TEMPERED STABLE DISTRIBUTIONS: PROPERTIES AND EXTENSIONS

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PROPERTIES AND EXTENSIONS

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It has been observed that data often appears to be well approximated by infinite variance stable distributions in some central region, but the tails of the distribution are actually lighter. Tempered stable distributions, which were introduced in [Ros07], are a rich class of models that attempt to capture this type of behavior. We will define certain generalizations of these models, which allow for more flexible structure.

We will then derive a number of important results about them. In particular, we will give necessary and sufficient conditions for when they have regularly varying tails. We will also classify the possible weak limits of sequences of tempered stable distributions, and give necessary and sufficient conditions for convergence. These two properties will help us to categorize the long and short time behavior of their corresponding Lévy processes.

We also attempt to explain why such models appear in applications. The use of stable distributions is justified by the central limit theorem, which says that stable distributions are the only possible limits of scaled and shifted sums of iid random variables. While this does not apply to tempered stable distributions, we will show that they may provide a good approximation to such sums for large, but not too large, aggregation levels. We base this explanation on the prelimit theorems of [KRS99] and [KRS00]. We then generalize them to d-dimensions.

BIOGRAPHICAL SKETCH

Michael Grabchak was born in 1982 in the port city of Odessa, Ukraine, then part of the Soviet Union. In 1988 he emigrated to the United States. There he grew up in the town of East Brunswick, NJ. After graduating from East Brunswick High School, he attended Rutgers University majoring in Mathematics and Computer Science. In 2004 he graduated with highest honors and went on to pursue a Ph.D. at the Department of Statistical Science at Cornell University. There he studied under Professor Gennady Samorodnitsky. He is excited to join the Department of Mathematics and Statistics at UNC Charlotte this Fall.

FOR MY PARENTS.

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NOTATION

Let $\mathbb{R}^d$ be the space of d-dimensional column vectors of real numbers. For any $a \in \mathbb{R}$, we will write $a_d = (a, \ldots, a)^T \in \mathbb{R}^d$. Likewise we will write $a_{d\times d}$ for the $d \times d$ matrix where all of the entries are a. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^d$. Thus if $x = (x_1, x_2, \ldots, x_d)^T, y = (y_1, y_2, \ldots, y_d)^T \in \mathbb{R}^d$ then $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. This induces the norm $|x| = \langle x, x\rangle^{1/2} = \sqrt{\sum_{i=1}^d x_i^2}$, which itself induces the metric $d(x, y) = |x - y|$. Let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ and let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. As usual, let $i = \sqrt{-1}$. We will use the relations $1/0 = \infty$ and $1/\infty = 0$. If (A_n) is a sequence of matrices, then, by $\lim A_n$, $\limsup A_n$, and $\liminf A_n$, we will mean that the lim, lim sup, or lim inf is taken componentwise.

If E is a metric space, f, g are real valued functions on E, $c \in \mathbb{R}$, and $a \in E$, we will use the notation

$$f(t) \sim cg(t) \text{ as } t \to a$$

to mean $\lim_{t\to a} \frac{f(t)}{g(t)} = c$. In particular, $f(t) \sim 0g(t)$ as $t \to a$ means $\lim_{t\to a} \frac{f(t)}{g(t)} = 0$. If E is a topological space we will denote the class of Borel sets on E by $\mathfrak{B}(E)$.

If μ is a probability measure, we denote its **characteristic function** by $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle x,z\rangle}\mu(dx)$. If $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$ then, by Lemma 7.6 in [Sat99], there is a unique continuous function $C_\mu(z)$ such that $\hat{\mu}(z) = \exp\{C_\mu(z)\}$. This function is called the **cumulant generating function**. When $X \sim \mu$ we will sometimes write $C_X \equiv C_\mu$.

For the notation RV_ρ^c, $RV_\rho^c(\sigma)$, and $LRV_\rho^c(B)$ see Appendix B, for $f^{\leftarrow}$ see (B.2) and (B.5), for $ID(A, M, b)$, $ID^0(A, M, b_0)$, and $ID^1(A, M, b_1)$ see Appendix C, for $S_\alpha(\sigma, b)$ and $N(b, A)$ see Appendix C.1, for $\mathfrak{M}^\beta$ see Appendix C.2, for ∂A see Appendix A.2, and for $\bar{\mathbb{R}}^d$ and $\bar{\mathbb{R}}_0^d$ see Appendix A.4.

