SHORTEST PATH POSET OF BRUHAT INTERVALS AND THE COMPLETE CD-INDEX

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SHORTEST PATH POSET OF BRUHAT INTERVALS AND THE COMPLETE CD-INDEX

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Let (W, S) be a Coxeter system, $[u, v]$ be a Bruhat interval and $B(u, v)$ be its corresponding Bruhat graph. The combinatorial and topological structures of the longest u-v paths of $B(u, v)$ have been studied extensively and is well-known. Nevertheless, not much is known of the remaining paths. Here we define the *shortest path poset* of $[u, v]$, denoted by $SP(u, v)$, which arises from the shortest u-v paths of $B(u, v)$. If W is finite, then $SP(e, w_0)$ is the union of Boolean posets, where w_0 is the longest-length word of W. Furthermore, if $SP(u, v)$ has a unique rising chain under a reflection order, then $SP(u, v)$ is EL-shellable.

The complete **cd**-index of a Bruhat interval is a non-homogeneous polynomial that encodes the descent-set distribution, under a reflection order, of paths of $B(u, v)$. The highest-degree terms of the complete **cd**-index correspond to the **cd**-index of $[u, v]$ (as an Eulerian poset). We study properties of the complete **cd**-index and compute it for some intervals utilizing an extension of the CL-labeling of Björner and Wachs that can be defined for dihedral intervals (which we characterize by their complete **cd**-index) and intervals in a universal Coxeter system. We also describe the lowest-degree terms of the complete **cd**-index for some intervals.

BIOGRAPHICAL SKETCH

Saúl Antonio Blanco Rodríguez was born in Zacatecoluca, a small town southwest of San Salvador, El Salvador. He moved to San Salvador to attend university for some time. He later moved to Rochester, NY to study English at RIT. He transferred to Cornell University where he obtained an undergraduate degree in mathematics and continued his graduate studies under the supervision of Professor Louis J. Billera. In the Fall 2011, he will be at DePaul University.

To my math mentors

ACKNOWLEDGMENTS

It used to amazed me how long the acknowledgment section in some theses were. I suppose it was hard for me to understand how some would find it necessary to thank a considerable number of people; after all, writing a thesis involves many hours of solitary work. Now I understand that it is a joint effort. Indeed, I am impressed by the number of people who, knowingly or not, helped me conclude this thesis, and I want to thank them deeply.

First and foremost, I thank my advisor Louis J. Billera. There are no words that can be used to described how much I owe him for the many, many years. He taught the very first lecture I attended at Cornell, and I have admired him ever since. Moreover, I thank the other members of my committee Ed Swartz and Éva Tardos. In my time at Cornell, I took three courses from each one, and I enjoyed them all. Furthermore, Matthew Dyer and Martin Kossabov were willing to have insightful discussions and were very patient and diligent during such discussions. I would not be where I am today if it were not for the math educators I had in El Salvador, in particular, Luis Edmundo Ramírez Fuentes, Carlos Mauricio Canjura Linares, Alba Lila Rico de Tejada, and Pedro Alfonso Tejada Díaz. I apologize to all of the above-mentioned people for being so hard to train. I am sure that a lot of patience was needed, and that speaks a great deal about their quality as teachers and mentors.

On a more personal side, I would like to thank David Crandall whose unconditional support has been paramount in my life. I would also like to thank my parents for their good wishes, even though they might not have always liked the idea of their son studying mathematics. I want to thank some of the lovely people who positively affected my time as a graduate student, especially in my last year. In alphabetical order, I thank Jason Anema, Hyungryul "Harry" Baik, Cristina Benea, Voula Collins, Kristine Jones, Peter Luthy, Fatima Mahmood, Surin Maneevitjit, Christine McMeekin, Kristen Pueschel, Peter Samuelson, Junping Shao, Michelle Snider, Russ Thompson, and Radozlov Zlatev. Even though Cocoa, Toto and Zuma are not people (the first one is a dog and the other two are parakeets), they provided hours of much needed entertainment. I have heard it said that people who have pets live longer, and now I know why.

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CHAPTER 1

INTRODUCTION

This thesis studies the combinatorial structure of certain paths in graphs arising from Coxeter groups. Understanding such structure could provide a better understanding of widely-studied objects in Coxeter groups, such as Kazhdan-Lusztig polynomials in representation theory, as well as more recently-created objects, such as the complete **cd**-index.

1.1 Thesis outline

The thesis is organized as follows. In the rest of this chapter we provide the background material that is needed. In Chapter 2 we describe cases where a certain labeling procedure, called BW-labeling, can be used to assign labels to all maximal paths in the Bruhat graph of a Bruhat interval. In Chapter 3 we recall the definition of the complete **cd**-index and use the BW-labeling to compute it for some intervals. In Chapter 4 we study the shortest path poset, $SP(u, v)$, whose elements are those contained in a shortest path in the Bruhat graph of $[u, v]$. This poset satisfies similar properties to $[u, v]$ in the case where there is a unique rising chain. In the case where there is more than one rising chain, we give an algorithm to split $SP(u, v)$ into subposets, each one of which satisfy properties resembling those of $[u, v]$. In Chapter 5 we concentrate on the poset $SP(e, w_0^W)$ for finite Coxeter group W. It turns out that the poset $SP(e, w_0^W)$ has a very simple structure: it is the union of Boolean posets.

1.2 Basic combinatorial objects

Definition 1.2.1. (I) A *partially ordered set*, or *poset* for short, is a pairing $(P, \leq)$ where P is a set and $\leq$, called *a partial order*, satisfies the following properties.

- $a \leq a$ for all $a \in P$,
- $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in P$, and
- $a \leq b$ and $b \leq a$ implies $a = b$.

It will be convenient to ignore $\leq$ whenever the partial order is clear from the context.

(II) Given $x, y \in P$, we denote the set $\{z \in P : x \leq z \leq y\}$ by $[x, y]$. Furthermore, $[x, y]$ is called an *interval* of P.

(III) An element y *covers* an element x if $x \leq y$ and there is no element $z \in P \setminus \{x, y\}$ with $x \leq z \leq y$. In this situation we say that $x \leq y$ is a *cover relation*, which we denote by $x \lessdot y$.

(IV) The sequence $x_1 \leq x_2 \leq \cdots \leq x_k$, for $x_i \in P$ and $1 \leq i \leq k$, is called a *chain* of P. Moreover, a chain of the form $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$ is called a *saturated* chain. Finally, a chain of the form $u \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot v$ is called a *maximal chain* of $[u, v]$.

(V) An element $m \in P$ is said to be *minimal* if there are no elements $x \in P \setminus \{m\}$ with $x \leq m$. An element $M \in P$ is said to be *maximal* if there are no elements $y \in P \setminus \{M\}$ with $M \leq y$.

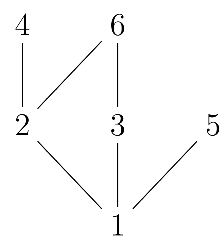
(VI) A *rank function* of P is a function $r : P \to \mathbb{Z}_{\geq 0}$ so that $r(x) = 0$ for every minimal element of P and $r(y) = r(x) + 1$ if $x \lessdot y$. P is said to be *ranked* if there exists a rank function of P. The *rank of* P is $\mathrm{rk}(P) \overset{\text{def}}{=} \max\{r(x) : x \in P\}$.

(VII) A poset P that is ranked and has a unique minimal and maximal element is said to be *graded*. We denote the unique minimal and maximal element by $\hat{0}$ and $\hat{1}$, respectively.

(VIII) A graded poset P is said to be Eulerian if every subinterval of P has as many elements of even rank as those of odd rank

(IX) The *Hasse diagram* of P is the undirected graph with vertex set P and edge set $E = \{(x, y) : x, y \in P$ and $x \lessdot y\}$.

Example 1.2.2. Consider the poset $P = (\{1, \ldots, 6\}, \leq)$ where $a \leq b$ if $a \mid b$. Then the Hasse diagram corresponding to this poset is depicted below.



Furthermore, notice that P is ranked, where the rank function r is given by $r(1) = 0, r(2) = r(3) = r(5) = 1$, and $r(4) = r(5) = 2$. On the other hand, since there is not a unique maximal element, P is *not* graded.

1.3 Poset topology

By "poset topology", one means the topology of a certain simplicial complex $\Delta(P)$, called the order complex, associated with the poset P. Wachs [27] provides a nice survey to the subject.

1.3.1 Order complex of a poset

We begin with the definition of a simplicial complex. This object will allow us to derive topological properties of a poset.

Definition 1.3.1. (I) An (abstract) *simplicial complex* Δ on a finite set V is a nonempty collection of subsets of V so that

(i) $\{v\} \in \Delta$ for all $v \in V$, and

(ii) if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$.

(II) The elements of V are called *vertices* of Δ.

(III) The elements of Δ are called *simplices* or *faces* of Δ. If F is a maximal element of Δ, then F is called a *facet* of Δ.

(IV) Given $F \in \Delta$, the *dimension of* F is defined as $\dim(F) \stackrel{\text{def}}{=} |F| + 1$. The dimension of Δ is defined as $\dim(\Delta) \stackrel{\text{def}}{=} \max\{\dim(F) : F \in \Delta\}$.

(V) If all the facets of Δ have the same dimension, then Δ is said to be *pure*.

We now construct a simplicial complex associated to a poset.

Definition 1.3.2. $\Delta(P)$ is the simplicial complex whose vertices are the elements of P and whose simplices are all chains in P.

Example 1.3.3. For the poset in Example 1.2.2, $\Delta(P)$ is depicted in Figure 1.1. The simplices formed by the vertices $\{1, 3, 6\}$, $\{1, 5\}$ and $\{1, 2, 4\}$ are correspond to the maximal chains of P.

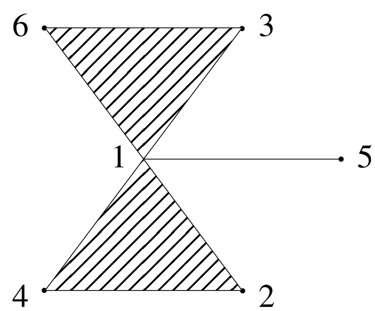
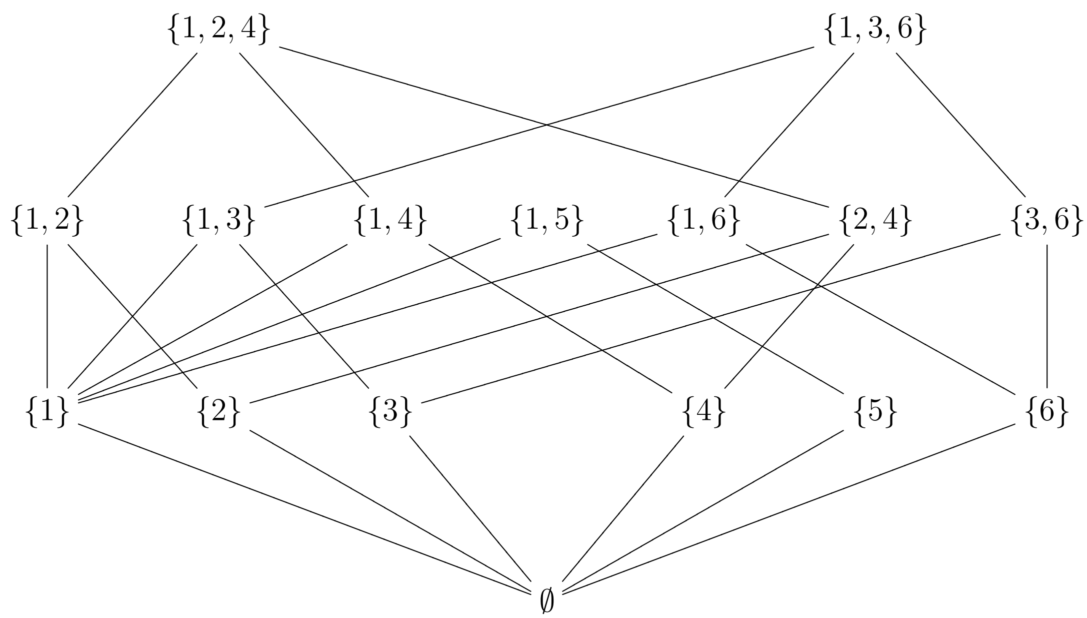


Figure 1.1: $\Delta(P)$ for the poset in Example 1.2.2.

Definition 1.3.4. The *face poset* $P(\Delta)$ of a simplicial complex Δ is the poset of the set of all simplices of Δ ordered by inclusion.

Example 1.3.5. The face poset of the simplicial complex of Figure 1.1 is depicted below.



Given a simplicial complex K, the complex $\Delta(P(K) \setminus \emptyset)$, that is, the order complex of the poset of nonempty faces of K, is called the *first barycentric subdivision* of K.

We now describe the concept of shellability, from which one can derive topological properties.

1.3.2 Shellability

Shellability is a tool used to derive topological complexes of posets (via its order complex).

Before defining shellability, we need the following notation: For a face F in a simplicial complex Δ, denote by $\langle F \rangle$ the set of subsets of F. That is,

$$\langle F \rangle \stackrel{\text{def}}{=} \{G : G \subset F\}.$$

Definition 1.3.6. (i) Let Δ be a pure simplicial complex. A *shelling order* of Δ is an ordering $F_1, F_2, \ldots, F_k$ of the facets of Δ so that for all $j \geq 2$, $\langle F_j \rangle \cap \bigcup_{i=1}^{j-1} \langle F_i \rangle$ is a pure simplicial complex of dimension $\dim(F_j) - 1$ for $1 \leq j \leq k$.
(ii) If the facets of Δ admit a shelling order, then Δ is said to be *shellable*.
(iii) A poset P is said to be *shellable* if its order complex $\Delta(P)$ is shellable.

Posets that are shellable are endowed with topological and algebraic properties. Unfortunately, sometimes it is not easy to construct a shelling directly. For the order complex $\Delta(P)$ of P, however, there is a method to construct a shelling by assigning labels to the Hasse diagram of P.

Definition 1.3.7. (i) Let $E(P)$ denote the edges of the Hasse diagram of P, i.e., the cover relations of P. An *edge labeling* is a map $\lambda : E(P) \to \mathbb{Z}$.
(ii) Let $c = (x_0 \lessdot x_1 \lessdot \cdots \lessdot x_s)$ be a maximal chain of P. Then c is *rising* in λ if

$$\lambda(x_0 \lessdot x_1) < \lambda(x_1 \lessdot x_2) < \cdots < \lambda(x_{s-1} \lessdot x_s)$$

Definition 1.3.8. (i) The poset P is said to be *EL-labelable* (**E**dge-wise **L**exicographically labelable) if there exists an edge labeling λ of P so that every subinterval $[x, y] \in P$ has a unique maximal chain that is rising and lexicographically earlier than any other maximal chains of $[x, y]$.
(ii) The λ above is called an *EL-labeling* of P.

The connection between EL-labelable posets and shellable posets is given by the following theorem.

Theorem 1.3.9 (cf. [27], Theorem 3.2.2)**.** *If P is EL-labelable, then $\Delta(P)$ is shellable. Furthermore, the shelling order of $\Delta(P)$ is given by the lexicographic order on the chains of P.*

If P is EL-labelable, we say that $\Delta(P)$ is *EL-shellable*.

We want to point out that the converse of the abouve theorem is not true. A counterexample was provided by Vince and Wachs [26].

1.3.3 Gorenstein* posets

We first need some definitions.

Definition 1.3.10. Let F be a face in a simplicial complex Δ. Then the *link of* F, denoted by $\text{link}_\Delta(F)$, is the set $\{G \in \Delta : F \cap G = \emptyset$ and $F \cup G \in \Delta\}$.

Now we are ready to define Cohen-Macaulay simplicial complexes.

Definition 1.3.11. A simplicial complex Δ is said to be Cohen-Macaulay over a field $\mathbb{K}$ if

$$\widetilde{H}_i(\text{link}_\Delta(F); \mathbb{K}) = 0$$

for all $F \in \Delta$ and $i < \dim(\text{link}_\Delta(F))$. Here $\widetilde{H}_i(\ \cdot\ ; \mathbb{K})$ denotes the *reduced homology* with coefficients in $\mathbb{K}$ (see [21]).

Cohen-Macaulay simplicial complexes are endowed with a rich topological structure. For instance, there are recursive techniques that can be used to compute their Betti numbers (see [27, Theorem 4.1.12]), homology, and *Whitney homology* (see [27, 4.4.1]).

We say that a poset P is *Cohen-Macaulay* if its order complex $\Delta(P)$ is Cohen-Macaulay. Checking Cohen-Macaulayness for posets can be carried out by finding a shelling of their order complex. That is,

Theorem 1.3.12 (cf. [27], Theorem 3.1.5 and Corollary 3.1.4)**.** *If* $\Delta(P)$ *is shellable, then* P *is Cohen-Macaulay.*

We are now ready to define Gorenstein* posets.

Definition 1.3.13. A poset P is *Gorenstein** if it is Eulerian and Cohen-Macaulay.

Examples of Gorenstein* posets include face posets of *regular CW-decompositions* of a sphere (cf. [21]), and so Bruhat intervals, which are defined later in this chapter.

1.4 cd-index of Eulerian posets

The definitions of this section follow [4]. We start with the *flag-f vector*.

Definition 1.4.1. Let P be a graded poset of rank $d+1$ with rank function r and smallest and largest elements $\widehat{0}$ and $\widehat{1}$, respectively. For any $S = \{s_1, \dots, s_k\} \subset [d]$, with $s_1 < \cdots < s_k$, define

$$f_S(P) \stackrel{\text{def}}{=} |\{\widehat{0} = x_0 < x_1 < \cdots x_{k+1} = \widehat{1} : x_i \in P_i \text{ with } r(x_i) = s_i\}|.$$

The set $\{f_S(P) : S \subset [d]\}$ is called the *flag f-vector* of P.

A closely related object, and sometimes more useful, is the *flag h-vector*. This is defined below by an inclusion-exclusion type of relation with the flag f-vector. That is,

$$h_S(P) = \sum_{T \subset S} (-1)^{|S \setminus T|} f_T(P) \quad \text{and} \quad f_S(P) = \sum_{T \subset S} h_T(P).$$

The set $\{h_S(P) : S \subset [d]\}$ is called the *flag h-vector* of P.

Notice that both the flag f-vector and flag h-vector carry the same information. The flag h-vector is used to compute the **cd**-index of an Eulerian poset, as described below.

Let **a** and **b** be noncommutative variables. For $S \subset [d]$, let $w(S) = u_1 u_2 \cdots u_d$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

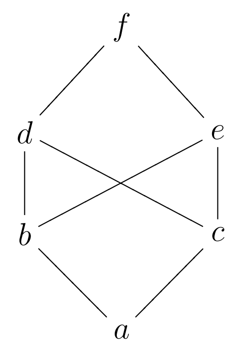
The polynomial

$$\Psi(\mathbf{a},\mathbf{b}) = \sum_{S\subset[d]} h_S(P)w(S)$$

is called the **ab**-index of P.

Bayer and Klapper [2] proved that, when P is Eulerian, the **ab**-index can be rewritten as a homogeneous polynomial in noncommutative variables **c** and **d**, where $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$. This polynomial, that we will denote by $\psi(P)$, is called the ***cd**-index* of P. We illustrate the definition with the example below.

Example 1.4.2. Consider the Eulerian poset P below.



Direct computation gives that

$$f_\emptyset = 1 \qquad h_\emptyset = 1$$

$$f_{\{1\}} = 2 \qquad h_{\{1\}} = f_{\{1\}} - f_\emptyset = 1$$

$$f_{\{2\}} = 2 \qquad h_{\{2\}} = f_{\{2\}} - f_\emptyset = 1$$

$$f_{\{1,2\}} = 4 \qquad h_{\{1,2\}} = f_{\{1,2\}} - f_{\{1\}} - f_{\{2\}} + f_\emptyset = 1.$$

Hence, $\psi(\mathbf{a},\mathbf{b}) = \mathbf{aa}+\mathbf{ba}+\mathbf{ab}+\mathbf{bb} = (\mathbf{a}+\mathbf{b})^2 = \mathbf{c}^2$. Thus $\psi(P) = \mathbf{c}^2$.

Notice that $\Psi(P)$, and thus $\psi(P)$, is an encoding of the flag h-vector, and hence the flag f-vector. In the example below, all the enumerative properties of P are given by the monomial $\mathbf{c}^2$.

A similar computation gives the following lemma.

Lemma 1.4.3. *If P is the face poset of an n-gon, then $\psi(P) = \boldsymbol{c}^2 + (n-2)\boldsymbol{d}$.*

Proof. Note that $f_\emptyset(P) = 1, f_{\{1\}}(P) = n, f_{\{2\}}(P) = n$, and $f_{\{1,2\}}(P) = 2n$, and so $h_\emptyset(P) = 1, h_{\{1\}}(P) = n-1, h_{\{2\}}(P) = n-1$, and $h_{\{1,2\}}(P) = 1$. Thus $\Psi(\mathbf{a}, \mathbf{b}) = \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{ab} + \mathbf{ba} + (n-2)(\mathbf{ab} + \mathbf{ba})$, and so $\psi(P) = \mathbf{c}^2 + (n-2)\mathbf{d}$. □

From the definition it follows that the coefficients of the **ab**-index are nonnegative; however, the coefficients of the **cd**-index need not be. Karu [24] showed that if P is Gorenstein*, then the coefficients $\psi(P)$ are nonnegative.

In addition to enumerating flag vectors, the **cd**-index has been shown to have connections to *quasisymetric functions* and the *peak algebra* (see [5]) and to Coxeter groups, (see [3]). This thesis contributes to the study of the latter.

