RECURSIVE RELATIONS OF SCATTERING AMPLITUDES IN GAUGE AND GRAVITY THEORIES

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RECURSIVE RELATIONS OF SCATTERING AMPLITUDES IN GAUGE AND GRAVITY THEORIES

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In this dissertation, we discussed the new development of scattering amplitudes in gauge and gravities theories. The LHC era requires the new development of scattering amplitude beyond the traditional Feynman diagram approach. We reviewed the new scattering amplitude methods, inspired by string theory, analyticity and supersymmetry. With these new methods, we [37][54] proved (1) the color/kinematics equivalence in Bern-Carrasco-Johansson (BCJ) recursive identity from the viewpoint of heterotic string theory and (2) the quadratic identities for Yang-Mills theory via the Kawai-Lewellen-Tye (KLT) relation. Both identities simplify the Yang-Mills amplitude calculation and illustrate deep structures in gauge and gravity theories.

BIOGRAPHICAL SKETCH

Yang Zhang was born in BengBu, China on the second day of November, 1983, and then grew up in Hefei, China. He developed enthusiasm in science from his childhood, under the influence of his family. His interest was in chemistry, then mathematics, and finally established in theoretical physics. He received his B.S. in 2005 from the department of modern physics of University of Science and Technology of China. After that, Yang Zhang entered the physics Ph.D. program at Cornell University on 2005. His research focused on theoretical high energy physics, including scattering amplitude, quantum field theory tunneling process and inflationary cosmology.

This dissertation is dedicated to my family.

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CHAPTER 1

INTRODUCTION

1.1 Background

Currently, gauge theories and the Einstein gravity theory describe our universe. The standard model (SM), as a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory, accurately predicted the electromagnetic, strong and weak interactions below the TeV scale [29]. On the other hand, Einstein's general relativity (GR) was verified in experiments of the solar system scale to the cosmic microwave background observation [55].

However, there are still many open problems in gauge theories and the Einstein gravity. The SM prediction has not been fully tested at the TeV scale, especially the Higgs boson in the SM, has not been discovered experimentally. And the hierarchy problems in the standard model imply that the SM is only an effective field theory for the low-energy physics. On the other hand, experimentally the Einstein gravity has not been tested in short distance or strong fields cases like the black holes. Furthermore, there is no consistent quantization method for the Einstein gravity. To answer these questions, we need to study the deeper aspects of gauge theories and the Einstein gravity themselves and to formulate theories of the new physics.

Scattering amplitudes play the central role in the study of gauge and gravity theories. Scattering amplitude, is the quantum transition amplitude between certain asymptotically free incoming states and outgoing states via interaction. Schematically, if the α and β are the initial state and final states, the scattering amplitude A can be expressed as,

$$_{out}\langle\beta|\alpha\rangle_{in} \equiv \langle\beta|S|\alpha\rangle = \delta_{\beta\alpha} + i(2\pi)^D\delta^D(p_\alpha - p_\beta)A \tag{1.1}$$

where D is the dimension of the spacetime.

Scattering amplitudes in gauge theories are the links between theories and particle physics experiments: in particle physics experiments, scattering amplitudes are directly related to the cross section measurement. In gauge theories, scattering amplitudes are often precisely predicted by the perturbative calculation. So it is crucial to verify a gauge theory by calculating its scattering amplitudes and comparing them with the experiments. For example, quantum electrodynamics (QED) in the SM was verified to the greatest accuracy in the history of physics. The QED four-loop scattering amplitude calculation agrees with the electron anomalous magnetic moment experiment to more than 10 significant figures [1].

Furthermore, the study of scattering amplitudes leads to the discovery of deeper structures in gauge theories. Sometimes, the scattering amplitude calculation showed unexpected properties or symmetries, and required theorists to reformulate gauge theories. A famous example is that, the quantum chromodynamics (QCD) n-gluon tree amplitude vanishes when all the gluons except one have the same helicity. This property is unexpected and it is hard to see this feature from the original Yang-Mills action. The reason is actually deep: the QCD tree amplitude coincides with the corresponding super-Yang-Mills amplitude, and the latter's symmetry forces the amplitude to vanish. So this is a hint that the tree-level QCD amplitude can be reformulated as a super-Yang-Mills amplitude.

In practice, as the Large Hadron Collider (LHC) started up in 2010, there are urgent demands for the development of gauge theory scattering amplitudes calculation. Usually, the QCD scattering dominates LHC processes. To extract new physics signals from the background QCD amplitudes, we have to calculate the QCD amplitudes to great accuracy with new methods.

The scattering amplitude is as important in Einstein gravity but more subtly. The Einstein gravity is a nonrenormalizable theory, which means to cancel the divergence in gravity loop amplitudes, we have to introduce an infinite number of coupling constants. So although the tree amplitude in the Einstein theory is well-defined, the loop amplitude is a big problem. However, in the view point of effective field theories, most coupling constants are highly suppressed by the Planck scale, and it is meaningful to discuss the gravity amplitude in the low energy limit. Also, the famous Kawai-Lewellen-Tye (KLT) relation implied that the gravity and gauge tree amplitudes are closely related. Hence the study of Einstein gravity amplitude provides the new insight for gauge theory amplitudes.

There are many candidate theories beyond the SM and the Einstein gravity theory, for example, supersymmetry, supergravity, extra-dimension theories and string theories. It is very important to study scattering amplitudes in the new physics, because their scattering amplitudes not only give predictions for future experiments but also provide new techniques for the calculation of scattering amplitudes in QCD.

This thesis discusses the new development of scattering amplitudes calculation in gauge and gravity theories, not limited to the SM or the Einstein gravity. We will focus on the recursive relations of scattering amplitudes, which not only greatly simplify the scattering amplitudes calculation, but also illustrate the deep structure in gauge and gravity theories.

1.2 Scattering amplitudes, traditional approaches

Recall that in quantum field theories, Feynman diagram is the standard way to calculate scattering amplitudes [48]: First, write down the Feynman rules for fields propagation and interaction according to the action. Then, draw all the possible Feynman diagrams up to some perturbative order and calculate each diagram by Feynman rules. Finally, the scattering amplitude is determined by,

$$iA = \sum (\text{Feynman diagrams}). \tag{1.2}$$

The problem of the Feynman diagram approach in gauge theory is that, there are too many diagrams and each diagram is not gauge invariant. For example, the 2 gluon → 4 gluon process has 220 tree-level Feynman diagrams. There is no clear physical meaning for each diagram, since it depends on the gauge choice. Only when the diagrams are summed, the physical result is obtained.

Since the invention of Feynman diagrams, a lot of new techniques were developed to simplify the scattering amplitude calculation. An important tool for gauge and gravity amplitudes is the *spinor helicity formalism*: The momentum and polarization vectors can be written as the products of *Weyl spinors*. In this way, the little group transformation is manifest, and we can make most of the gauge Ward-Takahashi identity. Furthermore, in spinor helicity formalism, the *Maximal-Helicity-Violated (MHV) amplitudes*, has a particular simple expression, the Parke-Taylor formula [47],

$$A^{partial,tree}(-,-,+,...+) = \frac{\langle 12\rangle^4}{\langle 12\rangle..\langle n-1,n\rangle\langle n1\rangle} \tag{1.3}$$

For gauge theories, the traditional Feynman diagrams contain color factors which made the amplitudes complicated. There is an efficient way to strip the color factors by introducing the *partial amplitudes* or *color-stripped amplitudes*. These amplitudes are important, since they contain no color factor and are gauge-invariant.

The calculation of the Einstein gravity tree amplitude is particularly complicated. Since the Einstein theory is a nonrenormalizable theory, there are infinity types of vertices in its Feynman rules. There is another way to calculate these amplitudes without referring to Feynman rules. Kawai, Lewellen and Tye [39] discovered that the closed string tree amplitude is the square of the corresponding open string amplitudes. Take the low energy limit, this relation ensures that the gravity amplitude is the square of Yang-Mills amplitudes so the gravity Feynman rules are not needs. This is the *KLT relation*.

1.3 New recursive relations in scattering amplitudes

After the invention of the spinor helicity formalism and partial amplitudes, several recursive relations were discovered to simplify the scattering amplitudes. They are inspired by the *unitarity*, *analytic properties*, *twistor space* and *string theory* techniques. These relations dramatically extend our ability in amplitude calculation.

Cachazo, Svrcek and Witten [20] found that the Yang-Mills tree amplitude is built out of diagrams whose vertices are MHV amplitudes. Britto, Cachazo, Feng and Witten (BCFW) [17][18] proved a recursive relation, which can rewrite a tree amplitude as the product of on-shell subamplitudes.

The extremely powerful Grassmannian formalism in twistor space was discovered, [4] [3] [46] [2] for $N = 4$ super Yang-Mills theory. It gives a simple and unified formalism for the $N = 4$ Yang-Mills tree amplitude,

$$\mathcal{A}_{n;k} = \frac{1}{\mathrm{Vol}(GL(k))} \int \frac{d^{k\times n} C_{\alpha a}}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a) \tag{1.4}$$

for N^{k-2}MHV tree amplitude. Its analytic structure also provides the loop amplitude information. All planer loop amplitudes in $N = 4$ Yang-Mills theory can be generated recursively. Furthermore, by the Grassmannian formalism, dual superconformal symmetry for $N = 4$ tree amplitude, is discovered.

Because the $N = 4$ Yang-Mills tree amplitude is the same as the non-supersymmetric Yang-Mills tree amplitude, the Grassmannian formalism is very important for the QCD tree amplitude calculation. However, the generalization of the Grassmannian formalism for non-supersymmetric loop amplitudes has not been proposed.

We focus on a new set of recursive relations, which reduce the number of terms in scattering amplitude by a large factor. Recently, a new set of recursive relations in Yang-Mills theory was conjectured by Bern, Carrasco and Johansson (BCJ) [9]. The Yang-Mills tree scattering amplitude can be written as the channel sum,

$$A^{YM} = \sum_i \frac{c_i n_i}{P_i} \tag{1.5}$$

where c_i is the color factor, n_i is the kinematic factor and P_i is the pole for each channel. If three channels satisfy the Jacobi identity,

$$c_i + c_j + c_k = 0 \tag{1.6}$$

BCJ [9] conjectured that the corresponding kinematics factors satisfy the dual identity,

$$n_i + n_j + n_k = 0. \tag{1.7}$$

For example, for the four-gluon scattering amplitudes, there are three channels, s, t and u. The Jacobi identity reads $c_s + c_t + c_u = 0$ while it can be checked explicitly that $n_s + n_t + n_u = 0$.

If BCJ conjecture is correct, then the number of independent terms in Yang-Mills amplitudes drops dramatically. For example, in the $2g \to 5g$ process, the number of kinematics factors drops from 945 to 120.

Furthermore, the BCJ conjecture is related to the gravity tree amplitude. [9] proposed that the graviton amplitude is related to the QCD kinematics factors,

$$A^{grav} = \sum_i \frac{n_i n_i}{P_i}. \tag{1.8}$$

The gauge-invariant form of (1.7) was proven by the open string monodromy relations in the low energy limit [15]. However, it is not easy to understand the duality between (1.6) and (1.7) and the duality between (1.5) and (1.8).

We used heterotic string theory techniques to prove the two dualities [37]. Inspired by the (KLT) relation [39], we can treat the color factors and kinematic factors equivalently by heterotic string theory techniques, and show the duality between (1.6) and (1.7). Furthermore, with the KLT relation, we proved the duality between (1.5) and (1.8). This work also provides an elegant way to study $N = 4$ supergravity scattering amplitude.

The BCJ relation has also been explicitly verified in several examples up to two-loop Yang amplitudes [9] [8]. The loop BCJ relation, if proved to be correct in general, would be a very powerful tool in QCD loop amplitude calculation.

Beyond the linear relations, surprising quadratic identities for QCD scattering amplitude were discovered by the BCFW methods [12][13]: products of two QCD tree amplitudes with particular helicities sum to zero. These identities are interesting, because they provide strong restrictions on the complicated NMHV tree amplitudes in terms of the known simple MHV amplitudes. However, their physical meaning is not transparent.

$$
\begin{aligned}
& s_{12} s_{34} A^{YM}(1^- 2^- 3^- 4^+ 5^+) A^{YM}(2^- 1^+ 4^+ 3^- 5^+) \\
& + s_{13} s_{24} A^{YM}(1^- 3^- 2^- 4^+ 5^+) A^{YM}(3^- 1^+ 4^+ 2^- 5^+) = 0,
\end{aligned} \tag{1.9}
$$

We [54] proved that these quadratic identities are the result of the global symmetries in gravity-scalar theory, via the KLT relation. Following this viewpoint, generalized quadratic identities, which can include fermionic amplitudes, were discovered [30] inspired by the R-symmetry in supergravity.

It is also promising that these quadratic identities would be used to study the gluon inferred (IR) limit in QCD scattering amplitude. We find that by the soft pion theorem in supergravity, the quadratic identities imply a new set of identities for QCD scattering amplitudes in the IR limit.

It is interesting to see how the KK and BCJ relation and KLT relation look like in the Grassmannian formalism. Because the Grassmanian formalism has larger explicit symmetry, these relations should manifest themselves in a more subtle way than the usual spacetime form. Furthermore, if the KLT relation can be embedded in the Grassmanian formalism, it would be straightforward to calculate $N = 8$ supergravity amplitudes and helpful for solving the long-lasting problem on the finiteness of $N = 8$ supergravity.

1.4 Outline

In this thesis, first the Yang-Mills theory and Einstein gravity theory is reviewed in the appendix. Then beyond standard model theories, especially string theories are introduced to create the new methods for scattering amplitude calculation in gauge and gravity theories. After that, we present a review of new scattering amplitude approaches beyond Feynman diagrams, like the color decomposition, the spin-helicity formalism, the BCFW recursive relations and etc.

Then we would focus on our results of recursive relations,

- The BCJ identities in the viewpoint of heterotic string theory,
- The proof of quadratic identities by the KLT relation and the generalized quadratic identities.

Finally, we conclude with an overview of the future directions of the development of scattering amplitude calculation.

CHAPTER 2

YANG-MILLS THEORY AND EINSTEIN GRAVITY THEORIES EMBEDDED IN STRING THEORY

To solve fundamental problems in standard model and quantum gravity, string theory was proposed in 1970's. Instead of considering fundamental particles, the fundamental object in string theory is one-dimensionally extended object, string. Basic gradients in string theory are [50][51],

- Consistent quantum gravity. The spin-2 graviton appears in string theory naturally and their interaction in string theory would reduce to Einstein's general relativity in the low energy limit. Furthermore, the quantum gravity in string theory is free of UV divergence and therefore, consistent.
- Fixed gauge group. String theory also has gauge interaction and in the low energy limit, it reduces to Yang-Mills theory. However, unlike the field theory, the gauge group in string theory is fixed by the anomaly-free condition.
- Critical Dimension. Strings can only be quantized without anomaly in *critical dimension*. This provides hints to the mysterious question, *Why is our spacetime four-dimensional?*. However, in general, the string theory's critical dimension is not four, and we need to *compactify the extra dimensions*.
- No free parameter. Unlike the standard model, there is no free continuous parameter in string theory.
- Supersymmetry. String theory requires supersymmetry to include fermions and remove the unphysical tachyon modes.

Until now (2011), there is no direct experimental proof of string theory. However, in practice, string theory provides new methods for perturbative scattering amplitude calculation, other than Feynman diagrams. Since the string interaction would reduce to Yang-Mills theory and Einstein gravity in the low energy limit, string amplitude techniques are very important for Yang-Mills theory and Einstein gravity amplitude calculation. String theory also illustrates the fundamental relation between gluon and graviton amplitude, by the famous *KLT relation*.

And string theory has rich non-perturbative physics, for example, the *AdS/CFT duality*. The $(d+1)$-dimensional superstring theory in curved space is dual to the d-dimensional super-Yang-Mills theory. This is a strong-weak duality. So take the weak coupling limit in string theory side, we get the supergravity. The latter is due to the strongly coupled Yang-Mills theory. Since the supergravity weakly coupled supergravity can be studied by the perturbative method, the same calculation provides the answer for the long-lasting problem, strongly-coupled Yang-Mills theory.

2.1 Quantization of string theories and string states

Basically, string theory, after quantization, has an infinite tower of quantum states. The lightest string states are just the photon, gluon, graviton and their supersymmetry partners. The other states are massive and have higher spins. They satisfy the *Regge behavior*,

$$s = \alpha' m^2 + const. \tag{2.1}$$

where α' is the *Regge slope*. In the low energy limit, we can neglect the higher-spin states, and study the effective theory of the lightest states.

2.1.1 Bosonic string theory

The simplest string theory is the *bosonic string theory*. The $2d$ world-sheet *Polyakov action* is,

$$S = -\frac{1}{4\pi\alpha'} \int_M d^2\sigma (-\gamma)^{ab} \partial_a X^\mu \partial_b X_\mu \tag{2.2}$$

where $a, b = 1, 2$ are the world-sheet coordinates. X^μ is the embedding coordinators of the bosonic string and γ_{ab} is the world sheet metric. Besides the spacetime Poincare symmetry for X^μ, the world-sheet covariance, this action further symmetry,

$$X^\mu(x) \mapsto X^\mu(x), \quad g_{ab}(x) \mapsto e^{2\omega(x)} g_{ab}(x) \tag{2.3}$$

which is called *Weyl symmetry*. Weyl symmetry is equivalent to a conformal transformation, specially, the *dilatation*. Hence (2.2) is a conformal field theory.

In a quantum theory, conformal symmetry is usually broken by the anomaly. For bosonic string theory, the anomaly cancels out only if the spacetime dimension is 26, which is the *critical dimension* for bosonic string theory.

A well-defined bosonic string theory should satisfy certain boundary condition in its extension, i.e. the σ direction. If the string world-sheet is periodical,

$$X^\mu(\tau, 0) = X^\mu(\tau, 2\pi), \tag{2.4}$$

then it is a *closed string*. Instead, if either the Neumann (2.5) or the Dirichlet boundary condition (2.6) is satisfied,

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0 \tag{2.5}$$

$$X^\mu(\tau,0) = X^\mu, \quad X^\mu(\tau,\pi) = X'^\mu \tag{2.6}$$

then it is an *open string* state.

The lowest open string state with Neumann boundary condition can be constructed from the *mode expansion*. In the conformal coordinators $(z,\bar{z})$,

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i\Big(\frac{\alpha'}{2}\Big)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{m}\Big(\frac{1}{z^m} + \frac{1}{\bar{z}^m}\Big). \tag{2.7}$$

where z is in the upper half plane. Open string states are constructed from the vacuum state by the raising operator α_n^μ, $(n < 0)$. The physical condition is,

$$(L_n + \delta_{n,0}A)|\psi\rangle = 0, \quad n \geq 0 \tag{2.8}$$

where L_n is the Virasoro generator and A is the sum of zero point energy of the transverse modes. In this case, $A = -1$. (2.8) determines the mass and the physical modes of string states. For the massless vector boson, the physical condition is the same as that in gauge theory.

The $d = 2$ CFT has a state/operator duality, so we can also use CFT operator to describe a string state. The lowest closed string states are summarized in Table 2.1. Note that the tachyon would be removed in superstring theory.

state	m^2	operator	physical condition	spacetime interpretation
$\lvert 0,k\rangle$	$-\frac{1}{\alpha'}$	$e^{ik\cdot X}$		tachyon
$\epsilon_\mu\alpha_{-1}^\mu\lvert 0,k\rangle$	0	$\epsilon_\mu\partial X^\mu e^{ik\cdot X}$	$k^\mu\epsilon_\mu = 0$, $\epsilon_\mu \sim \epsilon_\mu + k_\mu$	gauge vector boson

Table 2.1: Lowest open bosonic string states

To extend the $U(1)$ gauge symmetry to Non-Abelian gauge symmetry, we may introduce the *Chan-Paton* factor to the open string state. For the left end of the open string, we associate an integer index i. Similar we associate the index j to the right end.

$$|N,k\rangle \rightarrow |N,k,i,\,j\rangle \tag{2.9}$$

The string action and the physical condition is the same as before. Let $1 \leq i, j \leq N$ and $(\lambda^a)_{ij}$ be the Lie algebra matrices of $U(N)$. If $(\lambda^a)_{ij}$ is normalized as,

$$tr(\lambda^a\lambda^b) = \delta^{ab}, \tag{2.10}$$

then

$$|N,k,a\rangle \equiv \sum_{ij}(\lambda^a)_{ij}|N,k,i,\,j\rangle \tag{2.11}$$

is the $U(N)$-gluon with the color index a. Furthermore, we can introduce the gauge group $SO(n)$ or $Sp(k)$ in the unoriented open string theory. However, there is no way to introduce the exceptional Lie group by Chan-Paton factors.

The closed string have independent left and right-moving modes while the open string modes are always the combination of left and right-moving modes. Hence a close string mode with the momentum k^μ can be regarded as the tensor product of two open string modes, each of which has the momentum $k^\mu/2$. The factor $1/2$ comes from the normalization.

Similarly, the lowest closed string states are also constructed from the mode expansion,

$$X^\mu(z,\overline{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty}^{\infty}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\overline{z}^m}\right) \tag{2.12}$$

All the string states can be constructed from the vacuum by two sets of raising operators $\alpha_n^\mu, \tilde{\alpha}_n^\mu$, $(n < 0)$. And there are also two sets of physical conditions,

$$(L_n + \delta_{n,0}A)|\psi\rangle = 0, \quad (\tilde{L}_n + \delta_{n,0}\tilde{A})|\psi\rangle = 0, \quad n \geq 0 \tag{2.13}$$

where L_n and $\tilde{L}_n$ are the Virasoro generators. The lowest closed string states are summarized in Table. 2.2. The massless states contains both graviton, antisymmetric field and dilaton. For the graviton, the physical condition is the same as that in Einstein's gravity.

state	m^2	operator	physical condition	spacetime interpretation
$\|0,k\rangle$	$-\frac{4}{\alpha'}$	$e^{ik\cdot X}$		tachyon
$e_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{1}\|0,k\rangle$	0	$e_{\mu\nu}\partial X^{\mu}\bar{\partial}X^{\nu}e^{ik\cdot X}$	$k^{\mu}e_{\mu\nu} = k^{\nu}e_{\mu\nu} = 0$, $e_{\mu\nu} \sim e_{\mu\nu} + a_{\mu}k_{\nu} + k_{\mu}b_{\nu}$, $k \cdot a = k \cdot b = 0$	graviton, anti-symmetric field and dilaton

Table 2.2: Lowest closed bosonic string states

2.1.2 Type II and Type I string theories

Although the bosonic string theories contain gauge symmetry and gravity, they have some disadvantages: (1) lacks fermions (2) contains tachyons. Supersymmetry will solve these problems. The $(1,1)$ world-sheet-supersymmetric action is,

$$S = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^{\mu}\bar{\partial}X_{\mu} + \psi^{\mu}\bar{\partial}\psi_{\mu} + \tilde{\psi}^{\mu}\partial\tilde{\psi}_{\mu}\right) \tag{2.14}$$

where the fermionic ψ^{μ} and $\tilde{\psi}^{\mu}$ are the supersymmetric partners of X^{μ}. The addition fermionic fields change the total central charge, so *the critical dimension for superstring theory is 10*.

For closed superstring theory, although X^{μ} is periodical in σ, ψ^{μ} and $\tilde{\psi}^{\mu}$ could be either periodical or anti-periodical. For the periodical case, we call it *Ramond boundary condition* (R) while *Neveu-Schwartz boundary condition* (NS) for the anti-periodical. Again, the mode expansion and physical condition determine the physical spectrum, and both left and right-moving sectors are open string sectors with one half of the momentum.

For the left-moving sectors (X_L^μ and ψ^μ), in the light-cone gauge, the lowest states are listed in (2.3), where $\mathbf{8}_v$ is the spacetime vector while $\mathbf{8}$ and $\mathbf{8}'$ are Weyl

sector	spacetime spin	m^2
NS+	$\mathbf{8}_v$	0
NS-	$\mathbf{1}$	$-\frac{1}{2\alpha'}$
R+	$\mathbf{8}$	0
R-	$\mathbf{8}'$	0

Table 2.3: massless open superstring states

spinors with opposite helicities.

The left and right-moving sectors for closed strings could be in the same or different sectors. Furthermore, not all the tensor products of left and right-moving sectors have the consistent operator product expansion. The physical closed string states can be chosen by the Gliozzi-Scherk-Olive (GSO) projection. We have two types of closed superstring theories,

- Type IIA. (NS+,NS+), (R+, NS+), (NS+,R-), (R+,R-)
- Type IIB. (NS+,NS+), (R+, NS+), (NS+,R+), (R+,R+)

while the tensor product's spacetime spin can be determined by group theory. We have the following closed string states (2.4), Here C_0, C_1, C_2, C_3 and C_4 are *Ramond-Ramond* fields. The $(1,1)$ world-sheet supersymmetry generates $d = 10, N = 2$ spacetime supersymmetry. The type IIB closed string theory is a chiral theory while IIA closed string theory is non-chiral.

sector	spacetime spin	irreducible representation	interpretation
(NS+,NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	$\mathbf{1} + \mathbf{28} + \mathbf{35}$	dilaton + anti-symmetry tensor+graviton
(R+,R+)	$\mathbf{8} \times \mathbf{8}$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_+$	C_0, C_2, C_4
(R+,R-)	$\mathbf{8} \times \mathbf{8}'$	$\mathbf{8}_v + \mathbf{56}_t$	C_1, C_3
(NS+,R+)	$\mathbf{8}_v \times \mathbf{8}$	$\mathbf{8}' + \mathbf{56}$	dilatino + gravitino
(NS+,R-)	$\mathbf{8}_v \times \mathbf{8}'$	$\mathbf{8} + \mathbf{56}'$	dilatino + gravitino

Table 2.4: massless closed superstring states

The open superstring theory can also be constructed from the mode expansion. The GSO projection will select the following sectors in the massless level,

- Type I. NS+, R+

Again, the tachyon mode is removed by the GSO projection. The NS+ sector is a spacetime vector and the R+ sector is a Weyl spinor. We can also introduce the Chan-Paton factors, so the NS+ sector becomes the gluon while the R+ sector becomes the gaugino. The open string boundary condition removes one of the spacetime supercharge, so now the symmetry is $d = 10, N = 1$. In addition, we need to include the *unoriented* closed string states.

To get the four dimensional effective theory, we have to compactify the extra six dimensions. Different geometry for the extra dimension would constraint the number of supercharges. In particular, if the extra dimension is a *Calabi-Yau* manifold, then 3/4 supercharges are removed. So we get $d = 4, N = 1$ super-Yang-Mills theory from the Type I string theory in this case.

2.1.3 Heterotic string theory

There is another way to introduce the gauge symmetry, other than the Chan-Paton factors in open string theory: *heterotic string theory* [34][35]. It is a closed string theory and called heterotic, because the left-moving sector is a bosonic string and the right-moving sector is superstring. Some of the bosonic modes become the Lie algebra generators of the gauge symmetry. Heterotic string theory compactified on Calabi-Yau manifold is a promising phenomenology model for realizing the standard model physics in string theory. In our research, heterotic string theory plays a crucial role in our analysis of gauge theory scattering amplitude structure.

Instead of considering the $(1,1)$ world-sheet supersymmetry, $(0,1)$ supersymmetry is used: The left-moving sector is non-supersymmetric while the right-moving part is the same as before, with NS and R sectors. The same analysis implies that the spacetime dimension should be 10. However, if the left-moving sector only contains X_L^{μ}, $\mu = 0, ..., 9$, then the central charge c is not zero and the Weyl anomaly is not canceled. Hence we have to add more matter fields to the left-moving sector.

There are two ways to achieve that, which are called *bosonic construction* and *fermion construction*. The Lie algebra's Cartan structure is manifest in the former one, so throughout this thesis, we are using the bosonic construction: 16 extra field,

$$X_L^I, \quad I = 1, \ldots, 16 \tag{2.15}$$

are added to the left-moving sector, so the total central charge is zero. To get the deserved spacetime dimension, X_L^I are compactified on the torus. Hence the non-compact momenta,

$$k_L^\mu = k_R^\mu \tag{2.16}$$

are continuous, while k_L^I's are discrete and take values in a lattice. Mathematically, it is convenient to use the dimensionless momenta,

$$l^I \equiv \sqrt{\frac{\alpha'}{2}} k_L^I \tag{2.17}$$

and l takes value in Γ. The OPE locality condition and the modular invariance requires that Γ is *even self-dual*.

$$l \cdot l \in 2\mathbb{Z}, \quad l \in \Gamma \tag{2.18}$$

$$\Gamma = \Gamma^* \tag{2.19}$$

where the inner product is Euclidean. There are only two types of even self-dual lattices in 16-dimensional space:

- Γ_{16}, which is defined as follows. We will see that this lattice corresponds to the gauge group $SO(32)$.

$$(n_1, \ldots, n_{16}), \quad (n_1 + \frac{1}{2}, \ldots, n_{16} + \frac{1}{2}) \tag{2.20}$$

$$\sum_{i=1}^{16} n_i \in 2\mathbb{Z}, \quad n_i \in \mathbb{Z} \tag{2.21}$$

- $\Gamma_8 \times \Gamma_8$, where Γ_8 is defined as follows. This lattice corresponds to the gauge group $E_8 \times E_8$.

$$(n_1, \ldots, n_8), \quad (n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}) \tag{2.22}$$

$$\sum_{i=1}^{8} n_i \in 2\mathbb{Z}, \quad n_i \in \mathbb{Z} \tag{2.23}$$

The massless left-moving states are shown in (2.5), where $\hat{C}(l)$ is the cocycle

vertex operator	spacetime spin	interpretation
$\partial X^\mu e^{ik_\mu \cdot X_L^\mu(z)}$	$\mathbf{8}_v$	vector sector
$\partial X^I e^{ik_\mu \cdot X_L^\mu(z)}$	$\mathbf{1}$	color sector, Cartan subalgebra
$\hat{C}(l) e^{ik_\mu \cdot X_L^\mu(z) + i(\frac{\alpha'}{2})^{1/2} l_I \cdot X_L^I}$, $l^2 = 2$	$\mathbf{1}$	color sector, roots

Table 2.5: left-moving heterotic string states

operator for the normalization of the Lie algebra structure constant [39]. It is defined as follows: first, choose a basis $\{e_i^I\}$, $i = 1, \ldots, 16$ for the lattice Γ. (This basis may not be orthonormal.) For two normalized momenta l, v in Γ, $l = \sum_i n_i e_i$, $v = \sum_i m_i e_i$. We define,

$$l \star v = \sum_{i>j} n_i m_j (e_i \cdot e_j) \tag{2.24}$$

Note that the even lattice condition implies that,

$$l \star v + v \star l - l \cdot v = \sum_i n_i m_i e_i^2 \in 2\mathbb{Z} \tag{2.25}$$

Then the operator $\hat{C}(l)$ is defined as,

$$\hat{C}(l) = (-1)^{p \star l} \tag{2.26}$$

where p is the compact momentum operator. $p^I |l\rangle = l^I |l\rangle$. Now we can confirm the physical interpretation in (2.5). p^I serves as the Cartan subalgebra generators. The roots are defined by,

$$E(l) \equiv \int \frac{dz}{2\pi i z} \hat{C}(l) e^{i(\frac{\alpha'}{2})^{1/2} l_I \cdot X_L^I(z)}. \tag{2.27}$$

The commutators can be obtained by the OPE calculation and contour integral. Then the Lie algebra is explicitly,

$$[p^I, E(l)] = l^I E(l) \tag{2.28}$$

$$[E(l), E(v)] = \begin{cases} 0 & \text{if } l \cdot v \geq 0 \\ (-1)^{v \star l} E(l+v) & \text{if } l \cdot v = -1 \\ (-1)^{l \star l} l_I p^I & \text{if } l + v = 0\text{, i.e., } l \cdot v = -2 \end{cases} \tag{2.29}$$

Now we determine the g_{ab} in the $\{p^I, E_K\}$ basis. First, we define,

$$\mathrm{Tr}(p^I p^J) = \delta_{IJ},\ \mathrm{Tr}(p^I E_K) = 0 \tag{2.30}$$

Second, Plug in $T_a = E_K$, $T_b = E_{-K}$ and $T_c = p^I$ into the invariant relation $f^{abc} = f^{cab}$, it is easy to see that by Eq.(2.29)

$$\mathrm{Tr}(E_K E_{-K}) = (-1)^{K\star K}. \tag{2.31}$$

Third, if $K_1 + K_2 \neq 0$,

$$\mathrm{Tr}(E_{K_1} E_{K_2}) = 0. \tag{2.32}$$

Eq.(2.30), (2.31) and (2.32) completely fixed the g^{ab}. The Jacobi identity is

$$f^{ab}{}_e f^{cde} + f^{ca}{}_e f^{bde} + f^{bc}{}_e f^{ade} = 0. \tag{2.33}$$

The string tree amplitude can be viewed as the OPE's expectation value, say,

$$\langle 0|V(x_1; k_1, K_1)V(x_1; k_2, K_2)...V(x_n; k_n, K_n)|0\rangle \tag{2.34}$$

where we suppressed the integral over x_i. The co-cycle part gives,

$$co(12...n) \equiv (-1)^{K_1\star K_1 + \sum_{1<i<j\leq n} K_j\star K_i} \tag{2.35}$$

It is easy to check that when the discrete momentum is conserved, i.e., $\sum_{i=1}^n K_i = 0$, the notation $co(12...n)$ has the following properties:

- Cyclic permutation.

$$co(12...n) = co(n12...n-1) \tag{2.36}$$

- Adjacent transpositions.

$$co(12...ij...n) \cdot (-1)^{K_i \cdot K_j} = co(12...ji...n), \tag{2.37}$$

  when i and j is a pair of adjacent indices.

The left-moving sectors, combined with the right-moving sectors, form physical states (2.6). Again, the tachyon mode is removed by the level-matching condition. Note that the heterotic string theory has gauge symmetry $E_8 \times E_8$ or

sector	spacetime spin	irreducible representation	interpretation
(v,NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	$\mathbf{1} + \mathbf{28} + \mathbf{35}$	dilaton + anti-symmetry tensor+graviton
(c,NS+)	$\mathbf{1} \times \mathbf{8}_v$	$\mathbf{8}_v$	gluon
(v,R+)	$\mathbf{8}_v \times \mathbf{8}$	$\mathbf{8}' + \mathbf{56}$	dilatino + gravitino
(c,R+)	$\mathbf{1} \times \mathbf{8}$	$\mathbf{8}$	gaugino

Table 2.6: heterotic string massless states

$SO(32)$ and spacetime supersymmetry $d = 10, N = 1$.

2.2 String theory scattering amplitudes

String theory scattering amplitudes have many interesting new features, which are different from Feynman diagrams in field theory. In particular, the holomorphic and anti-holomorphic structures in string theory induce many amazing relations between scattering amplitudes [50] [51]. These relations are interesting not only theoretically but also practically, because the low energy limit of the string amplitude is Yang-Mills or Einstein gravity amplitude. Hence the study of string theory amplitudes provides lots of new relations for field theory amplitudes, especially for QCD calculation and graviton scattering.

The perturbative string interaction process is represented by the combined world-sheet of several incoming and outgoing strings. For example, the 2 open strings $\rightarrow$ 2 open string scattering amplitude could be represented by the diagrams in Fig.2.7 and more complicated diagrams. The long stripes represent the incoming and outgoing open string states and the "central part" is the interaction region. In this diagram, the interaction region is (a) a disk (b) an annulus. They have different topology: a disk's Euler characteristic $\chi_D = 1$ while an annulus' Euler characteristic $\chi_A = 0$. To make a perturbative series, we add weights to world-sheet action with different topology,

$$S'_p = S_p + \lambda\chi = S_p + \frac{\lambda}{4\pi}\int d\tau d\sigma g^{1/2} R, \tag{2.38}$$

where the *Gauss-Bonnet theorem* is used. Here,

$$\lambda = \Phi \tag{2.39}$$

is the v.e.v of the dilaton value. When λ is large, the contribution from diagrams with high χ (more holes) is suppressed, and we get a well-defined perturbative series. Fig.2.7(a) is an analogy to the gauge theory tree amplitude while Fig.2.7(a) is an analogy of the one-loop vacuum polarization gauge theory diagram.

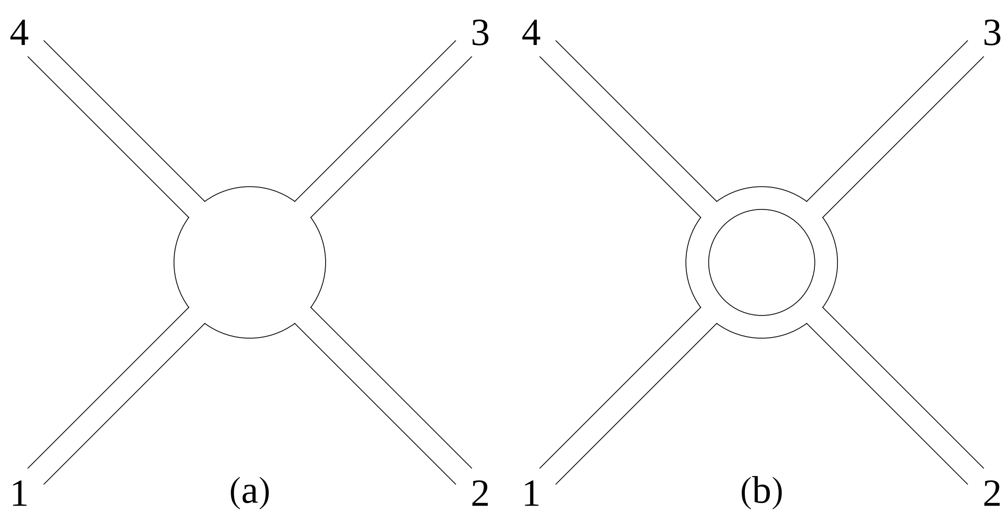


Table 2.7: 2 open strings → 2 open string scattering processes: (a) Disk diagram (b) Annulus diagram

Note that unlike Feynman diagrams, there is no particular interacting point in spacetime. It implies that the string theory interaction is "smoother" than field theory interaction, and has better UV behavior.

Closed strings scattering is represented by the world-sheet without boundary. Again, the 2 open strings → 2 open string scattering amplitude could be represented by the diagrams in Fig.2.8 and more complicated diagrams. The long tubes represent the incoming and outgoing open string states. (a) is a spheric diagram and the Euler characteristic $\chi_S = -2$, while (b) is a torus and the $\chi_T = 0$. Similarly, the diagrams with genus> 1 have larger Euler characteristic and are suppressed. Again (a) is a tree level diagram while (b) is a one-loop diagram.