CHAPTER 1

INTRODUCTION

Tempered stable distributions are a rich class of stochastic models that are interesting from both theoretical and applied perspectives. These models look like stable distributions in some central region, but they have lighter tails. There are a number of situations where such models appear to be appropriate. And, in fact, certain special cases have been used in finance [CGMY02], [CT04], biostatistics [Hou86], [All92], [PRM08], computer science [TG09], and physics [Kop95], [ZBR10].

The idea of using models that are "stable-like" in the center but possess lighter tails seems to have originated in the physics literature with the influential paper of Mantegna and Stanley [MS94]. There they introduce truncated Lévy flights[1]. These models start with a stable distribution and then truncate its tails. More formally, let $f(x)$ be the density of an infinite variance α-stable distribution, and let $T > 0$ be a truncation level. We define the **truncated Lévy flight (TLF)** with truncation level T as a probability distribution with the density

$$f_T(x) = c_T f(x) 1_{|x| \leq T}, \tag{1.1}$$

where c_T is a normalizing constant. If T is very large, then the two models may be statistically indistinguishable, even for very large datasets. However, the tail behavior of these models is vastly different. The stable distribution has an infinite variance, while the TLF has all moments finite.

Although these models are interesting and have been used in a variety of applications, they have a number of limitations. The two most important ones are that they are not infinitely divisible and that the truncation level may appear arbitrary. The lack of infinite divisibility may not seem like much of a problem. However,

[1] "Lévy flights" is a physics term for infinite variance stable distributions.

many models rely on the fact that stable distributions are infinitely divisible. In particular, it is common to use stable Lévy processes. However, we cannot define Lévy processes with marginals that are not infinitely divisible. The second issue has major ramifications for risk estimation. The problem is that, even if the data comes from such a truncated distribution, it is virtually impossible to estimate the parameter T. However, different values of T give very different estimates of risk. Aside for these considerations, it is generally desirable to allow for more flexible tail behavior than what is given by this model.

With issues such as these in mind, Koponen [Kop95] suggested a different approach to modifying the tails of stable distributions to make them lighter. His idea begins with observing that an infinite variance α-stable distribution is infinitely divisible with no Gaussian part and a Lévy measure given by

$$M(dx) = 1_{x<0} c_- |x|^{-1-\alpha} dx + 1_{x>0} c_+ x^{-1-\alpha} dx, \tag{1.2}$$

where $c_-, c_+ \geq 0$. Noting that the tails of the Lévy measure are intimately related to the tails of the distribution, his approach is to modify the tails of the Lévy measure to make them lighter and yet to keep the Lévy measure virtually unchanged in some central region. Thus he introduced an infinitely divisible distribution with a Lévy measure given by

$$M(dx) = 1_{x<0} c_- |x|^{-1-\alpha} e^{-|x|/\ell_-} dx + 1_{x>0} c_+ x^{-1-\alpha} e^{-x/\ell_+} dx, \tag{1.3}$$

where $c_-, c_+, \ell_-, \ell_+ \geq 0$. Clearly, if ℓ_- and ℓ_+ are large then the Lévy measure will be close to that of the corresponding α-stable distribution in the center and potentially quite far into the tails, but ultimately its tails will decay exponentially fast. This leads to exponential decay in the tails of the corresponding probability measure as well.

Infinitely divisible distributions with no Gaussian part and a Lévy measure given by (1.3) have since come to be known as **smoothly truncated Lévy flights**

(STLF). However, these models have also been called CGMY, KoBoL, and tempered stable. We will reserve the later term for a more general class of models and refer to these simply as STLFs.

It should be mentioned that some special cases of STLFs had previously appeared in the literature. These include the inverse Gaussian (see e.g. [CF89]) and some of its extensions given in [Hou86] and [All92]. However, in these cases, the distributions were not introduced from the perspective of modifying the tails of a stable distribution, but from other considerations.