1.5 Preamble to Coxeter groups

Consider two lines L_1 and L_2 through the origin in $\mathbb{R}^2$ so that the angle between them is $\frac{\pi}{n}$. Let $r_1, r_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal reflections through L_1 and L_2, respectively. Let $I_2(n)$ be the group generated by r_1, r_2 (the operation is composition of maps). Notice that the composition of r_1 and r_2, which we denote by $r_1 r_2$, rotates the plane through an angle of $\frac{2\pi}{n}$. Furthermore, one can see that the elements of $I_2(n)$ are the n rotations through angles $\frac{2\pi k}{n}$, $0 \leq k < n$ given by composing $r_1 r_2$ repeatedly, and by composing these rotations with r_1. In other words, $I_2(n) = \langle r_1, r_2 \mid r_1^2 = r_2^2 = (r_1 r_2)^n = e \rangle$. The group $I_2(n)$ is called the *dihedral group* of order $2n$. In fact, $I_2(n)$ is the group of symmetries of an n-gon (See Figure 1.2).

It is certainly possible that the angle between L_1 and L_2 is not a rational multiple of π. In this case the composition $r_1 r_2$ would have infinite order, and thus the group generated by r_1 and r_2 is also infinite. This group is called the *infinite dihedral group*, and it is denoted by $I_2(\infty)$.

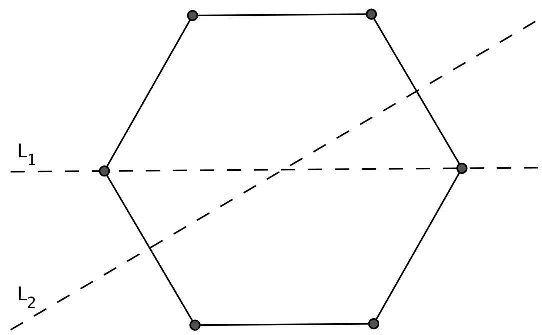


Figure 1.2: $I_2(6)$, the group of symmetries of the hexagon.

1.6 Coxeter groups

Coxeter groups are generalizations of $I_2(n)$. They have the following presentation

$$W = \langle S : (s_i s_j)^{m_{i,j}} = e, \text{ for all } s_i, s_j \in S \rangle,$$

where the numbers $m_{i,j}$ satisfy the following properties.

- $m_{i,j} \in \mathbb{Z}_{>0} \cup \{\infty\}$.
- $m_{i,i} = 1$. So the elements of S are involutions, which resemble the reflections in the dihedral group.
- $m_{i,j} = m_{j,i} \geq 2$ if $i \neq j$.

The pair (W, S) is called a *Coxeter system* and the group W is called a *Coxeter group*. From now on (W, S) will denote an arbitrary, but fixed, Coxeter system.

The set S is called the set of *generators*, or *simple reflections*. By definition, the elements of S are involutions (have order two), just like the r_1, r_2 described in the previous section.

Of special importance is the set $T(W)$, or simply T of *reflections* of (W, S), which is given by

$$T(W) \stackrel{\text{def}}{=} \{wsw^{-1} : w \in W, s \in S\}.$$

Symmetry groups of regular polytopes are Coxeter groups. These groups include the symmetric group on $n+1$ elements, denoted by A_n, which is the group of symmetries of the $(n+1)$-simplex. Finite, irreducible Coxeter groups have been classified, and we present this classification at the end of Section 1.7. In Chapter 5 we utilize this classification in a case-by-case proof of the main result of that chapter.

Definition 1.6.1. Let $w \in W$. Then there exists $s_1, s_2, \ldots s_k \in S$ so that $s_i \in S$ for $1 \leq i \leq k$, and $w = s_1 s_2 \cdots s_k$. Then,

- We call $s_1 s_2 \cdots s_k$ an *expression* for w.
- If k happens to be minimal, then $s_1 s_2 \cdots s_k$ is a *reduced expression* for w.
- If k is minimal, then we define the *length* of w, $\ell(w)$, to be k.

1.7 Finite Coxeter groups

The finite Coxeter groups were classified by Coxeter [13], and we depict such a classification in Figure 1.3. It is given using *Coxeter-Dynkin* diagrams. These diagrams provide a convenient way to denote Coxeter groups.

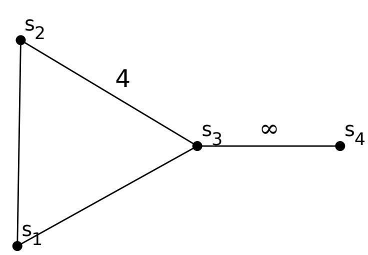
Definition 1.7.1. The *Coxeter-Dynkin diagram* of the Coxeter system (W, S) is the graph with vertex set S so that there is an edge between s_i and s_j if and only if $m_{i,j} \geq 3$. Furthermore, if $m_{i,j} > 3$, then the edge (s_i, s_j) is assigned the label $m_{i,j}$.

We illustrate the use of Coxeter-Dynkin diagrams in the following example.

Example 1.7.2. Consider the Coxeter system generated by $\{s_1, s_2, s_3, s_4\}$ and subject to the following relations

$$\begin{cases} s_1^2 = s_2^2 = s_3^2 = s_4^2 = e, \\ (s_1 s_2)^3 = (s_2 s_1)^3 = e, \\ (s_1 s_3)^3 = (s_3 s_1)^3 = e, \\ (s_1 s_4)^2 = (s_4 s_1)^2 = e, \\ (s_2 s_4)^2 = (s_4 s_2)^2 = e, \\ (s_2 s_3)^4 = (s_3 s_2)^4 = e, \text{ and} \\ (s_3 s_4)^\infty = (s_4 s_3)^\infty = e. \end{cases}$$

Then the corresponding Coxeter-Dynkin diagram is



Notice that the edges (s_2, s_3) and (s_3, s_4) are labeled with $m_{2,3}$ and $m_{3,4}$, respectively. Furthermore, there is no edge between s_1 and s_4, and thus these two elements commute. One notices relatively quickly the convenience of using Coxeter-Dynkin diagrams to denote Coxeter groups.

The finite, irreducible Coxeter groups are presented in Figure 1.3.

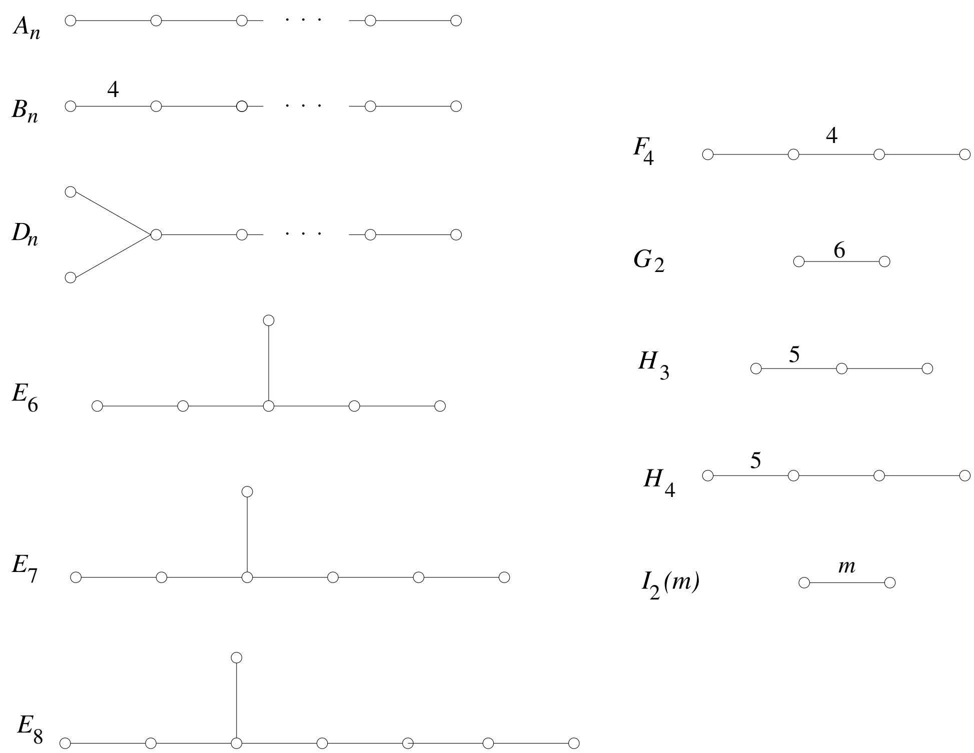


Figure 1.3: Dynkin-Coxeter diagrams for the finite, irreducible Coxeter groups. The subscripts indicate the number of generators

1.8 Bruhat graph and Bruhat order

The *Bruhat graph* $B(u, v)$ of a Bruhat interval $[u, v]$ is the directed graph with vertex set $[u, v]$ and edge set

$$\{(w_1, w_2) : w_1, w_2 \in [u, v],\ \ell(w_1) < \ell(w_2) \text{ and there exists a reflection } t \text{ with } w_1 t = w_2\}.$$

To illustrate the definition, consider Figure 1.4 depicting the Bruhat graph of $I_2(4)$, the dihedral group of order 8. The *length* of an x-y path is the number of edges between x and y. We denote the set of u-v paths of length k by $B_k(u, v)$. For instance, considering Figure 1.4 one has that $|B_2(e, abab)| = 4$.

Notice that when the direction of the maximal-length paths in Figure 1.4 are ignored, one obtains the Hasse diagram of the Bruhat order.

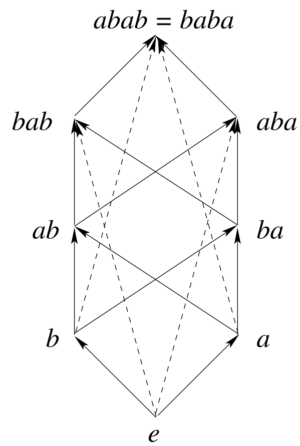


Figure 1.4: Bruhat graph of $I_2(4)$.

One can define a partial order, called the *Bruhat order* on W as follows: $w_1 \leq w_2$ if $B(x, y)$ is not the empty graph. Furthermore, we say that $u \leq_{\mathrm{w}} v$ in *weak Bruhat order* if there is a u-v paths in $B(u, v)$ formed by edges corresponding to *simple* reflections.

There is an equivalent way of defining the Bruhat order using reduced expressions. Let $s_1 s_2 \cdots s_k$ be a reduced expression for w_1. Then $w_1 \leq w_2$ if and only if there exists a reduced expression for w_2 of the form $s_{i_1} s_{i_2} \cdots s_{i_p}$ with $1 \leq i_1 < i_2 < \cdots < i_p \leq k$.

The Bruhat order satisfies the following condition, called the *Strong Exchange Condition*, hereinafter SEC.

Theorem 1.8.1 (Strong Exchange Condition, [22], Theorem 5.8.1)**.** *Let $s_1 s_2 \cdots s_r$ ($s_i \in S$) be an expression for $w \in W$, not necessarily reduced. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index i for which $wt = s_1 \cdots \widehat{s_i} \cdots s_r$ (omitting s_i). If the expression for w is reduced, then i is unique.*

In fact, the SEC serves as a characterization for Coxeter groups. Indeed, we have the following theorem.

Theorem 1.8.2 ([10], Theorem 1.6.1)**.** *(W, S) is a Coxeter system if and only if*

- *S is a set of involutions.*

- *(W, S) has the SEC.*

The Bruhat interval $[u, v]$ satisfies the following properties.

Theorem 1.8.3. *Let $[u, v]$ be a Bruhat interval of W, then*

(i) $[u, v]$, is an Eulerian, graded poset ([8, Theorem 5.3] and [7, Theorem 2.2.6]). In particular, if $\ell(v) - \ell(u) = 2$ then $[u, v]$ has two atoms (coatoms). That is, Bruhat intervals are *thin*.

(ii) $[u, v]$ is EL-shellable ([17, Proposition 4.3]).

In Chapter 4 and Chapter 5 we study posets arising from the Bruhat graph that satisfy the previous theorem.

1.9 Reflection Order

For $w \in W$, we define the *negative set* of w, denoted by $N(w)$, to be the set of reflections that shorten the length of w, i.e., $N(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$. Notice that if $s_1 \cdots s_k$ is a reduced expression for w, then $N(w) = \{t_1, \ldots, t_k\}$, where $t_i = s_k \cdots s_{k-i+2} s_{k-i+1} s_{k-i+2} \cdots s_k$ for $i = 1, \ldots, k$. Furthermore, the total order defined by

$$s_k = t_k <_w s_k s_{k-1} s_k = t_{k-1} <_w \ldots <_w s_k s_{k-1} \cdots s_2 s_1 s_2 \cdots s_{k-1} s_k = t_1$$

is said to be *induced* by the reduced expression $s_1 \cdots s_k$ for w.

A *reflection subgroup* W' is a subgroup of W generated by elements of T. In fact, they are Coxeter groups with simple reflections

$$S' = \{t \in T : N(t) \cap W' = \{t\}\},$$

i.e., (W', S') is a Coxeter system (see [14]). If $|S'| = 2$, then W' is called a *dihedral reflection sugroup* of W.

Dyer [17] showed the existence of linear orders $<_T$ on T that satisfy the following property. For any Coxeter system of the form $(W', \{t_1, t_2\})$ either

$$t_1 <_T t_1t_2t_1 <_T t_1t_2t_1t_2t_1 <_T \cdots <_T t_2t_1t_2t_1t_2 <_T t_2t_1t_2 <_T t_2, \text{ or}$$

$$t_2 <_T t_2t_1t_2 <_T t_2t_1t_2t_1t_2 <_T \cdots <_T t_1t_2t_1t_2t_1 <_T t_1t_2t_1 <_T t_1.$$

These linear orders are called *reflection orders*. Given a reflection order $<_T$, an *initial section* A_T of $<_T$ is a subset of T with $r <_T t$ for all $r \in A_T$ and $t \in T \setminus A_T$.

Dyer showed the following theorem.

Lemma 1.9.1 ([17], Lemma 2.11)**.** *A_T is a finite initial section of a reflection order if and only if $A_T = N(w)$ for some $w \in W$. In other words A_T is a finite initial section of a reflection order if and only if it is induced by a reduced expression for some $w \in W$.*

Notice that [7, Proposition 2.3.1(i)] gives the existence of a unique longest-length element w_0^W for finite W, that is, $w_0^W \geq w$ for all $w \in W$. Moreover, $|N(w_0^W)| = \ell(w_0^W) = |T|$ by [7, Proposition 2.3.2(iv)], and so we have the following corollary.

Corollary 1.9.2. *If W is finite, then all reflection orders on T are induced by a choice of reduced expression for w_0^W.*

In [18] and [17], Dyer proved two important consequences that follow from the existence of reflection orders. The statement of one such consequence is as follows.

Theorem 1.9.3 ([17], Proposition 4.3)**.** *Let $[u, v]$ be a Bruhat interval. Then $[u, v]$ is EL-labelable.*

The second consequence is an alternative, nonrecursive definition of the $\widetilde{R}$-polynomials. These polynomials are defined in Chapter 4.

CHAPTER 2

BW-LABELABLE INTERVALS

The generators of a Coxeter system (W, S) are subject to two types of relations, (cf. Section 3.3, [7]):

(i) *nil relations*, which are of the form $s^2 = e$ for $s \in S$, and

(ii) *braid relations*, which are of the form $\underbrace{s_i s_j s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{i,j}}$ for $s_i, s_j \in S$.

Definition 2.0.4. We say that an expression $s_1 s_2 \cdots s_k$ for $w \in W$ is *nil-reduced* if $s_i \neq s_{i+1}$ for $1 \leq i < k$.

By abuse of notation, we write $\Delta \in B(u, v)$ to indicate that Δ is a u-v path in the Bruhat graph of $[u, v]$. As a convention, Δ can be written in two ways:

(i) $(a_0 = u < a_1 < \cdots < a_k = v)$, with $a_i \in W$, when we want to refer to the vertices of Δ. If Δ is a maximal-length u-v path, then we write $(u = a_0 \lessdot a_1 \lessdot \cdots \lessdot a_{\mathrm{rk}([u,v])} = v)$ to emphasize that the edges of Δ represent cover relations. In particular, an edge in $B(u, v)$ can be thought of as a path of length one, and so the edge between w and w_1 with $\ell(w) < \ell(w_1)$ is denoted by $(w < w_1)$, and

(ii) $(t_1, \ldots, t_k)$, with $t_i \in T$ and $a_{i-1} t_i = a_i$, $i = 1, \ldots, k$, when we wish to refer to the edges that Δ traverses.

Given a path $\Delta = (t_1, \ldots, t_k) \in B(u, v)$ and a reflection order $<_T$ on $T(W, S)$, the *descent set of* Δ *with respect to* $<_T$, denoted by $D_{<_T}(\Delta)$, is the set

$$D_{<_T}(\Delta) \stackrel{\text{def}}{=} \{i : t_{i+1} <_T t_i\} \subset [k-1].$$

If $<_T$ is clear from the context, we omit it from the notation. The following theorem is a consequence of [7, Theorem 5.3.4].

Theorem 2.0.5. *Let $u, v \in W$ with $u \leq v$ and let $<_T$, $<'_T$ be two reflection orders. Then*

$$|\{\Gamma \in B(u, v) : D_{<_T}(\Gamma) = \emptyset\}| = |\{\Gamma \in B(u, v) : D_{<'_T}(\Gamma) = \emptyset\}|.$$

In view of the previous theorem, we do not need to indicate the reflection order used. Thus we simply write $|\{\Gamma \in B(u,v) : D(\Gamma) = \emptyset\}|$ when referring to the number of paths in $B(u,v)$ with empty descent set, i.e., rising paths. Furthermore, we are allowed to choose a specific reflection order when dealing with the number of rising paths; we do just that when it serves our purposes.

A *composition* of a positive integer n into t parts is a finite sequence of positive integers $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_t$ such that $\sum_{i=1}^{t} \alpha_i = n$. We write $\alpha \models n$ to mean that α is a composition of n. Given two compositions $\alpha = \alpha_1, \cdots, \alpha_r$ and $\beta = \beta_1, \cdots, \beta_s$ of n, we say that α *refines* β if and only if there exist $1 \leq i_1 < i_2 < \cdots < i_{s-1} < r$ such that $\sum_{j=i_{k-1}+1}^{i_k} \alpha_j = \beta_k$ for $k = 1, \ldots, s$. Here we define $i_0 = 0$ and $i_s = r$. If α refines β, we write $\alpha \preceq \beta$.

For $\Delta \in B_k(u,v)$, we define the *descent composition* of Δ to be the composition $\alpha_1, \cdots, \alpha_t \models k$ such that $\{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\alpha_2+\cdots+\alpha_{t-1}\} = D(\Delta)$. We denote the descent composition of Δ by $\mathcal{D}(\Delta)$. For $u, v \in W$ and $\alpha \models k$, let

$$c_\alpha(u,v) = |\{\Delta \in B_k(u,v) \mid \alpha \preceq \mathcal{D}(\Delta)\}|. \tag{2.1}$$

Notice that $D(\Delta) = \emptyset$ is equivalent to $\mathcal{D}(\Delta)$ having exactly one part.

We remark, in passing, that the numbers $c_\alpha(u,v)$ can be used to compute the Kazhdan-Lusztig polynomial of $[u,v]$. Details can be found in [7].

Of particular interest are the coefficients of the form $c_k(u,v)$, where k is a positive integer, which correspond to the number of paths in $B_k(u,v)$ that are rising. The quantities of the form $c_k(x,y)$ can be used to obtain all $c_\alpha(u,v)$, where $[x,y] \subset [u,v]$, due to the convolution-like formula

$$c_\alpha(u,v) = \sum_{u \leq x_1 \leq \cdots x_{n-1} \leq v} c_{\alpha_1}(u,x_1) c_{\alpha_2}(x_1,x_2) \cdots c_{\alpha_n}(x_{n-1},v), \tag{2.2}$$

where $\alpha = \alpha_1, \cdots, \alpha_n$ (see [7, Proposition 5.54]). It is shown in [11] that these coefficients do not depend on the choice of reflection order.

2.1 Björner and Wachs's CL-labeling

For this subsection, we set $n = \mathrm{rk}([u, v])$.

Björner and Wachs [8] defined a chain label on the edges of $B_n(u, v)$. The existence of such a labeling procedure depends on the SEC (Theorem 1.8.1).

Notice that once a reduced expression for v has been chosen, say $v = s_1 s_2 \cdots s_k$, one can obtain a reduced expression for any element (vertex) in a path $\Delta \in B_n(u, v)$ by simply removing generators from $s_1 s_2 \cdots s_k$. Thus one can label each edge of Δ with the index of the generator removed.

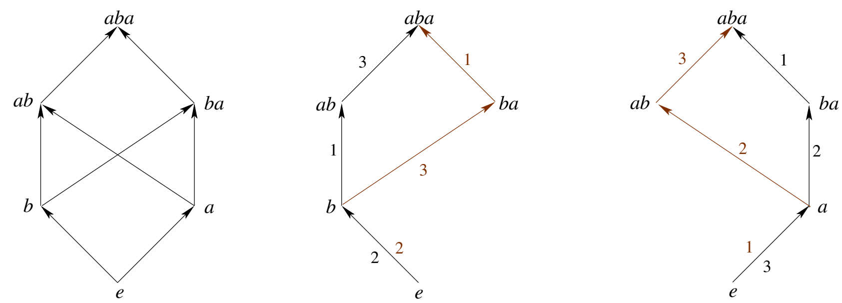


Figure 2.1: Björner and Wachs' labeling for maximum-lengh paths of $B(e, aba)$. Notice that the edge $(e < a)$ has two possible labels depending on which "a" is removed first from aba.

We now describe the labeling in more detail. Let $\Delta = (x_0 = u \lessdot x_1 \lessdot \cdots \lessdot x_n = v) \in B_n(u, v)$ and $s_1 s_2 \cdots s_k$ be a reduced expression for v. Theorem 1.8.1 guarantees the existence of a reduced expression for x_{n-1} of the form $s_1 \cdots \widehat{s}_{j_n} \cdots s_k$, where j_n is unique. Given this reduced expression for x_{n-1}, the same theorem yields the existence of an index j_{n-1} so that the removal of $s_{j_{n-1}}$ from $s_1 \cdots \widehat{s}_{j_n} \cdots s_k$ gives a reduced expression for x_{n-2}. Proceeding in this manner, there is a unique index $j_i \in [k]$ so that removing s_{j_i} from the reduced expression for x_i yields a reduced expression for x_{i-1}. Björner and Wachs' labeling associates Δ with $(\lambda_1(\Delta), \lambda_2(\Delta), \ldots, \lambda_n(\Delta))$, where $\lambda_i(\Delta) = j_i$.

