RANDOM WALKS AND SUBGROUP GEOMETRY

A Dissertation

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Doctor of Philosophy

by

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For Mandi, who has heard more of this than anyone.

BIOGRAPHICAL SKETCH

Russ Thompson was born in the backwoods and hill country of North Carolina. He always had some penchant for games, puzzles, and being a smarty-pants, though at the time it was not at all clear that he would ever produce a mathematical tome as you now hold.

That this would happen grew more likely when he spent the final two years of high school at the North Carolina School of Science and Mathematics. There he took an absurd number of mathematics courses, for reasons he would be hard pressed to explain. Perhaps it was his natural allergy to lab work, but he liked them all.

After that, he attended the University of North Carolina at Chapel Hill, where he spread his attention between mathematics, computer science, and english. Thanks in part to a dull survey course and a professor's poor readings, he chose mathematics over the belles-lettres. He supposes that the reader may be relieved to know that a treatise on the poetry of Wallace Stevens does not await in these pages, though the author would be happy to oblige. Poor computer science was never in the running at all.

That, upon choosing mathematics, he would end up at the interface between probability and geometric group theory seems thoroughly arbitrary and perfectly natural to him. Events accrued and there he was, and who was to know until it happened? The mind knows no destiny, only distribution.

Russ also spends a good deal of his time in making photographs and writing short stories and the stubs of novels. This provides him much time to think about math and much to think about when doing math. This trend will continue, first for a semester at MSRI, and then for a postdoc in the land where self-similar groups lie—Texas A&M.

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I have also found a great deal of solace and inspiration in the aeons-long flow of land and time here (even, on occasion, the winters). There is no one or thing to thank for the beauty that is and surrounds Ithaca, but it bears mentioning all the same.

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CHAPTER 1.

INTRODUCTION

1.1 Random walks

The principal focus of this work is the study of random walks. Classically random walks were considered on the integer lattice, but one can generalize the state space to other graphs or metric spaces. Let Γ denote a graph with vertex set V and edge set E. A random walk X_n on G is defined by a Markov operator, $P = \{p(u,v)\}_{u,v\in V}$, such that

1. for each each $u, v \in V$, $p(u,v) \geq 0$,
2. and for each $v \in V$, $\sum_{u\in V} p(v,u) = 1$.

To translate, P gives the transition probabilities for the Markov chain X_n:

$$P(X_{n+1} = u|X_{n-1} = v) = p(v,u). \tag{1.1}$$

The holy grail of a random walk is the *heat kernel*, given by the n-step transition probabilities of the walk, $p_n(v,u)$. If you have good estimates of these quantities for all vertices and all n, you can say accurately almost anything you could care to about a random walk. Unfortunately, we are much like the meekest of King Arthur's knights and may only find a piece of the way to the grail and should hope not to come away from our quest having lost too many years and limbs. We do have the fortune that many have set out on this quest before us so that for some examples the heat kernel is well understood in the mythic form of gaussian estimates, and for many more examples the on-diagonal heat kernel, $p_n(v, v)$, is well understood.

We will be derelict in our knightly duties, and attempt to study some properties of random walks without directly resorting to using or computing heat kernels. We can do this because of the intimate relationship between random walks and their state spaces. The classical example of this is that a simple symmetric random walk on the integer lattice are recurrent in dimension two or less and transient in higher dimensions. The combination of the words "simple" and "symmetric" indicates that these random walks are attentive to the geometry of the integer lattice. We now make this more precise.

Suppose

$$p(v, u) > 0 \text{ if and only if } (v, u) \in E. \tag{1.2}$$

From this perspective, the edges of Γ are thought of as being directed. If we are given a graph with undirected edges, for the purpose of the random walk, we will view such an edge as being bidirectional. Such a geometrically adapted random walk moves from its current vertex, v, to a new vertex by choosing at random an edge incident at v to travel along. Such a random walk is called a *simple random walk* on Γ.

In some graphs the edges are translation invariant, in that if you were to label all the edges incident at a given vertex you could use that labeling in a consistent fashion for all vertices. For instance, on the two dimensional lattice one can label the edges incident at a vertex as “up”, “down”, “left”, and “right”, and this labeling makes sense at every vertex. We can define a probability measure μ on this labeling and consider the random walk with transition probabilities given by

$$p(u,v) = \mu((u,v)). \tag{1.3}$$

Such a random walk will be denoted by (X_n, μ). These random walks are by default simple, and if $\mu((u,v)) = \mu((v,u))$ for all edges, then we say it is *symmetric* as well.

The n-step transition probabilities for X_n can now be expressed as an n-fold convolution of μ. Let $\gamma_n(v_1, v_n)$ denote the set of edge-paths in Γ of length n in Γ with initial vertex v_1 and terminal vertex v_n. We denote the ith vertex along this path as γ_i with $v_1 = \gamma_1$ and $v_n = \gamma_n$. Then

$$p_n(v_1, v_n) = \mu^{(n)}((v_1, v_n)) \tag{1.4}$$

$$= \sum_{\gamma \in \gamma_n(v_1, v_n)} \mu((\gamma_1, \gamma_2)) \cdots \mu((\gamma_{n-1}, \gamma_n)). \tag{1.5}$$

Now that we have an idea of the kinds of random walks we want to consider, we need a place to put them. Finitely generated groups naturally give rise to graphs with translation invariant edge labelings, and we shall cover such spaces in the following two sections.

1.2 Finitely generated groups

1.2.1 Groups as metric spaces

A group G is *finitely generated* if there exists a finite subset $S \subset G$ such that $\cup_{n\geq 0} S^n = G$. Such a set S is called a (finite) generating set of G. We will assume that generating sets are symmetric, i.e. if $s \in S$ then $s^{-1} \in S$. All the groups we will treat are finitely generated, and most are *finitely presentable* as well. A groups is finitely presented if it has a finite number of defining relations, i.e. any trivial word is freely equal to a product of conjugates of words chosen from a finite set.

Generating sets provide a framework for considering finitely generated groups as geometric objects. First we define the *word length* of an element of G relative to S as

$$|g|_S = \min\{n : \exists s_1, \ldots, s_n \in S; g = s_1 \cdots s_n\}.$$

By convention, we assume that the identity element, e, of G has length zero. The *word metric* on G relative to a generating set S is

$$d_S(x, y) = |x^{-1}y|_S.$$

For a generating set S the Cayley graph, $\Gamma(G, S)$, has vertex set G, and two vertices x and y are connected by an edge if $x^{-1}y \in S$. Thus, every word metric on G corresponds to the combinatorial metric on a *Cayley graph* of G. The translation invariant labeling of the edges of $\Gamma(G, S)$ is with elements of S or their inverses.

Note that we have defined the edges of the Cayley graph via multiplication on the right. Our random walks will follow this convention so that $X_n = X_{n-1}s$ where s is chosen according to the law of X_n.

Example 1.2.1. *Let $G = \mathbb{Z}$ and $S = \{1, -1\}$. The word length of an integer is its absolute value, $d_S(n, m) = |m - n|$, and $\Gamma(\mathbb{Z}, S)$ is a line.*

There is no canonical generating set of a group, though it may be useful to consider a generating set of minimal cardinality, and thus we would prefer to consider the geometry of a finitely generated group without reference to a specific generating set. We will do this by considering the *coarse geometry* of the group. This means we wish to suppress as much of the small scale geometry of the group as possible while retaining a grasp of what it looks like at large. Just as you do not need to know the location of every stroke of a painting to know what it looks like, you do not need to know the intimate details of a Cayley graph to know what a group looks like. Random walks may live on the small scale, but their behavior is (we hope) controlled by the large scale geometry of the group.

The coarse geometry of a metric space is reckoned up to *quasi-isometry*. Two metric spaces (M_1, d_1) and (M_2, d_2) are quasi-isometric if there exists $\phi : M_1 \to M_2$ and constants $C > 0$ and $c \geq 1$ such that

$$\frac{1}{c}d_1(x, y) - c \leq d_2(\phi(x), \phi(y)) \leq cd_1(x, y) + c,$$

for all $x, y \in M_1$, and for every $x_2 \in M_2$,

$$d_2(\phi(M_1), x_2) < C.$$

That is, ϕ distorts distances in X_1 in a uniform and linear manner and its image is spread evenly throughout X_2. It is easy to check that quasi-isometry forms an equivalence relation on metric spaces [14].

One can see that $\Gamma(G,S)$ and $\Gamma(G,T)$ are quasi-isometric for any choice of finite generating sets as $min_{s\in S}|s|_T, min_{t\in T}|t|_S < \infty$. Whenever possible we will talk about quantities related to a group or random walks on it up to quasi-isometry. This is done via the notion of the *coarse type* of a function [1]. For two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ we write $f \preceq g$ if there exists $c > 0$ such that

$$f(x) \leq cg(cx) \text{ for all } x \in \mathbb{R}.$$

In this case we say the coarse type of f is less than or equal to that of g. If $f \preceq g$ and $g \preceq f$ we write $f \simeq g$ and say f and g are *coarsely equivalent*. Frequently, the coarse equivalence of some quantity will be a quasi-isometric invariant of a group.

When studying groups from a geometric perspective, one generally wants to show that what one is studying is invariant up to quasi-isometry, or, failing that, that it is invariant up to change in generating set. Often, this is hard. We shall see this difficulty appear later for the rate of escape of random walk. Even certain basic algebraic properties of groups may not be quasi-isometric invariants. For instance, there are solvable groups which are quasi-isometric to not solvable groups [19]. This is a clear warning that one should be careful when trying to assess which properties of groups are truly geometric in nature.

[1] This terminology is non-standard as there is no standard in the literature. Other authors use the term asymptotic type or no term at all. The use of $\preceq$ and $\simeq$ is common, though they often used for similar but not identical relations between functions.

1.2.2 Volume growth

One of the most important geometric properties of a finitely generated group is its *volume growth*

$$V_{G,S}(n) = \#\{g \in G : |g|_S \leq n\}.$$

We will do away with mentioning S due to the following result.

Lemma 1.2.1. *Let G be a finitely generated group. Then for any generating sets S and T,*

$$V_S(n) \simeq V_T(n). \tag{1.6}$$

Furthermore, the coarse type of $V_S(n)$ is a quasi-isometric invariant of G.

Perhaps the most important theorem on the volume growth of groups is the following result of Gromov.

Theorem 1.2.1 (Gromov's Theorem [30])**.** *A finitely generated group G has a polynomial volume growth if and only if it is virtually nilpotent.*

We say a group is virtually $\mathcal{P}$, if it contains a finite index subgroup with property $\mathcal{P}$. One should be careful not to assume that being virtually $\mathcal{P}$ is a quasi-isometric invariant. This is true for nilpotency by Gromov's theorem but not for solvability by a result of Erschler [19]. We will return to nilpotency and its relation to volume growth in Section 1.2.3.

Example 1.2.2. *The Heisenberg group, H_3, is a groups of matrices of the form*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.7}$$

where $x, y, z \in \mathbb{Z}$. This group is nilpotent, and, while being composed of three subgroups isomorphic to $\mathbb{Z}$, it has quartic volume growth.

There are other important theorems related to volume growth.

Theorem 1.2.2 (Milnor-Wolf [14, 55])**.** *A finitely generated solvable group has either polynomial or exponential volume growth, and this depends on whether or not it contains F_2 as a subgroup.*

Theorem 1.2.3 (Milnor [14, 37])**.** *Let X be a Riemannian manifold with strictly negative sectional curvature, then $\pi_1 X$ exponential volume growth.*

These two theorems might lead one to think that we are hinting that groups only have polynomial or exponential volume growth, and for a long time it was open whether or not this was the case. There do exist groups with intermediate volume growth:

$$e^{n^{\gamma_1}} \leq V(n) \leq e^{n^{\gamma_2}},$$

for $0 < \gamma_1 \leq \gamma_2 < 1$.

The first example of such a group was found by Grigorchuk [28], and other examples have been found since then. The exact volume growth for certain examples has recently been computed by Ershcler and Bartholdi [5]. All known examples of groups with intermediate volume growth have a self-similar structure and posses many other interesting properties [39].

1.2.3 Some algebraic properties

We will now cover some of the algebraic properties and structures of groups that will be of use and interest to us. A group G is *nilpotent* if its lower central series is finite, that is there exists a series

$$G = G_1 > G_2 > \cdots > G_n = \{e\}, \tag{1.8}$$

where $G_{i-1} = [G_i, G]$. Furthermore G_i/G_{i+1} is abelian for $i = 1, \ldots, n-1$. The volume growth of a nilpotent group is determined by the groups in its lower central series.

Theorem 1.2.4 (Bass-Guivarch formula [14, 55]). *Let G be a nilpotent group with lower central series $G = G_1 > G_2 > \ldots > G_n = \{e\}$, where $G_{i+1} = [G_i, G]$, and with subgroup H. Then $V_G(n) \simeq n^d$ where*

$$d = \sum_{i=1}^{n-1} i \cdot \boldsymbol{rk}(G_i/G_{i+1}), \tag{1.9}$$

where $\boldsymbol{rk}$ *denotes the free abelian rank of a group.*

A group G is *polycyclic* if there exists a subgroups series

$$G = G_1 > G_2 > \cdots > G_n = \{e\} \tag{1.10}$$

such that G_i/G_{i+1} is cyclic for $i = 1, \ldots, n-1$. An important feature of polycyclic groups is that all their subgroups are finitely generated [48]. Every nilpotent group is polycyclic; however, there are polycyclic groups with exponential volume growth. The difference between these two classes of polycyclic groups lies entirely in how these cyclic quotient groups are glued together to make the entire group.

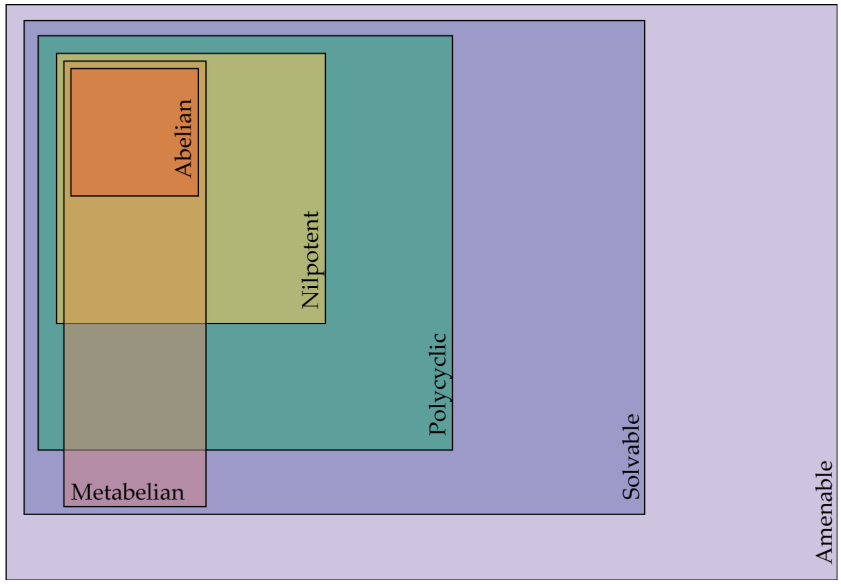

Figure 1.1: A map of some classes of groups.