Despite the usefulness of STLFs, they have a number of limitations. For one thing they have exponential tails. While this is useful in many situations, there are times when we would like more flexibility. In particular, there is some evidence that the distributions of financial returns have regularly varying tails (see e.g. [CT04]), yet this type of behavior cannot be captured by STLFs. Moreover, it is not immediately clear how to generalize them to d-dimensions, and they are not closed under convolution.

To deal with these limitations Rosiński [Ros02] introduced what are now called proper tempered stable distributions. A slightly more general class of tempered stable distributions was later introduced in [Ros07]. Tempered stable distributions which are not proper may lose the property of looking "stable-like" in some central region, but they serve to make the class larger and more flexible.

The idea comes from considering the sum of n STLFs. Let $X_1, \dots, X_n$ be independent random variables such that the distribution of X_i is that of a STLF with a Lévy measure given by (1.3) with parameters $c_+^i, c_-^i, \ell_-^i, \ell_+^i$. The distribution of the sum $\sum_{i=1}^n X_i$ is infinitely divisible with no Gaussian part and a Lévy measure given by

$$\begin{aligned} M(dx) &= 1_{x<0}|x|^{-1-\alpha}\left(\sum_{i=1}^{n} c_{-}^{i} e^{-|x|/\ell_{-}^{i}}\right)dx + 1_{x>0}x^{-1-\alpha}\left(\sum_{i=1}^{n} c_{+}^{i} e^{-x/\ell_{+}^{i}}\right)dx \\ &= 1_{x<0}|x|^{-1-\alpha}\int_0^\infty e^{-|x|t}Q(dt|-1)dx + 1_{x>0}x^{-1-\alpha}\int_0^\infty e^{-xt}Q(dt|1)dx, \end{aligned}$$

where $Q(dt|-1) = \sum_{i=1}^n c_-^i \delta_{1/\ell_-^i}(dt)$ and $Q(dt|1) = \sum_{i=1}^n c_+^i \delta_{1/\ell_+^i}(dt)$. Clearly, we can consider more general measures $Q(dt|-1)$ and $Q(dt|1)$. Moreover, this easily extends to d-dimensions by allowing a different measure $Q(\cdot|u)$ for each $u \in \mathbb{S}^{d-1}$. This motivates the following definition from [Ros07].

Definition 1.1. *Fix $\alpha \in (0,2)$ and let σ be a finite measure on $\mathbb{S}^{d-1}$. A probability measure μ on $\mathbb{R}^d$ is called **tempered α-stable** if it is infinitely divisible without Gaussian part and has Lévy measure M that can be written as*

$$M(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_B(ru)q(r,u)r^{-1-\alpha}dr\sigma(du), \quad B \in \mathfrak{B}(\mathbb{R}^d), \tag{1.4}$$

*where $q : (0,\infty)\times\mathbb{S}^{d-1} \mapsto (0,\infty)$ is a Borel function such that $q(\cdot,u)$ is completely monotone with $\lim_{r\to\infty} q(r,u) = 0$ for each $u \in \mathbb{S}^{d-1}$. μ is called a **proper tempered α-stable** distribution if, in addition to the above, $\lim_{r\downarrow 0} q(r,u) = 1$ for each $u \in \mathbb{S}^{d-1}$. The function q is called the **tempering function**.*

To see that these are the models that we had previously described note that, by Bernstein's Theorem (Theorem 1a in Section XIII.4 of [Fel71]), the complete monotonicity of $q(\cdot,u)$ implies that there is a family of measures $\{Q(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ such that

$$q(r,u) = \int_0^\infty e^{-rt}Q(dt|u). \tag{1.5}$$

Clearly, the class of tempered α-stable distributions is semiparametric. A number of interesting parametrizations are explored in [TW06].

While tempered stable distributions are much more general than STLFs, they nevertheless place a lot of structure on the tempering function. A number of other forms for the tempering function have been suggested in the literature (see [RS10] and the references therein). These lead to various generalizations. To place all of these into a common framework, [RS10] introduced generalized tempered stable distributions. These attempt to remove as much structure as possible, without losing "stable-like" behavior in some central region. They are defined in [RS10] as follows.