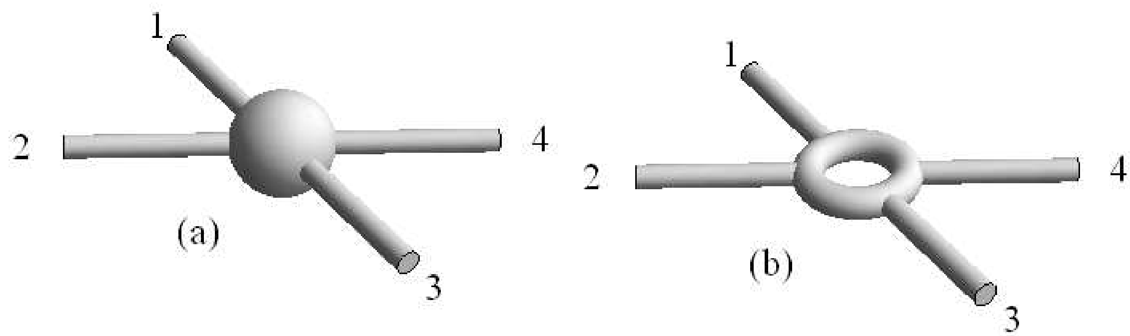


Table 2.8: 2 closed strings → 2 open string scattering processes: (a) Sphere diagram (b) torus diagram

The scattering amplitude can be calculated by path integral. For closed open string theory, the long tubes are replaced of vertex operator insertions. Schematically, if the scattering process is represented by a compact world-sheet M *without boundary*, the amplitude is,

$$\mathcal{A}(k_1, \ldots, k_n) = \int \frac{[dX][dg]}{V_{Diff \times Weyl}} \exp(-S_X - \lambda \chi_M) \prod_{i=1}^{n} \int_M d^2\sigma_i g(\sigma_i)^{1/2} V_i(k_i, \sigma_i) \quad (2.40)$$

where V_i is the vertex operator for the i-th incoming (outgoing) state. We integrate the position of a vertex operator over the whole compact manifold M. Therefore, there is no order for these incoming (outgoing) states. We also need to add coupling constants to the vertex operators. For example, the bosonic closed tachyon state, after integral, is now,

$$g_c \int d^2z\, e^{ik\cdot X}, \tag{2.41}$$

where g_c is the closed string coupling constant. We would see that it is determined by the Newton gravitational constant. The vertex insertion should be (Diff× Weyl) invariant: under the conformal transformation,

$$z \to z', \quad \bar{z} \to \bar{z}' \tag{2.42}$$

We have $d^2z \to (\partial_z z')(\partial_{\bar{z}}\bar{z}')d^2z$. To compensate this change, the vertex operator should transfer as,

$$V'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\tilde{h}}V(z,\bar{z}) \tag{2.43}$$

where the weight $(h,\tilde{h}) = (1,1)$. This is guaranteed by the on-shell condition.

The bosonic closed massless state is represented as,

$$\frac{2g_c}{\alpha'} \int d^2z\, e_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}, \tag{2.44}$$

where the normalization is fixed by the *unitarity condition*. Again, the weight condition is satisfied by the massless on-shell condition.

Similarly, open string (without Chan-Paton factor) amplitude on a compact world-sheet M *with boundary* is,

$$\mathcal{A}(k_1,\ldots,k_n) = \int \frac{[dX][dg]}{V_{Diff\times Weyl}} \exp(-S_X - \lambda\chi_M) \prod_{i=1}^{n} \int_{\partial M} ds_i V_i(k_i,s_i) \tag{2.45}$$

where we integrate over the boundary of M and s_i is the parameter of the vertex operator's position. It is not allowed to put the open string vertex operator inside M. The lowest bosonic states are now represented as,

$$\text{Tachyon: } (2\alpha')^{1/2} g \int_{\partial M} ds\, e^{ik\cdot X} \tag{2.46}$$

$$\text{Massless vector state: } -ig \int_{\partial M} ds\, \epsilon_\mu \frac{dX^\mu}{ds} e^{ik\cdot X} \tag{2.47}$$

where g is the open string coupling constant. We will see that g would serve as the gauge theory coupling.

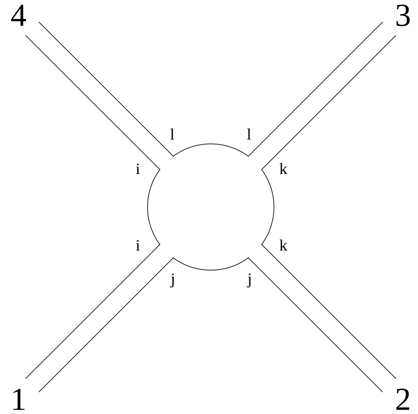
Since for compact M, its boundary ∂M is isomorphic to S^1 so it is possible to define the *cyclic order* of the vertex operators. We can separate the integral in (2.45) with n insertions into the sum of $(n-1)!$ integrals, each of which corresponds to a particular cyclic order. Although this separation seems trivial, it becomes crucial in open string theory with Chan-Paton factors.

The scattering amplitude for open strings with Chan-Paton factors is treated almost in the same way. The new feature is that the two adjacent Chan-Paton factors should be the same. For example, the four-point open string gauge interaction on a disk, for the ordering (1234) is, where the integral is the same as before. There are other 5 orderings and the sum gives the tree level string gauge interaction. We will see that this is a new way to organize the terms in gauge amplitude. Note that each cyclic-order term is *gauge invariant*. The gauge transformation $\epsilon_i^\mu \mapsto \epsilon_i^\mu + k_i^\mu$, the vertex operator transfers as,

$$\epsilon_\mu \frac{dX^\mu}{ds} e^{ik\cdot X} \mapsto \epsilon_\mu \frac{dX^\mu}{ds} e^{ik\cdot X} - i \frac{d}{ds}\left(e^{ik\cdot X}\right), \tag{2.48}$$

where the extra piece is a descent operator.

The above formalism is only formally defined, because the path integral has (Diff× Weyl) redundancy and is divergent in general. The usual *Faddeev-Popov*



$\propto (\lambda^{a_1})_{ij}(\lambda^{a_2})_{jk}(\lambda^{a_3})_{kl}(\lambda^{a_4})_{li} = tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4})$

Table 2.9: Open string scattering with Chan-Paton factor

quantization can be used here, and we need to add the ghost determinant to the scattering amplitude. However, even if the metric is fixed to be flat by Faddeev-Popov procedure, there are still some residual global symmetries, which are inside the *conformal Killing group* (CKG). For example, let $M = S^2$. The *Mobius transformation*,

$$z \to \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0 \tag{2.49}$$

is a diffeomorphism of S^2. This transformation is conformal, so an additional Weyl transformation would rescale the metric to the flat case. Hence this symmetry is not fixed by the Faddeev-Popov procedure. However, for the scattering amplitude with an enough number of incoming states, we can use CKG to fix the position of vertex insertions, so all the symmetries are fixed. For example, the Mobius group is complex 3-dimensional, so this symmetry is fixed when we fix three vertex operators on S^2.

Furthermore, for a world-sheet M, there may not exist a global (Diff× Weyl) symmetry which can transform M's metric to be flat. In the other, there are topologically equivalent M's with different complex structures. This happens in string theory loop amplitude. For example, the tori,

$$z \sim z+1, \quad z \sim z+\tau \tag{2.50}$$

have different complex structures if two τ's are not related by a Mobius transformation. Such τ parameterizes the inequivalent world-sheet with the same topology, and is called the *moduli*. So in the string scattering amplitude, we have to integrate over the *fundamental region* of the moduli.

2.2.1 Gauge and gravitational string amplitude

In this subsection, we present several tree-level string amplitude examples.

There is a trick for the OPE calculation of the vector boson string state. We can rewrite (2.47) as,

$$\epsilon_\mu \frac{dX^\mu}{ds} e^{ik\cdot X} \mapsto -ie^{ik\cdot X + i\epsilon\cdot\dot{X}} \tag{2.51}$$

and compute the OPE with this new operator. Now every operator is exponential and the OPE's have a concise form. Since the gauge scattering amplitude is always liner in the polarization, we just need to keep the linear term for every ϵ_i and the physical result is obtained. The same trick works for the graviton string state.

The gauge-vector-boson disk scattering amplitude can therefore be calculated as follows: for a particular cyclic ordering $(12\ldots n)$, we put the open string vertex operators along the *real axis with* ∞, $x_1 \le x_2 \le \ldots x_n$. The disk OPE calculation reads,

$$\mathcal{A}(12\dots n) = \frac{g^{n-2}}{2\alpha'^2}\int_{x_1\le x_2\le\dots x_n}\frac{dx^1dx^2\dots dx^n}{dx^a dx^b dx^c}(x_b-x_a)(x_c-x_b)(x_c-x_a)$$
$$\times\Big(\prod_{i\le i<j\le n}(x_j-x_i)^{2\alpha' k_i\cdot k_j}\Big)\exp\Big(\sum_{i<j}\frac{2\alpha'\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2}+\sum_{i\neq j}\frac{2\alpha' k_i\cdot\epsilon_j}{x_i-x_j}\Big)\Big|_{\text{linear}} \tag{2.52}$$

where the denominator $dx^a dx^b dx^c$ ($1\le a<b<c\le n$) means that the CKG of the disk fixes the position of three vertex operators. $x_a<x_b<x_c$, and $(x_b-x_a)(x_c-x_b)(x_c-x_a)$ is the ghost determinant. It is convenient to use $x_a=0$, $x_b=1$ and $x_c=\infty$. However, other choices are also useful in some calculation. The scattering amplitude is the sum of color-ordered amplitudes with the Chan-Paton factors,

$$\mathcal{A}=\sum_{\sigma\in S^{n-1}} tr(t^{a_{\sigma_1}}t^{a_{\sigma_2}}\dots t^{a_n})\mathcal{A}(\sigma_1\sigma_2\dots n) \tag{2.53}$$

where $\sigma\in S^{n-1}$ is a permutation of the first $(n-1)!$ vertices. $A(\sigma_1\sigma_2\dots n)$ is also called string partial amplitudes.

For this integral expression, the partial amplitude has the reverse symmetry,

$$\mathcal{A}(12\dots n) = (-1)^n\mathcal{A}(n\dots 21) \tag{2.54}$$

This should be understood as the charge conjugation symmetry of the gauge theory. Instead of (A.21), we consider the transformation on the generators,

$$\lambda^a\mapsto -(\lambda^a)^T,\quad A^a_\mu\mapsto A^a_\mu \tag{2.55}$$

For the closed string amplitude with incoming gravitons, anti-symmetric tensors or dilatons. We use the trick,

$$e_{\mu\nu}\partial X^\mu\bar\partial X^\mu e^{ik\cdot X}\to\epsilon_\mu\bar\epsilon_\nu\partial X^\mu\bar\partial X^\mu e^{ik\cdot X}\to -e^{ik\cdot X+i\epsilon_\mu\partial X^\mu+i\bar\epsilon_\mu\bar\partial X^\mu} \tag{2.56}$$

and the integral reads,

$$
\begin{aligned}
\mathcal{A}(12\ldots n) = g_c^{n-2}4\pi\Big(\frac{2}{\alpha'}\Big)^{n+1}\int \frac{dz^1dz^2\ldots dz^n}{dz^a dz^b dz^c}|z_b-z_a|^2|z_c-z_b|^2|z_c-z_a|^2 \\
\times\Big(\prod_{i\le i<j\le n}(z_j-z_i)^{\frac{\alpha'}{2}k_i\cdot k_j}\Big)\exp\Big(\sum_{i<j}\frac{\alpha'\epsilon_i\cdot\epsilon_j}{2(z_i-z_j)^2}+\sum_{i\neq j}\frac{\alpha' k_i\cdot\epsilon_j}{2(z_i-z_j)}\Big) \\
\times\Big(\prod_{i\le i<j\le n}(\bar{z}_j-\bar{z}_i)^{\frac{\alpha'}{2}k_i\cdot k_j}\Big)\exp\Big(\sum_{i<j}\frac{\alpha'\bar{\epsilon}_i\cdot\bar{\epsilon}_j}{2(\bar{z}_i-\bar{z}_j)^2}+\sum_{i\neq j}\frac{\alpha' k_i\cdot\bar{\epsilon}_j}{2(\bar{z}_i-\bar{z}_j)}\Big)\Big|_{\text{bilinear}}
\end{aligned}
\tag{2.57}
$$

where the subscript "bilinear" means that only the $\epsilon_{i,\mu}\bar{\epsilon}_{i,\nu}$ terms are kept in the final result and recombined as $\epsilon_{i,\mu}\bar{\epsilon}_{i,\nu} \to e_{i,\mu\nu}$. Note that there is no ordering of the closed string vertices.

Three-gluon string scattering

This is the simplest case, however, the string effect is already inside. By 2.52, for the ordering (123), we have

$$
\mathcal{A}(123) = ig\Big(\epsilon_1\cdot k_{23}\epsilon_2\cdot\epsilon_3+\epsilon_2\cdot k_{31}\epsilon_3\cdot\epsilon_1+\epsilon_3\cdot k_{12}\epsilon_2\cdot\epsilon_3+\frac{\alpha'}{2}\epsilon_1\cdot k_{23}\epsilon_2\cdot k_{31}\epsilon_3\cdot k_{12}\Big),
\tag{2.58}
$$

Furthermore, by the reflective symmetry, $\mathcal{A}(132) = -\mathcal{A}(123)$. So the scattering amplitude for three gluons is,

$$
\begin{aligned}
\mathcal{A} = ig\cdot tr(t^{a_1}[t^{a_2},t^{a_3}])\Big(\epsilon_1\cdot k_{23}\epsilon_2\cdot\epsilon_3+\epsilon_2\cdot k_{31}\epsilon_3\cdot\epsilon_1+\epsilon_3\cdot k_{12}\epsilon_2\cdot\epsilon_3 \\
+\frac{\alpha'}{2}\epsilon_1\cdot k_{23}\epsilon_2\cdot k_{31}\epsilon_3\cdot k_{12}\Big).
\end{aligned}
\tag{2.59}
$$

The $(\epsilon\cdot k)(k\cdot k)$ terms already appear in Yang-Mills tree amplitude. The last term, $(\epsilon\cdot k)(\epsilon\cdot k)(\epsilon\cdot k)$ is the string theory correction. Note that in the lower energy limit $\alpha' k^2 \to 0$, this term is negligible and the amplitude is reduced to Yang-Mills theory.

Four-gluon string scattering

Here we need to carry on one integral. We can fix $x_1 = 0$, $x_2 = 1$ and $x_4 = \infty$, and integrate over x_3. The evaluation of (2.52) is tedious but straightforward. It is explicitly calculated in [39]: the ordering (1234) is

$$A(1234) = \frac{ig^2}{2}\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(\alpha' u + 1)}K(1234) \tag{2.60}$$

Where $K(1234)$ is the kinematic factors containing the polarization vectors. Again, we used the Mandastem variables,

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_4)^2, \quad u = -(k_1 + k_3)^2 \tag{2.61}$$

$$s + t + u = 0. \tag{2.62}$$

And,

$$K(1234) = K^{\text{Type I}}(1234) + K^{\text{Bosonic}}(1234) \tag{2.63}$$

where $K^{\text{Type I}}$ is the same kinetic term as that in Type I open string theory.

$$\begin{aligned} K^{\text{Type I}}(1234) \quad = \quad & \alpha'^2(st\epsilon_1 \cdot \epsilon_3\epsilon_2 \cdot \epsilon_4 + su\epsilon_2 \cdot \epsilon_3\epsilon_1 \cdot \epsilon_4 + tu\epsilon_1 \cdot \epsilon_2\epsilon_3 \cdot \epsilon_4) \\ & -2\alpha'^2 s(\epsilon_1 \cdot k_4\epsilon_3 \cdot k_2\epsilon_2 \cdot \epsilon_4 + \epsilon_2 \cdot k_3\epsilon_4 \cdot k_1\epsilon_1 \cdot \epsilon_3 + \epsilon_1 \cdot k_3\epsilon_4 \cdot k_2\epsilon_2 \cdot \epsilon_3 \\ & +\epsilon_2 \cdot k_4\epsilon_3 \cdot k_1\epsilon_1 \cdot \epsilon_4) \\ & -2\alpha'^2 t(\epsilon_2 \cdot k_1\epsilon_4 \cdot k_3\epsilon_3 \cdot \epsilon_1 + \epsilon_3 \cdot k_4\epsilon_1 \cdot k_2\epsilon_2 \cdot \epsilon_4 + \epsilon_2 \cdot k_4\epsilon_1 \cdot k_3\epsilon_3 \cdot \epsilon_4 \\ & +\epsilon_3 \cdot k_1\epsilon_4 \cdot k_2\epsilon_2 \cdot \epsilon_1) \\ & -2\alpha'^2 u(\epsilon_1 \cdot k_2\epsilon_4 \cdot k_3\epsilon_3 \cdot \epsilon_2 + \epsilon_3 \cdot k_4\epsilon_2 \cdot k_1\epsilon_1 \cdot \epsilon_4 + \epsilon_1 \cdot k_4\epsilon_2 \cdot k_3\epsilon_3 \cdot \epsilon_4 \\ & +\epsilon_3 \cdot k_2\epsilon_4 \cdot k_1\epsilon_1 \cdot \epsilon_2) \end{aligned} \tag{2.64}$$

Supersymmetry puts strong constraints on the possible interactions, so the non-supersymmetric theory have much more terms,

$$
\begin{aligned}
K^{\text{Bosonic}}(1234) = 4\alpha'^3 s\Big[&\epsilon_1\cdot k_3\epsilon_2\cdot k_3(\epsilon_3\cdot k_1\epsilon_4\cdot k_1+\epsilon_3\cdot k_2\epsilon_4\cdot k_2)+\\
&\frac{1}{3}(\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1-\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1-\epsilon_4\cdot k_2)\Big]\\
&+4\alpha'^3 t\Big[\epsilon_2\cdot k_1\epsilon_3\cdot k_1(\epsilon_1\cdot k_3\epsilon_4\cdot k_3+\epsilon_1\cdot k_2\epsilon_4\cdot k_2)+\\
&\frac{1}{3}(\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1)(\epsilon_4\cdot k_3-\epsilon_4\cdot k_2)\Big]\\
&+\alpha'^3 u[\epsilon_1\cdot k_2\epsilon_3\cdot k_2(\epsilon_2\cdot k_1\epsilon_4\cdot k_1+\epsilon_2\cdot k_3\epsilon_4\cdot k_3)+\\
&\frac{1}{3}(\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1-\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3-\epsilon_4\cdot k_1)]\\
+(2\alpha')^2\frac{st}{4}\frac{1}{1+\alpha' u}&(\epsilon_1\cdot\epsilon_3-(2\alpha')\epsilon_1\cdot k_3\epsilon_3\cdot k_1)(\epsilon_2\cdot\epsilon_4-(2\alpha')\epsilon_2\cdot k_4\epsilon_4\cdot k_2)\\
+(2\alpha')^2\frac{tu}{4}\frac{1}{1+\alpha' s}&(\epsilon_1\cdot\epsilon_2-(2\alpha')\epsilon_1\cdot k_2\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_4-(2\alpha')\epsilon_3\cdot k_4\epsilon_4\cdot k_3)\\
+(2\alpha')^2\frac{su}{4}\frac{1}{1+\alpha' t}&(\epsilon_1\cdot\epsilon_4-(2\alpha')\epsilon_1\cdot k_4\epsilon_4\cdot k_1)(\epsilon_2\cdot\epsilon_3-(2\alpha')\epsilon_2\cdot k_3\epsilon_3\cdot k_2)\\
&-\alpha'^2(st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4-su\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4-tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4)
\end{aligned}
\tag{2.65}
$$

It is interesting to look at this amplitude in detail. First, from the Gamma function expansion, the only massless poles are s and t while the u pole is absent. This is consistent with the color-ordered Yang-Mills Feynman diagram analysis.

Second, $K^{\text{Type I}} \propto \alpha'^2$, while $K^{\text{Bosonic}} \propto \alpha'^3$. So in the low energy limit, the leading order of the scattering amplitude is,

$$\mathcal{A}(1234) \supset \frac{1}{\alpha'}\frac{1}{st}K^{\text{Type I}} = A(1234). \tag{2.66}$$

This term is the same as the Yang-Mills theory scattering amplitude.

Third, $\mathcal{A}(1234)$ and $A(1234)$ is cyclically symmetric, as it should be. However, $\mathcal{A}(1234)$ and $A(1234)$ have a stronger symmetry. Note that, explicitly, both the kinematic factors $K^{\text{Type I}}$ and K^{Bosonic} and totally symmetric in the incoming states, for example, under $1 \leftrightarrow 2$.

$$K^{\text{Type I}}(1234) = K^{\text{Type I}}(2134), \quad K^{\text{Bosonic}}(1234) = K^{\text{Bosonic}}(2134). \tag{2.67}$$

This permutation exchanges u and t. Hence,

$$\mathcal{A}(1234)\sin(\pi\alpha' t) = \mathcal{A}(2134)\sin(\pi\alpha' u) \tag{2.68}$$

$$tA(1234) = uA(2134), \tag{2.69}$$

where the second equality is the Bern, Carrasco and Johansson (BCJ) [9] identity for four-gluon scattering amplitude. The first equality is understood as the string version of the BCJ identity. There is no simple Feynman-diagram-like proof for these identities. We will later present the systematic string theory proof.

Three-graviton amplitude

There is no integral of the vertex operator. The result is,

$$A_c(123;\alpha) = \pi g_c/g^2 A(123;\alpha/4)\bar{A}(123;\alpha/4) \tag{2.70}$$

where $A(123;\alpha/4)$ is the open string three-gluon with the polarization vectors ϵ_i, while the Regge slope α' is replaced by $\alpha'/4$. Similarly, $A(123;\alpha/4)$ is the open string three-gluon with the polarization vectors $\bar{\epsilon_i}$ and the renormalized α'. Finally, we use the trick $\epsilon_{i,\mu}\bar{\epsilon}_{i,\nu} \mapsto \epsilon_{\mu\nu}$. The normalization of α' comes from the fact that, both the left and right-moving sectors only carry a half of the total momentum. Note that α' has the dimension L^2.

This explicit relation between open and closed string theory seems surprising. One may simply think that it is coming from the left-right moving sectors decomposition, and the closed string theory amplitude is always a simple product of two open string amplitudes. However, the statement is not accurate. For example, if,

$$A_c(1234;\alpha) \propto A(1234;\alpha/4)\bar{A}(1234;\alpha/4) \quad (?) \tag{2.71}$$

then $A_c(1234)$ would contain the $1/s^2$ pole. However, this is a contradiction to the effective theory analysis. The correct relation, Kawai-Lewellen-Tye relation, which involves delicate contour analysis, will be reviewed in the next section.

Effective action

From the previous examples, we see that in the low energy limit, the tree-level interaction of massless string state is the same as the Yang-Mills theory and Einstein gravity. The high order terms in string tree amplitude correspond to the high-derivate terms, which vanishes in the low energy limit.

For example, the bosonic string theory's gauge effective theory is,

$$S = \int d^d x \Big(-\frac{1}{4g^2} tr(F_{\mu\nu}F^{\mu\nu}) - \frac{2i\alpha'}{3g^2} tr(F_\mu{}^\nu F_\nu{}^\omega F_\omega{}^\mu) + \ldots \Big) \tag{2.72}$$

where the first term is the Yang-Mills action. All the other terms are high-derivatives. The low energy limit is equivalent to,

$$\alpha' \to 0, \tag{2.73}$$

so all high-derivatives are vanishing in low energy. Type I open string theory also contains the Yang-Mills action term. However, as the four-gluon example, the high-derivative terms in Type I theory are different from that in the bosonic string theory.

In the low energy limit, Type IIA and IIB super string theory would reduce to $d = 10$, Type IIA and IIB supergravity theory respectively.

2.3 Kawai-Lewellen-Tye relation

The tree level relation between open and closed string amplitude is discovered by Kawai, Lewellen and Tye (KLT) [39]. KLT relation illustrates the fundamental connection between gauge and gravity theory. It is crucial to the scattering amplitude calculation in this thesis.

The three point KLT relation is (2.70). Since there is no integral in the three point case, the closed string amplitude is just the product of two open string amplitudes.

The first nontrivial KLT relation is the four point scattering amplitudes. The real and imaginary part of the integral in closed string amplitude, can be treated as two independent real integrals. However, we have to take care of the contours. Let $z_1 = 0$, $z_2 = z$, $z_3 = 1$ and $z_4 \to \infty$, and (2.57) reads,

$$\mathcal{A}_c(1234) = 4\pi g_c^2 \Big(\frac{2}{\alpha'}\Big)^5 \int dz\, z^{\frac{\alpha'}{2}k_1\cdot k_2}(1-z)^{\frac{\alpha'}{2}k_2\cdot k_3} f(z)\, \bar{z}^{\frac{\alpha'}{2}k_1\cdot k_2}(1-\bar{z})^{\frac{\alpha'}{2}k_2\cdot k_3} \bar{f}(\bar{z}) \tag{2.74}$$

where $f(z)$ is a holomorphic function which contains the polarization vectors. Similar $\bar{f}(\bar{z})$ is the conjugation of $f(z)$.

$$\begin{aligned} f(z) &= \lim_{z_4\to\infty} z_4^2 \exp\Big(\sum_{i<j}\frac{\alpha'\epsilon_i\cdot\epsilon_j}{2(z_i-z_j)^2} + \sum_{i\neq j}\frac{\alpha' k_i\cdot\epsilon_j}{2(z_i-z_j)}\Big)\Big|_{\text{linear}} \\ \bar{f}(\bar{z}) &= \lim_{z_4\to\infty} \bar{z}_4^2 \exp\Big(\sum_{i<j}\frac{\alpha'\bar{\epsilon}_i\cdot\bar{\epsilon}_j}{2(\bar{z}_i-\bar{z}_j)^2} + \sum_{i\neq j}\frac{\alpha' k_i\cdot\bar{\epsilon}_j}{2(\bar{z}_i-\bar{z}_j)}\Big)\Big|_{\text{linear}} \end{aligned} \tag{2.75}$$

The integral is over the whole complex plane, and both z and $\bar{z}$ are complex. Let

$z = x + iy$,
$$\int d^2z \mapsto 2\int dx \int dy \tag{2.76}$$

and the integrand is analytic both in x and y. We would like to consider y on the whole complex plane. Note the possible poles of y are

$$y = ix, i(1-x), i(x-1), -ix \tag{2.77}$$

which are all on the imaginary axis. Hence we can use the Wick rotation, as Fig.(2.1). Now y is imaginary (up to the infinitesimal prescription), so both $x+iy$ and $x-iy$ are real. Define $\xi = x+iy$ and $\eta = x-iy$, and the integral (2.57) reads,

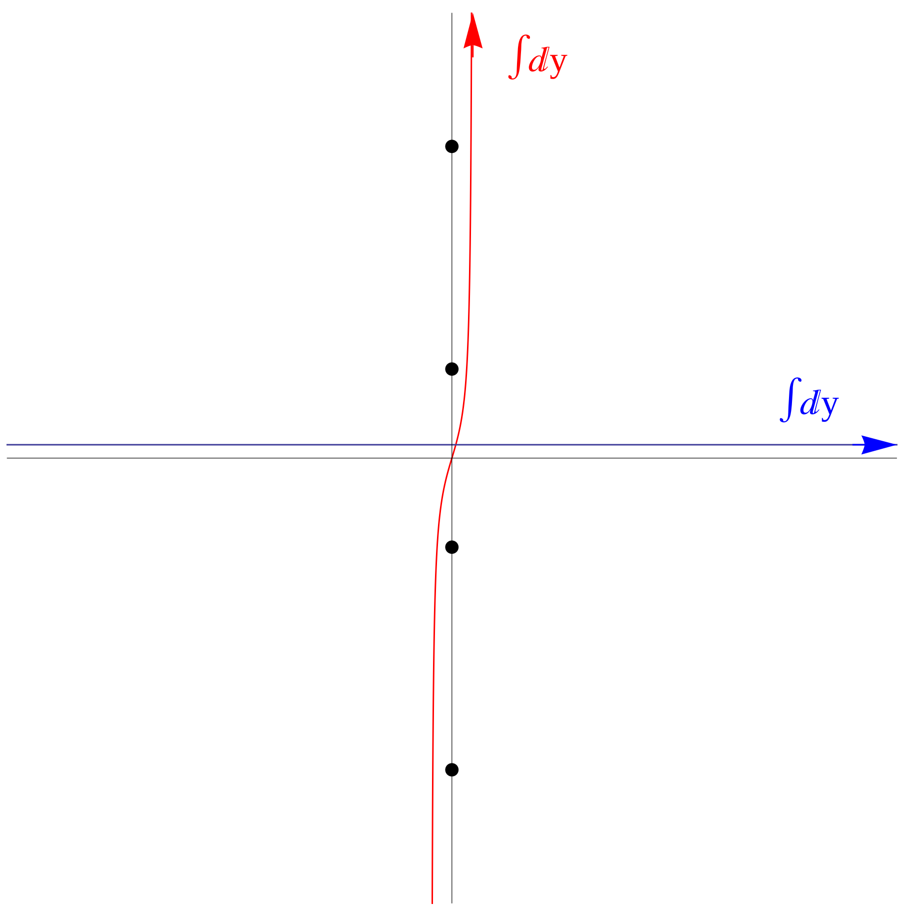


Figure 2.1: Analytical continuation of y.

$$\mathcal{A}_c(1234) = 4\pi i g_c^2\Big(\frac{2}{\alpha'}\Big)^5 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta |\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3} f(\xi)\, |\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)$$
$$\times \exp\Big(i\pi\frac{\alpha'}{2}k_1\cdot k_2\theta(-\xi\eta) + i\pi\frac{\alpha'}{2}k_2\cdot k_3\theta(-(1-\xi)(1-\eta))\Big) \qquad (2.78)$$

where the phase term comes from the prescription of the contour. $\theta(\ldots)$ is the Heaviside step function. (2.78) looks like a product of open string amplitudes.

However, there are 3 different orderings in the ξ integral while 3 orderings in the η integral. So it seems that (2.57) would be the sum of $3\times 3 = 9$ pairs of open string amplitudes. However, we can represent the phase as the contour of η integral. Only when $0 < \xi < 1$, the contour of η is nontrivial. It is shown in (2.2). If we use the contour C_1, (2.78) becomes,

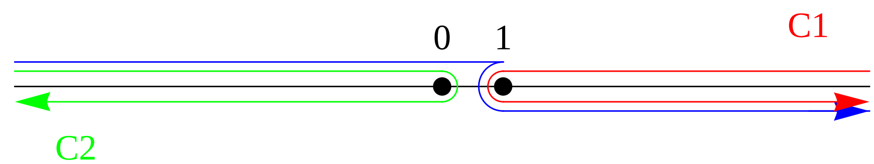


Figure 2.2: Contour integral for η when $0 < \xi < 1$. The original contour can be deformed to either C_1 or C_2.

$$\mathcal{A}_c(1234) = 8\pi g_c^2\Big(\frac{2}{\alpha'}\Big)^5 \sin\Big(\frac{\alpha'\pi k_2\cdot k_3}{2}\Big)\times \int_0^1 d\xi|\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3} f(\xi)$$
$$\times \int_1^{\infty} d\eta|\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)$$
$$= 4\pi\frac{g_c^2}{\alpha' g^4}\sin\Big(\frac{\alpha'\pi k_2\cdot k_3}{2}\Big)\mathcal{A}(1234;\frac{\alpha}{4})\mathcal{A}(1324;\frac{\alpha}{4}) \qquad (2.79)$$

and similarly if we use the C_2, the result is,

$$\mathcal{A}_c(1234) = 8\pi g_c^2\Big(\frac{2}{\alpha'}\Big)^5 \sin\Big(\frac{\alpha'\pi k_1\cdot k_2}{2}\Big)\times \int_0^1 d\xi|\xi|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\xi|^{\frac{\alpha'}{2}k_2\cdot k_3} f(\xi)$$
$$\times \int_{-\infty}^{0} d\eta|\eta|^{\frac{\alpha'}{2}k_1\cdot k_2}|1-\eta|^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(\eta)$$
$$= 4\pi\frac{g_c^2}{\alpha' g^4}\sin\Big(\frac{\alpha'\pi k_1\cdot k_2}{2}\Big)\mathcal{A}(1234;\frac{\alpha}{4})\bar{\mathcal{A}}(2134;\frac{\alpha}{4}) \qquad (2.80)$$

These two results are equivalent. There are also other contour choices, which are summarized in table. 2.10. The permutation symmetry of the closed string

	$\bar{\mathcal{A}}(1234)$	$\bar{\mathcal{A}}(1324)$	$\bar{\mathcal{A}}(2134)$
$\mathcal{A}(1234)$		t	s
$\mathcal{A}(1324)$	t		u
$\mathcal{A}(2134)$	s	u	

Table 2.10: Different expressions for four-point KLT relation. The kinematic factors in this table appear in the argument of the sine function, $\sin\left(\frac{\alpha'\pi\ldots}{4}\right)$.

amplitude is not manifest in (2.79) or (2.80). However, we can average out the six equivalent expressions in table. 2.10, and then the permutation symmetry is restored.

The equivalence of (2.79) and (2.80) implies that,

$$\sin\left(\frac{\alpha'\pi k_2\cdot k_3}{2}\right)\mathcal{A}(1324;\frac{\alpha}{4}) = \sin\left(\frac{\alpha'\pi k_1\cdot k_2}{2}\right)\bar{\mathcal{A}}(2134;\frac{\alpha}{4}) \tag{2.81}$$

or if α' is renormalized,

$$\sin(\alpha'\pi t)\mathcal{A}(1324) = \sin(\alpha'\pi s)\bar{\mathcal{A}}(2134). \tag{2.82}$$

This is the explicit identity in (2.69). Hence the contour argument in KLT relation proved the BCJ identity in four-point case.

It is also interesting to study the low energy limit of (2.79). Take $\alpha' \to 0$, we get,

$$\begin{aligned}\mathcal{A}_c(1234) &= -\pi^2\frac{g_c^2}{g^4}tA(1234)\bar{A}(1324)\\ &= -\frac{\kappa^2 t}{4g^4}A(1234)\bar{A}(1324)\end{aligned} \tag{2.83}$$

This identity has no α' dependence, because in the low energy limit, string theory is reduced to Yang-Mills theory and Einstein gravity. Here we used the normalization $\kappa = 2\pi g_c$. Since $A(1234)$ and $A(1324)$ can be calculated by field theory, this identity provides a effective way for graviton tree amplitude calculation. Note that the graviton amplitude is suppressed by a small factor $\kappa^2 t$, as it should be.

Similarly, n-point ($n > 4$) tree level closed string amplitude can be decomposed as the sum of open string amplitude products. For example, the five-point KLT relation is,

$$\begin{aligned}\mathcal{A}_c(12345) = &\frac{g_c^3}{g^6\alpha'^2}\sin\left(\frac{\alpha'\pi k_1\cdot k_2}{2}\right)\sin\left(\frac{\alpha'\pi k_3\cdot k_4}{2}\right)\mathcal{A}(12345)\bar{\mathcal{A}}(21435)\\ &+\frac{g_c^3}{g^6\alpha'^2}\sin\left(\frac{\alpha'\pi k_1\cdot k_3}{2}\right)\sin\left(\frac{\alpha'\pi k_2\cdot k_4}{2}\right)\mathcal{A}(13245)\bar{\mathcal{A}}(31425).\end{aligned} \tag{2.84}$$

Note that unlike the four point case, this identities are the sum of two pairs. [39] counted the minimum number of pairs for general KLT relations,

$$\begin{aligned}&(n-3)!(\frac{n-3}{2})!(\frac{n-3}{2})!, \quad n \text{ is odd}\\ &(n-3)!(\frac{n-4}{2})!(\frac{n-2}{2})!, \quad n \text{ is even}\end{aligned} \tag{2.85}$$

The other KLT expressions have more terms. They are systematically studied in [14].

CHAPTER 3

SCATTERING AMPLITUDES TECHNIQUES BEYOND FEYNMAN DIAGRAMS

In the previous chapter, we reviewed the string theory techniques in scattering amplitudes computation. To use the full power of string techniques, we also need recent new scattering amplitude techniques beyond Feynman diagrams. In this chapter, we review the color decomposition, spinor helicity formalism, BRST recursive relation and coherent symmetric space. We will also illustrate how these methods can be combined with the string techniques.

3.1 Color decomposition

In open string theory with Chan-Paton factors, the gauge tree amplitude is,

$$\mathcal{A} = \sum_{\sigma \in S^{n-1}} tr(t^{a_{\sigma_1}} t^{a_{\sigma_2}} \ldots t^{a_n}) \mathcal{A}(\sigma_1 \sigma_2 \ldots n). \tag{3.1}$$

where $\mathcal{A}(\sigma_1\sigma_2 \ldots n)$ is the partial amplitude with cyclic ordering $(\sigma_1\sigma_2 \ldots n)$. t^{a_i} is the generator associated with the i-th gluon. $\mathcal{A}(\sigma_1\sigma_2 \ldots n)$ is gauge invariant. Taking the limit $\alpha' \to 0$, the amplitudes reduce to Yang-Mills amplitudes,

$$\mathcal{A} \to \mathcal{A}_{YM}, \quad \mathcal{A}(\sigma_1\sigma_2 \ldots n) \to A(\sigma_1\sigma_2 \ldots n). \tag{3.2}$$

So we have the Yang-Mills color decomposition.

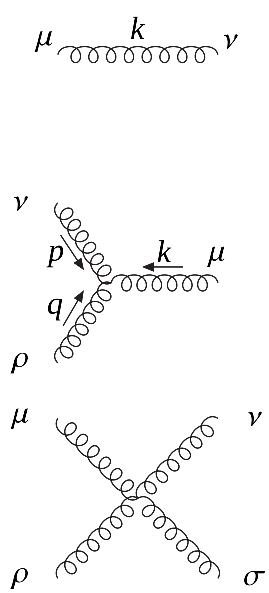
$$A = \sum_{\sigma \in S^{n-1}} tr(t^{a_{\sigma_1}} t^{a_{\sigma_2}} \ldots t^{a_n}) A(\sigma_1 \sigma_2 \ldots n), \tag{3.3}$$

The gauge-invariant, cyclic and reflective properties of string partial amplitude $\mathcal{A}(12 \ldots n)$ hold in each order of α'. Therefore, the Yang-Mills partial amplitudes satisfy,

- Gauge invariance. The longitudinal mode's change $\epsilon_\mu \mapsto \epsilon_\mu + ck_\mu$ or the change of ξ in the propagator does not change $A(12\ldots n)$.
- Cyclic symmetry. $A(12\ldots n) = A(2\ldots n1) = \ldots = A(n1\ldots n-1)$.
- Reflective symmetry. $A(12\ldots n) = (-1)^n A(n\ldots 21)$.

Since the Yang-Mills partial amplitude $A(12\ldots n) = A$ has these good properties and is independent of the color factors, it would be the main focus in our study of Yang-Mills theory.