The reason why the labeling procedure is called a chain labeling is because every *chain* (in our case a maximal-length path) is assigned a label, but not an individual edge. For instance, consider Figure 2.1. Notice that $(e < a < ab < aba)$ and $(e < a < ba < aba)$ are labeled $(1, 2, 3)$ and $(3, 2, 1)$, respectively. In particular, the edge $(e < a)$ is labeled either 1 or 3, depending on the index of the "a" that is first removed. In general, if $(u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = v) \in B_n(u, v)$ then the label of $\Delta_1 = (u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_j)$ is uniquely determined once the label of $\Delta_2 = (x_j \lessdot x_{i+1} \lessdot \cdots \lessdot x_n = v)$ has been chosen. In this situation we say that Δ_1 is a *rooted path* (of $B_n(u, v)$); and more precisely, that Δ_1 is *rooted* at Δ_2.

Björner and Wachs [8] showed that their labeling procedure yields a *CL-labeling*, that is, it satisfies (L1) and (L2) below.

(L1) Let $\Delta = (x_0 = u \lessdot x_1 \lessdot \cdots \lessdot x_n = v)$ and $\Delta' = (y_0 = u \lessdot y_1 \lessdot \cdots \lessdot y_n = v)$ be two paths in $B_n(u, v)$. If $x_i = y_i$ for $i \geq j$, with $1 \leq j \leq n-1$, then $\lambda_i(\Delta) = \lambda_i(\Delta')$ for $i \geq j$.

(L2) Let $\Gamma = (y = z_0 \lessdot \cdots \lessdot z_q = v)$ be a y-v path. Then there exists a unique rising (in λ), rooted path Δ in $B_{\text{rk}([u,y])}(u, y)$ that comes earlier in the lexicographic order than any other path Δ' in $B_{\text{rk}([u,y])}(u, y)$ rooted at Γ.

Björner and Wachs [8] utilized their CL-labeling to prove algebraic and topological properties of Bruhat intervals. Later, Dyer [17] would show the same properties using an EL-label, the reflection order. We will discuss this labeling in Chapter 3.

In general, one cannot use Björner and Wachs's procedure to label paths $\Gamma = (x_0 = u < x_1 < \cdots < x_k = v) \in B_k(u, v)$ if $k \neq n$. The reason is that by removing generators from a reduced expression for v one could obtain a nonreduced expression for some x_i, and hence the index in Theorem 1.8.1 need not be unique. For example, consider the the nonreduced expression $s_1s_2s_1s_2s_3s_2s_3s_2$ for $s_2s_1s_2s_3$ in A_3. If one removes the first or fourth generator one obtains nonreduced expressions for $s_2s_1s_3$.

Nevertheless, there are cases where the Björner and Wachs's labeling procedure can be used to label *all* the edges of paths in $B(u,v)$. Some of these cases are discussed in the following section.

2.2 BW-labelable Bruhat intervals

We recall that the generators of a Coxeter system (W,S) are subject to two types of relations, (cf. Section 3.3, [7]):
(i) *nil relations*, which are of the form $s^2 = e$ for all $s \in S$, and
(ii) *braid relations*, which are of the form $\underbrace{s_is_js_is_j\cdots}_{m_{i,j}} = \underbrace{s_js_is_js_i\cdots}_{m_{i,j}}$ for all $s_i, s_j \in S$, $i \neq j$.

Definition 2.2.1. We say that an expression $s_1s_2\cdots s_k$ for $w \in W$ is *nil-reduced* if $s_i \neq s_{i+1}$ for $1 \leq i < k$.

As it is customary, we use the notation $[n] \stackrel{\text{def}}{=} \{1,2,\ldots,n\}$.

Let $s_1s_2\cdots s_n$ be a nil-reduced expression for v. Given $A = \{i_1,\ldots,i_j\}$ we denote the expression $s_1\cdots\widehat{s}_{i_1}\cdots\widehat{s}_{i_j}\cdots s_n$ by $s_{[n]\setminus A}$. For any path $\Delta = (x_0 = u < x_1 < \cdots < x_k = v) \in B_k(u,v)$, the Strong Exchange Condition gives the existence of sets $A_k(\Delta), A_{k-1}(\Delta),\ldots,A_0(\Delta) \subset [n]$ that are constructed recursively: $A_k(\Delta) = \emptyset$ and for $0 \leq i < k$, $b \in [n]$ is an element of $A_i(\Delta)$ if and only if there exists a nil-reduced expression for x_i of the form $s_{[n]\setminus\left(\bigcup_{j>i}\{a_j\}\cup\{b\}\right)}$, where $a_j \in A_j(\Delta)$. We call the sets $A_i(\Delta)$ the *removal sets* of Δ. We remark that since the Björner and Wachs' procedure labels the edges from top to bottom, it is natural for our construction to start with A_k and end with A_0.

Definition 2.2.2. We say that $[u,v]$ is *BW-labelable* if $|A_i(\Delta)| = 1$ for all k and $\Delta \in B_k(u,v)$, $1 \leq i < k$. The *BW-label* of Δ is $(\lambda_1(\Delta),\ldots,\lambda_k(\Delta))$, where $\{\lambda_i(\Delta)\} =$

$A_i(\Delta)$. If every finite interval of a Coxeter group W is BW-labelable, then we say that W is BW-labelable. In other words, $[u, v]$ is BW-labelable if for all $\Delta \in B(u, v)$, the removal sets of Δ are singletons.

As an example, Figure 2.2 depicts the BW-labeling of $[e, aba]$, where the interval is the full dihedral group of order 6 with generators a, b. Furthermore, Figure 3.2 shows the labels $(4, 3, 2, 1)$ and $(1, 3)$ that correspond to the paths $(e < a < ba < aba < baba)$ and $(e < b < baba)$, respectively, where the intervals are in the dihedral group of order 8 with generators a, b.

Let $u \leq x \leq y \leq v$ be elements of W and consider a path $\Delta = (x_0 = x < x_1 < \cdots < x_k = y < \cdots < x_{k+m} = v) \in B_{k+m}(x, v)$. By the same reason as for maximum-length paths, the BW-label of $\Delta_1 = (x_0 = x < x_1 < \cdots < x_k = y) \in B_k(x, y)$ depends on the BW-label of $\Delta_2 = (x_k = y < \cdots < x_{k+m} = v) \in B_m(y, v)$. In this situation, we say that Δ_1 is *rooted* at Δ_2, or that Δ_1's *root* is Δ_2.

There are two consequences that follow directly from the definition.

Remark 2.2.3. (i) Notice that the BW-labeling on paths of $B_{\mathrm{rk}([u,v])}(u, v)$ is exactly Björner and Wachs's CL-labeling. In other words, the BW-labeling can always be given to paths in $B_{\mathrm{rk}([u,v)]}(u, v)$.

(ii) Let $[u, v]$ be BW-labelable, $P = (u < \cdots < x < \cdots < y < \cdots < v) \in B(u, v)$ and $P' = (y < \cdots < v) \in B(y, v)$, i.e., P and P' share all vertices from y to v. Then the interval $[x, y]$ is BW-labelable with labels given by setting P' as the root for all paths in $B(x, y)$. In this case, we say that $s_1 \cdots s_m$ (an expression for y) is given by *following* P'.

(iii) Let $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$, where W, W_1 are Coxeter groups. Suppose that $[u_1, v_1]$ and $[u_2, v_2]$ are BW-labelable, then $[u_1, v_1] \times [u_2, v_2]$ is a BW-labelable interval in $W \times W_1$. This follows by noticing that the removal set for any path in $B((u_1, u_2), (v_1, v_2))$ is contained in $C \times D$, where C is a removal set of a path in $B(u_1, v_1)$ and D is a removal set of a path in $B(u_2, v_2)$.

We remark that not all Bruhat intervals are BW-labelable, as can be seen in the example below.

Example 2.2.4. Consider the reduced expression $v = s_1s_2s_1s_4s_2s_3s_2s_4s_3s_2 \in \langle s_1, \ldots, s_4 : s_i^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_3)^2 = (s_js_4)^\infty = e, i \in [4], j \in [3]\rangle$. Now consider $\Delta = (u < s_2s_1s_2s_3 < s_2s_1s_3s_2s_4s_3s_2 < v) \in B_3(u, v)$, where $u = s_2s_1s_3$. Then $A_3(\Delta) = \emptyset$, $A_2(\Delta) = \{4\}$ (which yields the expression $s_1s_2s_1s_2s_3s_2s_4s_3s_2$), $A_1(\Delta) = \{8\}$ (which yields the expression $s_1s_2s_1s_2s_3s_2s_3s_2$) and $A_0(\Delta) = \{1, 5\}$ (which yields the expressions $s_2s_1s_2s_3s_2s_3s_2 = s_1s_2s_1s_3s_2s_3s_2 = u$).

Thus $[u, v]$ is not BW-labelable. However, the groups of interest here, namely dihedral and universal Coxeter groups, are BW-labelable. We utilize this fact in Section 3.2.

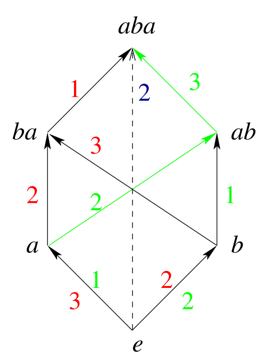


Figure 2.2: $[e, aba]$ is BW-labelable. The path $(e < aba)$ has label $\lambda_1((e < aba)) = 2$.

We now present examples of BW-labelable intervals and groups.

Example 2.2.5. (a) If $B(u, v)$ has paths of only two lengths then $[u, v]$ is BW-labelable. Indeed, the maximal-length u-v paths can be labeled with the Björner and Wachs' CL-labeling. Moreover, if $\Delta \in B_{\mathrm{rk}([u,v])-2}(u, v)$, then any expression for the vertices of Δ obtained by following Δ is either reduced or of the form $xs_1s_2s_1y$ with $\ell(xs_1\widehat{s}_2s_1y) = \ell(xy) = \ell(xs_1s_2s_1y) - 3$. Thus the edge $(xy < xs_1s_2s_1y)$ has a unique label.

(b) Similarly, it can be argued that intervals $[u, v]$ of rank up to 5 are BW-labelable, since paths of length 1 correspond to the reflection $u^{-1}v$ and thus there is a unique label assigned to it (see the edge $(e < aba)$ in Figure 2.2).

Example 2.2.6. Let $I_2(\infty)$ denotes the *infinite dihedral group* (which is the affine Weyl group $\widetilde{A}_1$). Let $[u, v]$ be an interval in $I_2(n)$ or $I_2(\infty)$ and $s_1 \cdots s_n$ be a reduced expression for v. Since any nil-reduced expression for x_i in $\Delta = (x_0 = u < x_1 < \cdots < x_k = v) \in B_k(u, v)$ obtained by removing generators of a reduced expression for v is reduced, $[u, v]$ is BW-labelable. Indeed, the Strong Exchange Condition guarantees that $|A_i(\Delta)| = 1$. So both $I_2(n)$ and $I_2(\infty)$ are BW-labelable.

Example 2.2.7. One says that (W, S) is *universal* if there are no braid relations. Let $[u, v]$ be an interval in a universal Coxeter system and $s_1 \cdots s_n$ be a reduced expression for v. Similar to the dihedral group case, any nil-reduced expression for an element in $[u, v]$ obtained by removing generators from $s_1 \cdots s_n$ is reduced, and so W is BW-labelable. This fact will allow us to compute the **cd**-index, as described in Section 3.2.

Definition 2.2.8. Let $[u, v]$ be a BW-labelable interval of a Coxeter system (W, S), $u \leq x \leq y \leq v$ be elements of W, $\Gamma_1 = (x_0 = u < x_1 < \cdots < x_m = x) \in B(u, x)$, $\Delta = (x_m = x < x_{m+1} < \cdots < x_k = y) \in B(x, y)$ and $\Gamma = (x_k = y < \cdots < x_n = v) \in B(y, v)$. Notice that Γ is a root of Δ.

(i) We denote the *concatenation* $(x_0 = u < \cdots < x_m = x < \cdots < x_k = y < \cdots < x_n = v)$ of Γ_1, Δ and Γ by $\Gamma_1\Delta\Gamma$.

(ii) We define the *BW-descent set* of Δ as

$$D^{BW}_{\Gamma_1,\Gamma}(\Delta) \stackrel{\text{def}}{=} \{i \in \{m, m+1, \ldots, k-1\} \mid \lambda_{i+1}(\Gamma_1\Delta\Gamma) < \lambda_i(\Gamma_1\Delta\Gamma)\}.$$

Notice that the label given to Δ only depends on the choice of root Γ. So we drop Γ_1 from the notation and write simply $D^{BW}_{\Gamma}(\Delta)$.

(iii) We further denote the *BW-descent composition* corresponding to $D^{BW}_{\Gamma}(\Delta)$ by $\mathcal{D}^{BW}_{\Gamma}(\Delta)$.

(iv) We say that $\Delta \in B(x, y)$ is *BW-rising with respect to* Γ if $D^{BW}_{\Gamma}(\Delta) = \emptyset$, or equivalently, $\mathcal{D}^{BW}_{\Gamma}(\Delta) = \ell(\Delta)$. When Γ is clear by context, we simply write *BW-rising*.

(v) Define $c^{BW}_{\alpha,\Gamma}(x, y) \stackrel{\text{def}}{=} |\{\Delta \in B_k(x, y) \mid \alpha \preceq \mathcal{D}^{BW}_{\Gamma}(\Delta)\}|$, where $\alpha \models k$.

If $y = v$, then Γ is the path with no edges. In this case, we ignore the reference to Γ in the notation and write $D^{BW}(\Delta)$, $\mathcal{D}^{BW}(\Delta)$ and $c^{BW}_{\alpha}(x, y)$, respectively.

We now prove that $c^{BW}_{k,\Gamma}(x, y) = c_k(x, y)$, for any $\Gamma \in B(y, v)$. This is the first step to show that $c^{BW}_{\alpha,\Gamma}(u, y) = c_\alpha(u, y)$.

Lemma 2.2.9. *Let $[u, v]$ be a BW-labelable interval with $u \leq x \leq y \leq v$. Then for $k > 0$, $c^{BW}_{k,\Gamma}(x, y) = c_k(x, y)$, regardless of the choice of $\Gamma \in B(y, v)$.*

Proof. Let $s_1 s_2 \cdots s_n$ be the expression for y given by following Γ. First let us assume that $s_1 s_2 \cdots s_n$ is reduced. and let $C = (x_0 = x < x_1 < \cdots < x_k = y)$ be BW-rising. By the Strong Exchange Condition we have that $x = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}} \cdots s_n$. Since C is BW-rising, the BW-label of C is $(i_1, i_2, \ldots, i_k)$, regardless of the choice of Γ. Let $t_j = s_n \cdots s_{j+1} s_j s_{j+1} \cdots s_n$. Then $N(y) = \{t_1, \ldots, t_n\}$, and so $t_n <_T t_{n-1} <_T \cdots <_T t_1$ is the initial section for some reflection order $<_T$, by [17, Lemma 2.11, Proposition 2.13 and Remark 2.4(i)]. Since $y = x t_{i_k} \cdots t_{i_1}$, the label of C with under $<_T$ is $(t_{i_k}, \ldots, t_{i_1})$, and so C is rising under $<_T$.

Conversely, let $C = (x = y_0 < y_1 < \cdots < y_k = y)$ be a rising path under the reflection order $<_T$. Then there exists a reflection t with $y_{k-1}t = y$. Thus $t_{j_1} \stackrel{\text{def}}{=} t = s_n \cdots s_{j_1} \cdots s_n$, where $yt = s_1 \cdots \widehat{s_{j_1}} \cdots s_n$, belongs to the initial section induced by $y = s_1 \cdots s_n$. Since C is rising, t_{j_1} is the highest-labeled reflection in the path, and the remaining reflections lie in that initial section. Thus C has (reflection order) label $(t_{j_k}, \ldots, t_{j_1})$, with $t_{j_m} <_T t_{j_{m'}}$ if and only if $m >_{\mathbb{Z}} m'$. Hence C's BW-label is $(j_1, \ldots, j_k)$, and so C is BW-rising.

The case where $y = s_1 s_2 \cdots s_n$ is not reduced is handled similarly. If $s_1 s_2 \cdots s_n$ is not reduced, then it is of the form $w_1 u w_2$, where w_1, w_2 are reduced and $u = s_i s_j s_i \cdots$ (h characters) is not reduced (so $\ell(u) < h < 2m_{i,j}$). This is because the procedure to obtain the BW-labeling removes characters from a reduced expression for v. Notice that $\ell(y) = \ell(w_1) + \ell(u) + \ell(w_2)$. Thus, the subexpression of $u = s_i s_j s_i \cdots$ formed by the "middle word" w of length $\ell(u)$ is reduced. That is, there exists u_1 and u_2 so that the concatenation of u_1, w and u_2 yields $u = s_i s_j s_i \cdots$ (h characters), and the number of characters in u_1 and u_2 is the same. Notice that if one removes a generator from w to produce w', then $u_1 w' u_2$ is reduced (after applying nil-moves). So by considering w instead of $u = s_i s_j s_i \cdots$ (h characters) one can construct the desired bijection in a similar manner as in the last paragraph. □

Theorem 2.2.10. *Let $[u, v]$ be a BW-labelable interval with $u \leq x \leq y \leq v$, let $\alpha = \alpha_1, \alpha_2 \ldots, \alpha_m \models k$ and $\Delta \in B(x, y)$. Then $c^{BW}_{\alpha,\Delta}(u, x) = c_\alpha(u, x)$.*

Proof. We proceed by induction on m. If $m = 1$, the statement follows from Lemma 2.2.9. If $m > 1$, let $\widehat{\alpha} = \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$. Furthermore, for $\Gamma \in B(u, x)$, let $A_n(\Gamma, z) = \{\Delta_{\Gamma,z} \in B_n(z,x) : u \leq z \leq x, D(\Delta_{\Gamma,z}) = \emptyset\}$. Then

$$\begin{aligned}
c^{BW}_{\alpha,\Delta}(u,x) &= \sum_{\substack{\Gamma \in B(u,x) \\ \Delta_{\Gamma,z} \in A_{\alpha_m}(\Gamma,z)}} \sum c^{BW}_{\widehat{\alpha},\Delta_{\Gamma,z}\Delta}(u,z) \\
&= \sum_{\substack{\Gamma \in B(u,x) \\ \Delta_{\Gamma,z} \in A_{\alpha_m}(\Gamma,z)}} \sum c_{\widehat{\alpha}}(u,z) \\
&= \sum_{u \leq z \leq x} c_{\widehat{\alpha}}(u,z) \sum_{\Delta_{\Gamma,z} \in A_{\alpha_m}(\Gamma,z)} 1 \\
&= \sum_{u \leq z \leq x} c_{\widehat{\alpha}}(u,z) c_{\alpha_m}(z,x) \\
&= c_\alpha(u,x),
\end{aligned}$$

where the second equality follows by induction and the last one from (2.2). □

In particular, if $[u,v]$ is BW-labelable, $c^{BW}_\alpha(u,v) = c_\alpha(u,v)$. Thus the BW-labeling and the reflection order yield the same descent-set distribution on paths in $B(u,v)$.

Example 2.2.11. Consider the interval $[e, s_2s_1s_3s_2s_1]$ in A_3 (which corresponds to $[1234, 4312]$ in one-line notation for permutations). This interval is BW-labelable, as its rank is 5. Using the "canonical" reflection order on A_3 (see Corollary 5.2.3), the elements of $B_3(e, s_2s_1s_3s_2s_1)$ are listed in the table below.

Notice that under either label, the descent sets are: $\emptyset$ (two of them), $\{1\}$ (three of them), $\{2\}$ (three of them), and $\{1,2\}$ (two of them). Notice that the labels used in the BW-labeling are taken from the set $\{1, 2, \ldots, \ell(s_2s_1s_3s_2s_1) = 5\}$.

Table 2.1: Elements of $B_3(e, s_2s_1s_3s_2s_1)$. The columns lists the paths in reflection order and BW-labeling, respectively.

Reflection Order	BW-labeling
134	134
143	143
235	245
251	341
423	413
462	425
514	431
521	452
625	524
652	542

CHAPTER 3

COMPLETE cd-INDEX OF BRUHAT INTERVALS

3.1 Complete cd-index of Bruhat intervals

Billera and Brenti [3] provided a way to encode the descents sets of paths in $B(u, v)$ with a nonhomogeneous polynomial on the noncommutative variables **c** and **d**. The encoding is done as follows: For a path $\Delta = (t_1, t_2, \cdots, t_k) \in B_k(u, v)$, let $w(\Delta) = u_1 u_2 \cdots u_{k-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } t_i < t_{i+1} \\ \mathbf{b} & \text{if } t_{i+1} < t_i. \end{cases}$$

A mnemotechnic resource is to remember the encoding is to set u_i to **a** if there is an **a**scent at position i.

We call the (nonhomogeneous) polynomial $\sum_{\Delta \in B(u,v)} w(\Delta)$ the *complete **ab**-index of* $[u, v]$, and we denote it by $\widetilde{\Psi}_{u,v}(\mathbf{a}, \mathbf{b})$.

Billera and Brenti showed that $\sum_{\Delta \in B(u,v)} w(\Delta)$ becomes a polynomial in the variables **c** and **d**, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the *complete* **cd**-*index* of $[u, v]$, and it is denoted by $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$.

Notice that the complete **cd**-index of $[u, v]$ is an encoding of the distribution of the descent sets of each path Δ in the Bruhat graph of $[u, v]$, and thus seems to depend on $<_T$. However, since the c_α (see equations (2.1) and (2.2)) do not depend on $<_T$, $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ does not either.

The *degree* of a term in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is given by noticing that $\deg(\mathbf{c}) = 1$ and $\deg(\mathbf{d}) = 2$. For instance, $\deg(\mathbf{d}^2\mathbf{c}) = 5$.