Definition 1.2. *Fix $\alpha \in (0,2)$ and let σ be a finite Borel measure on $\mathbb{S}^{d-1}$. An infinitely divisible distribution μ on $\mathbb{R}^d$ is said to be **generalized tempered α-stable** if μ has no Gaussian part and its Lévy measure can be represented as* (1.4)*, where the tempering function $q : (0,\infty) \times \mathbb{S}^{d-1} \mapsto (0,\infty)$ is a Borel function such that for some nonnegative function $g \in L^1(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}), \sigma)$*

$$\lim_{r \downarrow 0} \|q(r,\cdot) - g(\cdot)\|_{L^1(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}), \sigma)} = 0.$$

*The function g is called the **limiting function**.*

Proper tempered α-stable distributions are generalized tempered stable, while nonproper ones are not. Many interesting properties of these models are given in [RS10]. Perhaps the most important is that a Lévy process with generalized tempered stable marginals behaves like a stable process in a short time frame, but if it has a finite variance, then in a long time frame it approximates a Brownian motion.

Another extension of tempered stable distributions that will be important for us are the **tempered infinitely divisible distributions** and **proper tempered infinitely divisible distributions** of [BRKF11]. These are defined analogously to tempered stable and proper tempered stable distributions, except that now instead of the tempering function being completely monotone, the function $q(r^{1/2}, u)$, as a function of r, is completely monotone for each $u \in \mathbb{S}^{d-1}$. This allows for more flexible and lighter tails than those of tempered α-stable distributions

We will consider some further extensions of tempered stable distributions and derive some of their properties. In Chapter 2, we will introduce p-tempered α-stable distributions. These are defined analogously to tempered stable and tempered infinitely divisible distributions, but assuming that the function $q(r^{1/p}, u)$, as a function of r, is completely monotone for each $u \in \mathbb{S}^{d-1}$. Moreover, it turns out that we can define such distributions for any $\alpha < 2$. However, the case when $\alpha < 0$ has not been studied in much detail. Aside for certain integral representations given in [MN09], little has been done in general. Nevertheless, these appear to be useful and certain special cases have found a number of applications, see e.g. [All92], [CGMY02], and [CT04].

We derive a number of results about p-tempered α-stable distributions. One of the most important will be to categorize the long and short time behavior of their Lévy processes. Formally, fix $c \in \{0, \infty\}$ and let $\{X_t : t \geq 0\}$ be a d-dimensional Lévy process. We want to derive necessary and sufficient conditions under which there exists a function $a_t > 0$ such that

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to c \tag{1.6}$$

for some random vector Y with a distribution not concentrated at a point. When $c = 0$ we will call this **short time behavior** of the Lévy process and when $c = \infty$ we will call it **long time behavior** of the Lévy process.

An important consequence of long and short time behavior is that it can be extended to convergence at the level of processes. Specifically, by Theorem 15.17 in [Kal02] (1.6) implies that if $X^h = \{X_{th} : t \geq 0\}$ then there exist processes $\tilde{X}^h \stackrel{d}{=} X^h$ such that for all $t \geq 0$

$$\lim_{h\to c} \sup_{s\leq t} |a_h \tilde{X}^h_s - Y_s| \xrightarrow{p} 0, \tag{1.7}$$

where $\{Y_t : t \geq 0\}$ is a Lévy process with $Y_1 \stackrel{d}{=} Y$. Thus, in a sense, long time behavior corresponds to what the process looks like when we "zoom out" and short time behavior corresponds to what the process looks like when we "zoom in" on it.

In Section 4.2 we will derive long and short time behavior for p-tempered α-stable Lévy processes. For completeness, in Appendix D.2 we will give necessary and sufficient conditions for long and short time behavior of general Lévy processes. In this case, long time behavior is well known, but short time behavior does not appear to have been studied in the multivariate setting before.

Another important property, which does not seem to have been analyzed before will be given in Section 2.4. There we will give necessary and sufficient conditions for the tails of p-tempered α-stable distributions to be regularly varying.