Alternatively, we derive the Yang-Mills partial amplitude directly from Feynman diagrams [27]. The partial amplitude Feynman rules are summered in (3.1). To calculate the Yang-Mills partial amplitude $A(12\ldots n)$, we need to align



$$= \frac{-i}{k^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

$$= i\frac{g}{\sqrt{2}}[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu]$$

$$= -ig^2[\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{2}\eta^{\mu\sigma}\eta^{\nu\rho} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}]$$

Figure 3.1: Feynman rules for gluons in partial amplitude. (a) gluon propagator (b) three-point gluon vertex (c) four-point gluon vertex.

the incoming gluon states $1, 2\ldots n$ clockwise on a circle. The partial amplitude is the sum of Feynman diagrams with the new rules (3.1). *Note that unlike usual*

Feynman diagrams, it is not allowed to swap two legs of a vertex or to cross a propagator with another propagator. For example, the partial amplitude $A(1234)$ has only s and t poles, while u-pole the absent. The result is that the u-pole diagram would have propagator-crossing when $1, 2, 3, 4$ are aligned in cyclic ordering.

For n-gluon tree amplitude, there are $(n-1)!/2$ partial amplitudes by the cyclic and reflective symmetry. Each channel corresponds to a way to cut a convex polygon with n sides into triangles by connecting vertices with straight lines: Put the incoming gluons on the edges of the polygon in clockwise order. Then each triangle represents a three-gluon vertex and each straight line represents a propagator. Hence the number of Channels in one partial amplitude is C_n, the *Catalan number.*

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{3.4}$$

3.2 Spinor helicity formalism

The photon or gluon state is associated with the polarization vector ϵ_μ. The scattering amplitude is proportional to the polarization. However, ϵ_μ has d components while the photon or gluon just has $d-2$ degrees of freedom. The spinor helicity formalism manifestly gets rid of the redundancy of the unphysical degrees of freedom and organizes the amplitude in a compact form.

Spinor helicity formalism works only for the four-dimensional spacetime. This formalism rewrites all the momentum, polarization vector and vertices as the product of Weyl spinors. Instead of consider the amplitude,

$$A(12\ldots n) = \epsilon_{1\mu_1}\epsilon_{2\mu}\ldots\epsilon_{n\mu}A(12\ldots)^{\mu_1\mu_2\ldots\mu_n}, \tag{3.5}$$

we directly consider the *amplitudes with particular helicities* $A(1^{\pm}; 2^{\pm}...; n^{\pm})$, where $\pm$ means the helicity of the incoming photon or gluons. Similarly, spinor helicity formalism can also be used in gravity amplitude.

First, for light-like momentum p, we decompose it as the product of Weyl spinors.

$$p_{\mu}(\sigma^{\mu})_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \tag{3.6}$$

where $\sigma^{\mu} = (I_{2\times 2}, \sigma_x, \sigma_y, \sigma_z)$, λ_a is a Weyl spinor with $-1/2$ helicity while $\tilde{\lambda}_{\dot{a}}$ is a Weyl spinor with $+1/2$ helicity. Note that the components of λ_a and $\tilde{\lambda}_a$ are complex numbers, not Grassmannian. The definition of the two spinors has a rescale ambiguity,

$$\lambda \to t\lambda, \quad \tilde{\lambda} \to t^{-1}\tilde{\lambda} \tag{3.7}$$

For real momentum p in $(+,-,-,-)$ Minkowski space, we can further require that,

$$\lambda_a^* = \pm\tilde{\lambda}_a. \tag{3.8}$$

In Dirac spinor notations, λ and $\tilde{\lambda}$ can be written as,

$$\gamma^{\mu} = \begin{pmatrix} 0 & (\sigma^{\mu})_{a\dot{a}} \\ (\bar{\sigma}^{\mu})^{\dot{b}b} & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} \lambda_a \\ 0 \end{pmatrix}, \psi^+ = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{a}} \end{pmatrix} \tag{3.9}$$

Note that for the ψ^+ case, we have to use the upper-indice $\tilde{\lambda}^{\dot{a}}$. The usual spin sum rule $\sum_s u(p)\bar{u}(p) = \gamma^{\mu} p_{\mu}$ reads,

$$\begin{pmatrix} \lambda_a \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_{\dot{a}}^* \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{b}} \end{pmatrix} \begin{pmatrix} \tilde{\lambda}^{b*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (p_{\mu}\sigma^{\mu})_{a\dot{a}} \\ (p_{\mu}\bar{\sigma}^{\mu})^{\dot{b}b} & 0. \end{pmatrix} \tag{3.10}$$

A Lorentzian transformation parameterized by $\omega_{\mu\nu}$ acts on the spinors λ_a and $\tilde{\lambda}_{\dot{a}}$ as,

$$\lambda_a \to (e^{\frac{1}{2}w_{\mu\nu}\sigma^{\mu\nu}})_a^b \lambda_b, \quad \tilde{\lambda}^{\dot{a}} \to (e^{\frac{1}{2}w_{\mu\nu}\bar{\sigma}^{\mu\nu}})_{\dot{b}}^{\dot{a}} \tilde{\lambda}^{\dot{b}} \tag{3.11}$$

where

$$(\sigma^{\mu\nu})_a^b = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a^b, \quad (\bar{\sigma}^{\mu\nu})^{\dot{a}}_{\dot{b}} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}_{\dot{b}}, \tag{3.12}$$

and $(\sigma^{\mu\nu})^\dagger = -\bar{\sigma}^{\mu\nu}$. It is easy to check that by a θ-rotation along the momentum direction,

$$\lambda_a \to e^{-i\theta/2}\lambda_a, \quad \tilde{\lambda}_{\dot{a}} \to e^{+i\theta/2}\tilde{\lambda}_{\dot{a}}. \tag{3.13}$$

The inner products for λ_a is defined to be,

$$\langle\lambda, \lambda'\rangle = \epsilon_{ab}\lambda^a\lambda'^b \tag{3.14}$$

where $\epsilon_{12} = 1$ and $\lambda^a = \epsilon^{ab}\lambda_b$. It is *Lorentzian-invariant* because,

$$\begin{aligned} \epsilon_{ab}\lambda^a\lambda'^b &\to \epsilon_{ab}(e^{-\frac{1}{2}w_{\mu\nu}\sigma^{T\mu\nu}})^a_c\lambda^c(e^{-\frac{1}{2}w_{\mu\nu}\sigma^{T\mu\nu}})^b_d\lambda'^d \\ &= \epsilon_{cd}\lambda^c\lambda'^d \end{aligned} \tag{3.15}$$

Similarly, The inner products for $\tilde{\lambda}_a$ is,

$$[\tilde{\lambda}, \tilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{a}}\tilde{\lambda}'^{\dot{b}} \tag{3.16}$$

where $\epsilon_{\dot{1}\dot{2}} = 1$. Both $\langle,\rangle$ and $[,]$ are anti-symmetric. Since,

$$(p_\mu\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(p_\mu\sigma^\mu)_{\beta\dot{\beta}}, \tag{3.17}$$

the inner product of two momentum $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$ and $p'_{a\dot{a}} = \lambda'_a\tilde{\lambda}'_{\dot{a}}$ is,

$$p_\mu p'^\mu = \frac{1}{2}(p_\mu\sigma^\mu)_{a\dot{a}}(p'_\nu\bar{\sigma}^\nu)^{\dot{a}a} = \frac{1}{2}\langle\lambda, \lambda'\rangle[\tilde{\lambda}, \tilde{\lambda}'] \tag{3.18}$$

For a gluon with the light-like momentum $p^\mu p_\mu = 0$ and $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$, the polarization vector can be chosen as,

$$\epsilon^+_{a\dot{a}} = \frac{\mu_a\tilde{\lambda}_{\dot{a}}}{\langle\mu, \lambda\rangle}, \quad \epsilon^-_{a\dot{a}} = \frac{\lambda_a\tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \tag{3.19}$$

It is easy to check that $\epsilon^\pm \cdot p = 0$ because of the antisymmetric products $\langle,\rangle$ and $[,]$. The choice of μ and $\tilde{\mu}$ is arbitrary as long as the products are nonzero.

Particular choices of μ and $\tilde{\mu}$ would simplify the amplitude calculation. One practical choice is that:

- For the amplitude $A(1^{h_1}2^{h_2}\ldots n^{h_n})$, choose a negative-helicity incoming gluon, say i. Then for all gluons k with $h_k = +$, we choose $\mu_k = \lambda_i$.
- Choose a positive-helicity incoming gluon, say j. for all gluons l with $h_l = +$, we choose $\tilde{\mu}_l = \tilde{\lambda}_j$.

Then the product of polarization vectors with the same helicity vanishes, $\epsilon_k^+ \cdot \epsilon_{k'}^+ = \epsilon_l^- \cdot \epsilon_{l'}^- = 0$. Furthermore, $p_i \cdot \epsilon_k^+ = p_j \cdot \epsilon_k^- = 0$. So many terms in Feynman diagram calculation vanish automatically. For the calculation of scattering amplitude, we may drop the letter λ and the spinor indices.

The tree-level *partial amplitude with all gluons of the same helicity* vanishes.

$$A(1^+2^+...n^+) = A(1^-2^-...n^-) = 0 \tag{3.20}$$

Furthermore, the tree-level *partial amplitude with all gluons of the same helicity but one* also vanish.

$$A(1^-2^+...n^+) = A(1^+2^-...n^-) = 0, \quad n \geq 4 \tag{3.21}$$

These identities are hard to prove by Feynman diagrams. We will see that the simplest proof is based on supersymmetry. Since the tree-level all-gluon amplitude is the same as that in super-Yang-Mills theory, the supersymmetry identity holds in the tree-level non-supersymmetric case.

The first non-vanishing tree-level amplitude is the *partial amplitude with all gluons of the same helicity but two*. The amplitude with $n-2$ negative-helicity gluons and 2 positive-helicity gluon is called *Maximum-helicity-violating amplitude*, or *MHV* amplitude. Similarly, The amplitude with $n-2$ positive-helicity gluons and 2 negative-helicity gluon is called $\overline{MHV}$ amplitude. The amplitude with $n-2-k$ negative-helicity gluons and $2+k$ positive-helicity gluon is called N^kMHV amplitude.

3.3 BCFW recursive relation

BCFW relation is an extremely important recursive relation [17][18]. It relates the high point tree level scattering amplitude with the lower point scattering amplitudes. So it is possible to build all the scattering amplitude systematically from the lowest amplitude. The advantage is that all the scattering amplitudes involved are on-shell.

BCFW relation holds for (super/non-super)-Yang-Mills theory, (super/non-super)-gravity, but not for spin-0 field theories like ϕ^4. We illustrate the BCFW recursive relation by the Yang-Mills amplitude.

We consider a color-ordered tree level amplitude $A(p_1, h_1; ...; p_M, h_M)$. Here $h_i = \pm 1$ is the helicity. Because the color-ordered amplitude is gauge independent, the reference spinor μ is dropped out so we just need to specify the helicities. BCFW method complexifies two momenta, say, p_1 and p_i,

$$p_1 \to p_1(z) = p_1 + zq, \quad p_i \to p_i(z) = p_i - zq \tag{3.22}$$

And the complexified scattering amplitude is $A(z)$. *The key is:* $A(z)$ *vanishes at the infinity* $z \to \infty$.

$$\int_{|z|=\infty} \frac{A(z)dz}{z} = 0 \tag{3.23}$$

Note that (3.23) is highly nontrivial. For "simpler" quantum field theories, like $\lambda\phi^4$ theory, this identity is not valid. For example, the four-point $\lambda\phi^4$ tree-level scattering amplitude is approaching a constant, in the limit $z \to \infty$. In general, for a quantum field theory with the highest spin s, the asymptotic behavior of $A(z)$ is,

$$A(z) \sim \frac{1}{z^s}, \quad z \to \infty. \tag{3.24}$$

So BCFW recursive relation works for Yang-Mills theory, Super-Yang-Mills theory, Einstein gravity and supergravity.

Pick up the residue, we get,

$$\begin{aligned} A(0) &= -\sum_{\sigma}\sum_{h=\pm 1} A(p_{\sigma_1}, h_{\sigma_1}; p_{\sigma_2}, h_{\sigma_2}; \ldots; p_{l(\sigma)}, h_{l(\sigma)}; -P(z), -h)|_{z=-P^2/2p\cdot q} \\ &\times \frac{i}{P^2} \cdot A(P(z), h; p_{\tau_1}, h_{\tau_1}; p_{\tau_2}, h_{\tau_2}; \ldots; p_{l(\tau)}, h_{l(\tau)})|_{z=-P^2/2p\cdot q} \end{aligned} \tag{3.25}$$

where we separate the original color-ordered diagram into two subdiagrams. σ is the indices such that $(\sigma_1, \sigma_2, \ldots, \sigma_{l(\sigma)})$ is the indices of the subdiagram which contains 1 and $l(\sigma)$ is the number of indices in this subdiagram. P is the *uncomplexified* total momentum in the subdiagram. Similarly, τ is the indices for the other subdiagram.

BCFW relation implies that the tree level amplitude of arbitrary gluons can be generated by three-point amplitude, which we would calculate explicitly before considering any example of BCFW relation.

Simple examples

The three-gluon partial amplitude $A(p_1, h_1; p_2, h_2; p_3, h_3)$ vanishes if all the momenta are real and on-shell, so we need to consider the complex momenta. Because of $p_1 \cdot (p_2 + p_3) = 0$ and the similar identities, $p_i \cdot p_j = 0$ for arbitrary i and

j,

$$\langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\lambda}_1] = 0$$

$$\langle \lambda_2, \lambda_3 \rangle [\tilde{\lambda}_2, \tilde{\lambda}_3] = 0$$

$$\langle \lambda_3, \lambda_1 \rangle [\tilde{\lambda}_3, \tilde{\lambda}_1] = 0$$

If $\langle \lambda_1, \lambda_2 \rangle$ vanishes, then λ_1 and λ_2 are linear dependent. If two of the $\langle , \rangle$ products vanish, then the rest $\langle , \rangle$ also vanish. Therefore there are just two cases,

- $\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_2, \lambda_3 \rangle = \langle \lambda_3, \lambda_1 \rangle = 0$
- $[\tilde{\lambda}_1, \tilde{\lambda}_2] = [\tilde{\lambda}_2, \tilde{\lambda}_3] = [\tilde{\lambda}_3, \tilde{\lambda}_1] = 0$

So the amplitude does not contain the cross term, $\langle\rangle[]$. On the other hand, by dimension analysis, each term in the amplitude shoule have the form.

$$\frac{()^{m+1}}{()^m} \tag{3.26}$$

where $()$ should be either $\langle\rangle$ or $[]$, but not both.

Now we consider the different helicities,

- $(+ + +)$. In this case, even $()$ is $[]$, the amplitude just has $+1$ helicity, which is a contradiction to the $(+++)$ helicity. So $A(+ + +) = 0$.
- $(+ + -)$. We need $() = []$. The result is

$$A(p_1, -; p_2, +; p_3, +) = i\sqrt{2}\frac{[23]^3}{[12][13]}, \tag{3.27}$$

which is antisymmetric under $2 \leftrightarrow 3$, where $[12]$ is short for $[\tilde{\lambda}_1, \tilde{\lambda}_2]$. This result can also be explicitly, calculated,

$$\begin{aligned} \frac{i}{\sqrt{2}}\epsilon_3^+ \cdot \epsilon_1^-(p_3 - p_1) \cdot \epsilon_2^+ &= i\sqrt{2}\epsilon_3^+ \cdot \epsilon_1^- p_3 \cdot \epsilon_2^+ \\ &= i\sqrt{2}\frac{\langle \mu \lambda_1 \rangle [\tilde{\lambda}_3, \tilde{\lambda}_2]}{\langle \mu, \lambda_3 \rangle [\tilde{\lambda}_1, \tilde{\lambda}_2]} \frac{\langle \lambda_3, \mu \rangle [\tilde{\lambda}_3, \tilde{\lambda}_2]}{\langle \mu, \lambda_2 \rangle} \end{aligned} \tag{3.28}$$

where we choose $\epsilon_1^- = \lambda_1\tilde{\lambda}_2/[\tilde{\lambda}_1,\tilde{\lambda}_2]$, $\epsilon_2^+ = \mu\tilde{\lambda}_2/\langle\mu,\lambda_2\rangle$ and $\epsilon_2^+ = \mu\tilde{\lambda}_3/\langle\mu,\lambda_3\rangle$. Because $\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$,

$$\frac{\langle\mu,\lambda_1\rangle}{\langle\mu,\lambda_2\rangle} = \frac{[\tilde{\lambda}_2,\tilde{\lambda}_3]}{[\tilde{\lambda}_1,\tilde{\lambda}_3]}.$$

therefore

$$A(p_1,-;p_2,+;p_3,+) = i\sqrt{2}\frac{[23]^3}{[12][13]},$$

- $(+--)$. Similarly,

$$A(p_1,+;p_2,-;p_3,-) = i\sqrt{2}\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}, \tag{3.29}$$

- $(---)$. Similarly, $A(---) = 0$.

Now we can consider the four point amplitude for the helicity $(+-+-)$. p_1 and p_4 are complexified by,

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_4$$
$$\hat{\tilde{\lambda}}_4 = \tilde{\lambda}_4 - z\tilde{\lambda}_1 \tag{3.30}$$

So $q = \lambda_1\tilde{\lambda}_4$. There is only one way to separate the indices, $\sigma_1 = 1, \sigma_2 = 2$ and $\tau_1 = 3, \tau_2 = 4$. $P(z) = p_1 + p_2 + zq$. z can be determined as,

$$p_1\cdot p_2 + zp_2\cdot q = 0 \Rightarrow z = \frac{\langle 12\rangle}{\langle 24\rangle} \tag{3.31}$$

Therefore,

$$\hat{P} = -1\tilde{1} - 2\tilde{1} - z4\tilde{1} = -2(\tilde{2} + \frac{\langle 14\rangle}{\langle 24\rangle}\tilde{1}) \tag{3.32}$$

We define $P = p\tilde{p}$ where $p = 2$ and $\tilde{p} = \tilde{2} + \frac{\langle 14\rangle}{\langle 24\rangle}\tilde{1}$. Similarly,

$$\hat{1}\tilde{1} = \Big(\frac{\langle 14\rangle}{\langle 24\rangle}2\Big)\tilde{1}$$
$$4\hat{\tilde{4}} = 4\Big(\tilde{4} - \frac{\langle 12\rangle}{\langle 24\rangle}\tilde{1}\Big) \tag{3.33}$$

BCFW relation gives,

$$\frac{[\hat{1}\hat{P}]^3}{[\hat{P}2][2\hat{1}]}\frac{1}{\langle 12\rangle[12]}\frac{\langle\hat{4}\hat{P}\rangle}{\langle 3\hat{P}\rangle\langle 3\hat{4}\rangle}=\frac{[12]^3}{\frac{\langle 14\rangle}{\langle 24\rangle}[21][21]}\frac{1}{\langle 12\rangle[12]}\frac{\langle 42\rangle^3}{\langle 32\rangle\langle 34\rangle}=\frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

which is the MHV scattering amplitude for four point.

3.4 Coherent super space and super KLT relation

Supersymmetry can be used in scattering amplitude calculation. The tree-level gluon scattering amplitudes are the same in super or non-super Yang-Mills theory. (See the appendix for details.) In the other word, the tree-level Yang-Mills amplitude "knows it is supersymmetric."

There are many super-Yang-Mills theories. The simplest one is the $d = 4, N = 4$ super-Yang-Mills theory, which contains the maximum supersymmetry without graviton in $4D$. (See the appendix for the review of $d = 4, N = 4$ super-Yang-Mills theory.) Different scattering amplitudes are related by supersymmetry transformations. To manifest the supersymmetry, we can use the coherent representation of the states . This method also works for $d = 4, N = 8$ supergravity.

Again, let s be the highest spin in the theory, $s = 1$ for $N = 4$ super-Yang-Mills theory while $s = 2$ for $N = 8$ supergravity. All the *on-shell physical states* can be combined in a coherent state,

$$|\bar{\eta}, \lambda, \bar{\lambda}\rangle = e^{\bar{Q}^{I\dot{\alpha}}\bar{\omega}_{\dot{\alpha}}\bar{\eta}_I}| + s, \lambda, \bar{\lambda}\rangle, \quad |\eta, \lambda, \bar{\lambda}\rangle = e^{Q_{I\alpha}\omega^{\alpha}\eta^I}| - s, \lambda, \bar{\lambda}\rangle \tag{3.34}$$

where $Q_{I\alpha}$ and $\bar{Q}^{I\dot{\alpha}}$ are supercharges. $I = 1, \ldots, 4s$. λ and $\bar{\lambda}$ are the spinors for the massless particles in spinor helicity formalism. η and $\bar{\eta}$ are the Grassmann coordinate for superspace. w and $\bar{w}$ are spinors for the normalization,

$$\langle w, \lambda\rangle = 1, \quad [\bar{w}, \bar{\lambda}] = 1 \tag{3.35}$$

Note that since η and $\bar{\eta}$ are fermonic, the formal exponential series will truncate at the $4s$-th order. This is consistent with the helicities of one supermultiplet. Instead of using $|h, \lambda, \bar{\lambda}\rangle$ to label the incoming states, we would use $|\eta, \lambda, \bar{\lambda}\rangle$ or $|\bar{\eta}, \lambda, \bar{\lambda}\rangle$. Each physical state can be read off from the η or $\bar{\eta}$ expansion.

Then the supersymmetric transformation is manifest,

$$e^{Q_{I\alpha}\zeta^{I\alpha}}|\eta\rangle = |\eta + \langle\zeta\lambda\rangle\rangle, \quad e^{Q_{I\alpha}\zeta^{I\alpha}}|\bar{\eta}\rangle = e^{\bar{\eta}_J\langle\lambda\zeta^J\rangle}|\bar{\eta}\rangle, \tag{3.36}$$

where ζ is the parameter for the supersymmetric transformation. The above notations are summarized in Table. (3.1). The anti-holomorphic part is just the conjugate.

Notation	Abbreviation	Number of components	Statistics	Comment
λ_α	λ	2	complex number	momentum spinor
η^I	η	N	Grassmann number	coherent coordinate
$Q_{I\alpha}$	Q	$2N$	Fermionic operator	supercharge
$\zeta^{I\alpha}$	ζ	$2N$	Grassmann number	superspace coordinates

Table 3.1: Notation for Coherent States

The Grassmann variable η and $\bar{\eta}$ are Fourier conjugates of each other.

$$|\bar{\eta}\rangle = \int d^N\eta e^{\eta\bar{\eta}}|\eta\rangle, \quad |\eta\rangle = \int d^N e^{\bar{\eta}\eta}|\bar{\eta}\rangle \tag{3.37}$$

For each incoming particle, we can use either η or $\bar{\eta}$ representation, but not both. Careful choice in η and $\bar{\eta}$ may simplify computation.

We would consider the *superamplitude*. $M(\{\eta_i, \lambda_i, \bar{\lambda}_i\}, \{\bar{\eta}_{\bar{i}}, \lambda_{\bar{i}}, \bar{\lambda}_{\bar{i}}\})$. The supersymmetry transformations apply on the superamplitude as,

$$M(\eta_i, \overline{\eta_i}) = e^{\sum_j [\overline{\lambda}_j \overline{\zeta}_I] \eta_j^I} M(\eta_i + \langle \lambda_i \zeta \rangle, \overline{\eta_i}), \quad Q \text{ transformation} \tag{3.38}$$

$$M(\eta_i, \overline{\eta_i}) = e^{\sum_{\bar{j}} \langle \lambda_{\bar{j}} \zeta^I \rangle \overline{\eta}_{\bar{j},I}} M(\eta_i, \overline{\eta_i} + [\overline{\lambda}_{\bar{i}} \overline{\zeta}]), \quad \overline{Q} \text{ transformation} \tag{3.39}$$

Note that Q and $\overline{Q}$ do not commute, so the order of the two transformations is important. Because η_I has $2N$ components, so the Q transformation can translate 2 η_i to zero. Similarly, the $\overline{Q}$ transformation translates 2 $\overline{\eta_i}$ to zero.

Here can show that the amplitude $A(1^+, 2^+, \ldots, n^+)$ vanishes. By the Fourier transformation, and set $\overline{\eta_i} = 0$, we have,

$$A(1^+, 2^+, \ldots, n^+) = \int d^N\eta_1 \ldots d^N\eta_n M(\eta_1, \eta_2, \ldots, \eta_n) \tag{3.40}$$

By (3.39), $M(\eta_1, \eta_2, \ldots, \eta_n) = M(0, \eta_2, \ldots, \eta_n)$. Hence,

$$A(1^+, 2^+, \ldots, n^+) = \int d^N\eta_1 \ldots d^N\eta_n M(0, \eta_2, \ldots, \eta_n) = 0 \tag{3.41}$$

Similarly,

$$A(1^+, 2^+, \ldots, n^-) \quad = \quad \int d^N\eta_1 d^N\eta_2 \ldots d^N\overline{\eta}_n M(\eta_1, \eta_2, \ldots, \overline{\eta}_n). \tag{3.42}$$

Using (3.39), and the detailed Grassmann calculation, this integral is also zero.

3.4.1 Super KLT relation

In coherent state representation, the KLT relation also has a compact form. We can think that both the left and right-moving sectors have $N = k$ spacetime supersymmetry, so the closed string has $N = 2k$ spacetime supersymmetry. By the super KLT relation, a $N = 2k$-supersymmetric closed string superamplitude would be decomposed as the products of two $N = k$-supersymmetric open string superamplitudes.

Again, string KLT relation can be reduced to field theory KLT relations. The most interesting case is the $d = 4, N = 8$ supergravity, reduced to two copied of $d = 4, N = 4$ super-Yang-Mills theories. We use the indices $I = 1, 2, 3, 4$ to label the left-handed $N = 4$ supercharges and $I' = 5, 6, 7, 8$ for the right-handed $N = 4$ supercharges. Furthermore, we use $M = 1, 2, \ldots 8$ as the common label for all the 8 supercharges of the $N = 8$ supergravity.

Recall that, in the original KLT relation, the formalism is the same for amplitudes with different helicities. This fact implies that the super KLT relation has the same form of the original KLT relation. If for n-point amplitude, the KLT relation reads,

$$A_c(k_1, e_1; k_2, e_2; \ldots k_n, e_n) = \sum_{\sigma,\sigma'} f(\sigma, \sigma') A(k_{\sigma_1}, \epsilon_{\sigma_1}; k_{\sigma_2}, \epsilon_{\sigma_2}; \ldots k_{\sigma_n}, \epsilon_{\sigma_n}) \\ \times \bar{A}(k_{\sigma'_1}, \bar{\epsilon}_{\sigma'_1}; k_{\sigma'_2}, \bar{\epsilon}_{\sigma'_2}; \ldots k_{\sigma'_n}, \bar{\epsilon}_{\sigma'_n}), \tag{3.43}$$

where σ and σ' are the permutations of the gluon color orderings. $f(\sigma, \sigma')$ is a kinematic factor for the permutation pair (σ, σ'). Then the super KLT relation in η representation is,

$$A_c(k_1, \eta^c_1; k_2, \eta^c_2; \ldots k_n, \eta^c_n) = \sum_{\sigma,\sigma'} f(\sigma, \sigma') A(k_{\sigma_1}, \eta_{\sigma_1}; k_{\sigma_2}, \eta_{\sigma_2}; \ldots k_{\sigma_n}, \eta_{\sigma_n}) \\ \times \bar{A}(k_{\sigma'_1}, \eta'_{\sigma'_1}; k_{\sigma'_2}, \eta'_{\sigma'_2}; \ldots k_{\sigma'_n}, \eta'_{\sigma'_n}), \tag{3.44}$$

where η^c_i is a 8-component Grassmann vector, while both η_i and η'_i are 4-component Grassmann vectors. There is a simple relation,

$$\eta^c_i = (\eta_i, \eta'_i), \quad \forall i. \tag{3.45}$$

Similar, there is an equivalent super KLT relation for $\bar{\eta}$ representation or mixed representation.

CHAPTER 4

BCJ RECURSIVE RELATIONS FROM THE VIEWPOINT OF HETEROTIC STRING THEORY

In this chapter, we use [37] the new techniques mentioned above, to study the an interesting class of scattering amplitude identities, Bern-Carrasco-Johansson identity [9].

4.1 Introduction

Some years ago, Zhu showed that the terms in the 4-gluon tree amplitude obey an identity [57]. Recently, Bern, Carrasco and Johansson conjectured the presence of such identities in higher-order amplitude as well as loop amplitudes [9]. If these identities are true, the evaluation of the tree-level M-gluon amplitudes can simplify considerably. Furthermore, given the M-gluon tree amplitudes, M-graviton tree scattering amplitudes can be written down immediately. Loop amplitudes can be obtained from the tree amplitudes using the unitarity method [10] [11] and these identities can be carried over [9]. In this paper, we use the properties of the heterotic string and open string scattering amplitudes to refine and prove parts of the BCJ conjecture and to extend the identities to include scatterings of massless gluinos and gravitinos.

Consider the M-gluon tree scattering amplitude, which is a function of the external gluon momenta k_i^μ where $k_i^2 = 0$ (and $\sum_i k_i^\mu = 0$), polarizations ζ_i^μ where $\zeta_i \cdot k_i = 0$ and color a_i, $i = 1, 2, ..M$,

$$\mathcal{A}_M^{\rm YM}(k_i, \zeta_i, a_i) = g^{M-2} \sum_j \frac{c_j(a_i) n_j(k_i, \zeta_i)}{P_j} \tag{4.1}$$

where the sum is over all allowed channels (or terms) with different pole structures. There are $(2M-5)!!$ channels in $\mathcal{A}_M^{\mathrm{YM}}$. Each denominator is a product of $(M-3)$ pole factors : $P_j = \Pi_{m=1}^{M-3} p_{j,m}(k_i)$, where each pole factor $p_{j,m}(k_i)$ corresponds to the kinematic invariant of an internal gluon propagator. For example, a 2-particle channel pole $p_{j,m}$ takes the form $s_{ln} = -(k_l+k_n)^2$. The kinematic factor $n_j(k_i, \zeta_i)$ is a function of k_i^μ and ζ_i^μ. Although the choice of the set of n_j's is far from unique, $\mathcal{A}_M^{\mathrm{YM}}$ itself is independent of the specific choice of n_j's. In this paper, we shall discuss the choices of the n_j's in some detail. The color factor $c_j(a_i)$, a function of the colors a_i, is a product of the $(M-2)$ group structure constants $\tilde{f}^{abc}$ corresponding to the respective pole structure, where, for a given Lie algebra, $\mathrm{Tr}(T^a T^b) = \delta^{ab}$, $[T^a, T^b] = i\sqrt{2} f^{abc} T^c \equiv \tilde{f}^{abc} T^c$. As an illustration, we see that the 5-point diagram in Figure 1 has denominator $s_{13}s_{54}$ and $c_{(54)2(13)} = \tilde{f}^{a_5 a_4 b}\tilde{f}^{b a_2 d}\tilde{f}^{d a_1 a_3}$. Note that $\mathcal{A}_M^{\mathrm{YM}}$ is unchanged if we flip the signs of both c_j and n_j ($c_j \to -c_j$ and $n_j \to -n_j$) in any term in Eq.(4.1). At times, we shall set the coupling $g=1$.

The dual identities are best illustrated by the 4-gluon tree level scattering amplitude,

$$\mathcal{A}_4^{\mathrm{YM}}(k_1, \zeta_1, a_1, ...k_4, \zeta_4, a_4) = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}, \tag{4.2}$$

where s, t, u are Mandelstam variables, $s = s_{12} = -(k_1+k_2)^2$, $t = s_{14} = -(k_1+k_4)^2$, $u = s_{13} = -(k_1+k_3)^2$ and $s+t+u=0$. Here the color factors,

$$\begin{aligned} c_s &= \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} \\ c_t &= \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_1 a_4} \\ c_u &= \tilde{f}^{a_3 a_1 b}\tilde{f}^{b a_2 a_4} \end{aligned} \tag{4.3}$$

depend on the color indices. By the Jacobi identity,

$$c_s + c_u + c_t = 0. \tag{4.4}$$

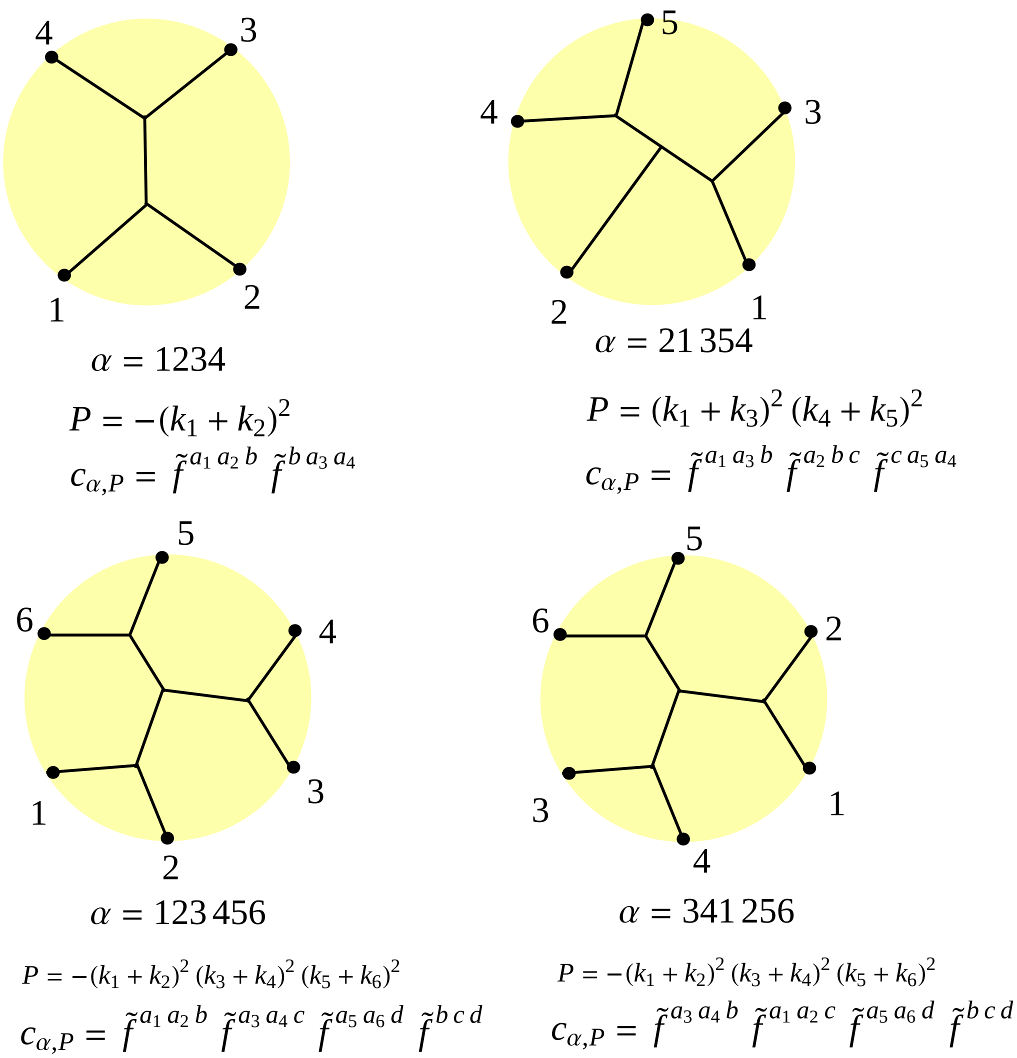


Figure 4.1: Several examples of the poles $P = \Pi_j s_{ln}$ and the color factors. The structure constants are labeled in the counter-clockwise direction. The field theory tree amplitudes A^{tree} are related to the zero-slope limit of the open string amplitudes A^{open}, which are given by the disc diagrams in open string theory. The (yellow) disc for each graph is shown to emphasize this feature.

It was shown [57] that the $n_j(k_i, \zeta_i)$'s satisfy the dual identity,

$$n_s + n_u + n_t = 0. \tag{4.5}$$

Note that $n_s(k_i, \zeta_i)$ is determined up to a term proportional to s:

$$n_s \to n'_s = n_s + s\eta(k_i, \zeta_i) \tag{4.6}$$

and similarly for n_u and n_t, where η is an arbitrary function of the kinematic variables. Following Ref.[9], we shall refer to this as a gauge transformation.

We shall consider only the η's that have no pole term, so this "gauge freedom" changes only the "contact" part but not the "residue" or "non-contact" part of n_s. A redistribution of the contact (4-point coupling) term among the 3 terms inside $\mathcal{A}_4^{\text{YM}}$ (4.2) will lead to such a change in the n_j's. However, the identity (4.5) is gauge-independent,

$$n'_s + n'_u + n'_t = (n_s + s\eta) + (n_u + u\eta) + (n_t + t\eta) = (s + t + u)\eta = 0.$$

Note that $\mathcal{A}_4^{\text{YM}}$ is gauge-invariant, as it should be.

Let us now consider the general $\mathcal{A}_M^{\text{YM}}(k_i, \zeta_i, a_i)$ (4.1). Here the n_j's are "gauge"-dependent, even though $\mathcal{A}_M^{\text{YM}}$ itself is invariant. Unless specified otherwise, the n_j's are chosen to have no poles, that is, they are local. A convenient symmetrized way of expressing them may be found in Ref.[21]. There are many triplets of c_j in $\mathcal{A}_M^{\text{YM}}$ (4.1) that satisfy,

$$c_j + c_l + c_k = 0. \tag{4.7}$$

Each color identity (4.7) is nothing but the Jacobi identity multiplied by an overall factor of a product of structure constants. One can take any 4 (internal and/or external) gluons in a diagram that are connected by a single internal gluon propagator. The 3 c_j's in a color identity (4.7) simply correspond to the 3 ways (i.e., the "s, t, u" channels) of connecting those 4 gluons. BCJ conjectured that whenever a set of 3 c_j in $\mathcal{A}_M^{\text{YM}}$ (4.1) satisfy the color identity (4.7), the corresponding 3

n_j's in the same $\mathcal{A}_M^{\text{YM}}$ (4.1) satisfy the kinematic identity

$$n_j + n_l + n_k = 0. \tag{4.8}$$

In general, there are many such dual pairs of identities for the M-point amplitude, not all of them are independent. One can explicitly check this for the 5-gluon tree amplitude with its 9 independent kinematic identities [9, 42], which we shall also discuss in some detail. BCJ also conjectured that these identities can be carried over to loop amplitudes using the unitarity method [10] [11].

There are 2 key properties in the BCJ relation :
(1) There is a set of kinematic identities (4.8) in $\mathcal{A}_M^{\text{YM}}$ (4.1) for an appropriate set of n_j's;
(2) There is a duality between a color identity (4.7) and the corresponding kinematic identity (4.8).
In this paper, we shall use the properties of the heterotic string model [34][35] to prove the duality property between a color identity (4.7) and the corresponding kinematic identity for the n_j's,

$$\left(n_j + n_l + n_k\right)\Big|_{\text{residue}} = 0. \tag{4.9}$$

where the "residue" refers to the residue of the product of the $(M-4)$ poles that are common among the n_j, n_l and n_k channels.

