FREE RESOLUTIONS OF MONOMIAL IDEALS

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Let k be a field and $S = k[x_1, \dots, x_n]$ a polynomial ring. This thesis considers the structure of minmial free resolutions of monomial ideals in S.

In Chapter 3 we study reverse lex ideals, and compare their properties to those of lex ideals. In particular we provide an analogue of Green's Theorem for reverse lex ideals. We also compare the Betti numbers of strongly stable and square-free strongly stable monomial ideals to those of reverse lex ideals.

In Chapter 5 we study the minimal free resolution of the edge ideal of the complement of the n-cycle for $n \geq 4$ and construct a regular cellular complex which supports this resolution.

BIOGRAPHICAL SKETCH

Jennifer Biermann was born in California and grew up in Sandy, Oregon. She attended Sandy High School before moving to Wisconsin to attend Lawrence University where she graduated in 2005 with a Bachelor of Arts in mathematics. Biermann then proceeded to Cornell University where she completed her graduate work under the direction of Professor Irena Peeva in 2011.

For my family, Bruce, Vicki and Sharon May Biermann and Christopher Scheper.

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CHAPTER 1

INTRODUCTION

Much of the modern study of commutative algebra is the study of modules over commutative rings. These can in turn be studied via their minimal free resolutions. Unlike the case of vector spaces, the generators of a module are not necessarily linearly independent. The minimal free resolution of a module contains all of the information about the generators of the module, the relations on the generators, the relations on the relations on the generators, and so on. Therefore studying the minimal free resolutions of modules is a good way to understand the modules of a ring.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field k. The ring S is graded by setting $\deg(x_i) = 1$. An S-module M is called *graded* if there is a decomposition of M as a direct sum of k-vector spaces

$$M = \bigoplus_{j \in \mathbb{N}} M_j$$

such that $S_i M_j \subseteq M_{i+j}$, where S_i is the k-vector space spanned by the degree i monomials of S. One particularly interesting invariant of such a module is its graded *minimal free resolution*. A minimal free resolution of a module M is an exact sequence of the form

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{p,j}} \longrightarrow \ldots \longrightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longrightarrow \bigoplus_j S(-j)^{\beta_{0,j}} \longrightarrow M \longrightarrow 0\ ,$$

where $S(-j)$ stands for the ring S graded so that the element 1 is in degree j.

One constructs a free resolution of a module M by taking in homological degree 0, a free module on the generators of M; in homological degree 1, a free module on the relations on those generators (called the first syzygies); in homological degree 2, a free module on the relations on the first syzygies (called the second syzygies); and so on. If at every step in this process we take a free module on a minimal generating set of the previous syzygies, then the free resolution is called minimal. We know by Hilbert's Syzygy Theorem that if M is a graded finitely generated S-module, then the minimal free resolution of M has length at most n; so this process is finite. The ranks of the free modules of the minimal free resolution of M are called the *Betti numbers* of M. Specifically, the rank of $S(-j)$ in the ith step of the minimal free resolution of M is called the graded Betti number of M in homological degree i and internal degree j, and is denoted $\beta_{i,j}(M)$. The Betti numbers of a module are often written in a table called the Betti table, where $\beta_{i,i+j}(M)$ is written in ith row and jth column (the shift is to save space since $\beta_{i,j}(M) = 0$ for $j < i$).

Since at every step we take a free module on the relations of the previous module, a minimal free resolution is a good way of looking at the structure of the original module. Therefore, it is of interest to study minimal free resolutions and the module invariants which come from them.

The study of minimal free resolutions of modules is a wide subject area and so we has focus on the case where the modules are monomial ideals in a polynomial ring. Focusing on monomial ideals allows one to immediately see the generators of the ideal, however even in this case the minimal free resolutions can be quite complicated and in general there the minimal free resolution of a monomial ideal is not known. Specializing even more, we can consider the class of squarefree quadratic monomial ideals. These ideals are called edge ideals and their study lies at the intersection of commutative algebra and graph theory.

CHAPTER 2

BACKGROUND

In this chapter we give basic background material necessary for our work on free resolutions of monomial ideals. In the first three sections of this chapter we give basic definitions and theorems on graded rings and modules and free resolutions of modules, as well as invariants derived from free resolutions. Later sections cover cellular resolutions and a tool from homological algebra called the mapping cone of a map.

All rings R in this work are commutative with unity. We will work most of the time over the polynomial ring $S = k[x_1, \ldots, x_n]$ where k is a field.

2.1 Graded Rings and Modules

Let R be a ring and let $(G, +)$ an abelian group. We say that R is *G-graded* if there is a decomposition of R as a $\mathbb{Z}$-module

$$R = \bigoplus_{i \in G} R_i$$

such that $R_i R_j \subseteq R_{i+j}$. An R-module M is G-graded if there is a decomposition

$$M = \bigoplus_{i \in G} M_i$$

such that $R_i M_j \subseteq M_{i+j}$.

An ideal of R is G-graded if it is G-graded as an R-module. If R is a graded ring and I a graded ideal of R, then the quotient ring R/I inherits a grading from that of R. That is, $(R/I)_i \cong R_i/I_i$.

If a ring R is graded by $\mathbb{Z}$ we say that R is *graded*. If R is graded by $\mathbb{N}^r$ we say that R is *multigraded*.

We say that an element f in R is *homogeneous* if it is an element of R_j for some j. If $f \in R_j$ then we defined the degree of f to be $\deg(f) = j$. The homogeneous elements of a ring, R, give us a criterion for when an ideal of R is graded.

Proposition 2.1.1. *An ideal I in a graded ring R is graded if there exists a system of homogeneous generators $f_1, \dots f_r$ of I.*

Construction 2.1.2. The ring $S = k[x_1, \dots, x_n]$ is graded by taking $S = \bigoplus_{i \in \mathbb{Z}} S_i$ where S_i is the k-vector space spanned by the monomials of degree i (note that S_i is empty for $i < 0$). This is called the *standard grading on S*. By Proposition 2.1.1, any ideal of S which is generated by homogeneous polynomials is graded. In particular, every monomial ideal of S is graded.

Construction 2.1.3. $S = k[x_1, \dots x_n]$ is also multigraded as follows

$$S = \bigoplus_{\alpha \in \mathbb{N}^n} S_\alpha$$

where S_α is a one-dimensional k-vector space which is spanned by the monomial $x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$. The multigraded ideals of S are exactly the monomial ideals of S.

2.2 Free Resolutions

An important homological tool for studying the modules of a commutative ring is the minimal free resolution of a module. These objects encode much of the information about the structure of the module as well as containing several important numerical invariants of the module.

One constructs a free resolution of a module M by taking in homological degree 0, a free module on the generators of M; in homological degree 1, a free module on the relations on those generators; in homological degree 2, a free module on the relations on the relations on the generators; and so on. In this way, a great deal of information about the structure of the module is encoded.

Definition 2.2.1. Let R be a ring and M an R-module. A *free resolution* of M over R is a complex of finitely generated free R-modules $\mathbf{F}$

$$\mathbf{F}: \qquad \cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

such that $H_i(\mathbf{F}) = 0$ for $i \geq 1$ and $H_0(\mathbf{F}) \cong M$. The collection of maps $\{d_i\}$ is called the differential of $\mathbf{F}$.

The exact sequence

$$\cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

is called an *augmented free resolution* of M.

Remark 2.2.2. When considering an ideal I of R we may take either a free resolution of I or a free resolution of R/I. They are related as follows: If

$$\mathbf{F}: \qquad \cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a free resolution of I over R then

$$\mathbf{F}: \qquad \cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow S \longrightarrow 0$$

is a free resolution of R/I over R. In this work we will generally work with free resolutions of R/I instead of free resolutions of I.

If R is a graded ring and M a graded R-module, then we may use the grading of M to grade the free modules in a free resolution $\mathbf{F}$ of M provided that the maps which make up the differential of $\mathbf{F}$ are all degree 0. In this case we write

$$F_i = \oplus_j R(-j)^{\beta_{i,j}}$$

where $R(-j)$ represents the ring R with the grading shifted so that the generator of R is in degree j. We call the graded version of $\mathbf{F}$ a *graded* free resolution of M. We can do a similar thing if R and M are multigraded.

Free resolutions of modules are not unique as we see in the next example.

Example 2.2.3. Let $I = (x^2, xy, y^3)$. I is an ideal in the polynomial ring $S = k[x, y]$. The following are both free resolutions of S/I

$$\mathbf{F}: \quad \begin{tikzcd} 0 \arrow[r] & S^2 \arrow[r, "\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}"] & S^3 \arrow[r, "\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}"] & S \arrow[r] & 0 \end{tikzcd}$$

$$\mathbf{G}: \quad \begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\begin{pmatrix} y^2 \\ x \\ -1 \end{pmatrix}"] & S^3 \arrow[r, "\begin{pmatrix} -y & 0 & -y^3 \\ x & -y^2 & 0 \\ 0 & x & x^2 \end{pmatrix}"] & S^3 \arrow[r, "\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}"] & S \arrow[r] & 0 \end{tikzcd}.$$

In order to avoid the problem of the non-uniqueness of free resolutions we introduce the concept of *minimal* free resolutions.

Definition 2.2.4. Let M be an R-module. A free resolution $\mathbf{F}$ of M over R is said to be *minimal* if the ranks of the free modules in $\mathbf{F}$ are less than or equal to the ranks of the corresponding free modules in an arbitrary free resolution of M over R.

We refer to the kernel of the differential map d_{i-1} in the minimal free resolution of M over R as the *ith syzygy module* of M over R.

One forms a minimal free resolution of an R-module M by choosing a minimal set (with respect to inclusion) of generators and then a minimal set of relations on those generators, and a minimal set of relations on the relations on the generators, etc.

Proposition 2.2.5. *A minimal free resolution*

$$\mathbf{F}: \qquad \cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

of an R-module M is unique up to a change of basis of the free modules F_i.

In light of this proposition, we often refer to *the* minimal free resolution of M over R.

In the case of the polynomial ring S, one can recognize the minimal free resolution $\mathbf{F}$ of M over S by the entries in the matrices of the differential of $\mathbf{F}$ as the next proposition shows.

Proposition 2.2.6. *Let M be an S-module and $\mathbf{F}$ a free resolution of M over s. Then $\mathbf{F}$ is the minimal free resolution of M over R if and only $d_i(F_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$.*

Example 2.2.7. We can see by examining the differentials of $\mathbf{F}$ and $\mathbf{G}$ in Example 2.2.3 that $\mathbf{F}$ is minimal while $\mathbf{G}$ is not.

2.3 Invariants from Free Resolutions

There are several useful numerical invariants of modules that one obtains from the minimal free resolution of a module. The first invariant we consider is the length of the minimal free resolution. We call this invariant the projective dimension.

Definition 2.3.1. Let M be an R-module and let

$$\mathbf{F}: \qquad \cdots \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

be the minimal free resolution of M over R. The *projective dimension* of M is the smallest i such that $F_i \neq 0$ and $F_j = 0$ for all $j > i$. We write $\mathrm{pdim}_R(M) = n$. When it is clear which ring we are working over, we write $\mathrm{pdim}(M) = n$. If there is no such i, we say $\mathrm{pdim}_R(M) = \infty$.

Note that we begin indexing the free modules in a free resolution at 0, so the projective dimension of M is one less than the number of non-zero free modules in the minimal free resolution of M.

It is important to note that when we are working over the polynomial ring S, the minimal free resolution of an S-module M is finite in length. This result is known as Hilbert's Syzygy Theorem (see [12]) and is stated more precisely as Theorem 2.3.2 below.

Theorem 2.3.2. *[12] Let $S = k[x_1, \ldots, x_n]$ and let M be an S-module. Then $\mathrm{pdim}_S(M) \leq n$.*

We see from Remark 2.2.2 that the projective dimension of I and S/I for I an ideal of S are related by

$$\mathrm{pdim}(S/I) = \mathrm{pdim}(I) + 1\,.$$

Example 2.3.3. Let $I = (x^2, xy, y^3)$. I is an ideal of $S = k[x,y]$ as in Example 2.2.3. We saw previously that the minimal free resolution of I over S/I is

$$\mathbf{F}: \qquad 0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} S \longrightarrow S \longrightarrow 0\,.$$

We can see from this that $\mathrm{pdim}(I) = 2$.

A finer set of invariants arising from the minimal free resolution of an R-module M are the Betti numbers of M.

Definition 2.3.4. The rank of the ith syzygy module of the minimal free resolution of M over R is known as the ith total Betti number (or simply the ith Betti number) of M over R. We write

$$\beta_i^R(M) = \mathrm{rank}_R(F_i)$$

where

$$\mathbf{F}: \qquad 0 \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

is the minimal free resolution of M over R.

If M is a graded R-module over a graded ring R, then we can obtain the graded Betti numbers of M from the graded minimal free resolution as follows.

Definition 2.3.5. Let R be a graded ring and M a graded R-module and let

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{p,j}} \longrightarrow \ldots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}} \longrightarrow M \longrightarrow 0\,,$$

be the graded minimal free resolution of M over R. The exponents $\beta_{i,j}$ of the shifted modules, $R(-j)$, are known as the graded Betti numbers of M over R.

We write $\beta^R_{i,j}(M)$ for the Betti number in homological degree i and inner degree j. As before, we omit the R and write $\beta_{i,j}(M)$ when it is clear which ring we are working over.

Since $\beta_{i,j}(M) = 0$ for $j < i$ we often save space by writing the graded Betti numbers of an S-module M in a matrix called the *Betti table* of M where the entry in the ith column and the jth row in the matrix is the Betti number $\beta_{i,i+j}(M)$. The Betti table of M is denoted $\beta(M)$ and has the form:

$$\beta(M) = \begin{pmatrix} \beta_{0,0} & \beta_{1,1} & \dots & \beta_{p,p} \\ \beta_{0,1} & \beta_{1,2} & \dots & \beta_{p,p+1} \\ & & \vdots & \\ \beta_{0,r} & \beta_{1,1+r} & \dots & \beta_{p,p+r} \end{pmatrix}.$$

Example 2.3.6. Let I be the ideal (x^2, xy, y^3) in the ring $S = k[x,y]$. We saw in example 2.2.3 that the minimal free resolution of S/I over S is

$$\mathbf{F}: \qquad 0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} S \longrightarrow 0\,.$$

The total Betti numbers of S/I over S are then

$$\beta_0(S/I) = 1, \quad \beta_1(S/I) = 3, \quad \beta_2(S/I) = 2\,.$$

If we want the graded Betti numbers of S/I we must look at the graded minimal free resolution of S/I:

$$0 \longrightarrow S(-3) \oplus S(-4) \longrightarrow S(-2)^2 \oplus S(-3) \longrightarrow S \longrightarrow S/I \longrightarrow 0\,.$$

The graded Betti numbers of S/I are given by the Betti table

$$\beta(S/I) = \begin{pmatrix} 1 & - & - \\ - & 2 & 1 \\ - & 1 & 1 \end{pmatrix}.$$

Note that the projective dimension of M is the width of the Betti table of M. The height of the Betti table is also an invariant and is called the regularity of M.

Definition 2.3.7. The regularity of an S-module M is the maximum j such that $\beta_{i,i+j}(M) \neq 0$ for some i.

Example 2.3.8. We see from the Betti table in Example 2.3.6 that the regularity of S/I is 2.

2.4 Cellular Resolutions

Part of the interest in the study of free resolutions of monomial ideals is that their nature lends itself to combinatorial techniques. In this section we study how the information of the minimal free resolution of a monomial ideal in a polynomial ring can be encoded in a CW-complex. The theory of regular cellular resolutions of monomial ideals was first developed by Bayer, Peeva, and Sturmfels in [3] and by Bayer, and Sturmfels in [4].

