s) = \lambda^{-1/3} l(s) = \lambda^{-2/3} x(s)$,

$$E\left[(A^0_{s,t})^2\right] \le 3\cdot\frac{27}{2}\left[x^2(s)\frac{(t-s)^4}{4} + l^2(s)(t-s)^2 + a^2(s)\right]$$

$$\le 243 a^2(s)\left[\frac{(t-s)^4}{4!}\lambda^{4/3} + \frac{(t-s)^2}{2!}\lambda^{2/3} + 1\right]. \tag{4.4.5}$$

Combining (5.8.1), (4.4.4) and (4.4.5) we get the result. □

To show uniform convergence of $C^k_{W,\psi(\cdot)}$ to $C_{\psi(\cdot)}$, we also need to bound the difference A_t and $A^k_{s,t}$ for suitable choices of s and t.

Lemma 4.4.2. *If* $T = S(\epsilon^{2/3})$*, where* $S(\cdot)$ *is as in* (4.1.8)*, then for any* $t > 0$

$$EA_{T+tN^{\alpha/3}} - EA^k_{T,T+tN^{\alpha/3}} \le 3\epsilon^{2/3} N^2 \sum_{j=k+1}^{\infty} \frac{t^j}{j!}.$$

Proof. By (4.4.2) $EA^0_{s,t} = EA_s + EL_s(t-s) + EX_s(t-s)^2/2$. If $X^k_{s,t}$ and $L^k_{s,t}$ denote the number of centers and sum of radii of all the balloons in $\mathcal{A}_t$ of generations $j \in \{1, 2, \ldots, k\}$ at time t, where generation 0 centers are those which are born before time s, then for $t > s$,

$$\frac{d}{dt}EX^1_{s,t} = N^{-\alpha} EA^0_{s,t},\ \frac{d}{dt}EL^1_{s,t} = EX^1_{s,t},\ \frac{d}{dt}EA^1_{s,t} = EL^1_{s,t}.$$

Integrating over $[s,t]$ and using (4.4.2) we have

$$\begin{aligned}
EX^1_{s,t} &= N^{-\alpha}\left[(t-s)EA_s + \frac{(t-s)^2}{2!}EL_s + \frac{(t-s)^3}{3!}EX_s\right],\\
EL^1_{s,t} &= N^{-\alpha}\left[\frac{(t-s)^2}{2!}EA_s + \frac{(t-s)^3}{3!}EL_s + \frac{(t-s)^4}{4!}EX_s\right],\\
EA^1_{s,t} &= N^{-\alpha}\left[\frac{(t-s)^3}{3!}EA_s + \frac{(t-s)^4}{4!}EL_s + \frac{(t-s)^5}{5!}EX_s\right].
\end{aligned}$$

Turning to other generations, for $k \geq 2$ and $t > s$,

$$\begin{aligned}
\frac{d}{dt}\left(EX^k_{s,t} - EX^{k-1}_{s,t}\right) &= N^{-\alpha}\left(EA^{k-1}_{s,t} - EA^{k-2}_{s,t}\right),\\
\frac{d}{dt}\left(EL^k_{s,t} - EL^{k-1}_{s,t}\right) &= \left(EX^k_{s,t} - EX^{k-1}_{s,t}\right),\\
\frac{d}{dt}\left(EA^k_{s,t} - EA^{k-1}_{s,t}\right) &= \left(EL^k_{s,t} - EL^{k-1}_{s,t}\right),
\end{aligned}$$

and using induction on k we have

$$EA^k_{s,t} = \sum_{j=0}^{k} N^{-\alpha j}\left[\frac{(t-s)^{3j}}{(3j)!}EA_s + \frac{(t-s)^{3j+1}}{(3j+1)!}EL_s + \frac{(t-s)^{3j+2}}{(3j+2)!}EX_s\right].$$

Since $A^k_{s,t} \uparrow A_t$ for any $s < t$, $EA_t = \lim_{k\to\infty} EA^k_{s,t}$ by Monotone Convergence Theorem. Replacing s by T and t by $T + tN^{\alpha/3}$,

$$\begin{aligned}
&EA_{T+tN^{\alpha/3}} - EA^k_{T,T+tN^{\alpha/3}} \qquad (4.4.6)\\
&= \sum_{j=k+1}^{\infty}\left[\frac{t^{3j}}{(3j)!}EA_T + \frac{t^{3j+1}}{(3j+1)!}N^{\alpha/3}EL_T + \frac{t^{3j+2}}{(3j+2)!}N^{2\alpha/3}EX_T\right].
\end{aligned}$$

Using the fact that $EA_T + N^{\alpha/3}EL_T + N^{2\alpha/3}EX_T - 3a(T) = 0$ and $a(T) = \epsilon^{2/3}N^2$, the right hand side of (4.4.6) is $\leq 3\epsilon^{2/3}N^2\sum_{j=k+1}^{\infty} t^j/j!$, which completes the proof. □

Recall the definitions of $\psi(\cdot), W$ and $I_{\epsilon,t}$ from the displays before Lemma 4.4.1 and that for $\log(3\epsilon) \leq t$,

$$g_0(t) = \epsilon\left[1 + (t - \log(3\epsilon) + \frac{(t-\log(3\epsilon))^2}{2}\right]. \qquad (4.4.7)$$

Lemma 4.4.3. *For any $t < \infty$, there is an $\epsilon_0 = \epsilon_0(t) > 0$ so that for $0 < \epsilon < \epsilon_0$,*

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left|N^{-2}A^0_{W,\psi(s)} - g_0(s)\right| > \eta\right) = 0 \text{ for any } \eta > 0,$$

$$P\left(\inf_{s\in I_{\epsilon,t}} N^{-2}\left(C^0_{W,\psi(s)} - A^0_{W,\psi(s)}\right) < -\epsilon^{7/6}\right) \le P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Proof. To prove the first result we use (4.4.2) to conclude

$$A^0_{W,\psi(t)} = \frac{(t-\log(3\epsilon))^2}{2}N^{2\alpha/3}X_W + (t-\log(3\epsilon))N^{\alpha/3}L_W + A_W.$$

Applying Lemma 4.3.2

$$\begin{aligned}
&\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left|N^{-2}A^0_{W,\psi(s)} - g_0(s)\right| > \eta\right)\\
&\quad\le \lim_{N\to\infty} P\left(|N^{-(2-2\alpha/3)}X_W - \epsilon| > \frac{2\eta}{3(t-\log(3\epsilon))^2}\right)\\
&\quad\quad+ \lim_{N\to\infty} P\left(|N^{-(2-\alpha/3)}L_W - \epsilon| > \frac{\eta}{3(t-\log(3\epsilon))}\right)\\
&\quad\quad+ \lim_{N\to\infty} P\left(|N^{-2}A_W - \epsilon| > \frac{\eta}{3}\right) = 0.
\end{aligned}$$

Let $\epsilon_0 = \epsilon_0(t)$ be such that $\epsilon_0^{1/12}p(t-\log(3\epsilon)) \le 1$, where $p(\cdot)$ is the polynomial in (4.4.1). Let $T = S(\epsilon^{2/3})$, where $S(\cdot)$ is defined in (4.1.8), and $T' = T + (t - \log(3\epsilon))N^{\alpha/3}$. Using the fact that $A^0_{s,s+t} - C^0_{s,s+t}$ is nondecreasing in s, Markov's inequality, and then Lemma 4.4.1 we see that

$$\begin{aligned}
&P\left(\sup_{s\in I_{\epsilon,t}} \left|A^0_{W,\psi(s)} - C^0_{W,\psi(s)}\right| > \epsilon^{7/6}N^2, W \le T\right)\\
&\quad\le P\left(|A^0_{T,T'} - C^0_{T,T'}| > \epsilon^{7/6}N^2\right) \le \frac{E|A^0_{T,T'} - C^0_{T,T'}|}{\epsilon^{7/6}N^2}\\
&\quad\le \frac{a^2(T)p(t-\log(3\epsilon))}{\epsilon^{7/6}N^4}.
\end{aligned}$$

Noting that $P(W > T) = P(M < \epsilon^{1/3})$, $a(T) = \epsilon^{2/3}N^2$, and $\epsilon^{1/12}p(t-\log(3\epsilon)) < 1$ for $\epsilon < \epsilon_0$ we have

$$P\left(\sup_{s\in I_{\epsilon,t}} \left|A_{W,\psi(s)} - C_{W,\psi(s)}\right| > \epsilon^{7/6}N^2\right) \le P\left(M < \epsilon^{1/3}\right) + \epsilon^{1/12},$$

which completes the proof. $\square$

Our next step is to improve the lower bound in Lemma 4.4.3. Let

$$\rho_t^0 = N^{-2} A_{W,\psi(t)} - \epsilon^{7/6}.$$

On the event

$$F = \left\{ \left| N^{-2} \mathcal{C}^0_{W,\psi(s)} \right| \geq \rho_s^0 \text{ for all } s \in I_{\epsilon,t} \right\}, \tag{4.4.8}$$

which has probability tending to 1 as $\epsilon \to 0$ by Lemma 4.4.3, $\mathcal{C}^0_{W,\psi(s)}$ can be coupled with a process $\mathcal{B}^0_{\psi(s)}$ so that $N^{-2}|\mathcal{B}^0_{\psi(s)}| = \rho_s^0$ and $\mathcal{C}^0_{W,\psi(s)} \supseteq \mathcal{B}^0_{\psi(s)}$ for $s \in I_{\epsilon,t}$. If for $k \geq 1$ $\mathcal{B}^k_{\psi(t)}$ is obtained from $\mathcal{B}^0_{\psi(t)}$ in the same way as $\mathcal{C}^k_{W,\psi(t)}$ is obtained from $\mathcal{C}^0_{W,\psi(t)}$, then, on F, $\mathcal{C}^k_{W,\psi(s)} \supseteq \mathcal{B}^k_{\psi(s)}$ for $s \in I_{\epsilon,t}$. For $k \geq 1$ let

$$\rho_s^k = N^{-2} |\mathcal{B}^k_{\psi(s)}|.$$

We begin with the case $k = 1$. For $f_0(t) = g_0(t) - \epsilon^{7/6}$, where g_0 is as in (4.4.7), let

$$f_1(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2} f_0(s)\, ds \right). \tag{4.4.9}$$

Lemma 4.4.4. *For any $t < \infty$ there is an $\epsilon_0 = \epsilon_0(t) > 0$ so that for $0 < \epsilon < \epsilon_0$ and any $\delta > 0$,*

$$\limsup_{N \to \infty} P\left[\inf_{s \in I_{\epsilon,t}} (N^{-2} C^1_{W,\psi(s)} - f_1(s)) < -\delta \right] \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Proof. As in Lemma 4.3.5, if $x \notin \mathcal{B}^0_{\psi(t)}$, then $x \notin \mathcal{B}^1_{\psi(t)}$ if and only if no generation 1 center is born in the space-time cone

$$K^\epsilon_{x,t} \equiv \left\{ (y,s) \in \Gamma(N) \times [W, \psi(t)] : |y - x| \leq (\psi(t) - s)/\sqrt{2\pi} \right\}.$$

Conditioning on $\mathcal{G}^0_t = \sigma\{\mathcal{B}^0_{\psi(s)} : s \in I_{\epsilon,t}\}$, the locations of generation 1 centers in $\mathcal{B}^1_t$ is a Poisson point process on $\Gamma(N) \times [W, \psi(t)]$ with intensity

$$N^{-2} \times |\mathcal{B}^0_s| N^{-\alpha} = \rho^0_{\psi^{-1}(s)} N^{-\alpha},$$

Using this and then changing variables $s = \psi(r)$, where $\psi(r) = R + N^{\alpha/3} r$,

$$P\left(x \notin \mathcal{B}^1_{\psi(t)} \middle| \mathcal{G}^0_t\right) = (1-\rho^0_t) \exp\left(-\int_W^{\psi(t)} \frac{(\psi(t)-s)^2}{2} \rho^0_{\psi^{-1}(s)} N^{-\alpha}\, ds\right)$$
$$= (1-\rho^0_t) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-r)^2}{2} \rho^0_r\, dr\right).$$

Let $E_{x,t} = \{x \notin \mathcal{B}^1_t\}$. Since $K^\epsilon_{x,t}$ and $K^\epsilon_{y,t}$ are disjoint if $|x-y| > 2(t-\log(3\epsilon))N^{\alpha/3}/\sqrt{2\pi}$, the events $E_{x,t}$ and $E_{y,t}$ are conditionally independent given $\mathcal{G}^0_t$ if this holds. Define the random variables Y_x, $x \in \Gamma(N)$, so that $Y_x = 1$ if $E_{x,t}$ occurs, and $Y_x = 0$ otherwise. From (4.4.10)

$$E\left(Y_x \middle| \mathcal{G}^0_t\right) = (1-\rho^0_t) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \rho^0_s\, ds\right). \tag{4.4.10}$$

Using independence of Y_x and Y_z for $|x-z| > 2(t-\log(3\epsilon))N^{\alpha/3}/\sqrt{2\pi}$, and the fact that $\left\{z : |x-z| \le 2(t-\log(3\epsilon))N^{\alpha/3}/\sqrt{2\pi}\right\}$ has area $2(t-\log(3\epsilon))^2 N^{2\alpha/3}$,

$$\begin{aligned}
&\text{var}\left(\int_{x\in\Gamma(N)} Y_x\, dx \middle| \mathcal{G}^0_t\right) \\
&= \int_{x,z\in\Gamma(N)} \left[E\left(Y_x Y_z \middle| \mathcal{G}^0_t\right) - E\left(Y_x \middle| \mathcal{G}^0_t\right) E\left(Y_z \middle| \mathcal{G}^0_t\right)\right] dx\, dz \\
&\qquad \le N^2 \cdot 2(t-\log(3\epsilon))^2 N^{2\alpha/3}.
\end{aligned} \tag{4.4.11}$$

Using Chebyshev's inequality, we see that

$$P\left(\left|\int_{x\in\Gamma(N)} \left(Y_x - E\left(Y_x \middle| \mathcal{G}^0_t\right)\right) dx\right| > \frac{\eta}{2} N^2 \middle| \mathcal{G}^0_t\right) \le \frac{4\text{var}\left(\int_{x\in\Gamma(N)} Y_x\, dx \middle| \mathcal{G}^0_t\right)}{\eta^2 N^4}. \tag{4.4.12}$$

Combining (4.4.10), (4.4.11), and (4.4.12) gives

$$P\left(\left|(1-\rho^1_t) - (1-\rho^0_t) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \rho^0_s\, ds\right)\right| > \frac{\eta}{2} \middle| \mathcal{G}^0_t\right) \le \frac{8(t-\log(3\epsilon))^2}{\eta^2 N^{2-2\alpha/3}}.$$

The same bound holds for the unconditional probability. By Lemma 4.4.3 if $\eta > 0$ and

$$F_{0,\eta} \equiv \left\{\sup_{s\in I_{\epsilon,t}} |\rho^0_s - f_0(s)| \le \eta\right\}, \quad \text{then} \quad \lim_{N\to\infty} P(F^c_{0,\eta}) = 0.$$

Let $\eta' = \eta\left[1 + (t - \log(3\epsilon))^3/3!\right]^{-1}/2$. Using (4.4.9) and the fact that for $x, y \geq 0$

$$|e^{-x} - e^{-y}| = \left|\int_x^y e^{-z}\,dz\right| \leq |x - y|, \tag{4.4.13}$$

we see that on the event $F_{0,\eta'}$, we have for any $s \in I_{\epsilon,t}$

$$\begin{aligned}
&\left|\left(1 - \rho_s^0\right)\exp\left(-\int_{\log(3\epsilon)}^s \frac{(s-r)^2}{2}\rho_r^0\,dr\right) - (1 - f_1(s))\right| \\
&\leq \left|\left(1 - \rho_s^0\right) - (1 - f_0(s))\right| + \eta'\int_{\log(3\epsilon)}^s \frac{(s-r)^2}{2}\,dr \leq \eta' + \eta'\frac{(s - \log(3\epsilon))^3}{3!} \leq \frac{\eta}{2}.
\end{aligned}$$

So for any $s \in I_{\epsilon,t}$

$$\begin{aligned}
&\lim_{N\to\infty} P\left(\left|\rho_s^1 - f_1(s)\right| > \eta\right) \leq \lim_{N\to\infty} P\left(F_{0,\eta'}^c\right) \\
&\quad + \lim_{N\to\infty} P\left(\left|\left(1 - \rho_s^1\right) - \left(1 - \rho_s^0\right)\exp\left(-\int_{\log(3\epsilon)}^s \frac{(s-r)^2}{2}\rho_r^0\,dr\right)\right| > \frac{\eta}{2}\right) = 0.
\end{aligned}$$

Since $\eta > 0$ is arbitrary, the two quantities being compared are increasing and continuous, and on the event F defined in (4.4.8) $N^{-2}C_{W,\psi(s)}^1 \geq \rho_s^1$ for $s \in I_{\epsilon,t}$,

$$\begin{aligned}
&\limsup_{N\to\infty} P\left[\inf_{s\in I_{\epsilon,t}}(N^{-2}C_{W,\psi(s)}^1 - f_1(s)) < -\delta\right] \\
\leq\ & P(F^c) + \limsup_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}}|\rho_s^1 - f_1(s)| > \delta\right) \leq P(F^c),
\end{aligned}$$

and the desired conclusion follows from Lemma 4.4.3. $\square$

To improve this we will let

$$f_{k+1}(t) = 1 - (1 - f_k(t))\exp\left(-\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2}(f_k(s) - f_{k-1}(s))\,ds\right), \tag{4.4.14}$$

and recall from (4.1.15) that as $k \uparrow \infty$, $f_k(t) \uparrow f_\epsilon(t)$.