To illustrate the difference between the $M = 4$ case and the $M > 4$ cases, let us look at a $M = 5$ open string amplitude identity which yields the following gauge-independent identity,

$$\frac{n_{(13)(42)5} - n_{2(13)(45)} + n_{(13)4(52)}}{s_{13}} + \frac{n_{1(32)(45)} - n_{(21)3(45)} + n_{(13)(45)2}}{s_{45}}$$
$$\frac{n_{(51)(32)4} - n_{2(51)(34)} + n_{(51)3(42)}}{s_{15}} + \frac{n_{(34)2(51)} - n_{(21)(34)5} + n_{1(34)(52)}}{s_{34}} = 0 \tag{4.10}$$

where $n_{(13)(42)5}$ is the numerator factor of the double pole $s_{13}s_{24}$ in $\mathcal{A}_5^{\rm YM}$ (4.1). The details of the $M = 5$ case will be explained in Section 5. Here it suffices to note that the 3 n_j's in any one of the 4 triplets has a common pole which appears in the respective denominator in the constraint (4.10). Actually, the corresponding color factors obey identical relations to the n_j's in Eq.(4.10). Since the c_j's have discrete values, $\left(c_j + c_l + c_k\right)\big|_{\rm residue} = 0$ implies the color identities (4.7). This is not the case for the $n_j(k_i, \zeta_i)$'s because the momenta k_i are continuous and because of the gauge freedom. Eq.(4.10) only implies that the residue of each pole term must vanish. For example, the residue of $(n_{(13)(42)5} - n_{2(13)(45)} + n_{(13)4(52)})$ must vanish, but its regular component that is proportional to s_{13} need not. On the other hand, the 4 regular pieces in Eq.(4.10) must sum to zero. This property generalizes to arbitrary M. There are $(M-3)!(M-3)$ independent open string identities each of which involves $2^{M-3}(M-3)(2M-7)!!/(M-2)!$ triplets, where each triplet of n_j's is the numerator of a product of $(M-4)$ poles that are common to the n_j channels in that triplet. This yields the set of kinematic identities (4.9), in one-to-one correspondence to the color identities (4.7).

As conjectured in Ref.[9], the kinematic identities (4.8) for $M > 4$ hold only in specific gauge choices. In proving this for $M = 5$, we reveal the underlying gauge choice issue. For larger M, we support (but do not prove) this part of the BCJ conjecture, that there always exist gauge choices such that (4.8) holds for the complete set of the kinematic identities. If true, the space of such gauge choices will have dimension $(M-3)!(M-3)$. On the other hand, the kinematic identity (4.9) refers to the "gauge"-invariant part of the n_j's and so may be more relevant. Since the string identities are among gauge-independent partial amplitudes, one should treat them as the defining identities.

Heterotic string model contains gauge fields, and their interactions agree with that of the Yang-Mills theory in the zero slope limit (as can be shown in the background field analysis). So their scattering yields tree amplitudes $\mathcal{A}^{\text{het}}_{\text{M-gluon}}$ that obey, in the zero Regge slope limit $\alpha' \to 0$,

$$\mathcal{A}^{\text{het}}_{\text{M-gluon}}(\alpha' = 0) = \mathcal{A}^{\text{YM}}_{M} \tag{4.11}$$

The amplitudes $\mathcal{A}^{\text{het}}_{\text{M-gluon}}$ are functions of open string amplitudes via the KLT relation [39]; these open string amplitudes obey identities that yield both the color identities and the kinematic identities on equal footings. As we shall see, the duality between the c_j and the n_j also corresponds precisely to the duality between M-gluon and M-graviton scattering amplitudes, proving yet another BCJ conjecture. The implications of this gauge-gravity duality remain to be further explored. In short, we see that there are 2 versions of duality, i.e, a double duality.

Let us briefly review Type I open string theory and explain first why heterotic string theory helps. We then summarize its key properties relevant for showing the duality property. If we treat the gluon field as a matrix, $A_\mu = A^a_\mu T^a$ in perturbation expansion, we obtain the M-gluon tree scattering amplitude as a sum of gauge invariant sub-amplitudes [5, 45, 43]

$$\mathcal{A}^{\text{YM}}_{M} = g^{M-2} \sum_{\sigma \in S^M/Z_M} \text{Tr}(T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}} ... T^{a_{\sigma_M}}) A^{tree}(\sigma_1 \sigma_2 \sigma_3 .... \sigma_M) \tag{4.12}$$

where S^M is the set of all permutations of M lines, and Z_M is the subset of cyclic permutations that preserve the color trace. The sum over the set S^M/Z_M is over all distinct cyclic orderings in the trace. The color-ordered sub-amplitudes A^{tree} are the partial amplitudes that receive contributions from diagrams with a particular cyclic ordering of the M external gluons, so the poles occur only in a limited set of momentum channels made out of sums of cyclically adjacent momenta. They also satisfy the cyclic and the reflection properties,

$$A^{tree}(1,2,3,...,M) = A^{tree}(2,3,...,M,1), \qquad A^{tree}(1,2,...,M) = (-1)^M A^{tree}(M,...,2,1) \tag{4.13}$$

so there are $(M-1)!/2$ different A^{tree}s in $\mathcal{A}_M$. Each A^{tree} is gauge-invariant and has $2^{M-2}(2M-5)!!/(M-1)!$ channels, i.e., terms of the form n_i/P_i given in Eq.(4.1). It is straightforward to show that $\mathcal{A}_M$ (4.12) is equal to $\mathcal{A}_M^{\rm YM}$ (4.1) by decomposing each partial amplitude into the channels and calculate the commutators of the matrices. For example, in the 4-gluon case, the terms in (4.12) which are related to the n_s/s is,

$$\begin{aligned} &\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3}T^{a_4}) \\ &- \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_4}T^{a_3}) = c_s. \end{aligned} \tag{4.14}$$

Next, we consider Type I open string M-gluon tree amplitudes,

$$\mathcal{A}_M^{open} = g^{M-2} \sum_{\sigma\in S^M/Z_M} Tr(T^{a_{\sigma_1}}T^{a_{\sigma_2}}T^{a_{\sigma_3}}...T^{a_{\sigma_M}}) A^{open}(\sigma_1\sigma_2\sigma_3....\sigma_M) \tag{4.15}$$

where the color properties are contained in the Chan-Paton factor (the trace) while A^{open} is a function of the kinematic variables only. Again, the cyclic and the reflection properties reduce the number of A^{open}s from $M!$ to $(M-1)!/2$. Now, relations among the A^{open}s follow from the analyticity properties of the open string amplitudes, so, among the $(M-1)!/2$ A^{open}s in $\mathcal{A}_M^{open}$, there are only $(M-3)!$ number of independent ones [39]. For a convenient set of the $(M-3)!$ basis amplitudes, we may choose $A^{open}(1,\sigma_2\sigma_3....\sigma_{M-2},M-1,M)$, where the first and the last 2 gluon positions are fixed, and the permutations involve the remaining $(M-3)$ gluons sandwiched between the first and the $(M-1)$th gluon. In the zero

Regge slope limit,

$$\lim_{\alpha' \to 0} A^{open}(\sigma_1\sigma_2\sigma_3....\sigma_M) \to A^{tree}(\sigma_1\sigma_2\sigma_3....\sigma_M), \tag{4.16}$$

so $\mathcal{A}_M^{open}$ reduces to the M-gluon amplitude $\mathcal{A}_M$. So it follows that there are only $(M-3)!$ number of independent A^{tree}s [15].

Consider the 4-gluon tree scattering amplitude in the open string case. In the zero slope limit,

$$\begin{aligned} A^{tree}(1234) &= +\frac{n_s}{s} - \frac{n_t}{t} \\ A^{tree}(2134) &= -\frac{n_s}{s} + \frac{n_u}{u} \\ A^{tree}(1324) &= -\frac{n_u}{u} + \frac{n_t}{t}. \end{aligned} \tag{4.17}$$

which are invariant under the transformation (4.6). Their analyticity properties yield the identities among A_4^{open}s. In the zero slope limit, they take the forms [15],

$$A^{tree}(1234) + A^{tree}(2134) + A^{tree}(1324) = 0 \tag{4.18}$$

$$sA^{tree}(2134) = tA^{tree}(1324) \iff n_s + n_u + n_t = 0 \tag{4.19}$$

The first one is obvious; it is the photon decoupling identity [44], or the Kleiss-Kuijf relation for $M = 4$ [41]. The second identity yields the kinematic identity (4.5). Note that the relations among A^{open} leads to relations among the gauge-invariant partial amplitudes A^{tree}. Using only the relation (4.18) and taking the zero-slope limit, we can express the 4-gluon amplitude from open string theory in terms of A^{tree} (4.17),

$$\mathcal{A}_4^{\rm YM} = c_s A^{tree}(1234) - c_u A^{tree}(1324) \tag{4.20}$$

Using the color identity (4.4), we see that this reproduces $\mathcal{A}_4^{\rm YM}$ (4.2), as expected. For general M-gluon amplitudes, the open string amplitudes identities that lead to only $(M-3)!$ independent partial amplitudes [39, 15, 53] among the A^{tree}s should also produce all the kinematic identities (4.9) given above. However, in open string theory, the color properties are in the Chan-Paton factors, so the duality between the color identities and the kinematic identities is not transparent at all.

In the heterotic string theory, on the other hand, there are both compactified dimensions and spacetime dimensions. Discrete momenta in the compactified directions correspond to color, so that the color properties are encoded in the string partial amplitudes. Now the string amplitude identities produce the color identities when we take the momenta in the compactified directions and produce the kinematic identities when we take the momenta in the spacetime directions. So the 2 sets of identities are now on equal footing. The emergence of one assures the emergence of the other. This duality property allows us to write down the kinematic identity (4.9) corresponding to each color identity (4.7). As we shall see, in general, the kinematic identities apply only to the residue part, which is gauge-invariant, but not to the "contact" part. However, we do believe the BCJ conjecture that there always exists a gauge choice such that the kinematic identities (4.8) are true.

A couple of comments are in order. Since we are not concerned with the finiteness of the string loop amplitudes, we do not have to restrict ourselves to 10 spacetime dimensions for superstrings (the right-movers of the heterotic string) or to 26 for bosonic strings (the left-movers of the heterotic string). We shall consider gauge groups other than those with even self-dual lattices. To simplify the discussion, we shall restrict our discussion to simply-laced Lie groups, in particular $U(N)$. A key fact we shall use is that the M-gluon heterotic tree scattering amplitude $\mathcal{A}^{\text{het}}_{\text{M-gluon}}$ equals the Yang-Mills M-gluon tree scattering amplitude in the zero slope limit.

Note that the spectrum in the Type I open string model is very different from that in the heterotic string model. However, both reproduce the M-gluon amplitude $\mathcal{A}^{\text{YM}}_M$ in the zero slope limit. So these 2 sets of stringy properties provide different relations for and insights into $\mathcal{A}^{\text{YM}}_M$.

To get a flavor of the properties of $\mathcal{A}^{\text{het}}(\alpha' = 0)$ from the heterotic string perspective, let us consider the 4-point tree amplitude. The heterotic string theory is a closed string model [34][35]. The KLT relation allows us to write the closed string amplitudes in terms of a sum of products of open string amplitudes [39]. Since there is only one $((M-3)! = 1)$ independent open string partial amplitude for $M = 4$, the 4-point amplitude $\mathcal{A}^{\text{het}}_4$ is simply a product of a 4-point open string amplitude A^L for the left-movers (the bosonic string) and an appropriate 4-point open string amplitude A^R for the right-movers (the superstring) multiplied by an appropriate sine factor. In the zero-slope limit (that is, keeping the lowest order in the α' expansion), the sine factor reduces to a Mandelstam variable that removes the double poles present in the product, leaving only single pole terms. More explicitly, we have the 3 left-moving partial amplitudes,

$$\begin{aligned}
A^L_{1234} &= +\frac{n^L_s}{s} - \frac{n^L_t}{t} \\
A^L_{2134} &= -\frac{n^L_s}{s} + \frac{n^L_u}{u} \\
A^L_{1324} &= -\frac{n^L_u}{u} + \frac{n^L_t}{t}.
\end{aligned} \tag{4.21}$$

and the 3 right-moving partial amplitudes,

$$\begin{aligned} A^R_{1234} &= +\frac{n^R_s}{s} - \frac{n^R_t}{t} \\ A^R_{2134} &= -\frac{n^R_s}{s} + \frac{n^R_u}{u} \\ A^R_{1324} &= -\frac{n^R_u}{u} + \frac{n^R_t}{t}. \end{aligned} \tag{4.22}$$

For $i = s, t, u$, $\mathcal{A}^{\rm het}_4(0)$ becomes

(1) the 4-gluon scattering amplitude $\mathcal{A}^{YM}_4$ if $n^L_j = c_j$ are the color factors and $n^R_j = n_j(k_i, \zeta_i)$ are the kinematic factors in (4.1). In this case, the A^Ls are simply the partial amplitudes in the scattering of 4 colored (in adjoint representation) massless scalar particles with only cubic couplings;

(2) the 4-graviton scattering amplitude, if $n^L_j = n_j(k_i, \xi_i)$ and $n^R_j = n_j(k_i, \zeta_i)$ so the graviton polarization $\epsilon_{\mu\nu}$ is the traceless symmetric part of the product $\xi_\mu \zeta_\nu$.

The open string amplitude identity (4.19) then yields

$$n^L_s + n^L_t + n^L_u = 0, \qquad n^R_s + n^R_t + n^R_u = 0 \tag{4.23}$$

The KLT relation tells us that there are 6 equivalent ways to write the full 4-point tree scattering amplitude, using $s + t + u = 0$,

$$\begin{aligned} \mathcal{A}^{\rm het}_4(0) &= -sA^L_{1234}A^R_{2134}\left(= \frac{n^L_s n^R_s}{s} - \frac{n^L_t(n^R_s + n^R_u)}{t} - \frac{(n^L_s + n^L_t)n^R_u}{u}\right) \\ &= -sA^L_{2134}A^R_{1234}\left(= \frac{n^L_s n^R_s}{s} - \frac{(n^L_s + n^L_u)n^R_t}{t} - \frac{n^L_u(n^R_s + n^R_t)}{u}\right) \\ &= -tA^L_{1234}A^R_{1324}\left(= -\frac{n^L_s(n^R_t + n^R_u)}{s} + \frac{n^L_t n^R_t}{t} - \frac{(n^L_s + n^L_t)n^R_u}{u}\right) \\ &= -tA^L_{1324}A^R_{1234}\left(= -\frac{(n^L_t + n^L_u)n^R_s}{s} + \frac{n^L_t n^R_t}{t} - \frac{n^L_u(n^R_s + n^R_t)}{u}\right) \\ &= -uA^L_{2134}A^R_{1324}\left(= -\frac{n^L_s(n^R_t + n^R_u)}{s} - \frac{(n^L_s + n^L_u)n^R_t}{t} + \frac{n^L_u n^R_u}{u}\right) \\ &= -uA^L_{1324}A^R_{2134}\left(= -\frac{(n^L_t + n^L_u)n^R_s}{s} - \frac{n^L_t(n^R_s + n^R_u)}{t} + \frac{n^L_u n^R_u}{u}\right) \\ &= \frac{n^L_s n^R_s}{s} + \frac{n^L_u n^R_u}{u} + \frac{n^L_t n^R_t}{t} \end{aligned} \tag{4.24}$$

where the identities (4.23) are used. This is the 4-gluon amplitude $\mathcal{A}_4^{YM}$ (4.1) for $n_j^L = c_j$ and $n_j^R = n_j(k_i, \zeta_i)$, $j = s, t, u$. Note that the way $\mathcal{A}_4^{YM}$ (4.1) is reproduced in the heterotic string approach (4.24) is very different from that in the open string approach (4.20).

Here, the specific functions $n_j(k_i, \zeta_i)$ can be extracted from the string theory amplitudes. Alternatively, we can start without knowing the identities (4.23). Demanding that the 6 ways to express $\mathcal{A}_4^{\text{het}}(0)$ (4.24) be equal now yields both the identities (4.23) and the diagonal form (4.2). The parallel (or dual) property between the left and the right movers is also clear.

More generally, open string amplitude identities provide relations among the $(M-1)!/2$ A_M^L so there are only $(M-3)!$ independent A_M^Ls (similarly for A^Rs). It is the freedom in choosing the set of independent partial amplitudes that allow us to express the heterotic M-point scattering amplitudes $\mathcal{A}_{\text{M}}^{\text{het}}$ in different but equivalent ways. In general, we may obtain the identities (4.7) and (4.8) if we use the relation (4.11) and compare (the many equivalent ways of expressing) $\mathcal{A}_{\text{M-gluon}}^{\text{het}}(0)$ to $\mathcal{A}_M^{\text{YM}}$ (4.1) directly. Notice that the identities are separate for left-movers and right-movers, that is, the left identity follows from the (left-moving) open string amplitude identities and the right identity follows from the (right-moving) open string amplitude identities.

It is important to note that the open string amplitude identities do not depend on the explicit forms of n_j^L and n_j^R. Choosing spacetime momenta instead of internal discrete momenta, the left (bosonic) amplitudes describe the scattering of massless vector particles, so the same set of (left-moving) open string identities yields the corresponding kinematic identity (4.8). Since the right (superstring) amplitudes also describe the scattering of massless vector particles, we have, in the zero slope limit, the same functional forms for n_j^R and n_j^L,

$$n_j^R = n_j^L = n_j \tag{4.25}$$

Now, the open string amplitude identities do not care about the explicit form of the kinematic factor n_j^L or n_j^R, so we can generalize the corresponding set of relations (4.9) or (4.7,4.8) to

$$n_j^R + n_l^R + n_k^R = 0, \qquad n_j^L + n_l^L + n_k^L = 0 \tag{4.26}$$

where at least the residue parts must hold. In general, each set of these identities are not necessarily independent, so we are free to select a subset of them as an independent set.

Although the open string identities are among the gauge-invariant partial amplitudes, these kinematic identities (4.26) (for $M > 4$) are gauge-dependent (that is, they are true only in specific gauges). This suggests that the open string amplitude identities among the gauge-invariant partial amplitudes may be more useful in general. In some applications, the knowledge of the existence of the kinematic identities is already sufficient. Although we are not able to prove this part of the BCJ conjecture, we do believe that there always exists a gauge choice where the complete set of kinematic identities (4.26) are exact.

In summary, the open string amplitude identities hold for general n_j^L and separately for n_j^R. Using Eq.(4.11), we see that, in the zero slope limit, $\mathcal{A}_{\mathrm{M}}^{\mathrm{het}}$ has the diagonal form for $n_i^L = c_j$ and $n_j^R = n_j$,

$$\mathcal{A}_{\mathrm{M}}^{\mathrm{het}}(0) = \sum_j \frac{n_j^L n_j^R}{P_j} \tag{4.27}$$

So, given the M-gluon amplitude $\mathcal{A}_M^{\mathrm{YM}}$ (4.1), the M-graviton amplitude A_M^{grav} can be written down immediately by replacing c_j by $n_j(k_i, \xi_i)$ (more accurately,

$c_j \to \alpha' n_j(k_i, \xi_i)$ and keeping the lowest order in α'), where ξ_i^μ are a new set of polarizations ($\xi_i \cdot k_i = 0$),

$$\mathcal{A}_M^{grav}(k_i, \epsilon_i) = \sum_j \frac{n_j(k_i, \xi_i) n_j(k_i, \zeta_i)}{P_j} \tag{4.28}$$

where the graviton tensor polarizations $\epsilon_i^{\mu\nu}$ is given by the $\mu\nu$-symmetrized product $\xi_i^\mu \zeta_i^\nu$. This form of $\mathcal{A}_M^{grav}$ is also conjectured by BCJ. We can also incorporate massless fermions f_i into the right movers, so that $n_j^R = n_j(k_i, \psi, \zeta_i)$ describes the fermion-vector particle scatterings $f + g \to f + g + ...$ and its cross channels. With $n_j^L = c_j$, $\mathcal{A}_{\rm M}^{\rm het}$ in the zero slope limit now becomes
(1) the gluon scattering ($n_j^R = n_j(k_i, \psi, \zeta_i)$) with gluinos;
(2) the graviton-gravitino scattering amplitude when $n_j^L = n_j(k_i, \zeta_i)$ and $n_j^R = n_j(k_i, \psi, \zeta_i)$.

Further generalization to identities in tree scattering amplitudes involving both gluon and gravitons as well as fermions and gravitinos is straightforward.

The rest of the chapter is organized as follows. Section 2 discusses properties of the heterotic string amplitudes in the zero slope limit that are relevant for understanding the identities and the duality property. The gauge choice issue and open string amplitude properties are also reviewed. In Section 3, we focus on the 4-gluon amplitude. In Section 4, we discuss general M-gluon amplitude, and we illustrate the issues with the 5-gluon amplitude in Section 5. Since some of the subtle issues appear only for $M > 4$, the reader may prefer to read parts of the discussion on the $M = 5$ case before the general M case in Section 4. Section 6 contains the discussion on the relation between BCJ identity and Schouten identiy.

4.2 Yang-Mills, Heterotic and Open String Scattering Tree Amplitudes

The heterotic string theory [34][35] is a closed string model that contains the bosonic string in the spacetime $R^{1,D-1} \times \Gamma^N$ as the left-moving part, and the superstring in the spacetime $R^{1,D-1}$ as the right-moving part. Here the internal discrete momenta span the N-dimensional torus Γ^N to form a lattice λ^N. Loop finiteness (modular invariance) requires $D = 10$ and $N = 16$ with an even self-dual lattice. Since we are not concerned with this important stringy property, we can choose other values of D, say $D = 4$ here, and λ^N for $U(N)$ or $SO(2N)$.

On the massless level, the left-movers contain the vector modes and the color modes, where the color modes contain either the discrete momenta K^I's, which correspond to the roots of the Lie algebra, or the polarizations ζ^I in the lattice λ^N, which correspond to vectors in the Cartan subalgebra. The right-movers contain the vector modes and the spinor modes. We shall use the superscript (v), (c), (s) to denote the vector, color and spinor sectors.

The gluons in the heterotic string are the product of a left-moving color mode and a right-moving vector mode, i.e., (color)×(vector). We shall use the fact that, in the zero slope limit, the M-gluon tree heterotic scattering amplitudes $\mathcal{A}^{\text{het}}_{\text{M-gluon}}$ equals the M-gluon amplitude in Yang-Mills theory: $\lim_{\alpha' \to 0} \mathcal{A}^{\text{het}}_{\text{M-gluon}} = \mathcal{A}^{\text{YM}}_M$.

Recall that $\mathcal{A}^{\text{YM}}_M$ is gauge-invariant. Let us take a closer look at this issue. Consider the terms inside the M-gluon amplitude (1.1) that have $(M-4)$ common channels (poles), with $\hat{P}$ as their product. It is easy to convince oneself that there are 3 and only 3 such terms for each choice of $\hat{P}$, so

$$P_j = \hat{P} s_j, \qquad P_k = \hat{P} s_k, \qquad P_l = \hat{P} s_l \tag{4.29}$$

where s_j, s_k and s_l label the the remaining pole in the c_j, the c_k and the c_l term respectively. As will be shown later, the corresponding color factors satisfy the color identity $c_j + c_k + c_l = 0$.

Now, under the gauge transformation

$$\begin{aligned} n_j &\rightarrow n'_j = n_j + \eta s_j \\ n_k &\rightarrow n'_k = n_k + \eta s_k \\ n_l &\rightarrow n'_l = n_l + \eta s_l \end{aligned} \tag{4.30}$$

where η is a local function of k_i and ζ_i, we have

$$\begin{aligned} \mathcal{A}_M^{\text{YM}} &= \frac{c_j n_j}{P_j} + \frac{c_k n_k}{P_k} + \frac{c_l n_l}{P_l} + \text{rest} = \frac{c_j n_j}{\hat{P} s_j} + \frac{c_k n_k}{\hat{P} s_k} + \frac{c_l n_l}{\hat{P} s_l} + \text{rest} \\ &\rightarrow \mathcal{A}_M^{\prime\text{YM}} = \frac{c_j n'_j}{\hat{P} s_j} + \frac{c_k n'_k}{\hat{P} s_k} + \frac{c_l n'_l}{\hat{P} s_l} + \text{rest} \\ &= \mathcal{A}_M^{\text{YM}} + \frac{\eta}{\hat{P}} \left(c_j + c_k + c_l \right) = \mathcal{A}_M^{\text{YM}} \end{aligned} \tag{4.31}$$

So we see that $\mathcal{A}_M^{\text{YM}}$ is invariant under this transformation. A general gauge transformation of interest here can be decomposed into $(M-3)(2M-5)!!/3$ (not all independent) transformations, each involving a triplet of terms inside $\mathcal{A}_M^{\text{YM}}$ as in the case just discussed. For the same product $\hat{P}$ of $(M-4)$ poles, either 2 or 0 terms with $\hat{P}$ in the denominator appear in each partial amplitude A^{tree}. For the partial amplitudes with 2 such terms appearing, these 2 terms always appear with opposite signs (in the sign convention where $c_j + c_k + c_l = 0$) so that the gauge terms $\propto \eta$ cancel. So A^{tree} is also gauge-invariant, as it should be.

An M-point L-loop heterotic string amplitude has only one closed string diagram. The KLT relation [39] shows that the heterotic string tree scattering amplitude can be written as a sum of terms, each of which is a product of a left-moving tree scattering amplitude, a right-moving tree scattering amplitude and a factor involving only momentum invariants. These left and right tree amplitudes can be expressed as open string amplitudes.

A typical M-point open string tree ordered amplitude is an integral with M Koba-Nielsen variables x_i. Mobius invariance allows us to fix any 3 of them, say $x_1 = 0$, $x_{M-1} = 1$ and $x_M = \infty$. So the ordered $A(1...M)$ takes the form (up to an x-independent factor in front)

$$A(1...M) = \int_0^1 \Pi_{i=2}^{M-2} dx_i \Theta(x_{i+1} - x_i) \Pi_{M>j>i\geq 1} (x_j - x_i)^{\alpha' k_i \cdot k_j/2 + m_{ji}} \tag{4.32}$$

where m_{ji} are integers. Extending any one of the variables from $-\infty$ to $+\infty$ and closing the the contour leads to a vanishing integral. For example, extending the integration of x_2 to $(-\infty, +\infty)$ and closing its contour, we have [49]

$$\int_{-\infty}^{\infty} dx_2 \int_0^1 \Pi_{i=3}^{M-2} dx_i \Theta(x_{i+1} - x_i) \Pi_{M>j>i\geq 1} (x_j - x_i)^{\alpha' k_i \cdot k_j/2 + m_{ji}} = 0 \tag{4.33}$$

Now we can break this x_2 integral into ordered pieces: $-\infty \to 0$, $0 \to x_3$, $x_3 \to x_4$, ... , and $1 \to \infty$. Up to a phase, each equals a different ordered open string amplitude. This way, we obtain a relation among the set of A^{open}'s. Extending the other x_i from $-\infty$ to $+\infty$ on other ordered amplitudes yields additional identities, not all of them are independent. As a result of these identities, there are only $(M-3)!$ number of independent ordered open string partial amplitudes A^{open}'s. For a convenient set of the basis amplitudes, we may choose $A(1, \sigma_2, \sigma_3, ..., \sigma_{M-2}, M-1, M)$, where the first and the last 2 particle positions are fixed, and the permutations involve the remaining $(M-3)$ particles sandwiched between the first and the $(M-1)$th ones [39].

In the zero slope limit, the phases drop out in the real part of the integral mentioned above so it yields a relation among the $(M-1)$ ordered amplitudes

[15],

$$A(213...(M-1)M)+A(123...(M-1)M)+A(132...(M-1)M)+...+A(13...(M-1)2M) = 0 \tag{4.34}$$

This and similar relations (real-sid, or the real parts of the open string identities) are known as the Kleiss-Kuijf relations [41]. These real-SID can be used to reduce the number of amplitudes to a smaller set with $(M-2)!$ A^{tree}s. This allows one to simplify the sum (4.35) into $(M-2)!$ terms [23],

$$\mathcal{A}_M = g^{M-2} \sum_{\sigma \in S^{M-2}} \tilde{f}^{a_1 a_{\sigma_2} x_1} \tilde{f}^{x_1 a_{\sigma_3} x_2} ... \tilde{f}^{x_{M-3} a_{\sigma_{M-1}} a_M} A^{tree}(\sigma_1 \sigma_2 \sigma_3 .... \sigma_M) \tag{4.35}$$

where S^{M-2} is the permutation group for $(2, ...M-1)$. Using the Jacobi identity repeatedly, one can show that it is equivalent to (4.1).

The imaginary part of the above integral yields another identity (im-SID) among all of them except $A(123...(M-1)M)$ which is real to start with. This yields [15],

$$(k_2 \cdot k_1) A(213...(M-1)M) - \sum_{i=3}^{M-1} \Big(\sum_{j=3}^{i} k_2 \cdot k_j \Big) A(13...i, 2, (i+1)...(M-1)M) = 0 \tag{4.36}$$

Extending the other x_i from $-\infty$ to $+\infty$ on other ordered amplitudes yields additional identities, not all them independent. These identities are among gauge-invariant A^{tree}s and so are gauge invariant themselves. As a result of these identities, there are now only $(M-3)!$ number of independent A^{tree}s. They form a set of basis amplitudes.

The open string identities from the contour integral of analytic expressions hold for both the left- and the right-moving parts. The residues of the left-moving string identities for the discrete momenta (color factor) will yield the color identities (4.7). The right-moving string identities for the partial amplitudes have exactly the same form as the left-movers. If we decompose the right-moving partial amplitudes into channels, with numerators n_j's, then the right-moving open string identities just give the identities for n_j (even when the n_i's are not gauge invariant). In particular, this leads to a set of kinematic identities (4.9). This is summarized in Table 1.

	Momenta	String identity	Comment
Left: $c_i(K^I, \zeta^I)$	discrete momenta	$c_i + c_j + c_k = 0$	color id.
Right: $n_i(k, \zeta)$	spacetime momenta	$n_i + n_j + n_k = 0$	kinematic id.

Table 4.1: Identities inside the M-gluon tree scattering amplitudes

The open string amplitude identities do not depend on the details of the numerator factors in the channel decomposition of the partial amplitudes. They can be the color factors c_j or the kinematic factors n_j. When applied to the left-movers, the open string amplitude identities yield the color identities when applied to the internal dimensions, λ^h, and yields the kinematic identities when applied to the spacetime dimensions. This one-to-one identity enables us to use the Jacobi identity to locate the color identity and hence the corresponding kinematic identities. Heterotic string also contains the graviton sector, which has both the left-moving and right-moving momenta non-compact and in the spacetime $R^{1,D-1}$. The graviton scattering amplitude can also be calculated by the KLT relation for the heterotic string, and the scheme is summarized as following:

Here $n_i(k, \xi)$ is simply $n_i(k, \zeta)$ with the polarizations ζ_i replaced by a new set of polarizations ξ_i. Note that there are 2 sets of dual pairs here:
(1) the c_j's and the n_j's in Table 1, which is present within YM amplitudes and
(2) the c_j's in Table 1 and the n_j's in the left-moving sector in Table 2.

	Momenta	String identity	Comment
Left: $n_i(k,\xi)$	spacetime momenta	$n_i + n_j + n_k = 0$	kinematic id.
Right: $n_i(k,\zeta)$	spacetime momenta	$n_i + n_j + n_k = 0$	kinematic id.

Table 4.2: Identities inside the M-graviton scattering amplitudes

So, if we replace the left-moving amplitude with discrete momenta and polarizations inside the lattice λ^N for the Lie algebra (say $SU(N)$) by the left-moving amplitude with spacetime momenta and polarizations, we convert the M-gluon scattering amplitude into the M-graviton scattering amplitude (up to a factor of α'^{M-3}),

$$\mathcal{A}_M^{\text{YM}} = \sum_i \frac{c_i n_i}{P_i} \iff \mathcal{A}_{M\text{-graviton}}^{grav} = \sum_i \frac{n_i(k,\xi) n_i(k,\zeta)}{P_i} \tag{4.37}$$

Now there are $(M-3)!$ independent left-moving partial amplitudes and $(M-3)!$ independent right-moving partial amplitudes. Since a heterotic string amplitude is a sum over the product of a left- and a right-moving amplitude, we can express it as a sum over $[(M-3)!]^2$ terms of a left-moving basis amplitude times a right-moving basis amplitude. However, a judicious choice of basis amplitudes can reduce the number of terms in the sum, especially when M is large. The resulting smallest number of terms known is given in Table 3, which also gives a summary of the counting of n_j's and other relevant quantities as well. The counting of c_j's is exactly the same as that for the n_j's. Taking the $(M-3)!$ independent $A^R = A^{tree}$ as the set of basis amplitudes, we can interpret the KLT formula for $\mathcal{A}_{\text{M}}^{\text{het}}(0) = \mathcal{A}_M^{\text{YM}}$ as expressing $\mathcal{A}_M^{\text{YM}}$ as the linear combination of the $(M-3)!$ basis amplitudes A^{tree}s.

# external gluons	M	4	5	6	7
# channels in $\mathcal{A}_M^{\rm YM}$ = # n_j	$(2M-5)!!$	3	15	105	945
# partial amplitudes A^{tree}	$(M-1)!/2$	3	12	60	360
# channels in each A^{tree}	$2^{M-2}(2M-5)!!/(M-1)!$	2	5	14	42
# independent im-SID	$(M-3)!(M-3)$	1	4	18	96
# A^{tree} in a real-SID	$M-1$	3	4	5	6
# A^{tree} in an im-SID	$M-2$	2	3	4	5
# triplets in each im-SID	$2^{M-3}(M-3)(2M-7)!!/(M-2)!$	1	4	15	56
# identities among n_j	$(M-3)(2M-5)!!/3$	1	10	105	1260
# indep. kin. identities	$(2M-5)!! - (M-2)!$	1	9	81	825
# independent n_j	$(M-2)!$	2	6	24	120
# basis A^{tree}s	$(M-3)!$	1	2	6	24
# of terms in the KLT relation	$(M-3)![\frac{1}{2}(M-3)]![\frac{1}{2}(M-3)]!$, M odd $(M-3)![\frac{1}{2}(M-4)]![\frac{1}{2}(M-2)]!$, M even	1	2	12	96

Table 4.3: Summary of the counting of the kinematic factors n_j or equivalently the color factors c_j. Note that the number of identities among the n_j's are not all independent. Here, real-SID refers to the real part of an open string amplitude identities (equivalent to the Kleiss-Kuijf relations) and im-SID refers to the imaginary part of an open string amplitude identities [15]. The number of A^{tree}'s refers to the number before the real-SID and the im-SID. Some entries are already given in Ref.[9, 39, 15].

4.3 The 4-Gluon Tree Amplitude from the Heterotic String Model

As an illustration, we consider the 4-gluon tree scattering amplitudes in heterotic string model. This is a long path in obtaining the color identity as well as the kinematic identity. However, its generalization to M-point is straightforward once we see the underlying properties.

Here, we shall take the following steps to prove the BCJ conjecture for $\mathcal{A}_4^{\text{YM}}$:

- We show that the color factor c_j's emerge as the residue of the different channels in the left-moving amplitude in the color sector. Using the contour integral for these left-moving open string amplitudes, we prove the color identity (4.4) for the c's.
- When applied to the right-moving open string amplitudes for the vector sector, the same contour integral argument yields the kinematic identity (4.5). In this manner, the kinematic identity (4.5) is dual to the color identity (4.4).
- Finally, the KLT relation is used to construct the complete 4-gluon amplitude and show its decomposition (4.2). Here, the duality between c_j and n_j are manifest.

Another way to see the duality property is to replace the c_j's in the left-movers by the n_j when we go from the compactified space to spacetime. This yields the 4-graviton scattering amplitude.

4.3.1 Left-moving amplitudes

The left-moving amplitude can be thought as open-string amplitudes with four vertex operator with either compact momentum K^I or the Cartan sub-Lie algebra vector ζ^I instead of the polarization ζ^μ. It is straightforward to write out the amplitudes for different orderings of the vertex operators,

$$\begin{aligned}\mathbf{A}^{L(c)}_{2134} &= i^2 \cdot co(2134) \cdot \left(-\frac{\alpha'}{4}\right) \\ &\cdot \int_{-\infty}^{0} dx_2\, (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2) \qquad (4.38)\\ \mathbf{A}^{L(c)}_{1234} &= i^2 \cdot co(1234) \cdot \left(-\frac{\alpha'}{4}\right) \\ &\cdot \int_{0}^{1} dx_2\, x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2) \qquad (4.39)\\ \mathbf{A}^{L(c)}_{1324} &= i^2 \cdot co(1324) \cdot \left(-\frac{\alpha'}{4}\right) \\ &\cdot \int_{1}^{\infty} dx_2\, x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2) \qquad (4.40)\end{aligned}$$

The factor i^2 comes from the vertex normalization, while for general M-point scattering amplitude, it would be i^{M-2}. The normalization factor $(-\frac{\alpha'}{4})$ is included to obtain the correct normalization for the color factors, which are dimensionless, cancel the undeserved overall factors of the c's. The coefficients $co(2134)$ etc. are cocycles for the root lattice,

$$co(2134) = (-1)^{K_2\star K_2+K_3\star K_1+K_4\star K_3+K_4\star K_1} \qquad (4.41)$$

$$co(1234) = (-1)^{K_1\star K_1+K_3\star K_2+K_4\star K_3+K_4\star K_2} \qquad (4.42)$$

$$co(1324) = (-1)^{K_1\star K_1+K_2\star K_3+K_4\star K_3+K_4\star K_2} \qquad (4.43)$$

The function $f(x_2)$ contains ζ^I, the "polarization" in the Cartan subalgebra in the color lattice,

$$f(x_2) = \exp\left(\sum_{i>j}\frac{\zeta_i^I\zeta_j^I}{(x_i-x_j)^2} - \sum_{i\neq j}\frac{\zeta_i\cdot K_j}{(x_i-x_j)}\right)\Bigg|_{\text{multiple-linear}}, \qquad (4.44)$$

where only the multi-linear terms in ζ_i^I's are kept. We have already shifted α' to $\alpha'/4$ in order to use the KLT relation later. However, the discrete momentum K^I just appears on the left-moving amplitude, so the exponent like $2\alpha' K_1\cdot K_2$ is not changed by this shift and we can just set $\alpha' = 1/2$ for this product in calculations here.