One limitation of p-tempered α-stable distributions is that they are not closed under weak convergence. In Chapter 3 we will introduce extended p-tempered α-stable distributions. This is the smallest class that contains all p-tempered α-stable distributions and is closed under weak convergence. We will give an explicit form for their Lévy measures and give necessary and sufficient conditions for weak convergence of sequences in this class.

Our entire discussion suggests the following important question. Why would distributions that are "stable-like" in some central region, but possessing lighter tails come up in applications? The main explanation for why stable distributions appear in applications is the generalized central limit theorem, which says that stable distributions (including the Gaussian) are the only possible limits of scaled and shifted sums of iid random vectors. Of course, this does not apply to tempered stable distributions. However, it turns out that even if the distribution of a random vector has light tails and is in the domain of attraction of the Gaussian, for large (but not too large) sums of iid copies of this random vector, the distribution of the sum may be well approximated by a stable distribution in some central region, although it, necessarily, has lighter tails. This result was first quantified by the prelimit theorems of [KRS99] and [KRS00]. In Chapter 5 we give a generalization of their results to d-dimensions.

Finally, in the appendix we collect some important results about weak and vague convergence, regular variation, and infinitely divisible distributions and their associated Lévy processes. We will use these results throughout. While most of them are well known, a number of them appear to be new. In particular, the concept of β-duals (introduced in Appendix C.2), their properties, and their use to derive short time behavior of Lévy processes does not seem to have been explored before.

CHAPTER 2

TEMPERED STABLE DISTRIBUTIONS

Tempered stable distributions will be defined as a subclass of infinitely divisible distribution. Recall that a probability measure μ is called **infinitely divisible** if for any n there is a probability measure μ_n such that if $X \sim \mu$ and $X_1, \dots, X_n \overset{\text{iid}}{\sim} \mu_n$ then $X \overset{d}{=} X_1 + \cdots + X_n$. The characteristic function of an infinitely divisible distribution μ on $\mathbb{R}^d$ never vanishes and is given by $\hat{\mu}(z) = \exp\{C_\mu(z)\}$ where

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\frac{\langle z, x\rangle}{1+|x|^2} \right) M(dx), \tag{2.1}$$

A is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and M satisfies

$$M(\{0_d\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) M(dx) < \infty. \tag{2.2}$$

We call A the **Gaussian part** and M the **Lêvy measure**. In fact, we will call any measure that satisfies (2.2) a Lévy measure. According to Theorem 8.1 in [Sat99], the measure μ is uniquely identified by the **Lêvy triplet** (A, M, b). We will write $\mu = ID(A, M, b)$. For more details see Appendix C.

Fix $\alpha \in (0, 2)$. Let M_σ^α be the Lévy measure of an α-stable distribution with spectral measure σ. By (C.20) it is given by

$$M_\sigma^\alpha(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(ru) r^{-\alpha-1} dr \sigma(du), \quad B \in \mathfrak{B}(\mathbb{R}^d). \tag{2.3}$$

Let $q : (0,\infty) \times \mathbb{S}^{d-1} \mapsto (0, \infty)$ be a Borel function. For all $u \in \mathbb{S}^{d-1}$ assume that $q(\cdot, u)$ is completely monotone with $\lim_{r\to\infty} q(r, u) = 0$. By the **complete monotonicity** of $q(\cdot, u)$ we mean that

$$(-1)^n \frac{\partial^n}{\partial r^n} q(r, u) > 0 \;\text{ for all } n \in \mathbb{N},\; r > 0,\; u \in \mathbb{S}^{d-1}.$$

In particular this means that, for a fixed $u \in \mathbb{S}^{d-1}$, $q(\cdot, u)$ is strictly decreasing and convex. Fix $p > 0$ and let

$$q_p(r, u) = q(r^p, u). \tag{2.4}$$

Define the Borel measure

$$M(B) \;=\; \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(ru) q_p(r,u) r^{-\alpha-1} dr \sigma(du), \quad B \in \mathfrak{B}(\mathbb{R}^d). \qquad (2.5)$$

Since $q(\cdot, u)$ is completely monotone, by Bernstein's Theorem (Theorem 1a in section XIII.4 of [Fel71]),

$$q_p(r,u) = \int_0^\infty e^{-r^p s} Q(\mathrm{d}s|u) \qquad (2.6)$$

for some measurable family $\{Q(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ of measures on $(0,\infty)$. For a guarantee that the family can be taken to be measurable see Remark 3.2 in [BNMS06]. It turns out that for any $\alpha < 2$, if $\{Q(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ satisfies certain conditions, (2.5) defines a Lévy measure. These conditions will be given in Corollary 2.4 below. This leads to the following definition.