Lemma 4.4.5. *For any $t < \infty$ there is an $\epsilon_0 = \epsilon_0(t) > 0$ so that for $0 < \epsilon < \epsilon_0$ and any $\delta > 0$,*

$$\limsup_{N\to\infty} P\left[\inf_{s\in I_{\epsilon,t}}(N^{-2}C_{\psi(s)} - f_\epsilon(s)) < -\delta\right] \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Proof. Conditioning on $\mathcal{G}_t^k = \sigma\left\{\mathcal{B}_{\psi(s)}^j : 0 \le j \le k, s \in I_{\epsilon,t}\right\}$, we have

$$P\left(x \notin \mathcal{B}_{\psi(t)}^{k+1} \middle| \mathcal{G}_t^k\right) = \left(1-\rho_t^k\right)\exp\left(-\int_0^t \frac{(t-s)^2}{2}\left(\rho_s^k - \rho_s^{k-1}\right)\,ds\right).$$

Let $F_{k,\eta} = \{\sup_{s\in I_{\epsilon,t}} |\rho_s^k - f_k(s)| \le \eta\}$, and $\eta' = \eta\left[1+2(t-\log(3\epsilon))^3/3!\right]^{-1}/2$. Using (4.4.14) and $|e^{-x}-e^{-y}| \le |x-y|$ for $x, y \ge 0$, we see that on the event $G_{k,\eta'} = F_{k,\eta'} \cap F_{k-1,\eta'}$, for any $s \in I_{\epsilon,t}$

$$\begin{aligned}
&\left|\left(1-\rho_t^k\right)\exp\left(-\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2}\left(\rho_s^k-\rho_s^{k-1}\right)\,ds\right) - (1-f_{k+1}(t))\right| \\
&\quad \le \left|\left(1-\rho_t^k\right) - (1-f_k(t))\right| + 2\eta'\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2}\,ds \\
&\quad = \eta' + 2\eta'(t-\log(3\epsilon))^3/3 \le \eta/2.
\end{aligned}$$

Bounding the variance as before we can conclude by induction on k that for any $\eta > 0$

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left|\rho_s^k - f_k(s)\right| > \eta\right) = 0. \tag{4.4.15}$$

Next we bound the difference between $f_k(t)$ and $f_\epsilon(t)$. Let $G(t) = t^3/3!$ for $t \ge 0$, and $G(t) = 0$ for $t < 0$. If $*k$ indicates the k-fold convolution, then for $k \ge 1$, using arguments similar to the ones in the proof of Lemma 4.2.2, $G^{*k}(t) = t^{3k}/(3k)!$ for $t \ge 0$, and $G^{*k}(t) = 0$ for $t < 0$. Now if $f * G^{*k}(t) = \int_0^t f(t-r)\,dG^{*k}(r)$, $\tilde{f}_k(\cdot) = f_k(\cdot + \log(3\epsilon))$ and $\tilde{f}_\epsilon(\cdot) = f_\epsilon(\cdot+\log(3\epsilon))$, then changing variables $s \mapsto t-r$ in (4.1.14) and (4.1.15), and using the inequality in (4.4.13),

$$\begin{aligned}
& |\tilde{f}_k(t-\log(3\epsilon)) - \tilde{f}_\epsilon(t-\log(3\epsilon))| \\
\le\ & \left|\exp(-\tilde{f}_{k-1} * G(t-\log(3\epsilon))) - \exp(-\tilde{f}_\epsilon * G(t-\log(3\epsilon)))\right| \\
\le\ & |\tilde{f}_{k-1} - \tilde{f}_\epsilon| * G(t-\log(3\epsilon)).
\end{aligned}$$

Iterating the above inequality and using $|\tilde{f}_\epsilon(s) - \tilde{f}_0(s)| = \tilde{f}_\epsilon(s) - \tilde{f}_0(s) \le 1$.

$$|f_k(t) - f_\epsilon(t)| = |\tilde{f}_k(t-\log(3\epsilon)) - \tilde{f}_\epsilon(t-\log(3\epsilon))|$$

$$\leq |\tilde{f}_0 - \tilde{f}_\epsilon| * G^{*k}(t - \log(3\epsilon)) \tag{4.4.16}$$
$$\leq G^{*k}(t - \log(3\epsilon)) = \frac{(t - \log(3\epsilon))^{3k}}{(3k)!}.$$

where the last equality comes from (4.2.1).

Choose $K = K(\epsilon, t)$ so that $(t - \log(3\epsilon))^{3K}/(3K)! < \delta/2$. Since $C_{\psi(t)} \geq C^k_{W,\psi(t)}$ for any $k \geq 0$, and on the event F defined in (4.4.8), we have $C^k_{W,\psi(t)} \geq |\mathcal{B}^k_{\psi(t)}|$, we have

$$P\left(\inf_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - f_\epsilon(s)\right) < -\delta\right) \leq P(F^c) + P\left(\sup_{s\in I_{\epsilon,t}} \left|\rho^K_s - f_K(s)\right| > \delta/2\right).$$

Using (4.4.15) and Lemma 4.4.3 we get the result. □

It is now time to get upper bounds on $C_{\psi(s)}$. Recall $g_0(t)$ defined in (4.4.7), let $g_{-1}(t) = 0$ and for $k \geq 1$ let

$$g_k(t) = 1 - (1 - g_{k-1}(t)) \exp\left(-\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2}(g_{k-1}(s) - g_{k-2}(s))\, ds\right) \tag{4.4.17}$$

As in the case of $f_k(t)$, the equations above imply

$$g_k(t) = 1 - (1 - g_0(t)) \exp\left(-\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2} g_{k-1}(s)\, ds\right),$$

so we have $g_k(t) \uparrow g_\epsilon(t)$ as $k \uparrow \infty$, where $g_\epsilon(t)$ satisfies

$$g_\epsilon(t) = 1 - (1 - g_0(t)) \exp\left(-\int_{\log(3\epsilon)}^t \frac{(t-s)^2}{2} g_\epsilon(s)\, ds\right).$$

Lemma 4.4.6. *For any $t < \infty$ there exists $\epsilon_0 = \epsilon_0(t) > 0$ such that for $0 < \epsilon < \epsilon_0$ and any $\delta > 0$,*

$$\limsup_{N\to\infty} P\left[\sup_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - g_\epsilon(s)\right) > \delta\right] \leq P(M < \epsilon^{1/3}) + \epsilon^{2/3}.$$

Proof. $C^0_{W,\psi(t)} \le A^0_{W,\psi(t)}$. If $\phi^0_t = N^{-2}A^0_{W,\psi(t)}$ is the fraction of area covered by generation 0 balloons at time $\psi(t)$, generation 1 centers are born at rate $N^{2-\alpha}\phi^0_{\psi^{-1}(\cdot)}$. Let ϕ^1_t denotes the fraction of area covered by centers of generations ≤ 1 at time $\psi(t)$, then using an argument similar to the one for Lemma 4.4.4 gives

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \phi^1_s - g_1(s) > \eta\right) = 0$$

for any $\eta > 0$. Continuing by induction, let ϕ^k_t be the fraction of area covered by centers of generations $0 \le j \le k$. Since (4.4.17) and (4.4.14) are the same except for the letter they use, then by an argument identical to the one for Lemma 4.4.5,

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left|\phi^k_s - g_k(s)\right| > \eta\right) = 0 \tag{4.4.18}$$

for any $\eta > 0$. Now using an argument similar to the one for (4.4.16)

$$\sup_{s\in I_{\epsilon,t}} |g_k(s) - g_\epsilon(s)| \le \frac{(t-\log(3\epsilon))^{3k}}{(3k)!}. \tag{4.4.19}$$

Next we bound the difference between $C^k_{W,\psi(t)}$ and $C_{\psi(t)}$. Let $T = S(\epsilon^{2/3})$, where $S(\cdot)$ is as in (4.1.8). Using the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$,

$$C_{\psi(t)} - C^k_{W,\psi(t)} \le A_{\psi(t)} - A^k_{W,\psi(t)}.$$

Using the fact that $EA_{s+t} - EA^k_{s,s+t}$ is nondecreasing in s, the definitions of W and T, Markov's inequality, and Lemma 4.4.2, we have for $T' = T + (t - \log(3\epsilon))N^{\alpha/3}$,

$$\begin{aligned} P\left(\sup_{s\in I_{\epsilon,t}} \left(C_{\psi(s)} - \mathcal{C}^k_{W,\psi(s)}\right) > \frac{\delta N^2}{4}\right) &\le P(W > T) + P\left(A_{T'} - A_{T,T'} > \frac{\delta N^2}{4}\right) \\ &\le P(M < \epsilon^{1/3}) + \frac{4}{\delta N^2} E(A_{T'} - A_{T,T'}) \\ &\le P(M < \epsilon^{1/3}) + \frac{12\epsilon^{2/3}}{\delta} \sum_{j=k+1}^{\infty} \frac{(t-\log(3\epsilon))^j}{j!}. \end{aligned}$$

Choose $K = K(\epsilon, t)$ large enough so that $\sum_{j=K+1}^{\infty}(t - \log(3\epsilon))^j/j! < \delta/12$. If we let

$$F_K = \left\{ \sup_{s \in I_{\epsilon,t}} \left(C_{\psi(s)} - C^K_{W,\psi(s)} \right) \leq (\delta/4)N^2 \right\}, \quad \text{then} \quad P(F_K^c) \leq P(M < \epsilon^{1/3}) + \epsilon^{2/3}.$$

By the choice of K and (4.4.19), $\sup_{s \in I_{\epsilon,t}} |g_K(s) - g_\epsilon(s)| \leq \delta/2$. Combining the last two inequalities and using the fact that $N^{-2}C^K_{W,\psi(s)} \leq \phi^K_s = N^{-2}A^K_{W,\psi(s)}$,

$$P\left(\sup_{s \in I_{\epsilon,t}} N^{-2}C_{\psi(s)} - g_\epsilon(s) > \delta \right) \leq P(F_K^c) + P\left(\sup_{s \in I_{\epsilon,t}} \left|\phi^K_s - g_K(s)\right| > \delta/4 \right).$$

So using (4.4.18) we have the desired result. $\square$

Our next goal is the

Proof of Lemma 5.8.5. We prove the result in two steps. To begin we consider a function $h_\epsilon(\cdot)$ satisfying $h_\epsilon(t) = e^t/3$ for $t < \log(3\epsilon)$.

$$h_\epsilon(t) = 1 - \exp\left(-\int_{-\infty}^{\log(3\epsilon)} \frac{(t-s)^2}{2}\frac{e^s}{3}\,ds - \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\,ds \right) \quad (4.4.20)$$

for $t \geq \log(3\epsilon)$, and prove that $h_\epsilon(\cdot)$ converges to some $h(\cdot)$ with the desired properties.

Lemma 4.4.7. *For fixed t, $h_\epsilon(t)$ in* (4.4.20) *is monotone decreasing in ϵ.*

Proof. If we change variables $s = t - u$ and integrate by parts, or remember the first two moments of the exponential with mean 1, then

$$\int_{-\infty}^{t} (t-s)e^s\,ds = \int_0^\infty u e^{t-u}\,du = e^t,$$
$$\int_{-\infty}^{t} \frac{(t-s)^2}{2} e^s\,ds = \int_0^\infty \frac{u^2}{2} e^{t-u}\,du = e^t \int_0^\infty u e^{-u}\,du = e^t. \quad (4.4.21)$$

Using $(t-s)^2/2 = (t-r)^2/2 + (t-r)(r-s) + (r-s)^2/2$ now gives the following identity.

$$\int_{-\infty}^{r} \frac{(t-s)^2}{2} e^s \, ds = e^r \left[\frac{(t-r)^2}{2} + (t-r) + 1 \right]. \tag{4.4.22}$$

Using (4.4.20), the inequality $1 - e^{-x} \leq x$, (4.4.21), and changing variables $s = t - u$,

$$\begin{aligned} h_\epsilon(t) - \frac{1}{3}e^t &\leq \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \left(h_\epsilon(s) - \frac{1}{3}e^s \right) ds \\ &= \int_0^{t-\log(3\epsilon)} \left(h_\epsilon(t-u) - \frac{1}{3}e^{t-u} \right) \frac{u^2}{2} \, du. \end{aligned}$$

Applying Lemma 4.3.4 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to $h_\epsilon(\cdot + \log(3\epsilon)) - \exp(\cdot + \log(3\epsilon))/3$,

$$h_\epsilon(t) - \frac{1}{3}e^t \leq 0 \text{ for any } t \geq \log(3\epsilon).$$

This shows that if $0 < \epsilon < \delta < 1$, then $h_\delta(t) \geq h_\epsilon(t)$ for $t \leq \log(3\delta)$. To compare the exponentials for $t > \log(3\delta)$, we note that

$$\begin{aligned} &\int_{\log(3\epsilon)}^{\log(3\delta)} \frac{(t-s)^2}{2} \left(h_\epsilon(s) - \frac{1}{3}e^s \right) ds + \int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} \left(h_\epsilon(s) - h_\delta(s) \right) ds \\ &\quad \leq 0 + \int_0^{t-\log(3\delta)} \left(h_\epsilon(t-u) - h_\delta(t-u) \right) \frac{u^2}{2} \, ds. \end{aligned}$$

Applying Lemma 4.3.4 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to $h_\epsilon(\cdot + \log(3\delta)) - h_\delta(\cdot + \log(3\delta))$, we see that $h_\epsilon(t) - h_\delta(t) \leq 0$ for $t \geq \log(3\delta)$. □

Lemma 4.4.8. $h(t) = \lim_{\epsilon \to 0} h_\epsilon(t)$ *exists. If* $h \not\equiv 0$ *then* h *has properties (a)–(d) in Lemma 5.8.5.*

Proof. Lemma 4.4.7 implies that the limit exists. Since $0 \leq h_\epsilon(t) \leq e^t/3$, $0 \leq h(t) \leq e^t/3$ and so $\lim_{t \to -\infty} h(t) = 0$. To show that

$$h(t) = 1 - \exp\left(-\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s) \, ds \right), \tag{4.4.23}$$

we need to show that as $\epsilon \to 0$

$$\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\, ds \to \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds. \tag{4.4.24}$$

Given $\eta > 0$, choose $\delta = \delta(\eta) > 0$ so that

$$\delta \left[1 + (t - \log(3\delta)) + (t - \log(3\delta))^2/2\right] < \eta/4.$$

By bounded convergence theorem, as $\epsilon \to 0$,

$$\int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\, ds \to \int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} h(s)\, ds.$$

So we can choose $\epsilon_0 = \epsilon_0(\eta)$ so that the difference between the two integrals is at most $\eta/2$ for any $\epsilon < \epsilon_0$. Therefore if $\epsilon < \epsilon_0$, then

$$\begin{aligned}&\left|\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\, ds - \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds\right| \\ &\leq \frac{\eta}{2} + 2\int_{-\infty}^{\log(3\delta)} \frac{(t-s)^2}{2} \frac{1}{3} e^s\, ds.\end{aligned}$$

Using the identity in (4.4.22) we conclude that the second term is

$$\leq 2\delta \left[1 + (t - \log(3\delta)) + (t - \log(3\delta))^2/2\right] \leq \frac{\eta}{2}.$$

This shows that (4.4.24) holds, and with (4.4.20) and (4.4.22) proves (4.4.23).

To prove $\lim_{t\to\infty} h(t) = 1$ note that if $h(\cdot) \not\equiv 0$, then there is an r with $h(r) > 0$, and so for $t > r$

$$\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds \geq h(r) \int_{r}^{t} \frac{(t-s)^2}{2}\, ds = h(r)\frac{(t-r)^3}{3!} \to \infty$$

as $t \to \infty$. So in view of (4.4.23), $h(t) \to 1$ as $t \to \infty$, if $h(\cdot) \not\equiv 0$.

The last detail is to show if $h(\cdot) \not\equiv 0$, then $h(t) \in (0,1)$ for all t. Suppose, if possible, $h(t_0) = 0$. (4.4.23) implies $\int_{-\infty}^{t_0} h(s)[(t-s)^2/2]\, ds = 0$, and hence

$h(s) = 0$ for $s \le t_0$. Changing variables $s \mapsto t - r$, and using (4.4.23) again with the inequality $1 - e^{-x} \le x$, imply that for any $t > t_0$

$$h(t) \le \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds = \int_{0}^{t-t_0} h(t-r) \frac{r^2}{2}\, dr.$$

Applying Lemma 4.3.4 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to the function $h(\cdot + t_0)$, we see that $h(t) \le 0$ for any $t > t_0$. But $h(t) \ge 0$ for any t, and hence $h \equiv 0$, a contradiction. □

To complete the proof of Lemma 5.8.5 it suffices to show that $|f_\epsilon(\cdot) - h_\epsilon(\cdot)|$ and $|g_\epsilon(\cdot) - h_\epsilon(\cdot)|$ converge to 0 as $\epsilon \to 0$. To do this, note that if

$$h_0(t) = 1 - \exp\left(- \int_{-\infty}^{\log(3\epsilon)} \frac{(t-s)^2}{2} \frac{e^s}{3}\, ds \right),$$

then

$$h_\epsilon(t) = 1 - (1 - h_0(t)) \exp\left(- \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s)\, ds \right),$$

and so using the inequality $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$,

$$|h_\epsilon(t) - g_\epsilon(t)| \le |h_0(t) - g_0(t)| + \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} \left| h_\epsilon(s) - g_\epsilon(s) \right| \, ds.$$

Using the inequality $0 \le e^{-x} - 1 + x \le x^2/2$, and the identity in (4.4.22),

$$\begin{aligned}
|h_0(t) - g_0(t)| &\le \frac{1}{2} \left[\epsilon + \epsilon(t - \log(3\epsilon)) + \epsilon \frac{(t - \log(3\epsilon))^2}{2} \right]^2 \\
&\le \frac{3}{2} \epsilon^2 \left[1 + (t - \log(3\epsilon))^2 + \frac{(t - \log(3\epsilon))^4}{4} \right].
\end{aligned}$$

Applying Lemma 4.3.4 with $\lambda = 1$ and $\beta(t) = 1 + t^2 + t^4/4$ to the function

$$\left| h_\epsilon(\cdot + \log(3\epsilon)) - g_\epsilon(\cdot + \log(3\epsilon)) \right|,$$

we have $|h_\epsilon(t) - g_\epsilon(t)| \le (3\epsilon^2/2)\beta * V(t - \log(3\epsilon))$, where $V(\cdot)$ is as in Lemma 4.2.2. Using $\lambda = 1$ in the expression of $V(\cdot)$ and Lemma 4.2.1,

$$\beta * V(t) = \beta(t) + \int_0^t \beta(t-s) V'(s)\, ds$$

$$= \sum_{k=0}^{\infty} \left[\frac{t^{3k}}{(3k)!} + 2\frac{t^{3k+2}}{(3k+2)!} + 6\frac{t^{3k+4}}{(3k+4)!} \right] \leq 6e^t.$$

So $|h_\epsilon(t) - g_\epsilon(t)| \leq (3\epsilon^2/2) \cdot 6 \exp(t - \log(3\epsilon))$, and so

$$\sup_{s \in I_{\epsilon,t}} |h_\epsilon(s) - g_\epsilon(s)| \leq 6\epsilon e^t/2.$$

Repeating the argument for $f_\epsilon(\cdot)$, and noting that $|h_0(t) - f_0(t)| = |h_0(t) - g_0(t)| + \epsilon^{7/6}$,

$$\sup_{s \in I_{\epsilon,t}} |h_\epsilon(s) - f_\epsilon(s)| \leq \left(6\frac{3}{2}\epsilon^2 + \epsilon^{7/6} \right) \exp(t - \log(3\epsilon)) = \left(\frac{1}{3}\epsilon^{1/6} + 3\epsilon \right) e^t.$$

This completes the second step and we have proved Lemma 5.8.5. □

Now we have all the ingredients to prove Theorem 4.1.4.