Example 3.1.1. Consider A_2, the symmetric group on 3 elements with generators $s_1 = (1\ \ 2)$ and $s_2 = (2\ \ 3)$. Then $t_1 = s_1 <_T t_2 = s_1 s_2 s_1 <_T t_3 = s_2$ is a reflection order. The paths of length 3 are: $(t_1, t_2, t_3), (t_1, t_3, t_1), (t_3, t_1, t_3)$, and (t_3, t_2, t_1), that encode to $\mathbf{a^2} + \mathbf{ab} + \mathbf{ba} + \mathbf{b^2} = \mathbf{c^2}$. There is one path of length 1, namely t_2, which encodes simply to 1. So $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \mathbf{c^2} + 1$.

We remark that [15, Proposition 3.3] shows that if $B_k(u, v) \neq \emptyset$ and $k \neq \mathrm{rk}([u, v])$, then $B_{k+2}(u, v) \neq \emptyset$. As a consequence, if $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ has terms of degree $k - 1$ (corresponding to paths of length k), then it also has terms of degree $k + 1$ (corresponding to paths of length $k + 2$).

There are some specializations of the complete **cd**-index that count paths with certain properties in $B(u, v)$. For instance, we have the proposition below.

Proposition 3.1.2. *Let $[u, v]$ be a Bruhat interval. Then,*
(i) $\widetilde{\psi}_{u,v}(2, 2) = |\{\Delta : \Delta \in B(u, v)\}|$, the number of paths of $B(u, v)$, and
(ii) $\widetilde{\psi}_{u,v}(1, 0) = |\{\Delta \in B(u, v) : D(\Delta) = \emptyset\}|$, the number of rising paths of $B(u, v)$.

Proof. (i) To each path $\Delta \in B(u, v)$ there is a corresponding $w(\Delta)$ as defined at the beginning of this section. Hence, the number of **ab**-monomials in the complete **ab**-index, $\widetilde{\Psi}_{u,v}(1, 1)$ equals the number of paths in $B(u, v)$. Since $\widetilde{\Psi}_{u,v}(1, 1) = \widetilde{\psi}_{u,v}(2, 2)$, we obtain the desired result.
(ii) By definition, $\widetilde{\Psi}_{u,v}(1, 0)$ gives the number of rising paths of $B(u, v)$. The result follows from noticing that $\widetilde{\Psi}_{u,v}(1, 1) = \widetilde{\psi}_{u,v}(1, 0)$. □

Other enumerative consequences of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ can be found in [3]. In particular, Proposition 5.4.

Notice that [3, Theorem 2.2 and Corollary 2.3] yields that $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ can be computed from the numbers $c_\alpha(u, v)$. Thus in view of Theorem 2.2.10, if $[u, v]$ is BW-labelable we can compute $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ using the identity $c^{BW}_\alpha(u, v) = c_\alpha(u, v)$. In the next two chapter, we use the BW-labeling to compute $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ for dihedral intervals and intervals in universal Coxeter groups.

3.1.1 Reason for the term "complete"

Recall that the Bruhat interval $[u, v]$ is Eulerian (see Theorem 1.8.3(i)). Thus the the **cd**-index of $[u, v]$, $\psi([u, v])$, is defined. It turns out that the highest-degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ coincide with the terms of $\psi([u, v])$. This is due to the following theorem, which we state in less generality but that suffices for our purposes.

Theorem 3.1.3 (cf. [6], Theorem 2.7)**.** *Suppose that P is an Eulerian poset of rank n that is EL-labelable. Then $h_S(P)$ is the number of maximal chains of P with descent set $S \subset [n-1]$.*

Hence, the highest-degree terms of the complete **ab**-index of $[u, v]$ do not depend on the labeling, as they are given by the flag-h vector of $[u, v]$. Furthermore, these terms coincide with the **ab**-index, which are also given by the flag-h vector of $[u, v]$. Hence, the the highest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ coincide with $\psi([u, v])$. So the term "complete" is used as the monomials of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ include the monomials of $\psi([u, v])$.

Karu [24] showed the nonnegativity of the the coefficients $\psi(P)$, where P is a Gorenstein* poset. In particular, we have the following theorem.

Theorem 3.1.4. *If $[u, v]$ is a Bruhat interval, then the highest-degree coefficients of $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d})$ are nonnegative.*

We utilize Karu's result to deduce nonnegativity of other coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. We also mention, in passing, that Karu [23] recently showed that the coefficients of the **cd**-terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ that contain a single **d** factor (such as $\mathbf{c}^2\mathbf{dc}$ and $\mathbf{dc}^4$, but not **dcd**) are non-negative.

3.2 Dihedral intervals

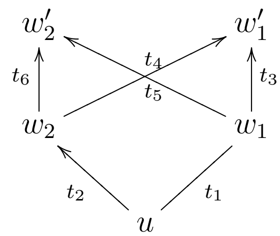
Let $u, v \in I_2(m)$ with $u \leq v$, then the isomorphism type of $B(u, v)$ is well known. For example, Figure 1.4 depicts $I_2(4)$.

Dyer [15] observed that if W_1 and W_2 are dihedral reflection subgroups and $W_1 \cap W_2$ contains a dihedral reflection subgroup W_3, then $\langle W_1, W_2 \rangle$ is a dihedral reflection subgroup. This observation is used in the proof of Lemma 3.2.1.

We say that a Bruhat interval $[u, v]$ is *dihedral* if it is isomorphic to an interval in a dihedral reflection subgroup. In this section, we compute the complete **cd**-index of dihedral intervals. The computation is simplified if the BW-labeling is utilized, and so we take this approach. It turns out that it is enough to consider the case where $u, v \in I_2(m)$ for some m. We make this explicit in the following lemma.

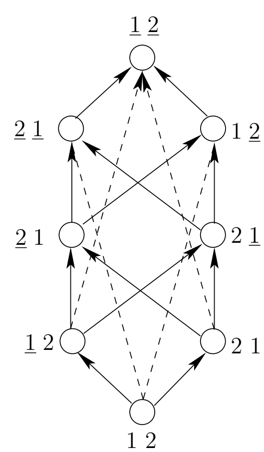
Lemma 3.2.1. *Let $[u, v]$ be a dihedral interval of (W, S), where the edges of $B(u, v)$ have been labeled by reflections. Then $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d}) = \widetilde{\psi}_{w,z}(\boldsymbol{c}, \boldsymbol{d})$, where $[w, z] \subset I_2(m)$ for some m.*

Proof. Let $[u, v]$ be a dihedral interval in $B(W)$ and let $t_1, t_2, \ldots, t_m$ be all the reflections that correspond to the labels of the edges of $B(u, v)$. Let $w_1, w_2, w_1', w_2' \in [u, v]$ with $u \lessdot w_1$, $u \lessdot w_2$, $w_1, w_2 \lessdot w_1'$, and $w_1, w_2 \lessdot w_2'$. Suppose that the labels of $B(u, w_1')$ and $B(u, w_2')$ are t_1, t_2, t_3, t_4 and t_1, t_2, t_5, t_6, respectively (see figure below). From Dyer [15, Lemma 3.1], we have that $W_1 \stackrel{\text{def}}{=} \langle t_1, t_2, t_3, t_4 \rangle$ and $W_2 \stackrel{\text{def}}{=} \langle t_1, t_2, t_5, t_6 \rangle$ are dihedral reflection subgroups of (W, S). Moreover, since $\langle t_1, t_2 \rangle \subset W_1 \cap W_2$, then $\langle t_1, t_2, \ldots, t_6 \rangle$ is a dihedral reflection subgroup of (W, S). Proceeding in a similar manner, we conclude that $W' \stackrel{\text{def}}{=} \langle t_1, t_2, \ldots, t_m \rangle$ is a dihedral reflection subgroup of (W, S).



Now by [15, Theorem 1.4] and [16, Proposition 1.4] there exists a label-preserving isomorphism (the labels are given by reflections) between $B_W(W')$ and $B_W(W'u)$, where $B_W(A)$ denotes the induced subgraph of $B(W)$ with vertex set $A \subset W$. Notice that $B(u, v)$ is an induced subgraph of $B_W(W'u)$. Moreover, by [15, Theorem 1.4(i)], $B_W(W') \cong B(W')$ as directed graphs, labeled graphs. Furthermore $B(W') \cong B(I_2(m))$ for some $m \in \mathbb{Z}_{>0} \cup \{\infty\}$, as directed graphs, and the reflection order on edges of $B_W(W')$ and $B(I_2(m))$ have the same descent-set distribution. The result now follows. $\square$

The previous result gives that $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \widetilde{\psi}_{w,z}(\mathbf{c}, \mathbf{d})$ and $\mathrm{rk}([w, z]) = \mathrm{rk}([u, v])$, where $[w, z] \subset I_2(m)$ for some m. On the other hand, since $[w, z]$ is BW-labelable, Theorem 2.2.10 gives that we can also compute $\widetilde{\psi}_{w,z}(\mathbf{c}, \mathbf{d})$ utilizing the BW-labeling. As it turns out, using the BW-labeling facilitates the computation.



We now describe the complete **cd**-index for dihedral intervals in terms of the *q-Fibonacci* polynomial of degree n, where $F_n(q)$ is defined by $F_1(q) = 1$, $F_2(q) = q$, and $F_n(q) = qF_{n-1}(q) + F_{n-2}(q)$ for $n > 2$.

Proposition 3.2.2. *If $[u, v]$ is a dihedral interval of rank n, then $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d}) = F_n(\boldsymbol{c})$.*

Proof. Lemma 3.2.1 gives that it is enough to consider the case $u, v \in I_2(m)$ for some m. Moreover, we can assume that paths in $B(u, v)$ are labeled with the BW-labeling.

We proceed by induction on $n \stackrel{\text{def}}{=} \mathrm{rk}([u, v])$. If $n = 1$ or $n = 2$, it is easy to verify that the result holds.

Let v_1, v_2 be the two elements of rank $n - 1$ in $[u, v]$. Notice that one of these, say v_1, is obtained from the chosen reduced expression of v by removing the last generator. Thus for any path $\Delta = (u < \cdots < v_1 < v) \in B_k(u, v)$, we have that $\lambda_{k-1}(\Delta) < \lambda_k(\Delta)$. Hence the contribution of the u-v paths through v_1 is $\mathbf{a}\widetilde{\psi}_{u,v_1}$. Similarly, v_2 is obtained from v by removing the first generator of the reduced expression chosen for v. In this case the contribution of all paths $\Gamma = (u < \cdots < v_2 < v)$ is $\mathbf{b}\widetilde{\psi}_{u,v_2}$, for $\lambda_k(\Gamma) < \lambda_{k-1}(\Gamma)$. Therefore, the contribution of all u-v passing through v_1 or v_2 is $\mathbf{a}\widetilde{\psi}_{u,v_1}(\mathbf{c}, \mathbf{d}) + \mathbf{b}\widetilde{\psi}_{u,v_2}(\mathbf{c}, \mathbf{d}) = \mathbf{a}F_{n-1}(\mathbf{c}) + \mathbf{b}F_{n-1}(\mathbf{c}) = \mathbf{c}F_{n-1}(\mathbf{c})$.

We are left with computing the contribution of paths that do not go through the vertices of rank $n - 1$. Notice that an expression for each of the vertices in these paths is obtained from the reduced expression of v by removing generators other than the first or last one. Thus the descent-set distribution of the paths in $B(u, v)$ that do not go through v_1 or v_2 is the same as that of an interval of the form $[u, x]$ with $\mathrm{rk}([u, x]) = n-2$. Hence, these paths' contribution is $\widetilde{\psi}_{u,x}(\mathbf{c}, \mathbf{d}) = F_{n-2}(\mathbf{c})$. □

Proposition 3.2.2 shows that the coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$, when $[u, v]$ is dihedral, are both nonnegative and combinatorially invariant (i.e., only depend on the isomorphism type of $[u, v]$). Non-negativity and combinatorial invariance are conjectured for $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ in general (cf. [3, Conjecture 6.1 and Remark 4.13]). An enumerative consequence follows immediately by setting $\mathbf{c} = 1$ in $F_n(\mathbf{c})$ and by Proposition 3.1.2.

Corollary 3.2.3. *If $[u, v]$ is a dihedral interval of rank n, then the number of u-v paths with increasing (decreasing) labels in $B(u, v)$ is the n-th Fibonacci number.*

The following theorem yields that dihedral intervals are the only ones that do not contain a $\mathbf{d}$ in their complete $\mathbf{cd}$-index.

Theorem 3.2.4. *Let $[u, v]$ be a Bruhat interval. Then $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d}) = \widetilde{\psi}_{u,v}(\boldsymbol{c}, 0)$ if and only if $[u, v]$ is a dihedral interval.*

Proof. The theorem is vacously true if the rank of $[u, v]$ is 1, so we can assume that $\mathrm{rk}[u, v] > 1$. If $[u, v]$ is dihedral, Proposition 3.2.2 gives that $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = F_{\mathrm{rk}([u,v])}(\mathbf{c}) = \widetilde{\psi}_{u,v}(\mathbf{c}, 0)$.

Suppose that $[u, v]$ is not dihedral. We show by contradiction that among the highest-degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ there must be a term containing a $\mathbf{d}$. Let $p_k(u, v)$ be the number of paths of length k from u to v, and define $n \stackrel{\text{def}}{=} \mathrm{rk}([u, v]) - 1$. Since $[u, v]$ is Eulerian, any element $w \in [u, v], w \neq v$, has at least two covers, and so there are at least two elements in each rank, except for the top and bottom elements. Moreover, since $[u, v]$ is not dihedral there are at least three elements of rank 1, and so $p_{n+1}(u, v) \geq 3 \cdot 2^{n-1}$. Nevertheless, [3, Proposition 5.3] states that

$$p_{n+1}(u, v) = \sum_{w:\deg(w)=n} 2^{n-|w|_{\mathbf{d}}}[w]_{u,v} \quad ,$$

where $|w|_{\mathbf{d}}$ is the number of $\mathbf{d}$'s in a $\mathbf{cd}$-word w and $[w]_{u,v}$ is the coefficient of w in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. Furthermore, if there are no $\mathbf{cd}$-terms containing a $\mathbf{d}$ then there is a unique **cd**-word of degree n which corresponds to the unique rising maximum-length path of length $n+1$. Hence $p_{n+1}(u,v) = 2^n$. This contradicts $p_{n+1}(u,v) > 2^n$, and the result follows. □

3.3 Universal Coxeter groups

Reflection orders are easy to understand for any finite Coxeter group W. Indeed, they are all induced by a choice of reduced expression for w_0^W, the longest element of W (see [17]). Furthermore, if W is of type A or B there are combinatorial descriptions of reflection orders for these groups in terms of permutations and signed permutations, respectively. For example, for type A, a reflection order is given by ordering the transpositions in lexicographic order (see [7]). In Chapter 5, we provide descriptions of the reflections and longest-length elements for groups of type A, B and D, as well as a description of a reflection order for the first two. Thus the complete **cd**-index for intervals in finite Coxeter groups can be easily computed. On the other hand, there is no known method to generate reflection orders for infinite Coxeter groups, not even in the "simple" case of universal Coxeter groups, where there are no braid relations. This difficulty makes the computation of the complete **cd**-index extremely difficult to carry out. In general, the BW-labeling allows us to compute $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ for intervals $[u,v]$ in universal Coxeter groups.

In general, the reflections used to label paths in $B(u,v)$ need not bear any connection with v. As an example, consider the interval $[e, s_1s_2s_1s_3s_2]$ of A_3, where $s_1s_2s_1s_3s_2$ is a reduced expression for the permutation 4312. There are five reflections induced by $s_1s_2s_1s_3s_2$, but the reflection order utilizes six reflections when labeling the edges of $B(e, s_1s_2s_1s_3s_2)$. Hence, knowing a reduced expression for $w \in W$ does not determine the reflections used to label the edges of paths in $B(e, s_1s_2s_1s_3s_2)$. This is not the case for the BW-labeling. Indeed, if $[u, v]$ is BW-labelable, all one needs to compute $D(\Delta)$ for $\Delta \in B(u, v)$ is a reduced expression for v. In particular, even if $|T(W, S)|$ is infinite (as is the case for universal Coxeter groups), the labels needed to compute the descent sets of paths in $B(u, v)$ are contained in the set $\{1, 2, \ldots, \ell(v)\}$.

Let us illustrate a computation of the complete **cd**-index for an interval in a universal Coxeter group.

Example 3.3.1. Consider the universal Coxeter group $W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = e\rangle$ and let $v = s_2s_1s_2s_3s_1$. Using the BW-labeling we obtain that the degree-two terms of $\widetilde{\psi}_{e,v}(\mathbf{c}, \mathbf{d})$ are $2\mathbf{c}^2 + \mathbf{d}$. Notice that v induces a reflection order $<_T$ with initial section

$$s_1 <_T s_1s_3s_1 <_T s_1s_3s_2s_3s_2 <_T s_1s_3s_2s_1s_2s_3s_1 <_T s_1s_3s_2s_1s_2s_1s_2s_3s_1.$$

However, such initial section would not suffice to compute $\widetilde{\psi}_{e,v}(\mathbf{c}, \mathbf{d})$, as some of the edges of paths in $B_3(e, v)$ are labeled with reflections that do not appear in that initial section. For example, the edge $(e < s_2)$ is labeled with s_2 and the edge $(s_1s_3 < s_1s_2s_3)$ is labeled with $s_3s_2s_3$. Thus one cannot compute $\widetilde{\psi}_{e,v}(\mathbf{c}, \mathbf{d})$ with a reflection order using information contained in $[u, v]$ (at least using reduced expressions of elements in $[u, v]$).

3.4 Existence of rising paths

Let $[u, v]$ have rank n, then it follows from [15, Proposition 3.3] that if $m < n$ and $B_m(u, v) \neq \emptyset$, then $B_{m+2}(u, v) \neq \emptyset$. So the lengths of paths in $B(u, v)$ increase by 2 and include all numbers congruent to n modulo 2 between the smallest number m such that $B_m(u, v) \neq \emptyset$ and n.

As we pointed out before, it is shown in [17] that there is a unique rising path in $B_{\mathrm{rk}([u,v])}(u,v)$ that is lexicographically first in the reflection order. In this subsection we show that the lexicographically first path in $B_k(u,v) \neq \emptyset$, where $k \equiv \mathrm{rank}([u,v]) \pmod 2$, is rising. In general, there might be more than one rising path in $B_k(u,v)$.

The following is [12, Theorem 1] and will be utilized in our proof of Proposition 3.4.2.

Theorem 3.4.1. *Let (W,S) be a finite or affine Coxeter group and let $<_T$ be a reflection order for W. If $\{t_1,t_2,\ldots,t_k\} \subset T(W,S)$ and $t_1 <_T t_2 <_T \cdots <_T t_k$ then $t_1t_2\cdots t_k \neq e$.*

We now follow [3] to define the *flip* of $\Gamma \in B_2(u,v)$. Let (t_1,t_2) and (r_1,r_2) be in $B_2(u,v)$. We say that $(t_1,t_2) \leq_{lex} (r_1,r_2)$ if $t_1 <_T r_1$ or if $t_1 = r_1$ and $t_2 <_T r_2$, or $t_2 = r_2$. The existence of the complete **cd**-index implies that there are as many paths with empty descent set in $B_2(u,v)$ as those with descent set $\{1\}$. Order all the paths in $B_2(u,v)$ lexicographically and let

$$r(\Gamma) = |\{\Delta \in B_2(u,v) \mid D(\Delta) = D(\Gamma), \Delta \leq_{lex} \Gamma\}|.$$

The *flip* of Γ is the $r(\Gamma)$-th Bruhat path in $\{\Delta \in B_2(u,v) \mid D(\Delta) \neq D(\Gamma)\}$ ordered by $\leq_{lex}$. We denote this path by *flip*(Γ).

The following was proved in the case of finite Coxeter and affine Weyl groups [3, Proposition 6.2]. We prove that the results holds for an arbitrary Coxeter group.

Proposition 3.4.2. *Let W be a Coxeter group, and let $u,v \in W$, $u < v$, $(u < y < v) \in B_2(u,v)$ be such that $D((u < y < v)) = \emptyset$ and $(u < x < v) \overset{def}{=}$ flip$((u < y < v))$. Then $u^{-1}y <_T u^{-1}x$ and $x^{-1}v <_T y^{-1}v$ for any reflection order $<_T$.*

Proof. Let $t_1 = u^{-1}y, t_2 = y^{-1}v, t_3 = u^{-1}x$, and $t_4 = x^{-1}v$. Notice that the reflection subgroup $W' = \langle t_1, t_2, t_3, t_4 \rangle$ is dihedral, by [15, Lemma 3.1]. Thus $\{t_1, t_2, t_3, t_4\} \subset T(W', \{a, b\})$, where $\langle a, b \rangle = \langle t_1, t_2, t_3, t_4 \rangle$ and $(W', \{a, b\})$ is a Coxeter system.

Suppose for the sake of contradiction that $t_3 <_T t_1$, then $t_4 <_T t_3 <_T t_1 <_T t_2$, since $D((u < y < v)) = \emptyset$ and $D((u < x < v)) = \{1\}$. Moreover since $t_1t_2 = t_3t_4$ one has that $t_4t_3t_1t_2 = e$. On the other hand $(W', \{a, b\})$ is either a finite or affine Coxeter system, and thus $t_4t_3t_1t_2 \neq e$ by Theorem 3.4.1. We have obtained our desired contradiction, and the result follows. The statement $t_4 <_T t_2$ is proved in a similar manner. □

We can now prove the following proposition.