Example 1.2.3 (Sol). *Let* $G = \mathbb{Z}^2 \rtimes_A Z$ *where* $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *The group multiplication is given by*

$$(x_1, z_1)(x_2, z_2) = (x_1 + A^{z_1} x_2, z_1 + z_2).$$

This group is polycyclic and has exponential volume growth.

As a capstone to these two properties, we have *solvability*. A group G is solvable if there exists a subgroup series

$$G = G_1 > G_2 > \cdots > G_n = \{e\} \tag{1.11}$$

such that G_i/G_{i+1} is abelian for $i = 1, \ldots, n-1$.

Example 1.2.4 (Lamplighters). *Consider* $G = H \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}^d} H \rtimes_\tau \mathbb{Z}^d$, *where* τ *represents translation by a* $\mathbb{Z}$ *action on the index of* $\bigoplus_{\mathbb{Z}^d} H$. *We will represent elements of* $\bigoplus_{\mathbb{Z}^d} H$ *as finitely supported functions mapping* $\mathbb{Z}$ *to* H. *Group multiplication is given by*

$$(f_1, z_1)(f_2, z_2) = (f_1 + \tau(z_1) \circ f_2, z_1 + z_2).$$

This group is solvable but not polycyclic as the subgroup $\bigoplus_{\mathbb{Z}^d} H$ is not finitely generated. The most common incarnations of lamplighter groups have H as either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$.

For any group property $\mathcal{P}$, a group G is meta-$\mathcal{P}$ if it possesses a normal subgroup N with $\mathcal{P}$ groups such that G/N also has $\mathcal{P}$. Another class of groups we will be interested in are the *metabelian* groups. These are the two-step solvable groups, and provide nice examples of groups in the classes described above.

All the classes of groups we have discussed thus far are examples of *amenable* groups. A group G is amenable if there is a finitely additive probability measure on G which is invariant under translation by left multiplication. There are many equivalent definitions of amenability, but the one most relevant to random walks is Kesten's criteria for the spectral radius:

$$\limsup_{n\to\infty} p_n(x,x)^{1/n} = 1. \tag{1.12}$$

for any non-degenerate symmetric random walk.

1.2.4 The structure of groups

Most of the types of groups we have described above have very nice structures; they are built in some fashion from abelian groups. If we are willing to exclude torsion, a finitely generated, infinite abelian group is just $\mathbb{Z}^d$ for some $d > 0$. Random walks on the integers are well understood and, if we can understand how these groups are built from abelian groups, we can hope to understand much about random walks on them.

The first notion we will need is that of a *group extension*. Suppose we have groups G, H, and K. We say G is an extension of H by K if there exists a short exact sequence

$$1 \to H \xrightarrow{f} G \xrightarrow{g} K \to 1. \tag{1.13}$$

This means that f is injective, g is surjective, $im(f) = ker(g)$, and $G/H \simeq K$. If $\mathcal{H}$ and $\mathcal{K}$ are properties of H and K, then we say that G is $\mathcal{H}$-by-$\mathcal{K}$. We will refer to H as the $\mathcal{H}$ kernel of G.

An extension *splits* if there exists $h : K \to G$ such that $h \circ g = id_K$. In this case, G is a *semidirect product* $H \rtimes_\phi K$ where $\phi : K \to Aut(H)$. In such a group multiplication is given by

$$(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1) \cdot h_2, k_1 k_2).$$

Example 1.2.5. *The Heisenberg group, introduced in Example 1.2.2, can be represented as the semidirect product* $\mathbb{Z}^2 \rtimes_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} \mathbb{Z}$. *Both Sol and the lamplighter groups are semidirect products as well.*

Example 1.2.6 (Baumsalg-Solitar groups)**.** *The Baumslag-Solitar group* $BS(1, n)$*,* $n \neq 0, 1$ *can be represented as* $\mathbb{Z}[1/n] \rtimes_n \mathbb{Z}$*. These groups are solvable but not polycyclic.*

Wreath products are a special case of semidirect products. The group $H \wr K$ is the semidirect product $\left(\bigoplus_K H\right) \rtimes K$ where K acts on $\bigoplus_K H$ by translation.

While the class of nilpotent groups extends that of abelian groups, it does not do so by very much. In particular, if a nilpotent group is finitely generated and torsion free then it is a subgroup of a the group of upper triangular matrices with ones on the diagonal over $\mathbb{Z}$ [48, Ch. 2]. This means that finitely generated torsion free nilpotent groups are built from semidirect products of abelian groups. However, not all such groups are nilpotent; rather, this requires something particular about the actions defining these semidirect products.

Polycyclic groups are also constructed out of abelian groups, but their defining series is much more open than the central series defining nilpotent groups. However, there are still very strong structure theorems for polycyclic groups [48, Ch. 2, Theorem 4].

Theorem 1.2.5 (Mal'cev's Theorem). *Every polycyclic group G contains a finite index torsion-free nilpotent-by-abelian subgroup G'. This group is again polycyclic and has a split exact sequence*

$$1 \to N \to G' \to \mathbb{Z}^r \to 1, \tag{1.14}$$

where N is a finitely generated , torsion-free, nilpotent group.

One additional point to take away from this theorem is that a finitely generated nilpotent-by-abelian group is polycyclic whenever the nilpotent kernel is finitely generated. A group like $\mathbb{Z} \wr \mathbb{Z}$ contains a non-finitely generated subgroup, namely $\bigoplus_{\mathbb{Z}} \mathbb{Z}$, and is thus not polycyclic. Furthermore, polycyclic groups are the solvable linear groups [48, Ch. 5].

Between Gromov's and Mal'cev's theorems we have enough structural information to develop a rich understanding of random walks on polycyclic groups. One of the main issues in understanding the geometry of polycyclic groups is the nature of the abelian action on the nilpotent kernel. A full classification is not yet known, though recent work on subgroup distortion and the quasi-isometric classification seems to offer a great deal of insight into the issue [18, 24, 41].

Later we will explore what differentiates the Heisenberg group and Sol more carefully. Both groups take the form $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ for $M \in \mathrm{SL}_2(\mathbb{Z})$. The eigenvalues of these matrices are what make the difference. The automorphism for the Heisenberg group has eigenvalue 1 with multiplicity 2, while the automorphism for Sol has two real eigenvalues distinct from 1. This means that the $\mathbb{Z}$-action does far more to distort the geometry of $\mathbb{Z}^2$ in Sol than in the Heisenberg group. In particular, this distortion leads to polynomial volume growth in the Heisenberg group but exponential volume growth in Sol. However, one should note that there is an additional atypical factor for the Heisenberg group: it has an eigenvalue of multiplicity greater than 1. This makes the Heisenberg group more complicated than a group for which M is a rotation matrix, and, in particular, this precludes the use of an eigendecompostion of $\mathbb{Z}^2$ in terms of M when studying the Heisenberg group. Thus we will often assume that M does not possess multiple eigenvalues.

For such groups it is straightforward to determine the volume growth based on the eigenvalues of the automorphism [16].

Theorem 1.2.6. *Suppose $G = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ for $M \in SL_2(\mathbb{Z})$. The volume growth of G is exponential iff M has an eigenvalue of modulus greater than 1, i.e. M is a hyperbolic automorphism of $\mathbb{Z}^2$.*

1.3 The geometry of subgroups and quotients

Throughout this work we will be concerned with how a random walk on a group interacts with subgroups or quotients of a group. Notice that we make no restrictions about the subgroup being normal. A group may be quotiented by any subgroup to produce what is called a *G-space*. A G-space is a set X along with a map $X \times G \to X$ which satisfies the following conditions

1. $(x \cdot g) \cdot h = x \cdot (gh)$ for all $x \in X$ and $g, h \in G$,
2. $x \cdot e = x$ for all $x \in X$.

We will restrict our attention to G-spaces where X is the set of cosets of G quotiented by a subgroup H. Our cosets will be *left* cosets since our random walks accrue increments on the right. We note that as left and right cosets are conjugate to each other, it may be convenient to study some questions on whichever side has a nicer presentation. In particular, this can be a useful technique when trying to determine the volume growth of a quotient (see Section 1.3.1 below). We denote our quotients by $H\backslash G$ so that their handedness is clear.

In this context, the analog of the Cayley graph $\Gamma(G, S)$ is the *Schreier graph* $\Gamma(G, H, S)$, where each edge is a coset of $H\backslash G$ and two cosets u, v are connected by an edge if there exists $s \in S$ such that $u = v \cdot s$. Every word metric on G naturally gives rise to a word metric on $H\backslash G$,

$$|Hg|_S = \min\{|hg|_S \ : \ h \in H\}, \tag{1.15}$$

which corresponds to the graph distance for the Schreier graph $\Gamma(G, H, S)$.

While Schreier graphs may seem rather specialized, they are in fact ubiquitous. Every even-regular graph is a Schreier graph, as are certain odd-regular graphs [36]. This does not mean that Schreier graphs are easy to understand. The difficulty of doing so is highlighted by the fact that the quasi-isometric classes of unlabeled Schreier graphs admit wildly different group/subgroup pairs. For instance, quotienting $\mathbb{Z}^{d+1}$ by $\mathbb{Z}^d$ for all $d \geq 0$ yields a graph quasi-isometric to a line, but each Schreier graph has d loops attached to each vertex. A more interesting example is given by the infinite dihedral group, $\mathbb{Z}_2 * \mathbb{Z}_2$, quotiented by a copy of $\mathbb{Z}_2$, and Grigorchuk's group $\mathcal{G}$ quotiented by the stabilizer

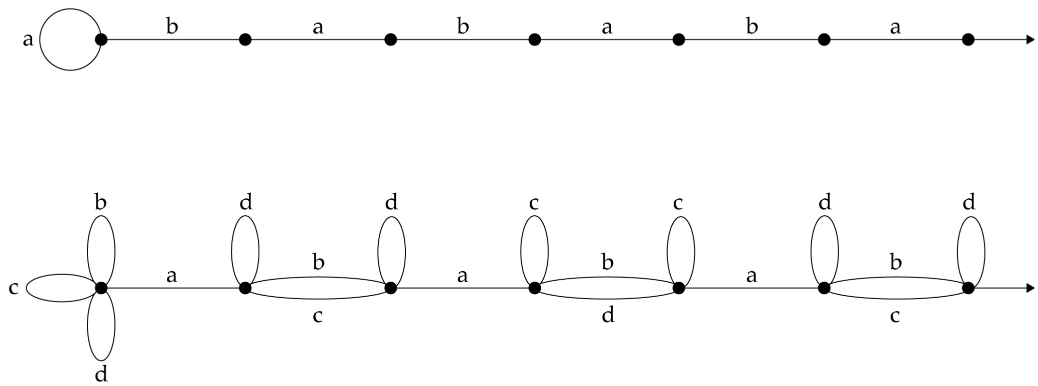

Figure 1.2: The Schreier graphs for $\mathbb{Z}_2\backslash(\mathbb{Z}_2 * \mathbb{Z}_2)$ and $\mathrm{Stab}1\backslash\mathcal{G}$

of 1[2] (see Figure 1.2).

1.3.1 Relative volume growth of subgroups and the volume growth of quotients

Let H be a subgroup of G and fix a finite generating set S of G. It is often useful to know the *relative volume growth* of H in G with respect to S:

$$V_{G,S}(n, H) = \#\{h \in H : |h|_S \leq n\}. \tag{1.16}$$

The relative volume growth of a subgroup is closely related to the distortion of that subgroup's intrinsic metric in the extrinsic metric of G. We will revisit this notion and how it relates to the speed of random walks in Chapter 2.

The volume growth of the Schreier graph or, equivalently, the quotient $H\backslash G$ is

$$V_{H\backslash G,S}(n) = \#\{x \in H\backslash G : \ |x|_S \leq n\}. \tag{1.17}$$

[2]See Appendix A

We will see in Chapter 3, that when G has polynomial volume growth $V_G(n, H)$ and $V_{H\backslash G}(n)$ are closely related.

The coarse type of the relative volume growth and the volume growth of the quotient space are independent of the choice of generating set, so we will suppress mention of the generating set S unless otherwise necessary. Likewise, the coarse type is invariant when moving to sub- or super-groups of finite index [44].

1.3.2 Induced random walks on quotients of groups

A random walk on a finitely generated group, G, naturally gives rise to random walks on quotients of the group. Let μ be a probability measure on G. The random walk X_n for the pair (G, μ) naturally induces a random walk, Y_n, on the pair $(H\backslash G, \mu)$ defined by

$$Y_n = \pi_H(X_n), \tag{1.18}$$

where π_H is the projection from G to $H\backslash G$ given by $g \mapsto Hg$. Note that π_H is only a homomorphism when H is normal in G. Instead, one has the following relation: for all $g_1, g_2 \in G$

$$\pi_H(g_1 g_2) = \pi_H(g_1) \cdot g_2. \tag{1.19}$$

Thus if $X_{n+1} = X_n s$, then

$$Y_{n+1} = \pi_H(X_{n+1}) = \pi_H(X_n) \cdot s = Y_n \cdot s. \tag{1.20}$$

When μ is supported on a finite generating set S of G, the induced random walk corresponds to a simple random walk on $\Gamma(G, H, S)$.

CHAPTER 2.

THE RATE OF ESCAPE OF RANDOM WALKS

2.1 Introduction

One of the most basic properties of a random walk is how fast it moves. The first notion one might consider is the *speed* of the random walk,

$$\lim_{n\to\infty} \frac{|X_n|_G}{n},$$

where $|\cdot|_G$ is the word length corresponding to the support of the law of X_n. This limit always exists, and, for non-amenable groups, any simple symmetric random walk has positive speed [15,31,34]. A random walk has positive speed if and only if there exist non-constant bounded harmonic functions with respect to the Markov operator associated to the random walk [52]. This also implies that the Poisson boundary of the random walk is non-trivial [33]. Thus, for non-amenable groups, the speed is positive. The situation is more complicated and much less well understood for amenable groups. Many random walks on amenable groups have zero speed, and so we need different notions to gauge how fast the walk is moving. Below we introduce three such notions, each of them parametrized by an exponent of the number of steps taken. We start with a definition introduced by Revelle [45].

A simple symmetric random walk, X_n, on a group is said to have *α-tight degree of escape* if

1. $\exists \gamma, \delta > 0$ such that $P(|X_n|_G > \gamma n^{\alpha}) \geq \delta$,
2. and $\exists \beta > 0$ such that $P(|X_n|_G > x n^{\alpha}) \leq c \exp(-c x^{\beta})$ for all $x \geq 0$,

hold for all $n \geq 0$.

We also consider the *expected displacement* of a random walk,

$$\mathbb{E}_{\mu}|X_n|_G. \tag{2.1}$$

It is important to keep the measure μ driving the random walk in mind when computing this quantity. It is evident that

$$\mathbb{E}_{\mu}|X_n|_S \simeq \mathbb{E}_{\mu}|X_n|_T \tag{2.2}$$

for any two distinct word metrics on G, but if μ and ν are supported on two distinct generating sets of G, it is unknown whether or not $\mathbb{E}_{\mu}|X_n|_G$ is coarsely equivalent to $\mathbb{E}_{\nu}|X_n|_G$. This is one of the significant open questions concerning the rate of escape. However, the expected displacement is not a quasi-isometric invariant of graphs due to an example of Benjamini and Revelle [11].