The 3 amplitudes (3.1)-(4.40) are related since we can consider the contour integral,

$$0 = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dx_2 \, x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2). \tag{4.45}$$

In terms of the string amplitudes, this reads,

$$\begin{aligned} 0 = {} & (-1)^{K_1\cdot K_2} e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)} \cdot co(2134)\mathbf{A}^{L(c)}_{2134} + co(1234)\mathbf{A}^{L(c)}_{1234} \\ & + (-1)^{K_2\cdot K_3} e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)} co(1324)\mathbf{A}^{L(c)}_{1324} \end{aligned} \tag{4.46}$$

However, it is easy to check that,

$$(-1)^{K_1\cdot K_2} co(2134) = co(1234) = (-1)^{K_2\cdot K_3} co(1324). \tag{4.47}$$

Therefore we get the string identity,

$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}^{L(c)}_{2134} + \mathbf{A}^{L(c)}_{1234} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}^{L(c)}_{1324} = 0. \tag{4.48}$$

In the low energy limit, we have $\mathbf{A}^{L(c)}_{1234}\big|_{\alpha'\to 0} \equiv A^{L(c)}_{1234}$ etc. Only the massless poles survive in this limit, so we have

$$\begin{aligned} A^{L(c)}_{2134} &= -\frac{\tilde{c}_s}{s} + \frac{c_u}{u} \\ A^{L(c)}_{1234} &= \frac{c_s}{s} - \frac{\tilde{c}_t}{t} \\ A^{L(c)}_{1324} &= -\frac{\tilde{c}_u}{u} + \frac{c_t}{t}. \end{aligned} \tag{4.49}$$

The lowest order of (4.48)'s real part (real-SID) yields [1]

$$A^{L(c)}_{2134} + A^{L(c)}_{1234} + A^{L(c)}_{1324} = 0 \tag{4.50}$$

which simplifies to the relations of the c_i coefficients,

$$\tilde{c}_s = c_s, \quad \tilde{c}_u = c_u, \quad \tilde{c}_t = c_t. \tag{4.51}$$

[1]Here the real (imaginary) part means that we choose the real (imaginary) part of the phases $e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}$. Because the left-moving partial amplitudes $A^{L(c)}$ are either pure real or pure imaginary for fixed M, this separation of the phases is valid.

Furthermore, the lowest order of the imaginary part (im-SID) of (4.48)'s gives,

$$sA^{L(c)}_{2134} = tA^{L(c)}_{1324} \tag{4.52}$$

which reduces to the Jacobi identity, $c_s + c_t + c_u = 0$ (4.4).

In Appendix B, we explicitly see that $c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$, $c_u = \tilde{f}^{a_3 a_1 b}\tilde{f}^{b a_2 a_4}$, and $c_t = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_1 a_4}$. So these c's defined in (4.49) are the same as that in [9]. Hence we see that the left-moving amplitude gives the color factors c's and the Jacobi identity (4.4) they satisfy. We readily admit that this is a complicated way to obtain a very simple result. The payoff is in the parallel derivations of the color identity and the kinematic identity, to which we now turn.

4.3.2 Explicit determination of the color factors for the 4-point amplitude

Because Eq.(4.24) gives the correct Yang-Mills 4-gluon scattering amplitude, we know that the factor c_s must contain the color index like $\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$. In this appendix, we explicit calculate the c's and hence check Eq.(4.3). The pattern for general M should be clear.

The emergence of the Lie group G in the heterotic string bosonic construction is interesting. To take advantage of the discrete momenta, we have to distinguish the generators in the Cartan-sub Lie algebra and those corresponding to vectors in the root lattice. The calculation of the color factors c's in the 2 cases are different. However, as expected, the end result puts all the color indices on an equal footing as claimed by Eq.(4.3).

- Four color indices as root vectors. In this case, all the vertex operators contain K^I but no ζ^I. As usual, the Mandelstam variables are defined

$$S = -(K_1 + K_2)^2, U = -(K_1 + K_3)^2, T = -(K_1 + K_4)^2, \tag{4.53}$$

and $S + T + U = -4/\alpha'$. For simplicity, we set $\alpha' = 1/2$ when it is combined with the discrete momenta but still keep α' when it is multiplying the spacetime momentum.

We can write the amplitude in terms of the Beta functions, in the zero slope limit,

$$\begin{aligned} A^{(c)}_{L,1234} &= \frac{\alpha'}{4} co(1234) \cdot B(-\frac{\alpha' s}{4} - \frac{1}{2}S - 1, -\frac{\alpha' t}{4} - \frac{1}{2}T - 1) \\ &\sim co(1234) \cdot -\frac{1}{s}(\delta_{S,-2} + K_1 \cdot K_3 \delta_{S,0}) \\ &\quad + co(1234) \cdot (-1)^{T/2} \frac{1}{t}(\delta_{T,-2} + K_1 \cdot K_2 \delta_{T,0}). \end{aligned} \tag{4.54}$$

Here "$\sim$" means the lowest order in energy, i.e., in $\alpha' s$ etc. Comparing with Eq.(4.49), we have,

$$c_s = -co(1234) \cdot (\delta_{S,-2} + K_1 \cdot K_3 \delta_{S,0}) \tag{4.55}$$

$$c_t = -co(1324) \cdot (\delta_{T,-2} + K_1 \cdot K_2 \delta_{T,0}). \tag{4.56}$$

Similarly, we have,

$$\begin{aligned} A^{(c)}_{L,2134} &= co(2134) \cdot B(-\frac{\alpha' s}{4} - \frac{1}{2}S - 1, -\frac{\alpha' u}{4} - \frac{1}{2}U - 1) \\ &\sim co(2134) \cdot -\frac{1}{s}(\delta_{S,-2} - K_1 \cdot K_3 \delta_{S,0}) \\ &\quad + co(2134) \cdot \frac{1}{u}(\delta_{U,-2} + K_2 \cdot K_3 \delta_{U,0}). \end{aligned} \tag{4.57}$$

Comparing with Eq.(4.49) again,

$$c_s = co(2134) \cdot (\delta_{S,-2} - K_1 \cdot K_3 \delta_{S,0}) \tag{4.58}$$

$$c_u = -co(2134) \cdot (\delta_{U,-2} + K_2 \cdot K_3 \delta_{U,0}). \tag{4.59}$$

Again, we get the factor c_s. Note that since $co(2134)\cdot(-1)^{S/2} = co(1234)$, the two results are identical. This is a consequence of the contour integral argument. Now we can compare them with the commutators. From the normalization convention in Eq.(2.29), (2.30) and (2.31),

$$\tilde{f}^{a_1a_2b}\tilde{f}^{ca_3a_4}g_{bc} = -co(1234)\cdot(\delta_{S,-2}+K_1\cdot K_3\delta_{S,0}) \quad (4.60)$$

$$\tilde{f}^{a_3a_1b}\tilde{f}^{ca_2a_4}g_{bc} = -co(1324)\cdot(\delta_{T,-2}+K_1\cdot K_2\delta_{T,0}) \quad (4.61)$$

$$\tilde{f}^{a_2a_3b}\tilde{f}^{ca_1a_4}g_{bc} = -co(2134)\cdot(\delta_{U,-2}+K_2\cdot K_3\delta_{U,0}). \quad (4.62)$$

Hence in this case, Eq.(4.3) is checked explicitly.

- Three color indices as root vectors. Without loss of generality, we set the first vertex operator has to have its color index in the Cartan subalgebra, i.e., $K_1 = 0$. The string amplitude calculation is straightforward; here we just keep the lowest order in $\alpha' s$ etc,

$$A^{(c)}_{2134} = (-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}\Big(\frac{1}{s}K_2\cdot\zeta_1 - \frac{1}{u}K_3\cdot\zeta_1\Big) \quad (4.63)$$

$$A^{(c)}_{1234} = (-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}\Big(-\frac{1}{s}K_2\cdot\zeta_1 - \frac{1}{t}(K_3+K_2)\cdot\zeta_1\Big) \quad (4.64)$$

$$A^{(c)}_{1324} = (-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}\Big(\frac{1}{t}(K_2+K_3)\cdot\zeta_1 - \frac{1}{u}K_3\cdot\zeta_1\Big) \quad (4.65)$$

Hence we can read the value of c's,

$$c_s = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_2\cdot\zeta_1 \quad (4.66)$$

$$c_u = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_3\cdot\zeta_1 \quad (4.67)$$

$$c_s = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_4\cdot\zeta_1 \quad (4.68)$$

It is clear that $c_s+c_u+c_t = 0$. We can compare the c's with the commutators,

$$\tilde{f}^{a_1a_2b}\tilde{f}^{ca_3a_4}g_{bc} = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_2\cdot\zeta_1 \quad (4.69)$$

$$\tilde{f}^{a_3a_1b}\tilde{f}^{ca_2a_4}g_{bc} = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_4\cdot\zeta_1 \quad (4.70)$$

$$\tilde{f}^{a_2a_3b}\tilde{f}^{ca_1a_4}g_{bc} = -(-1)^{K_4\star K_2+K_4\star K_3+K_3\star K_2}K_3\cdot\zeta_1 \quad (4.71)$$

Again, Eq.(4.3) is checked explicitly.

- Two generators in Cartan subalgebra

For this case, we can set the first and third generator in the Cartan subalgebra, $K_1 = K_3 = 0$. The same calculation gives

$$
\begin{aligned}
A^{(c)}_{2134} &= -\frac{1}{s}(-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \\
A^{(c)}_{1234} &= (-1)^{K_2\star K_2}\left\{\frac{1}{s}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3)+\frac{1}{t}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3)\right\} \\
A^{(c)}_{1324} &= -\frac{1}{t}(-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \qquad (4.72)
\end{aligned}
$$

so,

$$
\begin{aligned}
c_s &= (-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \\
c_s &= -(-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \\
c_u &= 0 \qquad (4.73)
\end{aligned}
$$

It is easy to get

$$\tilde{f}^{a_1a_2b}\tilde{f}^{ca_3a_4}g_{bc} = (-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \qquad (4.74)$$

$$\tilde{f}^{a_3a_1b}\tilde{f}^{ca_2a_4}g_{bc} = -(-1)^{K_2\star K_2}(K_2\cdot\zeta_1)(K_2\cdot\zeta_3) \qquad (4.75)$$

$$\tilde{f}^{a_2a_3b}\tilde{f}^{ca_1a_4}g_{bc} = 0 \qquad (4.76)$$

so Eq.(4.3) is again checked explicitly.

All the other cases are also straightforward. This completes the check of the color properties for 4-point scattering amplitude, that the c_j's are the correct color factors satisfying the Jacobi identity.

4.3.3 Right-moving amplitudes

The right-moving superstring amplitudes are obtained in the same way as the left-movers. Here we introduce the gluon polarizations ζ_i^μ and continuous momenta k_i^μ,

$$\mathbf{A}^{R(v)}_{2134} = \frac{8i}{\alpha'^2}\Big(\frac{1}{\sqrt{2}}\Big)^2 \int_{-\infty}^{0} dx_2\, (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}\,\bar{f}(x_2) \tag{4.77}$$

$$\mathbf{A}^{R(v)}_{1234} = \frac{8i}{\alpha'^2}\Big(\frac{1}{\sqrt{2}}\Big)^2 \int_{0}^{1} dx_2\, x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}\,\bar{f}(x_2) \tag{4.78}$$

$$\mathbf{A}^{R(v)}_{1324} = \frac{8i}{\alpha'^2}\Big(\frac{1}{\sqrt{2}}\Big)^2 \int_{1}^{\infty} dx_2\, x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3}\,\bar{f}(x_2) \tag{4.79}$$

where the $\bar{f}(x_2)$ contains the polarizations,

$$\bar{f}(x_2) = \exp\Big(\frac{\alpha'}{2}\sum_{i>j}\frac{\zeta_i\cdot\zeta_j}{(x_i-x_j)^2} - \frac{\alpha'}{2}\sum_{i\neq j}\frac{\zeta_i\cdot k_j}{x_i-x_j}\Big)\Big|_{\text{multiple-linear}}. \tag{4.80}$$

and we set $x_1 = 0$, $x_3 = 1$ and $x_4 = \infty$. The overall factor $8i/\alpha'^2$ comes from the sphere amplitude normalization which is the same for arbitrary M-point scattering amplitude. As the left-moving amplitude, we already replaced the α' in open string amplitude, by $\alpha'/4$, to match the close string decomposition. The factor $(1/\sqrt{2})^2$ comes from the commutator convention $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$, and exponent "2" in $(1/\sqrt{2})^2$ really means $M-2$ in the general case.

By the same argument for the left-moving part, we have the identity,

$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}^{R(v)}_{2134} + \mathbf{A}^{R(v)}_{1234} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}^{R(v)}_{1324} = 0. \tag{4.81}$$

The right hand amplitude like $A^{(v)}_{1234}$ etc. is just the open string amplitude A^{open}_{1234} with α' replaced by $\alpha'/4$. However, in the zero slope limit $\alpha' \to 0$, the open string amplitudes reduce to the Yang-Mills color-ordered partial amplitudes, which have no dependence on α'. Therefore, in the same limit,

$$\lim_{\alpha'\to 0}\mathbf{A}^{R(v)}_{1234} \equiv A^{R(v)}_{1234} = A^{\text{tree}}(1234) \tag{4.82}$$

etc., because the right moving amplitudes have the same forms as the open string amplitudes whose zero slope limit are the Yang-Mills color-ordered partial amplitudes. The lowest order in α' of (4.81) will determine the identities of the partial amplitude.

$$A^{R(v)}_{2134} + A^{R(v)}_{1234} + A^{R(v)}_{1324} = 0 \tag{4.83}$$

$$sA^{R(v)}_{2134} = tA^{R(v)}_{1324} \tag{4.84}$$

which was obtained in [15]. Note that these identities involve only the gauge invariant partial amplitudes. We can decompose the partial amplitudes A's into different channels to obtain

$$\begin{aligned} A^{R(v)}_{2134} &= -\frac{n_s}{s} + \frac{n_u}{u} \\ A^{R(v)}_{1234} &= \frac{n_s}{s} - \frac{n_t}{t} \\ A^{R(v)}_{1324} &= -\frac{n_u}{u} + \frac{n_t}{t}. \end{aligned} \tag{4.85}$$

so Eq.(4.84) yields the kinematic identity $n_s + n_t + n_u = 0$ (4.5). which is dual to the Jacobi identity $c_s + c_t + c_u = 0$. Although the partial amplitudes are gauge invariant, the n_j are not. However, the kinematic identity (4.5) is also gauge-invariant.

4.3.4 The Yang-Mills amplitude

Finally, we can use the KLT relation to find the 4-gluon string amplitude and its field theory limit. For 4-gluon amplitude, the KLT relation [39] reads,

$$\mathcal{A}^{\text{het}}_{\text{4-gluon}} = -\pi\left(\frac{g}{\pi}\right)^2 \sin\left(\pi\frac{\alpha'}{2}k_2 \cdot k_3\right) \cdot \left(-\frac{4}{\alpha'}\right)\mathbf{A}^{L(c)}_{1234}\mathbf{A}^{R(v)}_{1324}. \tag{4.86}$$

The low energy limit can be obtained by keeping the lowest order in α' in each term,

$$\mathcal{A}^{\text{het}}_{\text{4-gluon}}(0) = -\pi^2\left(\frac{g}{\pi}\right)^2 tA^{L(c)}_{L,1234}A^{R(v)}_{1324}. \tag{4.87}$$

Note that all α' cancel as they should. Using Eq.(4.49) and Eq.(4.85) we obtain

$$\begin{aligned}\mathcal{A}^{\text{het}}_{\text{4-gluon}}(0) &= -\pi^2\left(\frac{g}{\pi}\right)^2 t(\frac{c_s}{s} - \frac{c_t}{t})(-\frac{n_u}{u} + \frac{n_t}{t}) &(4.88)\\ &= g^2\left(\frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}\right), &(4.89)\end{aligned}$$

where the identities (4.4), (4.5) and $s + t + u = 0$ are used. This is the 4-gluon amplitude $\mathcal{A}^{\text{YM}}_4$ (4.2) or (4.24).

4.4 M-gluon Tree Scattering Amplitudes

The above analysis generalizes to the M-gluon amplitudes. However, for $M > 4$, the subtle issue of gauge dependence emerges, complicating the analysis. In this section, we consider the M-gluon heterotic string tree amplitude that yields the corresponding Yang-Mills scattering amplitude,

- The heterotic string left-moving amplitudes for the color sector introduces the c_i's, which are the color factors. By the contour integral argument, c_i's satisfy linear identities, which are shown to be the color (Jacobi) identities (4.7). The discreteness of the internal momenta makes the generalization to general M straightforward.
- The heterotic string right-moving amplitudes for the vector sector introduces the kinematic factors n_j's. The same contour integral arguments yield the identities among the A^{tree}s, and can be expressed into identities among the n_j's. Due to the continuous nature of the spacetime momenta,

the gauge dependence issue needs a more careful treatment. In particular, we obtain a refined version of the BCJ conjecture [9], namely the kinematic identities (4.9).

- By the KLT relations, the complete amplitude is a product of left and right moving-amplitudes, which reproduces the M-gluon amplitude via the relation (4.11).

4.4.1 Left-moving amplitudes

As before, the left-moving amplitude is the scattering amplitude of holomorphic vertex operators with spacetime momenta k_i^μ and color parts (discrete momentum K_i^I or discrete polarization ζ_i^I), but without the spacetime polarization.

For M-point scattering amplitude, up to cyclic symmetry, there are $(M-1)!$ vertex orderings, say, $\mathbf{A}^{L(c)}_{\sigma_1\sigma_2...\sigma_{M-1}M}$, where σ is a permutation of the first $M-1$ vertices. Again, its zero slope limit is $A^{L(c)}_{\sigma_1\sigma_2...\sigma_{M-1}M}$, which will be used for the Yang-Mills scatting amplitude. For the sake of simplicity, we shall use the Greek letters α, β etc. to represent the vertex order like "$\sigma_1\sigma_2...\sigma_{M-1}M$".

Each $A^{L(c)}_\alpha$ contains $2^{M-2}(2M-5)!!/(M-1)!$ channels [9],

$$A^{L(c)}_\alpha = \sum_P \frac{c_{\alpha,P}}{P}, \tag{4.90}$$

where P is a product of $(M-3)$ poles, going through these different $2^{M-2}(2M-5)!!/(M-1)!$ channels. (Note that the number of channels is $C(M-2)$, where $C(n)$ is simply the nth Catalan number.) For example, there are two channels for $A^{L(c)}_{1234}$, i.e., P is s or t.

For M points, the generalization of Eq.(4.3) is,

$$c_{\alpha,P} = \text{Contraction of the } \tilde{f}\text{ s.} \tag{4.91}$$

where the r.h.s. is determined by the rules in the channel decomposition subsection,

- Draw a color-ordered diagram according to the ordering α and the pole P by using the 3-point vertices only.
- For each 3-point vertex, read $\tilde{f}^{abc}$ if (abc) is CCW. For each propagator, read δ^{ab}.

Eq.(4.91) can be proven for general M by the unitarity relation of the tree amplitude and the induction on M. As in the $M = 4$ case, $c_{\alpha,P}$, which corresponds to the lowest order in α' in string amplitude, does not have spacetime momentum dependence. So there is no contact terms in the left-moving amplitude, and the expansion (4.90) is well-defined.

It is obvious that two different vertex-operator orderings, α and β, may contain a common channel and so there exist two factors $c_{\alpha,P}$ and $c_{\beta,P}$ with the same P. An example is the two 6-point diagrams in Fig.4.1 (disc diagrams in open string theory), where $A^{L(c)}_{123456}$ and $A^{L(c)}_{341256}$ contain the common channel which corresponds the same pole $P = -(k_1 + k_2)^2(k_3 + k_4)^2(k_5 + k_6)^2$, which are related by $A^{L(c)}_{123456,P} = -A^{L(c)}_{341256,P}$. In cases like this, we have,

$$c_{\alpha,P} = \pm c_{\beta,P}, \tag{4.92}$$

cases where a minus sign will appear each time when we flip the two legs of an internal vertex. The color factors satisfy the color identities which always involve 3 c's. This can be verified either by the explicit formula Eq.(4.91) and the Jacobi identity or the contour integral argument similar to the 4-point case, as we shall explain now.

4.4.2 Contour integral method and the color (Jacobi) identities

As in the $M = 4$ case, we can use the contour integral argument on the left-moving open string tree amplitudes to prove Eq.(4.92) and also all the identities for the color factors.

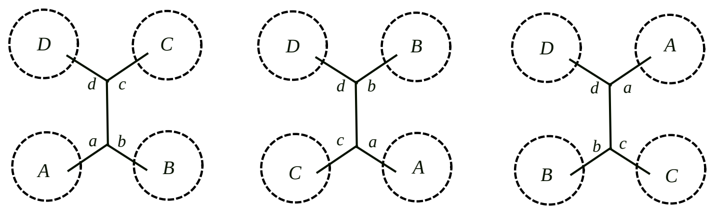
The M-point open string tree amplitude involves $M - 3$ integrals over the Koba-Nielsen variables (vertex operators' positions along the real axis), so there are many ways to use the contour integral argument. Here we just show a particular way which gives all the color (Jacobi) identities.

A general color identity for the color factors, which appears in the tree level amplitude, corresponds to the "s","t","u" channels of four sub-diagrams connected by an internal line. See the diagrams in Fig. 4.2, where A,B,C,D are four sub-diagrams. [A],[B],[C],[D] are the color factors of the correspondent sub-diagrams, while a, b, c, d are the "output" color indices of each diagram. The sum of the three diagram's color factors vanishes,

$$[A][B][C][D](\tilde{f}^{abe}\tilde{f}^{ecd} + \tilde{f}^{bce}\tilde{f}^{ead} + \tilde{f}^{cae}\tilde{f}^{ebd}) = 0 \tag{4.93}$$

because of the Jacobi identity.

Without loss of generality, we use the following notations for the vertex labels, orderings and poles,



$$[A][B][C][D]\,\tilde{f}^{abe}\,\tilde{f}^{ecd} \;+\; [A][B][C][D]\,\tilde{f}^{cae}\,\tilde{f}^{ebd} \;+\; [A][B][C][D]\,\tilde{f}^{bce}\,\tilde{f}^{ead} \;=\; 0$$

Figure 4.2: General color (Jacobi) identity for the color factors in tree diagrams. The discs A, B, C and D represent the sub-diagrams.

Sub-diagram	Vertex ordering	Pole	Color factor
A	$\alpha = 1, ..., p$	P(A)	[A]
B	$\beta = p + 1, ..., p + q$	P(B)	[B]
C	$\gamma = p + q + 1, ..., p + q + r$	P(C)	[C]
D	$\delta = p + q + r + 1, ..., p + q + r + s$	P(D)	[D]

Here $p + q + r + s = M$.

The string amplitude for the vertex ordering $1....M$ is,

$$\mathbf{A}^{L(c)}_{1...M} = i^{M-2}\Big(-\frac{\alpha'}{4}\Big)^{M-3} co(1...M)\cdot \int_{x_1<...<x_{p-1}<0} dx_1...dx_{p-1} \int_0^1 dx_{p+1}$$
$$\int_{x_{p+1}}^1 dx_{p+2}... \int_{x_{p+q+r-1}}^1 dx_{p+q+r} \int_{1<x_{p+q+r+2}<...x_M} dx_{p+q+r+2}...dx_M \; s(x)f(x), \quad (4.94)$$

where we fixed $x_p = 0$, $x_{p+q+r+1} = 1$ and $x_M = \infty$. As before,

$$s(x) = x_M(x_M - 1) \prod_{1\le i<j\le M} (x_j - x_i)^{\frac{\alpha'}{2}k_i\cdot k_j + 2\alpha' K_i\cdot K_j} \quad (4.95)$$

and

$$f(x) = \exp\Big(\sum_{1\le i<j\le M} \frac{\zeta_i\cdot\zeta_j}{(x_i - x_j)^2} - \sum_{1\le i\neq j\le M} \frac{\zeta_i\cdot K_j}{(x_i - x_j)}\Big)\Big|_{\text{multiple-linear}}, \quad (4.96)$$

again the ζ_i^{I}'s are the discrete polarizations in the Cartan Lie sub-algebra in the internal compactified space.

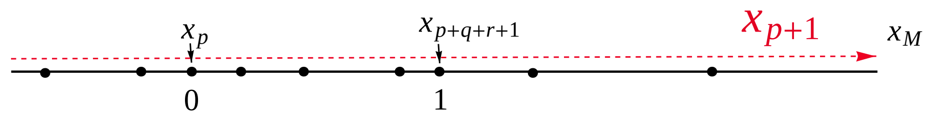


Figure 4.3: The contour integral over x_{p+1}.

Consider the contour-integral of the analytic function $s(x)f(x)$ in x_{p+1} over the straight line just above the real axis (see Fig. (4.3)),

$$\int_{-\infty}^{\infty} dx_{p+1} \left\{ i^{M-2}(-\frac{\alpha'}{4})^{M-3} co(1...M) \int dx\, s(x) f(x) \right\} = 0 \tag{4.97}$$

where $\int dx$ stands for all the other integrals appearing in (4.94). This equation is similar to the $M = 4$ case but with more pieces:

- $0 < x_{p+1} < 1$. Here, there is only one term which is the original vertex ordering,

$$\mathbf{A}^{L(c)}_{1...M} = \mathbf{A}^{L(c)}_{\alpha\beta\gamma\delta} \tag{4.98}$$

- $x_{p+1} < 0$. Here, the variables $x_{p+2}, ..., x_{p+q+r}$ are still larger than x_{p+1}. Although the orderings inside $\beta\gamma$ and α remain unchanged, the relative ordering between $\beta\gamma$ and α may change. If all the vertex operators in B and C are to the left of A, we have

$$\frac{co(1...M)}{co(\beta\gamma\alpha\delta)} \mathbf{A}^{L(c)}_{\beta\gamma\alpha\delta} \exp\left(\frac{i\pi\alpha'}{2} \sum_{i=p+1}^{p+q+r} \sum_{j=1}^{p} k_i \cdot k_j\right) (-1)^{2\alpha' \sum_{i=p+1}^{p+q+r} \sum_{j=1}^{p} K_i \cdot K_j}. \tag{4.99}$$

where the ratio between the two co-cycles appears because the two orderings $\alpha\beta\gamma\delta$ and $\beta\gamma\alpha\delta$ have different cocycles. The exponential term and the power of the (-1) factor comes from the changes of the vertex-operator orderings. By repeating the reduction of Eq.(2.37), it is clear that the power of the (−1) factor, which records every adjacent permutation of the vertices, would at the end cancel the ratio of the co-cycles. *This cancellation is general for any contour integral.* So this term simplifies to

$$\mathbf{A}^{L(c)}_{\beta\gamma\alpha\delta} \exp\left(\frac{i\pi\alpha'}{2}(k_B + k_C) \cdot k_A\right) \tag{4.100}$$

where we define

$$\sum_{i=1}^{p} k_i = k_A, \quad \sum_{i=p+1}^{p+q} k_i = k_B, \quad \sum_{i=p+q+1}^{p+q+r} k_i = k_C. \tag{4.101}$$

Similarly, when all the vertex operators in B are on the left of that in A and all the vertex operators in C are still on the right of A, we have,

$$\mathbf{A}^{L(c)}_{\beta\alpha\gamma\delta} \exp\left(\frac{i\pi\alpha'}{2} k_B \cdot k_A\right). \tag{4.102}$$

For the rest of the terms, the orderings α, β and γ are mixed, for example, and some operators in β are inserted into α. For these terms, we get,

$$\mathbf{A}^{L(c)}_{\sigma_1 \ldots \sigma_{p+q+r}, p+q+r+1, \ldots, M} \exp\left(\frac{i\pi\alpha'}{2} \sum_{i=p+1}^{p+q+r} \sum_{j=1}^{p} c(i, j; \sigma) k_i \cdot k_j\right) \tag{4.103}$$

where σ is a permutation of the vertices $\alpha\beta\gamma = 1 \ldots (p + q + r)$ which keeps intact the relative orderings inside α, β and γ, respectively. So $c(i, j; \sigma) = 1$ if $\sigma_i < \sigma_j$ and $c(i, j; \sigma) = 0$ if $\sigma_i > \sigma_j$.

- $x_{p+1} > 1$. In this case, because the integrals over all the other variables in β and γ have the original upper bound 1, we have to reverse their integrals, and obtain

$$(-1)^{q+r-1} \mathbf{A}^{L(c)}_{\alpha, \sigma_{p+1} \ldots \sigma_M} \exp\left(\frac{-i\pi\alpha'}{2} \sum_{i<j} c(i, j; \sigma) k_i \cdot k_j\right) \tag{4.104}$$

where σ is a permutation of the vertices $p+1, \ldots M$. Note that the orderings inside β and γ are reversed, i.e., for any two indices $i < j$ in $\beta\gamma$, $\sigma_i > \sigma_j$. Since we have fixed $x_M = \infty$, these terms have mixed indices, with the vertices in β are inserted into δ.

In summary, the contour integral identity (4.97) takes the form

$$\text{Eq.}(4.98) + \text{Eq.}(4.100) + \text{Eq.}(4.102) + \text{Eq.}(4.103) + \text{Eq.}(4.104) = 0 \qquad (4.105)$$

which is the generalization of Eq.(4.48). Again, as in Eq.(4.48), the co-cycles do not appear explicitly in the identity.

In the zero slope limit, the real part of Eq.(4.105) just gives Eq.(4.92). For example, Eq.(4.105) have two terms, which contain the pole $P \equiv -p(A)p(B)p(C)p(D)(k_A + k_B)^2$,

$$c_{\alpha\beta\gamma\delta,P} = -c_{\beta\alpha\gamma\delta,P} \qquad (4.106)$$

where the first terms comes from the amplitude $A^{L(c)}_{\alpha\beta\gamma\delta}$ while the second one comes from $A^{L(c)}_{\beta\alpha\gamma\delta}$ in the zero slope limit.

The imaginary part of Eq.(4.105) gives the color identities in the zero slope limit. The calculation is similar to the 4-point case, although we need to be more careful since the internal momenta may be off shell, i.e., $k_A^2 \neq 0$. For the zero slope limit, each one has the form, $k^2 A^{(c)}$. So we are not looking at the poles $-P(A)P(B)P(C)P(D)(k_A + k_B)^2$ of the order $k^{2(M-3)}$ but that of the order $k^{2(M-4)}$, say, $P' = p(A)p(B)p(C)p(D)$. The mixing terms (4.103) and (4.104) cannot give this pole when multiplied by a k^2 term. So the relevant terms are just from (4.100) and (4.102), $A^{L(c)}_{\beta\gamma\alpha\delta}(k_A \cdot k_B + k_A \cdot k_C)$, $A^{L(c)}_{\beta\alpha\gamma\delta} k_A \cdot k_B$ which gives the six possible terms,

$$c_{\beta\gamma\alpha\delta,-(k_B+k_C)^2P'} \frac{k_A \cdot k_B}{-(k_B + k_C)^2 P'}, \; c_{\beta\gamma\alpha\delta,-(k_B+k_C)^2P'} \frac{k_A \cdot k_C}{-(k_B + k_C)^2 P'} \qquad (4.107)$$

$$c_{\beta\gamma\alpha\delta,-(k_A+k_C)^2P'} \frac{k_A \cdot k_B}{-(k_A + k_C)^2 P'}, \; c_{\beta\gamma\alpha\delta,-(k_A+k_C)^2P'} \frac{k_A \cdot k_C}{-(k_A + k_C)^2 P'} \qquad (4.108)$$

$$c_{\beta\alpha\gamma\delta,-(k_A+k_B)^2p'} \frac{k_A \cdot k_B}{-(k_A + k_B)^2 P'}, \; c_{\beta\alpha\gamma\delta,-(k_A+k_C)^2p'} \frac{k_A \cdot k_B}{-(k_A + k_C)^2 P'} \qquad (4.109)$$

where the third term cancels the last term, i.e, $c_{\beta\gamma\alpha\delta,-(k_A+k_C)^2p'} = -c_{\beta\alpha\gamma\delta,-(k_A+k_C)^2p'}$. For the first two terms, we can rewrite the momentum invariants as,

$$k_A \cdot k_B + k_A \cdot k_C = -\frac{1}{2}k_A^2 + \frac{1}{2}k_D^2 - \frac{1}{2}(k_B + k_C)^2 \tag{4.110}$$

The last term will cancel the $(k_B+k_C)^2$ in the denominator, so we get the expected pole P'. The k_A^2 term is not involved in this numerator of the pole term because if k_A is on shell, then this term vanishes. If the k_A is off shell, then the k_A^2 appearing in $P(A)$ and also P' would be cancelled by this new k_A^2 factor and hence the P' pole structure would be changed. In this manner, we further simplify the identity to,

$$-c_{\beta\gamma\alpha\delta,-(k_B+k_C)^2P'} + c_{\beta\gamma\alpha\delta,-(k_A+k_C)^2P'} + c_{\beta\alpha\gamma\delta,-(k_A+k_B)^2P'} = 0 \tag{4.111}$$

which is the color (Jacobi) identity (4.93) by Eq.(4.91). Since the sub-diagrams A, B, C, D are completely general, we obtain all the possible color identities in the tree level by this analysis. There are $(2M-5)!!$ channels and each channel contains $(M-3)$ internal lines, so there are $(2M-5)!!(M-3)$ choices of (A)(B)(C)(D). Each color identity involves 3 terms, so there are $(2M-5)!!(M-3)/3$ color identities.

We now have the counting given in Table 4.2 (n_j's replaced by c_j's), where the number of c_i means the c_i identified by Eq.(4.92) but before the consideration of the color identities. The number of c_i is calculated from choosing one internal line of a tree diagram. An independent set may be chosen as [23]

$$f^{a_1 a_{\sigma_2} x_1} f^{x_1 a_{\sigma_3} x_2} \ldots f^{x_{M-3} a_{\sigma_{M-1}} a_M} \tag{4.112}$$

where σ's are the $(M-2)!$ permutations of $(2, 3, \ldots M-1)$. So there are $(M-2)!$ linearly independent c_i's.

4.4.3 Right-moving amplitudes

Here we choose the right-moving vertex-operators from the vector sector, so for example, the string amplitude for the vertex ordering $1....M$ is,

$$\mathbf{A}^{R(v)}_{1...M} = \frac{8i}{\alpha'^2}\left(\frac{1}{\sqrt{2}}\right)^{M-2} \cdot \int_{x_1<...<x_{p-1}<0} dx_1...dx_{p-1} \int_0^1 dx_{p+1}$$
$$\int_{x_{p+1}}^1 dx_{p+2}... \int_{x_{p+q+r-1}}^1 dx_{p+q+r} \int_{1<x_{p+q+r+2}<...x_M} dx_{p+q+r+2}...dx_M\ \bar{s}(x)\bar{f}(x), \quad (4.113)$$

where again we fixed $x_p = 0$, $x_{p+q+r+1} = 1$ and $x_M = \infty$. As before the function $\bar{s}(x)$ is,

$$\bar{s}(x) = x_M(x_M - 1) \prod_{1\leq i<j\leq M} (x_j - x_i)^{\frac{\alpha'}{2}k_i\cdot k_j} \quad (4.114)$$

and the function $\bar{f}(x)$ is,

$$\bar{f}(x) = \exp\left(\frac{\alpha'}{2}\sum_{1\leq i<j\leq M}\frac{\zeta_i\cdot\zeta_j}{(x_i-x_j)^2} - \frac{\alpha'}{2}\sum_{1\leq i\neq j\leq M}\frac{\zeta_i\cdot k_j}{x_i-x_j}\right)\Bigg|_{\text{multiple-linear}}, \quad (4.115)$$

where the ζ_i's are the gluon polarizations.

The right-moving amplitudes $\mathbf{A}^{R(v)}(k_i,\zeta_i)$ in the general M case satisfy the same open string identities as the $\mathbf{A}^{L(c)}$'s. More specifically, the contour integral method for the right-moving part will yield similar equations like Eq.(4.97), with all the $\mathbf{A}^{L(c)}$'s replaced by $\mathbf{A}^{R(v)}$. In the zero slope limit,

$$\mathbf{A}^{R(v)}_{a_{\sigma_1},a_{\sigma_2},...,a_n} \to A^{R(v)}_{a_{\sigma_1},a_{\sigma_2},...,a_n} = A^{tree}(a_{\sigma_1},a_{\sigma_2},...,a_n) \quad (4.116)$$

which are the Yang-Mills partial amplitudes. Therefore, the real (real-SID) and imaginary (im-SID) part of the string identities will generate the set of relations for the partial amplitudes. The key difference between color and kinematic identities become clear for $M > 4$.

Let us take a closer look at the general im-SID formula. We begin with the partial amplitude $A_{123...M-1,M}$ and consider the contour integral in x_2 while the relative order of $(13...M-1, M)$ are fixed. This yields the relation (4.35),

$$(k_2 \cdot k_1)A(213...(M-1)M) - \sum_{i=3}^{M-1}\Big(\sum_{j=3}^{i} k_2 \cdot k_j\Big)A(13...i, 2, (i+1)...(M-1)M) = 0 \quad (4.117)$$

Note that this contour is the same as (4.97) if we set $\beta = 2$. However, unlike the discussion following (4.97), here we cannot simply pick up the residues but have to work out all the terms inside this open string identity and find the relative signs. Besides β, the remaining 3 vertex orderings α, γ and δ, which correspond to the sub-diagram A, B and C, would combine into $\alpha\beta\gamma\delta = 12...M-1, M$, up to cyclic permutations. By the cyclic permutation, we can always put the index "1" inside α, so $\alpha = ...1, 3, ...i$.

These identities contain only the gauge independent amplitudes $A^{R(v)}$ so are very convenient for the KLT relation based on the left-moving $A^{L(c)}$ and right-moving $A^{R(v)}$. If we want to show that the dual kinematic identities have the same form as the color identities, we need to decompose the $A^{R(v)}$ into different channels,

$$A^{R(v)}_{\alpha} = \sum_{P} \frac{n_{\alpha,P}}{P} \quad (4.118)$$

where P goes through all the $2^{M-2}(2M-5)!!/(M-1)!$ channels within the ordering α. However, the choice of explicit expressions for the $n_{\alpha,P}$'s is not unique. (We may start with a particular choice, like the symmetric way according to the color factors [21].) In general, each n_j contains both a "residue" (or "non-contact') piece and a "contact" piece, and only the "residue" piece will obey the kinematic identities.