Definition 2.1. *Fix $\alpha < 2$, $p > 0$. An infinitely divisible probability measure μ is called a* **p-*tempered* α-*stable distribution*** *if it has no Gaussian part and its Lévy measure is given by* (2.5)*. If, in addition,* $\lim_{r\downarrow 0} q_p(r,u) = 1$ *for every $u \in \mathbb{S}^{d-1}$ then μ is called a* ***proper* p-*tempered* α-*stable distribution*.**

Note that the condition $q_p(0+,u) = 1$ for every $u \in \mathbb{S}^{d-1}$ is equivalent to the condition that $\{Q(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ is a family of probability measures.

When $p = 1$ and $\alpha \in (0,2)$ Definition 2.1 coincides with Rosiński's [Ros07] definitions of tempered α-stable and proper tempered α-stable distributions. When $p = 2$ and $\alpha \in [0,2)$ it coincides with the definitions of tempered infinitely divisible and proper tempered infinitely divisible distributions presented in [BRKF11]. If we allow for the distributions to have a Gaussian part, then we would have the class $J_{\alpha,p}$ in the notation of [MN09]. This, in turn, contains important subclasses including the Thorin class (when $p = 1$ and $\alpha = 0$) and the Goldie-Steutel-Bondesson class (when $p = 1$ and $\alpha = -1$). For more information about these classes see [BNMS06] and the references therein.

Note that, for proper p-tempered α-stable distributions, $|q_p(r,u)-1| \leq 2$. Thus by dominated convergence

$$\lim_{r \downarrow 0} \int_{\mathbb{S}^{d-1}} |q_p(r,u) - 1| \, \sigma(du) = 0.$$

By Definition 1.2, we get the following result.

Lemma 2.2. *Fix $\alpha \in (0,2)$ and $p > 0$. All proper p-tempered α-stable distributions are generalized tempered α-stable with tempering function q_p and limiting function $g \equiv 1$.*

Following [Ros07], we will reparametrize the measure M into a form that is easier to work with. Let Q be a measure on $\mathbb{R}^d$ given by

$$Q(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) Q(dr|u)\sigma(du), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{2.7}$$

Note that $Q(\{0_d\}) = 0$. Define a measure R on $\mathbb{R}^d$ by

$$R(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^{1+1/p}}\right) |x|^{\alpha/p} Q(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{2.8}$$

Note that $R(\{0_d\}) = 0$. To get the inverse transformation we have

$$Q(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^{p+1}}\right) |x|^{\alpha} R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{2.9}$$

From this we get

$$Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} |x|^\alpha R(dx). \tag{2.10}$$

All of this leads to the following theorem.

Theorem 2.3. *1. Let M be given by* (2.5) *and let R be given by* (2.8). *We can write*

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx) t^{-1-\alpha} e^{-t^p} dt R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d), \tag{2.11}$$

or equivalently,

$$M(A) = p^{-1} \int_{\mathbb{R}^d} \int_0^\infty 1_A(t^{1/p}x) t^{-1-\alpha/p} e^{-t} dt R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{2.12}$$

2. Fix $p > 0$. Let M be given by (2.11). It is the Lévy measure of some infinitely divisible distribution if and only if either $R(\mathbb{R}^d) = 0$ or the following hold: $\alpha < 2$,

$$R(\{0_d\}) = 0, \tag{2.13}$$

and

$$\begin{aligned} \int \left(|x|^2 \wedge |x|^\alpha\right) R(dx) < \infty \quad & \text{if } \alpha \in (0,2), \\ \int \left(|x|^2 \wedge [1 + \log^+ |x|]\right) R(dx) < \infty \quad & \text{if } \alpha = 0, \\ \int \left(|x|^2 \wedge 1\right) R(dx) < \infty \quad & \text{if } \alpha < 0. \end{aligned} \tag{2.14}$$

Moreover, when R satisfies these conditions, M is the Lévy measure of a p-tempered α-stable distribution and it uniquely determines R.