Definition 2.4.1. A finite regular CW-complex X is a space constructed in the inductively as follows:

1. We start with a discrete set X^0 which we call the 0-cells of X.
2. Inductively form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via embeddings $\varphi_\alpha : S^{n-1} \to X^{n-1}$ (where S^{n-1} denotes the $n-1$-dimensional sphere).

This process stops after finitely many steps.

Construction 2.4.2. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal of S with minimal monomial generators $\{m_1, \ldots, m_r\}$ and let Δ be a finite regular CW-complex with r vertices. We label the vertices of Δ by the monomials $m_1, \ldots, m_r$ and the faces of Δ by the lcm of the monomials labeling the vertices contained in that face. If f is a face of Δ we denote the label of f by u_f. Let $\mathbf{C}$ be the usual CW-chain complex of Δ with differential δ. Let $\mathbf{F}$ be a complex of free S modules obtained from $\mathbf{C}$ as follows.

The basis of F_i is given by the basis elements of C_i with multidegrees given by the labels of the faces in Δ.

Let f be a basis element of C_i and $\hat{f}$ the corresponding basis element of F_i. The differential of $\mathbf{F}$ is defined by

$$\partial(\hat{f}) = \sum_{j=1}^{s} \alpha_j \frac{u_f}{u_e} \cdot \hat{e}_j$$

where

$$\delta(f) = \sum_{j=1}^{s} \alpha_j \cdot e_j$$

Definition 2.4.3. If the complex $\mathbf{F}$ in Construction 2.1 is a free resolution of S/I then we say that the regular CW-complex Δ *supports a free resolution of* S/I. We say that a free resolution $\mathbf{F}$ is *regular cellular* (respectively *simplicial*) if there is a CW-cellular complex (respectively simplicial complex) that supports $\mathbf{F}$.

We have a criterion for determining when a CW-complex Δ supports a resolution of a monomial ideal I in terms of subcomplexes of Δ. We first define these subcomplexes and then give the criterion.

Definition 2.4.4. Let Δ be a CW-complex on r vertices labeled by the monomials $m_1, \dots, m_r$. Given a face, F, of Δ we label F with the monomial u_F which is the lcm of the labels of the vertices which are contained in F. For a monomial w we define $\Delta_{\leq w}$ to be the subcomplex of Δ which consists of all faces of Δ which are labeled by monomials which divide w.

Proposition 2.4.5. *Let I be a monomial ideal in the polynomial ring S with minimal monomial generators $m_1, \dots, m_r$, and let Δ be a CW-complex on r vertices. Δ supports a free resolution of S/I if and only if for all monomials w in I, the complex $\Delta_{\leq w}$ is acyclic.*

We illustrate this with the following example.

Example 2.4.6. Let $I = (x^2, xy, y^3)$ be an ideal in $S = k[x, y]$. Figure 2.1 shows two different simplicial complexes on three vertices. By Proposition 2.4.5 we can see that since $\Delta_{\leq x^2y^3}$ is not acyclic, Δ does not support a free resolution of S/I. On the other hand, it is easy to verify that $\Delta'_{\leq w}$ is acyclic for all monomials $w \in I$ and hence, Δ' does support a free resolution of S/I.

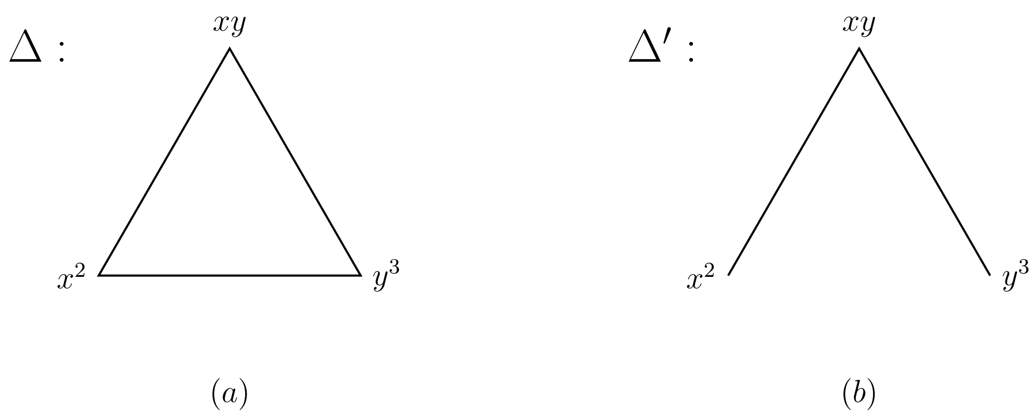


Figure 2.1: Two simplicial complexes on three vertices. (a) Δ does not support a free resolution of $S/(x^2, xy, y^3)$. (b) Δ' does support a free resolution of $S/(x^2, xy, y^3)$.

2.5 The Mapping Cone Construction

In this section we describe the mapping cone construction, a construction from homological algebra. This construction can be used in the context of free resolutions to build new free resolutions from previously known free resolutions. We begin by describing mapping cones and then show how they can be used in the context of free resolutions.

Let R be a ring and $(\mathbf{F}, d)$ and $(\mathbf{G}, d')$ be two complexes of R-modules. Further, let $\varphi : \mathbf{F} \to \mathbf{G}$ be a morphism of complexes. In other words, φ is a collection of maps $\varphi_i : F_i \to G_i$ which commutes with the differentials of the complexes $\varphi_{i-1} \circ d_i = d'_i \circ \varphi_i$.

Definition 2.5.1. Let $\varphi : (\mathbf{F}, d) \to (\mathbf{G}, d')$ be defined as above. The *mapping cone* of φ is the complex denoted $MC(\varphi)$ defined by $MC(\varphi)_i = F_{i-1} \oplus G_i$ with differential, ∂, given by

$$\partial_i = \begin{pmatrix} -d_{i-1} & 0 \\ \varphi_{i-1} & d'_i \end{pmatrix}.$$

Notice that there is a short exact sequence of complexes

$$0 \longrightarrow \mathbf{G} \longrightarrow MC(\varphi) \longrightarrow \mathbf{F}[-1] \longrightarrow 0$$

where $\mathbf{F}[-1]$ is the complex $\mathbf{F}$ shifted in homological degree so that $F[-1]_i = F_{i-1}$. This short exact sequence of complexes induces a long exact sequence on homology

$$\cdots \longrightarrow H_i(\mathbf{G}) \longrightarrow H_i(MC(\varphi)) \longrightarrow H_{i-1}(\mathbf{F}) \longrightarrow H_{i-1}(\mathbf{G}) \longrightarrow \cdots .$$

The connecting map $H_{i-1}(\mathbf{F}) \to H_{i-1}(\mathbf{G})$ is the map induced on homology by φ (see [30]).

Now suppose that

$$0 \longrightarrow M' \xrightarrow{\varphi} M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of R-modules and that $(\mathbf{F}, d)$ and $(\mathbf{G}, d')$ are free resolutions of M' and M respectively. Then we can lift φ to a morphism of complexes (also called φ) from $\mathbf{F}$ to $\mathbf{G}$. The long exact sequence on homology shows that $H_i(MC(\varphi)) = 0$ for $i \geq 0$ and that $H_0(MC(\varphi)) \cong M/M' \cong M''$. In other words, $MC(\varphi)$ is a free resolution of M''.

It is worth noting that even if $\mathbf{F}$ and $\mathbf{G}$ are minimal free resolutions of M' and M respectively, the mapping cone $MC(\varphi)$ may not be minimal.

2.6 Mapping Cones Applied to Monomial Ideals

We now consider the special case of monomial ideals. Let I be a monomial ideal in S and let $m_1, \ldots, m_r$ be the minimal monomial generators of I. Denote by I_i the ideal generated by $m_1, \ldots, m_i$ (so $I_r = I$). For each $1 \leq i \leq r-1$ we have a short exact sequence

$$0 \longrightarrow S/(I_i : m_{i+1})(-m_{i+1}) \xrightarrow{\cdot m_{i+1}} S/I_i \longrightarrow S/I_{i+1} \longrightarrow 0$$

where $S/(I_i : m_{i+1})(-m_{i+1})$ denotes the ring $S/(I_i : m_{i+1})$ with the multigrading shifted by m_{i+1}. This series of short exact sequences allows us to build up explicit free resolutions of monomial ideals if we know the free resolutions for the quotients of the colon ideals $S/(I_i : m_{i+1})$. One case where we can do this is when the ideal $(I_i : m_{i+1})$ is generated by variables. If this is true for all $0 \leq i \leq r-1$, we say that the ideal I has the *linear quotients property*. If $(I_i : m_{i+1})$ is generated by variables, then it is minimally resolved by the Koszul complex. Thus in the case where I has the linear quotients property we can form an explicit free resolution of S/I via a series of mapping cones.

CHAPTER 3

REVERSE LEX IDEALS

3.1 Introduction

In this chapter k stands for a field. We work over the polynomial ring $S = k[x_1, \dots, x_n]$ which is graded by setting the degree of each variable to be one. Throughout, I stands for a monomial ideal, and we denote by $I_j^{\#}$ the set of degree j monomials in I. We order the variables of S as follows: $x_1 > \cdots > x_n$.

An *initial lex segment* of length i in degree j is the set of monomials consisting of the first i monomials of degree j in the lexicographic order. Initial lex segments have the distinction of generating as little as possible in the next degree. A monomial ideal L is called *lexicographic* (or lex) if each space L_j is spanned by an initial lex segment. A monomial ideal B is called *strongly stable* if whenever m is a minimal monomial generator of B, x_i divides m, and $j < i$, we have that $x_j \cdot \frac{m}{x_i}$ is an element of B. Lex ideals are examples of strongly stable ideals. Both lex and strongly stable ideals play an important role in the study of Hilbert functions.

Given the importance of lex ideals, it is natural to think of defining a notion of a reverse lex ideal. In his paper [9] Todd Deery considers the following version of a reverse lex ideal. He calls a monomial ideal U a revlex segment ideal if $U_j^{\#}$ is an initial segment in the reverse lex order for each degree j. He proves [Dee96, Theorem 3.10] that such an ideal has smallest Betti numbers among all strongly stable ideals with the same Hilbert function. By [Dee96, Corollary 3.5] the Hilbert polynomial of a revlex segment ideal is constant, thus often there exists no revlex segment ideal attaining a given Hilbert function.

In their paper on the Betti numbers of monomial ideals [29], Nagel and Reiner began studying the situation in which we do not fix the Hilbert function, but only fix the number of minimal monomial generators and their degrees. Given a monomial ideal, we associate to it a reverse lex ideal (possibly in a bigger polynomial ring) as defined in Construction 3.1.1 below. The idea for this construction comes from [29].

Construction 3.1.1. Let $I \subseteq S$ be a monomial ideal and let q_j be the number of minimal generators of I in degree j (note that q_j may be 0). We construct a monomial ideal C by choosing the minimal generators as follows:

For each $j \geq 0$, the degree j minimal generators of C are the q_j largest monomials in the revlex order not in $\{x_1, \ldots, x_n\}(C)^{\#}_{j-1}$.

It is possible for the ring S not to have enough monomials in some degree in order to choose the minimal generators for C in this way. An example of this is Example 3.2.1 and we give a way to get around this difficulty by adding extra variables.

Definition 3.1.2. Let I be a monomial ideal in the ring S. The ideal C described in 3.1.1 is called the *reverse lex ideal associated to* I.

In [29] Nagel and Reiner work with square-free reverse lex ideals (defined below) rather than the reverse lex ideal which we have defined.

Definition 3.1.3. The *square-free reverse lex ideal* associated to a monomial ideal I is the monomial ideal D constructed as in Construction 3.1.1 with the modification that in each degree the generators of D are chosen to be the largest possible square-free monomials in the reverse lex order.

We prove in Section 3.4 that if I is a monomial ideal then the square-free reverse lex ideal associated to I and the reverse lex ideal associated to I have the same Betti numbers, and hence we use the two interchangeably.

Nagel and Reiner [29] proposed the idea that in some cases the total Betti numbers of a square-free reverse lex ideal are smaller than or equal to the total Betti numbers of ideals with the same fixed number of minimal generators in a single degree. In general, there are examples of Hilbert functions for which no ideal has minimal Betti numbers [31] [11]. There are techniques for finding upper bounds on Betti numbers; obtaining lower bounds is much harder. Therefore it is interesting to consider any construction which may give lower bounds on Betti numbers. Nagel and Reiner show in [29] that if I is a strongly stable ideal generated in one degree, then the Betti numbers of the square-free reverse lex ideal associated to I are smaller than or equal to those of I. At the beginning of Section 3.3 we provide two examples showing that this property does not hold if I is a strongly stable ideal generated in more than one degree. Both examples exist in a ring with four variables. In the first $\mathrm{pd}(I) < \mathrm{pd}(C)$ and in the second I is a lex ideal. In view of these examples we consider in Section 3.3 the special case where both I and C have minimal generators in several degrees but in at most three variables. We prove that in this special case the Betti numbers of C are indeed smaller than or equal to those of I.

In Section 3.4 we consider square-free strongly stable ideals. Nagel and Reiner showed that if J is a square-free strongly stable ideal generated in one degree then the square-free reverse lex ideal associated to J has smaller total

Betti numbers than J. By passing to the strongly stable case, we are able to prove results for square-free strongly stable ideals generated in several degrees which are analogous to those proved for strongly stable ideals.

A major theorem on Hilbert functions is Green's Theorem [20]. In order to formulate the theorem, we need some notation: For a monomial m in S, we set $\max(m) = \max\{i \mid x_i \text{ divides } m\}$.

Green's Theorem 3.1.4. [20] *If $I \subseteq S$ is a strongly stable ideal and L is the lexicographic ideal with the same Hilbert function as I, then for all p we have*

$$\left|\{m \in L_j^{\#} \mid \max(m) \leq p\}\right| \leq \left|\{m \in I_j^{\#} \mid \max(m) \leq p\}\right|.$$

We prove the following theorem which is analogous to Green's Theorem above.

Theorem 3.1.5. *Let I be a strongly stable ideal in S and C the corresponding revlex ideal. Then for all p we have*

$$\left|\{m \in I_j^{\#} \mid \max(m) \leq p\}\right| \leq \left|\{m \in C_j^{\#} \mid \max(m) \leq p\}\right|.$$

3.2 Green's Theorem for Reverse Lex Ideals

As stated in Section 3.1, the reverse lex ideal associated to a monomial ideal I does not always exist in the same polynomial ring as I. An example of this is provided below.

Example 3.2.1. Let $S = k[a,b,c]$ and $I = (a^2, ab, ac, b^3, b^2c, bc^2, c^4)$. Then following Construction 3.1.1 the minimal generators for C in degrees 2 and 3 are $\{a^2, ab, b^2, ac^2, bc^2, c^3\}$. There exist no monomials in degree 4 that are not divisible by these, so we cannot choose a degree 4 generator for C. The problem can be avoided by adding variables to the ring.

Proposition 3.2.2. *Let $I \subseteq S$ be a monomial ideal. After possibly adding variables to the ring S, the reverse lex ideal associated to I exists. It is a strongly stable ideal.*

For the remainder of this chapter we will assume the ring S has sufficiently many variables to construct C.

For any set of monomials M we define

$$W_{\leq p}(M) = \{m \in M \mid \max(m) \leq p\}$$

and

$$w_{\leq p}(M) = |\{m \in M \mid \max(m) \leq p\}| .$$

We will need the following lemma.

Lemma 3.2.3. [5, Proposition 1.2] *If I is a strongly stable ideal, then*

$$\{x_1, \dots, x_p\} \cdot W_{\leq p}(I_j^{\#}) = \bigcup_{i=1}^{p} x_i \cdot W_{\leq i}(I_j^{\#}) .$$

Now, we prove our main result:

Theorem 3.1.5. *Let I be a strongly stable ideal in S and C the corresponding revlex ideal. Then*

$$w_{\leq p}(I_j^{\#}) \leq w_{\leq p}(C_j^{\#}) \, .$$

Proof. We proceed by induction on j.