The terms inside (4.117) and related to the decomposition A, B, C and D are,

$$\begin{aligned}
&\{k_2 \cdot (k_3 + ... + k_i + k_C) - k_2 \cdot (k_3 + ... + k_i)\}\frac{n_{\alpha\gamma 2\delta,-P'(k_2+k_C)^2}}{-P'(k_2+k_C)^2} \\
&\{k_2 \cdot (k_3 + ... + k_i + k_C + k_D) - k_2 \cdot (k_3 + ... + k_i)\}\frac{n_{2\alpha\gamma\delta,-P'(k_2+k_A)^2}}{-P'(k_2+k_A)^2} \\
&\{k_2 \cdot (k_3 + ... + k_i + k_C + k_D) - k_2 \cdot (k_3 + ... + k_i + k_C)\}\frac{n_{2\alpha\gamma\delta,-P'(k_2+k_D)^2}}{-P'(k_2+k_D)^2}
\end{aligned} \tag{4.119}$$

where $P' = P(A)P(C)P(D)$. The related terms inside can be simplified into,

$$\frac{n_{\alpha\gamma 2\delta,-P'(k_2+k_C)^2} - n_{2\alpha\gamma\delta,-P'(k_2+k_A)^2} + n_{2\alpha\gamma\delta,-P'(k_2+k_D)^2}}{P(A)P(C)P(D)}. \tag{4.120}$$

Repeat this process, the open string identity (4.117) is reduced to the 'coupled" dual identity,

$$\sum_{(A,C,D)} \frac{n_{\alpha\gamma 2\delta,-P'(k_2+k_C)^2} - n_{2\alpha\gamma\delta,-P'(k_2+k_A)^2} + n_{2\alpha\gamma\delta,-P'(k_2+k_D)^2}}{P(A)P(C)P(D)} = 0 \tag{4.121}$$

where the sum is over all the possible decomposition (A, C, D) such that $\alpha 2\gamma\delta = 123...M-1, M$ up to cyclic permutation. Each (A, C, D) gives three terms, which have the same form of the color (Jacobi) identities. Because the three sub-diagrams A, C and D, can be interpreted as a $(M-1)$-point channel with one vertex removed, the number of (A, C, D) is

$$\frac{2^{M-3}(2M-7)!!(M-3)}{(M-2)!}, \tag{4.122}$$

so each open string identity contains $2^{M-3}(2M-7)!!(M-3)/(M-2)!$ triplets, each of which has the form of the color (Jacobi) identities. Eq.(4.121) is gauge invariant and holds in arbitrary choice of the n's. When we pick up the residues of (4.121), we get,

$$\{n_{\alpha\gamma 2\delta,-P'(k_2+k_C)^2} - n_{2\alpha\gamma\delta,-P'(k_2+k_A)^2} + n_{2\alpha\gamma\delta,-P'(k_2+k_D)^2}\}\big|_{\text{residue}} = 0 \tag{4.123}$$

However, when $M > 4$, the dual identities themselves with the non-residue terms do not hold for an arbitrary choice of the n's.

Let us give an example to illustrate the difference between the $M = 4$ case and the $M > 4$ cases. Let us consider the $M = 5$ case. Using Eq.(4.121), we can first draw the disc diagram with only (1345) in CCW direction, so there are two 4-point channels. For each channel, we can remove one of the 2 vertices to get the (A,C,D), so there are four choices of (A,C,D): ((13),4,5), (1,3,(45)),((51),3,4),(1,(34),5). Eq.(4.121) yields the gauge-independent identity (4.10),

$$0 = \frac{n_{(13)(42)5} - n_{2(13)(45)} + n_{(13)4(52)}}{s_{13}} + \frac{n_{1(32)(45)} - n_{(21)3(45)} + n_{(13)(45)2}}{s_{45}} \\ \frac{n_{(51)(32)4} - n_{2(51)(34)} + n_{(51)3(42)}}{s_{15}} + \frac{n_{(34)2(51)} - n_{(21)(34)5} + n_{1(34)(52)}}{s_{34}} \tag{4.124}$$

where $s_{13} = -(k_1+k_3)^2$ etc and $n_{(13)(42)5}$ is the numerator factor in the $s_{13}s_{24}$ channel in the partial amplitude A_{13425} in $A_\alpha^{R(v)}$ (4.118) or equivalently in $\mathcal{A}_5^{\text{YM}}$ (4.1). The details of the $M = 5$ case will be explained in Section 5. Here it suffices to note that the 3 n_j's in any of the 4 triplets has a common pole which appears in the respective denominator. If we replace the kinematic factors by the color factors, then each triplet of color factors must sum to zero. That is how the color identities appear. However, since the spacetime momenta are continuous, this im-SID only implies that the residue of each pole term must vanish. For general M, each im-SID involves a set of triplets, where each triplet of n_j's is the numerator of a product of $(M-4)$ poles that are common to the n_j's in that triplet. This yields the set of kinematic identities (4.9), in one-to-one correspondence to the color identities (4.7). For $M = 5$, we can prove that there always exists a gauge choice so that each triplet sums exactly to zero. We have not been able to extend the proof to general M. It is clear that even if such a gauge choice exists, it is hard to find and so the exact identity (4.8) may not be that useful. On the other hand, the im-SID in terms of the n_j's is gauge-independent and so may be more useful. However, the existence of the exact identity (4.8) (but without the explicit construction of the n_j's) can sometimes be useful.

To get a feeling of what to expect as M increases, we give an explicit example for (4.121) for the $M = 6$ case. Now we need to draw a 5-point disc diagram with (13456) in CCW direction, which contains 5 channels. For each channel, we can remove one of the 3 vertices to get a choice (A,C,D), so there are 15 choices, yielding

$$\begin{aligned} 0 \;=\; & \frac{n_{1((34)2)(56)} - n_{(21)(34)(56)} + n_{1(34)((56)2)}}{s_{34}s_{56}} + \frac{n_{(1(34))(52)6} - n_{(2(1(34)))56} + n_{(1(34))5(62)}}{s_{34}s_{134}} \\ & \frac{n_{((56)1)(32)4} - n_{(2((56)1))34} + n_{((56)1)3(42)}}{s_{56}s_{156}} + \frac{n_{(61)(32)(45)} - n_{(2(61))3(45)} + n_{(61)3((45)2)}}{s_{45}s_{16}} \\ & \frac{n_{((61)3)(42)5} - n_{(2((61)3))45} + n_{(6(13))4(52)}}{s_{16}s_{163}} + \frac{n_{1((3(45))2)6} - n_{(21)(3(45))6} + n_{1(3(45))(62)}}{s_{45}s_{345}} \\ & \frac{n_{(13)(42)(56)} - n_{(2(13))4(56)} + n_{(13)4((56)2)}}{s_{13}s_{56}} + \frac{n_{((13)4)(52)6} - n_{(2((13)4))56} + n_{(2((13)4))56}}{s_{13}s_{134}} \\ & \frac{n_{1(32)(4(56))} - n_{(21)3(4(56))} + n_{13((4(56))2)}}{s_{56}s_{456}} + \frac{n_{(61)((34)2)5} - n_{(2(61))(34)5} + n_{(61)(34)(52)}}{s_{61}s_{34}} \\ & \frac{n_{(5(61))(32)4} - n_{(2((56)1))34} + n_{(5(61))3(42)}}{s_{16}s_{165}} + \frac{n_{1(((34)5)2)6} - n_{(21)((34)5)6} + n_{1((34)5)(62)}}{s_{34}s_{345}} \\ & \frac{n_{(13)((45)2)6} - n_{(2(13))(45)6} + n_{(13)(45)(62)}}{s_{13}s_{45}} + \frac{n_{(6(13))(42)5} - n_{(2(6(13)))45} + n_{(6(13))4(52)}}{s_{13}s_{136}} \\ & \frac{n_{1(32)((45)6)} - n_{(21)3((45)6)} + n_{13(((45)6)2)}}{s_{45}s_{456}} \end{aligned} \tag{4.125}$$

where $s_{156} = -(k_1 + k_5 + k_6)^2$ etc.. Here, $n_{1((34)2)(56)}$ is the numerator factor in the $s_{34}s_{234}s_{56}$ channel in the partial amplitude A_{134256} in $A_\alpha^{R(v)}$ (4.118). For $M = 6$, there are 18 independent im-SID's and each im-SID contains 15 triplets.

4.4.4 Yang-Mills amplitudes

In this subsection, we determine the Yang-Mills amplitude for M gluons,

$$A_{\rm M}^{\rm YM} = g^{M-2} \sum_P \frac{c_P n_P}{P} \tag{4.126}$$

where P goes through all the $(2M-5)!!$ channels, i.e., all the different pole structures. The product $c_P n_P$ means $c_{\alpha,P} n_{\alpha,P}$, which actually has no dependence of the vertex ordering α. A different ordering choice β will introduce $\pm$ signs by Eq.(4.92) $c_{\alpha,P} = \pm c_{\beta,P}$, $n_{\alpha,P} = \pm n_{\beta,P}$, however, they always take the same sign so $c_{\alpha,p} n_{\alpha,p} = c_{\beta,p} n_{\beta,p}$.

We can use the KLT relation [39] to derive Eq.(4.126),

$$\mathcal{A}^{\text{het}}_{\text{M-gluon}} = \left(\frac{i}{2}\right)^{M-3} \pi \left(\frac{g}{\pi}\right)^{M-2} \left(-\frac{4}{\alpha'}\right)^{M-3} \sum_{\alpha,\beta} \mathbf{A}^{L(c)}_{\alpha} \mathbf{A}^{R(v)}_{\beta} F(\alpha,\beta) \tag{4.127}$$

where α, β are the $\frac{1}{2}(M-1)!$ vertex orderings since three vertices are fixed. $F(\alpha,\beta)$ is the phase factor,

$$F(\alpha,\beta) = \exp\left(i\pi \sum_{i>j} f(\frac{\alpha'}{2} k_i \cdot k_j; \alpha, \beta)\right) \tag{4.128}$$

where $(i > j)$

$$f(\frac{\alpha'}{2} k_i \cdot k_j; \alpha, \beta) = \begin{cases} \frac{\alpha'}{2} k_i \cdot k_j & \text{if } (\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0 \\ 0 & \text{if } (\alpha_i - \alpha_j)(\beta_i - \beta_j) > 0 \end{cases} \tag{4.129}$$

so $F(\alpha,\beta)$ is symmetric in α and β. Contour integral will simplify the expression (4.127) into a sum of the products of left-moving partial amplitudes and right-moving partial amplitude, where the number of terms is given in [39],

$$(M-3)! \quad [\frac{1}{2}(M-3)]![\frac{1}{2}(M-3)]!, \text{ if M is odd} \tag{4.130}$$

$$(M-3)! \quad [\frac{1}{2}(M-4)]![\frac{1}{2}(M-2)]!, \text{ if M is even.} \tag{4.131}$$

There is a large number of the channels involved in this sum: since each partial amplitude contains $2^{M-2}(2M-5)!!/(M-1)!$ channels a naive counting suggests that there are

$$\frac{2^{2M-4}}{(M-1)!(M-1)(M-2)} \left[(\frac{1}{2}(M-3))!(2M-5)!! \right]^2 \tag{4.132}$$

$c_i n_j$ terms in the Yang-Mills amplitude if M is odd and a similarly large number if M is even. However, the zero slope limit for the gluon sector is the same as Yang-Mills theory, so only the "diagonal" terms survive,

$$\mathcal{A}^{\text{het}}_{\text{M-gluon}}|_{\alpha'\to 0} = \mathcal{A}^{\text{YM}}_{M} = \sum_{P} \frac{c_{\alpha,P} n_{\alpha,P}}{P}. \tag{4.133}$$

As in the 4-point case, we conjecture that this simplification uses all the color identities for c_j's and all the independent kinematic identities for n_j's. This is explicitly checked for $M = 5$, to which we now turn.

4.5 The 5-gluon Tree Amplitude Example

Let us now illustrate the above discussion with the 5-point example. The properties of the 5-point open string amplitudes and their zero slope limit are discussed in Ref.[15, 42], while the Yang-Mills amplitudes are discussed in Ref.[9]. So here we shall emphasize the heterotic string properties. We shall also discuss the gauge choice issue to get a sense of the underlying structure.

4.5.1 Left-moving amplitudes and color identities

We fix the three points $x_1 = 0$, $x_4 = 1$ and $x_5 = \infty$, so the amplitude A_{12345} has the integration region

$$\int_0^1 dx_2 \int_{x_2}^1 dx_3 \tag{4.134}$$

As in Eq.(4.105), the contour integral in x_2 over the line above the real axis gives

$$0 = \mathbf{A}^{L(c)}_{12345} + \mathbf{A}^{L(c)}_{23145} e^{i\frac{\alpha'}{2}(k_1\cdot k_2 + k_1\cdot k_3)} + \mathbf{A}^{L(c)}_{21345} e^{i\frac{\alpha'}{2}(k_1\cdot k_2)} - \mathbf{A}^{L(c)}_{14325} e^{i\frac{\alpha'}{2}(k_1\cdot k_5)} \tag{4.135}$$

where in this case, $\alpha = 1$, $\beta = 2$, $\gamma = 3$ and $\delta = 34$. The correspond poles are $p_A = 1$, $p_B = 1$, $p_C = 1$ and $p_D = -(k_4 + k_5)^2$.

Here we follow the conventions in Ref.[9] in labeling the color factors within the left-moving amplitudes, in the limit $\alpha' \to 0$,

$$\begin{aligned}
A_5^{L(c)}(1,2,3,4,5) &\equiv \frac{c_1}{s_{12}s_{45}} + \frac{c_2}{s_{23}s_{51}} + \frac{c_3}{s_{34}s_{12}} + \frac{c_4}{s_{45}s_{23}} + \frac{c_5}{s_{51}s_{34}}, \\
A_5^{L(c)}(1,4,3,2,5) &\equiv \frac{c_6}{s_{14}s_{25}} + \frac{c_5}{s_{43}s_{51}} + \frac{c_7}{s_{32}s_{14}} + \frac{c_8}{s_{25}s_{43}} + \frac{c_2}{s_{51}s_{32}}, \\
A_5^{L(c)}(1,3,4,2,5) &\equiv \frac{c_9}{s_{13}s_{25}} - \frac{c_5}{s_{34}s_{51}} + \frac{c_{10}}{s_{42}s_{13}} - \frac{c_8}{s_{25}s_{34}} + \frac{c_{11}}{s_{51}s_{42}}, \\
A_5^{L(c)}(1,2,4,3,5) &\equiv \frac{c_{12}}{s_{12}s_{35}} + \frac{c_{11}}{s_{24}s_{51}} - \frac{c_3}{s_{43}s_{12}} + \frac{c_{13}}{s_{35}s_{24}} - \frac{c_5}{s_{51}s_{43}}, \\
A_5^{L(c)}(1,4,2,3,5) &\equiv \frac{c_{14}}{s_{14}s_{35}} - \frac{c_{11}}{s_{42}s_{51}} - \frac{c_7}{s_{23}s_{14}} - \frac{c_{13}}{s_{35}s_{42}} - \frac{c_2}{s_{51}s_{23}}, \\
A_5^{L(c)}(1,3,2,4,5) &\equiv \frac{c_{15}}{s_{13}s_{45}} - \frac{c_2}{s_{32}s_{51}} - \frac{c_{10}}{s_{24}s_{13}} - \frac{c_4}{s_{45}s_{32}} - \frac{c_{11}}{s_{51}s_{24}},
\end{aligned} \tag{4.136}$$

where $s_{ij} = s_{ji} = -(k_i + k_j)^2$. The rest left-moving partial amplitudes are not linear independent, so they do not contain new c's, for example,

$$A_{21345}^{L(c)} = \frac{c_8}{s_{25}s_{34}} + \frac{-c_{15}}{s_{13}s_{45}} + \frac{-c_1}{s_{12}s_{45}} + \frac{-c_9}{s_{25}s_{13}} + \frac{-c_3}{s_{12}s_{34}}. \tag{4.137}$$

The leading real part of the open string identity (4.135) yields

$$0 = A_{12345}^{L(c)} + A_{23145}^{L(c)} + A_{21345}^{L(c)} - A_{14325}^{L(c)} \tag{4.138}$$

which determines the relative sign for the c's appearing in different sub-amplitudes. Eq.(4.136) has already taken advantage of this identity.

The leading order of the imaginary part of the identity (4.135) is,

$$0 = A_{23145}^{L(c)}(s_{12} + s_{13}) + A_{21345}^{L(c)}s_{12} + A_{14325}^{L(c)}s_{15} \tag{4.139}$$

This identity contains 20 terms, each of which may contain a "single pole", like c_{15}/s_{45} or a "double pole", like $-s_{12}c_4/s_{23}s_{45}$. The basic strategy to simplify the identity is to combine the terms with the same c_i together and use the kinetic identities like,

$$s_{12} + s_{13} + s_{23} = s_{45} \tag{4.140}$$

to eliminate the double poles. Now each c_i is just the residue of a single pole. We are left with 3 single-pole terms for each of the 4 different single poles: s_{45}, s_{23}, s_{25} and s_{34}.

$$\frac{-c_6 + c_8 + c_9}{s_{25}} + \frac{-c_3 + c_5 - c_8}{s_{34}} - \frac{c_1 - c_4 - c_{15}}{s_{45}} - \frac{-c_2 + c_4 + c_7}{s_{23}} = 0. \tag{4.141}$$

Since the coefficient of each pole term must vanish, we get four color identities,

$$\begin{aligned} c_4 + c_{15} - c_1 &= 0 \\ c_4 + c_7 - c_2 &= 0 \\ c_8 + c_9 - c_6 &= 0 \\ c_3 + c_8 - c_5 &= 0 \end{aligned} \tag{4.142}$$

which correspond to residue of the four single poles s_{45}, s_{23}, s_{25} and s_{34}, respectively. Note that the first identity is the special case of the general identity (4.111) with $\alpha = 1, \beta = 2, \gamma = 3$ and $\delta = 34$.

We can repeat the above contour integral argument to get the rest of the color identities. Similarly, for the configuration $x_5 = 0$, $x_3 = 1$, $x_4 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_2$, the contour integral in x_1 gives,

$$\begin{aligned} c_6 - c_7 - c_{14} &= 0 \\ -c_2 + c_4 + c_7 &= 0 \\ -c_3 + c_5 - c_8 &= 0 \\ c_3 + c_{12} - c_1 &= 0 \end{aligned} \tag{4.143}$$

For the configuration $x_2 = 0, x_4 = 1, x_5 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_3$, the contour integral in x_1 gives,

$$
\begin{aligned}
-c_9 + c_{10} + c_{15} &= 0 \\
c_1 - c_4 - c_{15} &= 0 \\
-c_3 + c_5 - c_8 &= 0 \\
c_2 - c_5 - c_{11} &= 0.
\end{aligned}
\tag{4.144}
$$

while the configuration $x_3 = 0$, $x_4 = 1$, $x_5 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_2$, the contour integral in x_1 gives,

$$
\begin{aligned}
-c_{10} + c_{11} - c_{13} &= 0 \\
c_2 - c_5 + c_{11} &= 0 \\
c_3 + c_{12} - c_1 &= 0 \\
c_1 - c_4 - c_{15} &= 0.
\end{aligned}
\tag{4.145}
$$

So we get all the 10 color identities from four contour integrals. Other contour integrals do not yield additional identities. Since the sum of any 9 of the 10 identities yields the remaining one, there are 9 independent color identities.

Note that these color identities can be read off directly by looking at the diagrams. Consider any 4 (internal and/or external) lines that are connected by another internal line. These 4 lines can be connected 3 different ways, the equivalent of "s, t, u" channels. The corresponding 3 color factors obey a color identity, which is simply the Jacobi identity multiplied by a common factor of the structure constant. Actually, this is a more efficient way to find the color identities.

4.5.2 Right-moving amplitude and the gauge choices

Now turn to the right-moving part for the gluon momenta and polarizations. The analysis is the same as that for the color factors except for the crucial issue of the contact contributions or the gauge choice. Let us define the n_i's to be local, so that,

$$
\begin{aligned}
A_5^{tree}(1,2,3,4,5) &\equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}\,, \\
A_5^{tree}(1,4,3,2,5) &\equiv \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}\,, \\
A_5^{tree}(1,3,4,2,5) &\equiv \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}\,, \\
A_5^{tree}(1,2,4,3,5) &\equiv \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}, \\
A_5^{tree}(1,4,2,3,5) &\equiv \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}\,, \\
A_5^{tree}(1,3,2,4,5) &\equiv \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}\,,
\end{aligned}
\tag{4.146}
$$

The contour integral argument is exactly the same after we replace, in the partial amplitudes (4.136),

$$L \to R,\ (c) \to (v),\ c_j \to n_j \tag{4.147}$$

For example, now the contour integral in x_2 for $\mathbf{A}^{R(v)}_{12345}$, analogous to the relation (4.135), reads

$$0 = \mathbf{A}^{R(v)}_{12345} + \mathbf{A}^{R(v)}_{23145} e^{i\frac{\alpha'}{2}(k_1\cdot k_2 + k_1\cdot k_3)} + \mathbf{A}^{R(v)}_{21345} e^{i\frac{\alpha'}{2}(k_1\cdot k_2)} - \mathbf{A}^{R(v)}_{14325} e^{i\frac{\alpha'}{2}(k_1\cdot k_5)} \tag{4.148}$$

and its imaginary part in the zero slope limit yields,

$$\frac{-n_6+n_8+n_9}{s_{25}} + \frac{-n_3+n_5-n_8}{s_{34}} - \frac{n_1-n_4-n_{15}}{s_{45}} - \frac{-n_2+n_4+n_7}{s_{23}} = 0. \tag{4.149}$$

The residue of each pole term must vanish. However, each of the 4 terms do not have to vanish by itself. That is, the non-pole terms (the contact terms) can cancel among the 4 terms. In particular, $n_3 + n_8 - n_5 = \Delta(k_i, \zeta_i)s_{34}$. Consider the gauge transformation

$$n_3 \to n'_3 = n_3 + \beta s_{12} \tag{4.150}$$

where the kinematic function $\beta(k_i, \zeta_i)$ is local. Then invariance of

$$A^{R(v)}_{12345} = \frac{n_4}{s_{23}s_{45}} + \frac{n_5}{s_{34}s_{15}} + \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{15}} + \frac{n_3}{s_{12}s_{34}}$$

implies that

$$n_5 \to n'_5 = n_5 - \beta s_{15} \tag{4.151}$$

and invariance of all the remaining partial amplitudes means

$$n_8 \to n'_8 = n_8 + \beta s_{25} \tag{4.152}$$

It follows that $A^{\rm YM}_M$ is invariant under this triplet of simultaneous transformations. Under this gauge transformation, we can choose β such that

$$n'_3 + n'_8 - n'_5 = \Delta s_{34} + \beta(s_{12} + s_{15} + s_{25}) = (\Delta + \beta)s_{34} = 0 \tag{4.153}$$

Now we can repeat this process for $n_4 + n_{15} - n_1$ and $n_4 + n_7 - n_2$ to obtain

$$n'_4 + n'_{15} - n'_1 = 0, \qquad n'_4 + n'_7 - n'_2 = 0 \tag{4.154}$$

It then follows that $n'_8 + n_9 - n_6 = 0$. That is, we need to make 3 simultaneous gauge transformations to obtain the above 4 kinematic identities from the string identity (4.149). The key is that the gauge transformation always involves 3 n_j at a time. It is precisely such a triplet of n_j that appears in each kinematic identity.

With this preliminary discussion, we are now ready to consider the full set. Since there are $(M-3)! = 2$ basis amplitudes out of $(M-2)! = 6$ amplitudes A^{tree}s, there are 4 independent relations among them. We already obtained one in

(4.155). Similarly, as we did for the left-moving amplitude, for the configuration $x_5 = 0$, $x_3 = 1$, $x_4 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_2$, the contour integral in x_1 gives,

$$\frac{n_6 - n_7 - n_{14}}{s_{14}} + \frac{-n_2 + n_4 + n_7}{s_{23}} - \frac{-n_3 + n_5 - n_8}{s_{34}} - \frac{n_3 + n_{12} - n_1}{s_{12}} = 0 \qquad (4.155)$$

For the configuration $x_2 = 0$, $x_4 = 1$, $x_5 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_3$, the contour integral in x_1 gives the second equation in (4.166),

$$\frac{-n_9 + n_{10} + n_{15}}{s_{13}} + \frac{n_1 - n_4 - n_{15}}{s_{45}} - \frac{-n_3 + n_5 - n_8}{s_{34}} - \frac{n_2 - n_5 - n_{11}}{s_{15}} = 0. \qquad (4.156)$$

while the configuration $x_3 = 0$, $x_4 = 1$, $x_5 = \infty$ and $\int_0^1 dx_1 \int_{x_1}^1 dx_2$, the contour integral in x_1 gives the last equation in (4.166).

$$\frac{-n_{10} + n_{11} - n_{13}}{s_{24}} + \frac{n_2 - n_5 + n_{11}}{s_{15}} - \frac{n_3 + n_{12} - n_1}{s_{12}} - \frac{n_1 - n_4 - n_{15}}{s_{45}} = 0. \qquad (4.157)$$

Relations from other contour integral identities are redundant. Note that relations (4.149, 4.155, 4.156, 4.157) are gauge invariant. To avoid the gauge dependence issues, one may choose to consider relations among the gauge-invariant partial amplitudes only, which are equivalent to these relations.

It is clear that the residue of each pole term in these relations (4.149, 4.155, 4.156, 4.157) must vanish. This yields 10 relations, which are the 10 kinematic identities for the residues of the n_j's. Now we like to show that there exists a gauge choice such that every triplet vanishes completely, so we have the 10 kinematic identities,

$$
\begin{aligned}
&\tilde{n}_3 - \tilde{n}_5 + \tilde{n}_8 = 0\,,\\
&\tilde{n}_3 - \tilde{n}_1 + \tilde{n}_{12} = 0\,,\\
&\tilde{n}_4 - \tilde{n}_1 + \tilde{n}_{15} = 0\,,\\
&\tilde{n}_4 - \tilde{n}_2 + \tilde{n}_7 = 0\,,\\
&\tilde{n}_5 - \tilde{n}_2 + \tilde{n}_{11} = 0\,,\\
&\tilde{n}_7 - \tilde{n}_6 + \tilde{n}_{14} = 0\,,\\
&\tilde{n}_8 - \tilde{n}_6 + \tilde{n}_9 = 0\,,\\
&\tilde{n}_{10} - \tilde{n}_9 + \tilde{n}_{15} = 0\,,\\
&\tilde{n}_{10} - \tilde{n}_{11} + \tilde{n}_{13} = 0\,,\\
&\tilde{n}_{13} - \tilde{n}_{12} + \tilde{n}_{14} = 0
\end{aligned}
\tag{4.158}
$$

Note that one of these 10 identities is redundant.

Because there are 15 n_i's inside the 6 defining amplitudes (4.146) above, there are 9 degrees of freedom to redefine n_i's without affecting the A^{tree}'s, which can be realized as,

$$
\begin{aligned}
\tilde{n}_1 &= n_1 + a_{12}s_{45} - a_{45}s_{12}, & \tilde{n}_2 &= n_2 + a_{23}s_{15} - a_{15}s_{23}\\
\tilde{n}_3 &= n_3 + a_{34}s_{12} - a_{12}s_{34}, & \tilde{n}_4 &= n_4 + a_{45}s_{23} - a_{23}s_{45}\\
\tilde{n}_5 &= n_5 + a_{15}s_{34} - a_{34}s_{15}, & \tilde{n}_6 &= n_6 + a_{25}s_{14} - a_{14}s_{25}\\
\tilde{n}_7 &= n_7 + a_{14}s_{23} - a_{23}s_{14}, & \tilde{n}_8 &= n_8 + a_{34}s_{25} - a_{25}s_{34}\\
\tilde{n}_9 &= n_9 + a_{13}s_{25} - a_{25}s_{13}, & \tilde{n}_{10} &= n_{10} + a_{13}s_{24} - a_{24}s_{13}\\
\tilde{n}_{11} &= n_{11} + a_{15}s_{24} - a_{24}s_{15}, & \tilde{n}_{12} &= n_{12} + a_{35}s_{12} - a_{12}s_{35}\\
\tilde{n}_{13} &= n_{13} + a_{35}s_{12} - a_{12}s_{35}, & \tilde{n}_{14} &= n_{14} + a_{14}s_{35} - a_{35}s_{14}\\
n_{15} &= n_{15} + a_{45}s_{13} - a_{13}s_{45}.
\end{aligned}
\tag{4.159}
$$

where $a_{34}, a_{12}, a_{45}, a_{23}, a_{15}, a_{14}, a_{25}, a_{13}, a_{24}, a_{35}$ are arbitrary local functions of k_j and ζ_j. The signs are carefully chosen such that the partial amplitude is invariant. Although the number of a_{ij} is 10, a particular choice,

$$\begin{aligned} & (a_{34}, a_{12}, a_{45}, a_{23}, a_{15}, a_{14}, a_{25}, a_{13}, a_{24}, a_{35}) \\ = \quad & (s_{34}, s_{12}, s_{45}, s_{23}, s_{15}, s_{14}, s_{25}, s_{13}, s_{24}, s_{35}) \end{aligned} \tag{4.160}$$

does not change any n_i, so it is a trivial redefinition. Therefore we can simply set any one of them to zero. Let us say $a_{35} = 0$, so we end up with 9 degrees of gauge freedom. We already see that the non-contact terms inside n_i satisfy the dual identity $(n_i + n_j + n_k)|_{\text{residue}} = 0$, if the color factors with the same indices satisfy $c_i + c_j + c_k = 0$. We like to show that by using a proper redefinition of the n_i's, the dual identities (4.158) hold exactly.

Since the redefinition of the n_j's are realized by the a's in Eq.(4.159), and we like to see whether a choice of the a's exists for the set of kinematic identities (4.158) to hold, these equations can be understood as the equations for the a's. For example, $\tilde{n}_3 - \tilde{n}_5 + \tilde{n}_8 = 0$ reads,

$$a_{34} - a_{12} - a_{15} - a_{25} = \frac{-n_3 + n_5 - n_8}{s_{34}} \tag{4.161}$$

where we used $s_{15} + s_{25} + s_{12} = s_{34}$. Similarly, we can write all the 9 equations in the matrix form,

$$Ka = b \tag{4.162}$$

where

$$K = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} a_{34} \\ a_{12} \\ a_{45} \\ a_{23} \\ a_{15} \\ a_{14} \\ a_{25} \\ a_{13} \\ a_{24} \end{pmatrix} \tag{4.163}$$

and

$$\begin{aligned} b &= \Bigg(\frac{-n_3+n_5-n_8}{s_{34}}, \frac{n_3+n_{12}-n_1}{s_{12}}, \frac{n_1-n_4-n_{15}}{s_{45}}, \frac{-n_2+n_4+n_7}{s_{23}}, \frac{n_2-n_5-n_{11}}{s_{15}}, \\ &\quad \frac{n_6-n_7-n_{14}}{s_{14}}, \frac{-n_6+n_8+n_9}{s_{25}}, \frac{-n_9+n_{10}+n_{15}}{s_{13}}, \frac{-n_{10}+n_{11}-n_{13}}{s_{24}}\Bigg)^{\mathrm{T}} \end{aligned} \tag{4.164}$$

Here we are trying to find a solution to the a's so that Eq.(4.158) holds. Given an arbitrary b, a solution is always guaranteed if the rank of K equals its size (which is 9). However, here K is a degenerate matrix, with r$ank\ K = 5$, that is, r$ank\ K < 9$. So a solution of the a's exists only if 4 (=9-5) constraints among the components of the column vector b are satisfied. That is, only 5 equations are independent and they generate the remaining 4 equations. For example,

$$\begin{aligned} 0 &= (a_{25}-a_{14}-a_{34}-a_{13}) + (a_{34}-a_{12}-a_{15}-a_{25}) \\ &\quad -(a_{45}-a_{12}-a_{13}-a_{23}) - (a_{23}-a_{15}-a_{45}-a_{14}) \\ &= \frac{-n_6+n_8+n_9}{s_{25}} + \frac{-n_3+n_5-n_8}{s_{34}} - \frac{n_1-n_4-n_{15}}{s_{45}} - \frac{-n_2+n_4+n_7}{s_{23}} \end{aligned} \tag{4.165}$$

which is a constraint on the original n_j's. Similarly,

$$\frac{n_6-n_7-n_{14}}{s_{14}}+\frac{-n_2+n_4+n_7}{s_{23}}-\frac{-n_3+n_5-n_8}{s_{34}}-\frac{n_3+n_{12}-n_1}{s_{12}}=0,$$
$$\frac{-n_9+n_{10}+n_{15}}{s_{13}}+\frac{n_1-n_4-n_{15}}{s_{45}}-\frac{-n_3+n_5-n_8}{s_{34}}-\frac{n_2-n_5-n_{11}}{s_{15}}=0,$$
$$\frac{-n_{10}+n_{11}-n_{13}}{s_{24}}+\frac{n_2-n_5+n_{11}}{s_{15}}-\frac{n_3+n_{12}-n_1}{s_{12}}-\frac{n_1-n_4-n_{15}}{s_{45}}=0. \quad (4.166)$$

These 4 constraints (4.165,4.166) form the necessary and sufficient condition for the existence of a solution to Eq.(4.162). Naively, just from the Feynman diagram viewpoint, it is not clear why these conditions hold. However, we see that the open string amplitude identities (4.149, 4.155, 4.156, 4.157) yield precisely these 4 relations. Hence open string identities ensure that (4.162) has solutions. That is, there exists a gauge choice such that Eq.(4.158) is realized.

Since the column vector b has no pole (i.e., local), because $-n_3+n_5-n_8 \propto s_{34}$ etc., the solution yields a set of local a's. Because the solution for (4.162) exists, there are $9 - \text{rank}\, K = 4$ remaining transformations which keep the kinematic identities invariant. They take the forms

$$\begin{pmatrix} a_{34} \\ a_{12} \\ a_{45} \\ a_{23} \\ a_{15} \\ a_{14} \\ a_{25} \\ a_{13} \\ a_{24} \end{pmatrix} = f_1 \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; f_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; f_3 \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; f_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.167)$$

where f_i are arbitrary local functions of the kinematic variables. This concludes our proof that, for the $(M = 5)$-point case, the open string identities ensure that there exists choices of n_i's such that all the dual identities $n_i + n_j + n_k = 0$ (4.158) hold.

This pattern generalizes to the general M-point amplitudes. There are $(2M-5)!!$ n_i's which appeared in $(M-2)!$ partial amplitudes so there are $(2M-5)!!-(M-2)!$ degrees of freedom to redefine the n_i's. We can use $\frac{(M-3)(2M-5)!!}{3}$ parameters a_k modulo some trivial ones to realization the redefinitions. The number of the effective a_k's should be $(2M-5)!!-(M-2)!$, which are constraint by a linear equation like (4.162) if we want to get the kinematic identities. Again the matrix in the linear equation is degenerate with rank $(2M-5)!!-(2M-5)((M-3)!)$. The open string identities would ensure this linear equation has solutions. So the n_i choice for which the kinematic identities hold exist, and the "choice space" has the dimension $(M-2)!-(M-3)! = (M-3)!(M-3)$.

4.5.3 KLT relation and the 5-point amplitudes

We can now use the KLT relation in the zero slope limit to get the 5-point Yang-Mills tree amplitude. Since the heterotic amplitude is a sum over the product of a left-mover and a right mover, and since there are 2 independent partial amplitudes for A^L and 2 for A^R, we can express the full amplitude as a sum over $2 \times 2 = 4$ terms. As shown in Ref.[39], a judicious choice of basis amplitudes allows us to reduce the sum to only 2 terms. There are many equivalent ways to express the 5-point amplitude. For example,

$$\begin{aligned}
\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0) &= g^3 s_{12}s_{34}A^{L(c)}_{12345}A^{R(v)}_{21435} + g^3 s_{13}s_{24}A^{L(c)}_{13245}A^{R(v)}_{31425} \\
&= g^3\Big(\frac{c_4}{s_{23}s_{45}} + \frac{c_5}{s_{34}s_{15}} + \frac{c_1}{s_{12}s_{45}} + \frac{c_2}{s_{23}s_{15}} + \frac{c_3}{s_{12}s_{34}}\Big) \\
&\times\ s_{12}s_{34}\Big(\frac{-n_{12}}{s_{12}s_{35}} + \frac{-n_6}{s_{14}s_{25}} + \frac{n_3}{s_{12}s_{34}} + \frac{-n_{14}}{s_{14}s_{35}} + \frac{-n_8}{s_{34}s_{25}}\Big) \\
&+\ g^3\Big(\frac{c_{15}}{s_{13}s_{45}} + \frac{-c_2}{s_{23}s_{15}} + \frac{-c_{10}}{s_{13}s_{24}} + \frac{-c_4}{s_{45}s_{23}} + \frac{-c_{11}}{s_{15}s_{24}}\Big) \\
&\times\ s_{13}s_{24}\Big(\frac{-n_9}{s_{13}s_{25}} + \frac{-n_{14}}{s_{14}s_{35}} + \frac{-n_{10}}{s_{13}s_{24}} + \frac{-n_6}{s_{14}s_{25}} + \frac{n_{13}}{s_{24}s_{35}}\Big) \qquad (4.168)
\end{aligned}$$

First we go to the gauge choice where all the 9 independent kinematic identities hold. Then using all 18 of the 9 independent color identities (4.142, 4.143, 4.144,4.145) and the 9 independent kinematic identities (4.158), it is straightforward (but tedious) to show that the 5-point Yang-Mills tree amplitude is reproduced,

$$\begin{aligned}
\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0) &= \mathcal{A}^{\text{YM}}_5 = g^3 \sum_{j=1}^{15} \frac{c_j n_j}{P_j} \\
&= g^3\left(\frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{23}s_{15}} + \frac{c_3 n_3}{s_{12}s_{34}} + \ldots\right) \qquad (4.169)
\end{aligned}$$

If we want, we can now transform back to the original set of n_j we started with. In the choice of the particular way (4.168) to express $\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0)$, the presence of the diagonal term $c_3 n_3/(s_{12}s_{34})$ is obvious, but the other diagonal terms are not. Choosing a different basis to express $\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0)$, a different diagonal term will be obvious, but not the rest. It is the presence of the 9 + 9 identities that allows us write $\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0)$ in the diagonal form that is given in $\mathcal{A}^{\text{YM}}_5$ (4.1). On the other hand, knowing that $\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0) = \mathcal{A}^{\text{YM}}_5$, we can obtain the 9 + 9 identities as well, by exploiting the many different but equivalent ways to express $\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0)$.

On the other hand, instead of using the dual-Jacobi identities (4.158), one can show Eq.(4.168) is equivalent to the diagonal form (4.1) using only the gauge-invariant im-SID's (4.149, 4.155, 4.156, 4.157). First, we rewrite all the color factors c_i's in (4.168) in terms of $(M-2)! = 6$ of them, say, c_1, c_6, c_9, c_{12}, c_{14} and c_{15},

$$\begin{aligned}\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0) &= c_1 A^{R(v)}_{12345} + c_{12}\Big(-\frac{s_{34}}{s_{35}}A^{R(v)}_{12345} - A^{R(v)}_{12345} + \frac{s_{13}}{s_{35}}A^{R(v)}_{13245}\Big) \\ &+ c_6\Big(-\frac{s_{12}s_{34}}{s_{14}s_{25}}A^{R(v)}_{12345} - \frac{s_{12}}{s_{25}}A^{R(v)}_{12345} - \frac{s_{13}s_{34}}{s_{14}s_{25}}A^{R(v)}_{13245}\Big) + \ldots \end{aligned} \quad (4.170)$$

Using all the im-SID's,

$$-s_{34}A^{R(v)}_{12345} - s_{35}A^{R(v)}_{12345} + s_{13}A^{R(v)}_{13245} = s_{35}A^{R(v)}_{12435}, \ldots \quad (4.171)$$

to simply the expression so each coefficient contains only one $A^{R(v)}$,

$$\begin{aligned}\mathcal{A}^{\text{het}}_{\text{5-gluon}}(0) &= c_1 A^{R(v)}_{12345} + c_{15}A^{R(v)}_{13245} + c_{12}A^{R(v)}_{12435} \\ &\quad + c_9 A^{R(v)}_{13425} + c_{14}A^{R(v)}_{14235} + c_6 A^{R(v)}_{13245} \\ &= \sum_i \frac{c_i n_i}{P_i}, \end{aligned} \quad (4.172)$$

where the last equality is proven in [23] by using the color Jacobi identities only. So the diagonal form (4.1) is obtained, if only one set of the numerators (c_i here) satisfy the Jacobi identities while the $A^{R(v)}$s satisfy the im-SID. This property should extend to all M. The KLT relation simply expresses $\mathcal{A}^{\text{YM}}_M$ (4.1) in terms of the $(M-3)!$ basis amplitudes $A^{R(v)}$s. Furthermore, for the 5-graviton tree amplitude, in the same manner, we have two sets of numerators, n_i and $\tilde{n}_i$. As long as the n_i's satisfy the dual Jacobi identities, the tree amplitude is simplified to the diagonal form,

$$\mathcal{A}^{grav}_5 = \sum_i \frac{n_i \tilde{n}_i}{P_i}, \quad (4.173)$$

where the other set $\tilde{n}_i$ need not to satisfy the dual Jacobi identities.