Proposition 3.4.3. *Let Δ be the lexicographically-first path in $B_k(u, v)$. Then $D(\Delta) = \emptyset$, i.e, Δ is rising.*

Proof. Let $<_T$ be a reflection order and let $C = (x_0 = u < x_1 < \cdots < x_k = v)$ be the lexicographically-first path in $B_k(u, v)$. Let us suppose that $D(C) \neq \emptyset$, and consider the smallest i such that $x_i^{-1}x_{i+1} <_T x_{i-1}^{-1}x_i$. Let $(x_{i-1} < x_i' < x_{i+1}) \stackrel{\text{def}}{=}$ *flip*$((x_{i-1} < x_i < x_{i+1}))$, and define $C' = (x_0 < \cdots < x_{i-1} < x_i' < x_{i+1} < \cdots < x_k)$. Proposition 3.4.2 yields that $x_{i-1}^{-1}x_i' <_T x_{i-1}^{-1}x_i$, and so C' occurs earlier in the lexicographic order, contradicting the choice of C. □

Dyer [17, Proposition 4.3] showed that the lexicographically-first path in $B_{\text{rk}([u,v])}(u, v)$ is the unique rising path in that set. On the other hand, the above proposition shows that the lexicographically-first path in nonempty sets $B_k(u, v)$ with $k \equiv \text{rk}([u, v]) \pmod 2$ is rising. We remark that this is the best that can be done to extend Dyer's result for $k \neq \text{rk}([u, v])$, since there can be more than one rising path in $B_k(u, v)$; for instance, consider $I_2(m)$ with $m > 3$.

As an immediate consequence of Proposition 3.4.3, we have

Corollary 3.4.4. *If $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d})$ has a term of degree k, then $[\boldsymbol{c}^k]_{u,v} > 0$.*

We discuss further properties of the lowest-degree coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ in the next two chapters. In particular, the coefficients of degree k where k is the minimum integer with $[\mathbf{c}^k]_{e,w_0} > 0$, where w_0 is the longest-length element of a finite Coxeter group (this first appeared in [9]).

CHAPTER 4

SHORTEST PATH POSET OF BRUHAT INTERVALS

In this chapter we study the poset whose elements are those contained in a shortest u-v paths of $B(u, v)$.

4.1 Definition of the poset

We first need to have a notion of "distance" in $B(u, v)$.

Definition 4.1.1. (i) Let $\Delta = (x_0 = u \leq x_1 \leq \cdots \leq x_k = v) \in B(u, v)$. For each $x_i \in \Delta$, $0 \leq i \leq k$, the index i is the *distance of* x *on* Δ, which we denote by $d_\Delta(u, x_i)$. That is, if $w \in \Delta$, then $d_\Delta(w)$ is the number of vertices of Δ that come before w in Δ.

(ii)The *shortest distance* of $[u, v]$, denoted by $\ell_s(u, v)$, is the length of the shortest path of $B(u, v)$. That is, $\ell_s(u, v) \stackrel{\text{def}}{=} \min\{\ell : B_\ell(u, v) \neq \emptyset\}$. When the interval is clear from the context, we simply write ℓ_s.

Lemma 4.1.2. *Consider two paths* $\Gamma, \Gamma' \in B_{\ell_s}(u, v)$ *and let* $x \in [u, v]$ *be a vertex in both paths. Then* $d_\Gamma(u, x) = d_{\Gamma'}(u, x)$.

Proof. Let $\Gamma = (x_0 = u < x_1 < x_2 < \cdots < x_{\ell_s} = v)$ and $\Gamma' = (x'_0 = u < x'_1 < x'_2 < \cdots < x'_{\ell_s} = v)$. Since x is a vertex of both Γ and Γ', then $x_i = x$ and $x'_j = x$ for some $0 \leq i, j \leq \ell_s$.

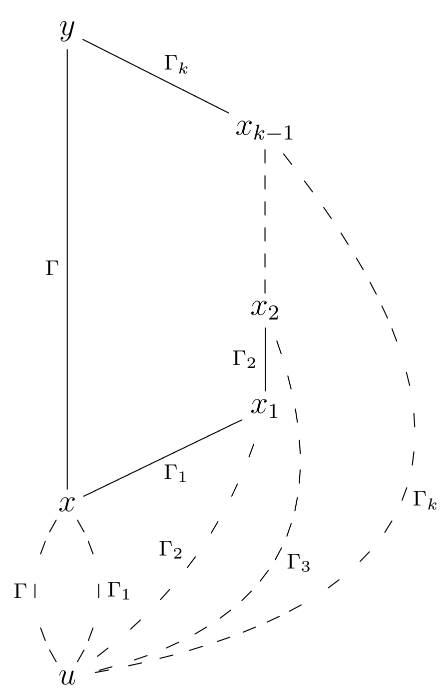
Notice that $d_\Gamma(u, x) = i$ and $d_{\Gamma'}(u, x) = j$. If the two distances are not equal, then one of them is bigger. Suppose without loss of generality that $i < j$. Then $(x_i < x'_{j+1})$ is an edge in the Bruhat graph, and the path $(x_0 = u < \cdots < x_i = x < x'_{j+1} < x'_{j+2} < x'_{\ell_s} = v)$ has length $i + (\ell_s - j) < \ell_s$. This contradicts the definition of ℓ_s. Thus $i = j$.

□

Proposition 4.1.3. *When ignoring directions, $B_{\ell_s}(u, v)$ is the Hasse diagram of a graded poset.*

Proof. The edges in $B_{\ell_s}(u, v)$ give a partial order $\leq_s$ on the elements of $[u, v]$ that are in a u-v path of length ℓ_s. This partial order is defined by $x \leq_s y$ if and only if $x = y$ or if there is a path $(x = y_0 < y_1 < \cdots < y_p = y) \in B(x, y)$ such that each edge $(y_{i-1} < y_i)$ is in a shortest u-v path, for $0 < i < p$. Since $B(u, v)$ is a directed, acyclic graph, $\leq_s$ is indeed a partial order.

Let $(x \leq_s y)$ be an edge in $B_{\ell_s}(u, v)$. Now, to prove the proposition we need to show that $x \lessdot_s y$. It suffices to show that *there is no* path $(x_0 = x < x_1 < x_2 < \cdots < x_k = y)$ with $k > 1$ such that each edge $(x_{i-1} < x_i)$ is in some path $\Gamma_i \in B_{\ell_s}(u, v)$ for $1 \leq i \leq k < \ell_s$. We point to the diagram below to make the situation more clear.



Notice that if such a path existed, then $B_{\ell_s}(u, v)$ (when ignoring directions) would not be a Hasse diagram, as there would be edges that would not represent cover relations. So let us assume for the sake of contradiction that such a path exists. Then

$$d_\Gamma(u,x) = d_{\Gamma_k}(u, x_{k-1}) \tag{4.1}$$

for otherwise one of them, say $d_\Gamma(u,x)$, is bigger than the other one. Thus there exits a u-v path Γ' formed by the edges of Γ_k up to y and then continue on the edges of Γ. Notice that the length of Γ' is $d_{\Gamma_k}(u,y) + (\ell_s - d_\Gamma(u,y)) < \ell_s$. This contradicts the definition of ℓ_s, and thus $d_\Gamma(u,x) = d_{\Gamma_k}(u, x_{k-1})$. Similarly, we obtain

$$\begin{aligned} d_{\Gamma_{k-1}}(u, x_{k-2}) &= d_{\Gamma_k}(u, x_{k-1}) - 1 \\ d_{\Gamma_{k-2}}(u, x_{k-3}) &= d_{\Gamma_{k-1}}(u, x_{k-2}) - 1 \\ \vdots \qquad & \quad \vdots \qquad\quad \vdots \\ d_{\Gamma_1}(u, x) &= d_{\Gamma_2}(u, x_1) - 1. \end{aligned}$$

Hence $d_\Gamma(u,x) = d_{\Gamma_1}(u,x) = d_{\Gamma_k}(u, x_{k-1}) - (k-1)$. However since $k > 1$ this contradicts (4.1). Thus the edges of $B_{\ell_s}(u,v)$ are the cover relations of a poset. Moreover, notice that this poset is graded by $r(x) \stackrel{\text{def}}{=} d_\Gamma(u,x)$ where $\Gamma \in B_{\ell_s}(u,v)$ contains the vertex x. This is a well-defined rank function by Lemma 4.1.2.

Finally, notice that if $(x < y)$ is an edge in $B_{\ell_s}(u,v)$ then there does not exist a x-y path containing an element other than x and y. Thus $x \lessdot y$ by definition. □

We call the poset in Proposition 4.1.3 the *shortest path poset* of u, v, which we denote by $SP(u,v)$. We consider the edges of $SP(u,v)$ to be labeled by the label of the corresponding edges in $B_{\ell_s}(u,v)$.

In the rest of the chapter we study further properties of $SP(u,v)$

4.2 Unique rising shortest path

In this section we will show that if there is a unique rising path in $B_{\ell_s}(u,v)$ then $SP(u,v)$ is a Gorenstein* poset. As a consequence, we derive nonnegativity of certain coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$.

4.2.1 $\widetilde{R}$-polynomials

In the study of Coxeter groups, it is common to encounter the following polynomials, which are defined in the proposition below.

Proposition 4.2.1 ([7], Proposition 5.3.2)**.** *Let $u,v \in W$ with $u \leq v$ and $\ell(vs) < \ell(v)$. Then there exists a monic polynomial $\widetilde{R}(\alpha)$ of degree $\ell(v) - \ell(u)$ given by*

$$\widetilde{R}_{u,v}(\alpha) = \begin{cases} \widetilde{R}_{us,vs}(\alpha) & \text{if } \ell(us) < \ell(u), \text{ and} \\ \widetilde{R}_{us,vs}(\alpha) + \alpha\widetilde{R}_{u,vs}(\alpha) & \text{otherwise} \end{cases}$$

The $\widetilde{R}$-polynomials are used, among other things, to define the R-*polynomials*, and these are used to define the *Kazhdan-Lusztig* polynomials from representation theory (see [7]). That is one of the reasons why the $\widetilde{R}$-polynomials are of interest.

Dyer used reflection orders to provide a nonrecursive definition of the $\widetilde{R}$-polynomials.

Theorem 4.2.2 ([18], Theorem 2.3)**.** *The $\widetilde{R}$-polynomials satisfy the following relation*

$$\widetilde{R}_{u,v}(\alpha) = \sum_{\substack{\Delta \in B(u,v) \\ D(\Delta)=\emptyset}} \alpha^{\ell(\Delta)}.$$

Dyer's theorem states that the $\widetilde{R}$-polynomial of $[u,v]$ is simply the generating function of the rising paths in $B(u,v)$. Using this interpretation, we are able to derive the following inequality.

Theorem 4.2.3. *If $u \le x \le v$, then $\widetilde{R}_{u,x}(\alpha)\widetilde{R}_{x,v}(\alpha) \le \widetilde{R}_{u,v}(\alpha)$ (coefficientwise).*

Proof. The inequality is equivalent to saying that there are more rising paths in $B(u,v)$ than rising paths in $B(u,x)$ times the number of rising paths of $B(x,v)$. So it is enough to find an injection

$$\varphi_x : \mathcal{R}(u,x) \times \mathcal{R}(x,v) \longrightarrow \mathcal{R}(u,v),$$

where $\mathcal{R}(y,z) = \{\Gamma \in B(y,z) : D(\Gamma) = \emptyset\}$.

Consider a reflection order $<_x$ with initial section $N(x)$. Let $(t_1, \cdots, t_p)$ be a $<_x$-rising path of $B(u,x)$ and let $(r_1, \cdots, r_q)$ be a $<_x$-rising path of $B(x,v)$. By choice of $<_x$, we have that $t_p <_x r_1$, as $t_p \in N(x)$ and $r_1 \notin N(x)$, as $N(x)$ is an initial section of $<_x$ (see Lemma 1.9.1). Hence the path $(t_1, \cdots t_p, r_1, \cdots r_q)$ is a $<_x$-rising path of $B(u,v)$. By Theorem 2.0.5, the number of rising chains is independent of the reflection order that has been chosen. Hence the desired bijection is given by concatenating a $<_x$-rising path in $B(u,x)$ and a $<_x$-rising path in $B(x,v)$. $\square$

The proposition above generalizes the following results of Brenti. All the inequalities are coefficientwise.

Corollary 4.2.4. *(1) [11, Corollary 5.5], $\alpha^{\ell(v)-\ell(y)}\widetilde{R}_{u,y}(\alpha) \le \widetilde{R}_{u,v}(\alpha)$, for $u \le y \le v$,*
(2) [11, Theorem 5.4], If W is finite and $u \le x \le y \le v$.

$$\alpha^{\ell(v)-\ell(y)+\ell(x)-\ell(u)}\widetilde{R}_{x,y}(\alpha) \le \widetilde{R}_{u,v}(\alpha).$$

(3) [11, Theorem 5.6]Let $x, y, z \in W$ be such that $y \le z$ in Bruhat order and $x \le_w y$ in weak Bruhat order. Then

$$\alpha^{\ell(y)-\ell(x)}\widetilde{R}_{y,z}(\alpha) \le \widetilde{R}_{x,z}(\alpha).$$

All these inequalities follow by appropriate substitution in Theorem 4.2.3. For instance, in the first inequality, Brenti's result only deals with terms of *some* degrees of $\widetilde{R}_{u,v}(\alpha)$, $\widetilde{R}_{u,y}(\alpha)$ and $\widetilde{R}_{y,v}(\alpha)$ while our result deals with *all* such terms.

Of special interest for our purposes is the following inequality.

Proposition 4.2.5. *Let $u \leq x \leq y \leq v$. Then*

$$|\mathcal{R}(x,y)| \leq |\mathcal{R}(u,v)|.$$

Proof. Since the interval $[u,v]$ is graded, it is enough to show that the result holds when $u \lessdot x \leq y$ or $u \leq y \lessdot v$. Either of these cases follow from Theorem 4.2.3 since then $\widetilde{R}_{u,x}(\alpha) = 1$ or $\widetilde{R}_{y,v}(\alpha) = 1$, respectively. □

Theorem 4.2.6. *Supose that $SP(u,v)$ has a unique maximal, rising chain, i.e., $[\boldsymbol{c}^{\ell_s-1}]\widetilde{\psi}_{u,v}(\boldsymbol{c},\boldsymbol{d}) = 1$. Then $SP(u,v)$ is a Gorenstein* poset.*

Proof. We verify that $SP(u,v)$ is EL-labelable (cf. Definition 1.3.8(i)). Proposition 3.4.3 gives that any subinterval of $SP(u,v)$ has at least one rising chain: the lexicographically-first one. Moreover, Proposition 4.2.5 states that the number of rising chains in any subinterval of $SP(u,v)$ can be at most one. Thus any subinterval of $SP(u,v)$ has a unique rising path that is lexicographically-first, and so $SP(u,v)$ is EL-shellable.

We just showed that $SP(u,v)$ is Cohen-Macaulay and only need to show that $SP(u,v)$ is Eulerian. Notice that any interval of rank 2 of $SP(u,v)$ has two atoms, for otherwise there must be more than one rising chain in some interval (of rank 2). Thus $SP(u,v)$ is thin (cf. Theorem 1.8.3(i)). Therefore the poset $P \stackrel{\text{def}}{=} SP(u,v) \setminus \{u,v\}$ is is a pure and thin poset. Hence by [27, Theorem 3.1.12], $SP(u,v)$ is the face poset of a regular CW- decomposition of an $(\ell_s - 2)$-sphere homeomorphic to $\Delta(P)$, the order complex of P. So $SP(u,v)$ must be Eulerian, and so it is a Gorenstein* poset. □

Corollary 4.2.7. *If* $[\boldsymbol{c}^{\ell_s-1}]\widetilde{\psi}_{u,v}(\boldsymbol{c},\boldsymbol{d}) = 1$*, the terms of* $\widetilde{\psi}_{u,v}(\boldsymbol{c},\boldsymbol{d})$ *of degree* $\ell_s - 1$ *are nonnegative.*

Proof. Since in this case $SP(u,v)$ is Eulerian and EL-shellable, the terms of smallest degree in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ coincide with the **cd**-index of $SP(u,v)$ as an Eulerian poset. Since the **cd**-index of Gorenstein* posets is nonnegative (see [24]), the terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ of degree $\ell_s - 1$ are also nonnegative. □

We point out that the above corollary is in agreement with [9, Theorem 4] applied to $[e, w_0^A]$, where w_0^A is the longest-length element of a Coxeter group of type A. Indeed, since there is a unique rising chain in $SP(e, w_0^A)$, the terms of degree $\ell_s(e, w_0^A)$ in $\widetilde{\psi}_{e,w_0^A}(\mathbf{c},\mathbf{d})$ are nonnegative. The same theorem gives that lowest-degree terms of $\widetilde{\psi}_{e,w_0^A}(\mathbf{c},\mathbf{d})$ form the **cd**-index of the Boolean poset of rank $\ell_T(w_0^A)$. This is discussed in more detail in Chapter 5.

We finish this section with the following two conjectures

Conjecture 4.2.8. *If* $SP(u,v)$ *has a unique rising chain, then* $SP(u,v)$ *is a lattice.*

This conjecture is inspired by an unpublished result of Dyer [19] stating that if *all* paths of the Bruhat graph of $[u,v]$ have length $\ell(v) - \ell(u)$, then $[u,v]$ is a lattice. In fact, $[u,v]$ is isomorphic to the face poset of a polytope.

A stronger conjecture is the following.

Conjecture 4.2.9. *If* $SP(u,v)$ *has a unique rising chain, then* $SP(u,v)$ *is isomorphic to a Bruhat interval.*

The above conjecture would imply Conjecture 4.2.8.

4.3 FLIP algorithm

In this section we aim to partition the paths of $SP(u,v)$ into posets P_i in such a way that the corresponding order complexes $\Delta(P_i \setminus \{u,v\})$ satisfy similar properties as $\Delta([u,v])$, where $[u,v]$ is a Bruhat interval. We have already seen that if $[\mathbf{c}^{\ell_s - 1}]_{u,v} = 1$, $SP(u,v)$ is a Gorenstein* poset, just like $[u,v]$. Thus it is a reasonable idea to separate the chains of $SP(u,v)$ into $[\mathbf{c}^{\ell_s-1}]_{u,v}$ subposets, each one of them containing exactly one rising chain.

In Section 3.4 we followed [3] and defined the flip of a path of length 2 in the Bruhat graph. This operation is key to the algorithm we are trying to construct.

First we define the operation FLIP$_i$. In a few words, given a path $\Delta = (u = x_0 < x_1 < \cdots < x_k = v) \in B_k(u,v)$, we take the path $(x_{i-1} < x_i < x_{i+1}) \in B_2(x_{i-1}, x_{i+1})$ and replace it by its flip to obtain a new path Δ'. The pseudocode is given in Algorithm 4.1

Algorithm 4.1 FLIP$_i(\Delta)$: Flips a path $\Delta \in B_k(u,v)$ at position i.

Given $\Delta = (u = x_0 < x_1 < \cdots < x_k = v)$ in $B_k(u,v)$ and $i \in [k-1]$
$(x_{i-1} < x'_i < x_{i+1}) := flip((x_{i-1} < x_i < x_{i+1}))$
$\Delta' := (x_0 < x_1 < \cdots < x_{i-1} < x'_i < x_{i+1} < \cdots < x_k)$,
return Δ'.

To separate the paths in $B_k(u,v)$ into $[\mathbf{c}^{k-1}]_{u,v}$ components, we define FLIP, whose pseudocode is given in Algorithm 4.2. Essentially for $\Delta \in B_k(u,v)$, FLIP(Δ) flips Δ at its first descent, from bottom to top, if it exists. Thus each path $\Delta \in B_k(u,v)$ with nonempty descent set is assigned the unique path $\Delta' \in B_k(u,v)$ obtained by flipping at its smallest descent. We write FLIP$^j(\Delta)$ to denote the path $\underbrace{\text{FLIP} \circ \cdots \circ \text{FLIP}}_{j}(\Delta)$.

Algorithm 4.2 FLIP(Δ), where $\Delta \in B_k(u,v)$

FLIP$(\Delta) := \Delta$
if $D(\Delta) \neq \emptyset$ **then**
    $i := \min D(\Delta)$
    FLIP$(\Delta) :=$ FLIP$_i(\Delta)$
end if
return FLIP(Δ)

We now define FLIP$(B_k(u,v))$. Basically it computes FLIP(Δ) for all paths in $B_k(u,v)$. The pseudocode is given in Algorithm 4.3. The output of FLIP is a directed graph.

Algorithm 4.3 FLIP$(B_k(u,v))$: Determine the FLIP-graph of $B_k(u,v)$

$G := (B_k(u,v), E)$, and $E := \emptyset$
for $\Delta = (x_0 < x_1 < \cdots < x_k) \in B_k(u,v)$ **do**
  if $D(\Delta) \neq \emptyset$ **then**
    Add (the directed) edge $(\Delta, \text{FLIP}(\Delta))$ to E.
  end if
end for
return G.

Definition 4.3.1. The output of FLIP$(B_k(u,v))$ is called the *flip-graph* of $B_k(u,v)$, and will be denoted by $FG(u,v;k)$.

Notice that the vertices of the flip-graph are labels corresponding to maximal chains of $B_k(u,v)$. FLIP is "reasonable" in the following sense.

Lemma 4.3.2. *(a)* FLIP *terminates.*

(b) $FG(u,v;k)$ has $[\mathbf{c}^{k-1}]_{u,v}$ connected components, each one of which is an arborescence.

Proof. (a) Notice that if $i \in D(\Delta)$, FLIP$_i(\Delta)$ is earlier in the lexicographic order than Δ by Proposition 3.4.2. Thus FLIP must terminate.

(b) If $D(\Delta) = \emptyset$, then FLIP(Δ) does not add a new edge to G. Furthermore, every path (vertex in the flip-graph) is associated to a unique rising path obtained by a iterating FLIP, and so there must be a component for each rising path (and there are $[\mathbf{c}^{k-1}]_{u,v}$ of them). Moreover, a vertex representing a rising path has out-degree zero, and thus each component is an arborescence. □

In particular, we have the following corollary.

Corollary 4.3.3. *(i) There is a minimum nonnegative integer j_0 so that* FLIP$^{j_0}(\Delta) =$ FLIP$^{j_0+1}(\Delta)$.

(ii) FLIP$^{j_0}(\Delta)$ *is rising.*

We now focus our attention to the case $B_{\ell_s}(u,v)$.