We will be interested in the case in which there are constants δ^- and δ^+, called the *upper and lower displacement exponents*, such that

$$n^{\delta^+} \preceq \mathbb{E}_{\mu}|X_n|_G \preceq n^{\delta^+}. \tag{2.3}$$

When $\delta^- = \delta^+$, we will say the random walk has *displacement exponent* $\delta := \delta^-$.

A generalization of the law of the iterated logarithm is also considered. We will say a random walk has a *law of iterated logarithm with exponent* α if

$$0 < \limsup_{n\to\infty} \frac{|X_n|_G}{(n \log \log n)^{\alpha}} < \infty. \tag{2.4}$$

In the literature, the above denominator is sometimes called an outer radius for the random walk [29] or the upper upper Lévy class [47].

These different exponents tend to come along with each other, and when a random walk has α-tight degree of escape, displacement exponent α, and an α law of iterated logarithm, we will say the random walk has *escape exponent* α. That is, having escape exponent α is equivalent to the following holding for a random walk:

1. tight degree of escape α,
2. $\mathbb{E}|X_n|_G \simeq n^{\alpha}$,
3. and $0 < \limsup \frac{|X_n|_G}{(n \log \log n)^{\alpha}} < \infty$.

The following theorem outlines the classical behavior of the escape exponent. For abelian groups this result is classical, while for virtually nilpotent groups it follows from the work of Hebisch and Saloff-Coste [32].

Theorem 2.1.1. *Let G be a finitely generated group with polynomial volume growth. Then any simple symmetric random walk on G has escape exponent* $1/2$.

Revelle showed that there exist random walks on Sol and $\mathrm{BS}(1,2)$ with escape exponent $1/2$. Later we will generalize these examples to determine the escape exponent for larger classes of groups and/or measures. Examples of groups with escape exponent different than either $1/2$ or 1 are rare and are restricted to iterated lamplighter groups.

Theorem 2.1.2. *Let F be any finite group, and let*

1. $L_1 = F \wr \mathbb{Z}$ *and* $L_i = L_{i-1} \wr \mathbb{Z}$*,*
2. $H_{1,i} = L_i \wr \mathbb{Z}^2$ *and* $H_{j,i} = H_{j-1,i} \wr \mathbb{Z}^2$*, and*
3. $G_1 = F \wr \mathbb{Z}^2$ *and* $G_k = G_{k-1} \wr \mathbb{Z}^2$*.*

There exist simple symmetric random walks for which:

1. L_i *has* $1/2$*-tight degree of escape and displacement exponent* $1 - 2^{-i}$ *for* $i \geq 1$ *[23, 45],*
2. $H_{j,i}$ *satisfies [20]*
$$\mathbb{E}|X_n|_{H_{j,i}} \simeq \frac{n}{(\log^{(j)} n)^{2^{-i}}}, \tag{2.5}$$
3. *and* G_k *satisfies [20]*
$$\mathbb{E}|X_n|_{G_k} \simeq \frac{n}{\log^{(k)} n}. \tag{2.6}$$

This class of examples has been effectively mined out, as if F is any non-trivial finite group, $F \wr \mathbb{Z}^d$, for $d \geq 3$ has non-trivial Poisson boundary for any simple symmetric random walk whose projection to $\mathbb{Z}^3$ is transient [33]. Thus these examples have positive speed, or, equivalently, escape exponent 1.

Lee and Peres have shown there is a universal lower bound of $1/2$ on the displacement exponent for all simple random walks on groups [35]. Their theorem covers Cayley graphs of finitely generated groups as a special case.

Theorem 2.1.3. *Let Γ be an infinite, connected, and amenable transitive d-regular graph. Any simple random walk on G satisfies*

$$\mathbb{E}d(X_0, X_t)^2 \leq t/d, \tag{2.7}$$

for all $t \geq 0$. Moreover, there exists a universal constant $C \geq 1$ such that for every $\epsilon > 0$ and $t \geq d/\epsilon^8$,

$$P(\ d(X_0, X_t) \leq \epsilon \sqrt{t/d}\ ) \leq C\epsilon. \tag{2.8}$$

Knowing the displacement exponent of a random walk has applications to other questions concerning the geometry of groups. One such application is an estimate on how accurately one can embed a group into L^p. The *compression exponent of G into L^p* is defined as

$$\alpha_p^*(G) := \sup\{\alpha > 0\ :\ \exists f \in \mathrm{Lip}(G \to L^p), c > 0;\ \forall x, y \in G, \|f(x) - f(y)\| \geq c d_G(x, y)^\alpha\}. \tag{2.9}$$

This quantity is related to the displacement exponent by the following theorem [3].

Theorem 2.1.4. *Let $\delta^*(G) = \sup \delta^-$, where δ^- is a lower displacement exponent for some simple symmetric random walk on G. Then*

$$\alpha_p^*(G) \leq \frac{1}{\min\{p, 2\}\delta^*(G)}. \tag{2.10}$$

There are more general versions of this theorem replacing L^p with an arbitrary Banach space and the constant on the right hand side of the conclusion with one depending on that space [38]. For more background and references, we recommend [38]. A similar estimate relating the upper escape exponent to critical constants for recurrence will be presented in Section 3.1.

2.1.1 Statement of results

We will show that certain polycyclic groups have escape exponent $1/2$ for any simple symmetric random walk. The principal assumption we need to demonstrate this regards how the intrinsic geometry of the nilpotent kernel is distorted in a polycyclic group. The notion of subgroup distortion has been much studied in geometric group theory [13, 26]; for polycyclic groups, the primary reference is Osin [41].

Let $\mathcal{F}$ be a coarse type of functions (the typical examples being polynomial and exponential). We say a subgroup $H < G$ has *upper $\mathcal{F}$ distortion* if there is an invertible function f of coarse type $\mathcal{F}$ such that there exists $c > 0$ such that

$$|h|_G \leq c f^{-1}(|h|_H) + c, \tag{2.11}$$

for all $h \in H$. We will be interested in the case of *upper exponential distortion*, which we write write out precisely for later reference:

$$|h|_G \leq c \log(|h|_H + 1) + c \tag{2.12}$$

for all $h \in H$. Note that upper $\mathcal{F}$ distortion of a group/subgroup pair is does not depend on the choice of word metrics for either the group or its subgroup. For convenience, we assume that the trivial group has every type of upper distortion.

Theorem 2.1.5. *Let G be a torsion-free polycyclic group satisfying a short exact sequence*

$$1 \to N \to G \to \mathbb{Z}^r \to 1. \tag{2.13}$$

If the nilpotent kernel of G, N, has upper exponential distortion, then any simple symmetric random walk on G has escape exponent $1/2$.

Upper exponential distortion of the nilpotent kernel is not necessary for a polycyclic group to have escape exponent $1/2$. In particular, the nilpotent kernel of a polycyclic group with polynomial volume growth has upper polynomial distortion, and all simple symmetric random walks on such groups have escape exponent 1/2. However, there are also cases with mixed distortion. For instance, the set of elements with upper exponential distortion do not necessarily form a subgroup in polycyclic groups [41]. This means that the type of upper distortion varies over different portions of the nilpotent kernel in a way that does not line up with any subgroup. We examine this phenomenon in abelian-by-cyclic groups in Section 2.4.3.

The upper distortion of the abelian kernel in abelian-by-cyclic groups can often be assessed using the characteristic polynomial of the associated automorphism. Let $G = \mathbb{Z}[\rho]^d \rtimes_\phi \mathbb{Z}$ be a torsion free abelian-by-cyclic group where ρ is an algebraic number and $\phi \in \mathrm{Aut}(\mathbb{Z}[\rho]^d)$. We will denote the characteristic polynomial of ϕ by p_ϕ.

Let $\max^{(k)} S$ denote the kth largest element of a set S. We will say a polynomial $p \in \mathbb{Z}[t, t^{-1}]$ has property *(EDP)* if

$$\max_i{}^{(1)}|p_i|\left(1 - \frac{1}{\max_i{}^{(2)}|p_i|}\right) > \sum_i |p_i|, \tag{2.14}$$

where p_i denotes the coefficient of t^i in p.

We will show in Section 2.5.2 that if the abelian kernel is finitely generated and p_ϕ has a multiple in $\mathbb{Z}[t, t^{-1}]$ which satisfies (*EDP*), then the abelian kernel has upper exponential distortion. However, when the abelian kernel is not finitely generated, i.e. if $A = \mathbb{Z}[q]^d$, $q \in \mathbb{Q}$, (*EDP*) can be viewed as a stand in for upper exponential distortion. The first result concerning (*EDP*) and the rate of escape is the following.

Theorem 2.1.6. *Let $G = \mathbb{Z}[\rho]^d \rtimes_\phi \mathbb{Z}$ where ρ is an algebraic number. If the character-*

istic polynomial, p_ϕ, of ϕ has a multiple with property (EDP) then there exist simple symmetric random walks on G with escape exponent $1/2$.

We will prove this theorem in Section 2.4.2. Often, the abelian kernel of a metabelian group will not have upper exponential distortion, or, if it is not finitely generated, the characteristic polynomial will not have property (EDP). This case is treated by the following theorem, which we prove Section 2.4.3.

Theorem 2.1.7. *Let $G = \mathbb{Z}[\rho]^d \rtimes_\phi \mathbb{Z}$ where ρ is an algebraic number. If the characteristic polynomial, p_ϕ, of ϕ can be factored over $\mathbb{Z}$ as $p_+ p_0$, where*

1. *p_+ has a multiple which satisfies (EDP),*
2. *the roots of p_+ are distinct with modulus distinct from 1, and*
3. *the roots of p_0 are distinct with modulus 1,*

then there exist simple symmetric random walks on G with escape exponent $1/2$.

A corollary of this result is the following.

Corollary 2.1.1. *There are 2-generated abelian-by-cyclic groups with exponential volume growth which admit a simple symmetric random walk with escape exponent* $1/2$ *whose abelian kernel has neither upper polynomial nor upper exponential distortion.*

With Theorem 2.3.1 it is easy to construct such groups which are not 2-generated; one just appends extra dimensions to the abelian kernel and makes the $\mathbb{Z}$ − *action* on these dimensions trivial. However, this requires adding extra generators to allow the random walk to reach these new dimensions. The examples we can derive from Theorem 2.1.7 are much more interesting.

Before dealing with specific groups, we present a simple result concerning the relationship between the rate of escape on a group and the rate of escape on quotients of that group. This result will help paint a picture of what rates of escape are possible in metabelian groups as well as hint at estimates for the rate of escape in other classes of groups. Finally, we will conclude this chapter with a summary of the results, as well as an extensive listing of groups to which they can be applied.

2.2 The rate of escape on quotients of finitely generated groups

The principal result of this section is the following lemma.

Lemma 2.2.1. *Let G be a finitely generated group with a subgroup H, and suppose X_n is a random walk on G driven by a measure μ. Set $\Gamma := H\backslash G$ and let $\pi : G \to \Gamma$ be the canonical projection. Then*

$$\mathbb{E}_\mu|\pi(X_n)|_\Gamma \leq \mathbb{E}_\mu|X_n|_G. \tag{2.15}$$

Proof. For each coset $\gamma \in H\backslash G$ we pick a unique g_γ such that $\gamma = Hg_\gamma$. Then we have

$$\begin{aligned}
\mathbb{E}_\mu|\pi(X_n)|_\Gamma &= \sum_{\gamma\in\Gamma} |\gamma|_\Gamma \mu^{(n)}(\gamma) \\
&= \sum_{g_\gamma:\gamma\in\Gamma} \min\{|hg_\gamma|_G \ : \ h \in H\} \mu^{(n)}(Hg_\gamma) \\
&= \sum_{g_\gamma:\gamma\in\Gamma} \left(\min\{|hg_\gamma|_G \ : \ h \in H\} \sum_{h\in H} \mu^{(n)}(hg_\gamma) \right) \\
&\leq \sum_{g_\gamma:\gamma\in\Gamma} \sum_{h\in H} |ng_\gamma|_G \mu^{(n)}(hg_\gamma).
\end{aligned}$$

For each $g \in G$, there exists $h \in H$ such that $g = hg_\gamma$ as the cosets of Γ partition G. Hence,

$$\sum_{g_\gamma : \gamma \in \Gamma} \sum_{h \in H} |hg_\gamma|_G \mu^{(n)}(hg_\gamma) = \sum_{g \in G} |g|_G \mu^{(n)}(g) \tag{2.16}$$

$$= \mathbb{E}_\mu |X_n|_G. \tag{2.17}$$

□

This lemma is no free lunch. To effectively use it one must often have a very good idea what the Schreier graph $\Gamma(G, H, \mathrm{supp}\{\mu\})$ looks like (see Section 1.3).

We now consider the potential application of this result to the Hanoi towers groups, $\mathcal{H}$ (see Appendix A). The Schreier graph $\Gamma(\mathcal{H}, \mathrm{Stab}(1), S)$ is homeomorphic to the infinite Sierpinski graph, and the corresponding limit space for the group is the Sierpinski Gasket. Barlow and Perkins have shown that Brownian motion, W_t, on the Sierpinski gasket satisfies

$$\mathbb{E}d(0, W_t) \simeq t^{1/d_w}, \tag{2.18}$$

where $d_w = \log(5)/\log(2)$ is the walk dimension [4]. Teufl has proven a refined version of this fact for a simple symmetric random walk on the infinite Sierpinski graph [50]. Similar estimates may be attainable on many fractal spaces, and one should be able to pass through the homeomorphism between the limit space and corresponding Schreier graph to prove the following conjectures.

Conjecture 2.2.1. *The Hanoi towers group, $\mathcal{H}$, admits a simple symmetric random walk on such that*

$$n^{\frac{\log 2}{\log 5}} \preceq \mathbb{E}|X_n|_{\mathcal{H}}. \tag{2.19}$$

Conjecture 2.2.2. *Let G be a self-similar group and let $\mathcal{X}$ be its limit space. If W_t is the Brownian motion of $\mathcal{X}$, then for any simple symmetric random walk X_n on G,*

$$\mathbb{E}d(0, W_n) \leq \mathbb{E}|X_n|_G. \tag{2.20}$$

2.3 Upper exponential distortion and the rate of escape

The goal of this section is to prove the following theorem, from which Theorem 2.1.5 follows as a corollary when H is nilpotent.

Theorem 2.3.1. *Let G be a torsion-free, finitely generated group with a short exact sequence,*

$$0 \to H \to G \to \mathbb{Z}^r \to 0, \tag{2.21}$$

where H is finitely generated and torsion-free. If H has upper exponential distortion in G, then any simple symmetric random walk on G has escape exponent $1/2$.

The idea underlying this theorem is, that under the assumption of upper exponential distortion of H in G, the distance travelled by a random walk after n steps can be controlled by the maximum of that random walk's projection to $\mathbb{Z}^r$. We will prove this theorem by generalizing to arbitrary generating sets of G some results obtained by Pittet and Saloff-Coste [42] for a particular class of generating sets of G.