4.6 BCJ relation and Schouten identity

The Kleiss-Kuijf (KK) [41] and BCJ relations [9] for YM helicity amplitudes can be proven by using Schouten identity repeatedly,

$$\langle ij\rangle\langle kl\rangle = \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle \tag{4.174}$$

Here we use the notation of [27, 19]: for lightlike momentum p_μ, $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$. The spinor products are defined to be $\langle \lambda, \lambda' \rangle = \epsilon_{ab}\lambda^a \lambda'^b$ and $[\tilde{\lambda}, \tilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{a}}\tilde{\lambda}'^{\dot{b}}$. Here $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle[ij]$. Note the similarity between Schouten identity and Jacobi identity of Lie algebra.

To simplify the discussion, let us consider only the maximal helicity-violating (MHV) amplitudes here. Checking the 4- and 5-point cases is straightforward. For example, the KK relation for 4-point, the photon decoupling identity

$$A(2^-1^-3^+4^+) + A(1^-2^-3^+4^+) + A(1^-3^+2^-4^+) = 0 \tag{4.175}$$

which reads [47] [6],

$$\frac{\langle 12\rangle^4(\langle 23\rangle\langle 41\rangle + \langle 13\rangle\langle 24\rangle - \langle 12\rangle\langle 34\rangle)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle\langle 24\rangle} = 0 \tag{4.176}$$

because of Schouten identity. The 4-point BCJ relation, $s_{12}A(2^-1^-3^+4^+) = s_{23}A(1^-3^+2^-4^+)$ is self-evident from the MHV amplitude expression. The 5-point case is similar: the KK relation is,

$$\begin{aligned} &A(2^-1^-3^+4^+5^+) + A(1^-2^-3^+4^+5^+) + \\ &A(1^-3^+2^-4^+5^+) + A(1^-3^+4^+2^-5^+) = 0 \end{aligned} \tag{4.177}$$

which reads,

$$\begin{aligned} &\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle\langle 13\rangle\langle 24\rangle\langle 25\rangle}(\langle 23\rangle\langle 51\rangle\langle 24\rangle + \langle 13\rangle \\ &\langle 24\rangle\langle 25\rangle - \langle 12\rangle\langle 34\rangle\langle 25\rangle - \langle 12\rangle\langle 23\rangle\langle 45\rangle) = 0 \end{aligned} \tag{4.178}$$

where the Schouten identity is used twice. The BCJ relation is,

$$s_{12}A(2^-1^-3^+4^+5^+) - s_{23}A(1^-3^+2^-4^+5^+)$$
$$-(s_{23}+s_{24})A(1^-3^+4^+2^-5^+) = 0 \tag{4.179}$$

whose left hand side is proportional to,

$$\begin{aligned} & \langle 32\rangle\langle 24\rangle\langle 51\rangle\langle 21\rangle[21] - \langle 21\rangle\langle 34\rangle\langle 52\rangle\langle 23\rangle[23] \\ - & \langle 21\rangle\langle 45\rangle\langle 32\rangle\langle 32\rangle[32] - \langle 21\rangle\langle 45\rangle\langle 32\rangle\langle 24\rangle[24] \\ = & \langle 32\rangle\langle 21\rangle\langle 24\rangle(\langle 51\rangle[21] + \langle 53\rangle[23] + \langle 54\rangle[24]) \\ = & 0 \end{aligned} \tag{4.180}$$

where the first step uses the Schouten identity and the last step follows from momentum conservation : $\sum_{j=1}^{M} p_j^{\mu} = 0 \to \sum_j \langle kj\rangle[jl] = 0$ and $\langle ii\rangle = 0$ and $[ii] = 0$. This feature can be generalized to the M-point MHV amplitudes. For example, consider a BCJ identity for M-points [9] [15] [37],

$$\sum_{i=3}^{M}\Big(\sum_{j=3}^{i} s_{2j}\Big)A(13...i, 2, (i+1)...M) = 0 \tag{4.181}$$

where the helicities are $(1^-2^-3^+....M^+)$ and the label $j = M+1$ should be identified with $j = 1$. The left hand side of (4.181) reads,

$$\begin{aligned} & \sum_{j=3}^{M} s_{2j}\Big(\sum_{i=j}^{M} A(13...i, 2, (i+1)...M)\Big) \\ = & \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle...\langle M1\rangle}\sum_{j=3}^{M} s_{2j}\Big(\sum_{i=j}^{M}\frac{\langle i, i+1\rangle}{\langle i2\rangle\langle 2, i+1\rangle}\Big) \end{aligned} \tag{4.182}$$

The sum over i can be calculated by using Schouten identity in $(M-j)$ steps, say,

$$\frac{\langle j, j+1\rangle}{\langle j, 2\rangle\langle 2, j+1\rangle} + \frac{\langle j+1, j+2\rangle}{\langle j+1, 2\rangle\langle 2, j+2\rangle} = \frac{\langle j, j+2\rangle}{\langle j, 2\rangle\langle 2, j+2\rangle} \tag{4.183}$$

and etc. The final result is,

$$\sum_{i=j}^{M} \frac{\langle i, i+1\rangle}{\langle i2\rangle\langle 2, i+1\rangle} = \frac{\langle j, 1\rangle}{\langle j, 2\rangle\langle 21\rangle}. \tag{4.184}$$

Therefore, the left hand side of (4.181) reads,

$$\begin{aligned} & \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle ... \langle M1\rangle} \sum_{j=3}^{M} \frac{s_{2j}\langle j, 1\rangle}{\langle j, 2\rangle\langle 21\rangle} \\ \propto \quad & \sum_{j=3}^{M} \frac{\langle j, 1\rangle[2, j]}{\langle 21\rangle} = 0, \end{aligned} \tag{4.185}$$

where we have used momentum conservation.

CHAPTER 5

QUADRATIC RECURSIVE RELATIONS FROM THE VIEWPOINT OF SUPERGRAVITY

Recently, Bjerrum-Bohr, Damgaard, Feng and Sondergaard derived a set of new interesting quadratic identities of the Yang-Mills tree scattering amplitudes, besides Bern-Carrasco-Johansson (BCJ) identities. We claim that these quadratic identities of YM amplitudes actually follow directly from the KLT relation for graviton-dilaton-axion scattering amplitudes (in 4 dimensional spacetime). This clarifies their physical origin and also provides a simpler version of the new identities.

In deriving the field theory version of the KLT relation, Bjerrum-Bohr, Damgaard, Feng and Sondergaard (BDFS) derived a new set of quadratic identities of YM amplitudes [12]. [31] gives its supersymmetric generalization. These identities are non-trivial, in the sense that they do not follow from the linear identities and very interesting, since they are unexpected, at least from the Yang-Mills theory point of view.

We like to point out that these quadratic identities actually follow directly from the KLT relation for the (massless) graviton-dilaton-axion scattering amplitudes. Recall that the KLT relation expresses the closed string tree scattering amplitudes in terms of the open string tree scattering amplitudes [39]. At the zero-slope limit, the massless open string modes can include the YM fields while the closed string modes can include the graviton, the dilaton and the anti-symmetric tensor field, which is equivalent to an axion in 4 dimensional spacetime. The symmetry of the extended gravity theory, mysteriously constraint the Yang-Mills amplitude in the form of quadratic identities.

5.1 The action of graviton-axion-dilaton system

The massless sector of a closed string theory contains the graviton, the dilaton and the antisymmetric tensor field $B_{\mu\nu}$. In 4-dimensional spacetime, $B_{\mu\nu}$ has only one degree of freedom, and one may identify it as an axion $a : \partial^\mu a = \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$.

To discuss the 4-dimensional polarization tensor structure for the graviton, the dilaton and the axion, we use the light cone gauge: consider a massless particle and the polarization vector or tensor which has only transverse components. In $4D$, a polarization tensor can be decomposed as,

$$e_{ij} = \frac{e_{ij} + e_{ji} - \delta_{ij}e_{kk}}{2} + \frac{e_{ij} - e_{ji}}{2} + \frac{\delta_{ij}e_{kk}}{2}$$

where $i = 1, 2$ labels the two transverse directions. The three terms correspond to the graviton, the axion (the antisymmetric tensor field) and the dilaton modes. We may choose the positive polarization vector $\epsilon^+ = (1, i)$ while the negative polarization $\epsilon^- = (1, -i)$. The addition of two spin 1 (with polarizations ϵ and $\tilde{\epsilon}$) is straightforward. It is easy to see that a graviton has polarization mode $\epsilon^+\tilde{\epsilon}^+$ or $\epsilon^-\tilde{\epsilon}^-$, an axion has polarization $\epsilon^+\tilde{\epsilon}^- - \epsilon^-\tilde{\epsilon}^+$ and the dilaton has polarization $\epsilon^+\tilde{\epsilon}^- + \epsilon^-\tilde{\epsilon}^+$.

The 4D graviton-axion-dilaton gravity action in Einstein frame,

$$S = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-G}(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-4\phi}H_{\mu\nu\rho}H^{\mu\nu\rho}) \tag{5.1}$$

where ϕ is the dilaton and $H_{\mu\nu\rho}$ is the strength of the antisymmetric field. We just keep the two-derivative term and neglect the higher-derivative terms from string theory correction. (So it is the low energy effective action for the gravity sector of any string theory.) The Poincare dual of $H_{\mu\nu\rho}$ is an axion,

$$\partial_\mu b = \frac{1}{6}e^{-4\phi}\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu\rho} \tag{5.2}$$

where $\epsilon_{0123} = (-\det G)^{1/2}$ and $\epsilon^{0123} = -(-detG)^{1/2}$. So the action can be rewritten as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{4\phi}\partial_\mu b\partial^\mu b) \tag{5.3}$$

Combining the axion and the dilaton, we have $S_\pm = b \pm ie^{-2\phi}$ and,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}(R - \frac{1}{2}\frac{|\partial_\mu S_+|^2}{(ImS_+)^2}) \tag{5.4}$$

Note that the axion b is characterized by the shift symmetry

$$b \mapsto b + c \tag{5.5}$$

where c is a dimensionful constant. Furthermore S_+ has $SL(2,\mathbb{R})$ global symmetry. The group $SL(2,R)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \tag{5.6}$$

acts on S_+ as,

$$S_+ \mapsto \frac{aS_+ + b}{cS_+ + d}. \tag{5.7}$$

S_+ takes value in the upper complex plane, which is the moduli space of this theory. We may choose $\langle S_+\rangle = i$. Then the unbroken symmetry is $SO(2)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.8}$$

We will see that this unbroken symmetry lead to the quadratic identities of Yang-Mills theory.

To study the perturbative theory of the axion-dilaton, we may define $S_\pm = \sqrt{2}\kappa\tau + i$ and $\tau = \tau_1 + i\tau_2$. Now $\langle\tau\rangle = 0$ and τ has the canonical kinetic terms,

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}R - \frac{1}{2}\int d^4x\sqrt{-G}\frac{\partial_\mu\tau\partial^\mu\overline{\tau}}{(1+\sqrt{2}\kappa\tau_2)^2} \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}R - \frac{1}{2}\int d^4x\sqrt{-G}\partial_\mu\tau\partial^\mu\overline{\tau}(1 - 2\sqrt{2}\kappa\tau_2 + 6\kappa^2\tau_2^2 \\ &\quad -8\sqrt{2}\kappa^3\tau_2^3 + ...) \end{aligned} \tag{5.9}$$

all the interaction terms contain positive powers of κ, as it should be. To the leading order, the symmetry 5.8 acts on τ as,

$$\tau \mapsto \tau + 2i\theta\tau, \tag{5.10}$$

So τ has an infinitesimal $U(1)$ symmetry. We may think that τ is positively charged and corresponds to the polarization $\epsilon^+\tilde{\epsilon}^-$, while τ is negatively charged and corresponds to the polarization $\epsilon^-\tilde{\epsilon}^+$.

It is possible to derive the Feynman rules for axion-dilaton system from (5.9). However, as the gravity case, there is an infinite number of vertices. So in the next section, we would use KLT relation to study its scattering amplitude and manifest the quadratic identities.

5.2 Quadratic identity in the viewpoint of KLT relation

Now we see that the dilaton and the axion combine to form a massless complex scalar field τ, which has a global conserved $U(1)$ charge associated with it. All scattering (tree or loop) amplitudes must obey this charge conservation.

Within helicity amplitudes, graviton j has helicity $\epsilon^\pm_j\tilde{\epsilon}^\pm_{\tilde{j}} = j^\pm\tilde{j}^\pm$ and an incoming positively charged scalar field j may be identified with helicity $j^+\tilde{j}^-$ while an incoming negatively charged scalar field may be identified with helicity $j^-\tilde{j}^+$. Any charge conservation-violating amplitude $\mathcal{A}$ must vanish. That is, any amplitude with unequal numbers of positively and negatively charged scalar fields will vanish. Let us start with a non-vanishing M-graviton scattering amplitude. Following the BDFS notation, let n_+ (n_-) be the number of "+" ("−") helicities in YM amplitude A that have been flipped in YM amplitude $\tilde{A}$. Then the resulting amplitude vanishes whenever $n_+ \neq n_-$.

Let us consider the 4-point case to establish some notation: the graviton-dilaton-axion scattering amplitude takes the form

$$\mathcal{A}_4 = -s_{12} A(1234) \tilde{A}(2134) \tag{5.11}$$

where both A and $\tilde{A}$ are YM amplitudes. For 4-graviton amplitudes, helicity conservation requires 2 with helicity $(++)$ and 2 with helicity $(--)$. So the only non-vanishing amplitude has the form

$$\mathcal{A}_4 = -s_{12} A(1^- 2^- 3^+ 4^+) \tilde{A}(2^- 1^- 3^+ 4^+) \tag{5.12}$$

Note that both A and $\tilde{A}$ are maximal helicity-violating amplitudes. For $(n_+, n_-) = (1, 1)$, say

$$\mathcal{A}_4 = -s_{12} A(1^- 2^- 3^+ 4^+) \tilde{A}(2^+ 1^- 3^- 4^+) \tag{5.13}$$

the amplitude describes the graviton-ϕ scattering. For $(n_+, n_-) = (2, 2)$, $\mathcal{A}_4$ describes the ϕ-ϕ scattering. For $n_+ - n_- \neq 0$, $\mathcal{A}_4 = 0$ because the charge conservation is violated. Mathematically, we see that it vanishes because $\tilde{A} = 0$. This case does not yield a quadratic identity. Rather it yields the well known results $\tilde{A}(1^+ 2^- 3^+ 4^+) = 0$ and $\tilde{A}(1^+ 2^+ 3^+ 4^+) = 0$ without having to look into the detailed structure of these amplitudes.

Next consider the 5-graviton scattering case,

$$\begin{aligned} \mathcal{A}_5 &= s_{12} s_{34} A(1^- 2^- 3^+ 4^+ 5^+) \tilde{A}(2^- 1^- 4^+ 3^+ 5^+) \\ &+ \quad s_{13} s_{24} A(1^- 3^+ 2^- 4^+ 5^+) \tilde{A}(3^+ 1^- 4^+ 2^- 5^+) \end{aligned} \tag{5.14}$$

For $n_+ - n_- \neq 0$, the resulting $\mathcal{A}_5 = 0$. For example, for $(n_+, n_-) = (1, 0)$, we have

$$\begin{aligned} 0 &= s_{12} s_{34} A(1^- 2^- 3^+ 4^+ 5^+) \tilde{A}(2^- 1^- 4^+ 3^- 5^+) + \\ & s_{13} s_{24} A(1^- 3^+ 2^- 4^+ 5^+) \tilde{A}(3^- 1^- 4^+ 2^- 5^+) \end{aligned} \tag{5.15}$$

It is easy to verify this quadratic identity by using the explicit formulae for the MHV amplitudes. For $M \geq 6$, non-MHV amplitudes appear in the quadratic identities.

Following from the KLT relation, each identity contains $(M-3)![\frac{1}{2}(M-3)]![\frac{1}{2}(M-3)]!$ (Modd) or $(M-3)![\frac{1}{2}(M-4)]![\frac{1}{2}(M-2)]!$ (Meven) terms quadratic in the M-point YM amplitudes. So the number of terms in each identity is substantially less than $[(M-3)!]^2$: 2 vs 4 for $M=5$, 12 vs 36 for $M=6$, 96 vs 576 for $M=7$ etc.. Of course, one may use the BCJ identity (repeatedly) to reduce the number of terms in the quadratic identities to that in the KLT relations.

5.2.1 Additional Quadratic Identities

Although ordering is important in YM amplitudes A_M and $\tilde{A}_M$, $\mathcal{A}_M$ is invariant. For example, besides Eq.(5.11), one can express the same $\mathcal{A}_4$ in other forms, $\mathcal{A}_4 = -s_{13}A(2134)\tilde{A}(1324) = ...$ Comparing this expression to Eq.(5.11), one obtains a quadratic identity,

$$s_{12}A(1234)\tilde{A}(2134) - s_{13}A(2134)\tilde{A}(1324) = 0$$

These identities are valid for any choice of helicities. Since the same $\mathcal{A}_M$ can be expressed in terms of A_M and $\tilde{A}_M$ in many different ways, we obtain a large set of new quadratic identities by equating any 2 different expressions. This result is known but not emphasized in earlier work. Presumably, these identities can be proven by using the BCJ identity (repeatedly); still their usefulness may follow from the relative ease in writing them down.

For example, expressing the same $\mathcal{A}_6$ in 2 different ways [39], we end up with a quadratic identity

$$\begin{aligned} & A(123456)[s_{35}\tilde{A}(215346) + s_{3(45)}\tilde{A}(215436)] \\ & -\tilde{A}(123456)[s_{13}A(231546) + s_{3(12)}A(321546)] + \text{permutations of } (234) = 0 \end{aligned} \quad (5.16)$$

where $s_{3(45)} = s_{34} + s_{35}$. Here, in each case where $\mathcal{A}_6 = 0$, we get a set of quadratic identities instead of only one. For example, besides

$$\begin{aligned} & [s_{35}\tilde{A}(2^-1^-5^+3^+4^+6^+) + s_{3(45)}\tilde{A}(2^-1^-5^+4^+3^+6^+)] \\ & \times A(1^-2^-3^-4^+5^+6^+) + \text{permutations of } (234) = 0 \end{aligned} \quad (5.17)$$

we also get

$$\begin{aligned} & [s_{13}A(2^-3^-1^-5^+4^+6^+) + s_{3(12)}A(3^-2^-1^-5^+4^+6^+)] \\ & \times\tilde{A}(1^-2^-3^+4^+5^+6^+) + \text{permutations of } (234) = 0 \end{aligned}$$

5.3 Super quadratic relations

We see that the global symmetry of gravity induced the quadratic identities for Yang-Mills theory. Furthermore, the global symmetry of supergravity induced a larger set of quadratic identities for super-Yang-Mills theory.

We may consider $d = 4, N = 8$ supergravity, which has the global $E_{7(7)}$ symmetry. This symmetry is reduced to $SU(8)$ symmetry by the v.e.v. of the moduli field. The elements in $SU(8)$ are R-symmetries which transform the 8 supercharges. A $N = 8$ supergravity amplitude would vanish if the $SU(8)_R$ symmetry is broken. However, by the super KLT relation, the supergravity amplitude can be decomposed as the product of super-Yang-Mills amplitudes. The latter may not vanish, so we obtain nontrivial identities of the super-Yang-Mills amplitudes.

For example, we may the scattering process which involves one positive-helicity graviton (++), two negative-helicity gravitons (−−) and two graviphoton with the R index 12, 34, respectively. These particles form the representation of $SU(8)_R$, $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$, which is not a singlet. Hence the amplitude vanishes,

$$A(1^{++}, 2^{\phi^{12}-}, 3^{\phi^{34}-}, 4^{--}, 5^{--}) = 0, \tag{5.18}$$

Then we rewrite it by the super KLT relation. For the graviphoton state $2^{\phi^{12}-}$, we decompose it as the left-handed scalar ϕ^{12} and the righ-handed negative-helicity photon. Similar decomposition works for $3^{\phi^{34}-}$. Then we get the identity,

$$\begin{aligned} 0 &= s_{12}s_{34}A(1^- 2^{\phi^{12}} 3^{\phi^{34}} 4^+ 5^+)\tilde{A}(2^- 1^- 4^+ 3^- 5^+) \\ &+ s_{13}s_{24}A(1^- 3^{\phi^{34}} 2^{\phi^{12}} 4^+ 5^+)\tilde{A}(3^- 1^- 4^+ 2^- 5^+) \end{aligned} \tag{5.19}$$

This is a quadratic identity for $N = 4$ super-Yang-Mills theory.

Note that the global symmetry in supergravity would be reduced to discrete symmetry in superstring theory. So the open string theory amplitudes are not constricted by this argument.

5.4 Pion theorem and its implication on Yang-Mills amplitudes

The above KLT treatment also implies that *the pion theorem for supergravity put constraints on Yang-Mills amplitude*. We consider $N = 8$ supergravity, which contains 70 massless scaler fields. These scaler fields can be understood as the Goldstone bosons (pion) from the symmetry breaking

$$E_{7(7)} \mapsto SU(8), \quad 133 - 63 = 70. \tag{5.20}$$

In effective action, Goldstone bosons only have derivative couplings. Pion theorem states that the scattering amplitude with pions vanishes, if the pion's momentum is zero. By the KLT relation, the vanishing of $N = 8$ supergravity amplitude in the soft pion limit.

Recall that the KLT formula gives

$$\begin{aligned}&(\text{graviton-scalar amplitude }) = \sum(\text{kinematic factors})\\&(\text{Yang-Mills amplitude}) \times (\text{Yang-Mills amplitude})\end{aligned} \tag{5.21}$$

where a pair of scalars may be labeled as π^{+-}, which are simply scalars with helicity $+-$ and $-+$ respectively.

It is interesting to see how, as the momentum of one of the scalars is becoming soft, the gluon amplitudes are blowing up but the particular combination of their products in KLT vanishes in this IR limit. For the soft momentum $\sim \epsilon^2$, usually both the left and right Yang-Mills amplitudes diverges as ϵ^{-2} and the kinematic factors may vanish as ϵ^2. So naively, it seems that an individual term in KLT diverges. It is interesting to see how such leading and as well as next to leading divergent terms in the product of the gluon amplitudes cancel out. This way, we shall find new constraints of the Yang-Mills amplitudes.

Assume that the particle 1 is soft, so we can find a reference frame such that, in spinor helicity formalism,

$$\mathbf{1} = \epsilon\lambda, \quad \tilde{\mathbf{1}} = \epsilon\tilde{\lambda} \tag{5.22}$$

where ϵ is the small parameter. Note that all the other momenta has ϵ dependence because of the momentum conservation,

$$\begin{aligned}
\mathbf{2} &= 2 + \epsilon^2 2_2 + \epsilon^4 2_4 + ..., \quad \tilde{\mathbf{2}} = \tilde{2} + \epsilon^2 \tilde{2}_2 + \epsilon^4 \tilde{2}_4 + ..., \\
\mathbf{3} &= 3 + \epsilon^2 3_2 + \epsilon^4 3_4 + ..., \quad \tilde{\mathbf{3}} = \tilde{3} + \epsilon^2 \tilde{3}_2 + \epsilon^4 \tilde{3}_4 + ..., \\
&\quad ...
\end{aligned} \tag{5.23}$$

Here we use the bold font to emphasize the momenta dependent on ϵ while the regular font for each order in the ϵ expansion. The momentum-conservation constraints are,

$$\sum_{i=2}^{M} i\tilde{i} = 0 \tag{5.24}$$

$$\lambda\tilde{\lambda} + \sum_{i=2}^{M} (i_2\tilde{i} + i\tilde{i}_2) = 0, \tag{5.25}$$

Since all the momenta p_i and the polarization vectors ϵ_i $(i = 1, ...M)$ contain only the even order of ϵ, the odd orders of ϵ in the amplitude are absent.

We consider the particle 1 to be the the pion $(+-)$, in $N = 8$ supergravity , i.e. $+1$ helicity from left-moving sector and -1 helicity from right-moving section. (Because of the $E_{7(7)}$ global symmetry, it is without generality to consider one of the pion $(+-)$.) In the soft pion limit,

$$\lim_{\epsilon \to 0} A(\mathbf{123...M}) = 0. \tag{5.26}$$

Take 5-point amplitude as an example, expand the supergravity amplitude $A(\mathbf{12345})$ in the KLT form, we have

$$A(\mathbf{12345}) = s_{\mathbf{12}} s_{\mathbf{34}} A^L(\mathbf{1}^+\mathbf{2345}) A^R(\mathbf{21}^-\mathbf{435}) + s_{\mathbf{13}} s_{\mathbf{24}} A^L(\mathbf{1}^+\mathbf{3245}) A^R(\mathbf{31}^-\mathbf{425}). \tag{5.27}$$

where A^L and A^R are left and right super Yang-Mills amplitudes. It is clear that

$$s_{\mathbf{12}} s_{\mathbf{34}} \sim \epsilon^2, \quad A^L(\mathbf{1}^+\mathbf{2345}) \sim \epsilon^{-2}, \quad A^R(\mathbf{21}^-\mathbf{435}) \sim \epsilon^{-2}, \tag{5.28}$$

so each term diverges as ϵ^{-2}. However, from (5.26), $A(\mathbf{12345})$ should vanish so it seems to be a contradiction. We need to look at (5.27) in detail.

5.4.1 Leading order

First, we show that the divergence terms in (5.27) cancel out. The super Yang-Mills amplitudes in the soft-pion limit read,

$$\begin{aligned} A^L(\mathbf{1}^+\mathbf{2345}) &= \frac{1}{\epsilon^2}\frac{\langle 52\rangle}{\langle 5\lambda\rangle\langle\lambda 2\rangle}(A^L(2345)+\epsilon^2 B^L(1^+2345))+\ldots \\ A^R(\mathbf{21}^-\mathbf{435}) &= \frac{1}{\epsilon^2}\frac{[24]}{[4\tilde{\lambda}][\tilde{\lambda}2]}(A^R(2435)+\epsilon^2 B^R(21^-435))+\ldots \end{aligned} \tag{5.29}$$

where both $A^L(2345)$ and $B^L(1^+2345)$ are indepedent of ϵ. Here we just need the leading order.

In the soft gluon limit, the divergent terms of (5.27) are reduced to,

$$\begin{aligned} & s_{12}s_{34}\frac{\langle 52\rangle}{\langle 5\lambda\rangle\langle\lambda 2\rangle}\frac{[24]}{[4\lambda][\lambda 2]}A^L(2345)A^R(2435) \\ & +s_{13}s_{24}\frac{\langle 53\rangle}{\langle 5\lambda\rangle\langle\lambda 3\rangle}\frac{[34]}{[4\lambda][\lambda 3]}A^L(3245)A^R(3425) \\ = \; & \frac{\langle 52\rangle}{\langle 5\lambda\rangle}\frac{[24]}{[4\lambda]}s_{34}A^L(2345)A^R(2435)+\frac{\langle 53\rangle}{\langle 5\lambda\rangle}\frac{[34]}{[4\lambda]}s_{24}A^L(3245)A^R(3425)=0 \end{aligned} \tag{5.30}$$

It is easy to show that this requirement holds because of the 4-point KLT and BCJ relations:

$$s_{34}A^L(2345)A^R(2435)=-A(2345), \quad s_{24}A^L(3245)A^R(3425)=-A(3245). \tag{5.31}$$

Because there is no order in the gravity amplitude, the divergent term is,

$$=\left(\frac{\langle 52\rangle}{\langle 5\lambda\rangle}\frac{[24]}{[4\lambda]}+\frac{\langle 53\rangle}{\langle 5\lambda\rangle}\frac{[34]}{[4\lambda]}\right)A(2345)=0 \tag{5.32}$$

where we used momentum conservation $\langle 52\rangle[24] + \langle 53\rangle[34] = 0$.

It is also straightforward to show that the 5-point KLT and BCJ relations ensure that in the soft pion limit, the 6-point sugergravity amplitude is not divergent.

5.4.2 The next to leading order

The above result still have a non-vanishing finite term. The condition for the vanishing ϵ^0 order in the graviton amplitude gives the constraint for the next-to-leading terms B^L and B^R,

$$\begin{aligned} A(2345)\cdot K &= \frac{\langle 52\rangle[24]}{\langle 5\lambda\rangle[4\lambda]}\cdot\Big[-s_{34}\Big(B^L(1^+2345)A^R(2435)+A^L(2345)B^R(21^-435)\Big) \\ &\quad +s_{24}\Big(B^L(1^+3245)A^R(3425)+A^L(3245)B^R(31^-425)\Big)\Big] \end{aligned} \tag{5.33}$$

where K is a kinematic factor indepedent of ϵ,

$$K = \Big(1-\frac{\langle 52\rangle[24]}{\langle 5\lambda\rangle[4\lambda]}\big(\frac{\delta s_{34}}{s_{34}}-\frac{\delta s_{24}}{s_{24}}\big)\Big) \tag{5.34}$$

where $\delta\langle 34\rangle = \langle 3_2 4\rangle + \langle 34_2\rangle$ and so on. Furthermore, we may use 4-point BCJ relation to simplify this above condition,

$$\begin{aligned} A(2345)\cdot K &= \frac{\langle 52\rangle[24]}{\langle 5\lambda\rangle[4\lambda]}\cdot s_{34}\cdot\Big[\Big(\frac{s_{24}}{s_{34}}B^L(1^+3245)-B^L(1^+2345)\Big)A^R(2435) \\ &\quad + A^L(2345)\Big(B^R(31^-425)-B^R(21^-435)\Big)\Big] \end{aligned} \tag{5.35}$$

- Example 1. $(1^{+-}2^{-+}3^{++}4^{++}5^{--})$. In this case, $A(\mathbf{2345}) = 0$ because of the global symmetry violation. On the other hand, $A^R(2435) = 0$ and explicit computation showes that

$$B^R(3^+1^-4^+2^+5^-) = 0, \quad B^R(2^+1^-4^+3^+5^-) = 0$$

Therefore (5.35) trivially holds.

- Example 2. $(1^{+-}2^{++}3^{++}4^{--}5^{--})$. The explcit computation gives,

$$B^L(1^+3^+2^+4^-5^-) = A^L(3^+2^+4^-5^-)\left[3\frac{\delta\langle 45\rangle}{\langle 45\rangle} - \frac{\delta\langle 32\rangle}{\langle 32\rangle} - \frac{\delta\langle 24\rangle}{\langle 24\rangle} - \frac{\delta\langle 53\rangle}{\langle 53\rangle}\right]$$
$$B^L(1^+2^+3^+4^-5^-) = A^L(2^+3^+4^-5^-)\left[3\frac{\delta\langle 45\rangle}{\langle 45\rangle} - \frac{\delta\langle 32\rangle}{\langle 32\rangle} - \frac{\delta\langle 34\rangle}{\langle 34\rangle} - \frac{\delta\langle 52\rangle}{\langle 52\rangle}\right] \quad (5.36)$$

and also $B^R(2^+1^-4^-3^+5^-) = B^R(3^+1^-4^-2^+5^-)$. So most terms in (5.35) vanished, and by using 4-point BCJ, (5.35) reads,

$$K = \frac{\langle 52\rangle[24]}{\langle 5\lambda\rangle[4\lambda]}\left[\frac{\delta\langle 24\rangle}{\langle 24\rangle} + \frac{\delta\langle 53\rangle}{\langle 53\rangle} - \frac{\delta\langle 34\rangle}{\langle 34\rangle} - \frac{\delta\langle 52\rangle}{\langle 52\rangle}\right] \quad (5.37)$$

which holds because of the kinematic identities like $\delta s_{34}/s_{34} = \delta\langle 34\rangle/\langle 34\rangle + \delta[34]/[34]$ and the momentum conservation.

Note that the example 2 is actually trivial since the supergravity amplitude $(1^{+-}2^{++}3^{++}4^{--}5^{--})$ is zero to all the orders of ϵ, because of the R-symmetry violation. The similar but non-trivial example would be $(1^{+-}2^{-+}3^{--}4^{--}5^{++}6^{++})$, which involves more calculation.

CHAPTER 6

SUMMARY

6.1 Discussion and Remarks

In this dissertation, we discussed the new development of scattering amplitude in gauge and gravities theories. We have shown that the study of LHC physics requires the improvement of scattering amplitude computation, while the traditional Feynman diagram approach becomes inefficient for scattering process with many particles. The string theory scattering amplitude calculation is reviewed, and we illustrated how to apply them for Yang-Mills and Einstein gravity amplitude. To make the most of string theory techniques, we also introduced other new techniques like spinor helicity formalism, BCFW recursive relations and coherent superspace.

With these new methods, we studied two classes of recursive relations,

- BCJ identities, which is a linear relation of Yang-Mills amplitudes.
- Quadratic identities.

In our work[37], BCJ identity is studied by heterotic string theory. The heterotic string contains the gauge sector, as (color) $\otimes$ (vector). Both sectors satisfy the same identity by the string monodromy relation. So the equivalence of the kinematic and color identities in BCJ conjecture is proven.

The heterotic string also contains the graviton sector, as (vector)$\otimes$(vector). So we can calculate the graviton scattering amplitudes in the heterotic string theory and then take the limit $\alpha' \to 0$ to get the graviton scattering amplitude in Einstein gravity. The graviton sector has both left and right-moving non-compact momenta, and the closed string amplitude can be separated into the product of left and right-moving open string amplitude, both of which are calculated in the Yang-Mill amplitude. As explained already, the M-graviton tree scattering amplitude is

$$A_{\text{M-graviton}} = \sum_j \frac{\tilde{n}_j n_j}{P_j} \tag{6.1}$$

where $n_j(k_i, \zeta_i)$ contains the polarizations ζ_i^μ while $\tilde{n}_j$ contains the polarization ξ_i^μ. Otherwise, they have identical functional forms, i.e., $\tilde{n}_j(k_i, \xi_i) = n_j(k_i, \xi_i)$. The graviton polarization $\epsilon_{\mu\nu}$ is the symmetrized traceless product in $\xi_\mu \zeta_\nu$.

We can easily incorporate the scattering of fermions since spinors are present in the right-moving superstring part of the heterotic string. This has been discussed in Ref.[28][16]. Keeping only the leading order in α', we have where the n_j^L and separately the n_j^R obey the same set of identities as the color factors. Here n_j^L can include both colors and/or vectors while n_j^R can include both vectors and fermions. These include scattering amplitudes involving gravitons, gluons, gravitinos and gluinos. The resulting identities should be very helpful in the evaluation of scattering amplitudes.

We can consider special polarizations to simplify the computation, for example, by using the spinor helicity formalism [7][22] [40] [32][56] [36]. where q_i is the reference momentum. In this convention, by careful choices of the q_i, many terms in the Yang-Mills amplitude vanish. For example, within the four-point partial amplitude $A(1^-, 2^-, 3^+, 4^+)$, $n_t = 0$. In this case, the kinematic identity $n_s + n_t + n_u = 0$ implies $n_s = -n_u$ and we just need to consider one channel.

Spinor helicity formalism is used for the graviton polarization [52], For example, since the Yang-Mills amplitude with only one gluon with opposite helicity vanishes: $\mathcal{A}_M^{\mathrm{YM}}(1^-, 2^+, 3^+, ..., M^+) = 0$, the corresponding $\mathcal{A}_M^{grav}(1^{--}, 2^{++}, 3^{++}, ..., M^{++})$ also vanishes.

The heterotic string involves modes in higher representations of the Lie group. Keeping them will introduce generalized "structure constants" f^{abC}, f^{aBC} and f^{ABC}, where capital letters A, B, C signify modes in higher representations (than the adjoint representation). Then the string amplitude identities (similar to (4.48)) will yield the corresponding *generalized Jacobi identities* among them.

As shown in Ref.[57], the kinematic identities can be extended to include the scattering of massive particles. Since the open string amplitudes are multivariable-integrals involving the Koba-Nielsen variables, such Koba-Nielsen integrals can be generalized to include massive particles with higher spins and to obtain the kinematic identities in the scattering of massive particles.

One can also start with $D = 10$ dimensions and compactify (toroidally) 6 of them. The resulting theory in the zero slope limit is a low energy effective $\mathcal{N} = 4$ supergravity theory. This allows us to study scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as well as in $\mathcal{N} = 4$ supergravity theory. The identities in the tree amplitudes will be carried over to the loop diagrams using the unitarity method. This should provide a better understanding of the loop amplitudes in the $\mathcal{N} = 4$ theory.

In this dissertation, we have restricted our discussions to the scattering of massless particles only. However, the analysis of Ref.[57, 33] strongly suggests that the kinematic identities can be generalized to massive particles as well.

We also used KLT relation to study the quadratic identities for Yang-Mills theory. The quadratic identities can be easily associated with the global symmetry in gravity theory. Furthermore, the symmetry in supergravity provides generalized quadratic identities for super-Yang-Mills amplitudes. It is possible to explore further properties of Yang-Mills theory amplitude by the global symmetry of gravity. In particular, the pion theorem in $d = 4, N = 8$ supergravity may put important constraint for $d = 4, N = 4$ super-Yang-Mills amplitudes.

There are many interesting future directions:

- $N = 8$ supergravity amplitudes. Is there a neat formalism for $N = 8$ supergravity like $N = 4$ Grassmannian formalism?
- Massive recursive identities. We would like to find a systematic set of recursive relations for amplitudes with massive fermion.
- Loop-level recursive identities. Although there are examples the BCJ identities hold in loop level, the systematic proof is not available right now.

The answer to these topics would lead a revolution in the understanding of Yang-Mills and gravity theories.

REVIEW OF YANG-MILLS THEORY AND EINSTEIN GRAVITY THEORY

In this chapter, we briefly review the Yang-Mills theory, Einstein gravity and their possible extends. We will use the natural units,

$$\hbar = 1, \quad c = 1, \quad k = 1 \tag{A.1}$$

Throughout this paper, we would use $(+, -, \ldots, -)$ metric signature for field theory calculation, while $(-, +, \ldots, +)$ signature for string and gauge theories.