4.3.1 FLIP$(SP(u,v))$

Let $G_1,\ldots,G_{r(u,v)}$ be the connected components of $FG(u,v) \overset{\text{def}}{=} FG(u,v;\ell_s)$, where $r(u,v) \overset{\text{def}}{=} [\mathbf{c}^{\ell_s-1}]_{u,v}$. For each G_i one can form a poset P_i, which we call a *flip-poset* of $SP(u,v)$, whose cover relations are given by the edges of any of the maximal chains that are vertices of G_i.

Example 4.3.4. Consider the 10 elements of $B_3(1234,4312)$ (in this interval, $\ell_s(u,v)=3$). The output of FLIP is shown in Figure 4.1 . In the first column we have the two components of G, and in the right column the posets P_i corresponding to each component.

Notice that the each component in Figure 4.1 has an even number of vertices. This is indeed the case in general.

Lemma 4.3.5. *Let G_i be a connected component of $FG(u,v;k)$, then G_i has an even number of vertices.*

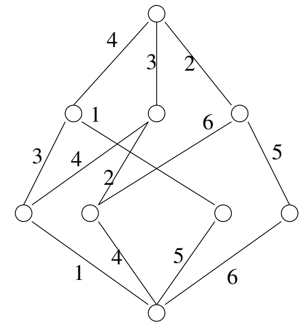
Proof. Let $C=(t_1,t_2,t_3,\ldots,t_{\ell_s})$ be a vertex of G_i (so $(t_1,t_2,t_3,\ldots,t_{\ell_s})$ is a maximal chain of $SP(u,v)$). Notice that $(t_1,t_2,t_3,\ldots,t_{\ell_s})$ is connected to $C'=(t'_1,t'_2,t_3,\ldots,t_{\ell_s})$, where $flip(t_1,t_2)=(t'_1,t'_2)$. Furthermore, since either (t_1,t_2) or (t'_1,t_2) has a descent, C' is also a vertex of G_i. Hence, one can establish a bijection

$$\varphi : \{C \in G_i : D(C)\cap\{1\}=\{1\}\} \longleftrightarrow \{C \in G_i : D(C)\cap\{1\}=\emptyset\}$$

given by $(r_1,r_2,r_3\ldots,r_{\ell_s}) \overset{\varphi}{\leftrightarrow} (r'_1,r'_2,r_3,\ldots,r_{\ell_s})$, with $flip(r_1,r_2)=(r'_1,r'_2)$. □

Each P_i has properties that resemble those of Bruhat intervals; for instance, consider the following lemma.

134 ← 143 ← 423 ← 462 ← 652
↑
514



235 ← 251 ← 521
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625

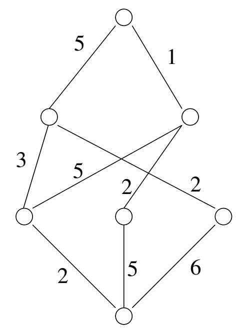

Figure 4.1: On the left, we find the output of FLIP: two connected components of $FG(1234, 4312)$. On the right the corresponding posets are depicted.

Lemma 4.3.6. *Let G_i, $1 \leq i \leq r(u,v)$, be a connected component of $FG(u,v)$ and P_i, $1 \leq i \leq r(u,v)$ the corresponding flip-poset. Then P_i is graded.*

Proof. We have already shown that $SP(u,v)$ is graded (see Proposition 4.1.3). We now prove that FLIP does not change the rank given in Proposition 4.1.3.

Let C be a maximal chain of $SP(u,v)$ and $w \in W$ be an element in C. Moreover, let $r_C(x)$ be the length of the u-x path in C. If $r_C(w) \neq r_{C'}(w)$ then either $r_C(w) < r_{C'}(w)$ or $r_{C'}(w) < r_C(w)$, but in either case there is a contradiction to C and C' being maximal chains of $SP(u,v)$. Indeed, there would exist a maximal chain D formed by concatenating either C_w and $(C')^w$ or C'_w and C^w, where C_w (or C'_w) and C^w (or $(C')^w$) denotes the chain C (or C') up to x and the chain C (or C') after x, respectively. So $r_C(w) = r_{C'}(w)$. Thus the function r defined in the proof of 4.1.3 is a rank function for P_i □

Notation:

(i) We define $[x, y]_s = \{z \in SP(u, v) : x \leq_s z \leq_s y\}$ for $x, y \in SP(u, v)$.

(ii) P_i is called the *flip-poset* corresponding to the connected component G_i of $FG(u, v)$. We also say that P_i is a flip-poset of $SP(u, v)$.

(iii) Let $[x, y]_{P_i} = \{z \in P_i : x \leq_s z \leq_s y\}$, for $x, y \in P_i$.

Definition 4.3.7. Let $u \leq_s w_1, w_2 \leq_s v$, and let $c = (u = z_0 <_s z_1 <_s \cdots <_s z_q = w_1)$ and $d = (w_2 = y_0 <_s y_1 <_s \cdots <_s y_p = v)$ be maximal chains of $SP(u, w_1)$ and $SP(w_2, v)$, respectively. Then

(i) Define $FG_c(u, v)$ to be the induced subgraph of $FG(u, v)$ whose vertex set V_c is the subset of vertices of $FG(u, v)$ that start with c. That is,

$$V_c \stackrel{\text{def}}{=} \{(u = x_0 <_s x_1 <_s \cdots <_s x_k = v) : (x_0 <_s x_1 <_s \cdots <_s x_q) = c\}.$$

(ii) Similarly, define $FG^d(u, v)$ to be the induced subgraph of $FG(u, v)$ whose vertex set V^d is the subset of vertices of $FG(u, v)$ that end with d. That is,

$$V^d = \{(u = x_0 <_s x_1 <_s \cdots <_s x_k = v) : (x_{k-p} <_s x_{k-p+1} <_s \cdots <_s x_k) = d\}.$$

Notice that one has the existence of a map $\varphi : V(FG_c(u, v)) \to V(FG(x, v))$ that assigns any vertex $V \in V(FG_c(u, v))$ to the vertex $v \in V(FG(x, v))$ by ignoring the segment in the label of V that starts with c. In other words if $V \in V_c(FG(u, v))$ then there exists $v \in FG(x, v)$ so that $cv = V$, and so we define φ by $V \stackrel{\varphi}{\mapsto} v$.

Proposition 4.3.8. *Let $x \in [u, v]_s$ and let $c = (u = w_0 <_s \cdots <_s w_q = x)$, $d = (x = y_0 <_s \cdots <_s y_p = v)$ be maximal chains of $SP(u, x)$ and $SP(x, v)$, respectively. Then*

(i) $FG^d(u, v) \cong FG(u, x)$ as graphs.

(ii) The map $\xi : E(FG_c(u, v)) \longrightarrow E(FG(x, v)))$ defined by $(C_1, C_2) \stackrel{\xi}{\mapsto} (\varphi(C_1), \varphi(C_2))$ is injective.

Proof. (i) Let (D_1, D_2) be an edge of $FG^d(u,v)$. We show that there exists an edge (d_1, d_2) of $FG(u,x)$, where $d_1, d_2 \in SP(u,x)$ are so that $d_1 d = D_1$ and $d_2 d = D_2$.

By definition, we can assume that $D_2 = \text{FLIP}(D_1)$. If the first descent from bottom to top of D_1 occurred in $(x_{p'-1} <_s x_{p'} = x = y_0 <_s \cdots <_s y_p = v)$, where $x_{p'-1} \in D_1$. Then $D_2 \notin V^d$. This contradicts the choice of D_2. So the first descent occurred in $(u = x_0 <_s \cdots <_s x_{p'-1})$, and therefore (d_1, d_2) is an edge of $FG(u,x)$.

Furthermore, if (d_1, d_2) is an edge of $FG(u,x)$ then (D_1, D_2) must be an edge of $FG^d(u,v)$, as the first descent from bottom to top of D_1 occurs in d_1.

(ii) Without loss of generality, we may assume that $C_2 = \text{FLIP}(C_1)$. Then the first descent from bottom to top of C_1 occurs after c, as $C_1, C_2 \in V_c$. Thus, $\varphi(C_2) = \text{FLIP}(\varphi(C_1))$, and the result follows. $\square$

Remark 4.3.9. (i) There are cases where $E(F_c(u,v)) \neq \emptyset$; for instance, consider the case where c can be extended to a rising chain. So part (ii) of the proposition is not vacuously true.
(ii) The map ξ need not be an isomorphism of graphs. For instance consider the output presented in Figure 4.1. We take $u = 1234$, $v = 4312$, $x = 1432$, and $c = (1234 <_s 1432)$, which corresponds to the edge labeled "5" in the figure. Then $FG_c(u,v)$ consists of two disconnected vertices, whereas $FG(x,v)$ has one connected component formed by an edge.

As a consequence, we have the following corollary.

Corollary 4.3.10. *Let P_i be a flip-poset of $SP(u,v)$ and let $x \in P_i$ with $u \leq_s x \leq_s v$. Then $[u,x]_{P_i}$ has a unique rising chain.*

Proof. Let $d = (x = x_0 <_s \cdots <_s x_k = v)$. Proposition 4.3.8(i) yields that $FG^d(u,v) \cong FG(u,x)$. Since each connected component of $FG(u,x)$ gives rise to a flip-poset, which has a unique rising chain, $[u, x]_{P_i}$ must have a unique rising chain. $\square$

Notice that lower intervals of flip-posets (those that start with he identity) are themselves flip-posets.

4.4 FLIP applied to $B_3(u, v)$

For the rest of the chapter, we assume that $B_3(u, v) \neq \emptyset$. For this case, we have been able to derive connections to the complete **cd**-index.

Lemma 4.4.1. *Let P_i be a flip-poset of $B_3(u, v)$. Then $\Delta(P_i \setminus \{u, v\})$ is a path or a polygon.*

Proof. The result follows if we show that each vertex (a maximal chain of $SP(u, v)$) in G_i at most two neighbors. Let $C = (u = x_0 <_s x_1 <_s x_2 <_s x_3 = v) \in G_i$. Notice that, by definition, only $C_1 = (u = x_0 <_s y_1 <_s x_2 <_s x_3 = v)$ or $C_2 = (u = x_0 <_s x_1 <_s z_2 <_s x_3 = v)$, where $flip(x_0 <_s x_1 <_s x_2) = (x_0 <_s y_1 <_s x_2)$ and $(x_1 <_s x_2 <_s x_3) = (x_1 <_s z_2 <_s x_3)$, could be neighbors (in G_i) of C. Thus C can have at most two neighbors, as desired. $\square$

In Figure 4.2 we depict the order complexes of the flip-posets of Figure 4.1. In this case, both complexes are paths.

Let Δ be the order complex $\Delta(P_i \setminus \{u, v\})$ for some flip-poset P_i. Let us assume that Δ is a path. One can construct another complex Δ' by identifying the two vertices in Δ of degree 1. Notice that Δ' is a polygon; for example, in the figure below one obtains, after identification, a hexagon and a square.

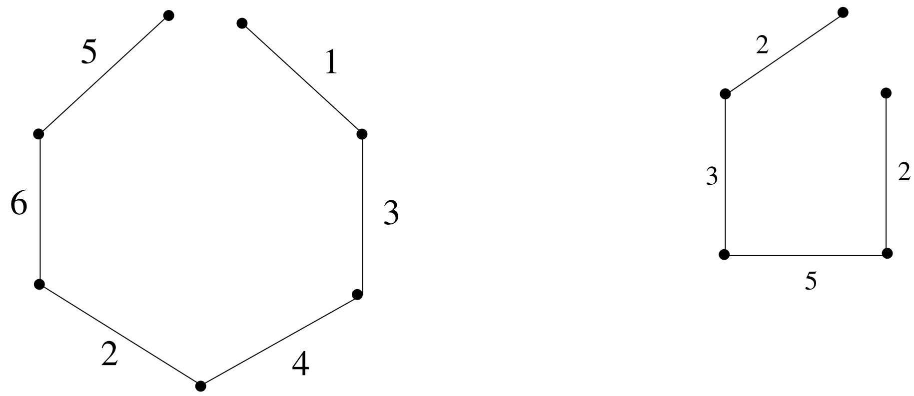

Figure 4.2: The edge-labels are the same as in the Hasse diagram of P_1, P_2 in Figure 4.1, respectively.

4.4.1 Degree-two terms of the complete cd-index

In this section we show how FLIP separates $B_3(u, v)$ into pieces whose **cd**-index add up to the degree two-terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$.

Let $\Delta_1, \ldots, \Delta_k$ be the order complexes of $P_1 \setminus \{u, v\}, \ldots, P_k \setminus \{u, v\}$, where $P_1, \ldots, P_k$ are the flip-posets of $B_3(u, v)$. Furthermore, let $\Delta'_1, \ldots, \Delta'_k$ be the corresponding Δ_i with identification (if Δ_i is already a polygon, then set Δ'_i to be Δ_i). Proposition 4.3.5 and Lemma 4.4.1 give that Δ'_i is an even polygon, say with $2m_i$ sides, for $1 \leq i \leq k$. Therefore each Δ'_i is the first barycentric subdivision of an m_i-gon C_i. Then we have the following proposition.

Proposition 4.4.2. *Let $[\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d})]_2$ denote the degree-two polynomial of $\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d})$. Then*

$$[\widetilde{\psi}_{u,v}(\boldsymbol{c}, \boldsymbol{d})]_2 = \sum_{i=1}^{k} \psi(P(C_i)),$$

where $P(C_i)$ is the face poset of C_i.

Proof. The the sum of the **cd**-index of all these k polygons is

$$\sum_{i=1}^{k} \psi(P(C_i)) = k\mathbf{c}^2 + \left(\sum_{i=1}^{k} m_i - 2k\right) \mathbf{d}$$
$$= k(\mathbf{a}^2 + \mathbf{b}^2) + \left(\sum_{i=1}^{k} m_i - k\right) (\mathbf{ab} + \mathbf{ba}),$$

where the first equality follows from Lemma 1.4.3.

Let us now compute the terms of degree 2 of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. There are exactly k raising chains and k falling chains. Thus,

$$[\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})]_2 = k(\mathbf{a}^2 + \mathbf{b}^2) + n_{ab}\mathbf{ab} + n_{ba}\mathbf{ba}, \tag{4.2}$$

$[\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})]_2$ denotes the terms of degree 2 of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$, n_{ab} and n_{ba} is the number of **ab** and **ba** monomials, respectively.

Notice that $n_{ab} + n_{ba}$ is the number of paths in $B_3(u, v)$ that are neither rising or falling. Thus

$$n_{ab} + n_{ba} = 2\sum_{i=1}^{k} m_i - 2k,$$

as the paths of B_3 are in one-to-one correspondence to the edges in the barycentric subdivisions $\Delta'_1, \ldots, \Delta'_k$. The existence of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ gives that $n_{ab} = n_{ba}$, and so (4.2) becomes

$$[\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})]_2 = k(\mathbf{a}^2 + \mathbf{b}^2) + \left(\sum_{i=1}^{k} m_i - k\right) (\mathbf{ab} + \mathbf{ba}),$$

and the result follows. □

We illustrate Proposition 4.4.2 with the following example.

Example 4.4.3. Let us consider the interval $[1234, 4312]$ in A_3. One can compute $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ to get

$$\widetilde{\psi}_{1234,4312}(\mathbf{c}, \mathbf{d}) = \mathbf{c}^4 + \mathbf{c}^2\mathbf{d} + \mathbf{d}\mathbf{c}^2 + \mathbf{d}^2 + 2\mathbf{c}^2 + \mathbf{d}.$$

Example 4.3.4 gives that Δ'_1 and Δ'_2 contribute $\mathbf{c}^2 + \mathbf{d}$ and $\mathbf{c}^2$, respectively. The sum gives the degree-two terms of $\widetilde{\psi}_{1234,4312}(\mathbf{c}, \mathbf{d})$.

CHAPTER 5

SHORTEST PATH POSET OF FINITE COXETER GROUPS

It is useful to have a combinatorial descriptions of the reflections for types A, B and D. We also include a description of a reflection order for those groups. The combinatorial descriptions of the reflections can be found in [7], and we include it in the next section. The key result used in our description is Theorem 1.9.1 that states that there exists a reflection order induced by a reduced expression of the longest word.

5.1 The poset $SP(e, w_0^W)$.

The rest of this chapter is essentially [9].

Here we are interested in $SP(W)$, where W is a finite Coxeter group. To illustrate the definition consider B_2 and $SP(B_2)$ as depicted below. The rank of $SP(B_2)$ is two since that is the length of the shortest paths in $B(B_2)$.

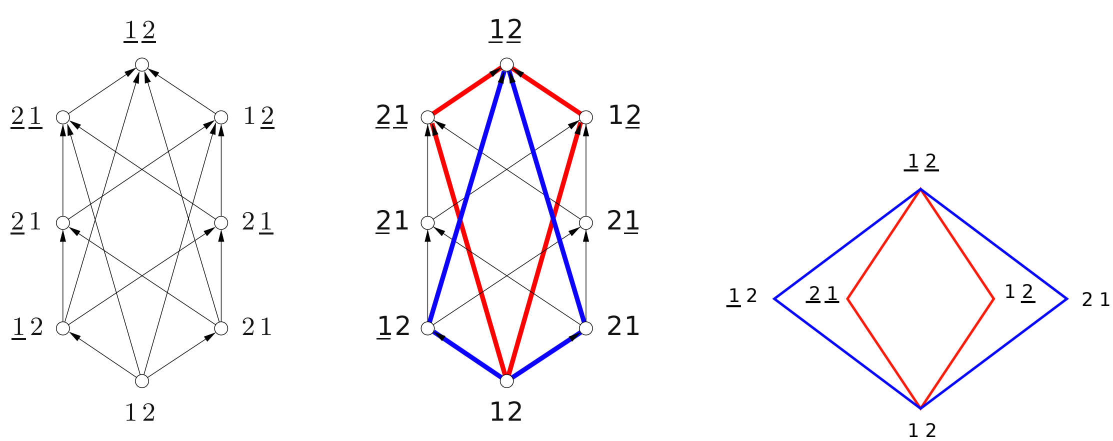

Figure 5.1: $B(B_2)$ and $SP(B_2)$.

For any finite Coxeter group, there is a word w_0^W of maximal length. It is a well known fact that $\ell(w_0^W) = |T(W)|$. For any $w \in W$, one can write $t_1 t_2 \cdots t_n =$

w for some $t_1, t_2, \ldots, t_n \in T(W)$. If n is minimal, then we say that w is $T(W)$-*reduced*, and that the *absolute length* of w is n. We write $\ell_{T(W)}(w) = n$, or simply $\ell_T(w) = n$.

Notice that for $w \in W$, if $\ell_T(w) = \ell$, then $t_1 t_2 \cdots t_\ell = w$ for some reflections in $T(W)$, but this *does not* mean that $(t_1, t_2, \ldots, t_\ell)$ is a (directed) path in $B([e, w])$. Nevertheless, we will show that for finite W and $w = w_0^W$, $(t_1, t_2, \ldots, t_\ell)$ and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(\ell)})$, $\tau \in A_{\ell-1}$, is a path in $B(W)$. To be more specific, we show the following theorem.

Theorem 5.1.1. *Let W be a finite Coxeter group and $\ell_0 = \ell_{T(W)}(w_0^W)$, the absolute length of the longest element of W. Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0.*

We point out that the union of the Boolean posets could share more elements than e and w_0^W. For instance, consider $SP(B_3)$ below.

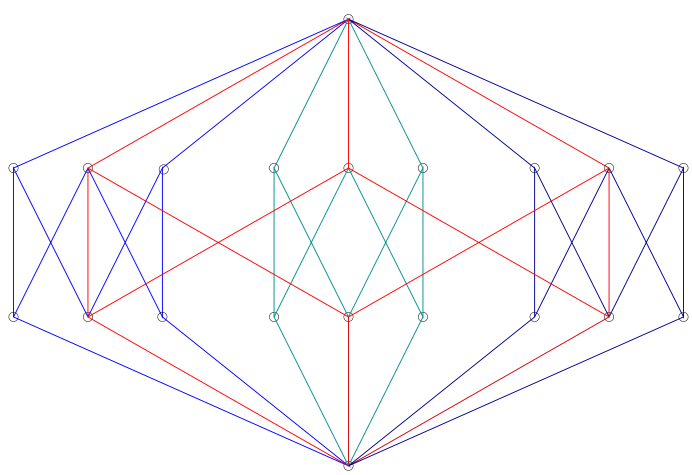

Figure 5.2: $SP(B_3)$ has 4 copies of $\mathcal{B}_3$. Notice these copies intersect, but each maximal chain is in a unique Boolean poset.

While some elements other than e and $w_0^{B_3}$ are shared by more than one Boolean poset, each maximal chain belongs to a *unique* Boolean poset.

In Section 5.2 we present the proof of the theorem for the infinite families (groups of type A, B, and D and dihedral groups). In Section 5.3 we discuss the validity of the Theorem for the exceptional groups. Computer search was used for F_4, H_3, H_4, and E_6, and a geometric argument was used to prove the case E_7 and E_8. We summarize the number of Boolean posets that form $SP(W)$ and the rank of $SP(W)$ for each finite Coxeter group in Table 5.1.

In Section 5.4 we discuss why Theorem 5.1.1 implies that the lowest-degree terms of the complete **cd**-index of W is given by $\alpha_W \widetilde{\psi}(\mathcal{B}_{\ell_0})$, where $\widetilde{\psi}(\mathcal{B}_{\ell_0})$ is the **cd**-index of the Boolean poset of rank $\ell_0 = \ell_{T(W)}(w_0^W)$, and α_W is the number of Boolean posets that form $SP(W)$.

The following lemma will be used in our proofs.

Lemma 5.1.2 (Shifting Lemma, [1], Lemma 2.5.1). *If $w = t_1 t_2 \cdots t_r$ is a $T(W)$-reduced expression for $w \in W$ and $1 \leq i < r$, then $w = t_1 t_2 \cdots t_{i-1}(t_i t_{i+1} t_i) t_i t_{i+2} \cdots t_r$ and $w = t_1 t_2 \cdots t_{i-2} t_i (t_i t_{i-1} t_i) t_{i+1} \cdots t_r$ are $T(W)$-reduced.*

As a consequence, there exists a $T(W)$-reduced expression for w having t_i as last reflection (or first), for $1 \leq i \leq r$. Furthermore, for any two reflections $t_i, t_j, i < j$ there exists a $T(W)$-reduced expression for w with t_i, t_j as the last two reflections (or the first two).