Let ℓ be the smallest degree in which the ideals I and C have minimal generators. The sets $W_{\leq p}(I_\ell^{\#})$ and $W_{\leq p}(C_\ell^{\#})$ consist only of minimal generators of I and C. If u and v are monomials of the same degree and max(u) < max(v), then $u > v$ in the reverse lex order. By construction, the minimal generators of C in degree ℓ form an initial segment in the reverse lex order. So since I and C have the same number of minimal generators in degree ℓ, we have the inequalities

$$w_{\leq p}(I_\ell^{\#}) \leq w_{\leq p}(C_\ell^{\#})$$

for all $1 \leq p \leq n$.

Now suppose that $w_{\leq p}(I_{j-1}^{\#}) \leq w_{\leq p}(C_{j-1}^{\#})$ for all $1 \leq p \leq n$. We next consider what happens in degree $j > \ell$. Fix a p between 1 and n.

The set $W_{\leq p}(I_j^{\#})$ consists of two kinds of monomials: minimal generators of I in degree j and monomials which are divisible by lower degree monomials in I. The latter group of monomials are exactly those in the set $\{x_1, \ldots, x_p\} \cdot W_{\leq p}(I_{j-1}^{\#})$.

We know

$$\begin{aligned}
\left|\{x_1,\ldots,x_p\}\cdot W_{\leq p}(I^{\#}_{j-1})\right| &= \sum_{i=1}^{p}|x_i\cdot W_{\leq i}(I^{\#}_{j-1})| \\
&= \sum_{i=1}^{p}w_{\leq i}(I^{\#}_{j-1}) \\
&\leq \sum_{i=1}^{p}w_{\leq i}(C^{\#}_{j-1}) \\
&= \sum_{i=1}^{p}|x_i\cdot W_{\leq i}(C^{\#}_{j-1})| \\
&= \left|\{x_1,\ldots,x_p\}\cdot W_{\leq p}(C^{\#}_{j-1})\right|
\end{aligned}$$

where Lemma 3.2.3 gives us the first and last equalities and the middle inequality holds by assumption. So all we need to consider are the degree j minimal generators of I and C.

By construction the degree j minimal generators of C were chosen to have the smallest possible maximum variables. So there are two possibilities for what happens in C:

Case 1. There are enough minimal generators in degree j to exhaust the monomials in $W_{\leq p}(S^{\#}_j)$ which are not already in $\{x_1,\ldots,x_p\}\cdot W_{\leq p}(C^{\#}_{j-1})$.

In other words we have the equality,

$$w_{\leq p}(C^{\#}_j) = w_{\leq p}(S^{\#}_j)\,.$$

This means that

$$w_{\le p}(I_j^{\#}) \le w_{\le p}(C_j^{\#})\,.$$

Case 2. There are not enough minimal generators in degree j to exhaust the monomials in $W_{\le p}(S_j^{\#})$.

Then all of the degree j minimal generators of C are in the set $W_{\le p}(C_j^{\#})$. Since the ideals I and C have the same number of degree j minimal generators and since

$$\left|\{x_1, \ldots, x_p\} \cdot W_{\le p}(I_{j-1}^{\#})\right| \le \left|\{x_1, \ldots, x_p\} \cdot W_{\le p}(C_{j-1}^{\#})\right|$$

again, we have

$$w_{\le p}(I_j^{\#}) \le w_{\le p}(C_j^{\#})\,.$$

□

The theorem and the previous lemma together imply the following proposition.

Proposition 3.2.4. *An initial reverse lex segment X in degree j generates as much as possible in degree $j+1$ among all sets of monomials in degree j with the strongly stable property and with the same cardinality as X.*

3.3 Betti Numbers

Nagel and Reiner showed [29] that if I is a strongly stable ideal generated in one degree and D the square-free reverse lex ideal associated to I, then $\beta_p^S(D) \leq \beta_p^S(I)$ for all p. We construct two examples which show this is not true if I is a strongly stable ideal generated in more than one degree.

Example 3.3.1. In the ring $A = k[a,b,c,d]$, let

$$I = (a^2, ab, ac, b^3, b^2c, bc^2, c^3)\,.$$

The corresponding revlex ideal is

$$C = (a^2, ab, b^2, ac^2, bc^2, c^3, acd)\,.$$

The Betti numbers of I and C are

$$\beta_0^A(I) = 7 \quad \beta_1^A(I) = 10 \quad \beta_2^A(I) = 4$$

$$\beta_0^A(C) = 7 \quad \beta_1^A(C) = 11 \quad \beta_2^A(C) = 6 \quad \beta_3^A(C) = 1\,.$$

This example also shows that the reverse lex ideal associated to a strongly stable ideal can have higher projective dimension than the original ideal.

Example 3.3.2. Let $A = k[a,b,c,d]$ and

$$I = (a^2, ab, ac, ad^2, b^3, b^2c, b^2d, bc^2, bcd, bd^2, c^3)\,.$$

The corresponding revlex ideal is

$$C = (a^2, ab, b^2, ac^2, bc^2, c^3, acd, bcd, c^2d, ad^2, bd^2)\,.$$

The Betti numbers of I and C are

$$\beta_0^A(I) = 11 \quad \beta_1^A(I) = 22 \quad \beta_2^A(I) = 16 \quad \beta_3^A(I) = 4$$

$$\beta_0^A(C) = 11 \quad \beta_1^A(C) = 23 \quad \beta_2^A(C) = 18 \quad \beta_3^A(C) = 5\,.$$

Note that in this example, the ideal I is a lexicographic ideal.

Proposition 3.3.3. *Let $I \subseteq S$ be a strongly stable ideal and C the reverse lex ideal associated to I. If $\max(m) \leq 3$ for all the minimal generators m of I and C then the following inequality holds for all p*

$$\beta_p^S(C) \leq \beta_p^S(I)\,.$$

Proof. Let $u_1, \ldots, u_r$ be the minimial generators of I and $v_1, \ldots, v_r$ the minimal generators of C. We may assume that these generators are ordered so that $\max(u_i) \leq \max(u_j)$ and $\max(v_i) \leq \max(v_j)$ for all $i < j$.

Our goal will be to use the formula for the Betti numbers of a strongly stable ideal given by the Eliahou-Kervaire resolution [14] to show the desired inequalities on the Betti numbers of I and C. The Eliahou-Kervaire resolution gives the following formula for Betti numbers of a strongly stable ideal I

$$\beta_p^S(I) = \sum_{i=1}^{r} \binom{\max(u_i) - 1}{p}\,.$$

Therefore, it will be sufficient to show that for all $1 \leq i \leq r$

$$\max(v_i) \leq \max(u_i)\,. \tag{**}$$

Since the ideals I and C are strongly stable, $\max(u_1) = 1$ and $\max(v_1) = 1$, and these are the only minimal generators in either ideal which have this property. This together with the assumption that $\max(u_i) \le 3$ and $\max(v_i) \le 3$ for all $1 \le i \le r$ means that all we need to show to prove (**) is

$$\left|\{u_i \mid 1 \le i \le r, \max(u_i) \le 2\}\right| \le \left|\{v_i \mid 1 \le i \le r, \max(v_i) \le 2\}\right| .$$

Let ℓ be the smallest and d the largest degree of a minimal generator of I. Then

$$\begin{aligned}
\left|\{u_i \mid 1 \le i \le r, \max(u_i) \le 2\}\right| &= w_{\le 2}(I_\ell^{\#}) + \sum_{j=\ell+1}^{d} \bigl(w_{\le 2}(I_j^{\#}) - |\{x_1, x_2\} \cdot W_{\le 2}(I_{j-1}^{\#})|\bigr) \\
&= w_{\le 2}(I_\ell^{\#}) + \sum_{\ell+1}^{d} \bigl(w_{\le 2}(I_j^{\#}) - w_{\le 2}(I_{j-1}^{\#}) - 1\bigr) \\
&= \sum_{\ell}^{d} w_{\le 2}(I_j^{\#}) - \sum_{\ell+1}^{d} w_{\le 2}(I_{j-1}^{\#}) - (d - (\ell + 1)) \\
&= w_{\le 2}(I_d^{\#}) - d + \ell + 1 .
\end{aligned}$$

The second equality above follows from Lemma 3.2.3. A similar formula holds for C, so by Theorem 3.1.5 we have the desired inequality.

$\square$

3.4 Square-free Strongly Stable Ideals

We will find it useful to be able to pass from a square-free strongly stable ideal to the case of a strongly stable ideal, which we have already considered. To this end we define a bijection between monomials and square-free monomials in $k[x_1, x_2, \ldots]$.

We think of a degree j monomial (in any number of variables) as a j-tuple of positive integers that correspond to the subscripts of the variables. In other words, the monomial $x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_j}$ is associated to $(\alpha_1, \alpha_2, \ldots, \alpha_j)$ where the α_i are not necessarily distinct. When representing a monomial this way we will always assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_j$. We use this notation to define a function from the set of monomials to the set of square-free monomials as follows

$$\varphi : \{\text{monomials}\} \longrightarrow \{\text{square-free monomials}\}$$

$$\varphi((\alpha_1, \ldots, \alpha_j)) = (\alpha_1, \alpha_2 + 1, \ldots, \alpha_i + i - 1, \ldots, \alpha_j + j - 1)\,.$$

Note that since we required $\alpha_1 \leq \alpha_2 \leq \ldots, \leq \alpha_j$, we know that

$$\alpha_1 < \alpha_2 + 1 < \alpha_3 + 2 \cdots < \alpha_j + j - 1\,.$$

so $\varphi((\alpha_1, \ldots, \alpha_j))$ is a square-free monomial.

The function φ is a bijection. Its inverse is given by

$$\varphi^{-1}((\beta_1, \ldots, \beta_j)) = (\beta_1, \beta_2 - 1, \ldots, \beta_i - (i-1), \ldots, \beta_j - (j-1))\,.$$

Suppose that m and m' are monomials in degree j such that $m \succ_{rlex} m'$. Let $(\alpha_1, \ldots, \alpha_j)$ and $(\beta_1, \ldots, \beta_j)$ be the j-tuples of the subscripts of the variables of m

and m' respectively. Let $i \in \{1, \ldots, j\}$ be the greatest integer such that $\alpha_i \neq \beta_i$. Since $m \succ_{rlex} m'$, $\alpha_i < \beta_i$. Then

$$\begin{aligned}\varphi((\alpha_1, \ldots, \alpha_j)) &= (\alpha_1, \alpha_2 + 1, \ldots, \alpha_i + (i-1), \alpha_{i+1} + i, \ldots, \alpha_j + (j-1)) \\ &= (\alpha_1, \alpha_2 + 1, \ldots, \alpha_i + (i-1), \beta_{i+1} + i, \ldots, \beta_j + (j-1))\end{aligned}$$

while

$$\varphi((\beta_1, \ldots, \beta_j)) = (\beta_1, \beta_2 + 1, \ldots, \beta_i + (i-1), \ldots, \beta_j + (j-1)).$$

Therefore, since $\alpha_i + (i-1) < \beta_i + (i-1)$ we know that $\varphi(m) \succ_{rlex} \varphi(m')$. In other words the function φ preserves the reverse lex order on monomials.

This bijection can be used to obtain square-free strongly stable ideals from strongly stable ideals and vice versa as the next proposition demonstrates. The examples at the beginning of this section were generated by applying φ to the generators of the examples in Section 3.3. The following two propositions were proved by Aramova, Herzog, and Hibi in [1]. We provide our own proofs.

Proposition 3.4.1. *Let $I = (u_1, \ldots, u_t)$ and $J = (v_1, \ldots, v_t)$ where $v_i = \varphi(u_i)$. Then*

a) *I is strongly stable if and only if J is square-free strongly stable.*

b) *If I is strongly stable (and hence J is square-free strongly stable), then $u_1, \ldots u_t$ are the minimal generators of I if and only if $v_1, \ldots, v_t$ are the minimal generators of J.*

Proof. a) Suppose that I is a strongly stable ideal, we will J is square-free strongly stable. Suppose that v_r is divisible by x_i and not divisible by x_{i-1}. It will be sufficient to show that $\frac{v_r}{x_i} x_{i-1}$ is in J. Say $\deg(v_r) = j$ and let $(\alpha_1, \ldots, \alpha_j)$ be the j-tuple consisting of the subscripts of the variables of v_r. Since x_i divides

v_r, there is some b such that $\alpha_b = i$ and since v_r is square-free, $\alpha_{b-1} < i-1$. Then $\frac{v_r}{x_i}x_{i-1} = (\alpha_1, \ldots, \alpha_{b-1}, i-1, \alpha_{b+1}, \ldots, \alpha_j)$.

We want to compare $\varphi^{-1}(v_r)$ and $\varphi^{-1}(\frac{v_r}{x_i}x_{i-1})$.

$$\begin{aligned}\varphi^{-1}(v_r) &= \varphi^{-1}((\alpha_1, \ldots, \alpha_{b-1}, i, \alpha_{b+1}, \ldots, \alpha_j)) \\ &= (\alpha_1, \alpha_2 - 1, \ldots, \alpha_{b_1} - (b-2), i-(b-1), \ldots, \alpha_j - (j-1))\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}\left(\frac{v_r}{x_i}x_{i-1}\right) &= \varphi^{-1}((\alpha_1, \ldots, \alpha_{b-1}, i-1, \alpha_b + 1, \ldots, \alpha_j)) \\ &= (\alpha_1, \alpha_2 - 1, \ldots, \alpha_{b-1} - (b-2), i-1-(b-1), \ldots, \alpha_j - (j-1)) \,.\end{aligned}$$

So $\varphi^{-1}(v_r)$ and $\varphi^{-1}(\frac{v_r}{x_i}x_{i-1})$ agree in every position except the b^{th} and therefore,

$$\begin{aligned}\varphi^{-1}\left(\frac{v_r}{x_i}x_{i-1}\right) &= \frac{\varphi^{-1}(v_r)}{x_{i-b+1}}x_{i-b} \\ &= \frac{u_r}{x_{i-b+1}}x_{i-b} \,.\end{aligned}$$

The monomial $\frac{u_r}{x_{i-b+1}}x_{i-b}$ is in I since I is a strongly stable ideal and therefore it is divisible by some u_s. In fact since I is strongly stable, we may write

$$\varphi^{-1}\left(\frac{v_r}{x_i}x_{i-1}\right) = u_s w$$

where $\max(u_s) \leq \min(w)$. This means that if we write $u_s w$ as a j-tuple of subscripts $(\beta_1, \ldots, \beta_j)$, then $(\beta_1, \ldots, \beta_{\deg(u_s)})$ are the subscripts of the variables in u_s and the rest are the subscripts of the variables in w. Therefore by the way φ is defined, $\varphi(u_s w) = \varphi(u_s)w'$ for some monomial w'. Hence,

$$\begin{aligned}\frac{v_r}{x_i}x_{i-1} &= \varphi(u_s w)\\ &= \varphi(u_s)w'\end{aligned}$$

so $\frac{v_r}{x_i}x_{i-1}$ is in the ideal J.