3. Let M be given by (2.11) with R satisfying (2.13) and (2.14). M is the Lévy measure of a proper p-tempered α-stable distribution if and only if $\int_{\mathbb{R}^d} |x|^\alpha R(dx) < \infty$.

Note that for all $\alpha < 2$ (2.14) implies the necessity of $\int \left(|x|^2 \wedge |x|^\alpha\right) R(dx) < \infty$ and $\int \left(|x|^2 \wedge 1\right) R(dx) < \infty$. In particular, this means that $R \in \mathfrak{M}^0$ the class of Lévy measures on $\mathbb{R}^d$ (see Appendix C.2 for the notation). Moreover, for $\alpha \in (0,2)$, R satisfies the assumptions of (2.14) if and only if $R \in \mathfrak{M}^\alpha$. Before proving Theorem 2.3, we will translate the integrability conditions on R into integrability conditions on $Q(\cdot|u)$ and σ.

Corollary 2.4. *Fix $p > 0$. Let M be given by (2.5). Then M is a Lévy measure if and only if either*

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty Q(dt|u)\sigma(du) = 0$$

or $\alpha < 2$ *and*

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-(2-\alpha)/p} \wedge 1\right) Q(dt|u)\sigma(du) < \infty \quad \alpha \in (0,2)$$
$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-2/p} \wedge \left[1 + \log^+(t^{-1/p})\right]\right) Q(dt|u)\sigma(du) < \infty \quad \alpha = 0$$
$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-(2-\alpha)/p} \wedge t^{\alpha/p}\right) Q(dt|u)\sigma(du) < \infty \quad \alpha < 0.$$

Note that, by Theorem 15.2 in [Bil95], these conditions guarantee that for any $p > 0$, $\int e^{-r^p s} Q(ds|u) < \infty$ for σ almost every u. Now to prove Theorem 2.3.

Proof of Theorem 2.3. First we show Part 1. By (2.5) we have

$$\begin{aligned}
M(A) &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) r^{-\alpha-1} q_p(r,u) dr \sigma(du) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(ru) r^{-\alpha-1} e^{-r^p s} dr Q(ds|u)\sigma(du) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(ts^{-1/p}u) t^{-1-\alpha} e^{-t^p} dt s^{\alpha/p} Q(ds|u)\sigma(du) \\
&= \int_{\mathbb{R}^d} \int_0^\infty 1_A\left(t \frac{x}{|x|^{1+1/p}}\right) t^{-1-\alpha} e^{-t^p} dt |x|^{\alpha/p} Q(dx) \\
&= \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx) t^{-1-\alpha} e^{-t^p} dt R(dx),
\end{aligned}$$

where the third equality follows by the substitution $t = rs^{1/p}$, and the last equality follows by (2.9). The substitution $u = t^p$ gives (2.12).

Now to show Part 2. By (2.2), M is a Lévy measure if and only if $M(\{0\}) = 0$ and $\int(|x|^2 \wedge 1)M(dx) < \infty$. Assume $R(\mathbb{R}^d) > 0$, since the other case is trivial. We have

$$M(\{0_d\}) = \int_{\mathbb{R}^d} \int_0^\infty 1_{\{0_d\}}(tx) t^{-\alpha-1} e^{-t^p} dt R(dx) = \int_{\{0_d\}} \int_0^\infty t^{-1-\alpha} e^{-t^p} dt R(dx),$$

which is equal to zero if and only if $R(\{0_d\}) = 0$. It remains to show the equivalence of the integrability conditions.