We now introduce some notation we will use throughout this section. Fix a finite symmetric generating sets B of H and let A be the canonical basis of $\mathbb{Z}^r$. For a generating set S of G, note that the projection $\pi_{\mathbb{Z}^r}(S)$ is a generating set of $\mathbb{Z}^r$, but the projection $\pi_H(S)$ is not necessarily a generating set of H. This means that $\mathbb{Z}^r$ is undistorted in G, but H may be distorted. We will show that H is at most exponentially distorted.

Since commutators of elements in $\mathbb{Z}^r$ may be non-trivial in G it will be helpful to fix a standard embedding of $\mathbb{Z}^r$ into G. Let $k = (k_1, \dots, k_r) \in \mathbb{Z}^r$, and set $\mathbf{k} = a_1^{k_1} \cdots a_r^{k_r}$. The set $K = \{\mathbf{k} : k \in \mathbb{Z}^r\}$ is a section of $\mathbb{Z}^r$ in G, and every $g \in G$ can be written uniquely in normal form: $g = h\mathbf{k} = ha_1^{k_1} \cdots a_r^{k_r}$, $h \in H$, $\mathbf{k} \in K$. By $|k|$ we denote the length of k in the canonical basis for $\mathbb{Z}^r$.

We start with an elementary lemma on the expansion factor of automorphisms [42].

Lemma 2.3.1. *Let G be a finitely generated group with generating set S, and let H be a finitely generated subgroup of $Aut(G)$ with generating set T. Then there exists $q \geq 1$ such that for all $h \in H$ and for all $g \in G$*

$$|h \cdot g|_S \leq q^{|h|_T} |g|_S. \tag{2.22}$$

Proof. Set $q = \sup_{s \in S, t \in T} |t \cdot s|_S$. The desired estimate follows via induction. □

The following lemma establishes an exponential upper bound on the distortion of H in G.

Lemma 2.3.2. *Under the above assumptions, for any generating set S of G there exists non-negative integers q and C such that, given $g = h\mathbf{k}$,*

$$|h|_B \leq q^{|k|}, \tag{2.23}$$

and

$$|k| \leq C|g|_S. \tag{2.24}$$

The proof of the lemma relies on the following lemma taken directly from Pittet and Saloff-Coste. We exclude the proof as no changes are necessary to bring it into our present setting. Note that the word metric on G does not appear in this lemma.

Lemma 2.3.3. *There exists an integer $q > \max\{|[a, a']|_B \ : \ a, a' \in A\}$ such that for each $\epsilon \in \{-1, 1\}$, each $k \in \mathbb{Z}^r$, and each $i \in \{1, \ldots, r\}$, there exists $h \in H$ such that*

$$a_1^{k_1} \cdots a_r^{k_r} a_i^{\epsilon} = h a_1^{k_1} \cdots a_i^{k_i+\epsilon} \cdots a_r^{k_r} \tag{2.25}$$

and

$$|h|_B \leq q^{|k|}. \tag{2.26}$$

Proof of 2.3.2. In the normal form, we can write each $s \in S$ as $s = s_H s_K$, where the factors are the projections to H and K. Let $M_H = \max_{s \in H}\{|s_H|_B\}$ and $M_K = \max_{s \in K}\{|s_K|_A\}$. Fix q such that lemmas 2.3.3 and 2.3.1 hold and fix C such that $q^C \geq 1 + M_H + M_K q^{M_K}$. We will induct on $|g|_S$.

The result is trivial when $|g|_S = 0$. We will assume the estimates hold for group element with length at most l relative to S. Let $|g|_S = l + 1$. Then, for any $s \in S$ such that there exists $g' \in G$ with $g = g's$ and $|g'|_S = l$, we have by the induction hypothesis

$$g' = h'\mathbf{k}' \tag{2.27}$$

where $|h'|_B \leq q^{Cl}$ and $|k'| \leq Cl$.

Observe that

$$g = h'\mathbf{k}' s_H s_K \tag{2.28}$$

$$= h'(\mathbf{k}' s \mathbf{k}'^{-1})\mathbf{k}' s_K \tag{2.29}$$

$$\tag{2.30}$$

Set $x = \mathbf{k}' s \mathbf{k}'^{-1} \in H$. By Lemma 2.3.1, $|x|_B \leq q^{Cl} |s_H|_B \leq M_H q^{Cl}$.

We now apply Lemma 2.3.3 $|s_K|_A$-times to find $y_i \in H$, $i \in \{1, \ldots, |s_K|_A\}$ such that $\mathbf{k}' s_K = y_1 \cdots y_{|s_K|_A} \mathbf{k}$. We have

$$|y_i|_B \leq q^{|k'|+i} \leq q^{Cl+M_K}, \tag{2.31}$$

and $|k| = |k'| + |s_K|_A \leq Cl + M_K \leq C(l+1)$.

Thus $g = h' x y_1 \cdots y_{|s_K|_A} \mathbf{k}$, where

$$|h' x y_1 \cdots y_{|s_K|_A}|_B \leq q^{Cl} + M_H q^{Cl} + M_K q^{Cl+M_K} \tag{2.32}$$

$$\leq q^{C(l+1)}, \tag{2.33}$$

which completes the proof. □

We next adapt this result to paths in the group.

Lemma 2.3.4. *Under the conditions of Lemma 2.3.2, consider a sequence $\sigma = s_1 \cdots s_l \in S^l$. Let*

$$k(i) = \pi_{\mathbb{Z}^r}(s_1 \cdots s_l) = (k_1(i), \ldots, k_r(i)), \tag{2.34}$$

and set

$$M(\sigma) = \max_{1 \leq i \leq l} |k(i)|. \tag{2.35}$$

Then there exist constants q and C such that, for any l and any sequence $\sigma \in S^l$,

$$s_1 \cdots s_l = h\boldsymbol{k} \tag{2.36}$$

with $h \in H$ and $\boldsymbol{k} \in K$,

$$|h|_B \leq Cl q^{CM(\sigma)}, \tag{2.37}$$

and $|k| \leq M(\sigma)$.

Proof. Let $M_H = \max_{s\in H}\{|s_H|_B\}$ and $M_K = \max_{s\in K}\{|s_K|_A\}$. Fix q such that Lemmas 2.3.3 and 2.3.1 hold and fix C such that $C \geq M_H + M_K q^{M_K}$.

We will induct on l. If $l = 0$ the result is trivial. Assume the result holds for any sequence of length l. Set $\sigma = (s_1, \ldots, s_{l+1})$ and $\sigma' = (s_1, \ldots, s_l)$ with $s_i \in S$. Set $g' = s_1 \cdots s_l$ and $g = g' s_{l+1}$. From the induction hypothesis,

1. $g' = h'\mathbf{k}'$,
2. $|h'|_B \leq Clq^{CM(\sigma')}$, and
3. $|k'| \leq M(\sigma')$.

Observe that

$$g = h'\mathbf{k}' s_H s_K \tag{2.38}$$

$$= h'(\mathbf{k}' s_H \mathbf{k}'^{-1})\mathbf{k}' s_K. \tag{2.39}$$

Set $x = k' s k'^{-1} \in H$. By Lemma 2.3.1,

$$|x|_B \leq q^{C|k'|}|s_H|_B \leq M_H q^{CM(\sigma')}. \tag{2.40}$$

We now apply Lemma 2.3.3 $|s_K|_A$-times to find $y_i \in H$, $i \in \{1, \ldots, |s_K|_A\}$ such that $\mathbf{k}' s_K = y_1 \cdots y_{|s_K|_A}\mathbf{k}$. We have

$$|y_i|_B \leq q^{|k'|+i} \leq q^{CM(\sigma')+M_K}. \tag{2.41}$$

It follows from the definition of $M(\sigma)$ and the uniqueness of the normal form that $|k| \leq M(\sigma)$.

Finally,

$$|h'xy_1 \cdots y_{|s_K|_A}|_B \leq Clq^{CM(\sigma')} + M_H q^{CM(\sigma')} + M_K q^{CM(\sigma')+M_K} \tag{2.42}$$

$$= (Cl + M_H + M_K q^{M_K}) q^{CM(\sigma')} \tag{2.43}$$

$$\leq C(l+1) q^{CM(\sigma')} \tag{2.44}$$

$$\leq C(l+1) q^{CM(\sigma)}. \tag{2.45}$$

□

We have now assembled the tools need to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. We apply Lemma 2.3.4 and the hypothesis to see that if $X_n = h_n \mathbf{k}_n$ then

$$|\mathbf{k}_n|_S \leq |X_N|_S \leq |h_n|_S + |\mathbf{k}_n|_S \tag{2.46}$$

$$\leq C' \log(|h_n|_N + 1) + |\mathbf{k}_n|_S \tag{2.47}$$

$$\leq C(M_n + \log(n)) + |\mathbf{k}_n|_S, \tag{2.48}$$

for some $C > 0$.

To see that X_n has $1/2$-tight degree of escape, first observe that, for a simple symmetric random walk on $\mathbb{Z}^r$, there exist $\gamma, \delta > 0$ such that $P(\,|k_n| > \gamma \sqrt(n)\,) \geq \delta$, which establishes the lower bound for $|X_n|_S$. Next, note that there exists $c > 0$ such that $P(M_n > x) \leq c' \exp(-c'x^2/n)$ [47], which implies an upper bound of the form

$$P(|X_n|_S > xn^{1/2}) \leq c \exp(-cx^2) \tag{2.49}$$

exists for some $c > 0$.

The remaining results follow from classical results on simple symmetric random walks and their maxima on $\mathbb{Z}^r$ [47]. □

2.4 Metabelian groups

The behavior of the rate of escape for metabelian groups is more complex than that of polycyclic groups. Besides displaying a broader range of known behaviors[1], metabelian groups are more restrictive in terms of the generating sets for which we can determine the rate of escape. However, by using Theorem 2.1.2 and Lemma 2.2.1 we can get a rough picture of what rates of escape are possible. This is enabled by the following lemma of Baumslag, which tells us that wreath products essentially serve as universal objects in the class of metabelian groups [9].

Lemma 2.4.1. *Let G be a finitely generated metabelian group. Then there exists a free abelian group A of finite rank and a finitely generated abelian group T such that G is isomorphic to a subgroup of W/N, where $W = A \wr T$ and N is a normal subgroup of W contained in $\oplus_T A$.*

This leads to the following observation.

Lemma 2.4.2. *Let G be a finitely generated, torsion free metabelian group. Let W be as in Lemma 2.4.1. Let S_W be the generating set for W described in Theorem 2.1.2. Then for the simple symmetric random walk, X_n, on G induced by the projection of this generating set we have, depending on the dimension of T, d:*

1. *if $d = 1$, $n^{1/2} \preceq \mathbb{E}|X_n| \preceq n^{3/4}$,*
2. *if $d = 2$, $n^{1/2} \preceq \mathbb{E}|X_n| \preceq n/\log(n)$, and*
3. *if $d \geq 3$, $n^{1/2} \preceq \mathbb{E}|X_n| \preceq n$.*

[1] Presumably, any simple symmetric random walk on any polycyclic group has escape exponent 1/2.

We will explore the first case of the above lemma more in the following sections.

2.4.1 Abelian by cyclic groups and polynomials

Considering a random walk on an abelian-by-cyclic group leads very quickly to an important observation about the relationship between such groups and the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. Let $G = A \rtimes_\phi \mathbb{Z}$, where A is torsion free. We will treat A as a vector space and denote elements thereof in boldface. We will assume that $\phi : \mathbb{Z} \to \mathrm{SL}_d(A)$ is *irreducible*, that is, the characteristic polynomial, p_ϕ, of ϕ is irreducible. This allows us to assume that G is 2-generated by a basis element of A and a generator of $\mathbb{Z}$. Explicitly, these generating sets are of the form

$$S = \{(\pm\mathbf{w}, 0)\} \cup \{(e, \pm 1)\}, \tag{2.50}$$

for some basis element $\mathbf{w}$ of A. We will concern ourselves only with random walks on such generating sets, but note that one can add elements of the form $(\pm\mathbf{v}, 0)$ to the generating sets and still apply the techniques described below.

Remark If p_ϕ were reducible we might need to admit additional basis vectors of A into our generating set. In particular, this happens when A is the direct sum of at least two ϕ-invariant subspaces. However, it is possible for ϕ to be reducible without requiring additional elements in the generating set. This occurs when the direct sum of the ϕ-invariant subspaces is of finite index in A. We treat such an example in Section 2.5.3.

Consider the random walk X_n on G driven by μ_S. Let $X_n = (\mathbf{W}_n, Y_n)$, where $\mathbf{W}_n \in A$ and $Y_n \in \mathbb{Z}$. Let $\xi_i = (\mathbf{w}_i, y_i)$ denote the increments of (X_n, μ_S), and denote the distributions of $\mathbf{w}_i$ and y_i as $\pi_A(\mu_S)$ and $\pi_{\mathbb{Z}}(\mu_S)$, respectively. Then $Y_n = \sum_{i=1}^n y_i$ is a simple symmetric random walk on $\mathbb{Z}$ with distribution $\pi_{\mathbb{Z}}(\mu_S)$, but the behavior of $\mathbf{W}_n$ is more complicated. Observe that

$$\mathbf{W}_n = \left(\sum_{i=1}^{n} \phi^{Y_{i-1}} \mathbf{w}_i\right) \tag{2.51}$$

$$= \left(\sum_{i\in\mathbb{Z}} \omega_i(n)\phi^i\right)\mathbf{w}, \tag{2.52}$$

where $\omega_i(n)$ are i.i.d. random variables equal in distribution to a simple random walk on $\mathbb{Z}$ with distribution $\pi_{\mathbb{Z}}(\mu_S)$, stopped at the random time

$$\theta_i(n) = \#\{0 < k < n \mid Y_k = i\}, \tag{2.53}$$

which is the local time of Y_n, ignoring endpoints. The order of $\sum_{i\in\mathbb{Z}} |\omega_i(n)|$ determines the rate of escape for $\mathbb{Z}^d \wr \mathbb{Z}$, and thus provides an upper bound on the rate of escape for abelian-by-cyclic groups.

From (2.52), it is clear that we can represent elements of A by a polynomial in $\mathbb{Z}[t, t^{-1}]$ evaluated at ϕ and acting on $\mathbf{w}$. We will denote this random polynomial by P_n. Thus the map $\mathbb{Z}[t, t^{-1}] \to A$ corresponds to the composition of the evaluation map $P_n \mapsto P_n(\phi)$ with the action of $\mathrm{SL}d(A)$ on A.

We now define some properties we will use in the study of P_n. The length of a polynomial p is given by $\|p\|_{\mathcal{P}} := \sum_i |p_i|$ where p_i is the coefficient of the ith degree term of p. We let $M(p) := \max\{i : |p_i| > 0\}$ and $m(p) := \min\{i : |p_i| > 0\}$. We will refer to $d(p) = M(p) - m(p)$ as the *diameter* of p. We also set $K(p) := \max_i |p_i|$.

For $\mathbb{Z}\wr\mathbb{Z}$ each element of $\mathbb{Z}[t, t^{-1}]$ corresponds to a distinct lamp configuration, while in G multiple elements of $\mathbb{Z}[t, t^{-1}]$ may represent the same element of A. In particular, if p_ϕ is the characteristic polynomial of ϕ, then, by the Cayley-Hamilton theorem[2], the following diagram commutes.