Conversely, suppose that J is a square-free strongly stable ideal and suppose that u_q is divisible by some x_ℓ. To show I is strongly stable, it is sufficient to show that $\frac{u_q}{x_\ell}x_{\ell-1}$ is in I as well. Let $j = \deg(u_q)$ and let $(\beta_1, \ldots, \beta_j)$ be the j-tuple consisting of the subscripts of u_q. Since x_ℓ divides u_s, some of the $\beta_i's$ are equal to ℓ. Let c be such that $\beta_c = \ell$ and $\beta_{c+1} \leq \ell - 1$ (since $\beta_1 \leq \cdots \leq \beta_j$, there is a unique such c). Then

$$\begin{aligned}\varphi(u_q) &= \varphi((\beta_1, \ldots, \beta_{c-1}, \ell, \beta_{c+1}, \ldots, \beta_j))\\ &= (\beta_1, \beta_2 + 1, \ldots, \beta_{c-1} + (c-2), \ell + (c-1), \ldots, \beta_j + (j-1))\end{aligned}$$

and

$$\begin{aligned}\varphi\left(\frac{u_q}{x_\ell}x_{\ell-1}\right) &= \varphi((\beta_1, \ldots, \beta_{c-1}, \ell - 1, \beta_{c+1}, \ldots, \beta_j))\\ &= (\beta_1, \beta_2 + 1, \ldots, \beta_{c-1} + (c-2), \ell - 1 + (c-1), \ldots, \beta_j + (j-1)).\end{aligned}$$

Thus since $\varphi(u_q) = v_q$, we have $\varphi(\frac{u_q}{x_\ell}x_{\ell-1}) = \frac{v_q}{x_{\ell+c-1}}x_{\ell+c-2}$. Since J is square-free strongly stable, $\frac{v_s}{x_{\ell+c-1}}x_{\ell+c-2}$ is in J so we may write

$$\frac{v_s}{x_{\ell+c-1}}x_{\ell+c-2} = v_p m$$

where $\max(v_p) < \min(m)$.

But then

$$\begin{aligned}\frac{u_q}{x_\ell}x_{\ell+1} &= \varphi^{-1}(\frac{v_q}{x_{\ell+c-1}}x_{\ell+c-2}) \\ &= \varphi^{-1}(v_p m) \\ &= \varphi^{-1}(v_p)m'\end{aligned}$$

for some monomial m'. Therefore $\frac{u_q}{x_\ell}x_{\ell+1}$ is in I and hence I is strongly stable.

b) Let I be strongly stable (and thus by part a, J is square-free strongly stable). Suppose that $u_1, \ldots, u_t$ are the minimal monomial generators of I. and suppose that $v_1, \ldots, v_t$ is not a minimal set of generators for J. So suppose v_r is redundant. Then there is some r, s such that v_s divides v_r. Since J is square-free strongly stable, we may assume $v_r = v_s w$ and $\max(v_s) < \min(w)$. Therefore

$$\begin{aligned}\varphi^{-1}(v_r) &= \varphi^{-1}(v_s w) \\ &= \varphi^{-1}(v_s)w'\end{aligned}$$

for some monomial w'. This means that $u_r = u_s w'$ which is a contradition. Therefore $v_1, \ldots, v_t$ are the minimal monomial generators of J.

Conversely, suppose that $v_1, \ldots, v_t$ are the minimal generators of J and that $u_1, \ldots, u_t$ are not a minimal set of generators of I. Then there is some u_q that is divisible by some u_p. We may assume that $u_q = u_p m$ with $\max(u_p) \leq \min(m)$. Then

$$\begin{aligned}v_q &= \varphi(u_q) \\ &= \varphi(u_p m) \\ &= \varphi(u_p)m' \\ &= v_p m'\end{aligned}$$

for some monomial m'. But this contradicts the fact that $v_1, \ldots, v_t$ are the minimal generators of J. Therefore $u_1, \ldots, u_t$ are the minimal generators of I.

$\square$

Proposition 3.4.2. *Let I be a strongly stable ideal with minimal generators $u_1, \ldots u_t$ and $J = (\varphi(u_1), \ldots, \varphi(u_t))$. Then for all p*

$$\beta_p^S(I) = \beta_p^S(J)\,.$$

Proof. By the previous proposition, since I is strongly stable, J is square-free strongly stable. Let m be any monomial of degree j and let $(\alpha_1, \ldots, \alpha_j)$ be the j-tuple consisting of the subscripts of the variables in m. Then $\varphi((\alpha_1, \ldots, \alpha_j)) = (\alpha_1, \alpha_2+1, \ldots, \alpha_j+(j-1))$. So $\max(m) = \alpha_j$ and $\max(\varphi(m)) = \alpha_j + j - 1$. Hence we have for any monomial m,

$$\max(m) - 1 = \max(\varphi(m)) - \deg(\varphi(m))\,.$$

Therefore,

$$\begin{aligned}
b_p^S(I) &= \sum_{i=1}^{t} \binom{\max(u_i) - 1}{p} \\
&= \sum_{i=1}^{t} \binom{\max(\varphi(u_i)) - \deg(\varphi(u_i))}{p} \\
&= \sum_{i=1}^{t} \binom{\max(v_i) - \deg(v_i)}{p} \\
&= b_p^S(J)\,.
\end{aligned}$$

$\square$

We list two examples that illustrate the fact that if J is a square-free strongly stable ideal generated in more than one degree and D the square-free reverse lex ideal associated to J, then it is not necessarily true that the Betti numbers of D are smaller than or equal to those of J.

Example 3.4.3. Let $A = k[a, \ldots, f]$. We will apply the function φ to the ideals in Examples 3.3.1 and 3.3.2. Proposition 3.4.2 tells us that the Betti numbers of these ideals are the same as the Betti numbers of the ideals in Examples 3.3.1 and 3.3.2.

From Example 3.3.1 we get the square-free strongly stable ideal

$$J = (ab, ac, ad, bcd, bce, bde, cde)\,,$$

and the square-free reverse lex ideal associated to J

$$D = (ab, ac, bc, ade, bde, cde, adf)\,.$$

From Example 3.3.2 we get

$$J = (ab, ac, ad, aef, bcd, bce, bcf, bde, bdf, bef, cde)\,.$$

and the square-free reverse lex ideal

$$D = (ab, ac, bc, ade, bde, cde, adf, bdf, cdf, aef, bef)\,.$$

Theorem 3.4.4. *Let I be any monomial ideal and let C be the reverse lex ideal associated to I and D the square-free reverse lex ideal associated to I. Then C and D have the same Betti numbers.*

Proof. By the previous proposition, it will be sufficient to show that if $u_1, \ldots, u_t$ are the minimal generators of C, then $D = (\varphi(u_1), \ldots, \varphi(u_t))$. This is easily checked. For completeness we include the argument. We assume that $u_1, \ldots, u_t$ are ordered so that $\deg(u_i) \leq \deg(u_{i+1})$ and if $\deg(u_i) = \deg(u_{i+1})$, then $u_i \succ_{rlex} u_{i+1}$. It is well known that φ preserves the reverse lex order (see [1], so the same order applies to $\varphi(u_1), \ldots, \varphi(u_t)$. In other words, $\deg(\varphi(u_i)) \leq \deg(\varphi(u_{i+1}))$ and if $\deg(\varphi(u_i)) = \deg(\varphi(u_{i+1}))$, then $\varphi(u_i) \succ_{rlex} \varphi(u_{i+1})$.

$(\varphi(u_1), \ldots, \varphi(u_t))$ has the right number of minimal generators in each degree so the only possible problem is if there were some s such that $\deg(\varphi(u_s)) = j$ and some square-free degree j monomial m such that $m \succ_{rlex} \varphi(u_s)$ and $m \notin (\varphi(u_1), \ldots, \varphi(u_{s-1}))$. Then $\varphi^{-1}(m) \succ_{rlex} u_s$ which implies by the construction of C that $\varphi^{-1}(m) \in (u_1, \ldots, u_{s-1})$. Since $C = (u_1, \ldots, u_t)$ is strongly stable and by the way $u_1, \ldots, u_t$ are ordered, $(u_1, \ldots, u_{s-1})$ is strongly stable also, so $\varphi^{-1}(m) = u_r w$ for some monomial w and some $1 \leq r \leq s-1$ and such that $\max(u_r) \leq \min(w)$. Thus $m = \varphi(u_r w) = \varphi(u_r)w'$ which is a contradition. Therefore for any $1 \leq s \leq t$ if $m \succ_{rlex} \varphi(u_s)$ and $\deg(m) = \deg(\varphi(u_s))$ then $m \in (\varphi(u_1), \ldots, \varphi(u_{s-1}))$. This is the defining property of D so $D = (\varphi(u_1), \ldots, \varphi(u_t))$ and hence D and C have the same Betti numbers.

□

Corollary 3.4.5. *Let J be a square-free strongly stable ideal and D the square-free reverse lex ideal associated to J. If $\max(m) - \deg(m) \leq 2$ for all minimal generators m of both J and D then*

$$\beta_p^S(D) \leq \beta_p^S(J)$$

for all p.

Proof. Let $I = \varphi^{-1}(J)$ and $C = \varphi^{-1}(D)$. Then the assumption $\max(m) - \deg(m) \leq 2$ for all minimial generators of J and D means that the generators of I and C involve at most 3 variables. Since C is the reverse lex ideal associated to I, the claim follows by Proposition 3.3.3.

□

CHAPTER 4

EDGE IDEALS

In this chapter we give an overview of the theory of edge ideals. The study of edge ideals is the endeavor to find connections between the algebraic properties of quadratic square-free monomial ideals and the graph theoretic properties of finite simple graphs. This is a rich area of study of which we have collected here a small part of the results known in order to give the reader a taste of the varied nature of the subject.

4.1 Graphs

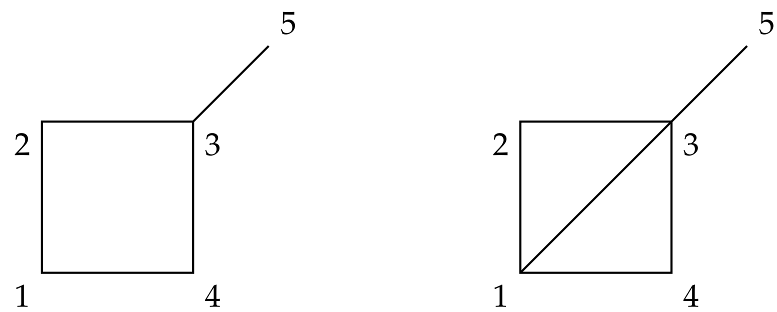
We begin this chapter with some basic graph theory definitions we will need in order to be able to state algebraic theorems about edge ideals.

Let G be a graph with vertex set V and edge set $E \subset V \times V$. We will always assume that G is a finite simple graphs, that is the vertex set of G is finite and G contains no loops and no multiple edges.

By an *induced subgraph* of G we mean the graph H with vertex set a subset S of V with edges all the edges of G which connect elements of S. A *cycle* of length $s$of a graph G is a set of edges of G of the form $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_s, v_1\}$. An induced cycle is an induced subgraph of G that is a cycle.

Definition 4.1.1. A graph, G, is called *chordal* if it contains no induced n-cycles for $n \geq 4$.

Example 4.1.2. The graph G on vertex set $\{1, 2, 3, 4, 5\}$ with edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{3, 5\}$ is not chordal since the subgraph induced by the vertex set $\{1,2,3,4\}$ is a cycle of length four (see Figure 4.1 (a)). On the other hand, the graph G' on the same vertex set but with edges $\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{1,3\}$ (see Figure 4.1 (b)) is chordal since it contains no induced cycles of length greater than three.

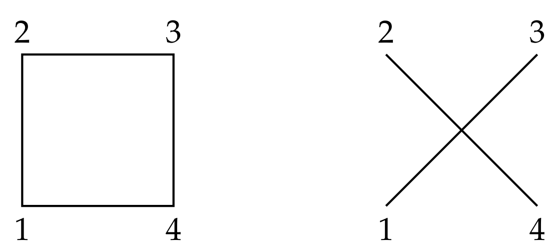


(a) The graph G (b) The graph G′

Figure 4.1: A non-chordal graph G and a chordal graph G'.

Definition 4.1.3. Let G be a graph on vertex set V and with edge set E. The *complement* of G is the graph G^c on the same vertex set V and with edge set $E^c = \{\{i,j\} | \{i,j\} \notin E\}$.

Example 4.1.4. Let G be the graph on vertex set $\{1,2,3,4\}$ with edge set $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ (see Figure 4.2 (a)). The complement graph is the graph G^c with edges $\{\{1,3\},\{2,4\}\}$ (see Figure 4.2 (b)).



(a) The graph G. (b) The graph G^c.

Figure 4.2: A graph G and its complement G^c.

4.2 Edge ideals and their Free Resolutions

We are interested in the minimal free resolutions of edge ideals and hence the numerical invariants of these free resolutions such as the Betti numbers and the regularity. Many of the results in this section are gathered in the survey paper [21] by Hà and Van Tuyl.

Let k be a field. To each simple graph on finite vertex set $\{1, \ldots, n\}$ we associate an ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ as follows:

Definition 4.2.1. Let G be a simple graph on vertex set $\{1, \ldots, n\}$ with edge set $E \subset V \times V$. The *edge ideal* associated to G is the ideal $I_G \subseteq S$ defined by

$$I_G = (x_i x_j | (i, j) \in E) .$$

A classical example of the connection between properties of the edge ideal I_G and the graph G is the following result of Fröberg [19].

Theorem 4.2.2. *Let G be a finite simple graph and I_G the edge ideal associated to G. The minimal free resolution of I_G is linear if and only if the graph complement G^c is chordal.*

Later Herzog, Hibi, and Zheng [25] extended this theorem to the following theorem on the powers of edge ideals of chordal graphs.

Theorem 4.2.3. *If I_G is the edge ideal of a graph G which has a linear minimal free resolution, then all powers of I_G also have linear minimal free resolutions.*

We can calculate many of the numerical invariants of the edge ideal I_G by passing to the Stanley-Reisner complex of I_G.

Definition 4.2.4. Given a square-free monomial ideal I in the polynomial ring $S = k[x_1, \ldots, x_n]$, we associate to I a simplicial complex Δ called the Stanley-Reisner complex which is defined as follows:

$$\Delta = \{F \subseteq \{1, \ldots, n\} | x^F \notin I\} .$$

For an edge ideal I_G, the Stanley-Reisner complex of I_G is given by the clique closure of the graph complement, G^c. That is, the simplicial complex whose faces are given by the induced complete subgraphs of G.

We can calculate the Betti numbers of the edge ideal I_G via Hochster's formula [27] which gives them in terms of the homology of subcomplexes of the Stanley-Reisner complex. Given a monomial m, we denote by Δ_m the subcomplex of Δ defined by $\Delta_m = \{F \subseteq \Delta |$ if $i \in F$, then x_i divides $m\}$.

Theorem 4.2.5. *(Hochster's Formula) Let I_G be an edge ideal in the ring S and let Δ be the clique closure of the graph G^c. Then the multigraded Betti numbers of the ideal I_G are given by the formula*

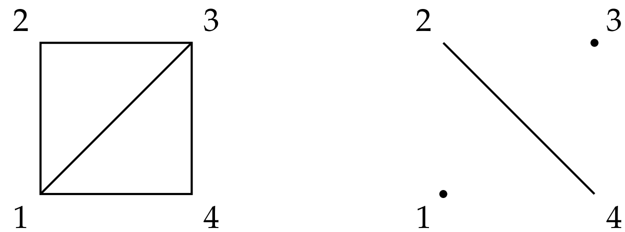
$$\beta_{i,m}(I_G) = \dim_k(\tilde{H}_{\deg(m)-i-2}(\Delta_m; k)) .$$

Summing over all multidegrees of the same degree we get the following formula for the graded Betti numbers.

Theorem 4.2.6. *Let I_G be the edge ideal of the graph G and Δ the clique closure of the complement G^c. The graded Betti numbers of I_G are given by*

$$\beta_{i,j}(I_G) = \sum_{deg(m)=j} \dim_k(\tilde{H}_{j-i-2}(\Delta_m; k)) .$$

Example 4.2.7. Let G be the graph depicted in Figure 4.3 (a) and let I_G be the edge ideal of G. G^c is shows in Figure 4.3 (b), in this case G^c is equal to its clique closure. We can see from Figure 4.3(b) that every subcomplex of G^c only has non-zero homology in degree zero, so the only non-zero Betti numbers of I_G are of the form $\beta_{i,i+2}$.