A.1 Yang-Mills theory

Yang-Mills theory was invented to describe the gauge interactions.

A.1.1 Abelian gauge theory

In 1940s, quantum electrodynamics (QED), the quantum theory of the electromagnetic field and the electron was discovered by H. Bethe, S. Tomonaga, J. Schwinger, R. Feynman, F. Dyson and etc. QED precisely predicted the Lamb shift in hydrogen atoms and the anomalous magnetic dipole of electrons. The action of QED is,

$$\mathcal{L}_{QED} = \int d^4x \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\psi \Big) \tag{A.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and ψ is the Dirac field for electrons. QED has the $U(1)$ gauge symmetry,

$$A_\mu \mapsto A_\mu - \frac{i}{e} e^{-i\alpha(x)} \partial_\mu e^{i\alpha(x)}, \quad \psi(x) \mapsto e^{-i\alpha(x)} \psi(x) \tag{A.3}$$

for a local function $\alpha(x)$. $e^{-i\alpha(x)}\partial_\mu e^{i\alpha(x)}$ is called the *Maurer-Cartan form* of $U(1)$. We keep this form in the transformation of A, to emphasize that it is the group element $e^{i\alpha(x)}$, not $\alpha(x)$ itself, that should satisfy certain boundary conditions if we study topological defects.

The interaction term in QED action is $-e\bar{\psi}\not{A}\psi$, where e is the charge of the fermion. The leading order QED interaction is proportional to the *fine structure constant* α,

$$\alpha = \frac{e^2}{4\pi}. \tag{A.4}$$

In the energy scale of the electron mass $\sim 511keV$, $\alpha(m_e) \sim 1/137$, while in the energy scale of the Z boson mass $\sim 90GeV$, $\alpha(m_e) \sim 1/128$. These values are small enough to allow the perturbative calculation, which can be effectively organized as the QED Feynman diagrams series. The QED Feynman rules are shown in Fig.A.1, where we used the Lorentz-like gauge condition $\partial_\mu A^\mu(x) = \omega(x)$ and the

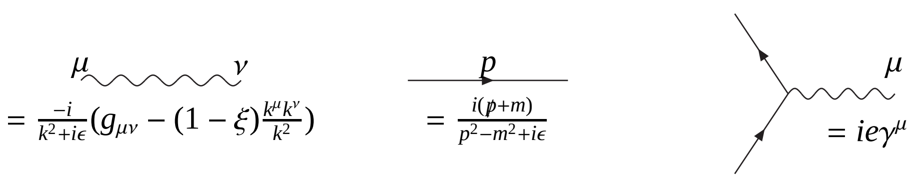


Figure A.1: QED Feynman rules. (a) photon propagator (b) fermion propagator (c) Electromagnetic vertex.

parameter ξ is the weight for ω. The choice $\xi = 1$ is the *Feynman gauge*, while $\xi = 0$ is the *Landau gauge*. The equivalence of all the choices of ξ is proven by the following *Ward-Takahashi identity*.

The U(1) gauge symmetry manifests itself in QED amplitude as the Ward-Takahashi identity. In general, for the off-shell Green function G with m photons, n incoming electrons and n outgoing electrons, if the one photon's polarization is taken to be longitudinal, i.e., $\epsilon_1 = k_1$, this Green function is related to green functions with one-fewer photons,

$$\begin{aligned}
& G(k_1, \epsilon_1 = k_1; ...; k_m; \epsilon_m, p_1, ...p_n, q_1, ...q_n) \\
&= e \sum_{i=1}^{n} G(k_2, \epsilon_2; ...; k_m, \epsilon_m; p_1, ...p_n, q_1, ...(q_i - k), ...q_n) \\
&- e \sum_{i=1}^{n} G(k_2, \epsilon_2; ...; k_m, \epsilon_m; p_1, ...(p_i + k_1), ..., p_n, q_1, ...q_n)
\end{aligned} \tag{A.5}$$

In particular, the corresponding scattering amplitude is the residue of the pole $1/(\prod_i(\not{p}_i - m)\prod_i(\not{q}_i - m))$. However, because the right-hand-side of (A.5) has no such pole, the corresponding scattering amplitude is zero.

$$A(k_1, \epsilon_1 = k_1; ...; k_m; \epsilon_m, p_1, ...p_n, q_1, ...q_n) = 0. \tag{A.6}$$

This identity would play an important role for the study of gauge theory amplitudes.

QED also satisfies the discrete symmetries, C, P and T. The charge conjugation symmetry C implies an important relation, the Furry theorem: since the photon has negative C parity,

$$C\psi C = -i(\gamma^2)^T \psi^*, \quad CA_\mu(x)C = -A_\mu(x) \tag{A.7}$$

the scattering amplitudes with an odd number of photons and without electron vanishes to all orders,

$$A(k_1, \epsilon_1; ...; k_m; \epsilon_m) = 0, \quad m \text{ is odd.} \tag{A.8}$$

Perturbatively, the beta function for e in QED is positive,

$$\mu \frac{de}{d\mu} = \frac{e^3}{12\pi^2} + ... \tag{A.9}$$

So the value of e increases as the energy scale increases.

A.1.2 Non-Abelian gauge theory

After the success of QED, great effort was spent on the gauge theory description of the hadron interactions. M. Gell-Mann and K. Nishijima proposed the quark model for hadrons. In this model, the meson is made of quark and anti-quark pairs while the baryon is made of three quarks. It leads to the discovery of *colors*. For example, the Δ^{++} baryon consists three u quarks and their spacetime and spin wavefunctions are totally symmetric. It seems that the *Pauli exclusive princlple* for fermions is violated. To resolve this puzzle, we need to add the three color degrees of freedom to quarks and assume the three quarks are in color totally antisymmetric state.

The three color degrees of freedom have a $SU(3)$ symmetry, which can be gauged as a local symmetry in *Yang-Mills theory*, i.e. non-Abelian gauge theory. The gauge vector boson is *gluon*. Different from photons, gluons are not neutral under the gauge symmetry. The action for Yang-Mills theory with the gauge group G is,

$$\mathcal{L}_{YM} = -\frac{1}{2}tr(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\partial\!\!\!/ - m + gA\!\!\!/)\psi \tag{A.10}$$

where the *Lie-algebra valued* one-form $A_\mu = A^a_\mu T^a$, are the gluon fields. T^a's are the generators of G. The trace, tr, actually means the Killing-form for G and the normalization is,

$$tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{A.11}$$

The field strength $F_{\mu\nu}$ is related with the field potential A_μ as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{A.12}$$

or equivalently,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{A.13}$$

where $[T^a, T^b] = if^{abc}T^c$ and f^{abc} is the structure constant for G. ψ are the spinors in a specific representation of G. For example, in the SM, quarks are in the fundamental representation of $G = SU(3)$. g is the coupling constant of Yang-Mills theory. In general, if G is not a simple Lie group, i.e., G's Lie algebra $\mathfrak{g}$ can be decomposed as,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n, \tag{A.14}$$

then we can decompose (A.10) into n Yang-Mills terms, and we can give different coupling constants g_i's for each sub Lie algebra.

The gauge transformation of (A.10) is,

$$A_\mu(x) \mapsto U(x)^{-1}A(x)U(x) + \frac{i}{g}U(x)^{-1}\partial_\mu U(x), \quad \psi(x) \mapsto U(x)^{-1}\psi(x), \tag{A.15}$$

where $U(x) \in G$. Again $U(x)^{-1}\partial_\mu U(x)$ is the Maurer-Cartan form of G.

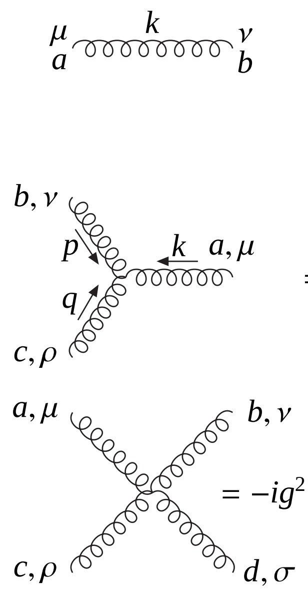
The interaction terms in (A.10) are,

$$-gf^{abc}\partial_\mu A^a_\nu A^{\mu b}A^{\nu c} + g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} + ig\bar{\psi}\not{A}^a_\mu t^a\psi \tag{A.16}$$

where $t^a \equiv (t^a)_{ij}$ which is the representation matrix of T^a. So unlike photons in QED, gluons in Yang-Mills theory are self-interacting. The corresponding Feynman rules for gluons are shown in Fig. (A.2).

Note that each vertex contains a color factor and a kinematic factor. Because gluons are bosonic, the vertices are totally symmetric in the legs. In particular, for three-gluon vertices, both the color and kinematic factor are antisymmetric so the product is symmetric.

The fermion propagator and vertex in shown in Fig.(A.3). Furthermore, in QED, the Faddeev-Popov ghost is decoupled from the theory, so there is no

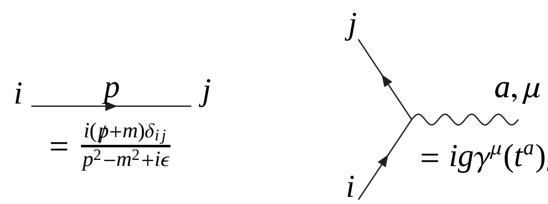


$$= \frac{-i\delta_{ab}}{k^2+i\epsilon}(\eta_{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2})$$

$$= g f^{abc}[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu]$$

$$= -ig^2[f^{abc}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\rho}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})]$$

Figure A.2: Feynman rules for gluons. (a) gluon propagator (b) three-point gluon vertex (c) four-point gluon vertex.



$$= \frac{i(\not{p}+m)\delta_{ij}}{p^2-m^2+i\epsilon} \qquad = ig\gamma^\mu(t^a)_{ij}$$

Figure A.3: Feynman rules for fermions in Yang-Mills theory. (a) fermion propagator (b) QCD vertex.

ghost Feynman rule in QED. However, in Yang-Mills theory, the Faddeev-Popov ghost is in the adjoint representation of G,

$$\mathcal{L}_{ghost} = \partial_\mu \bar{c}^a \partial^\mu c^a - g\bar{c}^a f^{abc} \partial^\mu (A^b_\mu c^c), \tag{A.17}$$

where the ghost c^a is a *spin-zero but fermonic* field. Ghosts cancel the unphysical degrees of freedom of gluons. The Feynman rules for ghosts are shown in Fig. (A.4). Since ghosts do not appear in the incoming or outgoing physical states, ghost propagators will form loops in any Yang-Mills Feynman diagram. Hence in tree level Yang-Mills amplitudes, we do not include the ghosts.

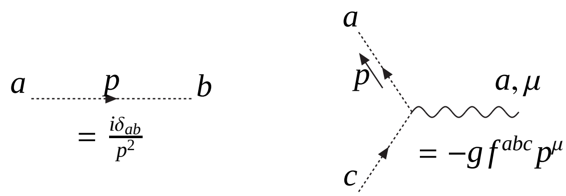


Figure A.4: Feynman rules for ghosts in Yang-Mills theory. (a) ghost propagator (b) ghost vertex.

Yang-Mills theory with $G = SU(3)$ and quarks in the fundamental representation is used to describe QCD. However in the energy scale of hadrons' mass ($\sim 100MeV$), the $SU(3)$ coupling constant $\alpha_s = g^2/(4\pi)$ is not a small parameter. So the Feynman diagram expansion does not work for the low energy QCD, for instance, the nuclear physics. In this energy scale, the degrees of freedom of QCD are not gluon or quarks, but the color-neutral particles like the pions and baryons. We can define the Yang-Mills theory on a lattice and calculate the QCD processes in lattice by numerical methods.

On the other hand, Yang-Mills theory with certain number of matter fields is *asymptotically free*, i.e., the coupling constant decreases as the energy scale increases. Perturbatively, to the one-loop order,

$$\mu\frac{dg}{d\mu} = -\frac{g^3}{(4\pi)^2}[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)] \tag{A.18}$$

where $C_2(G)$ is the *quadratic Casimir operator* for the adjoint representation, and $C(r)$ is the representation constant for the matter field. n_f is the number of matter fields. For example, if $G = SU(3)$ with n_f fundamental quarks, $C_2(G) = 3$, $C(r) = 1/2$, so as long as $n_f < 17$, the beta function is negative in the leading order and the theory is asymptotically free. Experimentally, we know that at the Z mass ($\sim 90GeV$),

$$\alpha_s(m_Z) = 0.1 \tag{A.19}$$

So in that energy scale or above, the Feynman diagram expansion is available.

Like the QED, the local non-Abelian gauge symmetry induces a set of identities, the *Slavnov-Taylor identities*. They are similar to QED's Ward-Takahashi identities but more complicated.

Again, if the fermions are Dirac spinors, the Yang-Mills theory has charge conjugation symmetry. Note that the naive flip $A_\mu^a \mapsto -A_\mu^a$ is not a symmetry, because the Yang-Mills action contains cubic terms in A and also under the charge conjugation of ψ (A.7), the representation becomes the conjugate representation,

$$\bar{\psi}_i t_{ij} \gamma^\mu \psi_j \mapsto -\bar{\psi}_i t_{ji} \gamma^\mu \psi_j. \tag{A.20}$$

So in the analog of the QED case, we may pick up the Cartan sub-Lie algebra h of G so we have r $U(1)$ gauge bosons A_μ^s, $s = 1, ...r$ and $r = \dim h$. All the other gluons are paired and charged by these $U(1)$'s. We denote them as $B_\mu^{l\pm}$, $l = 1, ...(\dim G - r)/2$. Then the correct charge conjugation symmetry is

$$A_\mu^s \mapsto -A_\mu^s, \quad B_\mu^{l\pm} \mapsto -B_\mu^{l\mp}, \quad \psi \mapsto -i(\gamma^2)^T \psi^*. \tag{A.21}$$

For example, consider $G = SU(2)$ and the scattering amplitude with five incoming gluons with the color indices, $a_1 = 1$, $a_2 = a_3 = 2$ and $a_4 = a_5 = 3$. Choose σ_3 as the generator for $SU(2)$'s Cartan Lie-algebra. σ_2 is antisymmetric while σ_1 and σ_3 are symmetric, so the charge conjugation reads

$$A_\mu^1 \mapsto -A_\mu^1, \quad A_\mu^2 \mapsto A_\mu^2, \quad A_\mu^3 \mapsto -A_\mu^3 \tag{A.22}$$

Hence the configuration is odd under the charge conjugation so the amplitude vanishes to all orders,

$$A(k_1, \epsilon_1, a_1 = 1; k_2, \epsilon_2, a_2 = 2; k_3, \epsilon_3, a_3 = 2; k_4, \epsilon_4, a_4 = 3; k_5, \epsilon_5, a_5 = 3) = 0 \tag{A.23}$$

Yang-Mills gauge bosons can also coupled to chiral fermions. For example, in the SM, the weak isospin gauge group $SU(2)_L$ coupled to left-handed quarks and leptons. In this case, both C and P parities are broken, but the CP parity is conserved up to the generation mixing effect (*CKM matrix*). The chiral fermions one-loop diagram will induce *anomalies* which may break the gauge symmetry. However, in the SM, all the gauge anomalies cancel out, and the gauge theory is consistent.

It is possible to add the *Yang-Mills angle* term in the Yang-Mills action,

$$\frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{A.24}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$ is the dual of Yang-Mills field strength. This comes from the nontrivial vacua structure of Yang-Mills theory, i.e., *θ-vacua*. This term is topological, i.e., a closed form but may not be exact globally. So it does not change the Yang-Mills Feynman rules. The Yang-Mills angle explicitly breaks CP parity.

A.2 Einstein gravity theory

Currently, gravity is described by Einstein's *general relativity*. General relativity is based on the *equivalence principle*: at any point in spacetime, we can locally choose a reference frame in which the matter satisfies the physical laws in special relativity. So we can first write down the special-relativistic equations of motion in the local frame, and then transfer them back to the original frame. The transition law of the two frames determines the *metric* in the original frame,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{A.25}$$

where ds is the proper time. The metric tensor $g_{\mu\nu}$ contains all the information of gravity. We may treat general relativity in the analogy of *Riemann geometry* and regard (A.25) as the metric for spacetime manifold.

Free particles travel in the geodesic curves of spacetime,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{A.26}$$

Note that the mass does not appear in this equation, since by the equivalence principle, the *inertial mass* equals the *gravity mass*. Furthermore, gravity couples to matters via the minimal coupling,

$$\partial_\mu \mapsto D_\mu, \quad \eta^{\mu\nu} \mapsto g^{\mu\nu}, \quad \int d^dx \mapsto \int d^dx\,\sqrt{-g}. \tag{A.27}$$

So gravity couples to all the SM particles.

The action of Einstein gravity is,

$$I = \int d^dx\,\sqrt{-g}\mathcal{L}_{EH} + \int d^dx\,\sqrt{-g}\mathcal{L}_{matter} \tag{A.28}$$

$\mathcal{L}_{EH} = -\frac{1}{16\pi G}R$ is the Einstein-Hilbert action and G is the Newton gravitational constant. The fundamental energy scale of gravity m_p, the *reduced Planck mass*, is related to G as, $(\hbar = c = 1)$,

$$m_p = \sqrt{\frac{1}{8\pi G}} \sim 2.43 \times 10^{18} GeV \tag{A.29}$$

The variance of this action gives the *Einstein equation*,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{A.30}$$

where $R^{\mu\nu}$ and R are the Ricci tensor and scalar curvature, respectively. $T_{\mu\nu}$ is the energy-momentum tensor defined as,

$$T^{\mu\nu}(x) = 2\frac{1}{\sqrt{-g}}\frac{\delta I_M}{\delta g_{\mu\nu}(x)}. \tag{A.31}$$

Note the energy-momentum tensor is symmetric by this definition.

As the electrodynamics, we want to find the plane-wave-like solution in the vacuum. We defined the flat background as the Minkowski space $eta_{\mu\nu}$ and perturb it,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \tag{A.32}$$

where $\kappa \equiv \sqrt{8\pi G} = m_p^{-1}$. We will see that the factor 2κ provides the correct normalization for the canonical kinetic term and κ serves as the coupling constant for Einstein gravity. Note that as the electrodynamics, $h_{\mu\nu}$ also has the gauge symmetry: as we apply a infinitesimal transformation, $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$,

$$h_{\mu\nu} \mapsto h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu. \tag{A.33}$$

So we can choose the *de Donder gauge* $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu (h^\mu{}_\mu)$, and the equation of motion is

$$\partial^2 h_{\mu\nu} = 0 \tag{A.34}$$

As the electrodynamics, the plane-wave solution is $h_{\mu\nu}(x) = e_{\mu\nu} e^{ik\cdot x} + e^*_{\mu\nu} e^{-ik\cdot x}$. The gauge and on-shell condition read,

$$k^2 = 0, \quad e_{\mu\nu} = e_{\nu\mu}, \quad k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu. \tag{A.35}$$

Furthermore, a gauge transformation will remove the trace mode in $e_{\mu\nu}$, so the gauge condition for $e_{\mu\nu}$ is simply,

$$e_{\mu\nu} = e_{\nu\mu}, \quad e^\mu{}_\mu = 0, \quad k^\mu e_{\mu\nu} = k^\nu e_{\mu\nu} = 0, \quad e_{\mu\nu} \sim e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu \tag{A.36}$$

where $k \cdot \epsilon = 0$. Therefore, the number of physical modes in $h_{\mu\nu}$ is $\frac{d(d+1)}{2} - d - 1 - (d-1) = d(d-3)/2$. By a Lorentz transformation on $h_{\mu\nu}$, it is clear that the helicity of the gravitational wave is ± 2. The unphysical helicity-zero mode is removed by the traceless condition, and the helicity-one modes are removed by the gauge transformation $e_{\mu\nu} \sim e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu$.

Since Einstein, people spent great effort on the quantization of the gravity. We may expand the Einstein-Hilbert action, and treat $h_{\mu\nu}$ as the particle *graviton*. It is tedious but straightforward to expand the action in $h_{\mu\nu}$. [24][25][26][38]

$$g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + 4\kappa^2 h^{\mu\lambda} h^{\nu}_{\lambda} - 8\kappa^3 h^{\mu\lambda} h_{\lambda\alpha} h^{\alpha\nu} + \ldots, \tag{A.37}$$

and,

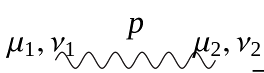
$$\sqrt{-g} = 1 + \kappa h + \frac{\kappa^2}{2}(h^2 - 2h^{\mu\nu}h_{\mu\nu}) + \frac{\kappa^3}{6}(h^3 - 6hh^{\mu\nu}h_{\mu\nu} + 8h^{\mu}_{\nu}h^{\nu}_{\lambda}h^{\lambda}_{\mu}) + \ldots \tag{A.38}$$

Hence the action would be,

$$\mathcal{L}_{EH} = \frac{1}{2\kappa^2}\sqrt{-g}R \tag{A.39}$$

$$\begin{aligned} &= \frac{1}{2}\left(\partial^{\mu}h^{\nu\lambda}\partial_{\mu}h_{\nu\lambda} - \frac{1}{2}\partial^{\mu}h\partial_{\mu}h\right) + 2\kappa\left(\frac{1}{2}h^{\alpha}_{\beta}\partial^{\mu}h^{\beta}_{\alpha}\partial_{\mu}h - \frac{1}{2}h^{\alpha}_{\beta}\partial_{\alpha}h^{\mu}_{\nu}\partial^{\beta}h^{\nu}_{\mu} - h^{\alpha}_{\beta}\partial_{\mu}h^{\nu}_{\alpha}\partial^{\mu}h^{\beta}_{\nu}\right. \\ &\quad \left. + \frac{1}{4}h\partial^{\alpha}h^{\mu}_{\nu}\partial_{\alpha}h^{\nu}_{\mu} + h^{\beta}_{\mu}\partial_{\nu}h^{\alpha}_{\beta}\partial^{\mu}h^{\nu}_{\alpha} - \frac{1}{8}h\partial^{\mu}h\partial_{\mu}h\right) + \ldots\ , \end{aligned} \tag{A.40}$$

From the quadratic terms, we can read the graviton propagator, However, the



$$= \frac{-i}{p^2+i\epsilon}\left(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} - \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \frac{2}{D-2}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\right)$$

Figure A.5: propagator for gravitons in de Donder gauge.

three-graviton vertex is very complicated. Furthermore, unlike the Yang-Mills theory, the action expansion in $h_{\mu\nu}$ does not truncate at finite order. Therefore, the quantized Einstein gravity contains infinite types of vertices.

Note that the three-graviton vertex is proportional to κ, which has the dimension $1 - d/2$. So for $d > 2$, this interaction term is non-renormalizable. Similarly, all the other vertices in Einstein gravity are non-renormalizable. So to cancel the loop divergences in scattering amplitude, we have to introduce an infinite number of counter terms and also renormalization conditions. So even to the one-loop order, it is impossible to determine all the counter terms at once. In this sense, the quantum Einstein gravity is not well-defined.

However, we may treat the quantum Einstein gravity as an effective theory, i.e. quantum Einstein gravity is well-defined in the energy scale far below the Planck mass. Since the high-order vertices are suppressed by powers of (E/m_p), for the cases if $E << m_p$, we may neglect their contribution. Then in practice, we can use quantum Einstein gravity to calculate low-energy scattering processes. For example, it is possible, to get the one-loop order correction for the classical Newton gravity equation. But the effective theory will break down near the Planck scale, and the new quantum gravity theory is needed.

Even the quantum Einstein gravity can be treated as an effective theory, the scattering amplitude calculation is still extremely complicated. For example, although the tree-level amplitude in quantum gravity is well defined, there are too many Feynman diagramsn and each diagram contains complicated vertices. However, there is a surprising relation between the gravity tree amplitude and the Yang-Mills gluon amplitude, *KLT relation*. [39].

$$\text{(graviton tree amplitude)} = \sum \text{(kinematic factors)}\times \tag{A.41}$$

$$\text{(gluon tree amplitude)} \times \text{(gluon tree amplitude)} \tag{A.42}$$

So the quantum gravity amplitude calculation is greatly simplified and readily calculable. This relation is inspired from the open/closed string theories and difficult to find from the field theory viewpoint.

The KLT relation may imply a fundamental relation between the gravity and the gauge theories: "gravity is gauge theory squared". However, until now (2011), the loop-level extension of the KLT relation was not found yet. However, the KLT relation inspires the discover of *AdS/CFT duality*, which is the duality between the gauge theory and the gravity theory in different dimensions.

A.3 Supersymmetric extension of the Yang-Mill and gravity theories

We know that symmetries, global or local, play the central role in Yang-Mill and gravity theories. It is a natural question to ask: is there a symmetry beyond the known Lorentz, CPT, conformal, chiral and gauge symmetries? The famous Coleman-Mandula theorem claimed that there is no mixing between an internal symmetries T's and Poincare symmetries P's, for a theory with mass gap. Schematically,

$$[T, T] \sim T, \quad [P, P] \sim P, \quad [T, P] = 0 \tag{A.43}$$

So the T's are decoupled from the Poincare symmetry and just the usual global or local *bosonic symmetries*. However, this theorem does not constrain the *fermionic* symmetries. It is possible construct fermionic generators Q's such that,

$$\{Q, Q\} \sim P \tag{A.44}$$

In particular, in $d = 4$, we can construct a Weyl spinor operators Q such that,

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_\mu \sigma^\mu, \quad [P_\mu, Q] = 0 \tag{A.45}$$

where $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$ and P_μ is the momentum operator. Q is called the *supercharge*. Since in $4D$ the Weyl spinor is the minimal spinor representation, (A.45) is the minimal possible representation in $4D$. Equivalently, we can represent the supercharges as *Majorana spinor* in $4D$.

$$\{Q, \bar{Q}\} = 2P_\mu \gamma^\mu, \quad [P_\mu, Q] = 0 \tag{A.46}$$

where Q is a Dirac spinor satisfying the Majorana condition, $Q = -i\gamma^2 Q^*$. That is the same supersymmetric algebra.

The supercharge would change the helicity of a particle by $\pm\frac{1}{2}$, so it is a symmetry between a boson and a fermion. This pair of particles is called the *supersymmetry multiplet*. Because the supercharge commutes with momentum and gauge transformation,

$$[P, Q] = 0, \quad [T^a, Q] = 0, \tag{A.47}$$

particles in the same multiplet have the same mass and in the same gauge representation. So it is clear that the standard model itself does not have supersymmetry, i.e. the partners of SM particles are outside the SM. And supersymmetry, if exists, must be broken at some energy scalar higher than TeV so the partner particles are heavy enough to avoid current experimental detection.

Exact supersymmetry forces the scalar one-loop quadratic divergence to vanish. So the SM hierarchy problem can be solved by supersymmetry. In real life, supersymmetry must be broken. However, in this manner, the *hierarchy problem* between the Planck scale and the electroweak scale is greatly reduced to the small hierarchy between the supersymmetry breaking scale and the EW scale.

A.3.1 $d = 4$ $N = 1$ super-Yang Mills theory

The $N = 1$ supersymmetric gauge theory action can be constructed from the original Yang-Mills action. The chiral quarks are extended to the *chiral supermultiplets*, $\psi_\alpha \to (\psi_\alpha, \phi)$. We may call ϕ *squark*. ψ is a Weyl spinor and ϕ is a complex scalar. And the gluons are extended to *vector supermultiplets*, $A^a_\mu \to (A^a_\mu, \lambda^a)$. λ^a is the *gaugino*, which is a Weyl spinor in the adjoint representation.

It seems that the action, including (ψ_α, ϕ) and (A^a_μ, λ^a) would be very complicated. However, the supersymmetry put strong constraints on the action. To manifest the supersymmetry, we can introduce the *superspace* and *superfields*. Let the Grassmann θ_α and $\overline{\theta}_{\dot\alpha} = (\theta_\alpha)^\dagger$ be the coordinates for the superspace. The supercharge can be realized as,

$$Q_\alpha = \left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial x^\mu}\right), \quad \overline{Q}_{\dot\alpha} = \left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\right) \tag{A.48}$$

. A superfield $F(x, \theta, \overline{\theta})$ is a field in both spacetime and superspace. The expansion in the Grassmann variables θ, $\overline{\theta}$ would give the *component fields* in one supermultiplet. The supersymmetry transformation is explicitly on a superfield is,

$$\delta F(x, \theta, \overline{\theta}) = (\xi Q + \overline{\xi}\overline{Q})F(x, \theta, \overline{\theta}). \tag{A.49}$$

Again, the expansion in θ, $\overline{\theta}$ gives the supersymmetric transformation of each component.

The chiral supermultiplet (ψ_α, ϕ), is embedded in the *chiral supermultiplet*,

$$\Phi(x, \theta, \overline{\theta}) = \phi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta^2\overline{\theta}^2F(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\overline{\theta} + \theta^2F(x) \tag{A.50}$$

where $F(x)$ is the auxiliary field to compensate the off-shell degrees of freedom. The superfield Φ satisfies the *chiral condition*,

$$\overline{D}_{\dot{\alpha}}\Phi(x,\theta,\overline{\theta}) = 0. \tag{A.51}$$

where the differential operator $\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}$. $\overline{D}_{\dot{\alpha}}$ commutes with Q and $\overline{Q}$, so this constraint is supersymmetric. The product of chiral fields and furthermore the analytic function of Φ's are still chiral superfield. However $\phi^{\dagger}$ is not a superfield.

The kinetic term for the chiral superfield is

$$\int d^4x\, d\theta^2 d\overline{\theta}^2 \Phi_i^{\dagger}\Phi_i = \int d^4x \left(i\psi_i^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\psi_i + \partial^{\mu}\phi_i^{*}\partial_{\mu}\phi_i + F_i^{*}F_i \right) \tag{A.52}$$

Here i is the index for the copies of chiral supermultiplets. Note that the Grassmann integral only picks up the $\theta^2\overline{\theta}^2$ term. This term, under the (A.49), is changed by a total derivative term. Hence the action is supersymmetric. In general, if we allow the non-renormalizable terms, the kinetic term can be chosen as,

$$\int d^4x\, d\theta^2 d\overline{\theta}^2 K(\Phi^{\dagger},\Phi) \tag{A.53}$$

where K is a real function which is called *Kähler potential*. The interaction is generated by the *superpotential*,

$$\int d^4x\, d\theta^2 W(\Phi) + h.c. = W_i(\phi)F_i - \frac{1}{2}W_{ij}(\phi)\psi_i\psi_j + h.c. \tag{A.54}$$

where $W(\Phi)$ is an analytic function. Since the product of chiral superfields are still chiral superfields, $W(\Phi)$ is a chiral superfield. Again, under the (A.49), $W(\Phi)$ is changed by a total derivative term so this term is supersymmetric.

Integrate out the auxiliary field F_i, i.e., $F_i = -W_i^{*}$, finally we get the action for the chiral supermultiplets,

$$\mathcal{L} = i\psi_i^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\psi_i + \partial^{\mu}\phi_i^{*}\partial_{\mu}\phi_i - \frac{1}{2}(W_{jk}(\phi)\psi_j\psi_k + h.c.) - W_i(\phi)W_i^{*}(\phi) \tag{A.55}$$

where the potential $W_i(\phi)W_i^*(\phi) = F_i F_i^*$ is called the *F-term*.

The vector superfield is a superfield V which satisfies the condition $V = V^\dagger$. In *Wess Zumino* gauge, the vector superfield is,

$$V^a = \theta\overline{\sigma}^\mu A_\mu^a + i\theta^2\overline{\theta}^2\lambda^{\dagger a} - i\theta\overline{\theta}^2\lambda^a + \frac{1}{2}\theta^2\overline{\theta}^2 D^a, \tag{A.56}$$

where a is the Lie-algebra index. Again, D^a is the auxiliary field. The field strength, which is a chiral superfield, is,

$$T^a W_\alpha^a = -\frac{1}{4}\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}(e^{-T^a V^a} D_\alpha e^{T^a V^a}), \tag{A.57}$$

where,

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - \theta^2\sigma^\mu D_\mu \lambda^{\dagger a}(y). \tag{A.58}$$

So the gauge field action is,

$$\frac{1}{4g^2}\int d^4x d^2\theta W^{a\alpha} W_\alpha^a + h.c., \tag{A.59}$$

and the kinetic term for the chiral matter is promoted to,

$$\int d^4\theta\, \Phi^\dagger e^{T^a V^a}\Phi. \tag{A.60}$$

and the superpotential is constrained to be gauge invariant. Again, integrate out the auxiliary field D^a, we get the *D-term* potential,

$$\frac{1}{2}(\phi^* T^a \phi)^2 \tag{A.61}$$

The Feynman rules for super-Yang-Mills theory contains all the Yang-Mills vertices and many new vertices. The new gluon interaction vertices are,

Furthermore, by the supersymmetric transformation, the gluon vertices generate other gauge vertices. Note that when the auxiliary field D^a is integrated

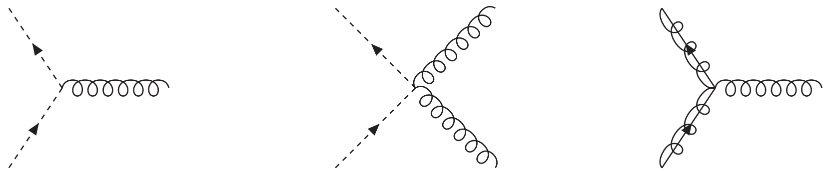


Figure A.6: Super-Yang-Mills gluon interaction. (a) two-scaler-one-gluon vertex (b) two-scalar-two-gluon vertex (c) two-gaugino-one-gluon vetex.

Figure A.7: Other gauge vertices in Super-Yang-Mills theory (a) gaugino-scalar-fermion vertex (b) D-term vertex

out, the D-term vertex becomes the D-term potential interaction. And the superpotential $W(\Phi)$ for chiral supermultiplets would generate the squark and squark-quark vertices.

The one-loop beta function for super-Yang-Mills' gauge coupling can be calculated identically as the Yang-Mills case (A.18). The new ingredient is the gaugino and squark gauge interaction, so the beta function is

$$\begin{aligned}\mu\frac{dg}{d\mu} &= -\frac{g^3}{(4\pi)^2}[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r) - \frac{2}{3}C_2(G) - \frac{2}{3}n_f C(r)] \\ &= -\frac{g^3}{(4\pi)^2}[3C_2(G) - 2n_f C(r)] \qquad \text{(A.62)}\end{aligned}$$

where in the first line, the last two term comes from the gaugino and squark loops, respectively. Again, n_f is the number of quarks, as Dirac spinors. For $G = SU(N)$, this result is $\beta_g = -\frac{g^3}{(4\pi)^2}(3N - n_f)$.

However, unlike Yang-Mills theory, the beta function in super-Yang-Mills can be determined to all loops. The exact beta function, or *NSVZ beta function* is,

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{(4\pi)^2} \frac{3C_2(G) - \sum_i 2C(r_i)(1-\gamma_i)}{1 - C_2(G)g^2/8\pi^2}. \tag{A.63}$$

This result can be proven by the holomorphic method, which treats the coupling constant g as a superfield. The difference between (A.62) and (A.63) is from the wavefunction renormalization. (A.62) is also called the *holomorphic beta function*.

A.3.2 $d = 4$, $N = 4$ super-Yang-Mills theory

It is possible to have several supercharges,

$$\{Q^A_\alpha, \overline{Q}^B_{\dot{\alpha}}\} = 2\delta^{AB} P_\mu \sigma^\mu_{\alpha\dot{\alpha}}, \tag{A.64}$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2Z^{AB}\epsilon_{\alpha\beta}, \tag{A.65}$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2Z^{*AB}\epsilon_{\alpha\beta}, \tag{A.66}$$

where Z^{AB} is an antisymmetric matrix which is called the central charge. We can diagonalize Z^{AB} and just consider the eigenvalues $Z_1, ...Z_k$. In one supermultiplet, $|Z_i| \leq M$, $\forall i$, otherwise there would be negative-mode states. If one $|Z_i| = M$, we get a short supermultiplet, which is called the *Bogomol'ny-Prasad-Sommerfield state*, or simply the BPS state.

In $d = 4$, the maximum supersymmetry without gravitons is $N = 4$ supersymmetry. $N = 2$ super-Yang-Mills theory has rich moduli structure and leads to the famous *Seiberg-Witten theory*. $N = 3$ supersymmetry can be decomposed to $N = 1$ and $N = 2$ supersymmetry. In the subsection, we focus on the $d = 4$, $N = 1$ super-Yang-Mills theory.

The vector supermultiplet in $N = 4$ super-Yang-Mills theory contains particles with the following helicities,

$$\left(1, \frac{1}{2}^4, 0^6, -\frac{1}{2}^4, -1^6\right) \tag{A.67}$$

so there are four gauginos and six scalars. Because of the strong constraint of the $N = 4$ supersymmetry, the gauge action is very simple. $d = 4$, $N = 4$ super-Yang-Mills has the same number of supercharges as the $d = 10$, $N = 1$ super-Yang-Mills theory. The latter's action is

$$-\frac{1}{4g^2} tr(F_{MN}F^{MN}) - \frac{i}{2g^2} tr(\bar{\lambda}\Gamma^M D_M \lambda) \tag{A.68}$$

where $M = 0, \ldots, 9$. By dimension reduction, $\mu = 0, \ldots, 3$, $m = 4, \ldots, 9$, and the six gluons in extra dimension A_m becomes the scalars ϕ_m. The ten-dimensional Weyl-Majorana spinor becomes four four-dimensional Weyl spinors. So the action is

$$-\frac{1}{4g^2} tr(F_{\mu\nu}F^{\mu\nu} + 2D_\mu\phi_m D^\mu\phi_m - [\phi_m, \phi_n]^2) - \frac{i}{2g^2} tr(\bar{\lambda}\gamma^\mu D_\mu \lambda + i\bar{\lambda}\Gamma_m[\phi_m, \lambda]). \tag{A.69}$$

The potential for scalars is minimal when,

$$[\phi_m, \phi_n] = 0, \forall m, n \tag{A.70}$$

so the v.e.v of ϕ_m can be taken simultaneously diagonal and confined in Cartan sub-Lie algebra. Hence the moduli space is $6 \times rank(G)$-dimensional. Outside the moduli space, the gauge symmetry is spontaneously broken.

$N = 4$ super-Yang-Mills theory can be treated as $N = 1$ super-Yang-Mills theory. Since there are 4 adjoint chiral fermions (gaugino) and 6 adjoint scalars, the holomorphic beta function vanishes.

$$\mu\frac{dg}{d\mu} \propto \left(\frac{11}{3}C_2(G) - \frac{2}{3}C(Ad) \times 4 - \frac{1}{6}C(Ad) \times 6\right) = 0. \tag{A.71}$$

Note here we have real scalars instead of complex scalars. Furthermore, the beta function not just vanishes perturbatively but also exactly, by the moduli argument.

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