[2]The theorem states that for any square matrix over a commutative ring, the matrix is a

$$
\begin{tikzcd}
\mathbb{Z}[t,t^{-1}] \arrow[r] \arrow[dr] & \mathbb{Z}[t,t^{-1}]/\langle p_\phi \rangle \arrow[d] \\
& A
\end{tikzcd}
$$

Hence, rather than using $||P_n||_{\mathcal{P}}$ to estimate $|\mathbf{W}_n|_G$, we can obtain more accurate estimates by using a representative in $\mathbb{Z}[t,t^{-1}]/ < p_\phi >$ with smaller length. We reduce P_n modulo p_ϕ via a process akin to a division algorithm which lowers $K(P_n)$ at the expense of increasing $d(P_n)$.

2.4.2 The flattening lemma

We now restrict our attention solely to the modification of polynomials in $\mathbb{Z}[t,t^{-1}]$. The ability to reduce constant polynomials is also sufficient to establish the upper exponential distortion of the (finitely generated) abelian kernel of an abelian-by-cyclic group, so we present this case separately. Once constant polynomials can be reduced, we can also reduce arbitrary polynomials.

Throughout this section, we will use log to denote the positive part of the logarithm function, $\log^{(k)}$ to denote the kth iterate of the positive part of the logarithm function, and $\log^*$ to denote number of iterates of the logarithm needed to to produce an output less than 1.

We recall for the reader that a polynomial p has property (EDP) if

$$
\max_i{}^{(1)}|p_i|\left(1-\frac{1}{\max_i{}^{(2)}|p_i|}\right) > \sum_i |p_i|, \tag{2.54}
$$

where p_i denotes the coefficient of t^i in p.

solution of its characteristic polynomial.

Lemma 2.4.3 (Flattening Lemma)**.** *Fix $K \in \mathbb{Z}$. If $y \in \mathbb{Z}[t, t^{-1}]$ has property (EDP), then there exists $x \in \mathbb{Z}[t, t^{-1}]$ such that $p(t) = K + x(t)y(t)$ satisfies the following*

1. *$K(p) < cy_0$ for some $c > 0$, and*
2. *$M(p) - m(p) \leq C(b - a)\log K$ for some $C > 0$.*

In particular, $\|p\|_{\mathcal{P}} = O(\log K)$.

Proof. We assume without loss of generality that K is positive and $\max_i^{(1)} |y_i| = y_0$. Let $-a$ and b denote the minimal and maximal degrees such that y_i is non-zero. Set $\delta := \|y\|_{\mathcal{P}} - y_0$ and $r := \frac{\max_{i \neq 0} |y_i|}{y_0}$. Note that (EDP) implies that $\delta r < 1$.

Set $p^{(1)} = K - \lfloor K/y_0 \rfloor y$. It is clear that $K(p^{(1)}) \leq rK$, $M(p^{(1)}) = b$, and $m(p^{(1)}) = a$. However, the constant term is $K \mod y_0$.

We now proceed by induction on the diameter of $p^{(i)}$, $d(p^{(i)}) = M(p^{(i)}) - m(p^{(i)})$. We have just done the $d = 1$ case. We will keep the salient information about our updates to p in terms of $K_i := K(p^{(i)})$ and $d_i := d(p^{(i)})$.

Suppose we are given K_i and d_i. Applying the base case to each term in $p^{(i)}$ yields $d_{i+1} = d_i + a + b$ and

$$\begin{aligned} K_{i+1} &\leq \delta r K_i + y_0 && (2.55)\\ &= (\delta r)^i rK + \sum_{j=0}^{i-1} (\delta r)^j y_0 && (2.56)\\ &= (\delta r)^i rK + \frac{1 - (\delta r)^i}{1 - \delta r} y_0. && (2.57) \end{aligned}$$

Thus for i on the order of $\log K$, K_{i+1} will be bounded by some constant multiple of y_0, and the desired estimates hold. ◻

Lemma 2.4.3 can also be used to produce an upper bound on the length of an arbitrary polynomial. This is useful when one does not necessarily have upper exponential distortion of the abelian kernel in an abelian-by-cyclic group.

Lemma 2.4.4. *Suppose $y \in \mathbb{Z}[t, t^{-1}]$ has property (EDP). If $P \in \mathbb{Z}[t, t^{-1}]$, then there exists $Q \in \mathbb{Z}[t, t^{-1}]$ such that*

1. $P(A) - Q(A) \in \langle y \rangle$,
2. *there exists $C > 0$, independent of P, such that $K(Q) \leq C$, and*
3. $d(Q) \leq d(P) + O\left((\log^* K(P))(\log K(P))\right)$.

Proof. To prove this we will apply Lemma 2.4.3 iteratively to $P^{(0)} := P$. This ensures (1). For the first pass we apply Lemma 2.4.3 to each term of P. This produces a polynomial $P^{(1)}$ such that

1. $K(P^{(1)}) \leq |y_0| \log K(P^{(0)})$, and
2. $d(P^{(1)}) \leq d(P^{(0)}) + C' \log K(P^{(0)})$ for some $C' > 0$.

On the nth pass we have

1. $K(P^{(n)}) \leq |y_0| \sum_{i=1}^{n-1} \log^{(i)} |y_0| + \log_n K(P^{(0)})$, and
2. $d(P^{(n)}) \leq d(P^{(0)}) + C \sum_{i=0}^{n-1} \log K(P^{(i)})$, for some $C > 0$.

It takes on the order of $\log^* K(P^{(0)})$) iterations to bring $K(P^{(n)})$ below some fixed constant C (which depends on $|y_0|$), from which the desired estimates follow. □

We can now apply these results to the rate of escape.

Proof of Theorem 2.1.6. Note that

$$|Y_n|_G \leq |X_n|_G \leq |\mathbf{W}_n|_G + |Y_n|_G. \tag{2.58}$$

By dint of the polynomial representation of $\mathbf{W}_n$, Lemma 2.4.4 implies that

$$|\mathbf{W}_n|_G \leq \|Q_n\|_{\mathcal{P}} \tag{2.59}$$

$$= O(d(P_n)), \tag{2.60}$$

where Q_n is the reduced version of P_n. One can observe that $d(P_n)$ is on the order of the maximum of Y_n, and so X_n has escape exponent $1/2$ via the reasoning in the proof of Theorem 2.3.1. □

2.4.3 The expanding and neutral eigenspaces

We no turn our attention towards proving Theorem 2.1.7. We will continue to operate under the notation and assumptions introduced in Section 2.4.1. Implicitly, the characteristic polynomial of an automorphism of an abelian group will satisfy (*EDP*) when all of the eigenvalues of the automorphism have modulus distinct from one. However, the inclusion of eigenvalues of modulus one need not be an impediment to determining the rate of escape. We now introduce some notions and notation that will help us talk more accurately about $\mathbb{Z}$-actions on $\mathbb{Z}^d$.

Fix $\phi \in \mathrm{SL}(A)$, and let d denote the abelian rank of A. The *expanding eigenspace*, E_+, of ϕ is the span of the eigenvectors whose eigenvalues have modulus distinct from 1. Note that the expanding eigenspace includes the directions in which ϕ is expanding and the directions in which it is contracting. We lump these two sets of directions together because our random walks can move via both ϕ and

ϕ^{-1}, so they can move fast relative to the intrinsic geometry of $\mathbb{Z}^d$ in each of these directions.

The *neutral eigenspace*, E_0, of ϕ is the span of the eigenvectors whose eigenvalues have modulus 1. Points in E_0 are not necessarily undistorted in G; for instance, if some of the corresponding eigenvalues have multiplicities greater than 1, we may see polynomial distortion as in the Heisenberg group. Outside of exceptions like this, it is convenient to think of ϕ as acting like a rotation on E_0.

We will need to index over the eigenvectors which span these eigenspaces. We will denote their index sets by I_+ and I_0. We will refer to the eigenvalues as $\{\lambda_i\}$ and the corresponding eigenvectors by $\{\mathbf{v}_i\}$. Furthermore, we will assume the eigenvalues satisfy

$$|\lambda_1| \geq \cdots \geq |\lambda_d|. \tag{2.61}$$

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{C}^d$, which projects isometrically to the Euclidean norm on $\mathbb{R}^d$, $\mathbb{Z}^d$, or $\mathbb{Z}[\rho]^d$. Note that the Euclidean metric on $\mathbb{Z}^d$ is equivalent to any intrinsic metric of $\mathbb{Z}^d$ as a finitely generated group. We will also use $\|\cdot\|$ to denote the extension of this norm to linear operators on $\mathbb{C}^d$ (i.e. the L^2 norm).

Let π_+ denote the projection to E_+ and π_0 the projection to E_0. We denote the distance of a vector $\mathbf{v} \in \mathbb{C}^d$ from E_+ as

$$d_+(\mathbf{v}) := \|(I - \pi_+)\mathbf{v}\|. \tag{2.62}$$

Likewise, we denote the distance from E_0 as d_0.

Lemma 2.4.5. *There exists $C > 0$ such that for all $k \in \mathbb{Z}$ and any $\mathbf{v} \in \mathbb{C}^d$*

$$d_0(\phi^k \mathbf{v}) \leq C|\lambda_1|^k, \tag{2.63}$$

and

$$d_+(\phi^k \mathbf{v}) \leq C. \tag{2.64}$$

Proof. Let w_i denote the coefficients of $\mathbf{w}$ in its eigendecomposition. Then

$$(I - \pi_0)\phi^k \mathbf{w} = \sum_{i \in I_+} \lambda_i^k w_i (I - \pi_0)\mathbf{v}_i, \tag{2.65}$$

and

$$(I - \pi_+)\phi^k \mathbf{w} = \sum_{i \in I_0} \lambda_i^k w_i (I - \pi_+)\mathbf{v}_i. \tag{2.66}$$

The desired estimates follow by our assumptions on the moduli of the eigenvalues. □

The above lemma establishes how far the increments of $\mathbf{W}_n$ can move the random walk from E_0 and E_+. This result can be seen as a more precise version of Lemma 2.3.1 for abelian-by-cyclic groups, which tells us when the exponential upper bound is accurate. When $\mathbb{Z}^d$ has upper exponential distortion $E_0 = \{0\}$, so in this case the exponential upper bound is sharp.

Using the above lemma, we can see that a random walk will move away from the expanding eigenspace at rate $n^{1/2}$.

Lemma 2.4.6. *There exists $C > 0$ such that*

$$\mathbb{E}_\mu d_+(\mathbf{W}_n) \leq C n^{1/2} \tag{2.67}$$

for all $n \geq 0$.

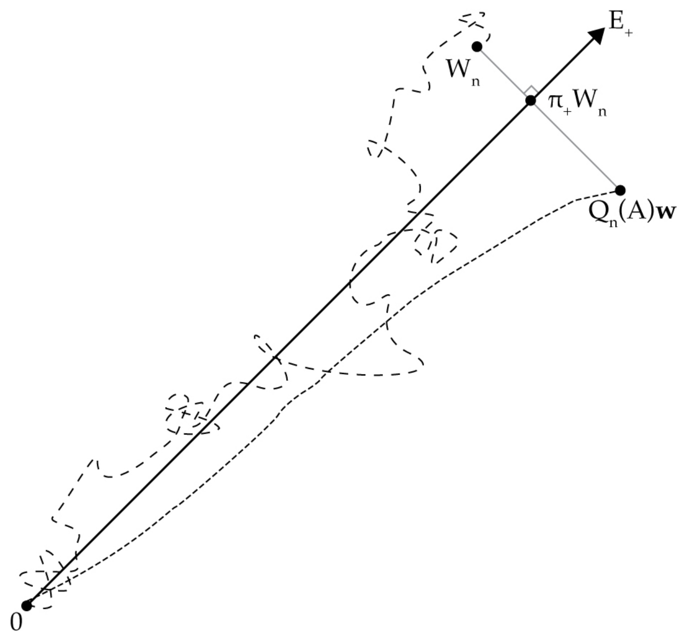

Figure 2.1: The model of an efficient path in $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ projected to $\mathbb{Z}^d$.

Proof. Consider the process $\mathbf{W}_n^0 = (I - \pi_+)\mathbf{W}_n$, which gives the deviation of $\mathbf{W}_n$ from E_+. This process is a random walk in $\mathbb{C}^d$ whose increments have bounded length. The increments are of the form

$$\xi_{k_n}^{\epsilon} = \epsilon\phi^{k_n}\mathbf{w}, \tag{2.68}$$

where $k_n \in \mathbb{Z}$, $\epsilon \in \{-1, 1\}$, and $\mathbf{w} = \sum_{i \in I_0} w_i(I - \pi_0)\mathbf{v}_i$. The distribution on $\epsilon\mathbf{w}$ is uniform, but the distribution of k_n is equal to that of Y_n. However, if we let μ_n denote the distribution on the nth increment of the process, we have

$$\sum_{\epsilon}\sum_{k} \xi_k^{\epsilon}\mu_n(\xi_k^{\epsilon}) = 0. \tag{2.69}$$

As the increments of $\mathbf{W}_n^0$ are centered and have bounded length, the desired result holds by an application of the Lindeberg-Feller theorem [17]. □

We now give a heuristic as to how we plan to take advantage of the above lemma. Suppose $\mathbf{W}_n$ is typical for the process. We know it takes on the order of $n^{1/2}$ steps in $\mathbb{Z}^d$ to get from $\mathbf{W}_n$ to E_+. Next we will show that it takes on the order of $n^{1/2}$ steps utilizing the action of ϕ to get to a point with the same projection to E_+ as $\mathbf{W}_n$. In doing so, we will generally pick up some distance from $\mathbf{W}_n$, but so long as this is on the order of $n^{1/2}$ we will have shown that X_n is roughly distance $n^{1/2}$ from the origin. We will do this by using Lemma 2.4.4 to reduce P_n relative to $p_+ := \sum_{i \in I_+} p_i t^i$.

To do this we will need the following generalization of the Cayley-Hamilton theorem. The proof is a direct computation and is left to the reader.

Lemma 2.4.7. *Let $\phi \in SL_d(\mathbb{C})$. Denote the eigenvalues of ϕ by λ_i and the corresponding eigenvectors by v_i. Fix an index set $I \subseteq \{1, \ldots, d\}$. Let $E_I = span_{i \in I}\{v_i\}$. Then ϕ is a solution of*

$$p_I(x) := \prod_{i \in I}(x - \lambda_i) = 0 \tag{2.70}$$

over E_I.

Lemma 2.4.8. *Given P_n, if p_+ has property (EDP), then there exists $Q_n \in \mathbb{Z}[t, t^{-1}]$ such that*

1. *$P_n - Q_n \in \langle p_+ \rangle$,*
2. *$P_n(\phi) - Q_n(\phi)$ acts trivially on E_+,*
3. *$d_+(Q_n(\phi)\mathbf{w}) \leq O(\|Q_n\|_{\mathcal{P}})$ for all $\mathbf{w}$, and*
4. *there exists $C > 0$, independent of n, such that $\|Q_n\|_{\mathcal{P}} \leq M_n - m_n + C(\log^* K_n)(\log K_n)$.*

Proof. We apply Lemma 2.4.4 to P_n using $y = p_+$. Item (1) follows from the construction used in Lemma 2.4.4, and item (4) is the second conclusion of Lemma 2.4.4.