Proof of Theorem 4.1.4. Let $h(\cdot)$ be as in Lemma 5.8.5. Choose $\epsilon \in (0, \delta/6)$ small enough so that

$$\sup_{s \in I_{\epsilon,t}} |g_\epsilon(s) - h(s)| < \delta/2, \quad \sup_{s \in I_{\epsilon,t}} |f_\epsilon(s) - h(s)| < \delta/2.$$

Let $D = \left\{ M \leq 3\epsilon N^{2-2\alpha/3} \right\}$. On the event D, $W = \psi(\log(3\epsilon)) > 0$. So

$$\begin{aligned} P\left(\sup_{s \leq t} \left| N^{-2} C_{\psi(s)} - h(s) \right| > \delta \right) &\leq P(D^c) + P\left(N^{-2} C_W + h(\log(3\epsilon)) > \delta \right) \\ &+ P\left(\sup_{s \in I_{\epsilon,t}} \left(N^{-2} C_{\psi(s)} - h(s) \right) > \delta \right) + P\left(\inf_{s \in I_{\epsilon,t}} \left(N^{-2} C_{\psi(s)} - h(s) \right) < -\delta \right). \end{aligned} \tag{4.4.25}$$

To estimate the second term in (4.4.25) note that $h(\log(3\epsilon)) \leq (1/3)\exp(\log(3\epsilon)) < \delta/2$, and

$$P\left(N^{-2} C_W > \delta/2 \right) \leq P\left(A_W > (\delta/2) N^2 \right) \to 0$$

as $N \to \infty$ by Lemma 4.3.2. To estimate the third term in (4.4.25) we use Lemma 4.4.6 to get

$$\limsup_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - h(s)\right) > \delta\right)$$
$$\leq \limsup_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - g_\epsilon(s)\right) > \delta/2\right) \leq P(M < \epsilon^{1/3}) + \epsilon^{2/3}.$$

For the fourth term in (4.4.25) use Lemma 4.4.5 to get

$$\limsup_{N\to\infty} P\left(\inf_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - h(s)\right) < -\delta\right)$$
$$\leq \limsup_{N\to\infty} P\left(\inf_{s\in I_{\epsilon,t}} \left(N^{-2}C_{\psi(s)} - f_\epsilon(s)\right) < -\delta/2\right) \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Letting $\epsilon \to 0$, we see that for any $\delta > 0$,

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\epsilon,t}} \left|N^{-2}C_{\psi(s)} - h(s)\right| > \delta\right) = 0. \tag{4.4.26}$$

It remains to show that $h(\cdot) \not\equiv 0$. Let ϵ, γ be such that

$$P[M \leq (1+\gamma)\epsilon^{1/3}] + 11\frac{\epsilon^{1/3}}{\gamma} < 1.$$

Fix any $\eta > 0$ and let $t_0 = \log(3\epsilon(1+\gamma) + 3\eta)$. Using Lemma 4.3.2 and 4.3.6

$$\limsup_{N\to\infty} P\left(N^{-2}C_{\psi(t_0)} < \epsilon\right) = \limsup_{N\to\infty} P(\tau(\epsilon) > \psi(t_0))$$
$$\leq \limsup_{N\to\infty} P[\tau(\epsilon) > \sigma(\epsilon(1+\gamma))] + \limsup_{N\to\infty} P[\sigma(\epsilon(1+\gamma)) > \psi(t_0)]$$
$$\leq \limsup_{N\to\infty} P[\tau(\epsilon) > \sigma(\epsilon(1+\gamma))] + \limsup_{N\to\infty} P\left(\left|N^{-2}A_{W_{\epsilon(1+\gamma)+\eta}} - \epsilon(1+\gamma) - \eta\right| > \eta\right)$$
$$\leq P[M \leq (1+\gamma)\epsilon^{1/3}] + 11\frac{\epsilon^{1/3}}{\gamma} < 1.$$

But if $h(t_0) = 0$, we get a contradiction to (4.4.26). This proves $h(\cdot) \not\equiv 0$. □

4.5 Asymptotics for the cover time

Proof of Theorem 4.1.5. Theorem 4.1.4 gives a lower bound on the area covered whcih implies that if $\delta > 0$ and N is large, then with high probability the number of centers in $\mathcal{C}_{\psi(0)}$ dominates a Poisson random variable with mean $\lambda(\delta)N^{2-(2\alpha/3)}$, where

$$\lambda(\delta) = \int_{-\infty}^{0} (h(s) - \delta)^+ \, ds.$$

If δ_0 is small enough, $\lambda_0 \equiv \lambda(\delta_0) > 0$. Dividing the torus into disjoint squares of size $\kappa N^{\alpha/3}\sqrt{\log N}$, where κ is a large constant, the probability that a given square is vacant is $\exp(-\lambda_0 \kappa^2 \log N)$. If $\kappa\sqrt{\log N} \geq 1$, the number of squares is $\leq N^{2-(2\alpha/3)}$ So if $\lambda_0 \kappa^2 \geq 2$, then with high probability none of our squares is vacant. Thus even if no more births of new centers occur then the entire square will be covered by a time $\psi(0) + O(N^{\alpha/3}\sqrt{\log N})$. $\square$

Chapter 5

Threshold-two contact process on random regular graphs

5.1 Introduction

Interacting particle systems are often formulated on the d-dimensional integer lattice $\mathbf{Z}^d$. See e.g. [34] or [35]. However, if one is considering the spread of influenza in a town, infections occur not only between individuals who live close to each other, but also over long distances due to social contacts at school or at work. Because of this, one should consider how these stochastic spatial processes change when the regular lattice is replaced by the random graphs that have been used to model social networks.

[18] considers the contact process on a small world graph $\mathcal{S}$. In the contact process, each vertex is either occupied or vacant. Occupied vertices become vacant at rate 1, while vacant vertices become occupied at rate λ times the number of occupied neighbors. The small world random graph, which [18] considers, is a modification of the d-dimensional torus $\mathcal{T}_L := (\mathbf{Z} \bmod L)^d$ in which each vertex has exactly one long-distance neighbor, where the long-distance neighbors are defined by a random pairing of the vertices of the torus.

The contact process on the small world (or on any finite graph) cannot have a nontrivial stationary distribution, because it is a finite state Markov chain with an absorbing state. However, on the small world and many other graphs, there is a "quasi-stationary distribution" which persists for a long time. To explain the concept in quotes, we recall the situation for the contact process on the d-dimensional torus $\mathcal{T}_L$. Let $\zeta_t^0 \subseteq \mathbf{Z}^d$ denote the contact process on $\mathbf{Z}^d$ starting with single occupied vertex at the origin and let

$$\lambda_c := \inf\{\lambda : P(\Omega_\infty) > 0\}, \text{ where } \Omega_\infty := \{\zeta_t^0 \neq \emptyset \text{ for all } t\}.$$

Let $\zeta_t^1 \subseteq \mathbf{Z}^d$ denote the contact process on $\mathbf{Z}^d$ starting with all vertices occupied. Monotonicity and self-duality imply that (see [35]) if $\lambda > \lambda_c$ and $\zeta_\infty^1 := \lim_{t\to\infty} \zeta_t^1$, where the limit is in distribution, then ζ_∞^1 is a translation invariant stationary distribution with $P(x \in \zeta_\infty^1) = P(\Omega_\infty)$.

Returning to the torus $\mathcal{T}_L$ and letting $\zeta_t^{1,\mathcal{T}_L} \subseteq \mathcal{T}_L$ denote the contact process on it starting from all vertices occupied, if $\lambda < \lambda_c$, then there is a $k_1(\lambda) > 0$ so that $P(\zeta_{k_1(\lambda)\log n}^{1,\mathcal{T}_L} \neq \emptyset) \to 0$ as $n \to \infty$, where $n = L^d$ is the number of vertices in $\mathcal{T}_L$. If $\lambda > \lambda_c$, then with high probability $\zeta_t^{1,\mathcal{T}_L}$ persists to time $\exp(k_2(\lambda)n)$ for some $k_2(\lambda) > 0$. Furthermore, at times $1 \ll t \leq \exp(k_2(\lambda)n)$ the finite dimensional distributions of $\zeta_t^{1,\mathcal{T}_L}$ are close to those of ζ_∞^1 (see [35]). Thus the quasi-stationary distribution for the contact process on the finite graph is like the stationary distribution for the contact process on the associated infinite graph.

Locally, the small world graph $\mathcal{S}$ looks like an infinite graph that is called the big world $\mathcal{B}$ in [18]. In this graph, traversing a long range edge brings one to another copy of $\mathbf{Z}^d$. Sophisticates will recognize this as the free product $\mathbf{Z}^d * \{0,1\}$, where the second factor is $\mathbf{Z} \bmod 2$. Like the contact process on the homogeneous tree, the contact process on $\mathcal{B}$ has two phase transitions $\lambda_1 < \lambda_2$, which correspond to global and local survival respectively. That is, if $\zeta_t^{0,\mathcal{B}} \subseteq \mathcal{B}$ denotes the contact process on $\mathcal{B}$ starting with single occupied vertex at the origin, then

$$\lambda_1 := \inf\{\lambda : P(\Omega_\infty^{\mathcal{B}}) > 0\} \text{ and}$$
$$\lambda_2 := \inf\left\{\lambda : \liminf_{t\to\infty} P\left(0 \in \zeta_t^{0,\mathcal{B}}\right) > 0\right\},$$

where as earlier $\Omega_\infty^{\mathcal{B}} = \{\zeta_t^{0,\mathcal{B}} \neq \emptyset \text{ for all } t\}$. Let $\zeta_t^{1,\mathcal{B}}$ denote the contact process on $\mathcal{B}$ starting with all vertices occupied. Monotonicity and duality imply that if $\lambda > \lambda_1$ and $\zeta_\infty^{1,\mathcal{B}} := \lim_{t\to\infty} \zeta_t^{1,\mathcal{B}}$, where the limit is in distribution, then $\zeta_\infty^{1,\mathcal{B}}$ is a translation invariant stationary distribution with $P(x \in \zeta_\infty^{1,\mathcal{B}}) = P(\Omega_\infty^{\mathcal{B}})$.

In order to study the persistence of the contact process $\zeta_t^{1,\mathcal{S}} \subseteq \mathcal{S}$ on the small world $\mathcal{S}$, [18] introduces births at a rate γ from each vertex, which go from an occupied vertex to a randomly chosen vertex. With this modification it is shown that if $\lambda > \lambda_1$, then there is a constant $k_3 = k_3(\lambda, \gamma) > 0$ so that for $n = L^d$, $\zeta_t^{1,\mathcal{S}}$ persists to time $\exp(k_3 n)$ with high probability.

In this paper, we study the behavior of the discrete time *threshold-two contact process* on a random r-regular graph on n vertices. We construct our random graph G_n on the vertex set $V_n := \{1, 2, \ldots n\}$ by assigning r "half-edges" to each of the vertices, and then pairing the half-edges at random. If r is odd, then n must be even so that the number of half-edges, rn, is even to have a valid degree sequence. Let $\mathbb{P}$ denote the distribution of G_n. We condition on the event E_n that the graph is simple, i.e. it does not contain a self-loop at any vertex, or more than one edge between two vertices. It can be shown (see e.g. Corollary 9.7 on page 239 of [28]) that $\mathbb{P}(E_n)$ is bounded away from 0, and hence for large enough n,

$$\text{if } \tilde{\mathbb{P}} := \mathbb{P}(\cdot|E_n), \text{ then } \tilde{\mathbb{P}}(\cdot) \leq c\mathbb{P}(\cdot) \text{ for some constant } c = c(r) > 0. \tag{5.1.1}$$

So the conditioning on the event E_n will not have much effect on the distribution of G_n. Since the resulting graph remains the same under any permutation of the half-edges corresponding to any vertex, the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all undirected r-regular graphs on the vertex set V_n. We choose G_n according to the distribution $\tilde{\mathbb{P}}$ on simple graphs, and once chosen the graph remains fixed through time.

We write $x \sim y$ to mean that x is a neighbor of y, and let

$$\mathcal{N}_y := \{x \in V_n : x \sim y\} \tag{5.1.2}$$

be the set of neighbors of y. The distribution $P_{G_n,p}$ of the (discrete time) threshold-two contact process $\xi_t \subseteq V_n$ with parameter p conditioned on G_n can be described as follows:

$$P_{G_n,p}\,(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| \geq 2\,) = p \text{ and}$$
$$P_{G_n,p}\,(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| < 2\,) = 0,$$

where the decisions for different vertices at time $t+1$ are taken independently. If $\mathbf{P}_p$ denotes the distribution of the threshold-two contact process ξ_t on the random graph G_n having distribution $\tilde{\mathbb{P}}$, then

$$\mathbf{P}_p(\cdot) = \tilde{\mathbb{E}} P_{G_n,p}(\cdot),$$

where $\tilde{\mathbb{E}}$ is the expectation corresponding to the probability distribution $\tilde{\mathbb{P}}$.

Let $\xi_t^A \subseteq V_n$ denote the threshold-two contact process starting from $\xi_0^A = A$, and let ξ_t^1 denote the special case when $A = V_n$. In the long history of the contact process the first step was to study whether the critical value of the parameter lies in the interior of the parameter-space or not. Based on results for the threshold contact process on random directed graph in [**?**], and basic contact process on the small world $\mathcal{S}$ in [18], it is natural to expect the existence of a critical value $p_c \in (0,1)$ defining the boundary between rapid convergence within logarithmically small time to all-zero configuration for $p < p_c$, and exponentially prolonged persistence of changes for $p > p_c$. We define the boundary p_c between convergence to the all-zero configuration within time $C(p)\log n$, and exponentially prolonged persistence as

$$p_c := \inf\left\{p \in [0,1] : \lim_{n\to\infty} \mathbf{P}_p\left(\inf_{t\le\exp(k(p)n)} \frac{|\xi_t^1|}{n} > u(p)\right) = 1 \text{ for some } k(p), u(p) > 0\right\}. \tag{5.1.3}$$

In order to show that $p_c < 1$, it suffices to show that if p is sufficiently close to 1, then ξ_t^1 maintains a positive fraction of occupied vertices for time $\ge \exp(c_1 n)$ for some constant $c_1 > 0$.

Theorem 5.1.1. *If $r \ge 4$ and $\eta \in (0, 1/4)$, then there is an $\epsilon_1 = \epsilon_1(\eta) \in (0,1)$ such that for*

$$\frac{1-\epsilon_1}{1-\left(\frac{3}{2r-4}+\eta\right)\epsilon_1} < p \le 1, \tag{5.1.4}$$

and for some positive constants C_1 and $c_1(\eta, p)$,

$$\mathbf{P}_p\left(\inf_{t\le\exp(c_1(\eta,p)n)} \frac{|\xi_t^1|}{n} < 1-\epsilon_1\right) \le C_1 \exp(-c_1(\eta,p)n).$$

In words, if p is sufficiently close to 1 and r is larger than 3, then the fraction of occupied vertices in the threshold-two contact process starting from all-one configuration remains close to 1 for exponentially long time with probability $1-o(1)$. Here and later $o(1)$ denotes a quantity that goes to 0 as n goes to ∞. So Theorem 5.1.1 confirms that $p_c < 1$ for $r \ge 4$. The argument does not work for

$r = 3$, as the lower bound in (5.1.4) is higher than 1 if we put $r = 3$. We believe that similar result holds for $r = 3$, but the problem remains open. The key to the proof of Theorem 5.1.1 is an 'isoperimetric inequality' (see Proposition 5.1.6 below).

Next we study the behavior of ξ_t^A, when $|A|$ is small.

Theorem 5.1.2. *There is a decreasing continuous function $\epsilon_2 : (0,1) \mapsto (0,1)$ and a collection $\mathcal{G}$ of simple r-regular graphs on n vertices such that for any $p \in (0,1)$, $C_0(p) := 2/\log(2/(1+p))$, and any subset $A \subset V_n$ with $|A| \le \epsilon_2(p)n$,*

$$\begin{array}{lll} (i) & \sup_{G_n\in\mathcal{G}} P_{G_n,p}\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) & = o(1), \\ (ii) & \tilde{\mathbb{P}}(\mathcal{G}^c) & = o(1). \end{array}$$

Hence $\lim_{n\to\infty} \mathbf{P}_p\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) = 0$.

In words, for any value of $p \in (0,1)$, whenever the fraction of occupied vertices drops below a certain level depending on p, all vertices of G_n become vacant within logarithmically small time with probability $1 - o(1)$. Thus the density of occupied vertices doesn't stay in the interval $(0, \epsilon_2(p))$ for long time. The key to the proof of Theorem 5.1.2 is another 'isoperimetric inequality' (see Proposition 5.1.5 below). As a consequence of Theorem 5.1.2, we have:

Corollary 5.1.3. *There is a $p_0 \in (0, 2/3)$ such that for $0 \le p < p_0$,*

$$\lim_{n\to\infty} \mathbf{P}_p\left(\xi^1_{\lceil (C_0(p)+1)\log n\rceil} \neq \emptyset\right) = 0, \text{ where } C_0(p) \text{ is as in Theorem 5.1.2.}$$

That is, if p is sufficiently close to 0, then starting from all-one configuration all vertices of G_n become vacant within logarithmically small time with probability $1 - o(1)$. So Corollary 5.1.3 confirms that $p_c > 0$.

Theorem 5.1.1 shows that $p_c < 1$, and so for $p \in (p_c, 1)$ the fraction of occupied vertices in the graph G_n is bounded away from zero for a time longer than $\exp(n^{1/2})$. So we can now define a quasi-stationary measure ξ^1_∞, which is an analogue of the upper invariant measure, as follows. For any $A \subset V_n$, $\xi^1_\infty\{B : B \cap A \neq \emptyset\} := \mathbf{P}_p(\xi^1_{\lceil \exp(n^{1/2})\rceil} \cap A \neq \emptyset)$. Let X_n be uniformly distributed on V_n, and let

$$\rho_n := \xi^1_\infty\{B : X_n \in B\} = \frac{1}{n}\left|\xi^1_{\lceil \exp(n^{1/2})\rceil}\right|.$$

So ρ_n is the quasi-stationary density of occupied vertices in the threshold-two contact process on the random graph G_n. Note that ρ_n is an analogue of the density of occupied vertices in the upper invariant measure for the contact process with sexual reproduction on regular lattices, which is conjectured to have a continuous phase transition (see Conjecture 1 and heuristic argument following that in [20]). As we now explain, things are different in the threshold-two contact process on a random regular graph.

First observe that if $p > p_c$, then ρ_n is bounded away from zero with high probability, because if $\rho_n < \epsilon_2(p)$, where $\epsilon_2(\cdot)$ is as in Theorem 5.1.2, then $|\xi^1_{\lceil \exp(n^{1/2})\rceil}| \leq n\epsilon_2(p)$. In that case, for $\sigma = \lceil \exp(n^{1/2})\rceil + \lceil C_0 \log n\rceil$, either $\xi^1_\sigma \neq \emptyset$, which has $\mathbf{P}_p$-probability $o(1)$ by Theorem 5.1.2, or $\xi^1_\sigma = \emptyset$, which has $\mathbf{P}_p$-probability $o(1)$ by the definition of p_c in (5.1.3) and the fact that $p > p_c$. Therefore, for $p > p_c$, $\rho_n \geq \epsilon_2(p)$ with $\mathbf{P}_p$-probability $1 - o(1)$.