5.2 Groups of type A, B and D

5.2.1 The poset $SP(A_{n-1})$

Define $\mathrm{inv}(w) \stackrel{\text{def}}{=} |\{(i,j) \in [n] \times [n] \mid i < j, w(i) > w(j)\}|$. Furthermore, if $i < j, w(i) > w(j)$ then we say that (i,j) is an *inversion pair* of w.

The importance of the inversions pair of a permutation becomes clear by the following proposition.

Proposition 5.2.1 (Proposition 1.5.2, [7]). *If $w \in A_{n-1}$ then $\ell(w) = \mathrm{inv}(w)$.*

The reflections A_{n-1} are all the elements of the set

$$T(A_{n-1}) = \{(i \ \ j) : 1 \leq i < j \leq n\},$$

that is, the set of *transpositions* of S_n. In particular, $|T(A_{n-1})| = \binom{n}{2}$.

Lemma 5.2.2. *(i) In one-line notation, $w_0^{A_{n-1}} = n \ \ n-1 \ \cdots \ 2 \ \ 1$*

(ii) Let $w_i = s_i s_{i+1} \cdots s_1$ for $i \geq 1$. Then $w_0^{A_n} = w_1 \cdots w_{n-1}$.

Proof. (i) This follows from observing that any pair (i, j) with $1 \leq i < j \leq n$ is an inversion pair of $n \ \ n-1 \ \cdots \ 2 \ \ 1$, and so

$$\ell(n \ \ n-1 \ \cdots \ 2 \ \ 1) = \binom{n}{2} = |T(A_n)| = \ell(w_0^{A_n}).$$

The last equality is [7, Proposition 2.3.2(iv)]. Thus by uniqueness of the longest-length element, one obtains that $w = w_0^{A_n}$.

(ii) In one-line notation, one has that $w_{n-1} = n \ 1 \ 2 \ \cdots \ n-1$ and for $i \leq n-2$ and $j \in [n]$,

$$w_i w_{i+1} \cdots w_{n-1}(j) = \begin{cases} n - j + 1 & \text{if } j \leq n - i, \text{and} \\ w_{i+1} w_{i+2} \cdots w_{n-1}(j) - 1 & \text{if } j \geq n - i + 1. \end{cases}$$

Hence, in one-line notation, we have that $w = n \ \ n-1 \ \cdots \ 2 \ \ 1 = w_0^{A_{n-1}}$. □

Corollary 5.2.3. *A reflection order on $T(A_{n-1})$ is given by*

$$(1 \ \ 2) < (1 \ \ 3) < \cdots < (n-1 \ \ n), \tag{5.1}$$

that is, the lexicographic order on the transpositions of S_n.

Proof. By uniqueness of $w_0^{A_{n-1}}$, one has that $w_0^{A_{n-1}} = s_1 \cdots s_{n-1} \cdots s_1 s_2 s_3 s_1 s_2 s_1$ (this is the reverse of the expression we gave in Lemma 5.2.2). Now, Lemma 1.9.1, yields that

$$s_1 < s_1 s_2 s_1 < s_1 s_2 s_3 s_2 s_1 < \cdots .$$

Direct computation gives (5.1). □

Lemma 5.2.4. $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) = \lfloor \frac{n}{2} \rfloor$.

Proof. We just showed that $w_0^{A_{n-1}}$ is the reverse of the identity $123\dots n$, i.e., $w_0^{A_{n-1}} = n\ (n-1)\ (n-2)\dots 21$. So $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) \geq \lfloor \frac{n}{2} \rfloor$ since a reflection in A_{n-1} is just a transposition, and thus cannot permute more than two elements of $[n]$ at a time.

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor = k$, let r_i be the transposition permuting i and $n+1-i$; that is, $r_i = (i \quad n+1-i)$. Notice that $r_1 \cdots r_k = w_0^{A_{n-1}}$, and so $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) \leq \lfloor \frac{n}{2} \rfloor$. □

Lemma 5.2.5. *For $\sigma \in A_{n-1}$, let $k = \lfloor \frac{n}{2} \rfloor$,*

$$f^A(\sigma) = \left\lfloor \frac{|\{i \in [n] \mid \sigma(i) = w_0^{A_{n-1}}(i)\}|}{2} \right\rfloor$$

and

$$g^A(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \dots, t_\ell \in T(A_{n-1}) \text{ with } t_1 t_2 \dots t_\ell \sigma = w_0^{A_{n-1}}\}.$$

Then $f^A(\sigma) = i \implies g^A(\sigma) \geq \lfloor \frac{n}{2} \rfloor - i$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Proof. We proceed by reverse (downward) induction. The case $i = k$ holds, since g^A is a nonnegative function. Suppose that the statement holds for i. We now show that it also holds for $i-1$. Let $\sigma \in A_{n-1}$ with $f^A(\sigma) = i-1$. Consider $t_1, t_2, \dots, t_\ell \in T(A_{n-1})$ with $t_1 t_2 \cdots t_\ell \sigma = w_0^{A_{n-1}}$ and $\ell = g^A(\sigma)$. Notice that there exists an positive integer m with $f^A(t_{\ell-m+1} t_{\ell-m+2} \cdots t_\ell \sigma) = i$, since $f^A(t_1 t_2 \cdots t_\ell \sigma) = k$ and a reflection can fix at most two elements in their position in $w_0^{A_{n-1}}$, and so $f^A(t\tau) \leq f^A(\tau) + 1$ for $t \in T(A_{n-1})$ and $\tau \in A_{n-1}$. The last equality yields $g^A(\sigma) = \ell \geq k + m - i \geq k + 1 - i$. □

We can now show the proposition below, which gives Theorem 5.1.1 for type A.

Proposition 5.2.6. *Let $k = \lfloor \frac{n}{2} \rfloor$, and $R = \{r_1, r_2, \ldots, r_k\}$, where $r_i = (i \quad n+1-i)$ is the transposition permuting $i \in [k]$ and $n+1-i$. If $t_1 t_2 \cdots t_k = w_0^{A_{n-1}}$, then:*

(a) $\{t_1, \ldots, t_k\} = R$.

(b) $t_i t_j = t_j t_i$ for all $i, j \in [k]$.

(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(k)})$ is a path in $B(A_{n-1})$ for all $\tau \in A_{k-1}$.

Proof. (a) Suppose that there exists $t_i \in T \setminus R$. Without loss of generality, using the Shifting Lemma, we can assume that $i = k$. Say $t_k = (m \ \ j)$ where $m < j \leq n$ and $j \neq n+1-k$. Hence $f^A(t_k) = 0$, and thus by Lemma 5.2.5 we have that $g^B(t_k) \geq k$. But this contradicts $t_1 t_2 \cdots t_{k-1} t_k = w_0^{A_{n-1}}$, which gives that $g^A(t_k) \leq k-1$.

(b) Notice that r_i and r_j are disjoint transpositions for $i \neq j$, and thus commute.

(c) By (b) it is enough to show that $\ell(t_1 t_2 \cdots t_m) > \ell(t_1 t_2 \cdots t_{m-1})$ for $1 < m \leq n$. For this, we recall [7, Proposition 1.5.2]: If $w \in A_{n-1}$ then $\ell(w) = \mathrm{inv}(w)$.

Suppose that $w_m = t_1 t_2 \cdots t_m$; we now compare $\mathrm{inv}(w_m)$ and $\mathrm{inv}(w_{m-1})$. By (a) we have that the t_i's are in R, so $t_m = (i \quad n+1-i)$ for some $i \in [k]$. Moreover, $w_{m-1}(i) = i$, $w_{m-1}(n+1-i) = n+1-i$ and $w_m(l) = w_{m-1}(l)$ for all $l \in [n] \setminus \{i, n+1-i\}$. Now consider that:

1. If (l, i) is an inversion pair of w_{m-1} then $l < i$ and $w_{m-1}(l) > i$. If $w_{m-1}(l) > n+1-i$ then (l, i) and $(l, n+1-i)$ are inversion pairs of both w_{m-1} and w_m. If $w_{m-1}(l) \leq n+1-i$, then $(l, n+1-i)$ is not an inversion pair of w_{m-1}, but since $w_m(n+1-i) = i$, it is an inversion pair of w_m.

2. If $(l, n+1-i)$ in an inversion pair of w_{m-1} then $l < n+1-i$ and $w_m(l) = w_{m-1}(l) > n+1-i > i = w_m(n+1-i)$. Hence $(l, n+1-i)$ is also an inversion pair of w_m

3. If (i, l) an inversion pair of w_{m-1} then $i < l$ and $i > w_{m-1}(l)$. Since $w_m(i) = n + 1 - i > i > w_{m-1}(l) = w_m(l)$, (i, l) is also an inversion pair of w_m.

4. If $(n+1-i, l)$ is an inversion pair of w_{m-1} then $n+1-i < l$ and $n+1-i > w_{m-1}(l)$. If $i > w_{m-1}(l)$ then (i, l) and $(n + 1 - i, l)$ are inversion pairs of both w_{m-1} and w_m. If $i \leq w_{m-1}(l)$, then (i, l) is not an inversion pair of w_{m-1}, but since $w_m(i) = n + 1 - i$, it is an inversion pair of w_m.

Thus $\mathrm{inv}(w_m) \geq \mathrm{inv}(w_{m-1})$. To show that $\mathrm{inv}(w_m) \geq \mathrm{inv}(w_{m-1}) + 1$, consider the pair $(i \quad n + 1 - i)$ which *is not* an inversion pair of w_{m-1}. But since $w_m(i) = n + 1 - i > i = w_m(n + 1 - i)$, this is an inversion pair of w_m. □

We remark that the above proposition shows that $SP(A_{n-1})$ is isomorphic to the Boolean poset of rank k. Moreover, $SP(A_{n-1})$ is the poset of subsets of R ordered by inclusion.

5.2.2 The poset $SP(B_n)$

We used the combinatorial description of B_n and $T(B_n)$ in [7], Section 8.1.

B_n is the group of *signed permutations*, i.e, it is the group of permutations σ of the set $[\pm n] = \{-n, -n + 1, \ldots, -1, 1, 2, \ldots, n - 1, n\}$ with the property $\sigma(-i) = -\sigma(i)$ for all $i \in [\pm n]$. We use the notation $\underline{i}$ to denote $-i$ for $i \in [\pm n]$. The reflections of B_n are given by the set

$$T(B_n) = \{(i \ \ \underline{i}) : i \in [n]\} \cup \{(i \ \ j)(\underline{i} \ \ \underline{j}) : 1 \leq i < |j| \leq n\}.$$

We call the set $\{(i \ \ \underline{i}) : i \in [n]\}$ *reflections of type I* and the set $\{(i \ \ j)(\underline{i} \ \ \underline{j}) : 1 \leq i < |j| \leq n\}$ *reflections of type II*. In particular, notice that $|T(B_n)| = n^2$.

There is also a combinatorial description of the length function for type B. Such an interpretaion is given in [7, Proposition 8.1.1] and is as follows: if $w \in$

B_n then

$$\ell(w) = \text{inv}(w) + \text{Neg}(w) \tag{5.2}$$

where

$$\text{inv}(w) = \text{inv}(w(1), w(2), \ldots, w(n)) \quad \text{and} \quad \text{Neg}(w) = -\sum_{j \in [n] : w(j) < 0} w(j).$$

This interpretation of length is handy to prove the following lemma.

Lemma 5.2.7. *(i) In one-line notation, $w_0^{B_n} = \underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}$.*

(ii) If we let $s_0 = (1\ \ \underline{1})$ and $s_i = (i\ \ i+1)(\underline{i+1}\ \ \underline{i})$. Then $w_0^{B_n} = (w_n)^n$, where $w_n = s_0 s_1 \cdots s_{n-1}$ and the exponent indicates that w_n is concatenated n times.

Proof. (i) This follows from noticing that $\underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}$ maximizes both $\text{inv}(w)$ and $\text{Neg}(w)$ for $w \in B_n$. Indeed, since every pair (i,j) with $1 \le i < j \le n$ is an inversion pair and $\text{Neg}(\underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}) = \sum_{i=1}^{n} i = \binom{n+1}{2}$, we have that

$$\begin{aligned} \ell(\underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}) &= \binom{n}{2} + \binom{n+1}{2} \\ &= n^2 \\ &= |T(B_n)|. \end{aligned}$$

Therefore, by [7, Proposition 2.3.2(iv)] we have $\underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n} = w_0^{B_n}$.

(ii) In one-line notation, $w_n = 2\ 3\ \cdots\ n\ \underline{1}$, and for $i \ge 2$, $j \in [n]$, we have

$$(w_n)^i(j) = \begin{cases} (w_n)^{i-1}(j) + 1 & \text{if } 1 \le j \le n - i, \\ \underline{1} & \text{if } j = n - i + 1, \text{ and} \\ (w_n)^{i-1}(j) - 1 & \text{if } n - i + 2 \le j \le n. \end{cases}$$

Thus $(w_n)^n = \underline{1}\ \underline{2} \cdots \underline{n} = w_0^{B_n}$. □

One can find a relatively nice description of a reflection order on $T(B_n)$, in the same spirit as the one presented for type A. Such a description is given by the following lemma.

Lemma 5.2.8. *Let $a \in [n]$ and let T_a be the linear order given by*

$$(a\ \ \underline{a}) < (a\ \ \underline{a+1})(\underline{a}\ \ a+1) < \cdots < (a\ \ \underline{n})(\underline{a}\ \ n) <$$
$$< (a\ \ 1)(\underline{1}\ \ \underline{a}) < (a\ \ 2) < (\underline{2}\ \ \underline{a}) < \cdots < (a\ \ a-1)(\underline{a-1}\ \ \underline{a}).$$

In particular, T_1 is

$$(1\ \underline{1}) < (1\ \underline{2})\ (2\ \underline{1}) < (1\ \underline{3})(3\ \underline{1}) < \cdots < (1\ \underline{n})(\underline{n}\ 1),$$

and T_2 is

$$(2\ \underline{2}) < (2\ \underline{3})\ (3\ \underline{2}) < (2\ \underline{4})(4\ \underline{2}) < \cdots < (2\ \underline{n})(\underline{n}\ 2) < (2\ 1)(\underline{1}\ \underline{2}).$$

Then

$$T_1 < T_2 < \cdots < T_n$$

is a reflection order on $T(B_n)$.

Proof. Consider the reverse of the reduced expression we found for $w_0^{B_n}$ in the previous lemma. This is also a reduced expression for $w_0^{B_n}$ by uniqueness of the longest-length element. The order above is induced by said expression. □

Proposition 5.2.9. $\ell_{T(B_n)}(w_0^{B_n}) = n$.

Proof. Notice that a reflection of type II changes the sign of either zero or two elements in $[n]$ and swaps them. So at least another reflection must be used to place them back in their respective order. That is, at least two reflections of type II are needed to place two elements in $[n]$ in their positions in $w_0^{B_n}$. Hence at least $2m$ reflections of type II are needed to place $2m$ elements of $[n]$ in their position in $w_0^{B_n}$, with $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$, and after that $n - 2m$ reflections of type I are needed to place the remaining $n - 2m$ elements in their position in $w_0^{B_n}$. So $\ell_{T(B_n)} \geq n$.

Now, notice that $(1\ \ \underline{1})(2\ \ \underline{2}) \cdots (n\ \ \underline{n}) = w_0^{B_n}$, and so $\ell_{T(B_n)}(w_0^{B_n}) \leq n$. □

Lemma 5.2.10. *For $\sigma \in B_n$, let*

$$f^B(\sigma) = |\{i \in [n] \mid \sigma(i) = w_0^{B_n}(i) = \underline{i}\}| + \\ + |\{(i,j) \in [n] \times [n], i < j \mid (\sigma(i), \sigma(j)) \in \{(j,i), (\underline{j}, \underline{i})\}|$$

and

$$g^B(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \ldots, t_\ell \text{ with } t_1 t_2 \ldots t_\ell \sigma = w_0^{B_n}\}.$$

Then $f^B(\sigma) = i \implies g^B(\sigma) \geq n - i$ for $0 \leq i \leq n$.

Proof. We proceed by reverse induction. The case $i = n$ holds, since g^B is a nonnegative function. Suppose that the statement holds for i. We now show that it also holds for $i-1$. Let $\sigma \in B_n$ with $f^B(\sigma) = i-1$. Consider $t_1, t_2, \ldots, t_\ell \in T(B_n)$ with $t_1 t_2 \cdots t_\ell \sigma = w_0^{B_n}$ and $\ell = g^B(\sigma)$. Notice that there exists m with $f^B(t_{\ell-m+1} \cdots t_{\ell-1} t_\ell \sigma) = i$, since $f^B(t_1 t_2 \cdots t_\ell \sigma) = n$ and a reflection can fix at most one element in its position in $w_0^{B_n}$ or create a pair (i,j) that is sent to $(\underline{j}, \underline{i})$ or (j,i). The last equality yields $g^B(\sigma) = \ell \geq k + m - i \geq k + 1 - i$. □

Let t_1, t_2 be two reflections of type II satisfying $\{t_1, t_2\} = \{(k\ \ \underline{l})(\underline{k}\ \ l), (k\ \ l)(\underline{k}\ \ \underline{l})\}$ for some k, l with $1 \leq k < l \leq n$. Then we see that $t_1 t_2(k) = t_2 t_1(k) = \underline{k}$ and $t_1 t_2(l) = t_2 t_1(l) = \underline{l}$. We call the pair t_1, t_2 a *good pair*. Good pairs play a special role in the shortest paths in $B(B_n)$, as seen in the theorem below.

Proposition 5.2.11. *If $t_1 t_2 \ldots t_n = w_0^{B_n}$, then:*

(a) For every $i \in [n]$ either t_i is of type I or there exists $j \in [n]$, $i \neq j$ so that t_i, t_j is a good pair.

(b) $t_i t_j = t_j t_i$ for all $i, j \in [n]$.

(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(n)})$ is a path in $B(B_n)$ for all $\tau \in A_{n-1}$.

Proof. (a) Suppose that some reflection in $\{t_1, \ldots, t_n\}$ is of type II, say $t_i = (k\ \ l)(\underline{k}\ \ \underline{l})$, and suppose that there is no t_j so that t_i, t_j is a good pair. Since

$w_0^{B_n}(k) = \underline{k}$ and $w_0^{B_n}(l) = \underline{l}$, there must be another reflection t_m that is not disjoint from t_i. Without loss of generality, we can assume that $\{t_i, t_m\} = \{t_{n-1}, t_n\}$. Since t_{n-1}, t_n is not a good pair, then $f^B(t_{n-1}t_n) = 0$. Hence $g^B(t_{n-1}t_n) \geq n$, which contradicts $t_1t_2 \cdots t_n = w_0^{B_n}$.

(b) Notice that since all the reflections in $t_1 \cdots t_n = w_0^{B_n}$ of type I are distinct, they commute with each other. Furthermore, if t_i, t_j are a good pair, then they also commute. We need to verify that (i) if t_i, t_j are of type II and *not* a good pair, then they are commuting reflections, and (ii) if t_i, t_j are of mixed types, then they commute. Using the Shifting Lemma again, we can assume that the reflections in both cases are t_{n-1} and t_n. Suppose that t_{n-1} and t_n do not commute. In both (i) and (ii) we see that $f^B(t_{n-1}t_n) = 0$, and so $g^B(t_{n-1}t_n) \geq n$ by Lemma 5.2.10, which contradicts $t_1t_2 \cdots t_{n-1}t_n = w_0^{B_n}$.

(c) For $i \in [n]$, let $w_i = t_1t_2 \cdots t_i$. Notice that from (b) it is enough to prove that $\ell(w_m) > \ell(w_{m-1})$ for $1 < m \leq n$. We have the following cases:

1. t_m is of type I, say $t_m = (j \;\; \underline{j})$, with $j \in [n]$. (a) and (b) give that no other reflection involves the element j, and so $w_{m-1}(j) = j$. Furthermore, we have that $w_m(k) = w_{m-1}(k)$ for $k \in [n] \setminus \{j\}$. Now,

    - If (i, j) is an inversion pair of w_{m-1}, then $i < j$ and $w_{m-1}(i) > w_{m-1}(j) = j$, which gives that $w_{m-1}(i) > 0$. So $w_m(i) = w_{m-1}(i) > \underline{j} = w_m(j)$, and the pair (i, j) is also an inversion pair of w_m. Since $\operatorname{Neg}(w_m) = \operatorname{Neg}(w_{m-1}) + j$, we have that $\ell(w_{m-1}) < \ell(w_m)$.

    - If (j, i) is an inversion pair of w_{m-1}, then $j < i$ and $w_{m-1}(j) = j > w_{m-1}(i)$. Suppose that (j, i) is not an inversion pair of w_m. There are at most $j - 1$ such inversion pairs (j, i) of w_{m-1}, since $1 < w_{m-1}(i) < j$.

On the other hand, notice that $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + j$. So

$$\begin{aligned}\ell(w_m) - \ell(w_{m-1}) &= \mathrm{inv}(w_m) + \mathrm{Neg}(w_m) - (\mathrm{inv}(w_{m-1}) + \mathrm{Neg}(w_{m-1}))\\ &\geq \mathrm{inv}(w_{m-1}) - (j-1) + (\mathrm{Neg}(w_{m-1}) + j) - (\mathrm{inv}(w_{m-1}) +\\ &\qquad + \mathrm{Neg}(w_{m-1}))\\ &\geq 1.\end{aligned}$$

2. t_m is of type II but does not change any element's signs, say $t_m = (i\ \ j)(\underline{i}\ \ \underline{j})$ with $1 \leq i < j \leq n$. Then by the same argument as in the proof of Proposition 5.2.6 (c), we have that $\ell(w_m) > \ell(w_{m-1})$.