Item (2) follows from (1) and the following observation: if $u \in E_+$, then

$$p_+(\phi)\mathbf{u} = \sum_{i \in I_+} \prod_{i \in I_+} (A - \lambda_i I) u_i \mathbf{v}_i \tag{2.71}$$

$$= 0. \tag{2.72}$$

For (3) we compute that

$$d_+(Q_n(\phi)\mathbf{w}) = \|(I - \pi_+)Q_n(\phi)\mathbf{w}\| \tag{2.73}$$

$$= \|(I - \pi_+) \sum_i Q_n(\lambda_i) w_i \mathbf{v}_i\| \tag{2.74}$$

$$= \| \sum_{i \in I_-} Q_n(\lambda_i) w_i \mathbf{v}_i\| \tag{2.75}$$

$$\leq \sum_{i \in I_-} \|Q_n(\lambda_i) w_i \mathbf{v}_i\| \tag{2.76}$$

$$\leq C \sum_{i \in I_-} \|Q_n(\lambda_i)\|. \tag{2.77}$$

As λ_i has modulus 1 for each $i \in I_-$, the final sum is on the order of $\|Q_n\|_{\mathcal{P}}$. □

We can now conclude that the random walk on G has escape exponent $1/2$.

Proof of Theorem 2.1.7. Recall that $\mathbf{W}_n = P_n(\phi)\mathbf{w}$. For the conclusion to hold we only need to show that $|\mathbf{W}_n|_G$ has the desired behavior. This is done by showing that $|\mathbf{W}_n|_G$ is on the order of M_n plus a term that behaves like the displacement of random walk with bounded increments on a Euclidean space. As in the proof of Theorem 2.1.5, this will be sufficient to imply escape exponent $1/2$.

Apply Lemma 2.4.8 to obtain Q_n. Observe that

$$\mathbf{W}_n = Q_n(\phi)\mathbf{w} - (P_n(\phi) - Q_n(\phi))\mathbf{w}, \tag{2.78}$$

and so

$$|\mathbf{W}_n|_G \leq |Q_n(\phi)\mathbf{w}|_G + |(P_n(\phi) - Q_n(\phi))\mathbf{w}|_G. \tag{2.79}$$

As $(P_n(\phi) - Q_n(\phi))\mathbf{w}$ lies in E_0, we have, by an application of the triangle inequality and Lemmas 2.4.6 and 2.4.8,

$$|(P_n(\phi) - Q_n(\phi))\mathbf{w}|_G = O(\|Q_n\|_{\mathcal{P}} + n^{1/2}). \tag{2.80}$$

Combining this with the estimate on $|Q_n(\phi)\mathbf{w}|_G$ to be had from Lemma 2.4.8, we have

$$|\mathbf{W}_n|_G = O(\|Q_n\|_{\mathcal{P}} + n^{1/2}), \tag{2.81}$$

which is on the order of the maximum of Y_n. Escape exponent $1/2$ follows from this observation. □

2.5 Summary of results

2.5.1 Polycyclic groups

Combining Theorems 2.1.5 and 2.1.1 we have the following.

Theorem 2.5.1. *Let G be a torsion-free polycyclic group. If the nilpotent kernel of G has either upper polynomial distortion or upper exponential distortion then any simple symmetric random walk on G has escape exponent $1/2$. Furthermore, the former case corresponds to G having polynomial volume growth, while the latter corresponds to G having exponential volume growth.*

We now provide some tools for determining when the nilpotent kernel of a polycyclic group has upper exponential distortion. Sol and similar groups make for a good place to begin.

Lemma 2.5.1. *Let $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $A \in SL_2(\mathbb{Z})$ with $|tr(A)| > 2$. Then $\mathbb{Z}^2$ is strictly exponentially distorted in G.*

Proof. Let $B = b_1, b_2$ be the canonical basis element of $\mathbb{Z}^2$, and let z be the standard basis element of $\mathbb{Z}$. We compute that

$$z^k b_i z^{-2k} b_i^{\det(A)^k} = A^k b_i + \det(A)^k M^- k b_i \tag{2.82}$$

$$= b_i^{\mathrm{tr}(A^k)}. \tag{2.83}$$

The left hand side above has length $3k+2$ in the word metric corresponding to S, while $b_i^{\mathrm{tr}(A^k)}$ has length exponential in k in terms on the word metric corresponding to B. Thus any $w \in \mathbb{Z}^2$ can be written in G as a word with length on the order of $\log(|w|_B + 1)$. □

We can move beyond abelian-by-cyclic polycyclic groups using the following observation.

Lemma 2.5.2. *Let $G = N \rtimes \mathbb{Z}^r$ be a torsion-free polycyclic group. Consider N as a group of upper-triangular matrices with ones on the diagonal. If two coordinate subgroups, H_1 and H_2, of N have upper exponential distortion in G, then so does $K = [H_1, H_2]$.*

Proof. By hypothesis either $H_1 \cong H_2 \cong K \cong \mathbb{Z}$ or K is trivial. We have assumed that the trivial group has upper exponential distortion so we will suppose that $K \cong \mathbb{Z}$. Let $k \in K$. Then for any $h_1 \in H_1$ and $h_2 \in H_2$ such that $k = [h_1, h_2]$,

$$|k|_G \leq 2(|h_1|_G + |h_2|_G) \tag{2.84}$$

$$\leq c_1 \log(|h_1|_{H_1} + 1) + c_2 \log(|h_2|_{H_2} + 1) + c_1 + c_2. \tag{2.85}$$

The upper exponential distortion of K follows from this as $|k|_K \geq |h_1|_{H_1}, |h_2|_{H_2}$. □

Combining, the prior lemmas lets us construct nilpotent-by-cyclic groups whose nilpotent kernels have upper exponential distortion.

Corollary 2.5.1. *Let $H_3(\mathbb{Z})$ denote the three dimensional Heisenberg group over $\mathbb{Z}$ and set $G = H_3(\mathbb{Z}) \rtimes_\phi \mathbb{Z}$ where ϕ acts on the abelianization of $H_3(\mathbb{Z})$ as an element of $SL_2(\mathbb{Z})$, with trace greater than 2 in absolute value. Then $H_3(\mathbb{Z})$ has upper exponential distortion in G.*

2.5.2 Metabelian groups

In what follows we will assume that G is a two-generated abelian-by-cyclic group group. This is done for ease of presentation rather than for any failure of the techniques used. It is clear that we may take any symmetric generating set in the cyclic subgroup without changing the rate of escape as the behavior of the maximum is changed only up to a constant multiple. Admitting additional generators in the abelian kernel is also not problematic. If we have generators $\{\mathbf{w}^{(i)}, ..., \mathbf{w}^{(k)}\}$, then $W_n = \sum_{i=1}^{k} P^(i)_n(\phi)\mathbf{w}^{(i)}$, and we now just have to apply the Flattening Lemma independently to each of these polynomials. Thus, so long as our generating set contains no elements with non-trivial projections to both the abelian kernel and the cyclic subgroup, the corresponding random walk will have escape exponent 1/2. We view this as merely a technical restriction.

It is clear that the results on metabelian groups apply without hassle to groups whose abelian kernel is $\mathbb{Z}^d$. When can we apply these results to groups whose abelian kernel is of the form $\mathbb{Z}[\rho]^d$, where ρ is an algebraic number? If ρ is rational, then some integer multiple of p_ϕ will lie in $\mathbb{Z}[t, t^{-1}]$, so our results readily apply to the Baumslag-Solitar groups $BS(1, n)$ and higher dimensional generalizations thereof. If ρ is not rational, one needs to find a multiple of p_ϕ with (EDP) (which implicitly must have integer coefficients).

Since metabelian groups may have non-finitely generated subgroups, subgroup distortion is not the appropriate tool to analyze the behavior of random walks. Instead, for abelian-by-cyclic groups we can generalize to the characteristic polynomial of ϕ having property (EDP).

Lemma 2.5.3. *Let $G = A \rtimes_\phi \mathbb{Z}$ where A is finitely generated. If a multiple of the characteristic polynomial of ϕ satisfies (EDP), then A has upper exponential distortion in G.*

Proof. By the hypothesis, Lemma 2.4.3 states that the polynomial $x(t) = K$ can be represented in $\langle p_\phi \rangle$ by a polynomial p with $\|p\|_{\mathcal{P}} = O(\log K)$. Via the polynomial model for G, p represents a word in G with length $\|p\|_{\mathcal{P}}$. This implies that any d-tuple in A, where d is the abelian rank of A, can be represented in G by a word whose length is logarithmic in terms of the length in A, and so upper exponential distortion holds. $\square$

We now highlight a specific case of the above lemma. For $\phi \in \mathrm{SL}_2(\mathbb{Z})$, there are hyperbolic ϕ with $|\mathrm{tr}(\phi)| \leq 2$. One can check that in this case $\det(\phi) = -1$. Then

$$(x^2 + \mathrm{tr}(\phi)x - 1)p_\phi(x) = x^4 - (2 + \mathrm{tr}(\phi)^2)x^2 + 1, \tag{2.86}$$

so these examples have upper exponential distortion as well. A similar analysis to this was done by Warshall to study the existence of dead-ends in the Cayley graphs of abelian-by-cyclic groups [54]. This observation, along with the fact that non-hyperbolic automorphisms imply polynomial volume growth[3], leads to the following corollary

Corollary 2.5.2. *Let ϕ be a hyperbolic automorphism of $\mathbb{Z}^2$. Then $\mathbb{Z}^2$ has upper exponential distortion in $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ if and only if ϕ is hyperbolic.*

For groups with higher dimensional abelian kernels the situation is more complicated. We treat this case in the following section. We close this section with an example where G is not 2-generated.

Example. Let $\lambda = \frac{1+\sqrt{5}}{2}$, and note that $1/\rho = \rho - 1$. Consider the group $G = \mathbb{Z}[\rho]^2 \rtimes_\phi \mathbb{Z}$, where

$$\phi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{2.87}$$

Since ϕ is diagonal we need to take two generators from $\mathbb{Z}[\rho]^2$, namely $\mathbf{e}_1$ and $\mathbf{e}_2$, the standard basis vectors of $\mathbb{Z}^2$. Note that $\mathbb{Z}[\rho]^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \sqrt{5}\mathbf{e}_1, \sqrt{5}\mathbf{e}_2\}$. The characteristic polynomial of ϕ is $x^2 - \sqrt{5}x + 1$, which does not lie in $\mathbb{Z}[t, t^{-1}]$. However,

$$(x^2 + \sqrt{5}x + 1)p_\phi = x^4 - 3x^2 + 1, \tag{2.88}$$

so property (EDP) is satisfied, and we can conclude that any symmetric random walk on G whose law has support $\{(\pm\mathbf{e}_1, 0), (\pm\mathbf{e}_2, 0), (\mathbf{0}, \pm 1)\}$ has escape exponent $1/2$.

It is interesting to note that $\mathbb{Z}[\rho]^2$ is isomorphic to $\mathbb{Z}^4$, and that the action induced on $\mathbb{Z}^4$ by ϕ corresponds to the matrix (with coordinates ordered as

[3]See Theorem 1.2.6.

$(\mathbf{e}_1, \sqrt{5}\mathbf{e}_1, \mathbf{e}_2 \sqrt{5}\mathbf{e}_2))$,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.89}$$

which has characteristic polynomial $p_M(x) = x^4 - 3x^2 + 1$. The moral of this example is that if one wishes to consider abelian kernels of the form $\mathbb{Z}[\rho]^d$ where ρ is an algebraic number, it may pay rewrite the group in terms of an action on $\mathbb{Z}^{d'}$ for suitably chosen d'.

2.5.3 Abelian kernels without upper exponential distortion

There are two examples we wish to highlight. The first example is due to Conner [12]. Let $G_1 = \mathbb{Z}^4 \rtimes_\phi \mathbb{Z}$ where

$$\phi = \begin{pmatrix} 2 & -1 & 2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.90}$$

G has exponential volume growth, but every cyclic subgroup is undistorted in G [12, 41].

The matrix ϕ has two positive, real eigenvalues distinct from 1 and two complex eigenvalues of modulus 1. The characteristic polynomial factors as

$$p_\phi(x) = (x^2 - (1 + \sqrt{2})x + 1)(x^2 - (1 - \sqrt{2})x + 1), \tag{2.91}$$

where the first factor corresponds to the eigenvalues with roots of modulus distinct from 1. That the characteristic polynomial does not factor over $\mathbb{Z}$ makes this example difficult to work with.

We also consider $G_2 = \mathbb{Z}^4 \rtimes_\psi \mathbb{Z}$ for

$$\psi = \begin{pmatrix} 3 & -2 & 3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.92}$$

This group has two real eigenvalues distinct from 1 and two complex eigenvalues of modulus 1, namely i and $-i$. The characteristic polynomial factors as

$$p_\psi(x) = (x^2 - 3x + 1)(x^2 + 1), \tag{2.93}$$

The first factor corresponds to the eigenvalues with modulus distinct from 1 while the other factor corresponds to the eigenvalues with modulus 1.

Every cyclic subgroup of G_1 is undistorted [12]. This is because the E_+ and E_0 for ϕ only intersect $\mathbb{Z}^4$ at the origin. However, the characteristic polynomial of ψ splits over $\mathbb{Z}$ so E_+ and E_0 have non-trivial intersections with $\mathbb{Z}^4$. In particular, $(E_+ \oplus E_0) \cap \mathbb{Z}^4$ has finite index in $\mathbb{Z}^4$. One can also check that G_2 is two generated if $\mathbf{w}$ is taken to be one of the canonical basis vectors of $\mathbb{Z}^4$; this happens because the basis vectors do not lie in $(E_+ \oplus E_0) \cap \mathbb{Z}^4$. This establishes Corollary 2.1.1.

CHAPTER 3.

CRITICAL CONSTANTS FOR RECURRENCE

3.1 Introduction

Whether or not a Markov chain is recurrent is one of the most basic probabilistic questions we can ask. On a countably infinite state space the problem was first treated by Polya, who showed that the integer lattice carries a recurrent random walk only in dimensions one and two [43]. Later this question was generalized to groups in the form of Kesten's conjecture: groups with at most quadratic volume growth are the only ones that admit recurrent, non-degenerate simple random walks. This result was proven by Varopoulos [53], utilizing Gromov's theorem that polynomial volume growth is equivalent to containing a nilpotent subgroup of finite index [30]. More recently, others have examined the question of recurrence for subgroups or quotients of finitely generated groups [21,22,45]. Gallardo and Schott showed that a homogeneous space of a simply connected nilpotent Lie group is recurrent if and only if it has polynomial growth of degree at most two [25]. Later, in Theorem 3.2.1 we will see an extension of this result to finitely generated nilpotent groups.

In general, the recurrence or transience of quotients is hard to determine as we lack good heat kernel estimates and structural theorems. However, progress has been made for certain classes of groups. Nilpotent groups have a very strong algebraic structure which is inherited by their subgroups, and using this fact Revelle showed that an analog of Varoupoulos's theorem holds for quotients of groups of polynomial growth [44]. Another important class of examples comes from self-similar groups, where many of the quotients are fractal [39].