(a) The graph G (b) The graph G^c

Figure 4.3: Graphs G and G^c.

By counting the connected components of induced subgraphs we get the following Betti table for the edge ideal of G

$$\beta(S/I_g) = \begin{pmatrix} 1 & - & - & - \\ - & 5 & 6 & 2 \end{pmatrix}.$$

Many other results are given in [32] giving formulas for the Betti numbers of edge ideals which are obtained either by restricting to a subset of the Betti numbers or by restricting to special cases of graphs.

Since it is often difficult (except in special cases) to calculate the Betti numbers of an edge ideal it is useful to consider results on global invariants of the minimal free resolution of an edge ideal. We have the following result, due to Hà and Van Tuyl [23], on the regularity of an edge ideal I_G.

Theorem 4.2.8. *Let G be a graph and I_G the edge ideal of G. Then, if j is the maximal number of pairwise disconnected edges in G, then*

$$reg(I_G) \geq j + 1\,.$$

If G is a chordal graph, then the above inequality is an equality.

If G is not a chordal graph then the inequality in Theorem 4.2.8 is not an equality as we can see by taking G to be the five cycle (this example is due to Zheng [35]).

We can also get an upper bound on the regularity of the edge ideal I_G in terms of properties of the graph G.

Definition 4.2.9. Let G be a graph. A *matching* of G is a set of pairwise disjoint edges of G. The largest size of a maximal matching is called the matching number of G. We denote the matching number of G by $\alpha(G)$.

The following bound on the regularity of I_G is again due to Hà and Van Tuyl and is found in [23]

Theorem 4.2.10. *Let G be a finite simple graph. Then*

$$\mathrm{reg}(S/I_G) \leq \alpha(G)\,.$$

4.3 Cohen-Macaulay Graphs

A nice class of commutative rings are Cohen-Macaulay rings. A local ring is Cohen-Macaulay if the depth of the ring is equal to the Krull dimension of the ring. A general commutative ring is called Cohen-Macaulay if the localization of the ring at every prime ideal is Cohen-Macaulay.

Cohen-Macaulay rings have nice properties and so we want to know when the quotient ring S/I_G of S by the edge ideal I_G is Cohen-Macaulay.

Definition 4.3.1. We say that a graph G is Cohen-Macaulay if the ring S/I_G is Cohen-Macaulay. Villarreal characterized all Cohen-Macaulay trees (see [34]), and in [24] Herzog and Hibi generalized Villarreal's work to characterize all Cohen-Macaulay bipartite graphs. This led to the following result by Herzog, Hibi, and Zheng [26]:

Theorem 4.3.2. *Let G be a chordal graph then G is Cohen-Macaulay if and only if G is height unmixed.*

As it is intractable to classify all Cohen-Macaulay graphs, work has also been done to determine which graphs are sequentially Cohen-Macaulay, a weaker property. Classifying graphs which are sequentially Cohen-Macaulay is still a difficult problem and there are several partial results.

Francisco and Van Tuyl [18] showed in 2007 that all chordal graphs are sequentially Cohen-Macaulay. Another proof of this fact is shown in [10]

Later Van Tuyl and Villarreal [33] showed that if G is a bipartite graph then G is sequentially Cohen-Macaulay if and only if G is shellable. In that paper they also provided a recursive criteria for determining when a bipartite graph is shellable.

Finally, extending work of Francisco and Hà in [16], Dochtermann and Engström showed in [10] that if G is any graph and G' is the graph obtained from G by adding a whisker at every vertex, then G' is sequentially Cohen-Macaulay.

4.4 Graph Coloring

One interesting area of connections between graph theory and commutative algebra is in the study of graph colorings. We say that a graph G is n-colorable if there is a labeling of the vertices of G with the integers $\{1, \ldots, n\}$ such that no two adjacent vertices are the same color.

Definition 4.4.1. Let G be a graph. The chromatic number of G is the smallest integer n such that G is n-colorable.

Example 4.4.2. The graph G shown in Figure 4.4 has chromatic number 3 since it is 3-colorable but not 2-colorable.

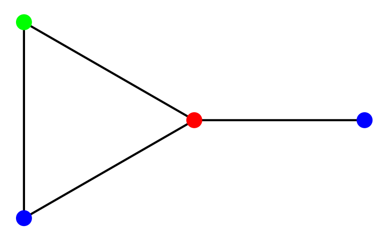


Figure 4.4: A graph G which is 3-colorable, but not 2-colorable.

Definition 4.4.3. A perfect graph G is one for which the chromatic number of every induced subgraph is equal to the size of the largest clique (i.e. induced complete graph) in that subgraph.

It is easy to see that the size of the largest clique of a graph is a lower bound on the chromatic number, so perfect graphs are graphs for which this lower bound is an equality. Perfect graphs include several important classes of graphs such as bipartite graphs and chordal graphs. Chudnovsky, Robertson, Seymour and Thomas [6] proved the following characterization of perfect graphs, known as the Strong Perfect Graph Theorem.

Theorem 4.4.4. *A graph G is perfect if and only if neither G nor G^c contain an induced cycle of odd length with length greater than or equal to five.*

In view of this theorem, it is interesting to find ways of detecting induced cycles of odd length. Recently Francisco, Hà and Van Tuyl [17] proved that one can detect cycles of odd length in a graph from algebraic properties of the edge ideal of the Alexander dual.

Theorem 4.4.5. *Let G be a graph and I_G the edge ideal of G. Let $J = {I_G}^{\vee}$ (J is the Alexander dual of I_G). A prime ideal $P = (x_{i_1}, \ldots, x_{i_s})$ is in Ass(R/J^2), the set of associated prime ideals of R/J^2, if and only if:*

(1) $P = (x_{i_1}, x_{i_2})$, and $\{x_{i_1}, x_{i_2}\}$ is an edge of G, or

(2) s is odd, and the induced graph on $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$ is an induced cycle of G.

This allows us to search for induced cycles of odd length using algebraic tools. The theorem is more useful than a theorem which merely identifies perfect graphs, since it tells us exactly where the cycles of odd length occur.

CHAPTER 5

THE COMPLEMENT OF THE N-CYCLE

5.1 Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field k. We are interested in the structure of the minimal free resolutions of quadratic monomial ideals of S. The method of polarization allows us to narrow our considerations to square-free quadratic monomial ideals. The minimal monomial generators of such an ideal can be easily encoded in a graph as follows: let G be a graph with vertex set $\{1, \ldots, n\}$, the *edge ideal of* G is the monomial ideal I_G of S whose minimal monomial generators are the monomials $x_i x_j$ where (i, j) is an edge of G. Much work has been done to discover connections between the combinatorial properties of the graph G and the algebraic properties of its edge ideal I_G. The properties of the complement graph G^c have turned out to be useful in this endeavour; recall that the complement of G is the graph G^c such that the vertex set of G^c the same as the vertex set of G and the edges of G^c are the non-edges of G. One of the main results about resolutions of edge ideals was proved by Fröberg [19] and states that an edge ideal I_G has a linear minimal free resolution if and only if the complement graph G^c is chordal.

We consider the question of whether there exists a regular cellular structure which supports the minimal free resolution of an edge ideal. In [2], Batzies and Welker showed that if an edge ideal has a linear minimal free resolution then there is a CW-cellular complex which supports that resolution. Their proof is non-constructive, however. Corso and Nagel in [7] and [8] and Horwitz in [28] construct explicit regular cellular structures for several classes of edge ideals with linear minimal free resolutions. In view of these results, we focus on edge ideals whose minimal free resolutions are not linear, but are close to being linear. The simplest non-chordal graphs are cycles of length four or greater and the simplest examples of edge ideals with non-linear resolutions are the edge ideals of the complements of such cycles. We study the minimal free resolutions of such ideals. By [13] and [15] we know that the minimal free resolution of the edge ideal of the complement of the n-cycle is linear until homological degree $n-4$ and that the only non-zero Betti number in homological degree greater than $n-4$ is $\beta_{n-3,n} = 1$.

Let $I_n \subset S$ be the edge ideal of the complement of the n-cycle. That is,

$$I_n = (x_1x_3, x_1x_4, \ldots, x_1x_{n-1}, x_2x_4, x_2x_5, \ldots, x_2x_n, \ldots, x_{n-2}x_n)\,.$$

Let $J_n = I_n + (x_1x_n)$. We study the minimal free resolution of S/I_n by first considering the minimal free resolution of S/J_n. In Section 5.2 we construct an explicit resolution for S/J_n and a regular cellular complex which supports this resolution (a different cellular complex is constructed in [7]; see Remark 5.3.8). Then in Section 5.3 we obtain a regular cell complex which supports the resolution of S/I_n from that which we constructed for S/J_n.

5.2 The Resolution of S/J_n

We begin this section by constructing a basis and differential maps for the minimal free resolution of S/J_n. The minimal free resolution of S/J_n has basis $\{1\}$ in homological degree 0 and basis in homological degree $f+1$ the set of symbols

$$x = (x_ix_j; e_1, \ldots, e_f)$$

where $x_i x_j$ is a minimal monomial generator of J_n with $i < j$,

$$e_1 < e_2 < \cdots < e_f < j\,,$$

and $e_p \neq i, i+1$ for all $1 \leq p \leq f$.

The differential is made up of three maps, ∂, μ_1, and μ_2 which we define below. First define $b(m)$ for a monomial m to be the largest (in the lex order with $x_1 > x_2 > \cdots > x_n$) minimal generator of the ideal J_n that divides m.

Then we define

$$\partial(x) = \sum_{p=1}^{f} \partial^{e_p}(x)$$

where

$$\partial^{e_p}(x) = (-1)^p x_{e_p}(x_i x_j; e_1, \ldots, \widehat{e}_p, \ldots e_f)\,.$$

The second map is defined by

$$\mu_1(x) = \sum_{q=1}^{f} \mu_1^{e_q}(x)$$

where

$$\mu_1^{e_q}(x) = (-1)^{q+1} \frac{x_i x_j x_q}{b(x_i x_j x_q)}(b(x_i x_j x_q); e_1, \ldots, \widehat{e}_q, \ldots, e_f)\,,$$

Finally, assume that x has the form $(x_i x_j; e_1, \ldots, e_c, e_{c+1}, \ldots, e_{c+r}, \ldots, e_f)$ such that $e_{c+1} = i - r, e_{c+2} = i - r + 1, \ldots, e_{c+r} = i - 1$ and $e_c \neq i - r - 1$. We define

$$\mu_2(x) = \sum_{s=c+1}^{c+r-1} \mu_2^{e_s}(x)$$

where $\mu_2^{e_s}(x)$ is

$$(-1)^{c+r+1} x_{e_{s+1}}(x_{e_s} x_j; e_1, \ldots, e_c, e_{c+1}, \ldots, \widehat{e}_s, \widehat{e}_{s+1}, \ldots, e_{c+r}, i, e_{c+r+1}, \ldots, e_f)\,.$$

It will sometimes be the case that the symbols appearing in $\mu_1(x)$ are not valid elements the basis as defined above. It is understood in this case that those terms of $\mu_1(x)$ are zero.

Define $d(x) = \partial(x) + \mu_1(x) + \mu_2(x)$ for x in homological degree 2 or higher. In homological degree 1 define $d(x_i x_j; \emptyset) = x_i x_j$. Then $d(x)$ is the differential of the minimal free resolution of S/J_n with the basis described above. Before proving that the minimal free resolution of S/J_n has basis and differential as described, we prove the following lemma.

Lemma 5.2.1. *Let*

$$x = (x_i x_j; e_1, \ldots, e_c, e_{c+1} \ldots, e_{c+r}, \ldots, e_f)$$

as above. Then $d^2(x) = 0$.

Proof. Every term of $d^2(x)$ has the form $(-1)^t x_u x_v \cdot y$ where y is the symbol for some basis element in homological degree f. We call $x_u x_v$ the *coefficient* of this term and we proceed by considering all the terms of $d^2(x)$ with the same coefficient $x_u x_v$ and show that these terms cancel.

First note that for all $p, q \in \{1, \ldots, f\}$ (assume without loss of generality that $p < q$), we have the following equality

$$\begin{aligned} \partial^{e_p} \circ \partial^{e_q}(x) &= (-1)^{p+q} x_{e_p} x_{e_q} (x_i x_j; e_1, \ldots, \widehat{e_p}, \ldots, \widehat{e_q}, \ldots, e_f) \\ &= -\partial^{e_q} \circ \partial^{e_p}(x)\,. \end{aligned}$$

In other words, $\partial^2(x) = 0$ for all x. In view of this, in the following we consider only the terms of $d^2(x)$ which do not come from $\partial^2(x)$.

There are several cases to consider, but first we make the following observations:

1. $b(x_i x_j x_{e_p}) = x_i x_{e_p}$ if $e_p \neq i - 1$
   $b(x_i x_j x_{e_p}) = x_j x_{e_p}$ if $e_p = i - 1$
   So μ_1 always contributes x_i or x_j to the coefficient of a term of $d(x)$

2. μ_2 always contributes x_{e_p} with $c+2 \leq p \leq c+r$

3. ∂ always contributes x_{e_p} with $1 \leq p \leq f$.

Case 1. Consider the terms of $d^2(x)$ with the coefficient $x_{e_p}x_{e_q}$ where $p<q$ and

$$p-1, q-1 \notin \{c+1, \dots, c+r-1\} .$$

The only terms with this coefficient come from $\partial^{e_p} \circ \partial^{e_q}$ and $\partial^{e_q} \circ \partial^{e_p}$. We have already shown that $\partial^{e_p} \circ \partial^{e_q}(x) = -\partial^{e_q} \circ \partial^{e_p}(x)$, so we are done.

Case 2. Consider terms of $d^2(x)$ with the coefficient $x_{e_p}x_{e_q}$ where

$$p-1 \in \{c+1, \dots, c+r-1\}$$
$$q-1 \notin \{c+1, \dots, c+r-1\} ,$$

again with $p<q$.

In this case μ_2 can also contribute to the coefficient $x_{e_p}x_{e_q}$ so we also have the terms

$$\begin{aligned}
&\partial^{e_q} \circ \mu_2^{e_{p-1}}(x) \\
&= (-1)^{c+r+q} x_{e_p}x_{e_q}(x_{e_{p-1}}x_j; e_1, \dots, e_{c+1} \dots, \widehat{e_{p-1}}, \widehat{e_p}, \dots, e_{c+r}, i, \dots, \widehat{e_q}, \dots, e_f) \\
&= -\mu_2^{e_{p-1}} \circ \partial^{e_q}(x) .
\end{aligned}$$

The case where $p \in \{c+1, \dots, c+r-1\}$ and $q \notin \{c+1, \dots, c+r-1\}$ but $q<p$ is analogous and results in the same relation

$$\partial^{e_q} \circ \mu_2^{e_{p-1}}(x) = -\mu_2^{e_{p-1}} \circ \partial^{e_q}(x) .$$

Case 3. Next we consider terms of $d^2(x)$ with the coefficient $x_{e_p}x_{e_q}$ where

$$p-1, q-1 \in \{c+1, \dots, c+r-1\}$$

and $p < q$.