Next observe that for any $p_1, p_2 \in [0,1]$ with $p_1 < p_2$, the random variables $Z_i \sim Bernoulli(p_i), i = 1, 2$, can be coupled so that $Z_1 \leq Z_2$. Using this coupling for all the Bernoulli random variables, which are used in deciding whether $x \in$

ξ_t for $x \in V_n, t = 1, 2, \ldots$, it is easy to see that

$$P_{G_n,p_1} \le P_{G_n,p_2}, \text{ i.e. for any increasing event } B, P_{G_n,p_1}(B) \le P_{G_n,p_2}(B).$$

The same inequality holds for the unconditional probability distributions $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$. Since $\{\rho_n \ge \epsilon\} = \{|\xi^1_{\lceil \exp(n^{1/2})\rceil}| \ge \epsilon n\}$ is an increasing event, it follows that for any $p > p' > p_c$

$$\mathbf{P}_p(\rho_n \ge \epsilon_2(p')) \ge \mathbf{P}_{p'}(\rho_n \ge \epsilon_2(p')) = 1 - o(1)$$

by the above discussion. Taking p' sufficiently close to p_c and noting that $\epsilon_2(\cdot)$ is a decreasing continuous function, we get the result of this paper that the threshold-two contact process on the random graph G_n has a discontinuous phase transition at the critical value p_c.

Theorem 5.1.4. *Let $\rho := \epsilon_2(p_c)$, where $\epsilon_2(\cdot)$ is as in Theorem 5.1.2 and p_c is as in* (5.1.3). *Then $\rho > 0$. For any $p > p_c$ and $\delta > 0$,*

$$\lim_{n\to\infty} \mathbf{P}_p(\rho_n > \rho - \delta) = 1.$$

The key to the proof of Theorem 5.1.2 is an "isoperimetric inequality". Given a subset $U \subset V_n$, let

$$U^{*2} := \{y \in V_n : y \sim x \text{ and } y \sim z \text{ for some } x, z \in U \text{ with } x \ne z\}. \tag{5.1.5}$$

The idea behind this definition is that if $U = \xi_t$ for some t, then U^{*2} is the set of vertices which have a chance of being occupied at time $t+1$. Note that U^{*2} can contain vertices of U.

Proposition 5.1.5. *Let $E(m,k)$ be the event that there is a subset $U \subset V_n$ with size $|U| = m$ so that $|U^{*2}| \ge k$. Then there is an increasing positive function $\epsilon_3(\cdot)$ so that* *for any* $\eta > 0$ *and* $m \le \epsilon_3(\eta)n$,

$$\mathbb{P}\left[E(m,(1+\eta)m)\right] \le C_3 \exp\left(-\frac{\eta^2}{8r} m \log(n/m)\right)$$

for some constant $C_3 = C_3(r)$.

In words, if U is a small set, then for any $\eta > 0$, $|U^{*2}| \le (1+\eta)|U|$ with high probability. Now if $E_{G_{n,p}}$ is the expectation corresponding to the probability distribution $P_{G_{n,p}}$, then $E_{G_{n,p}}(|\xi_{t+1}|\ |\xi_t) = p|\xi_t^{*2}|$. Given $p \in (0,1)$, we can choose $\eta(p) > 0$ so that $p(1+\eta(p)) < (1+p)/2$. So using Proposition 5.1.5, if $|\xi_t|$ is small, $E_{G_{n,p}}(|\xi_{t+1}|\ |\xi_t) < |\xi_t|(1+p)/2$ with high probability. This observation together with large deviation results for the Binomial distribution implies that $|\xi_{t+1}| \le |\xi_t|(1+p)/2$ with high probability if $|\xi_t|$ is small. Finally if the number of occupied vertices reduces by a fraction at each time, all vertices will be vacant by time $O(\log n)$ and so Theorem 5.1.2 follows.

The key to the proof of Theorem 5.1.1 is another 'isoperimetric inequality'. If $W = V_n \setminus \xi_t$ is the set of vacant vertices at time t, then $(W^c)^{*2}$ is the set of vertices which have a chance of being occupied at time $t+1$, and so $((W^c)^{*2})^c$ is the set of vertices which will surely be vacant at time $t+1$.

Proposition 5.1.6. *Let* $F(m,k)$ *be the event that there is a subset* $W \subset V_n$ *with* $|W| = m$ *so that* $|((W^c)^{*2})^c| > k$. *Given* $\eta > 0$, *there are positive constants* $\epsilon_4(r,\eta)$ *and* $C_4(r)$ *so that for* $m \le \epsilon_4 n$,

$$\mathbb{P}\left[F\left(m, \left(\frac{3}{2(r-2)} + \eta\right)m\right)\right] \le C_4 \exp(-(\eta/8) m \log(n/m)).$$

In words, if W is a small set, then for any $\eta > 0$, $|((W^c)^{*2})^c| \le (3/(2r-4) + \eta)|W|$ with high probability. As noted above, $E_{G_{n,p}}(|\xi_{t+1}|\ |\xi_t) = p|\xi_t^{*2}|$. For p as in

(5.1.4), we can choose $\delta(p,\eta) > 0$ so that $(p-\delta)(1-(3/(2r-4)+\eta)\epsilon_1) > 1-\epsilon_1$. So using Proposition 5.1.6 with $W = V_n \setminus \xi_t$, if $|\xi_t|/n \geq 1-\epsilon_1$, then $E_{G_n,p}(|\xi_{t+1}|\ |\xi_t) \geq p(1-(3/(2r-4)+\eta)\epsilon_1)n > (1-\epsilon_1)np/(p-\delta)$ with high probability. This observation together with large deviation results for the Binomial distribution implies that $|\xi_{t+1}| \leq (1-\epsilon_1)n$ with exponentially small probability if $|\xi_t|/n \geq 1-\epsilon_1$. Thus if τ is the first time the fraction of occupied vertices drops below $1-\epsilon_1$, then $\tau > \exp(c_1(\eta,p)n)$ with high probability for a suitable choice of $c_1(\eta,p)$, and so Theorem 5.1.1 follows.

The remainder of the paper is organized as follows. In section 5.2, we present sketches of the proofs of Proposition 5.1.5 and 5.1.6. In section 5.3 and 5.4, we use the propositions to study the behavior of ξ_t starting from a small occupied set and the fact that $p_c \in (0,1)$ respectively, while in section 5.6 and 5.7 we present the proofs of the propositions. Section 5.5 is about the first order phase transition at p_c. Finally in section 5.8 we prove several probability estimates, which are needed in the proof of the propositions.

5.2 Sketch of the proofs of the isoperimetric inequalities.

Recall the definition of U^{*2} from (5.1.5). We need some more definitions and notations. For any vertex $x \in V_n$ and subsets $U, W \subset V_n$ let ∂U be the boundary of the set U, U^{*1} be the set of vertices which have at least one neighbor in U, $e(U,W)$ be the number of edges between U and W. Also let U_0 be the set of vertices in U which have all their neighbors in U^c, and U_1 be the complement of

U_0. So

$$\partial U := \{y \in U^c : y \sim x \text{ for some } x \in U\}, \quad U^{*1} := \{y \in V_n : y \sim x \text{ for some } x \in U\},$$
$$e(U, W) := |\{(x, y) : x \in U \text{ and } y \in W\}|,$$
$$U_0 := \{x \in U : y \sim x \text{ implies } y \in U^c\}, \qquad U_1 := U \cap U_0^c. \tag{5.2.1}$$

5.2.1 Isoperimetric inequality in Proposition 5.1.5

From the definitions in (5.2.1) it is easy to see that if $|U| = m$, then

$$rm \geq \sum_{y \in U^{*1}} e(\{y\}, U) \geq |U^{*1} \setminus U^{*2}| + 2|U^{*2}| = |U^{*1}| + |U^{*2}|.$$

So for any subset U of vertices of size m,

$$\text{if } |U^{*2}| \geq k, \text{ then } |U^{*1}| \leq rm - k. \tag{5.2.2}$$

In view of (5.2.2), for proving Proposition 5.1.5 it suffices to estimate the probability

$$\mathbb{P}\left[H(m, (r - 1 - \eta)m)\right], \text{ where } H(m, k) = \cup_{\{U \subset V_n : |U| = m\}} \left\{|U^{*1}| \leq k\right\} \tag{5.2.3}$$

is the event that there is a subset U of vertices of size m with $|U^{*1}| \leq k$.

Note that U^{*1} is a disjoint union of ∂U and U_1. Our first step in estimating (5.2.3), taken in Lemma 5.8.2, is to show the following.

I. *For $|U| = m$ and any $\eta > 0$, $e(U, U^c) \geq (r - 2 - \eta)|U|$ with probability at least $1 - \exp\left(-(1 + \eta/2)m \log(n/m) + \Delta_1 m\right)$ for some constant Δ_1.*

Take $\alpha = (r - 2 - \eta)/r$ in Lemma 5.8.2 so that $(1 - \alpha)r/2 = 1 + \eta/2$. We cannot hope to do better than $r - 2$. Consider a tree in which all vertices have degree

r and let U be the set of vertices within distance k of a fixed vertex. If $s = r-1$, then $|U| = 1 + r + rs + \cdots + rs^{k-1} \approx rs^k/(s-1)$ and $e(U, U^c) = rs^k$, so $e(U, U^c)/|U| \approx s - 1 = r - 2$.

In the next step, see Lemma 5.8.4, we show the following.

II. *Given* $e(U, U^c) = u|U|$ *for some constant* u *and* $\eta > 0$, *if* $m = |U| \leq \epsilon_5(\eta)n$, *then* $|\partial U| \geq (u - \eta)|U|$ *with probability* $\geq 1 - \exp(-\eta m \log(n/m) + \Delta_2 m)$ *for some constant* Δ_2.

Considering all possible values of $u \geq r - 2 - \eta$ and using *I* and *II*,

$$|\partial U| \geq (r-2-2\eta)|U| \quad \text{with probability } \geq 1 - 2\exp\left(-(1+\eta)m\log(n/m) + (\Delta_1 + \Delta_2)m\right).$$

Using the fact (see Lemma 5.8.1) that

III. *the number of subsets of* V_n *of size* m *is at most* $\exp(m \log(n/m) + m)$,

the expected number of subsets U of size m with $|\partial U| < (r - 2 - 2\eta)|U|$ is exponentially small if $m \leq \epsilon(\eta)n$ for some small fraction $\epsilon(\eta)$. Therefore,

$$\text{with high probability } |U^{*1}| \geq |\partial U| \geq (r - 2 - 2\eta)|U|, \text{ whenever } |U| \leq \epsilon(\eta)n. \tag{5.2.4}$$

But this is not good enough, so we need to work to improve the first inequality above.

Recall the definitions of U_0 and U_1 from (5.2.1). There are two possibilities based on $|U_1|$. Given $\eta > 0$, if $|U_1| \leq (\eta/2r)|U|$, then $e(U, U^c) \geq r|U_0| \geq (r - \eta/2)|U|$. So using *II*,

if $|U| = m,$ then $|\partial U| < (r - 1 - \eta)|U|$ and $|U_1| \leq (\eta/2r)|U|$ with probability at most

$$\exp\left(-(1+\eta/2)m\log(n/m)+\Delta_2 m\right).$$

Combining with *III* the expected number of subsets of size m with the above property is exponentially small, if $m \le \epsilon(\eta)n$. Therefore,

$$\text{with high probability } |U^{*1}| \ge |\partial U| \ge (r-1-\eta)|U| \quad \text{whenever } |U_1| \le (\eta/2r)|U|.$$

Next we look at the other possibility $|U_1| > (\eta/2r)|U|$. Using an argument similar to the one leading to (5.2.4),

$$\text{with high probability } e(U_1, U_1^c) \ge (r-2-\eta)|U_1| \text{ whenever } |U| \le \epsilon(\eta)n \text{ and } |U_1| > (\eta/2r)|U|. \tag{5.2.5}$$

Using the equalities $e(U_0, U^c) = e(U_0, U_0^c) = r|U_0|$ and $e(U_1, U^c) = e(U_1, U_1^c)$, we have $e(U, U^c) = r|U_0| + e(U_1, U_1^c)$. Combining this with another equality $|U^{*1}| = |U_1| + |\partial U|$ and a little algebra give that $\{|U^{*1}| \le (r-1-\eta)|U|\} = \{e(U, U^c) - |\partial U| \ge (1+\eta)|U_0| + e(U_1, U_1^c) - (r-2-\eta)|U_1|\}$. In view of (5.2.5), the probability of the last events is estimated to be small enough (see (5.6.14) for details), so that using *III* the expected number of subsets U of size m with the above property is exponentially small. Combining the last two arguments,

$$\text{with high probability } |U^{*1}| \ge (r-1-\eta)|U| \quad \text{whenever } |U_1| > (\eta/2r)|U|.$$

This completes the argument to estimate the probability in (5.2.3) and thereby proves Proposition 5.1.5.

5.2.2 Isoperimetric inequality in Proposition 5.1.6

Recall the definition of $\mathcal{N}_y$ from (5.1.2). We need some more notations for Proposition 5.1.6. For any subset W of V_n, let W^0 be the subset of vertices which are in

W and have at most 1 neighbor in W^c, and W^1 be the subset of vertices which are in W^c and have at most 1 neighbor in W^c. So

$$W^0 := \{y \in W : |\mathcal{N}_y \cap W| \geq r-1\}, \quad \beta_0(W) := |W^0|/|W|,$$
$$W^1 := \{y \in W^c : |\mathcal{N}_y \cap W| \geq r-1\}, \quad \beta_1(W) := |W^1|/|W|. \tag{5.2.6}$$

The idea behind these definitions is that if W^c is occupied at tine t in the threshold-two contact process, then the subset of V_n, which cannot be occupied at time $t+1$, is

$$\left((W^c)^{*2}\right)^c = W^0 \cup W^1, \text{ and } \left|\left((W^c)^{*2}\right)^c\right| = |W^0| + |W^1|.$$

By I, $e(W^0, (W^0)^c) \geq (r-2-(2r-4)\eta)|W^0|$ with high probability if $|W| \leq \epsilon(\eta)n$. But $e(W^0, W^c) \leq |W^0|$ by the definition of W^0. So if $e(W^0, (W^0)^c) \geq (r-2-(2r-4)\eta)|W^0|$, then

$$\begin{aligned}
e(W^0, W \setminus W^0) &= e(W_0, (W^0)^c) - e(W_0, W^c) \\
&\geq (r-2-(2r-4)\eta)|W^0| - |W^0|.
\end{aligned}$$

Using $e(W^0, W^c) \leq |W^0|$ again with $W_0 \subset W$ and the last inequality, we have

$$\begin{aligned}
e(W, W^c) &= e(W \setminus W^0, W^c) + e(W^0, W^c) \\
&\leq r|W \setminus W^0| - e(W \setminus W^0, W_0) + |W^0| \\
&= [r - (2r-4)(1-\eta)\beta_0(W)]|W|.
\end{aligned}$$

Each $x \in \partial W$ has $e(\{x\}, W) \geq 1$ while each $x \in W^1$ has $e(\{x\}, W) \geq r-1$. So using the previous result and the definition of $\beta_i(W)$,

$$\begin{aligned}
|\partial W| &\leq e(W, W^c) - (r-2)|W^1| \\
&\leq [r - (2r-4)(1-\eta)\beta_0(W) - (r-2)\beta_1(W)]|W|.
\end{aligned}$$

Now if $(2r-4)(1-\eta)\beta_0(W)+(r-2)\beta_1(W)>2+\eta$, then the above implies that $|\partial W|\le(r-2-\eta)|W|$, which has a small probability as mentioned earlier. From another viewpoint,

$$(r-2)|W^1|\le e(W,W^c)-|\partial W|. \tag{5.2.7}$$

By *II*, if $|W|=m$, then

$$e(W,W^c)-|\partial W|\le(1+2\eta)|W| \text{ with probability } \ge 1-\exp\left(-(1+2\eta)m\log(n/m)+\Delta_2 m\right),$$

and combining with *III* the expected number of subsets W with $e(W,W^c)-|\partial W|>(1+2\eta)|W|$ is exponentially small if $|W|\le\epsilon(\eta)n$. Therefore,

$$\text{with high probability } e(W,W^c)-|\partial W|\le(1+2\eta)|W| \text{ whenever } |W|\le\epsilon(\eta)n.$$

From (5.2.7), if $e(W,W^c)-|\partial W|\le(1+2\eta)|W|$, then $\beta_1(W)\le(1+2\eta)/(r-2)$.

Combining the last two observations, and noting that the maximum value of $\beta_0+\beta_1$ under the constraints (i) $2(1-\eta)\beta_0+\beta_1\le(2+\eta)/(r-2)$ and (ii) $\beta_1\le(1+2\eta)/(r-2)$ is achieved when both constraints are equalities, we see that with high probability

$$\beta_0+\beta_1\le\frac{1}{2(r-2)}+\frac{1+2\eta}{r-2}\le\frac{3}{2(r-2)}+\frac{2}{r-2}\eta\le\frac{3}{2(r-2)}+\eta$$

for $r\ge4$, and Proposition 5.1.6 is established.

5.3 Behavior of ξ_t starting from a small occupied set

In this section, we will use Proposition 5.1.5 to prove Theorem 5.1.2.

Proof of Theorem 5.1.2. If $p \in (0,1)$, we can choose $\eta > 0$ so that $(p+\eta)(1+\eta)$ equals any value between p and 1. To fix idea, we want to choose $\eta > 0$ so that $(p+\eta)(1+\eta) = (1+p)/2$. The roots of the quadratic equation $\eta^2 + (1+p)\eta + p = (1+p)/2$ are $\eta_\pm = (-(1+p) \pm \sqrt{3+p^2})/2$. Clearly $\eta_- < 0$. Since $p \in (0,1)$, $(1+p)^2 \leq 3+p^2$, which implies $(1+p) < \sqrt{3+p^2}$ and so $\eta_+ > 0$. We choose

$$\eta = \eta(p) := \frac{\sqrt{3+p^2} - (1+p)}{2} > 0 \text{ so that } (p+\eta)(1+\eta) = \frac{1+p}{2} < 1. \quad (5.3.1)$$

Next we take $\epsilon_2(p) := \epsilon_3(\eta(p))$, where $\epsilon_3(\cdot)$ is as in Proposition 5.1.5 and $\eta(\cdot)$ is as in (5.3.1). Since $\epsilon_3(\cdot)$ and $\eta(\cdot)$ are continuous, so is $\epsilon_2(\cdot)$. Also note that $\epsilon_3(\cdot)$ is increasing by Proposition 5.1.5, and

$$\frac{\partial \eta}{\partial p} = \frac{p}{2\sqrt{3+p^2}} - \frac{1}{2} < 0, \text{ as } p < \sqrt{3+p^2},$$

which implies that $\eta(\cdot)$ is decreasing. Combining these two observations, $\epsilon_2(\cdot)$ is decreasing. Having chosen ϵ_2, let

$$\mathcal{G} := \cap_{m=1}^{\lfloor \epsilon_2(p) n \rfloor} E_m^c, \text{ where } E_m = E(m, (1+\eta)m) \text{ is the event defined in Proposition 5.1.5.} \quad (5.3.2)$$

The argument for (i) consists of two steps.