One way to study recurrence is to ask what conditions we can place on the law of a random walk to ensure recurrence. For example there are classical results that a finite first moment guarantees recurrence on $\mathbb{Z}$ and a finite second moment guarantees recurrence on $\mathbb{Z}^2$ (see Chapter 2, section 8 in [49]). To study this question we will use the critical constant for recurrence, c_{rt}, introduced by Erschler in [21].

Definition 3.1.1. *For a finitely generated group G with subgroup H the critical constant for recurrence, c_{rt}, is defined as the supremum of the $\beta \geq 0$ such that there exists a measure μ on G with*

$$\sum_{g \in G} |g|^{\beta} \mu(g) < \infty, \tag{3.1}$$

whose induced random walk on $H \backslash G$ is transient.

Let $B_G(R)$ denote the ball of radius R in G. For the purpose of the above definition, the moment condition (3.1) can be replaced by the following *tail condition*:

$$\mu(G \backslash B_G(R)) \leq C R^{-\beta}, \tag{3.2}$$

for $R \geq 1$ and some constant $C > 0$. Replacing one condition by the other produces the same value for the critical constant. Often, it is easier to check the tail condition than the moment condition.

From the definition, it is clear that $c_{rt}(G, H) = 0$ if H is of finite index in G and that $c_{rt}(G, H) = \infty$ if any simple random walk on the quotient is transient. On $\mathbb{Z}$ and $\mathbb{Z}^2$ one can construct measures with moments $1 - \epsilon$ and $2 - \epsilon$, respectively, for any $\epsilon > 0$ such that the random walks are transient (see Example 8.2 in [49]). Thus $c_{rt}(\mathbb{Z}, 1) = 1$ and $c_{rt}(\mathbb{Z}^2, 1) = 2$. This essentially exhausts the behavior of c_{rt} for quotients of groups of polynomial volume growth, as per the following theorem.

Theorem 3.1.1. *Let G be an infinite group with a subgroup H which is recurrent in G. If G has polynomial volume growth, then $c_{rt}(G, H)$ is an integer and at most 2.*

An immediate consequence of this result is the following corollary.

Corollary 3.1.1. *Let H be a recurrent subgroup of G. If $c_{rt}(G, H)$ is non-integral then G has super-polynomial volume growth.*

Section 3.2 will be dedicated to proving Theorem 3.1.1.

Determining the possible values c_{rt} can take on in a given class of groups should be quite hard. For instance, the free metabelian group of rank 2[1] has a continuum of non-isomorphic subgroups [8], so one does not have to go very far to have difficulties even coming to terms with which subgroups to consider. However, if the abelian kernel of a metabelian group has finite rank then there are only countably many non-isomorphic subgroups [10], so the situation is not entirely bleak.

[1]That is $F_2/[[F_2, F_2], [F_2, F_2]]$.

3.1.1 Basic properties of c_{rt}

The critical constant for recurrence is well-behaved with respect to moving to subgroups in either of its inputs. The proof of the following lemma given in [22] is incomplete; we present a complete proof below.

Lemma 3.1.1 (Ladder Lemma)**.** *If $H \leq K \leq G$ then*

1. *$c_{rt}(K, H) \leq c_{rt}(G, H)$, where equality holds if $[G : K] < \infty$.*
2. *$c_{rt}(G, K) \leq c_{rt}(G, H)$, where equality holds if $[K : H] < \infty$.*

Proof. (1) Let S be a generating set of G and $T \subset S$ a generating set of K. A measure μ on K is also a measure on G, so if

$$\mu(K \backslash B_{K,T}(r)) \leq Cr^{-\alpha},$$

then

$$\mu(G \backslash B_{G,S}(r)) \leq Cr^{-\alpha}.$$

It is also clear that if $(H\backslash K, \mu)$ is transient then so is $(H\backslash G, \mu)$. Thus $c_{rt}(K, H) \leq c_{rt}(G, H)$.

Now, suppose that $[G : K] \leq \infty$, and fix a measure μ on G. Since K is of finite index it will be visited infinitely often by the random walk, X_n, driven by μ. Thus the first entry distribution, ν, of K is well defined. In particular for any $k \in K$, we have $\nu(k) = P_e(X_{\tau_K^+} = k)$ where τ_K^+ is the first positive time X_n is in K.

The tail of ν decays exponentially when μ is finitely supported. Note that τ_K^+ is also the first positive hitting time of the identity coset for $(K\backslash G, \mu)$. For finitely supported μ we have

$$\{Z_{\tau_K^+} \in K\backslash B_G(r)\} \subset \{\tau_K^+ > r\}. \tag{3.3}$$

The probability of the first event is $\nu(K - B_G(r))$, and the second is the tail of the hitting time for a finite reversible Markov Chain. This decays exponentially in r (see Chapter 2, section 4.3 in [1]). Thus ν has all polynomial moments.

If μ is not finitely supported, then

$$\{Z_{\tau_K} \in K\backslash B_G(r)\} \subseteq \{X_i \in B_G(r/\tau_K),\ i \leq \tau_K\}^c. \tag{3.4}$$

Thus

$$P(Z_{\tau_K^+} \in G\backslash B_G(r)) \leq P\left(\bigcup_{i=1}^{\tau_K^+} \{|X_i| \leq r/\tau_K^+\}^c\right) \tag{3.5}$$

$$\leq \tau_K^+ P(X_1 \in G\backslash B_G(r/\tau_K^+)). \tag{3.6}$$

The left hand side is $\nu(G\backslash B_G(r))$ which for some $c > 0$ bounds

$$\nu(K\backslash B_K(ct)),$$

and the probability on the right hand side is bounded by $c_2\mu(G - B_G(c_1 r))$ for some constants $c_1, c_2 > 0$. We then take the expectation to see that the tail of ν decays at least as fast of that of μ. Thus

$$c_{rt}(K, H) \geq c_{rt}(G, H),$$

which suffices to show $c_{rt}(K, H) = c_{rt}(G, H)$.

(2) For a given measure μ on G, if $(K\backslash G, \mu)$ is transient then $(H\backslash G, \mu)$ is transient. Hence $c_{rt}(G, K) \geq c_{rt}(G, H)$. Now suppose $[K : H] < \infty$. A random walk visiting K infinitely often will also visit H infinitely often, and so transience of H implies transience of K. Hence $c_{rt}(G, K) \leq c_{rt}(G, H)$, and we get that $c_{rt}(G, K) = c_{rt}(G, H)$. $\square$

Also of interest are general lower bounds on the possible values of c_{rt}. The following two results are due to Erschler [22]. We will prove generalized versions of them in Section 3.4.

Theorem 3.1.2. *For any infinite index subgroup H of a finitely generated group G*

$$c_{rt}(G, H) \geq \frac{1}{2}. \tag{3.7}$$

Theorem 3.1.3. *Let ν be a symmetric probability measure on G whose support generates G. If $\mathbb{E}_\nu|X_n| \leq Cn^{\delta^+}$ for all $n \in \mathbb{N}$, some $C > 0$, then for any infinite index subgroup H of G*

$$c_{rt}(G, H) \geq \frac{1}{2\delta^+}. \tag{3.8}$$

Note the similarity between Theorem 3.1.3 and Theorem 2.1.4. We can surmise that the rate of escape of a random walk provides a significant amount of information about the geometry of its state space. Additionally, one can conclude from Theorem 3.1.3 that there is a "classical" behavior for c_{rt} in terms of escape exponents: If there exists a random walk on G with escape exponent $1/2$, then c_{rt} does not take on any values between 0 and 1.

Erschler has shown that there do exist examples where $c_{rt} \in [1/2, 1)$. In particular, if $\mathcal{G}$ is a Grigorchuk group then $1/2 \leq c_{rt}(\mathcal{G}, \mathrm{Stab}(1)) < 1$. This result is highly non-trivial and relies on the non-triviality of the Poisson boundary of $\mathcal{G}$ for some non-simple random walks.

3.2 c_{rt} on virtually nilpotent groups

To obtain Theorem 3.1.1 we need a classification of recurrent subgroups of groups of polynomial growth. This was done by Revelle in [44]. For a mildly different treatment see [46]. We remind the reader that relative volume growth is defined in Section 1.3.1.

Theorem 3.2.1 (Recurrent Subgroups of groups of polynomial volume growth)**.** *Let G be a finitely generated group with polynomial volume growth. A subgroup $H \leq G$ is recurrent if and only if its relative degree of growth is within two of the growth rate of G. In this case there exists H' and G' such that*

(i) $H < H'$ *and* $[H' : H] < \infty$*,*

(ii) $G' < G$ *and* $[G : G'] < \infty$*,*

(iii) $H' \cap G' \lhd G'$*, and*

(iv) $(H' \cap G')\backslash G'$ *is isomorphic to* $\mathbb{Z}^d$ *for some* $d \leq 2$*.*

With this in hand we can classify c_{rt} on virtually nilpotent groups.

Proof of Theorem 3.1.1. Let G be a group with recurrent subgroup H. Suppose G has polynomial volume growth of degree d. We apply 3.2.1 to obtain subgroups $G' \leq G$ with $[G : G'] < \infty$ and $H' \leq H$ with $[H : H'] < \infty$ such that $c_{rt}(G', H' \cap G') = s$ for some $s \in \{0, 1, 2\}$.

To apply the equality cases of the Ladder Lemma we need to show that $[H : H \cap G'] < \infty$ and $[H' \cap G' : H \cap G'] < \infty$. For the first pair, if $h \in H$,

$$(H \cap G')h = H \cap Gh',$$

and thus a coset of $(H \cap G')\backslash H$ corresponds to a coset of $G'\backslash G$. As G' is of finite index in G there are only finitely many such cosets. For the second pair, take $h' \in H' \cap G'$. Then

$$(H \cap G')h' = Hh' \cap G'h',$$

so a coset in $(H \cap G')\backslash(H' \cap G')$ corresponds to the intersection of a coset of $H\backslash H'$ with a coset of $G'\backslash G$. As there are only finitely many cosets in both quotients we conclude that $(H \cap G')\backslash(H' \cap G')$ is finite.

We now apply lemma 3.1.1. By (2) to $(H \cap G') \leq H \leq G$ we see that

$$c_{rt}(G, H) = c_{rt}(G, H \cap G'). \tag{3.9}$$

Next we apply (1) to $(H \cap G') \leq G' \leq G$ to get

$$c_{rt}(G, H \cap G') = c_{rt}(G', H \cap G'). \tag{3.10}$$

Finally, we apply (2) to $(H \cap G') < (H' \cap G') < G'$ to see that

$$c_{rt}(G', H \cap G') = c_{rt}(G', H' \cap G'). \tag{3.11}$$

Combining these identities gives $c_{rt}(G, H) = c_{rt}(G', H' \cap G')$ as desired. $\square$

The classification of c_{rt} can be made more precise using theorem 3.2.1, giving us the following corollary.

Corollary 3.2.1. *If G has polynomial growth and H is a subgroup such that $c_{rt}(G, H) < \infty$, then $c_{rt}(G, H)$ is the growth rate of the quotient $H\backslash G$. In particular, $c_{rt}(G, H) = d - r$ where d is the growth rate of G and r is the relative growth rate of H in G.*

3.3 Critical constants for recurrence on polycyclic groups

For a polycyclic group G, the Hirsch length of G, $h(G)$, is the number of infinite cyclic factors in a cyclic series for G. This number is a quasi-isometric invariant of the group [48].

Theorem 3.3.1. *Let G be polycyclic and let H be a subgroup of G. Set $k = h(G) - h(H)$, where h is the Hirsch length. Then*

1. *if $k = 0$, $c_{rt}(G, H) = 0$, and*
2. *if $k \geq 1$, $c_{rt}(G, H) \geq 1$*

Proof. If $h(G) = h(H)$ then H is of finite index in G [48], which establishes (1). For (2), suppose $h(G) - h(H) > 0$. We will assume without loss of generality that H and G are torsion free. Since G is polycyclic, H will also be polycyclic and if

$$G = G_1 \rhd \cdots \rhd G_{n+1} = \{e\} \tag{3.12}$$

is a cyclic series for G, then

$$H = H \cap G_1 \rhd \cdots \rhd H \cap G_{n+1} = \{e\} \tag{3.13}$$

is a cyclic series for H. By assumption, there exist k integers $i \in \{1, ..., n\}$ such that $[H \cap G_i : H \cap G_{i+1}] < \infty$ while $G_i/G_{i+1} \cong \mathbb{Z}$.

Take $g_i \in G_i$ such that $G_i/G_{i+1} = \langle g_i \rangle$. Suppose there exists $l > 0$ such that $g_i^l \in H \cap G_i$. Then $g_i^{lk} \in H \cap G_{i+1}$, which is a contradiction. Hence $\langle g_i \rangle \cap H$ is empty. Thus $c_{rt}(\langle H, g_i \rangle, H) = 1$. We then apply the Ladder Lemma to $H < \langle H, g_i \rangle < G$ to get the desired estimate. □

Remark. In the above argument, one can conclude that if $h(G) - h(H) = k$, then there are k such g_i which generate cyclic subgroups that intersect H only at the identity. Additionally, $\langle g_i \rangle \cap \langle g_j \rangle = \{e\}$ for $i \neq j$. This information is compelling, but unenlightening. Let I denote the set of such i. One simply does not know very much about $\langle H, g_i; i \in I \rangle \backslash \langle g_i; i \in I \rangle$. The nature of this quotient space depends finely on how each $\langle g_i \rangle$ acts on G_{i+1}, and so we can in general say little about relevant properties of the quotient. Further difficulties arise because there are no general theorems for random walks on graphs which preclude graphs with super-quadratic volume growth from being recurrent [55]. However, we are still willing to offer the following conjecture.

Conjecture 3.3.1. *Let G be polycyclic and let H be a subgroup of G. Suppose $k = h(G) - h(H)$, then*

1. *if $k = 2$, $c_{rt}(G, H) \geq 2$, and*
2. *if $k \geq 3$, $c_{rt}(G, H) = \infty$.*

3.4 Extended critical constants for recurrence

3.4.1 Moments of the heat kernel and extended critical constants for recurrence

While the notion of recurrence has played a significant role in the study of probability it is not *a priori* very useful when studying random walks on groups. This is because only groups with at most quadratic volume growth admit nondegenerate recurrent random walks, and there are very few of these groups up to quasi-isometry[2].

The class of recurrent graphs is much richer than the space of recurrent groups, and this is in part what motiates us to study random walks on quotient spaces.

By the Borel-Cantelli lemma, transience is equivalent to the convergence of the series $\sum_{n=1}^{\infty} p_n(e,e)$, which can be viewed as the 0th moment of the on-diagonal heat kernel. Rather than restrain ourselves to classical notions of recurrence and transience, we can consider higher moments of the heat kernel. We will say a random walk is *β-transient* if

$$\sum_{n=1}^{\infty} n^{\beta} p_n(e,e) = \infty, \tag{3.14}$$

i.e. the heat kernel has a β-moment. Note that β-transience implies β'-transience for all $\beta' > \beta$. A recurrent random walk is 0-transient.

Our notion of β-transience generalizes what is called "strong transience" by Yamamuro [56]. In particular, strong transience implies that the process is at least 1-transient.

The dimension of an abelian group determines its degree of transience.

Lemma 3.4.1. *Suppose that G contains $\mathbb{Z}^d$ as a finite index subgroup. Then the infimum β such that a simple symmetric random walk on G is β-transient is max*$\{(d-2)/2, 0\}$.