3. If $t_m = (i\ \ \underline{j})(\underline{i}\ \ j)$, with $1 \leq i < j \leq n$; that is, t_m swaps i and j and changes their sign. (a) and (b) give that $(w_{m-1}(i), w_{m-1}(j)) \in \{(i,j),(j,i)\}$, $(w_m(i), w_m(j)) \in \{(\underline{j},\underline{i}),(\underline{i},\underline{j})\}$, and $w_{m-1}(k) = w_m(k)$ for $k \in [\pm n] \setminus \{\pm i, \pm j\}$. Then

    - If (k,i) is an inversion pair of w_{m-1} then $k < i$ and either $w_{m-1}(k) > i$ or $w_{m-1}(k) > j$. In either case (k,i) is also an inversion pair of w_m since $w_m(k) = w_{m-1}(k) > 0$ and $w_m(i) < 0$. Further, $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, and so $\ell(w_{m-1}) < \ell(w_m)$.

    - If (i,k) is an inversion pair of w_{m-1} then $i < k$ and either $i > w_{m-1}(k)$ or $j > w_{m-1}(k)$. If we assume that (i,k) *is not* an inversion pair of w_m, then in the former case, there are at most $i-1$ pairs lost, and in the latter there are at most $j-1$. However, since $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, we still have that $\ell(w_{m-1}) < \ell(w_m)$.

    - If (j,k) is an inversion pair of w_{m-1} then $j < k$ and either $j > w_{m-1}(k)$ or $i > w_{m-1}(k)$. If we assume that (j,k) *is not* an inversion pair of w_m, then in the former case, there are at most $j-1$ pairs lost, and in the latter there are at most $i-1$. However, since $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, we still have that $\ell(w_{m-1}) < \ell(w_m)$.

- If (k, j) is an inversion pair of w_{m-1} then $k < j$ and either $w_{m-1}(k) > j$ or $w_{m-1}(k) > i$. In either case (k, j) is also an inversion pair of w_m since $w_m(k) = w_{m-1}(k) > 0$ and $w_m(j) < 0$. Further, $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, and so $\ell(w_{m-1}) < \ell(w_m)$.

In all cases, we have the desired result. □

The previous proposition says that $SP(B_n)$ is (isomorphic to) the union of Boolean posets of rank n, one for each set $\{t_1, t_2, \ldots, t_n\}$ with $t_1 t_2 \cdots t_n = w_0^{B_n}$. As an example, Figure 5.1 illustrates that $SP(B_2)$ is the union of two Boolean posets. In general, one can compute the number of Boolean posets in $SP(B_n)$.

Number of Boolean posets in $SP(B_n)$

Let b_n be the number of Boolean posets in B_n. We obtain a Boolean poset for each set $\{t_1, \ldots, t_n\}$ with $t_1 t_2 \cdots t_n = w_0^{B_n}$. It is easy to see that $b_1 = 1$ and $b_2 = 2$ (see Figure 5.1). For $n \geq 2$, notice that if $t_1 t_2 \cdots t_n(1) = \underline{1}$, then by Proposition 5.2.11 there are two possible cases: (i) there exists $t_j = (1\ \ \underline{1})$ or there exists a good pair of reflections of the form $(1\ \ \underline{k})(\underline{k}\ \ 1), (1\ \ k)(\underline{k}\ \ \underline{1})$. There are b_{n-1} such reflections in case (i) and $(n-1)b_{n-2}$ in case (ii). So b_n satisfies the recurrence relation

$$b_n = b_{n-1} + (n-1)b_{n-2}$$

with initial conditions $b_1 = 1$ and $b_2 = 2$. Notice that this count is the same as the number of partitions of a set of n distinguishable elements into sets of size 1 and 2.

It is easy to see that

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}.$$

5.2.3 The poset $SP(D_n)$

As in the previous section, we use the combinatorial description of D_n and $T(D_n)$ in [7] Section 8.2.

D_n, with $n > 1$, is the group of *signed permutations* with an *even* number of negative elements (e,g, $\underline{1}\ \underline{2}\ 3$ is an element in D_3 whereas $\underline{1}\ \underline{2}\ \underline{3}$ is not). Like B_n, if $\sigma \in D_n$ then $\sigma(-i) = -\sigma(i)$ for $i \in [\pm n]$.

The reflections of D_n are given by

$$T(D_n) = \{(i\ \ j)(\underline{i}\ \ \underline{j}) : 1 \leq i < |j| \leq n\},$$

that is, the reflections of D_n are the reflections of B_n of type II. In particular, notice that $|T(D_n)| = n^2 - n$.

Lemma 5.2.12. *(i)* $w_0^{D_n}$ *is given by*

$$w_0^{D_n} = \begin{cases} \underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n} & \text{if } n \text{ is even, and} \\ 1\ \underline{2}\ \underline{3}\ \cdots\ \underline{n} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) Let $s_0 = (1\ \ \underline{2})(2\ \ \underline{1})$ *and* $s_i = (i\ \ i+1)(\underline{i+1}\ \ \underline{i})$ *for* $1 \leq i < j \leq n$. *Then* $w_0^{D_n} = (w_n)^{n-1}$, *where* $w_n = s_0 s_1 \cdots s_{n-1}$.

Proof. (i) By [7, Proposition 8.2.1], $w \in D_n$ is given by

$$\ell(w) = \mathrm{inv}(w) + \mathrm{Neg}(w) - \mathbf{neg}(w),$$

where $\mathrm{neg}(w) \stackrel{\text{def}}{=} |\{i \in [n] : w(i) < 0\}|$. Hence

$$\begin{aligned}\ell(\underline{1}\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}) &= \binom{n}{2} + \binom{n+1}{2} - n \\ &= n^2 - n \\ &= |T(D_n)|.\end{aligned}$$

Similarly,

$$\begin{aligned}\ell(1\ \underline{2}\ \underline{3}\ \cdots\ \underline{n}) &= \binom{n}{2} + \left(\binom{n+2}{2} - 1\right) - (n-1) \\ &= n^2 - n \\ &= |T(D_n)|,\end{aligned}$$

and the result follows.

(ii) In one-line notation, $w_n = \underline{1}\ 3\ 4\ \cdots\ n\ \underline{2}$. Then for $i \geq 2$ and $j \in [n]$, we have that

$$(w_n)^i(j) = \begin{cases} \underline{1} & \text{if } j = 1 \text{ with } i \text{ odd,} \\ 1 & \text{if } j = 1 \text{ with } i \text{ even,} \\ (w_n)^{i-1}(j) + 1 & \text{if } 2 \leq j \leq n - i, \\ \underline{2} & \text{if } j = n - i + 1, \text{ and} \\ (w_n)^{i-1}(j) - 1 & \text{if } n - i + 2 \leq j \leq n. \end{cases}$$

Therefore, it follows hat $(w_n)^{n-1} = w_0^{D_n}$. □

Proposition 5.2.13. *$\ell_{T(D_n)}(w_0^{D_n}) = n$ if n is even, and $\ell_{T(D_n)} = n - 1$ if n is odd.*

Proof. Same as for Proposition 5.2.9, but only using reflections of type II. Notice that that for n even, $r_1 r_1' r_2 r_2' \cdots r_k r_k' = w_0^{D_n}$, where $k = n/2$ and $r_i = (2i-1\ \underline{2i})(\underline{2i}\ 2i-1)$, $r_i' = (2i-1\ 2i)(\underline{2i}\ \underline{2i-1})$ $1 \leq i \leq n/2$. Similarly, for n odd, we have that $t_1 t_1' t_2 t_2' \cdots t_k t_k' = w_0^{D_n}$, where $k = (n-1)/2$ and $r_i = (2i\ \underline{2i+1})(\underline{2i+1}\ 2i)$, $r_i' = (2i\ 2i+1)(\underline{2i+1}\ \underline{2i})$, $1 \leq i \leq (n-1)/2$. □

Lemma 5.2.14. *For $\sigma \in D_n$ with n even, define*

$$f^D(\sigma) = |\{i \in [n] \mid \sigma(i) = w_0^{D_n}(i) = \underline{i}\}| + \\ + |\{(i,j) \in [n] \times [n], i < j \mid (\sigma(i), \sigma(j)) \in \{(j,i), (\underline{j}, \underline{i})\}|$$

and for n odd, define

$$f^D(\sigma) = |\{i \in [n] \setminus \{1\} \mid \sigma(i) = w_0^{D_n}(i) = \underline{i}\}| + \\ + |\{(i,j) \in [n] \times [n], i < j \mid (\sigma(i), \sigma(j)) \in \{(j,i), (\underline{j}, \underline{i})\}|.$$

Moreover, let

$$g^D(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \ldots, t_\ell \text{ wih } t_1 t_2 \ldots t_\ell \sigma = w_0^{D_n}\}.$$

Then $f^D(\sigma) = i \implies g^D(\sigma) \geq m - i$ for $0 \leq i \leq m$ and $m = n$ if n is even, $m = n - 1$ if n is odd.

Proof. Same as in Lemma 5.2.10, using only reflections of type II. □

Proposition 5.2.15. *Suppose that $t_1 t_2 \ldots t_m = w_0^{D_n}$, where $m = n$ if n is even and $m = n - 1$ if n is odd. Then:*

(a) For every $i \in [m]$ there exists $j \in [m]$, $i \neq j$ so that t_i, t_j is a good pair.

(b) $t_i t_j = t_j t_i$ for all $i, j \in [m]$.

(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(m)})$ is a path in $B(D_n)$ for all $\tau \in A_{m-1}$.

Proof. The proof for (a) and (b) is the same as in Proposition 5.2.11, but only using reflections of type II.

For (c), even though the length function is not the same as described in the proof of Proposition 5.2.11(c), we recall that $B(D_n)$ is the induced graph of $B(B_n)$ on the elements of D_n by Proposition 8.2.6 in [7]. □

Number of Boolean posets in $SP(D_n)$

Let d_n be the number of Boolean posets in $SP(D_n)$ for each set $\{t_1, t_2, \ldots, t_n\} \subset T(D_n)$ with $t_1 t_2 \cdots t_n = w_0^{D_n}$. Counting these subsets is equivalent to counting the partitions of $[n]$, if n is even, or $[n-1]$, if n is odd, into subsets of two elements (these represents the good pairs). That is,

$$d_m = \frac{1}{\lfloor \frac{m}{2} \rfloor !} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-2i}{2}$$

where $m = n$ if n is even, and $m = n-1$ if n is odd. Since m is even, notice that this is the same as counting the number of partitions of $[m]$ into sets of size 2.

5.2.4 Finite dihedral groups

Let $I_2(m), m \geq 1$ be the dihedral group of order $2m$ with generating set $\{s_1, s_2\}$, and let $T = T(I_2(m))$ its reflection set. If n is odd, then

$$w_0^{I_2(m)} = \underbrace{s_1 s_2 s_1 \cdots s_1}_{m} = \underbrace{s_2 s_1 s_2 \ldots s_2}_{m}$$

is a reflection, and so $\ell_T(w_0^{I_2(m)}) = 1$. Hence $SP(I_2(m))$ is isomorphic to the Boolean poset of rank 1, if m is odd.

The case where m is even is more interesting, as $w_0^{I_2(m)} \notin T$. We readily see that $\ell_T(w_0^{I_2(m)}) = 2$, since for instance $w_0^{I_2(m)} = s_1 \underbrace{s_2 s_1 \cdots s_2}_{m-1}$. Thus $SP(I_2(m))$ is the union of Boolean posets of rank 2, if m is even.

Fix $w_0^{I_2(m)}$ to start with s_1. We now count number of Boolean posets in $SP(I_2(m))$ for m even. This number is the same as the number of sets $\{t_1, t_2\}$ with $t_1 t_2 = w_0^{I_2(m)}$. There is one such set for each element of odd rank that starts with s_1, since for each such element t_1 there exists a unique element t_2 with $t_1 t_2 = w_0^{I_2(m)}$. Since there are $\frac{m}{2}$ such elements, there are $\frac{m}{2}$ Boolean posets in $SP(I_2(m))$.

5.3 Exceptional Coxeter groups

5.3.1 F_4, H_3, H_4 and E_6

We were able to verify through computer search that the the results in the previous sections also worked for the following exceptional groups: F_4, H_3, H_4, E_6. That is, the shortest path posets for these groups are the union of Boolean posets of rank the absolute length of the longest word w_0^W. We summarize the results in Table 5.1. The computer search was done using Stembridge's `coxeter` Maple package [25], and it basically consisted of finding all shortest paths and verifying the analogous of Propositions 5.2.6, 5.2.11, 5.2.15 for those groups; that is, that the paths are given by reflections that are fully commutative.

An interesting observation is that the three Boolean posets that form $SP(E_6)$ are almost disjoint, sharing only e and $w_0^{E_6}$ (the bottom and top elements of each poset).

5.3.2 E_7 and E_8

For E_7 and E_8 we were not able to verify by computer that the shortest paths form a union of Boolean posets, since it involved more computer power (or a better code) than was available to us. However, we can argue that this is indeed the case using geometric arguments. Let (W, S) be Coxeter system, and consider the *geometric representation* of W, $\sigma : W \hookrightarrow GL(V)$, where V is a vector space with basis $\Pi = \{\alpha_s \mid s \in S\}$ (Π is called the set of *simple roots*). It is shown in [22] Section 5.4 that σ is a faithful representation.

The *root system* of the Coxeter system (W, S) is the set $\Phi = \{\sigma(w)(\alpha_s) : s \in S, w \in W\}$. Let $\beta \in \Phi$, then $\beta = \sum_{s\in S} c_s\alpha_s$. It is a well-known result that either $c_s \geq 0$ or $c_s \leq 0$ for all $s \in S$. In the former case we say that β is a *positive root*, and in the latter case we say that β is *negative root*. The set of positive roots is denoted by Φ^+ and the set of negative roots is denoted by Φ^-. It is also a well known fact (Proposition 4.4.5 in [7]) that there is a bijection between the set of reflections of W, $T(W)$ and Φ^+ given by $t = wsw^{-1} \mapsto \sigma(w)(\alpha_s)$.

Finally , we shall use the fact that $\sigma(w_0^{E_n}) = -\text{Id}$, where Id is the identity matrix of dimension n, and $n = 7, 8$. We point out that $\sigma(w_0^{E_6}) \neq -\text{Id}$, and thus rank$(SP(E_6)) < 6$. For details, see [10] Chapter VI, §4.10 and §.11. With this in mind we can show the following proposition.

Proposition 5.3.1. *For E_n, where $n = 7, 8$ we have that:*
(a) $\ell_T(w_0^{E_n}) = n$.
(b) If $w_0^{E_n} = t_1 t_2 \cdots t_n$ then $t_i t_j = t_j t_i$ for all $i, j \in [n]$.
(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(n)})$ is a path in $B(E_n)$ for all $\tau \in A_{n-1}$.

Proof. (a) Since a reflection fixes a hyperplane, the product of k reflections fixes the intersection of the k hyperplanes that are fixed by each reflection. This intersection has codimension at most k, and so it's not empty unless $k \geq n$. In particular, $\sigma(w_0^{E_n}) = -\text{Id}$ leaves no points fixed (except for $\mathbf{0}$) and so cannot be written as a product of fewer than n reflections; that is $\ell_T(w_0^{E_n}) \geq n$. Moreover by Carter's Lemma (Lemma 2.4.5 in [1]), we have that $\ell_T(w_0^{E_n}) \leq n$. Thus $\ell_T(w_0^{E_n}) = n$.

(b) Now consider $-\text{Id} = s_{t_1} s_{t_2} \cdots s_{t_n}$, where $\sigma(t_i) = s_{t_i}$ for $1 \leq i \leq n$ are the reflections (in V) with respect to the hyperplanes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ that are perpendicular to the unit vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The space fixed by the product $s_{t_1} s_{t_2} \cdots s_{t_{n-1}}$ is $\mathbb{R}\mathbf{v}_n$ (since the product of everything is $-\text{Id}$) which has codimension $n-1$ and then by the previous argument,

$$\bigcap_{i<n} \mathcal{H}_i = \mathbb{R}\mathbf{v}_n,$$

that is, $v_n \in \mathcal{H}_i$ for all $i < n$. Hence, $\mathbf{v}_i \perp \mathbf{v}_n$, which means that t_n commutes with t_i for $i < n$. By the Shifting Lemma, we have that any two reflections t_i, t_j commute.

(c) Let $t_1 \cdots t_n = w_0^{E_n}$. We are going to show that $\ell(t_1 t_2 \cdots t_k) > \ell(t_1 t_2 \cdots t_{k-1})$ for $1 < k \leq n$. As before, let $s_{t_i} = \sigma(t_i)$ be the reflection on V corresponding to t_i about the hyperplane $\mathcal{H}_i$, and let $\mathbf{v}_i$ be the normal vector to $\mathcal{H}_i$. Since $\mathbf{v}_i \perp \mathbf{v}_j$ for all $i \neq j$, we have that $s_{t_1} s_{t_2} \cdots s_{t_{i-1}}(\alpha_i) = \alpha_i$, where $\alpha_i \in \Phi^+$ is the positive root corresponding to t_i. Thus by Proposition 4.4.6 in [7], we have that $\ell(t_1 t_2 \cdots t_i) > \ell(t_1 t_2 \cdots t_{i-1})$ for $1 \leq i \leq n$. □

As a consequence of the above theorem, $SP(E_7)$ and $SP(E_8)$ are both formed by the union of Boolean posets that share at least the bottom and top elements. We are now done with the proof of Theorem 5.1.1.

Number of Boolean posets in $SP(E_7)$ and $SP(E_8)$

To count the number of paths (chains) in $SP(E_n)$ where $n = 7, 8$ we simply count the number n-tuples of perpendicular roots, since $\sigma(w_0^{E_n}) = -\text{Id}$. Each one of these n-tuples up to signs and permutations represents a Boolean poset. Direct computation yields 135 Boolean posets in $SP(E_7)$ and 2025 Boolean posets in $SP(E_8)$. These results are included in Table 5.1.

Remark 5.3.2. The above geometric argument can be used to obtain the results proven in Section 5.2. As was the case in the previous proofs, each group type requires its own geometric argument, since $\sigma(w_0^W)$ is different for each case (see [10] Chapter VI, §4.10 and §.11).

5.4 Lowest-degree terms of the complete cd-index of finite Coxeter groups

Recall that the highest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ coincide with the terms of $\psi([u,v])$, as discussed in Subsection 3.1.1. Here we compute the lowest degree terms of $\widetilde{\psi}_{e,w^{W_0}}(\mathbf{c},\mathbf{d})$.

Let $\psi(B_n)$ be **cd**-index of the Boolean poset $\mathcal{B}_n$ (so $\mathcal{B}_n$ is the poset of subsets of $[n]$ ordered by inclusion). We can use Theorem 5.2 in [20] to compute $\psi(\mathcal{B}_n)$. First $\psi(\mathcal{B}_1) = 1$ and for $n > 1$,

$$\psi(\mathcal{B}_n) = \psi(\mathcal{B}_{n-1}) \cdot \mathbf{c} + G(\psi(\mathcal{B}_{n-1})) \tag{5.3}$$

where G is is the derivation $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$. In particular, we have that $\psi(\mathcal{B}_2) = \mathbf{c}$, $\psi(\mathcal{B}_3) = \mathbf{c^2} + \mathbf{d}$, $\psi(\mathcal{B}_4) = \mathbf{c^3} + 2\mathbf{cd} + 2\mathbf{dc}$, and so on.

Propositions 5.2.6, 5.2.11 and 5.2.15, and the results and computer search of Section 5.3 give that for a finite Coxeter group W, the corresponding $SP(W)$ is the union of Boolean posets (that share at least the bottom and top elements). So any interval in $SP(W)$ belongs to a Boolean poset corresponding to a set $R = \{t_1, t_2, \ldots, t_\ell\} \subset T(W)$ with $\ell_{T(W)}(w_0^W) = \ell$ and $t_1 t_2 \cdots t_\ell = w_0^W$. Thus any interval of $SP(W)$ (thought of as paths in $B(W)$ labeled with $T(W)$, where $T(W)$ is ordered by a reflection ordering) has a unique chain (path) with empty descent set. Hence counting descent sets in the chains given by R is the same as counting the flag h-vector of the Boolean poset of rank ℓ.

As a consequence, the lowest-degree terms in the complete **cd**-index of W add up to a multiple N of the **cd**-index of the Boolean poset of ranks $\ell_{T(W)}(w_0^W)$. N is the number of Boolean posets in $SP(W)$; that is, the number of sets $\{t_1, \ldots, t_{\ell_T(w_0^W)}\}$ with $t_1 t_2 \cdots t_{\ell_{T(w_0^W)}} = w_0^W$. These terms can be computed with the information provided in (5.3) and Table 5.1. So we have

Theorem 5.4.1. *Let W be a finite Coxeter group, α_W is the number of Boolean posets that form $SP(W)$ and $\ell_0 = \ell_T(w_0^W)$. Then lowest degree terms of $\widetilde{\psi}_{e,w_0^W}(\boldsymbol{c}, \boldsymbol{d})$ are given by $\alpha_W \psi(\mathcal{B}_{\ell_0})$.*

In particular, the lowest-degree terms of $\widetilde{\psi}_{e,w_0^W}$ are minimized by $\psi(\mathcal{B}_{\ell_0})$.

Table 5.1: Finite coxeter groups W, $\mathrm{rk}(SP(W))$, and the number of Boolean posets in $SP(W)$

W	$\mathrm{rk}(SP(W))$	# of Boolean posets in $SP(W)$
A_n	$\lfloor \frac{n+1}{2} \rfloor$	1
B_n	n	b_n
D_n	n if n is even; $n-1$ if n is odd	d_n
$I_2(m)$	2 if m is even; 1 if m is odd	$\frac{m}{2}$ if m is even; 1 if m is odd
F_4	4	24
H_3	3	5
H_4	4	75
E_6	4	3
E_7	7	135
E_8	8	2025

We finish this chapter with the following remark: Each connected component given by FLIP$(SP(W))$ corresponds to exactly one Boolean copy of $SP(u, v)$. For instance for Figure 5.1, FLIP generates four connected components corresponding to the four copies of $\mathcal{B}_3$. The colors in the figure represents the 4 different components.

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