If $p < q-1$, then as in Case 2 we have

$$\partial^{e_p} \circ \mu_2^{e_{q-1}}(x) = -\mu_2^{e_{q-1}} \circ \partial^{e_p}(x)\,.$$

In this case ($p < q-1$), we also have the following relation

$$\begin{aligned}
&\partial^{e_q} \circ \mu_2^{e_{p-1}}(x) \\
&= (-1)^{c+r+q-1} x_{e_q} x_{e_p} (x_{e_{p-1}} x_j; e_1, \ldots, e_{c+1} \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, \widehat{e_q}, \ldots, e_{c+r}, i, \ldots, e_f) \\
&= -\mu_2^{e_{p-1}} \circ \mu_2^{e_{q-1}}(x)\,.
\end{aligned}$$

Finally, if instead we have $p = q-1$, then

$$\begin{aligned}
&\mu_1^{q-2} \circ \mu_2^{q-1}(x) \\
&= (-1)^{c+r+q} x_{e_q} x_{e_{q-1}} (x_{q-2} x_j; e_1, \ldots, e_{c+1}, \ldots, \widehat{e_{q-2}}, \widehat{e_{q-1}}, \widehat{e_q}, \ldots, e_{c+r}, i, \ldots, e_f) \\
&= -\partial^{e_q} \circ \mu_2^{q-2}(x)\,.
\end{aligned}$$

Case 4. Consider the terms of $d^2(x)$ with the coefficient $x_i x_{e_p}$ for $p \in \{1, \ldots, f\}$. The only terms of $d^2(x)$ with x_i in the coefficient come from μ_1^{i-1} or from ∂^i.

If $p \neq c+r$ (recall that $e_{c+r} = i-1$), then we have

$$\partial^{e_p} \circ \mu_1^{i-1}(x) = -\mu_1^{i-1} \circ \partial^{e_p}(x)\,.$$

To see this in the case where $p < c+r$, note that

$$\begin{aligned}
\partial^{e_p} \circ \mu_1^{i-1}(x) &= (-1)^{c+r+1+p} x_i x_{e_p} (x_{i-1} x_j; e_1, \ldots, \widehat{e_p} \ldots, \widehat{e_{c+r}}, \ldots, e_f) \\
&= -\mu_1^{i-1} \circ \partial^{e_p}(x)\,.
\end{aligned}$$

On the other hand, if $p = i-1$

$$\begin{aligned}
&\mu_1^{i-2} \circ \mu_1^{i-1}(x) \\
&= (-1)^{c+r+1+c+r} x_i x_{i-1}(x_{i-2}x_j; e_1, \ldots, e_c, e_{c+1}, \ldots, \widehat{e_{c+r-1}}, \widehat{e_{c+r}}, \ldots, e_f) \\
&= -\partial^i \circ \mu_2^{i-2}(x)\,.
\end{aligned}$$

Finally, if $p-1 \in \{c+1, \ldots, c+r-2\}$, then we also have the relation

$$\begin{aligned}
&\partial^i \circ \mu_2^{e_{p-1}}(x) \\
&= (-1)^{c+r+c+r} x_i x_{e_p}(x_{e_{p-1}}x_j; e_1, \ldots, e_c, e_{c+1}, \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, e_{c+r}, \ldots, e_f) \\
&= -\mu_2^{e_{p-1}} \circ \mu_1^{i-1}(x)\,.
\end{aligned}$$

Case 5. Now we consider terms of $d^2(x)$ with the coefficient $x_j x_{e_p}$ for p in the set $\{1, \ldots, f\}$. There are two ways that x_j can be part of the coefficient. The first is that x_j comes from μ_1^f.

If $p \neq f$ and $e_f \neq i-1$, then we have

$$\begin{aligned}
\partial^{e_p} \circ \mu_1^{e_f}(x) &= (-1)^{f+1+p} x_{e_p} x_j(x_i x_f; e_1, \ldots, \widehat{e_p}, \ldots, e_{f-1}) \\
&= -\mu_1^{e_f} \circ \partial^{e_p}(x)\,.
\end{aligned}$$

If instead we have $p = f$, and $e_f, e_{f-1} \neq i-1$ then

$$\begin{aligned}
\mu_1^{e_{f-1}} \circ \partial^{e_f}(x) &= (-1)^{2f} x_{e_f} x_j(x_i x_{f-1}; e_1, \ldots, e_{f-2}) \\
&= -\mu_1^{e_{f-1}} \circ \mu_1^{e_f}(x)\,.
\end{aligned}$$

Finally, if we have $p-1 \in \{c+1, \ldots, c+r-1\}$ and $e_f > i+1$,

$$\begin{aligned}
&\mu_2^{e_{p-1}} \circ \mu_1^{e_f}(x) \\
&= (-1)^{f+c+r+2} x_p x_j(x_{p-1}x_f; e_1, \ldots, e_{c+1}, \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, e_{c+r}, i, e_{c+r+1} \ldots, e_{f-1}) \\
&= -\mu_1^{e_f} \circ \mu_2^{e_{p-1}}(x)\,.
\end{aligned}$$

The other way that x_j can be part of the coefficient of a term of $d^2(x)$ is that it comes from $\mu_1^{e_q}$ where $e_{q+1} \neq e_q + 1$ and where $f = c + r$.

In this case, if $p < q$, we have

$$\begin{aligned} \partial^{e_p} \circ \mu_1^{e_q}(x) &= (-1)^{q+1+p} x_{e_p} x_j (x_{e_q} x_i; e_1, \ldots, \widehat{e_p}, \ldots, \widehat{e_q}, \ldots, e_{c+1}, \ldots, e_{c+r}) \\ &= -\mu_1^{e_q} \circ \partial^{e_p}(x)\,. \end{aligned}$$

The case where $p > q$ is similar and results in the same relation.

In addition, if $p < q$ and $e_p = e_{p-1} + 1$, $e_{p+1} = e_p + 1$, …, $e_q = e_{q-1} + 1$, then we have

$$\begin{aligned} &\mu_2^{e_{p-1}} \circ \mu_1^{e_q}(x) \\ &= (-1)^{2q+1} x_{e_p} x_j (x_{e_{p-1}} x_i; e_1, \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, e_q, \ldots, e_{c+1}, \ldots, e_{c+r})) \\ &= -\mu_1^{e_{p-1}} \circ \partial^{e_p}(x)\,. \end{aligned}$$

Case 6. Finally we consider terms of $d^2(x)$ whose coefficients are $x_i x_j$. First note that the variable x_i only divides the coefficient of terms which come from μ_1 or terms which come from $\partial^i \circ \mu_2$. However, the coefficient of $\partial^i \circ \mu_2^{e_p}(x) \neq x_i x_j$ for any p. This together with the fact that x_j only appears as part of a coefficient via the map μ_1 means that $x_i x_j$ only appears as the coefficient of terms of μ_1^2.

Hence the only terms of $d(x)^2$ which have coefficient $x_i x_j$ appear in two cases. The first case is when $f = c + r$, $e_{q+1} \neq e_q + 1$. In this case we have

$$\begin{aligned} \mu_1^{e_{c+r}} \circ \mu_1^{e_q}(x) &= (-1)^{q+1+c+r} x_i x_j (x_{e_q} x_{i-1}; e_1, \ldots, \widehat{e_q}, \ldots, e_{c+1}, \ldots, e_{c+r-1}) \\ &= -\mu_1^{e_q} \circ \mu_1^{e_{c+r}}(x)\,. \end{aligned}$$

The other case in which we have terms with the coefficient $x_i x_j$ is when

$f \neq c + r$. In this case we have

$$\begin{aligned}\mu_1^{e_f} \circ \mu_1^{e_{c+r}}(x) &= (-1)^{c+r+1+f} x_i x_j (x_{i-1} x_{e_f}; e_1, \ldots, \widehat{e_{c+r}}, \ldots, e_{f-1}) \\ &= -\mu_1^{e_{c+r}} \circ \mu_1^{e_f}(x) \, .\end{aligned}$$

$\square$

Theorem 5.2.2. *The minimal free resolution of S/J_n has basis 1 in homological degree 0 and basis $(x_i x_j; e_1, \ldots, e_f)$ in homological degree $f+1$ where $x_i x_j$ is a minimal generator of J_n, $e_1 < e_2 < \cdots < e_f < j$, and $e_p \neq i, i+1$ for all $1 \leq p \leq f$. The differential of the resolution is the map d defined above.*

Proof. We prove this by induction on n. First consider the case where $n = 4$. The minimal free resolution, $\mathbf{G}$, of S/J_4 is the following

$$0 \longrightarrow S^2 \xrightarrow{d_1} S^3 \xrightarrow{d_0} S \longrightarrow 0 \, ,$$

where the basis of G_1 is

$$\Big\{(x_1x_3; \emptyset), (x_1x_4; \emptyset), (x_2x_4; \emptyset), \Big\} \, ,$$

and the basis of G_2 is

$$\Big\{(x_1x_4; 3), (x_2x_4; 1)\Big\} \, .$$

The differential of $\mathbf{G}$ is given by the following two maps:

$$d_0 = \begin{pmatrix} x_1x_3 & x_1x_4 & x_2x_4 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} x_4 & 0 \\ -x_3 & x_2 \\ 0 & -x_1 \end{pmatrix} \, .$$

It is easily checked that this is exact and hence it is the minimal free resolution of S/J_4.

Now assume that the minimal free resolution of S/J_{n-1} is as stated. Call this minimal free resolution $\mathbf{F}$. We will construct the minimal free resolution of S/J_n by using a series of mapping cones; one for each of the minimal monomial generators $x_1x_n, \ldots, x_{n-2}x_n$. First we have the following short exact sequence:

$$0 \longrightarrow S/(J_{n-1} : x_1x_n) \xrightarrow{x_1x_n} S/J_{n-1} \longrightarrow S/(J_{n-1} + (x_1x_n)) \longrightarrow 0\,.$$

The ideal $J_n + (x_1x_n)$ is the edge ideal of the complement of the n-cycle with the additional edge $\{1, n\}$. In this graph, the edge $\{1, n\}$ is a splitting edge as defined by Hà and Van Tuyl in [22]. In their paper they study the effect on the edge ideal of removing a splitting edge from a graph as in the short exact sequence above. In the remainder of this proof we will have similar short exact sequences for each minimal monomial generator x_ux_n, however only this first short exact sequence and the last, (that corresponding to the final minimal generator $x_{n-2}x_n$), are examples of short exact sequences representing the removal of a splitting edge.

Note that the ideal $(J_{n-1} : x_1x_n)$ is equal to the ideal $(x_3, x_4, \ldots, x_{n-1})$. Then the minimal free resolution of $S/(J_{n-1} : x_1x_n)$ is the Koszul complex on the variables $\{x_3, x_4, \ldots, x_{n-1}\}$. Call this Koszul complex $\mathbf{K}^{(1)}$ and shift the multigrading so that the generator in homological degree 0 has multidegree x_1x_n. We denote the generator in homological degree 0 of $\mathbf{K}^{(1)}$ by $(x_1x_n;\emptyset)$, and the basis in homological degree $f \geq 1$ by

$$\left\{(x_1x_n; e_1, e_2, \ldots, e_f)\middle|\, 3 \leq e_1 < \cdots < e_f \leq n-1\right\}.$$

The differential of $\mathbf{K}^{(1)}$ is given by ∂ as we have defined it above.

Let $\mu = \mu_1 + \mu_2$, and extend μ so that $\mu((x_1x_n;\emptyset)) = -x_1x_n$. It is easy to see that $\partial^2 = 0$, so by Lemma 5.2.1 we have $\mu \circ \partial = -d \circ \mu$. Thus the map

$$(-\mu) : \mathbf{K}^{(1)} \longrightarrow \mathbf{F}$$

is a map of complexes of degree 0 which lifts the map

$$S/(J_{n-1} : x_1x_n) \xrightarrow{x_1x_n} S/J_{n-1}\,.$$

The mapping cone of $(-\mu) : \mathbf{K}^{(1)} \to \mathbf{F}$ gives us a minimal free resolution of $S/(J_{n-1} + (x_1x_n))$ with differential $\partial + \mu$. Call this resolution $\mathbf{F}^{(1)}$.

For each of the minimal monomial generators $x_1x_n, x_2x_n, \ldots, x_{n-2}x_n$ of J_n we have a similar short exact sequence and mapping cone. We show the step which adds the minimal monomial generator x_ux_n. Let $\mathbf{F}^{(u-1)}$ be the minimal free resolution of $S/(J_{n-1} + (x_1x_n, x_2x_n, \ldots, x_{u-1}x_n))$ obtained in the previous step. The basis of $\mathbf{F}^{(u-1)}$ in degree $f+1$ is

$$\Big\{(x_ix_j; e_1, \ldots, e_c, \ldots, e_f)\Big| e_1 < e_2 < \cdots < e_c < i,\ i+1 < e_{c+1} < \cdots < e_f\Big\}$$

where x_ix_j is a minimal generator of the ideal $(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n))$.

We have the short exact sequence:

$$0 \to S/((J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n)) : x_ux_n) \to S/(J_{n-1} + (x_1x_n \ldots, x_{u-1}x_n))$$
$$\to S/(J_{n-1} + (x_1x_n, \ldots, x_ux_n)) \to 0\,.$$

Note that the ideal $(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n) : x_ux_n)$ is equal to the ideal $(x_1, x_2, \ldots, x_{u-1}, x_{u+2}, \ldots, x_n)$. Let $\mathbf{K}^{(u)}$ be the Koszul complex on the elements $\{x_1, x_2, \ldots, x_{u-1}, x_{u+2}, \ldots, x_n\}$. We multigrade this complex so that the basis element in homological degree 0 has multidegree x_ux_n. $\mathbf{K}^{(u)}$ has differential ∂ and basis in homological degree f given by

$$\left\{(x_u x_n; e_1, \ldots, e_c \ldots, e_f) \middle| e_1 < e_2 < \cdots < e_c < u, u+1 < e_{c+1} < \cdots < e_f < n\right\}.$$

As before, we define $\mu(x_u x_n; \emptyset) = -x_u x_n$. Then the map

$$(-\mu) : \mathbf{K}^{(u)} \longrightarrow \mathbf{F}^{(u-1)}$$

is a map of complexes of degree 0 which lifts the map

$$S/(J_{n-1} + (x_1 x_n, \ldots, x_{u-1} x_n) : x_u x_n) \xrightarrow{x_u x_n} S/(J_{n-1} + (x_1 x_n, \ldots, x_{u-1} x_n)) .$$

Let $\mathbf{F}^{(u)}$ be the mapping cone complex of this map of complexes. $\mathbf{F}^{(u)}$ is a free resolution of $S/(J_{n-1} + (x_1 x_n, \ldots, x_u x_n))$. This resolution is minimal since the basis elements in homological degree $f > 0$ all have multidegree a monomial of degree $f + 1$.

□

Next we construct a regular cellular structure which supports the minimal free resolution of S/J_n which we have just constructed.

Theorem 5.2.3. *There exists a regular cell complex supporting the minimal free resolution of the ideal S/J_n for all $n \geq 4$.*

Proof. We proceed by induction on n. A regular cell complex supporting the minimal free resolution of $S/J_4 = S/(x_1 x_3, x_1 x_4, x_2 x_4)$ is shown in Figure 5.1.

We use the same notation as in the proof of Theorem 5.2.2: $\mathbf{F}$ is the minimal free resolution of S/J_{n-1} with basis and differential as in Theorem 5.2.2, $\mathbf{F}^{(u)}$ the minimal free resolution of $S/(J_{n-1} + (x_1 x_n, \ldots, x_u x_n))$, and $\mathbf{K}^{(u)}$ the Koszul complex on the variables $\{x_1, \ldots, x_{u-1}, x_{u+2}, \ldots, x_{n-1}\}$ shifted so that the generator in homological degree 0 has multidegree $x_u x_n$.

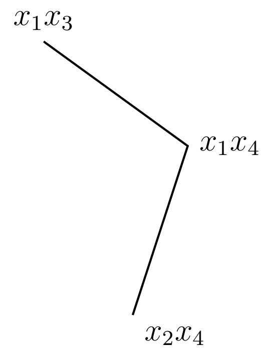


Figure 5.1: A regular cell complex supporting the minimal free resolution of S/J_4.

Let X_{n-1} be a regular cellular complex supporting S/J_{n-1}. We will construct a regular cellular complex supporting the minimal free resolution of S/J_n by constructing a regular cellular complex $X_{n-1}^{(u)}$ supporting the resolution $\mathbf{F}^{(u)}$ for each $1 \leq u \leq n-2$ in turn.