Proof. Fix $\beta \in \mathbb{R}$. Observe that

$$\sum_{n=1}^{\infty} n^{\beta} p_n(e,e) \simeq \sum_{n=1}^{\infty} n^{\frac{2\beta-d}{2}}. \tag{3.15}$$

[2]Gromov's theorem restricts us to virtually nilpotent groups, and once there the Bass-Guivarch formula severely limits the structure of groups with at most quadratic volume growth.

This sum diverges when $2\beta - d \geq -2$, i.e. when $\beta \geq (d-2)/2$. $\square$

Note that groups with super-polynomial decay of the heat kernel are not β-transient for any $\beta < \infty$. We will say such a walk is ∞-transient.

We now use the notion of β-transience to define the *extended critical constant for recurrence*.

Definition 3.4.1. *For a finitely generated group G with subgroup H, the β-critical constant for recurrence, c_{rt}^{β}, is defined as the supremum of the $\alpha \geq 0$ such that there exists a measure μ on G with*

$$\sum_{g \in G} |g|^{\alpha} \mu(g) < \infty, \tag{3.16}$$

and whose induced random walk on $H\backslash G$ is β-transient.

Remark. Note that classifying the extended critical constants for even Abelian groups is more difficult than for the original critical constant. This is because one needs more information about the behavior of the heat kernel for spread out measures. This prevents us from generalizing the classification of the critical constant on groups of polynomial volume growth. However, Theorem 3.3.1 generalizes to extended critical constants without fuss.

Theorem 3.4.1. *Let G be polycyclic and let H be a subgroup of G. Set $k = h(G) - h(H)$, where h is the Hirsch length. Then*

1. *if $k = 0$, $c_{rt}^{\beta}(G, H) = 0$,*
2. *and if $k \geq 1$, $c_{rt}^{\beta}(G, H) \geq c_{rt}^{\beta}(\mathbb{Z}, 1)$*

Theorem 3.1.1 is a different matter entirely as it requires actually knowing which subgroups are recurrent. To generalize this result to the extended critical constants, one would need to generalize the classification of recurrent subgroups of nilpotent groups to determine the β-transient subgroups. We leave this task to the ambitious reader.

3.4.2 Basic results for extended critical constants

In this section, we will prove generalizationss of some of the lemmas and theorems in Section 3.1 for extended critical constants for recurrence. The Ladder Lemma carries over as is by replacing "transient" with "β-transient" in the proof. That one can do this is clear, as β-transience implies that the heat kernel has a β'-moment for all $0 \leq \beta' < \beta$. The other results take an epsilon more work; the techniques used below are essentially those employed by Erschler in [22].

Let ν be a finitely supported, symmetric, probability measure on G whose support generates G, and let $\bar{\mu}_\alpha(i) = ci^{-\alpha}$, for $1 < \alpha$, be a probability measure on $\mathbb{N}$. By the local limit theorem for stable laws (see [27]) we have that there exists $C > 0$ such that

$$\overline{\mu}_\alpha^{(n)}(i) \leq Cn^{-1/\alpha} \tag{3.17}$$

for all $i, n \in \mathbb{N}$.

We will consider the probability measure on G defined by

$$\mu_\alpha = c \sum_{i=1}^{\infty} \overline{\mu}_\alpha(i) \nu^{(i)}.$$

When referring to a word metric on G (and thus an induced metric on a Schreier graph of G) we use the word metric associated with the support of ν.

We now present some lemmas which will be used to obtain the lower bounds.

Lemma 3.4.2. *Let $H < G$ be of infinite index. Then for $\alpha \in (1, 3/2]$ the induced random walk on $H\backslash G$ with respect to μ_α is $\frac{3-2\alpha}{2(\alpha-1)}$-transient.*

Proof. First we note that the return probability for a simple random walk on an infinite regular graph Γ satisfies

$$p_n(v, v) = O(n^{-1/2}) \tag{3.18}$$

for any vertex v [51]. Plainly, this implies that $\nu^{(n)}(H) = O(n^{-1/2})$. We can then see that

$$\begin{aligned} \mu_\alpha^{(n)}(H) &= c \sum_{i=1}^{\infty} \overline{\mu}_\alpha^{(n)}(i)\nu^{(i)}(H) &(3.19)\\ &\leq c' \sum_{i=1}^{\infty} \frac{\overline{\mu}_\alpha^{(n)}(i)}{i^{1/2}} &(3.20)\end{aligned}$$

for some $c' > 0$. We apply the local limit theorem to $\overline{\mu}_\alpha$, to see that, for some $c_3, c_4 > 0$,

$$\begin{aligned} \mu_\alpha^{(n)}(H) &\leq c_3 n^{-\frac{1}{\alpha-1}} \sum_{i=1}^{n^{-\frac{1}{\alpha-1}}} \frac{1}{i^{1/2}} &(3.21)\\ &\leq c_4 n^{-\frac{1}{2\alpha-2}}. &(3.22)\end{aligned}$$

Thus the sum

$$\sum_{n=1}^{\infty} n^\beta \mu_\alpha^{(n)}(H) \tag{3.23}$$

converges for $\beta < \frac{3-2\alpha}{2(\alpha-1)}$, and the lemma follows. □

If we assume $p_n(v, v) = O(n^\rho)$ for $\rho \in (0, 1/2]$, then we get $\frac{2-\rho-\alpha}{\alpha-1}$-transience. Thus, smaller estimates for the return time on the Schreier graph lead to a smaller degree of transience. In particular, this allows us to work with $\alpha \in (1, 2-\rho]$. Note that 2 is a natural upper bound on α due the nature of local limit theorems [27].

Lemma 3.4.3. *For $\alpha > 1$,*

$$\mu_\alpha(G \backslash B(e,R)) \leq CR^{1-\alpha}. \tag{3.24}$$

Proof. Note that $\mathrm{supp}\nu^{(n)} \subset B(e,R)$ for $n < R$. Thus there exists $C > 0$ such that

$$\mu_\alpha(G \backslash B(e,R)) \leq \sum_{n=R}^{\infty} \overline{\mu}_\alpha(i) \tag{3.25}$$

$$= \sum_{n=R}^{\infty} \frac{1}{n^\alpha} \tag{3.26}$$

$$\leq CR^{1-\alpha}. \tag{3.27}$$

□

Lemma 3.4.4. *Suppose the random walk on G with respect to ν has upper escape exponent δ^+. Then, for any $\beta \leq 1$ such that $\beta < (\alpha-1)/\delta^+$, the β-moment of μ_α is finite.*

Proof. Since $\beta \leq 1$, there exists $C > 0$ such that

$$\sum_{g \in G} \nu^{(i)}(g)|g|^\beta \leq Ci^{\delta^+\beta}. \tag{3.28}$$

Then

$$\sum_{g \in G} \mu_\alpha(g)|g|^\beta = \sum_{i=1}^{\infty} \left(\frac{1}{i^\alpha} \sum_{g \in G} \nu^{(i)}(g)|g|^\beta \right) \tag{3.29}$$

$$\leq C \sum_{i=1}^{\infty} \frac{i^{\delta^+\beta}}{i^\alpha}. \tag{3.30}$$

The final sum converges when $\beta < (\alpha-1)/\delta^+$. □

We now obtain the lower bounds on the extended critical constants.

Theorem 3.4.2. *Fix $\beta \geq 0$. Then, for any infinite index subgroup H of a finitely generated group G,*

$$c_{rt}^{\beta}(G, H) \geq \frac{1}{2(1+\beta)}. \tag{3.31}$$

Proof. This follows from Lemmas 3.4.2 and 3.4.3. □

Theorem 3.4.3. *Fix $\beta \geq 0$ and suppose that $\mathbb{E}_{\nu}|X_n| \leq Cn^{\delta^+}$ for all $n \in \mathbb{N}$ and some $C > 0$. Then for any infinite index subgroup H of G*

$$c_{rt}^{\beta}(G, H) \geq \frac{1}{2\delta^+(1+\beta)}. \tag{3.32}$$

Proof. This follows from Lemmas 3.4.2 and 3.4.4. □

CHAPTER A.

SELF-SIMILAR GROUPS

In recent years self-similar groups (or more generally, automata groups) have been a topic of much interest for both group theorists and those probabilists concerned with random walks on groups. For a general survey we refer the reader to Nekrashevych's excellent book [39]. Erschler has done work on the Poisson boundary and critical constants for recurrence of a class of Grigorchuk groups [21, 22]. A large part of what makes these groups interesting is that many of them have intermediate volume growth. Groups of polynomial volume growth are amenable, and so a natural question is whether or not this extends to groups of sub-exponential volume growth. Thus much attention has also been payed to the amenability of such groups. For certain groups, amenability has been shown via computation of the heat kernel or through techniques using the entropy of random walks [2, 6, 7].

A self-similar group is an automorphism group of a rooted, regular tree whose action can be defined recursively. More precisely, let X be a finite alphabet with $\#X = d$. We identify the d-regular rooted tree with X^*, the set of words over X in the natural way. A group $G < \mathrm{Aut}(X^*)$ is self-similar if $G \leq G \wr S(X)$, where $S(X)$ is the permutation group of X. This means that on any level of the tree, G acts as a permutation on that level and then assigns another element of itself to act on each subtree of that level.

Elements of self-similar groups are generally denoted via the following *wreath recursion*

$$g = \langle g_{x_1}, \dots, g_{x_d} \rangle \sigma_g, \tag{A.1}$$

so that if $w \in X^*$ and $x \in X$, then $g(xw) = \sigma_g(x) g_x(w)$. Many self-similar groups are *nuclear*, which means that no matter which element of g we choose, g_{x_i}, lies within a fixed, finite subset of G, called the nucleus [39]. The nucleus often makes for a natural generating set of the group. The examples we present below are nuclear.

Part of what makes self-similar groups so interesting is their connection to fractals. Every self-similar group G posses a limit space $\mathcal{X}_G$ determined by the action of G on the boundary of the tree. These limit spaces are closely related to the Schreier graphs of the action of G on each level of the tree. This limit space is self-similar, and the Schreier graphs can often be viewed as finite approximations to this limiting fractal. This allows for many interesting connection between self-similar groups and fractal geometry, especially in terms of analysis on fractals [40].

Some familiar groups, such as $\mathbb{Z}$ and $\mathbb{Z}_2 \wr \mathbb{Z}$, are self-similar, but other self-similar groups are more exotic. Below, we give some examples which appear in the text.

A.0.3 Griogorchuk groups

The first Grigorchuk is generated by the following automorphisms of the rooted binary tree, where $\sigma = (01)$

$$a = \langle 1, 1 \rangle \sigma \tag{A.2}$$

$$b = \langle a, c \rangle e \tag{A.3}$$

$$c = \langle a, d \rangle e \tag{A.4}$$

$$d = \langle 1, b \rangle e. \tag{A.5}$$

One can observe that each of these generators has order two, $b = cd$, and $\langle b, c, d \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. This definition can be generalized to an infinite family of Grigorchuk groups [21], while preserving the aforementioned properties.

A.0.4 Hanoi towers group

The Hanoi towers group is generated by the following automorphisms of the rooted trinary tree:

$$a = \langle a, 1, 1 \rangle (12) \tag{A.6}$$

$$b = \langle 1, b, 1 \rangle (02) \tag{A.7}$$

$$c = \langle 1, 1, c \rangle (12). \tag{A.8}$$

The groups is named so because the Schreier graph of the action on the nth level of the tree can be identified with the Hanoi towers game on three posts with n washers. In this game there are three posts and n washers of increasing sizes. All of the washers start on a single post with the largest washer on the

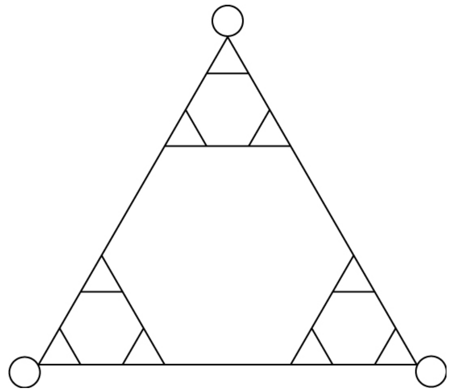

Figure A.1: The third level Schreier graph of the Hanoi towers groups.

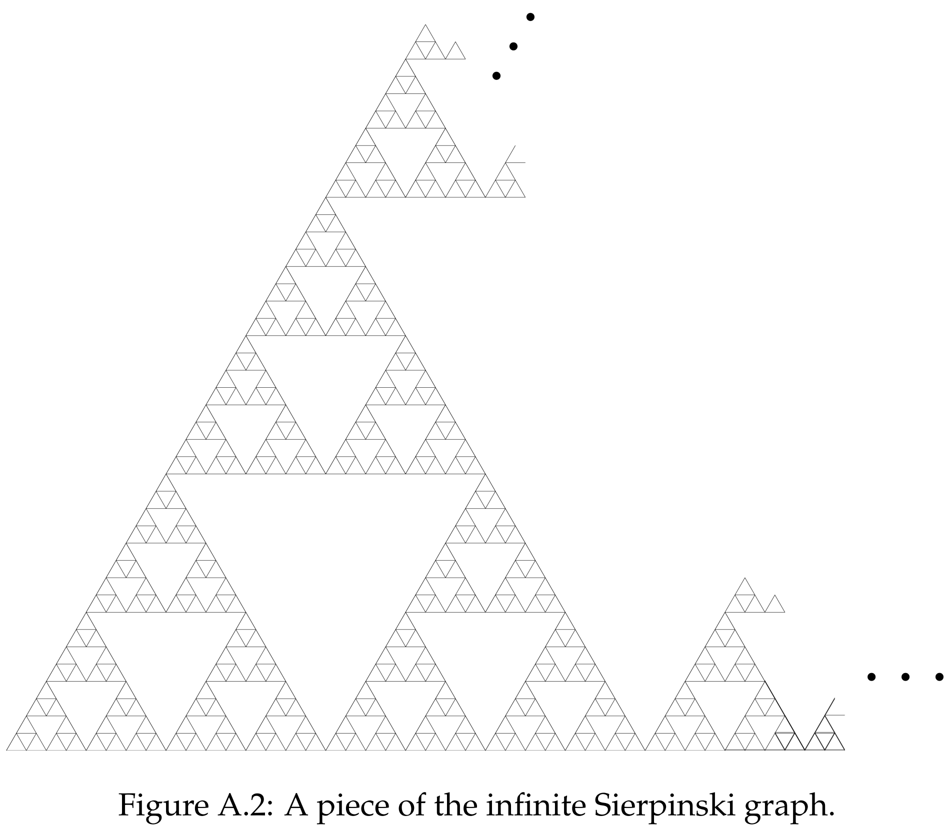

Figure A.2: A piece of the infinite Sierpinski graph.

bottom, proceeding to the smallest washer on top. The rules are simple: washers are moved one at a time from peg to peg, only the top washer on a peg may be moved, and a washer cannot be placed on top of a smaller peg. The goal is to move the entire stack of washers to a new peg. One can readily see the recursive structure of this game. The Schreier graphs are essentially finite approximations of the Sierpinski gasket (see Figures A.1 and A.2).

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