Step 1: In the first step we show that for suitable choices of $C_{01} > 0$ and $b \in (0,1)$, if $|A| \leq \epsilon_2 n$, then the number of occupied vertices in the threshold-two contact process ξ_t^A reduces to n^b within time $C_{01} \log n$. The argument of this step goes through for any choice of $b \in (0,1)$. But for future benefits we will choose b with the following desirable property.

First note that using the inequality $(1+p) < \sqrt{3+p^2}$,

$$\frac{\sqrt{3+p^2}}{2} < \frac{3+p^2}{2(1+p)} = \frac{(1+p)^2 + 2(1-p)}{2(1+p)}, \text{ which implies } \eta < \frac{1-p}{1+p}.$$

By the last inequality,

$$1+\eta < \frac{2}{1+p}, \text{ so that } \frac{\log(1+\eta)}{\log(2/(1+p))} < 1 \text{ and } \frac{\log(2/(1+p)) - \log(1+\eta)}{\log(2/(1+p)) + \log(1+\eta)} \in (0,1). \tag{5.3.3}$$

The assertion in (5.3.3) suggests that we can choose

$$b = b(p) \in (0,1) \text{ small enough, so that } b+(b+1)\frac{\log(1+\eta)}{\log(2/(1+p))} < 1 \text{ and } b \le \eta^2/16r. \tag{5.3.4}$$

Having chosen b, let A be any subset of vertices with $|A| \le \epsilon_2 n$, and define

$$\begin{aligned}
\nu &:= \min\left\{t : \left|\xi_t^A\right| \le n^b\right\}, \\
J_t &:= \left\{\left|\xi_t^A\right| \le \left(\frac{1+p}{2}\right)\left|\xi_{t-1}^A\right|\right\}, N_t := \cap_{s=1}^t J_s \text{ for } t \ge 1, N_0 := \{|\xi_0^A| \le \epsilon_2 n\}, \\
L_t &:= \left\{\text{at most } (p+\eta)(1+\eta)\left|\xi_t^A\right| \text{ many vertices of } \left(\xi_t^A\right)^{*2} \text{ are occupied at time } t+1\right\}.
\end{aligned}$$

Now if L_t occur, then by the choice of η,

$$\left|\xi_{t+1}^A\right| \le (p+\eta)(1+\eta)\left|\xi_t^A\right| = \left(\frac{1+p}{2}\right)\left|\xi_t^A\right|. \text{ So } J_{t+1} \supset L_t. \tag{5.3.5}$$

By the definition of $(\xi_t^A)^{*2}$, each vertex of $(\xi_t^A)^{*2}$ will be in ξ_{t+1}^A with probability p, and for $G_n \in \mathcal{G}$, $|(\xi_t^A)^{*2}| \le (1+\eta)|\xi_t^A|$ on the event N_t. So using the binomial large deviations, see Lemma 2.3.3 on page 40 in [16], and the stochastic monotonicity property of the Binomial distribution,

$$\begin{aligned}
P_{G_n,p}(L_t^c \cap N_t|\xi_t^A) &\le P(Binomial((1+\eta)|\xi_t^A|, p) > (p+\eta)(1+\eta)|\xi_t^A|) \\
&\le \exp\left(-\Gamma((p+\eta)/p)p(1+\eta)\left|\xi_t^A\right|\right),
\end{aligned} \tag{5.3.6}$$

where $\Gamma(x) = x\log x - x + 1 > 0$ for $x \neq 1$. Since $|\xi_t^A| \ge n^b$ on $\{t < \nu\}$, we can replace $|\xi_t^A|$ in the right side of (5.3.6) by n^b to have

$$P_{G_n,p}(L_t^c \cap N_t \cap \{t<\nu\}) \le P_{G_n,p}\left(L_t^c \cap N_t \cap \left\{\left|\xi_t^A\right| \ge n^b\right\}\right) \le \exp(-\Gamma((p+\eta)/p)pn^b). \tag{5.3.7}$$

Combining (5.3.5) and (5.3.7) we get

$$\begin{aligned} P_{G_n,p}(J^c_{t+1} \cap N_t \cap \{t < \nu\}) &\leq P_{G_n,p}(L^c_t \cap N_t \cap \{t < \nu\}) \\ &\leq \exp(-\Gamma((p+\eta)/p)\; pn^b). \end{aligned} \tag{5.3.8}$$

We choose

$$C_{01}(p) := (1-b(p))/\log(2/(1+p)) \text{ to satisfy } \left(\frac{1+p}{2}\right)^{C_{01}\log n} n = n^b, \tag{5.3.9}$$

so that $N_{\lceil C_{01}\log n\rceil} \subset \{|\xi^A_{\lceil C_{01}\log n\rceil}| \leq [(1+p)/2]^{C_{01}\log n}|A| < n^b\}$. Hence $\{\nu > \lceil C_{01}\log n\rceil\} \subset N^c_{\lceil C_{01}\log n\rceil}$. Therefore, recalling the definition of N_t and noting that N^c_t is the disjoint union $\cup_{s=1}^t (J^c_s \cap N_{s-1})$,

$$\begin{aligned} P_{G_n,p}(\nu > \lceil C_{01}\log n\rceil) &= P_{G_n,p}\left(\{\nu > \lceil C_{01}\log n\rceil\} \cap N^c_{\lceil C_{01}\log n\rceil}\right) \\ &\leq P_{G_n,p}\left[\cup_{t=1}^{\lceil C_{01}\log n\rceil}(J^c_t \cap N_{t-1} \cap \{\nu > t-1\})\right] \\ &\leq \sum_{t=1}^{\lceil C_{01}\log n\rceil} P_{G_n,p}(J^c_t \cap N_{t-1} \cap \{\nu > t-1\}). \end{aligned}$$

Using (5.3.8) we can bound the summands of the above sum, and have

$$P_{G_n,p}(\nu > \lceil C_{01}\log n\rceil) \leq \lceil C_{01}\log n\rceil \exp(-\Gamma((p+\eta)/p)\, pn^b) \leq \exp(-\Gamma((p+\eta)/p)pn^b/2) \tag{5.3.10}$$

for large enough n.

Step2: Our next goal is to show that starting from any subset B of size $|B| \leq n^b$, the threshold-two contact process ξ^B_t dies out within time $C_{02}\log n$ for a suitable choice of $C_{02} > 0$. Note that we always have $|\xi^B_{t+1}| \leq |(\xi^B_t)^{*2}|$. In addition, for $G_n \in \mathcal{G}$ we have $|(\xi^B_t)^{*2}| \leq (1+\eta)|\xi^B_t|$ only when $|\xi^B_t| \leq \epsilon_2(p)n$. Keeping this in mind, we recall the choice of b from (5.3.4) and choose

$$C_{02}(p) := (b+1)/\log(2/(1+p)) \text{ to satisfy } b + C_{02}\log(1+\eta) < 1, \tag{5.3.11}$$

so that for $G_n \in \mathcal{G}$ and $t \leq C_{02} \log n$ and large enough n,

$$\left|\xi_t^B\right| \leq (1+\eta)\left|\xi_{t-1}^B\right| \leq \cdots \leq (1+\eta)^t \left|\xi_0^B\right| \leq (1+\eta)^t n^b \leq n^{b+C_{02}\log(1+\eta)} < \epsilon_2(p)n.$$

Now if $\mathcal{F}_t = \sigma\{\xi_s^B : 0 \leq s \leq t\}$, then

$$E_{G_n,p}\left(\left|\xi_{t+1}^B\right| \,\middle|\, \mathcal{F}_t\right) = p\left|\left(\xi_t^B\right)^{*2}\right|, \text{ and so} \tag{5.3.12}$$

$$\text{for } t \leq C_{02}\log n \text{ and } G_n \in \mathcal{G}, E_{G_n,p}\left(\left|\xi_{t+1}^B\right| \,\middle|\, \mathcal{F}_t\right) \leq p(1+\eta)|\xi_t^B|.$$

Iterating the above inequality,

$$E_{G_n,p}\left(\left|\xi_{\lceil C_{02}\log n\rceil}^B\right|\right) \leq [p(1+\eta)]^{C_{02}\log n}|\xi_0^B| \text{ for } G_n \in \mathcal{G}.$$

Now by the choices of η in (5.3.3), $p(1+\eta) < (1+p)/2$, and by the choice of C_{02} in (5.3.11), $[(1+p)/2]^{C_{02}\log n} = n^{-(1+b)}$. So

$$[p(1+\eta)]^{C_{02}\log n}|\xi_0^B| \leq \left(\frac{1+p}{2}\right)^{C_{02}\log n} n^b = 1/n.$$

Combining the last two inequalities,

$$E_{G_n,p}\left(\left|\xi_{\lceil C_{02}\log n\rceil}^B\right|\right) \leq \frac{1}{n} \text{ for } G_n \in \mathcal{G}.$$

Finally using Markov inequality,

$$P_{G_n,p}\left(\left|\xi_{\lceil C_{02}\log n\rceil}^B\right| \geq 1\right) \leq E_{G_n,p}\left(\left|\xi_{\lceil C_{02}\log n\rceil}^B\right|\right) \leq \frac{1}{n} \text{ for } G_n \in \mathcal{G}.$$

Combining with (5.3.10), and using the Markov property of the threshold-two contact process under the probability distribution $P_{G_n,p}$, we get the result in (i) for $C_0(p) := C_{01}(p) + C_{02}(p)$, where C_{01} is as in (5.3.9) and C_{02} is as in (5.3.11).

To show (ii) we use Proposition 5.1.5 and the fact from (5.1.1) that $\tilde{\mathbb{P}}(\cdot) \leq c\mathbb{P}(\cdot)$ to have

$$\tilde{\mathbb{P}}(\mathcal{G}^c) \leq \sum_{m=1}^{\lfloor\epsilon_2(p)n\rfloor} \tilde{\mathbb{P}}(E_m) \leq cC_3\left[\sum_{m=\lceil n^b\rceil}^{\lfloor\epsilon_2(p)n\rfloor} \exp\left(-\frac{\eta^2}{8r}m\log\frac{n}{m}\right) + \sum_{m=1}^{\lceil n^b\rceil-1} \exp\left(-\frac{\eta^2}{8r}m\log\frac{n}{m}\right)\right].$$

Noting that the function $\phi(\eta) = \eta \log(1/\eta)$ is increasing for $\eta \in (0, 1/e)$ (see (5.8.2)) and recalling that $\epsilon_2(p) \le 1/e$ by its definition, $m \log(n/m) = n\phi(m/n)$ is an increasing function of m for $m \le \epsilon_2(p)n$. So we can bound the summands of the last display by the first terms of the respective sums to have

$$\tilde{\mathbb{P}}(\mathcal{G}^c) \le cC_3 \left[(n - n^b) \exp\left(-\frac{\eta^2}{8r} n^b \log(n/n^b) \right) + n^b \exp(-(\eta^2/8r) \log n) \right] = o(1),$$

as $b \le \eta^2/16r$ by our choice in (5.3.4). $\square$

5.4 The critical value p_c

In this section, we show that the critical value p_c is in the interval $(0, 1)$. The fact that $p_c > 0$ follows as a consequence of Theorem 5.1.2.

Proof of Corollary 5.1.3. If $\mathcal{H}_t := \sigma\{\xi_s^1 : 0 \le s \le t\}$, then, as observed in (5.3.12), $E_{G_n,p}(|\xi_{t+1}^1| \, |\mathcal{H}_t) = p|(\xi_t^1)^{*2}| \le np$. So using Markov inequality,

$$\text{if } K_t := \{|\xi_t^1| \ge 3np/2\}, \text{ then } P_{G_n,p}(K_{t+1}|\mathcal{H}_t) \le \frac{2}{3}.$$

Using properties of the conditional expectation,

$$E_{G_n,p}\left(\mathbf{1}_{\cap_{s=1}^{t+1} K_s} \middle| \mathcal{H}_t\right) = \mathbf{1}_{\cap_{s=1}^{t} K_s} E_{G_n,p}(\mathbf{1}_{K_{t+1}}|\mathcal{H}_t) \le \frac{2}{3} \mathbf{1}_{\cap_{s=1}^{t} K_s},$$

so that $E_{G_n,p}\mathbf{1}_{\cap_{s=1}^{t+1} K_s} \le \frac{2}{3} E_{G_n,p}\mathbf{1}_{\cap_{s=1}^{t} K_s}$. Iterating the last inequality,

$$P_{G_n,p}(\cap_{s=1}^{\lfloor \log n \rfloor} K_s) \le (2/3)^{\lfloor \log n \rfloor} \le (3/2) n^{-\log(3/2)}. \tag{5.4.1}$$

Now since $\epsilon_2 : (0, 1) \mapsto (0, 1)$ is decreasing and continuous, by intermediate value theorem there is a unique $p_0 \in (0, 2/3)$ such that $\epsilon_2(p_0) = 3p_0/2$ and for $p \in [0, p_0)$, $\epsilon_2(p) > 3p/2$. So if $p \in [0, p_0)$, then (5.4.1) suggests that $|\xi_s^1|/n$ drops below

$\epsilon_2(p)$ for some $s \le \log n$ with $P_{G_n,p}$-probability $\ge 1-(3/2)n^{-\log(3/2)}$. Combining this with (i) of Theorem 5.1.2, noting that $\lfloor \log n \rfloor + \lceil C_0(p)\log n \rceil \le \lceil (C_0(p)+1)\log n \rceil$, and using Markov property of $P_{G_n,p}$, we have

$$\sup_{G_n\in\mathcal{G}} P_{G_n,p}\left(\xi^1_{\lceil (C_0(p)+1)\log n\rceil} \neq \emptyset\right) = o(1) \text{ for } p\in[0,p_0) \text{ and } G_n\in\mathcal{G}.$$

This together with (ii) of Theorem 5.1.2 proves the desired result. □

Now we show that $p_c < 1$ using Proposition 5.1.6.

Proof of Theorem 5.1.1. Given $\eta \in (0,1/4)$ let $\epsilon_4(\eta)$ be the constant in Proposition 5.1.6 and take $\epsilon_1 := \epsilon_4$. Since $r \ge 4$ and $\eta < 1/4$, $3/(2r-4) \le 3/4 < 1-\eta$ so that the fraction in (5.1.4) is < 1. For p between this fraction and 1, we can choose $\delta = \delta(\eta,p) > 0$ such that

$$(p-\delta)\left(1-\left(\frac{3}{2r-4}+\eta\right)\epsilon_1\right) > 1-\epsilon_1. \tag{5.4.2}$$

For $t=0,1,\dots$ if $|\xi^1_t| \le \lfloor (1-\epsilon_1/2)n\rfloor$, then let $U_t = \xi^1_t$, and if $|\xi^1_t| > \lfloor (1-\epsilon_1/2)n\rfloor$, we have too many vertices to use Proposition 5.1.6, so we let U_t be the subset of ξ^1_t consisting of $\lfloor (1-\epsilon_1/2)n\rfloor$ many vertices with smallest indices. Thus $|U^c_t| \ge \epsilon_1 n/2$ for any $t \ge 0$. We begin with some notations. For $t \ge 0$ let

$$\begin{aligned}
I_t &:= \left\{|\xi^1_t| \ge (1-\epsilon_1)n\right\}, \quad O_t := \cap_{s=0}^t I_s,\\
S_t &:= \left\{\left|U^{*2}_t\right| \ge n - \left(\frac{3}{2r-4}+\eta\right)|U^c_t|\right\},\\
T_t &:= \left\{\text{ at least } (p-\delta)\left|U^{*2}_t\right| \text{ many vertices of } U^{*2}_t \text{ are occupied at time } t+1\right\}.
\end{aligned}$$

On the event $S_t \cap T_t$, $|\xi^1_{t+1}| \ge (p-\delta)|U^{*2}_t| \ge (p-\delta)[n-(3/(2r-4)+\eta)|U^c_t|]$, and on the event O_t, $|\xi^1_t| \ge (1-\epsilon_1)n$ so that $|U_t| = \min\{|\xi^1_t|, \lfloor (1-\epsilon_1/2)n\rfloor\} \ge (1-\epsilon_1)n$

and hence $|U_t^c| \le \epsilon_1 n$. Therefore, using (5.4.2) it is easy to see that on the event $S_t \cap T_t \cap O_t$,

$$\left|\xi_{t+1}^1\right| \ge (p-\delta)\left(1 - \left(\frac{3}{2r-4} + \eta\right)\epsilon_1\right) n > (1-\epsilon_1)n.$$

So $I_{t+1} \cap O_t \supset S_t \cap T_t \cap O_t$ for any $t \ge 0$. Next we see that if we take $F_t := F(|U_t^c|, (3/(2r-4)+\eta)|U_t^c|)$, where $F(\cdot,\cdot)$ is defined in Proposition 5.1.6, then $P_{G_n,p}(S_t|U_t) \ge \mathbf{1}_{F_t^c}$, since $|(U_t^{*2})^c| \le (3/(2r-4)+\eta)|U_t^c|$ on the event S_t. Taking expectation with respect to the distribution of G_n, $\mathbf{P}_p(S_t|U_t) \ge \tilde{\mathbb{P}}(F_t^c)$. As noted above, $|U_t^c| \le \epsilon_1 n$ on the event O_t. So, recalling from (5.1.1) that $\tilde{\mathbb{P}}(\cdot) \le c\mathbb{P}(\cdot)$, we can apply Proposition 5.1.6 with $m = |U_t^c|$ to have

$$\mathbf{P}_p(S_t^c \cap O_t|U_t) \le \mathbf{P}_p(S_t^c \cap \{|U_t^c| \le \epsilon_1 n\}|U_t) \le cC_4 \exp(-(\eta/8)|U_t^c| \log(n/|U_t^c|)).$$

Since $\epsilon_1 = \epsilon_4 \le 1/e$ by (5.7.10), combining the facts that the function $\phi(\eta) = \eta\log(1/\eta)$ is increasing on $(0, 1/e)$ (see (5.8.2)) and $|U_t^c|$ is always $\ge \epsilon_1 n/2$ by its definition, we have $\phi(|U_t^c|/n) > \phi(\epsilon_1/2)$ or equivalently $|U_t^c|\log(n/|U_t^c|) \ge (\epsilon_1/2)n\log(2/\epsilon_1)$ on the event O_t. Keeping this in mind, we can increase the upper bound in the last display to have

$$\mathbf{P}_p(S_t^c \cap O_t) \le \mathbf{P}_p\left(S_t^c \cap \{\epsilon_1 n/2 \le |U_t^c| \le \epsilon_1 n\}\right) \le cC_4 \exp\left(-\frac{\eta}{8}\frac{\epsilon_1}{2}\log(2/\epsilon_1)n\right). \tag{5.4.3}$$