Recall from the proof of Theorem 5.2.2 that $\mathbf{F}^{(1)}$ is the mapping cone of the map

$$(-\mu) : \mathbf{K}^{(1)} \longrightarrow \mathbf{F}$$

where $\mathbf{K}^{(1)}$ is the Koszul complex on the variables $\{x_3, \ldots, x_{n-1}\}$. The Koszul complex $\mathbf{K}^{(1)}$ is supported on an $(n-4)$-dimensional simplex with vertices labeled by the basis elements $(x_1x_n; x_3), \ldots, (x_1x_n; x_{n-1})$. Since the mapping cone construction shifts the basis elements of $\mathbf{K}^{(1)}$ up a homological degree, these vertices become the new one-dimensional cells. The 1-cell $(x_1x_n; x_i)$ has endpoints $(x_1x_n; \emptyset)$ and $(x_1x_i; \emptyset)$. Thus adding $\mathbf{K}^{(1)}$ to $\mathbf{F}$ corresponds to adding a cone over the point $(x_1x_n; \emptyset)$ to X_{n-1}. This cone is attached to X_{n-1} at the cell $(x_1x_{n-1}; 3, \ldots n-2)$ since

$$\mu((x_1x_n; 3, \ldots, n-1)) = (-1)^{n-2}x_n(x_1x_{n-1}; 3, \ldots, n-2)\,.$$

Let $X_{n-1}^{(1)}$ be X_{n-1} together with this cone over the point $(x_1x_n; \emptyset)$ with base the cell $(x_1x_{n-1}; 3, \ldots, n-2)$. Since X_{n-1} was regular and since the base of the cone we have just added is a single $(n-4)$-dimensional cell, the complex X^1_{n-1} is a regular cell complex which supports the resolution $\mathbf{F}^{(1)}$.

Now suppose that we have constructed a regular cell complex, $X^{(u-1)}_{n-1}$ supporting the resolution $\mathbf{F}^{(u-1)}$. We wish to construct a regular cellular complex $X^{(u)}_{n-1}$ supporting $\mathbf{F}^{(u)}$. We obtain $\mathbf{F}^{(u)}$ from the mapping cone of the map

$$(-\mu) : \mathbf{K}^{(u)} \longrightarrow \mathbf{F}^{(u-1)} .$$

The Kozsul complex $\mathbf{K}^{(u)}$ is supported on an $(n-4)$-dimensional simplex with vertices labeled by the basis elements

$$\{(x_ux_n; j) | j \in \{1, \ldots, u-1, u+2, \ldots, n-1\}\} .$$

Again, the mapping cone construction shifts the basis elements of $\mathbf{K}^{(u)}$ up a homological degree so that these vertices become the new 1-cells. The 1-cell $(x_ux_n; j)$ has endpoints $(x_ux_n; \emptyset)$ and $(x_jx_u; \emptyset)$ for $j \neq u-1$ and for $j = u-1$ the cell $(x_ux_n; j)$ has endpoints $(x_ux_n\emptyset)$ and $(x_jx_n; \emptyset)$. Adding $\mathbf{K}^{(u)}$ to $\mathbf{F}^{(u-1)}$ thus corresponds to adding a cone over the point $(x_ux_n; \emptyset)$. The base of this cone is the collection of cells in $X^{(u-1)}_{n-1}$ which are labelled by the basis elements of $\mathbf{F}^{(u-1)}$ which make up

$$\mu(x_ux_n; 1, 2, \ldots, u-1, u+2, \ldots, n-1) .$$

In other words, the base of the cone is the collection of cells

$$(x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2),$$
$$(x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1),$$
$$(x_2 x_n; 1, 4, \ldots, u-1, u, u+2, \ldots, n-1),$$
$$\vdots$$
$$(x_{u-2} x_n; 1, \ldots, u-3, u, u+2, \ldots, n-1),$$
$$(x_{u-1} x_n; 1, \ldots, u-2, u+2, \ldots, n-1).$$

Let $X_{n-1}^{(u)}$ be the regular cell complex $X_{n-1}^{(u-1)}$ together with this cone. In order to show that $X_{n-1}^{(u)}$ is regular we need only show that the union of the cells labelled by

$$(x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2),$$
$$(x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1),$$
$$(x_2 x_n; 1, 4, \ldots, u-1, u, u+2, \ldots, n-1),$$
$$\vdots$$
$$(x_{u-2} x_n; 1, \ldots, u-3, u, u+2, \ldots, n-1),$$
$$(x_{u-1} x_n; 1, \ldots, u-2, u+2, \ldots, n-1)$$

in the cell complex $X_{n-1}^{(u-1)}$ is homeomorphic to an $(n-4)$-dimensional ball.

First consider just the first two elements in this list. The intersection of these two elements is

$$\begin{aligned}
&\mu_2^1((x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2)) \\
&= (x_1 x_{n-1}; 3, 4, \ldots, u-1, u, u+2, \ldots, n-2) \\
&= \partial^{n-1}((x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1))
\end{aligned}$$

if $u > 2$, and

$$\begin{aligned}
\mu_1^1((x_2 x_{n-1}; 1, 4, \ldots, n-2)) &= (x_1 x_{n-1}; 4, \ldots, n-2) \\
&= \partial^{n-1}((x_1 x_n; 4 \ldots, n-1))
\end{aligned}$$

if $u = 2$. (We have already considered the case where $u = 1$). In either case the intersection consists of a single cell of dimension $n-5$. This is homeomorphic to an $(n-5)$-ball and thus the union of the two elements

$$(x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2)$$

and

$$(x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1)$$

is homeomorphic to an $(n-4)$-ball.

Now suppose that we know that the union of the first p elements in the list are homeomorphic to an $(n-4)$-ball. Explicitly, we assume that the union of the cells

$$\begin{gathered}(x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2), \\ (x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1), \\ (x_2 x_n; 1, 4, \ldots, u-1, u, u+2, \ldots, n-1), \\ \vdots \\ (x_{p-1} x_n; 1, \ldots, p-2, p+1, \ldots, u-1, u, u+2, \ldots, n-1)\end{gathered}$$

is homeomorphic to an $(n-4)$-dimensional ball.

The intersection of the cell $(x_p x_n; 1, \ldots, p-1, p+2, \ldots, u-1, u, u+2, \ldots, n-1)$ with the union of cells listed above is the following union of cells:

$$\begin{gathered}(x_p x_{n-1}; 1, \ldots, p-1, p+2, \ldots, u, u+2, \ldots, n-2), \\ (x_1 x_n; 3, \ldots, p, p+2, \ldots, u, u+2, \ldots, n-1), \\ (x_2 x_n; 1, 4, \ldots, p, p+2, \ldots, u, u+2, \ldots, n-1), \\ \vdots \\ (x_{p-1} x_n; 1, \ldots, p-2, p+2, \ldots, u, u+2, \ldots, n-1)\,.\end{gathered}$$

These cells are the collection of cells which come from

$$\mu(\partial^{u+1}(x_p x_n; 1, \ldots, p-1, p+2, \ldots, n-1))\,.$$

Since

$$(x_p x_n; 1, \ldots, p-1, p+2, \ldots, n-1)$$

is a regular cell which is a cone over the point $(x_p x_n; \emptyset)$, the face

$$\partial^{u+1}((x_p x_n; 1, \ldots, p-1, p+2, \ldots, n-1))$$

is also a regular cell which is a cone over the point $(x_p x_n; \emptyset)$. Therefore the base cells of this cone (i.e. the cells of $\mu(\partial^{u+1}(x_p x_n; 1, \ldots, p-1, p+2, \ldots, n-1)))$ must be homeomorphic to an $(n-5)$-ball and thus the union of the set of cells

$$(x_u x_{n-1}; 1, 2, \ldots, u-1, u+2, \ldots, n-2),$$
$$(x_1 x_n; 3, \ldots, u, u+2, \ldots, n-1),$$
$$(x_2 x_n; 1, 4, \ldots, u-1, u, u+2, \ldots, n-1),$$
$$\vdots$$
$$(x_{p-1} x_n; 1, \ldots, p-2, p+1, \ldots, u-1, u, u+2, \ldots, n-1)$$

and the cell

$$(x_p x_n; 1, \ldots, p-1, p+2, \ldots, u-1, u, u+2, \ldots, n-1)$$

is homeomorphic to an $(n-4)$-ball.

□

Example 5.2.4. In Figure 5.2 we show the steps in constructing the regular cell structure supporting the minimal free resolution of S/J_5 from that supporting the minimal free resolution of S/J_4 (shown in Figure 5.1). Part (a) of Figure 5.2 shows the regular cell structure supporting the minimal free resolution of S/J_4. The first step in the construction adds a cone over the point $(x_1 x_5; \emptyset)$ with base the cell $(x_1 x_4; 3)$. This step is shown in Figure 5.2 (b).

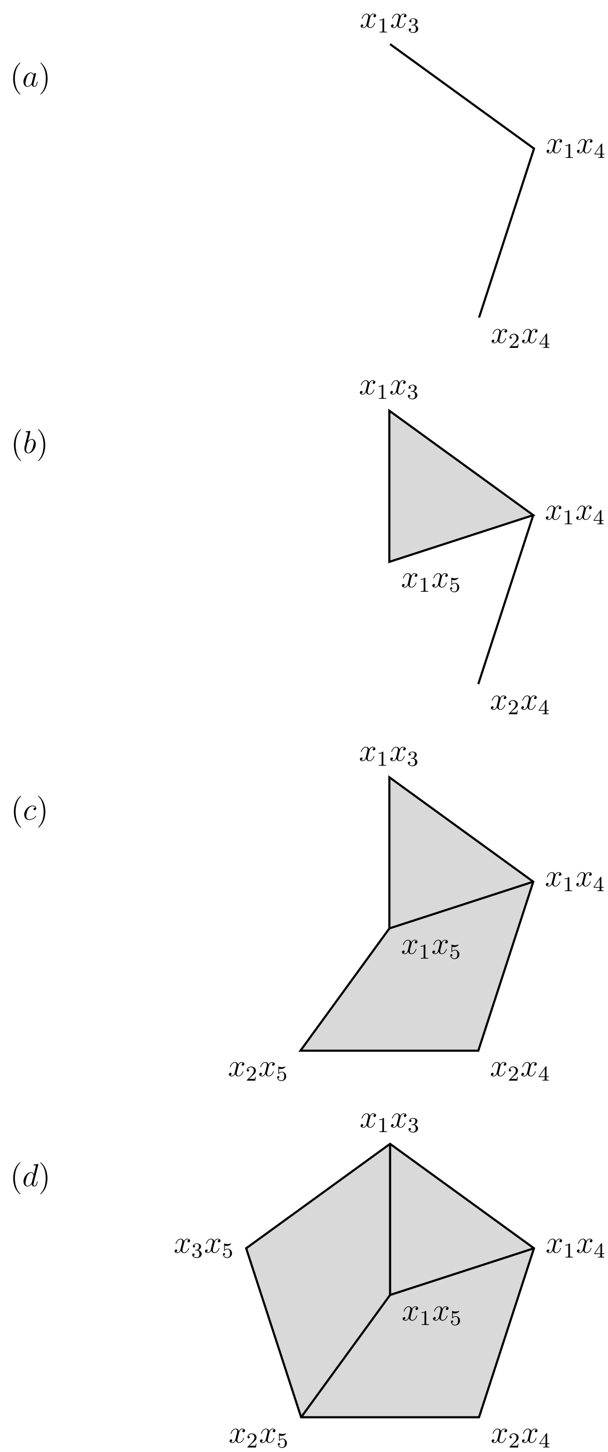


Figure 5.2: The construction of a regular cell complex supporting the minimal free resolution of S/J_5.

The next step of the construction adds a cone over the point $(x_2x_5;\emptyset)$ with base the union of the cells $(x_2x_4;1)$ and $(x_1x_5;4)$. This step is shown in Figure 5.2 (c).

The final step in the construction is shown in Figure 5.2 (d). It adds a cone over the point $(x_3x_5;\emptyset)$ with base the union of the cells $(x_2x_5;1)$ and $(x_1x_5;3)$.

Definition 5.2.5. We say a CW-complex, X, is *pure of dimension* d if every cell of X is contained in the boundary of a cell of dimension d.

Proposition 5.2.6. *The regular cell complex X_n constructed in Theorem 5.2.3 which supports the minimal free resolution of S/J_n is pure of dimension $n-3$.*

Proof. We prove this by induction on n. It is clear from Figure 5.1 that the regular cell complex supporting the minimal free resolution of S/J_4 is pure of dimension 1. Now let X_n be the regular cell complex supporting the minimal free resolution of S/J_n and suppose that the regular cell complex X_{n-1} supporting S/J_{n-1} is pure of dimension $n-4$. By the way we constructed X_n from X_{n-1} every cell of X_n which was not in X_{n-1} is contained in the boundary of an $(n-3)$-dimensional cell. Therefore, to finish the proof we need to show that every cell of X_{n-1} is contained in an $(n-3)$-dimensional cell in X_n. Since X_{n-1} is pure of dimension $n-4$, we only need to consider the $(n-4)$-dimensional cells of X_{n-1}.

Every $(n-4)$-dimensional cell of X_{n-1} has the form

$$(x_ix_{n-1};1,2,\dots,i-1,i+2,\dots,n-2)$$

for some $1\le i\le n-3$. Then

$$\begin{aligned}&\mu_1^{n-1}((x_ix_n;1,2,\dots,i-1,i+2,\dots,n-1))\\ &=(-1)^{n-2}x_n(x_ix_{n-1};1,2,\dots,i-1,i+2,\dots,n-2)\,,\end{aligned}$$

so $(x_ix_{n-1}; 1, 2, \ldots, i-1, i+2, \ldots, n-2)$ is part of the boundary of the $(n-3)$-dimensional cell $(x_ix_n; 1, 2, \ldots, i-1, i+2, \ldots, n-1)$ in X_n. Hence X_n is pure of dimension $n-3$.

□

5.3 The Resolution of S/I_n

In this section we construct a regular cell complex which supports the minimal free resolution of S/I_n. We do this by taking the cells from the regular cell complex supporting S/J_n which we have already constructed which do not contain the point x_1x_n and then adding an additional cell. We then show that the resulting complex satisfies the necessary acyclicity conditions so that it supports the minimal free resolution of S/I_n.

Before we construct the regular cell complex supporting the minimal free resolution of S/I_n, we need to know more about the structure of the regular cell complex we constructed to support the minimal free resolution of S/J_n. To this end, we need the following lemma and proposition.

Lemma 5.3.1. *The cells of X_n which contain as part of their boundary the point $(x_1x_n; \emptyset)$ are exactly those cells which are labeled by symbols of the form*

$$(x_ix_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$$

where $i+2 \leq e_i < e_{i+1} < \cdots < e_f < n$.

Proof. One direction of this claim is easy. Any cell of the form

$$(x_ix_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$$

contains in its boundary a cell of the form $(x_1x_n; t_1, \ldots, t_d)$. To see this, note that if $i = 1$ then the original cell is already of this form. If not, then applying μ_1^1 (if $i = 2$), or μ_2^1 (if $i > 2$) yields a cell of the desired form. Then repeated applications of ∂ to $(x_1x_n; t_1, \ldots, t_d)$ will eventually yield $(x_1x_n; \emptyset)$.

We prove the opposite direction by induction on the dimension of the cell. Clearly the only 1-dimensional cells which contain $(x_1x_n; \emptyset)$ in their boundary are cells of the form $(x_1x_n; j)$ for some $3 \leq j \leq n-1$ and the cell $(x_2x_n; 1)$.