On the other hand, using the binomial large deviation, see Lemma 2.3.3 on page 40 in [16],

$$P_{G_n,p}(T_t \mid U_t^{*2}) \ge 1 - \exp\left(-\Gamma((p-\delta)/p)p\left|U_t^{*2}\right|\right), \tag{5.4.4}$$

where $\Gamma(x) = x\log x - x + 1 > 0$ for $x \ne 1$. As noted earlier in the proof, on the event O_t, $|\xi_t^1| \ge (1-\epsilon_1)n$ so that $|U_t| = \min\{|\xi_t^1|, \lfloor(1-\epsilon_1/2)n\rfloor\} \ge (1-\epsilon_1)n$. Therefore, on the event $S_t \cap O_t$, $|U_t^{*2}| \ge [1-(3/(2r-4)+\eta)\epsilon_1]n$. Keeping this in mind, we can replace $|U_t^{*2}|$ in the right hand side of (5.4.4) by $[1-(3/(2r-4)+\eta)\epsilon_1]n$ to have

$$\begin{aligned} P_{G_n,p}(T_t^c \cap S_t \cap O_t) &\leq P_{G_n,p}(T_t^c \cap \{|U_t^{*2}| \geq [1-(3/(2r-4)+\eta)\epsilon_1]n\}) \\ &\leq \exp\left(-\Gamma((p-\delta)/p)p\left\{1-\left(\frac{3}{2r-4}+\eta\right)\epsilon_1\right\}n\right) \end{aligned} \quad (5.4.5)$$

The same bound also works for the unconditional probability distribution $\mathbf{P}_p$. Combining these two bounds of (5.4.3) and (5.4.5), and recalling that $I_{t+1}\cap O_t \supset S_t \cap T_t \cap O_t$,

$$\mathbf{P}_p(I_{t+1}^c \cap O_t) \leq \mathbf{P}_p((S_t \cap T_t)^c \cap O_t) \leq \mathbf{P}_p(S_t^c \cap O_t) + \mathbf{P}_p(T_t^c \cap S_t \cap O_t) \leq C_1 \exp(-2c_1(\eta,p)n),$$

where $C_1 = 2\max\{1, cC_4\}$ and

$$c_1(\eta,p) = \frac{1}{2}\min\left\{\frac{\eta\epsilon_1}{16}\log(2/\epsilon_1), \Gamma((p-\delta)/p)p\left(1-\frac{3\epsilon_1}{2r-4}-\eta\epsilon_1\right)\right\}.$$

Hence for $\tau = \exp(c_1(\eta,p)n)$, we use the above estimate of $\mathbf{P}_p(I_{t+1}^c \cap O_t)$ and the relation between O_t and I_t to have

$$\begin{aligned} \mathbf{P}_p\left(\inf_{t\leq\tau}|\xi_t^1| < (1-\epsilon_1)n\right) &= \mathbf{P}_p\left(\cup_{t=1}^{\lfloor\tau\rfloor} I_t^c\right) \\ &= \sum_{t=0}^{\lfloor\tau\rfloor-1} \mathbf{P}_p(I_{t+1}^c \cap O_t) \leq C_1\tau\ \exp(-2c_1(\eta,p)n) = C_1\exp(-c_1(\eta,p)n), \end{aligned}$$

and we get the desired result. $\square$

5.5 First order phase transition at p_c

In this section, we use Theorem 5.1.1 and 5.1.2 to prove Theorem 5.1.4.

Proof of Theorem 5.1.4. First we estimate the probability $\mathbf{P}_p(\rho_n \geq \epsilon_2(p))$ for $p \in (p_c, 1)$. Let $\sigma_1 = \lceil \exp(n^{1/2}) \rceil$ and $\sigma_2(p) = \lceil C_0(p)\log n \rceil$, where $C_0(p)$ is as in Theorem 5.1.2. Depending on the fate of the process ξ_t^1 at time $\sigma_1+\sigma_2$ and whether $G_n \in \mathcal{G}$ or not, where $\mathcal{G}$ is defined in Theorem 5.1.2, we have

$$\begin{aligned}
&\mathbf{P}_p(\rho_n < \epsilon_2(p)) = \mathbf{P}_p(|\xi_{\sigma_1}^1| < \epsilon_2(p)n) \\
\le\ & \mathbf{P}_p(\xi_{\sigma_1+\sigma_2}^1 = \emptyset) + \tilde{\mathbb{E}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p)n\ ,\ \xi_{\sigma_1+\sigma_2}^1 \ne \emptyset) \\
\le\ & \mathbf{P}_p(\xi_{\sigma_1+\sigma_2}^1 = \emptyset) + \tilde{\mathbb{E}} \mathbf{1}_{\mathcal{G}^c} + \tilde{\mathbb{E}}\left[\mathbf{1}_{\mathcal{G}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p)n\ ,\ \xi_{\sigma_1+\sigma_2}^1 \ne \emptyset)\right] \quad (5.5.1)
\end{aligned}$$

By the definition of p_c in (5.1.3), the first term in the right side of (5.5.1) is $o(1)$ for $p \in (p_c, 1)$. By the estimate in (ii) of Theorem 5.1.2, the second term is also $o(1)$. To bound the third term in (5.5.1) we use Markov property of $P_{G_n,p}$ and the estimate in (i) of Theorem 5.1.2 to have

$$\begin{aligned}
\mathbf{1}_{\mathcal{G}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p)\ ,\ \xi_{\sigma_1+\sigma_2}^1 \ne \emptyset) &= \sum_{A:|A|<\epsilon_2(p)n} P_{G_m,p}(\xi_{\sigma_1}^1 = A)\mathbf{1}_{\mathcal{G}} P_{G_n,p}(\xi_{\sigma_2}^A \ne \emptyset) \\
&\le o(1) \sum_{A:|A|<\epsilon_2(p)n} P_{G_m,p}(\xi_{\sigma_1}^1 = A).
\end{aligned}$$

Combining the last three observations,

$$\mathbf{P}_p(\rho_n < \epsilon_2(p)) \le o(1) + o(1) + o(1) \sum_{A:|A|<n\epsilon_2(p)} \tilde{\mathbb{E}} P_{G_n,p}(\xi_{\sigma_1}^1 = A) = o(1).$$

Since $p_c < 1$ by Theorem 5.1.1 and $\epsilon_2(p) > 0$ for $p \in (0,1)$ and $\epsilon_2(\cdot)$ is a decreasing continuous function by Theorem 5.1.2, $\epsilon_2(p_c) > 0$ and for any $\delta \in (0, \epsilon_2(p_c))$, there exists $p' > p_c$ such that $\epsilon_2(p') > \epsilon_2(p_c) - \delta$.

Therefore, using the fact that $\epsilon_2(\cdot)$ is a decreasing function and the stochastic monotonicity of the probability distributions $\mathbf{P}_p, p \in [0,1]$, which is discussed in the introduction before Theorem 5.1.4, for any $p \in (p_c, 1]$

$$\begin{aligned}
\mathbf{P}_p(\rho_n > \epsilon_2(p_c) - \delta) &\ge \mathbf{P}_p(\rho_n > \epsilon_2(p')) \\
&\ge \mathbf{P}_{p \wedge p'}(\rho_n \ge \epsilon_2(p \wedge p')) = 1 - o(1),
\end{aligned}$$

where $p \wedge p' = \min\{p, p'\} > p_c$. So letting $n \to \infty$ the desired result follows. $\square$

5.6 Proof of the first isoperimetric inequality

In this section, we present the proof of the isoperimetric inequality in Proposition 5.1.5.

Proof of Proposition 5.1.5. In view of (5.2.2), it suffices to estimate the probability $\mathbb{P}[H(m,(r-1-\eta)m)$, where $H(m,k) = \{\exists U \subset V_n : |U| = m, |U^{*1}| \leq k\}$. Recall the definitions of U_0 and U_1 from (5.2.1). We need some more notations to proceed. Given $\eta > 0$ define the following events for a subset $U \subset V_n$.

$$\begin{aligned} &A_U := \{|U_1| \geq (\eta/2r)|U|\}, \qquad B_U := \{|U^{*1}| \leq (r-1-\eta)|U|\}, \\ &D_U := \{e(U,U^c) \leq (r-2-\eta)|U|\}. \end{aligned} \tag{5.6.1}$$

There are three steps in the proof.

Step 1: Our first step is to estimate the probability that there is a subset U of vertices of size m for which $B_U \cap A_U^c$ occurs. On the event A_U^c, $|U_0| > (1-\eta/2r)|U|$ and so $e(U,U^c) \geq r|U_0| \geq (r-\eta/2)|U|$. Also on the event B_U, $|\partial U| \leq |U^{*1}| \leq (r-1-\eta)|U|$. From these two observations we have

$$\begin{aligned} \mathbb{P}(B_U \cap A_U^c) &\leq \mathbb{P}(\{|\partial U| \leq (r-1-\eta)|U|\} \cap \{e(U,U^c) \geq (r-\eta/2)|U|\}) \\ &\leq \mathbb{P}(e(U,U^c) - |\partial U| \geq (1+\eta/2)|U|). \end{aligned} \tag{5.6.2}$$

Combining (5.6.2) with the bound in (ii) of Lemma 5.8.4,

$$\text{if } |U| = m \leq \epsilon_5 n, \text{ then } \mathbb{P}(B_U \cap A_U^c) \leq \exp\left[-(1+\eta/2)m\log(n/m) + \Delta_2 m\right]. \tag{5.6.3}$$

Suppose

$$F_1 := \cup_{\{U \subset V_n : |U| = m\}} \left(B_U \cap A_U^c\right).$$

Using (5.6.3) and the inequality in Lemma 5.8.1,

$$\begin{aligned}\text{if } m \le \epsilon_5 n, \text{ then } \mathbb{P}(F_1) &\le \binom{n}{m} \exp\left[-(1+\eta/2)m\log(n/m) + \Delta_2 m\right] \\ &\le \exp\left[-(\eta/2)m\log(n/m) + (1+\Delta_2)m\right]. \qquad (5.6.4)\end{aligned}$$

If m is small enough, then the above estimate is exponentially small, and so with high probability there is no subset U of size m for which $B_U \cap A_U^c$ occurs.

Step 2: Our next step is to estimate the probability that there is a subset U of vertices for which A_U occurs and $e(U_1, U_1^c) \le (r-2-\eta)|U_1|$. If A_U occurs for some subset U of size m, then $|U_1| \in [\eta m/2r, m]$. So we consider all possible subsets having size in that range, and let

$$F_2 := \cup_{\{W:(\eta/2r)m \le |W| \le m\}} D_W.$$

Then using Lemma 5.8.2 with $\alpha = 1 - (2+\eta)/r$ and the inequality in Lemma 5.8.1,

$$\begin{aligned}\mathbb{P}(F_2) &= \mathbb{P}\left(\cup_{m' \in [\eta m/2r, m]} \cup_{\{W:|W|=m'\}} \{e(W,W^c) \le (r-2-\eta)m'\}\right) \\ &\le \sum_{m' \in [\eta m/2r, m]} \binom{n}{m'} C_5 \exp\left[-\left(\frac{2+\eta}{2}\right) m' \log(n/m') + \Delta_1 m'\right] \\ &\le \sum_{m' \in [\eta m/2r, m]} C_5 \exp\left(-(\eta/2)m'\log(n/m') + (1+\Delta_1)m'\right). \qquad (5.6.5)\end{aligned}$$

Noting that the function $\phi(\eta) = \eta\log(1/\eta)$ is increasing on $(0, 1/e)$ (see (5.8.2)), if $m \le n/e$, then for $m' \in [\eta m/2r, m]$, $m'\log(n/m') \ge (\eta m/2r)\log(2rn/\eta m)$. Using this inequality and the fact that $(\eta/2r)\log(2r/\eta) > 0$, we can bound each summand in (5.6.5) by $C_5\exp(-(\eta/2)(\eta/2r)m\log(n/m) + (1+\Delta_1)m)$. As there are fewer than m terms in the sum over m' in (5.6.5), if we use the inequality

$m \leq e^m$ for $m \geq 0$, and

$$\text{if } m \leq n/e, \text{ then } \mathbb{P}(F_2) \leq C_5 \exp\left(-(\eta/2)(\eta/2r)m\log(n/m) + (2+\Delta_1)m\right). \tag{5.6.6}$$

If m is small enough, then the right-hand side of (5.6.6) is exponentially small, and so with high probability there is no subset U of size m for which A_U occurs and $e(U_1, U_1^c) \leq (r-2-\eta)|U_1|$.

Step 3: Our final step is to estimate the probability that there is a subset U of size m for which B_U occurs assuming F_1 and F_2 do not occur. Noting that U^{*1} is a disjoint union of U_1 and ∂U, and $|U| = |U_0| + |U_1|$, a little arithmetic gives

$$\begin{aligned} |U^{*1}| =& |U_1| + |\partial U| \\ =& (r-1-\eta)|U| + |\partial U| - (r-2-\eta)|U_1| - (r-1-\eta)|U_0|. \end{aligned}$$

Letting

$$\Delta(U) = |\partial U| - (r-2-\eta)|U_1| - (r-1-\eta)|U_0|, \tag{5.6.7}$$

we see that if B_U occurs, then $\Delta(U)$ has to be negative. Also if $|U| = m$, then by the definition of F_1, $B_U \cap F_1^c \subset B_U \cap A_U$, and on the event $A_U \cap F_2^c$, $|U_1| \geq (\eta/2r)|U|$ and so $e(U_1, U_1^c) > (r-2-\eta)|U_1|$. Combining these two observations,

$$\mathbb{P}(B_U \cap F_1^c \cap F_2^c) \leq \mathbb{P}(B_U \cap A_U \cap F_2^c) \leq \mathbb{P}(\{\Delta(U) \leq 0\} \cap \{e(U_1, U_1^c) > (r-2-\eta)|U_1|\}). \tag{5.6.8}$$

Now by the definitions of U_0 and U_1,

$$\begin{aligned} &e(U_0, U_0^c) = e(U_0, U^c) = r|U_0| \text{ and } e(U_1, U_1^c) = e(U_1, U^c), \text{ so that} \\ &e(U, U^c) = e(U_0, U^c) + e(U_1, U^c) = r|U_0| + e(U_1, U_1^c), \end{aligned} \tag{5.6.9}$$

and a little algebra shows that $\{\Delta(U) \leq 0\} = \{e(U, U^c) - |\partial U| \geq (1+\eta)|U_0| + e(U_1, U_1^c) - (r-2-\eta)|U_1|\}$. Also $e(U_1, U_1^c) < r|U_1|$. So

$$\begin{aligned} &\mathbb{P}(\{\Delta(U) \leq 0\} \cap \{e(U_1, U_1^c) > (r-2-\eta)|U_1|\}) \qquad (5.6.10) \\ &= \sum_{\gamma \in (0, 2+\eta)} \mathbb{P}\left(\{e(U_1, U_1^c) = (r-2-\eta+\gamma)|U_1|\} \cap \{e(U, U^c) - |\partial U| \geq (1+\eta)|U_0| + \gamma|U_1|\}\right). \end{aligned}$$

Combining (5.6.8) and (5.6.10), and recalling that $|U_1| \in [\eta m/2r, m]$,

if we write $R = r - 2 - \eta$,

and if $r(\gamma, k) := \mathbb{P}(e(U_1, U_1^c) = (R+\gamma)|U_1|, |U_1| = k)$ and

$s(\gamma, k) := \mathbb{P}\left(e(U, U^c) - |\partial U| \geq (1+\eta)|U_0| + \gamma|U_1| \ \mid e(U_1, U_1^c) = (R+\gamma)|U_1|, |U_1| = k\right)$,

then $\mathbb{P}(B_U \cap F_1^c \cap F_2^c) = \displaystyle\sum_{\gamma \in (0, 2+\eta)} \sum_{k \in [\eta m/2r, m]} r(\gamma, k)\, s(\gamma, k).$ (5.6.11)

In view of (5.6.9), if $L = (R+\gamma)k + r(m-k)$, then $\{e(U_1, U_1^c) = (R+\gamma)|U_1|\} \cap \{|U_1| = k\} = \{e(U, U^c) = L\} \cap \{|U_1| = k\}$. So

$$s(\gamma, k) = \mathbb{P}(e(U, U^c) - |\partial U| \geq \gamma k + (1+\eta)(m-k) \mid e(U, U^c) = L, |U_1| = k).$$

Since under the conditional distribution $\mathbb{P}(\cdot | e(U, U^c) = L)$ all the size-L subsets of half-edges corresponding to U^c are equally likely to be paired with those corresponding to U, the conditional distribution of $e(U, U^c) - |\partial U|$ given $e(U, U^c)$ and $|U_1|$ does not depend on $|U_1|$. So we can drop the event $\{|U_1| = k\}$ from the last display and use (i) of Lemma 5.8.4 with η replaced by $(\gamma k + (1+\eta)(m-k))/m$ to have

$$s(\gamma, k) \leq \exp\left(-\{\gamma k + (1+\eta)(m-k)\} \log(n/m) + \Delta_2 m\right), \text{ when } m \leq \epsilon_5 n. \tag{5.6.12}$$

In order to estimate $r(\gamma, k)$, we again use (5.6.9) and recall that $R = (r-2-\eta)$

to have

$$\begin{aligned} r(\gamma,k) &= \mathbb{P}(e(U_1,U_1^c)=(R+\gamma)k, |U_1|=k) \\ &= \mathbb{P}(e(U,U^c)=(R+\gamma)k+r(m-k), |U_1|=k) \\ &\leq \mathbb{P}(e(U,U^c)=rm-(2+\eta-\gamma)k), \end{aligned}$$

Using Lemma 5.8.2 with $\alpha = 1-(2+\eta-\gamma)k/rm$,

$$r(\gamma,k) \leq C_5 \exp\left(-\frac{2+\eta-\gamma}{2} k\log(n/m) + \Delta_1 m\right). \tag{5.6.13}$$

Combining (5.6.11), (5.6.12) and (5.6.13), if $m \leq \epsilon_5 n$, then

$$\begin{aligned} &\mathbb{P}(B_U \cap F_1^c \cap F_2^c) \\ \leq &\sum_{\gamma\in(0,2+\eta)} \sum_{k\in[\eta m/2r,m]} C_5 \exp\left[-\left\{\left(\frac{2+\eta+\gamma}{2}\right)k + (1+\eta)(m-k)\right\}\log(n/m) + (\Delta_1+\Delta_2)m\right]. \end{aligned}$$

Noting that there are fewer than rm terms in the sum over γ and at most m terms in the sum over k, and using the inequality $m^2 \leq e^m$ for $m \geq 0$, the above is

$$\begin{aligned} &\leq C_5 rm^2 \exp\left[-(1+\eta/2)m\log(n/m) + (\Delta_1+\Delta_2)m\right] \\ &\leq C_5 r \exp\left[-(1+\eta/2)m\log(n/m) + (1+\Delta_1+\Delta_2)m\right]. \end{aligned} \tag{5.6.14}$$

Recalling the definition of $H(m,(r-1-\eta)m)$ and considering whether the events $F_i, i=1,2$, occur or not,

$$\begin{aligned} \mathbb{P}(H(m,(r-1-\eta)m)) &= \mathbb{P}\left(\cup_{\{U:|U|=m\}} B_U\right) \\ &\leq \mathbb{P}(F_1)+\mathbb{P}(F_2)+\mathbb{P}\left(\cup_{\{U:|U|=m\}}(B_U\cap F_1^c\cap F_2^c)\right) \\ &\leq \mathbb{P}(F_1)+\mathbb{P}(F_2)+\sum_{\{U:|U|=m\}} \mathbb{P}(B_U\cap F_1^c\cap F_2^c). \end{aligned}$$

Combining (5.6.4), (5.6.6) and (5.6.14), and using the inequality in Lemma 5.8.1 to estimate the number of terms in the sum, if $m \leq \min\{1/e, \epsilon_5(\eta)\}n$, then

$$\mathbb{P}(H(m,(r-1-\eta)m))$$

$$\begin{aligned}
&\leq \ \mathbb{P}(F_1) + \mathbb{P}(F_2) + \binom{n}{m} C_5 r \exp\left[-(1+\eta/2)m\log(n/m) + (1+\Delta_1+\Delta_2)m\right] \\
&\leq \ \mathbb{P}(F_1) + \mathbb{P}(F_2) + C_5 r \exp\left[-(\eta/2)m\log(n/m) + (2+\Delta_1+\Delta_2)m\right] \\
&\leq \ C_3 \exp[-(\eta^2/4r)m\log(n/m) + (2+\Delta_1+\Delta_2)m], \qquad (5.6.15)
\end{aligned}$$

where $C_3 = 3\max\{1, C_5 r\}$. To clean up the result to have the one given in Proposition 5.1.5, choose ϵ_3' such that

$$(\eta^2/4r)\log(1/\epsilon_3')/2 = 2 + \Delta_1 + \Delta_2, \quad \text{and} \quad \epsilon_3(\eta) := \min\{1/e, \epsilon_5(\eta), \epsilon_3'(\eta)\}, \qquad (5.6.16)$$

where ϵ_5 is defined in (5.8.8). So for any $m \leq \epsilon_3 n$, the estimate in (5.6.15) holds, and

$$(\eta^2/4r)\log(n/m)/2 \geq (\eta^2/4r)\log(1/\epsilon_3')/2 = 2 + \Delta_1 + \Delta_2,$$

which gives the desired estimate for the probability in (5.2.3), and thereby, in view of (5.2.2), provides the required bound for the probability in Proposition 5.1.5.

To finish the proof of Proposition 5.1.5 it remains to check that $\epsilon_3(\cdot)$ is increasing. By the definition of $\epsilon_5(\cdot)$ in (5.8.8) and the properties of $\beta(\cdot,\cdot)$ in Lemma 5.8.3, $\epsilon_5(\cdot)$ is increasing. Also by the definition of ϵ_3' in (5.6.16), $\log(1/\epsilon_3'(\cdot))$ is decreasing and hence $\epsilon_3'(\cdot)$ is increasing. Since minimum of increasing functions is still increasing, we conclude from (5.6.16) that $\epsilon_3(\cdot)$ is increasing. □

5.7 Proof of the second isoperimetric inequality

In this section, we present the proof of the isoperimetric inequality in Proposition 5.1.6.

Proof of Proposition 5.1.6. Recall the definitions of W^i and $\beta_i(W)$ from (5.2.6). We need some more notations to proceed. Given $\eta > 0$, let

$$Q_W := \left\{ \beta_0(W) + \beta_1(W) > \frac{3}{2(r-2)} + \eta \right\}, \quad R_W := \left\{ \beta_1(W) > \frac{1+2\eta}{r-2} \right\}.$$

We divide the argument into three steps.

Step 1: Our first step is to estimate the probability that there is a subset $W \subset V_n$ of size m for which R_W occurs. Since each $x \in \partial W$ has $e(\{x\}, W) \geq 1$ and each $x \in W^1$ has $e(\{x\}, W) \geq r-1$,

$$e(W, W^c) \geq (r-1)|W^1| + (|\partial W| - |W^1|) = (r-2)|W^1| + |\partial W|, \tag{5.7.1}$$

and so $R_W \subset \{e(W, W^c) - |\partial W| \geq (1+2\eta)|W|\}$. Therefore, using (ii) of Lemma 5.8.4

$$\text{if } |W| = m \leq \epsilon_5 n, \text{ then } \mathbb{P}(R_W) \leq \exp[-(1+2\eta)m\log(n/m) + \Delta_2 m]. \tag{5.7.2}$$

Now if

$$M_1 := \cup_{\{W:|W|=m\}} R_W,$$

then using (5.7.2) and the inequality in Lemma 5.8.1,

$$\begin{aligned} \text{if } m \leq \epsilon_5 n, \text{ then } \mathbb{P}(M_1) &\leq \binom{n}{m} \exp[-(1+2\eta)m\log(n/m) + \Delta_2 m] \\ &\leq \exp[-2\eta m\log(n/m) + (1+\Delta_2)m]. \end{aligned} \tag{5.7.3}$$

If m is small enough, the above estimate is exponentially small, which implies that with high probability there is no subset W of size m for which R_W occurs.

Step 2: Our next step is to estimate the probability that there is a subset $W \subset V_n$ for which $Q_W \cap R_W^c$ occurs and $e(W^0, (W^0)^c) \leq (r-2-(2r-4)\eta)|W^0|$. If $Q_W \cap R_W^c$

occurs for some subset W of size m, then a little algebra shows that for $r \geq 4$,

$$\beta_0(W) \geq \frac{3}{2(r-2)} + \eta - \frac{1+2\eta}{r-2} \geq \frac{1}{2(r-2)},$$

and so $|W^0| \in [m/(2r-4), m]$. For this reason we consider all possible subsets having size in that range and let

$$M_2 := \cup_{\{U:|U|\in[m/(2r-4),m]\}}\{e(U,U^c) \leq (r-2-(2r-4)\eta)|U|\}.$$

Applying Lemma 5.8.2, using the inequality in Lemma 5.8.1, and then using an argument similar to the one leading to (5.6.6),

if $m \leq n/e$, then

$$\begin{aligned} \mathbb{P}(M_2) &= \mathbb{P}\left(\cup_{m'\in[m/(2r-4),m]} \cup_{\{U:|U|=m'\}} \{e(U,U^c) \leq (r-2-(2r-4)\eta)|U|\}\right) \\ &\leq \sum_{m'\in[m/(2r-4),m]} \binom{n}{m'} C_5 \exp\left[-\frac{2+(2r-4)\eta}{2} m' \log(n/m') + \Delta_1 m'\right] \\ &\leq C_5 \exp\left[-\frac{\eta}{2} m \log(n/m) + (2+\Delta_1)m\right]. \end{aligned} \tag{5.7.4}$$

If m is small enough, then the right hand side of (5.7.4) is exponentially small, and so with high probability there is no subset W of size m for which $Q_W \cap R_W^c$ occurs, and $e(W^0,(W^0)^c) \leq (r-2-(2r-4)\eta)|W^0|$.

Step 3: Our final step is to estimate the probability that there is a subset $W \subset V_n$ for which Q_W occurs assuming M_1 and M_2 do not occur. If $|W| = m$, then by the definition of M_1, $Q_W \cap M_1^c \subset Q_W \cap R_W^c$. On the event $Q_W \cap R_W^c \cap M_2^c$, $|W^0| \in [m/(2r-4), m]$ and so $e(W^0,(W^0)^c) > (r-2-(2r-4)\eta)|W^0|$. Also by the definition of W^0, $e(W^0, W^c) \leq |W^0|$. Combining these three observations with the fact that $W^0 \subset W$, on the event $Q_W \cap M_1^c \cap M_2^c$,

$$e(W^0, W \setminus W^0) = e(W^0,(W^0)^c) - e(W^0,W^c)$$

$$\geq (r-2-(2r-4)\eta)|W^0| - |W^0|$$
$$= (r-3-(2r-4)\eta)\beta_0(W)|W|. \quad (5.7.5)$$

Next we see that W is a disjoint union of W^0 and $W \setminus W^0$, and $(W \setminus W^0)^c$ is a disjoint union of W^0 and W^c. So

$$\begin{aligned} e(W, W^c) &= e(W \setminus W^0, W^c) + e(W^0, W^c) \\ &= e(W \setminus W^0, (W \setminus W^0)^c) - e(W \setminus W^0, W^0) + e(W^0, W^c). \quad (5.7.6) \end{aligned}$$

Combining the inequalities in (5.7.5) and (5.7.6), recalling that $e(W \setminus W^0, (W \setminus W^0)^c) \leq r|W \setminus W^0|$, and again using the inequality $e(W^0, W^c) \leq |W^0|$, we see that on the event $Q_W \cap M_1^c \cap M_2^c$,

$$\begin{aligned} e(W, W^c) &\leq r|W \setminus W^0| - e(W^0, W \setminus W^0) + e(W^0, W^c) \\ &\leq [r - (2r-4)(1-\eta)\beta_0(W)]|W|. \end{aligned}$$

Therefore by (5.7.1),

$$\begin{aligned} |\partial W| &\leq e(W, W^c) - (r-2)|W^1| \\ &\leq [r - (2r-4)(1-\eta)\beta_0(W) - (r-2)\beta_1(W)]|W|. \quad (5.7.7) \end{aligned}$$

Now we show that $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) > 2+\eta$ on the event $Q_W \cap M_1^c \cap M_2^c$. By the definition of M_1, $\beta_1(W) \leq (1+2\eta)/(r-2)$ on the event $Q_W \cap M_1^c \cap M_2^c$. So if $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) \leq 2+\eta$ on the same event, then noting that the maximum value of $\beta_0 + \beta_1$ under the constraints (i) $(2r-4)(1-\eta)\beta_0 + (r-2)\beta_1 \leq 2+\eta$ and (ii) $\beta_1 \leq (1+2\eta)/(r-2)$ is attained when both constraints hold with equality, a little algebra shows that

$$\beta_1 + \beta_0 \leq \frac{1+2\eta}{r-2} + \frac{1}{2(r-2)} = \frac{3}{2(r-2)} + \frac{2}{r-2}\eta \leq \frac{3}{2(r-2)} + \eta$$

on the event $Q_W \cap M_1^c \cap M_2^c$. But the definition of Q_W contradicts that. So, on the event $Q_W \cap M_1^c \cap M_2^c$, we must have $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) > 2+\eta$ and hence $|\partial W| < (r-2-\eta)|W|$ by (5.7.7). Thus $\mathbb{P}(Q_W \cap M_1^c \cap M_2^c) \le \mathbb{P}(|\partial W| < (r-2-\eta)|W|)$. In order to estimate the right-hand side of the last inequality we apply Lemma 5.8.5 to have

$$\text{if } |W| = m \le \epsilon_5 n, \text{ then } \mathbb{P}(Q_W \cap M_1^c \cap M_2^c) \le C_7 \exp(-(1+\eta/4)m\log(n/m) + (1+\Delta_1+\Delta_2)m). \tag{5.7.8}$$

Recalling the definition of $F(m,k)$ and considering whether the events $M_i, i = 1, 2,$ occur or not,

$$\begin{aligned}
\mathbb{P}[F(m,[3/(2r-4)+\eta]m)] &= \mathbb{P}\left(\cup_{\{W:|W|=m\}} Q_W\right) \\
&\le \mathbb{P}(M_1) + \mathbb{P}(M_2) + \mathbb{P}\left(\cup_{\{W:|W|=m\}}(Q_W \cap M_1^c \cap M_2^c)\right) \\
&\le \mathbb{P}(M_1) + \mathbb{P}(M_2) + \sum_{\{W:|W|=m\}} \mathbb{P}(Q_W \cap M_1^c \cap M_2^c).
\end{aligned}$$

Combining the probability bounds in (5.7.3), (5.7.4) and (5.7.8), using the inequality in Lemma 5.8.1 to estimate the number of terms in the sum, if $m \le \min\{1/e, \epsilon_5(\eta)\}n$, then

$$\begin{aligned}
&\mathbb{P}[F(m,[3/(2r-4)+\eta]m)] \\
\le\ & \mathbb{P}(M_1) + \mathbb{P}(M_2) + \binom{n}{m} C_7 \exp\left[-(1+\eta/4)m\log(n/m) + (1+\Delta_1+\Delta_2)m\right] \\
\le\ & \mathbb{P}(M_1) + \mathbb{P}(M_2) + C_7 \exp\left[-(\eta/4)m\log(n/m) + (2+\Delta_1+\Delta_2)m\right] \\
\le\ & C_4 \exp[-(\eta/4)m\log(n/m) + (2+\Delta_1+\Delta_2)m],
\end{aligned} \tag{5.7.9}$$

where $C_4 = 3\max\{1, C_7\}$. To clean up the result to have the one given in Proposition 5.1.6, choose $\epsilon_4'(\eta)$ such that

$$(\eta/8)\log(1/\epsilon_4') = (2+\Delta_1+\Delta_2), \quad \text{and} \quad \epsilon_4 := \min\{1/e, \epsilon_5(\eta), \epsilon_4'(\eta)\}, \tag{5.7.10}$$

where ϵ_5 is defined in (5.8.8). So for any $m \leq \epsilon_4 n$, the estimate in (5.7.9) holds, and

$$(\eta/8)\log(n/m) \geq (\eta/8)\log(1/\epsilon_4') = (2 + \Delta_1 + \Delta_2),$$

which gives the desired result. □

5.8 Probability estimates for $e(U, U^c)$ and $|\partial U|$

We begin with a simple estimate for the number of subsets of V_n of size m.

Lemma 5.8.1. *The number of subsets of V_n of size m is at most* $\exp(m\log(n/m) + m)$.

Proof. The number of subsets of V_n of size m is $\binom{n}{m}$. Using the inequalities $n(n-1)\cdots(n-m+1) \leq n^m$ and $e^m > m^m/m!$,

$$\binom{n}{m} \leq \frac{n^m}{m!} \leq \left(\frac{ne}{m}\right)^m = \exp(m\log(n/m) + m).$$

□

In order to study the distribution of $|\partial U|$, the first step is to estimate $e(U, U^c)$. Because of the symmetries of our random graph G_n, the distribution of $e(U, U^c)$ under $\mathbb{P}$ depends on U only through $|U|$.

Lemma 5.8.2. *Let U be any subset of V_n with $|U| = m$. Then for any $\alpha \in (0, 1)$,*

$$\mathbb{P}(e(U, U^c) \leq \alpha r|U|) \leq C_5 \exp\left(-\frac{r}{2}(1-\alpha)m\log(n/m) + \Delta_1 m\right)$$

for some constants C_5 and Δ_1.

Proof. Let $f(u)$ be the number of ways of pairing u objects. Then

$$f(u) = \frac{u!}{(u/2)!2^{u/2}}.$$

If $p(m,s) = \mathbb{P}(e(U,U^c) = s)$, then we have

$$p(m,s) \le \binom{rm}{s}\binom{r(n-m)}{s} s! \frac{f(rm-s)f(r(n-m)-s)}{f(rn)}.$$

To see this, recall that we construct the graph G_n by pairing the half-edges at random, which can be done in $f(rn)$ many ways as there are rn half-edges. We can choose the left endpoints of the edges from U in $\binom{rm}{s}$ many ways, the right endpoints from U^c in $\binom{r(n-m)}{s}$ many ways, and pair them in $s!$ many ways. The remaining $(rm-s)$ many half-edges of U can be paired among themselves in $f(rm-s)$ many ways. Similarly the remaining $(r(n-m)-s)$ many half-edges of U^c can be paired among themselves in $f(r(n-m)-s)$ many ways.

Write $D = rn, k = rm$ and $s = \eta k$ for $\eta \in [0,1]$. Combining the bounds of (6.3.4) and (6.3.5) of [16] we get

$$p(m,s) \le C_6 k^{1/2} \left(\frac{e^2}{\eta}\right)^{\eta k} \left(\frac{k}{D}\right)^{k(1-\eta)/2} \left(1 - \frac{(1+\eta)k}{D}\right)^{(D-(1+\eta)k)/2} \tag{5.8.1}$$

for some constant C_6. Now

$$\text{if } \phi(\eta) = \eta \log(1/\eta), \text{ then } \phi'(\eta) = -(1+\log\eta) \text{ and } \phi''(\eta) = -\frac{1}{\eta}. \tag{5.8.2}$$

So $\phi(\cdot)$ is a concave function and its derivative vanishes at $1/e$. This shows that the function $\phi(\cdot)$ is maximized at $1/e$, and hence $(1/\eta)^\eta = e^{\phi(\eta)} \le e^{1/e}$ for $\eta \in [0,1]$. So $(e^2/\eta)^{\eta k} \le C^k$ for $C = \exp(2 + 1/e)$. If we ignore the last term of (5.8.1), which is ≤ 1, then we have

$$\mathbb{P}(e(U,U^c) \le \alpha rm) = \sum_{s=1}^{\lfloor \alpha rm \rfloor} p(m,s) \le \sum_{\{\eta:\, \eta rm \in \mathbb{N},\, \eta \le \alpha\}} C_6 (rm)^{1/2} C^{rm} \left(\frac{m}{n}\right)^{rm(1-\eta)/2}$$

$$\leq C_6 r^{3/2} m^{3/2} C^{rm} \left(\frac{m}{n}\right)^{r(1-\alpha)m/2},$$

as there are at most rm terms in the sum and $(m/n)^{1-\eta} \leq (m/n)^{1-\alpha}$ for $\eta \leq \alpha$. The above bound is

$$\leq C_5 \exp\Big(-\frac{r}{2}(1-\alpha)m\log(n/m) + rm\log C + 3m/2\Big),$$

and we get the desired result with $C_5 = C_6 r^{3/2}$ and $\Delta_1 = r\log C + 3/2$. □

Lemma 5.8.2 gives an upper bound for the probability that $e(U, U^c)$ is small. Our next goal is to estimate the difference between $e(U, U^c)$ and $|\partial U|$. In order to do that, first we need the following large deviation probability estimate.

Lemma 5.8.3. *If $T_1, T_2, \ldots$ are independent random variables and $T_i \sim Geometric(p_i)$ with $p_i = (n-i+1)/n$, then for any $u > 0$ and $\eta \in (0, u)$ there are positive constants Δ_2 and $\beta = \beta(u, \eta)$ such that for large enough n and any $m < \beta n$,*

$$P\left(T_1 + T_2 + \cdots + T_{\lfloor (u-\eta)m \rfloor} > um\right) \leq \exp\big[-\eta m \log(n/m) + \Delta_2 m\big].$$

Moreover, $\beta(u, \eta) \downarrow 0$ as $\eta \downarrow 0$ and for fixed η, $\beta(u, \eta)$ is a decreasing function of u.