Now suppose that the claim holds for cells of dimension $f-1$. Let

$$x = (x_ix_j; e_1, \ldots, e_f)$$

be a cell of dimension f which contains $(x_1x_n; \emptyset)$ as part of its boundary. In order for $(x_1x_n; \emptyset)$ to be part of the boundary of x it must be part of the boundary of one of the cells which make up $d(x)$. Let y be a cell which contains $(x_1x_n; \emptyset)$ and appears as a term of $d(x)$. Since y is a cell of dimension $f-1$, by the induction hypothesis it must be of the form

$$y = (x_ux_n; 1, 2, \ldots, u-1, t_u, t_{u+1}, \ldots, t_{f-1})$$

with $u + 1 < t_u < t_{u+1} < \cdots < t_{f-1} < n$.

In order for y to be a term of $d(x)$, either y is a term of $\partial(x)$ or y is a term of $\mu(x)$. If y is a term of $\partial(x)$, x must have the form

$$(x_ux_n; 1, 2, \ldots, u-1, e_u, \ldots, e_f)$$

with $\{t_u, t_{u+1}, \ldots, t_{f-1}\} \subset \{e_u, e_{u+1}, \ldots, e_f\}$.

Since x_n divides the multidegree of y, if y is a term of $\mu(x)$ then

$$x = (x_ix_n; e_1, \ldots, e_f)\,.$$

In order for $\mu_2(x)$ to be non-zero, x must have the form

$$(x_i x_j; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)\,.$$

So if y is a term of $\mu_2(x)$,

$$x = (x_i x_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$$

with $i > u+1$. Finally, since $b(x_u x_i x_n) = x_u x_n$ if and only if $i = u+1$, if y is a term of $\mu_1(x)$ then we must have

$$x = (x_{u+1} x_n; 1, 2, \ldots, v, e_{u+1}, \ldots, e_f)\,.$$

Thus, in order for $(x_1 x_n; \emptyset)$ to be contained in the boundary of x, x must be of the form

$$x = (x_i x_n; 1, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)\,.$$

□

Proposition 5.3.2. *Let X_n be the regular cell complex supporting the minimal free resolution of S/J_n, constructed in Theorem 5.2.3. Then the boundary of the union of the $(n-3)$-dimensional cells of X_n is homeomorphic to a sphere of dimension $n-4$.*

Proof. The dimension $n-3$ cells of X_n correspond to the following basis elements of the minimal free resolution of S/J_n:

$$\begin{gathered}(x_1 x_n; 3, 4, \ldots, n-1),\\ (x_2 x_n; 1, 4, \ldots, n-1),\\ \vdots\\ (x_p x_n; 1, 2, \ldots, p-1, p+2, \ldots, n-1),\\ \vdots\\ (x_{n-2} x_n; 1, 2, \ldots, n-3)\,.\end{gathered}$$

By Lemma 5.3.1, all of these cells contain the point $(x_1x_n;\emptyset)$. Any two of these $(n-3)$-dimensional cells intersect in exactly one $(n-4)$-dimensional cell which also contains the point $(x_1x_n;\emptyset)$. More explicitly, the intersection of the cells

$$(x_px_n;1,2,\ldots,p-1,p+2,\ldots,n-1)$$

and

$$(x_qx_n;1,2,\ldots,q-1,q+2,\ldots,n-1)$$

where $p<q$ is exactly the $(n-4)$-dimensional cell

$$(x_px_n;1,2,\ldots,p-1,p+2,\ldots,\widehat{q+1},\ldots,n-1)\,.$$

Conversely, every $(n-4)$-dimensional cell which contains the point $(x_1x_n;\emptyset)$ is of the form

$$(x_px_n;1,2,\ldots,p-1,p+2,\ldots,\widehat{q+1},\ldots,n-1)$$

and thus is contained in boundary of exactly two $(n-3)$-dimensional cells. On the other hand, an $(n-4)$-dimensional cell which does not contain $(x_1x_n;\emptyset)$ can have two forms. It is either of the form

$$(x_px_{n-1};1,2,\ldots,p-1,p+2,\ldots,n-2)$$

or of the form

$$(x_px_n;1,\ldots,\widehat{q},\ldots,p-1,p+2,\ldots,n-1)\,.$$

In either of these cases the $(n-4)$-dimensional cells is contained in exactly one $(n-3)$-cell,

$$(x_px_n;1,2,\ldots,p-1,p+2,\ldots,n-1)\,.$$

This structure together with the fact that X_n is contractible means that X_n is homeomorphic to an $(n-3)$-dimensional ball. Therefore the boundary of X_n, by which we mean the $(n-4)$-cells which are contained in only one $(n-3)$-dimensional cell is homeomorphic to an $(n-4)$-sphere.

□

Now we are ready to construct a CW-complex which supports the minimal free resolutionof S/I_n.

Construction 5.3.3. Define a CW-complex Y_n as follows:

The dimension 0 cells of Y_n are the dimension 0 cells of X_n minus the 0-cell $(x_1x_n; \emptyset)$. The dimension f cells of Y_n are the dimension f cells of X_n which do not contain the point $(x_1x_n; \emptyset)$ in their boundary for $1 \leq f \leq n-4$. There is one dimension $n-3$ cell of Y_n whose boundary is the union of all the dimension $n-4$ cells of Y_n.

Before proving that Y_n supports the minimal free resolution of S/I_n we will need the following definition.

Definition 5.3.4. Let X be a CW-complex whose 0-cells are labeled by monomials and whose higher dimensional cells are labeled by the lcm of the monomials labeling the 0-cells contained in the boundary of the given cell. For a monomial m define $X_{\leq m}$ to be the subcomplex of X consisting of all cells labeled by monomials which divide m.

Theorem 5.3.5. *The minimal free resolution of S/I_n is supported on the regular cellular complex Y_n.*

Proof. We need only show that for every monomial m in the lcm lattice of I_n, the subcomplex $(Y_n)_{\leq m}$ is acyclic. Let m be an element of the lcm lattice of I_n which is not the product of all the variables $x_1, \ldots, x_n$. If x_1x_n does not divide m, then $(Y_n)_{\leq m} = (X_n)_{\leq m}$. Since the CW-complex X_n supports the minimal free resolution of S/J_n, we know that $(X_n) \leq m$ is acyclic.

Now suppose that $x_1 x_n$ does divide m. Let

$$m = x_1 x_2 \dots x_i x_{e_{i+1}} \dots x_{e_f} .$$

Then by Lemma 5.3.1 the only cell of X_n which has multidegree m and contains the point $x_1 x_n$ in its boundary is the cell

$$x = (x_i x_n; 1, 2, \dots, i-1, e_{i+1}, \dots, e_f)n .$$

Since we got Y_n from X_n by taking the cells which did not contain the point $(x_1 x_n; \emptyset)$, $(X_n)_{\leq m} = (Y_n)_{\leq m} \cup x$ where x is attached to $(Y_n)_{\leq m}$ along the cells of the boundary of x which do not contain the point $(x_1 x_n; \emptyset)$. Since $(X_n)_{\leq m}$ is contractible, if we knew that the intersection of the cell x with $(Y_n)_{\leq}$ was also contractible, then $(Y_n)_{\leq m}$ would have to be contractible as well.

To see that the union of the cells of the boundary of x containing the point $(x_1 x_n; \emptyset)$ is contractible, suppose that y and z are two cells contained in the boundary of x such that $(x_1 x_n; \emptyset)$ is contained in the boundary of both y and z. Further suppose that $w = (x_u x_v; p_1, \dots, p_c)$ is a cell contained in the intersection of y and z. Since both y and z contain $(x_1 x_n; \emptyset)$, they have the form

$$y = (x_{j_1} x_n; 1, 2, \dots, i_1 - 1, t_1, \dots, t_{f_1})$$

and

$$z = (x_{j_2} x_n; 1, 2, \dots, i_2 - 1, s_1, \dots, s_{f_2}) .$$

Let the lcm of the multidegree of y and the multidegree of z be

$$x_1 x_2 \dots x_j x_{e_1} \dots x_{e_f}$$

where $j = \min\{j_1, j_2\}$. Then it is not hard to check that w is contained in the cell $(x_j x_n; 1, 2, \dots, j-1, e_1, \dots, e_f)$ which is also contained in the intersection of y and z. Since all of the cells in the boundary of x which contain the point $(x_1x_n;\emptyset)$ intersect in cells which also contain $(x_1x_n;\emptyset)$, the union of cells in the boundary of x which contain $(x_1x_n;\emptyset)$ is contractible. Since X_n is a regular CW-complex, the boundary of x is homeomorphic to a sphere, therefore the union of the cells of the boundary of x which do not contain $(x_1x_n;\emptyset)$ is also contractible.

Finally, we must check that $(Y_n)_{\leq x_1\ldots x_n}$ is acyclic as well. By construction of Y_n, we know $(Y_n)_{\leq x_1\ldots x_n} = Y_n$, which consists of a single $(n-3)$-dimensional cell whose boundary is homeomorphic to a sphere. Therefore, Y_n is acyclic, and we are done.

$\square$

We end with two examples of regular cell complexes which support the minimal free resolution of S/I_4 and S/I_5.

Example 5.3.6. The regular cell complex which we constructed in Theorem 5.2.3 which supports the minimal free resolution of S/J_4 is shown in Figure 5.3 (a).

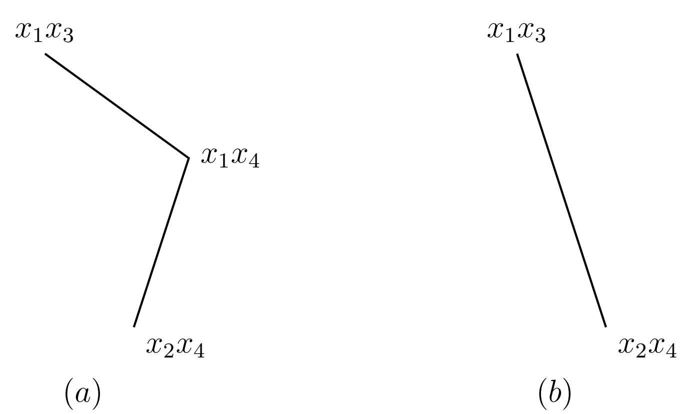


Figure 5.3: Regular cell complexes which support the minimal free resolutions of (a) S/J_4 and (b) S/I_4.

A regular cell complex which supports the minimal free resolution of S/I_4 is obtained from this cell complex by removing the cells which contain the 0-cell $(x_1x_4;\emptyset)$ (for simplicity, in Figure 5.3 this cell is labeled by its multidegree x_1x_4), and adding a 1-cell whose boundary made up of the cells $(x_1x_3;\emptyset)$ and $(x_2x_4;\emptyset)$. This is shown in Figure 5.3 (b).

Example 5.3.7. The regular cell complex which we constructed in Theorem 5.2.3 which supports the minimal free resolution of S/J_5 is shown in Figure 5.4 (a).

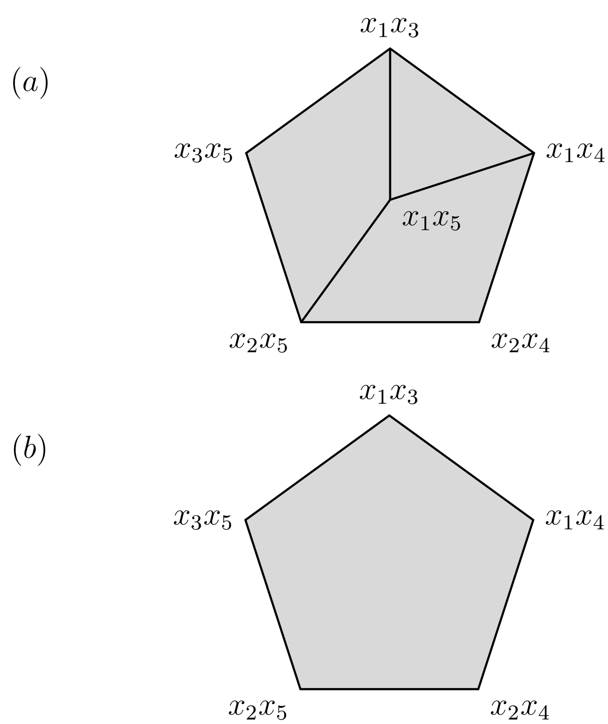


Figure 5.4: Regular cell complexes which support the minimal free resolutions of (a) S/J_5 and (b) S/I_5.

A regular cell complex which supports the minimal free resolution of S/I_5 is obtained from this cell complex by removing the cells which contain the 0-cell $(x_1x_5;\emptyset)$, and adding a 1-cell whose boundary made up of the cells

$$(x_1x_4;3),(x_2x_4;1),(x_2x_5;4),(x_3x_5;2),(x_3x_5;1)\,.$$

This complex is shown in Figure 5.4 (b).

Remark 5.3.8. Let M be a monomial ideal in S. The minimal free resolution $\mathbf{F}_M$ of S/M can have more than one cellular structure. A cellular structure uses a fixed basis, and different choices of basis in $\mathbf{F}_M$ can yield different cellular structures.

The ideal J_n is an example of a specialization of a Ferrers ideal. Corso and Nagel showed in [7] that such an ideal is supported on a regular cell complex. However for $n \geq 5$ the regular cell complex they constructed is different than that constructed in this chapter. For example, Figure 5.5 (a) shows the regular cell complex which supports the minimal free resolution of S/J_5 constructed in [7], and (b) shows that constructed in this chapter.

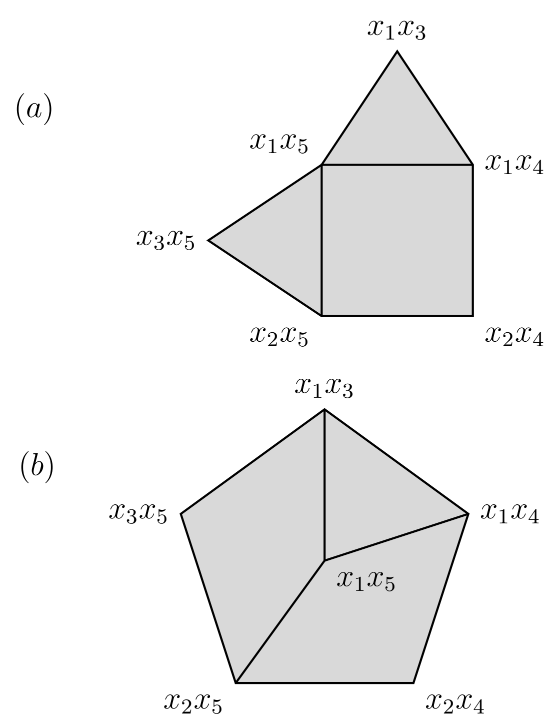


Figure 5.5: Two different regular cellular structures which support the minimal free resolution of S/J_n.

The goal of this chapter is to construct a cellular resolution of S/I_n. The cellular structure on the minimal free resolution of S/J_n is just used as a tool. The cellular structure in [7] cannot be used as a tool in the proof of Theorem 3.5 in the same way as we use our cellular structure. Consider how the proof of Theorem 3.5 works in the example in Figure 5.5. We use the cellular complex in Figure 5.5 (b) by removing all the cells containing the vertex x_1x_5 and then gluing a new two-dimensional cell to the remaining pentagon (the pentagon is the boundary of the new cell). If we remove the cells containing the vertex x_1x_5 from the cellular complex in Figure 5.5 (a), then we get the four edges

$$\{\{x_1x_3, x_1x_4\}, \{x_1x_4, x_2x_4\}, \{x_2x_4, x_2x_5\}, \{x_2x_5, x_3x_5\}\}$$

which don't form a cycle, so we cannot glue a new two-dimensional cell to them.

Note that for small numbers of n the resolution constructed here is the same as that constructed by Horwitz in [28], however for $n \geq 9$ Horwitz's resolution cannot be applied to the ideals J_n (see Example 3.18 in [28]).

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