Proof. Let $q_i = 1 - p_i = (i-1)/n$. Then for $\theta < \log(1/q_i)$,

$$E\left[e^{\theta T_i}\right] = \sum_{k=1}^{\infty} p_i q_i^{k-1} e^{\theta k} = \frac{p_i e^{\theta}}{1 - q_i e^{\theta}}.$$

Let $\epsilon = m/n$, $\theta > 0$ and $\epsilon e^{\theta} < 1/(u-\eta)$ so that $Ee^{\theta T_i}$ is finite for $i = 1, 2, \ldots, \lfloor (u-\eta)m \rfloor$. Using Markov inequality

$$P\left(T_1 + \cdots + T_{\lfloor (u-\eta)m \rfloor} > um\right) \leq \exp\left[-\theta um\right] \prod_{i=1}^{\lfloor (u-\eta)m \rfloor} Ee^{\theta T_i}.$$

Using $\epsilon = m/n$ and the formula for $E\exp(\theta T_i)$, a little arithmetic shows that the above is

$$\begin{aligned}
&\leq \exp\left[-\theta u\epsilon n + \sum_{i=1}^{\lfloor (u-\eta)\epsilon n\rfloor} \log\frac{p_i e^\theta}{1-q_i e^\theta}\right] \\
&\leq \exp\left[-\theta\eta\epsilon n + n\cdot\frac{1}{n}\sum_{i=1}^{\lfloor (u-\eta)\epsilon n\rfloor} \log\frac{1-(i-1)/n}{1-(i-1)e^\theta/n}\right]. \qquad (5.8.3)
\end{aligned}$$

Since $e^\theta > 1$, it can be verified that the function $g(x) = \log[(1-x)/(1-xe^\theta)]$ is increasing, so that we can bound the Riemann sum for the function $g(x)$ in (5.8.3) by the corresponding integral. Thus the above is

$$\leq \exp\left[-\theta\eta\epsilon n + n\left(\int_0^{(u-\eta)\epsilon}\log(1-x)dx - \int_0^{(u-\eta)\epsilon}\log(1-xe^\theta)dx\right)\right]. \qquad (5.8.4)$$

To bound the last quantity we let

$$h(\theta,u,\eta,\epsilon) = \theta\eta\epsilon - \left(\int_0^{(u-\eta)\epsilon}\log(1-x)dx - \int_0^{(u-\eta)\epsilon}\log(1-xe^\theta)dx\right).$$

Clearly $h(0,u,\eta,\epsilon) = 0$. We want to maximize h with respect to θ keeping all the other parameters fixed. Changing the variables $y = 1-x$ and $z = 1-xe^\theta$,

$$\begin{aligned}
h &= \theta\eta\epsilon - \left(\int_{1-(u-\eta)\epsilon}^{1}\log y\,dy - e^{-\theta}\int_{1-(u-\eta)\epsilon e^\theta}^{1}\log z\,dz\right) \\
&= \theta\eta\epsilon - \big(-(1-(u-\eta)\epsilon)\log(1-(u-\eta)\epsilon) \\
&\qquad + e^{-\theta}\left(1-(u-\eta)\epsilon e^\theta\right)\log\left(1-(u-\eta)\epsilon e^\theta\right)\big), \qquad (5.8.5)
\end{aligned}$$

where to evaluate the integrals we recall $(x\log x - x)' = \log x$.

$$\begin{aligned}
\partial h/\partial\theta &= \eta\epsilon + e^{-\theta}\left(1-(u-\eta)\epsilon e^\theta\right)\log\left(1-(u-\eta)\epsilon e^\theta\right) \\
&\quad + e^{-\theta}(u-\eta)\epsilon e^\theta\log\left(1-(u-\eta)\epsilon e^\theta\right) - e^{-\theta}\left(1-(u-\eta)\epsilon e^\theta\right)\frac{-(u-\eta)\epsilon e^\theta}{1-(u-\eta)\epsilon e^\theta} \\
&= \eta\epsilon + (u-\eta)\epsilon + e^{-\theta}\log\left(1-(u-\eta)\epsilon e^\theta\right) = u\epsilon + e^{-\theta}\log\left(1-(u-\eta)\epsilon e^\theta\right).
\end{aligned}$$

$\partial h/\partial\theta = 0$ implies $\exp(-u\epsilon e^{\theta}) = 1-(u-\eta)\epsilon e^{\theta}$. Letting

$$\beta = \beta(u,\eta) \text{ be the unique positive number satisfying } e^{-u\beta} = 1-(u-\eta)\beta, \tag{5.8.6}$$

$\partial h/\partial\theta > 0$ if $\epsilon e^{\theta} < \beta$. (5.8.6) suggests that $\beta \in (0, 1/(u-\eta))$. So for fixed u, η, $\epsilon < \beta(u,\eta)$ and $\theta^* := \log(\beta/\epsilon)$, $\theta^* > 0$ with $\epsilon e^{\theta^*} < 1/(u-\eta)$, and the function h is maximized at θ^*. Plugging the value of θ^* in (5.8.5),

$$h = \eta\epsilon\log(\beta/\epsilon) + (1-(u-\eta)\epsilon)\log(1-(u-\eta)\epsilon) - \frac{\epsilon}{\beta}(1-(u-\eta)\beta)\log(1-(u-\eta)\beta).$$

Noting that the function

$$\psi(\delta) := \frac{(1-\delta)\log(1-\delta)}{\delta} \text{ satisfies } \psi'(\delta) = \frac{-\delta-\log(1-\delta)}{\delta} > 0, \text{ and } \psi(\delta) \to \begin{cases} -1 \text{ if } \delta \to 0 \\ 0 \text{ if } \delta \to 1 \end{cases},$$

$[\psi(\delta) - \psi(\delta')] \geq -1$ for $\delta, \delta' \in (0,1)$, and so

$$\begin{aligned} h &= \eta\epsilon\log(1/\epsilon) + \eta\epsilon\log\beta + (u-\eta)\epsilon[\psi((u-\eta)\epsilon) - \psi((u-\eta)\beta)] \\ &\geq \eta\epsilon\log(1/\epsilon) - c_2(u,\eta)\epsilon, \end{aligned} \tag{5.8.7}$$

where $c_2(u,\eta) = u-\eta+\eta\log(1/\beta(u,\eta))$.

To see that $\beta(u,\eta)$ has the desired properties, note that if $\varphi_x(u,\eta) := e^{-ux} - 1 + (u-\eta)x$, then for $x > 0$, $\partial\varphi_x/\partial u = -xe^{-ux} + x > 0$ and $\partial\varphi_x/\partial\eta = -x < 0$. If we put $x = \beta(u,\eta)$, use (5.8.6), and note that $\varphi_x(u,\eta) \leq 0$ if and only if $0 \leq x \leq \beta(u,\eta)$, then

$$\text{for } u' > u, \varphi_{\beta(u,\eta)}(u',\eta) > \varphi_{\beta(u,\eta)}(u,\eta) = 0, \text{ and so we must have } \beta(u',\eta) < \beta(u,\eta),$$

$$\text{for } \eta' < \eta, \varphi_{\beta(u,\eta)}(u,\eta') > \varphi_{\beta(u,\eta)}(u,\eta) = 0, \text{ and so we must have } \beta(u,\eta') < \beta(u,\eta).$$

To ensure that $\beta(u,\eta) \downarrow 0$ as $\eta \downarrow 0$, see that if $\beta(u,0) := \lim_{\eta\to 0}\beta(u,\eta)$, then using continuity of $\beta(u,\eta)$ and (5.8.6), $\exp(-u\beta(u,0)) = 1-u\beta(u,0)$ and so $\beta(u,0) = 0$.

Using the properties of $\beta(u,\eta)$ we can show that $c_2(u,\beta)$ is bounded above as η and u vary. From the inequality $e^{-y} \geq 1-y$ we have $1-e^{-x} = \int_0^x e^{-y}\,dy \geq \int_0^x (1-y)\,dy = x - x^2/2$ for any $x \geq 0$. In view of (5.8.6), using the last inequality we see that

$$1-(u-\eta)\beta = e^{-u\beta} \leq 1-u\beta+\frac{u^2\beta^2}{2}, \text{ which implies } \beta \geq \frac{2\eta}{u^2} \text{ and so } c_2(u,\eta) \leq u-\eta+\eta\log\left(\frac{u^2}{2\eta}\right),$$

and $\limsup_{\eta\to 0} c_2(u,\eta) \leq u$. In the other direction, $\beta(u,\eta) \to \infty$ as $\eta \to u$, since for any $\beta_0 > 0$ we can choose $\eta_0 \in (0,u)$ so that $1-(u-\eta_0)\beta_0 > e^{-u\beta_0}$ (e.g. choose η_0 satisfying $1-(u-\eta_0)\beta_0 = (1+e^{-u\beta_0})/2$) to make sure $\beta(u,\eta_0) > \beta_0$. Thus $c_2(u,\eta) \to -\infty$ as $\eta \to u$. From the behavior of $c_2(u,\eta)$ when η is close to 0 and u , and noting that $c_2(u,\eta)$ depends continuously on η,

$$c_0(u) := \max\{c_2(u,\eta) : \eta \in (o,u)\} < \infty.$$

Next we recall that $e(U,U^c) \leq r|U|$ so that $u \in [0,r]$. Since $\beta(u,\eta)$ is decreasing in u, recalling the definitions of $c_2(u,\eta)$ and $c_0(u)$ it is easy to see that for fixed η, $c_2(u,\eta)$ is increasing in u, and hence so is $c_0(u)$. Therefore,

$$\text{if } \Delta_2 := c_0(r), \text{ then } c_2(u,\eta) \leq c_0(u) \leq \Delta_2 \text{ for any } 0 < \eta < u \leq r.$$

Coming back to estimate h, we can convert (5.8.7) to

$$h \geq \eta\epsilon\log(1/\epsilon) - \Delta_2\epsilon.$$

Plugging the bound on h and $\epsilon = m/n$ in (5.8.4) we get

$$P\left(T_1 + \cdots + T_{\lfloor (u-\eta)m \rfloor} > um\right) \leq \exp(-\eta m \log(n/m) + \Delta_2 m).$$

which completes the proof of Lemma 5.8.3 $\square$

Now we use Lemma 5.8.3 to get an upper bound for the probability that the difference between $e(U, U^c)$ and $|\partial U|$ is large.

Lemma 5.8.4. *If U is a subset of vertices of G_n such that $|U| = m$, then for any $\eta > 0$, $u \in (\eta, r]$ and Δ_2 as in Lemma 5.8.3, there is a constant $\epsilon_5 = \epsilon_5(\eta) > 0$ such that for large enough n and $m < \epsilon_5 n$,*

$$\begin{aligned}
&(i) \quad \mathbb{P}\left(|\partial U| \le (u-\eta)|U| \mid e(U,U^c) = u|U|\right) &&\le \exp(-\eta m \log(n/m) + \Delta_2 m),\\
&(ii) \quad \mathbb{P}\left(e(U,U^c) - |\partial U| > \eta |U|\right) &&\le \exp(-\eta m \log(n/m) + \Delta_2 m).
\end{aligned}$$

Proof. Since $|U^c| = n - m$, there are $r(n-m)$ many half-edges corresponding to U^c. In order to have $e(U, U^c) = um$, we need to choose um half-edges corresponding to U^c and pair them with the same number of half-edges corresponding to U. Since the half-edges are paired randomly under the probability distribution $\mathbb{P}$, all the subsets of half-edges corresponding to U^c of size um are equally likely to be chosen under the conditional probability distribution $\mathbb{P}(\cdot | e(U, U^c) = um)$. Noting that the subset of size um, which is obtained by choosing um objects one at a time from a set of size $r(n-m)$ uniformly at random without replacement, has uniform distribution over all possible subsets of that size, we can assume that the half-edges corresponding to U^c mentioned above are chosen one by one uniformly at random without replacement.

Suppose R_i half-edges are chosen by the time i many distinct vertices are chosen. Let $T_1' = R_1 = 1$ and $T_i' = R_i - R_{i-1}$ for $i \ge 2$. Since each vertex has r half-edges, $R_{i+1} \le 1 + ri$ and $e(U, U^c) \le r|U|$ so that $u \le r$. A little arithmetic gives that for large enough n,

$\dfrac{n}{r^2+r+1} \le \dfrac{n-1}{r^2+r} \le \dfrac{n-1}{ru+r}$ so that for $m \le \dfrac{n}{r^2+r+1}$ and $i = 1, \ldots, um, ri{+}1{+}rm \le n.$

Combining these inequalities, after choosing the i^{th} distinct vertex the failure probability to choose the $(i+1)^{th}$ distinct vertex at any step is

$$\leq \frac{ri-i}{r(n-m)-ri-1} \leq \frac{i}{n} \text{ for } i \leq um.$$

Then, on the event $\{e(U,U^c)=um\}$, the T_i' can be coupled with geometric random variables T_i with failure probability $(i-1)/n$ so that $T_i' \leq T_i$. So

$$\begin{aligned}\mathbb{P}(R_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um) &= \mathbb{P}\left(T_1' + \cdots + T'_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um\right) \\ &\leq P\left(T_1 + \cdots + T_{(u-\eta)m} > um\right),\end{aligned}$$

when $m \leq n/(1+r+r^2)$. If we let

$$\epsilon_5(\eta) = \min\{1/(1+r+r^2), \beta(r,\eta/2)\}, \tag{5.8.8}$$

where β is defined in Lemma 5.8.3, then for $m \leq \epsilon_5 n$ we have the above inequality and can use the probability estimate of Lemma 5.8.3 as $\beta(u,\eta) > \beta(r,\eta)$. From those two inequalities we conclude that

$$\begin{aligned}\mathbb{P}\left(|\partial U| < (u-\eta)m \mid e(U,U^c)=um\right) &\leq \mathbb{P}(R_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um) \\ &\leq \exp(-\eta m \log(n/m) + \Delta_2 m)\end{aligned}$$

for $m \leq \epsilon_5 n$, which completes the proof of (i).

To prove (ii), recall that $e(U,U^c) \leq rm$. So based on $e(U,U^c)$ we have

$$\begin{aligned}&\mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m) \\ \leq &\sum_{u\in(\eta,r]:\, um\in\mathbb{N}} \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m, e(U,U^c)=um) \\ = &\sum_{u\in(\eta,r]:\, um\in\mathbb{N}} \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m | e(U,U^c)=um)\mathbb{P}(e(U,U^c)=um). \end{aligned} \tag{5.8.9}$$

If $m \leq \epsilon_5 n$, we can use (i) to bound the first terms of the summands in the right-hand side of (5.8.9) and have

$$\begin{aligned} & \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m) \\ \leq \; & \exp(-\eta m \log(n/m) + \Delta_2 m) \sum_{u \in (\eta, r]:\, um \in \mathbb{N}} \mathbb{P}(e(U,U^c) = um) \\ \leq \; & \exp(-\eta m \log(n/m) + \Delta_2 m). \end{aligned}$$

□

Lemma 5.8.4 gives an upper bound for the probability that the difference between $|\partial U|$ and $e(U,U^c)$ is large. Now we use Lemma 5.8.2 and 5.8.4 to estimate the probability that $|\partial U|$ is smaller than $(r-2)|U|$.

Lemma 5.8.5. *Let $U \subset V_n$ be such that $|U| = m$ and $\eta > 0$. For the constants Δ_1 of Lemma 5.8.2, ϵ_5 and Δ_2 of Lemma 5.8.4, if n is large enough and $m \leq \epsilon_5 n$, then*

$$\mathbb{P}(|\partial U| \leq (r-2-\eta)|U|) \leq C_7 \exp[-(1+\eta/4)m\log(n/m) + (1+\Delta_1+\Delta_2)m]$$

for some constant C_7.

Proof. First we estimate the probability $\mathbb{P}(|\partial U| = (r-2-\eta')|U|)$ when $\eta' \geq \eta$. Noting that $|\partial U| \leq e(U,U^c) \leq r|U|$ for any $U \subset V_n$,

$$\begin{aligned} & \mathbb{P}(|\partial U| = (r-2-\eta')|U|) \\ = & \sum_{\gamma \in [0, 2+\eta']} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|). \end{aligned} \tag{5.8.10}$$

For the summands with $\gamma \geq \eta'/2$, we write each summand as the product of two terms

$$\mathbb{P}(e(U,U^c) = (r-2-\eta'+\gamma)|U|)\, \mathbb{P}(|\partial U| = (r-2-\eta')|U| | e(U,U^c) = (r-2-\eta'+\gamma)|U|).$$

We can use Lemmas 5.8.2 to estimate the first term above. For the second term, note that by the definition of ϵ_5 in (5.8.8) and the properties of $\beta(\cdot,\cdot)$ in Lemma 5.8.3, if $\gamma \geq \eta'/2$, then $\beta(r-2-\eta'+\gamma,\gamma) \geq \beta(r,\eta/2) \geq \epsilon_5$. So if $|U| = m \leq \epsilon_5 n$, we can use (i) of Lemma 5.8.4 to estimate the second term in the last display, and have

$$\begin{aligned}&\mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|)\\ \leq & C_5 \exp\left[-\left(\frac{2+\eta'-\gamma}{2}\right) m\log(n/m) + \Delta_1 m\right]\cdot \exp(-\gamma m\log(n/m) + \Delta_2 m).\end{aligned}$$

As there are fewer than rm terms in the sum over γ and each term has the same upper bound $C_5\exp(-(1+\eta'/2)m\, log(n/m) + (\Delta_1+\Delta_2)m)$, noting that $m \leq e^{m/2}$ for $m \geq 0$,

$$\begin{aligned}&\sum_{\gamma\in[\eta'/2,2+\eta']} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|)\\ \leq\ & rC_5\exp\left[-\left(\frac{2+\eta}{2}\right) m\log(n/m) + (1/2+\Delta_1+\Delta_2)m\right]. \end{aligned} \qquad (5.8.11)$$

For the summands in (5.8.10) with $\gamma < \eta'/2$, we can ignore one of the two events and use Lemma 5.8.2 to have

$$\begin{aligned}&\sum_{\gamma\in[0,\eta'/2)} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|) \qquad (5.8.12)\\ \leq\ & \mathbb{P}(e(U,U^c) \leq (r-2-\eta'/2)|U|) \leq C_5\exp\left(-\frac{2+\eta'/2}{2} m\log(n/m) + \Delta_1 m\right).\end{aligned}$$

Combining (5.8.11) and (5.8.13), noting that there are at most rm terms in the sum over η' below, and again using the inequality $m \leq e^{m/2}$ for $m \geq 0$,

$$\begin{aligned}\mathbb{P}(|\partial U| \leq (r-2-\eta)|U|) &= \sum_{\eta'\in[\eta,r-2]} \mathbb{P}(|\partial U| = (r-2-\eta')|U|)\\ &\leq \sum_{\eta'\in[\eta,r-2]} (C_5+C_5 r)\exp(-(1+\eta'/4)m\log(n/m) + (1/2+\Delta_1+\Delta_2)m)\end{aligned}$$

$$\leq rC_5(1+r)\exp(-(1+\eta/4)m\log(n/m)+(1+\Delta_1+\Delta_2)m),$$

and we get the desired result with $C_7 = C_5r(1+r)$. $\square$

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