We begin by showing necessity. Thus, assume that $\int(|x|^2 \wedge 1)M(dx) < \infty$. Fix

$\epsilon > 0$. We have

$$\begin{aligned} \infty &> \int_{|x|\le 1} |x|^2 M(dx) = \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} e^{-t^p} dt R(dx) \\ &\ge \int_{|x|\le 1/\epsilon} |x|^2 \int_0^{\epsilon} t^{1-\alpha} e^{-t^p} dt R(dx) \ge e^{-\epsilon^p} \int_{|x|\le 1/\epsilon} |x|^2 R(dx) \int_0^{\epsilon} t^{1-\alpha} dt. \end{aligned}$$

Since $R(\mathbb{R}^d) > 0$, for this be finite for all $\epsilon > 0$ it is necessary that $\alpha < 2$. Taking $\epsilon = 1$ gives the necessity of $\int_{|x|\le 1} |x|^2 R(dx) < \infty$.

Note that

$$\begin{aligned} \infty &> \int_{|x|\ge 1} M(dx) = \int_{\mathbb{R}^d} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\ge \int_{|x|\ge 1} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\ge \int_1^{\infty} t^{-1-\alpha} e^{-t^p} dt \int_{|x|\ge 1} R(dx) + e^{-1} \int_{|x|\ge 1} \int_{|x|^{-1}}^{1} t^{-1-\alpha} dt R(dx). \end{aligned}$$

This implies the necessity of $\int_{|x|\ge 1} R(dx) < \infty$ and $\int_{|x|\ge 1} \int_{|x|^{-1}}^{1} t^{-1-\alpha} dt R(dx) < \infty$. When $\alpha < 0$ we are done. When $\alpha = 0$ we have

$$\int_{|x|\ge 1} \int_{|x|^{-1}}^{1} t^{-1-\alpha} dt R(dx) = \int_{|x|\ge 1} \log |x| R(dx).$$

When $\alpha \in (0, 2)$ we have

$$\int_{|x|\ge 1} \int_{|x|^{-1}}^{1} t^{-1-\alpha} dt R(dx) = \frac{1}{\alpha} \int_{|x|\ge 1} \left(|x|^{\alpha} - 1\right) R(dx),$$

which together with the necessity of $\int_{|x|\ge 1} R(dx) < \infty$ gives (2.14).

We will now show sufficiency, so assume that R satisfies (2.14). We have

$$\begin{aligned} \int_{|x|\le 1} |x|^2 M(dx) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} e^{-t^p} dt R(dx) \\ &\le \int_{|x|\le 1} |x|^2 R(dx) \int_0^{\infty} t^{1-\alpha} e^{-t^p} dt \\ &\quad + \int_{|x|>1} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} dt R(dx) \\ &= p^{-1} \Gamma\left(\frac{2-\alpha}{p}\right) \int_{|x|\le 1} |x|^2 R(dx) + (2-\alpha)^{-1} \int_{|x|>1} |x|^{\alpha} R(dx). \end{aligned}$$

Note that for $\int_{|x|>1} |x|^\alpha R(dx)$ to be finite it is sufficient that $\int_{|x|>1} R(dx) < \infty$ when $\alpha < 0$ and $\int_{|x|>1} \log |x| R(dx) < \infty$ when $\alpha = 0$.

Let $D = \sup_{t\geq 1} t^{2-\alpha} e^{-t^p}$. Consider

$$\begin{aligned}\int_{|x|\geq 1} M(\mathrm{d}x) &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\leq D \int_{|x|\leq 1} \int_{|x|^{-1}}^{\infty} t^{-3} dt R(dx) + \int_{|x|>1} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx).\end{aligned}$$

The first integral in the above equals $.5D \int_{|x|\leq 1} |x|^2 R(dx)$, which is assumed finite. The second integral can be written as

$$\int_{|x|>1} \int_{|x|^{-1}}^{1} t^{-1-\alpha} e^{-t^p} dt R(dx) + \int_1^\infty t^{-1-\alpha} e^{-t^p} dt \int_{|x|>1} R(dx)$$

Of these, the second integral is finite when $\int_{|x|>1} R(dx) < \infty$. The first is bounded by $\int_{|x|>1} \frac{|x|^\alpha - 1}{\alpha} R(dx)$ when $\alpha \neq 0$ and by $\int_{|x|>1} \log |x| R(dx)$, when $\alpha = 0$.

Now assume that R satisfies (2.13) and (2.14) and M is given by (2.11). We will show that this implies that M is the Lévy measure of a p-tempered α-stable distribution. Define Q by (2.9). We can get R back from Q by (2.8). Note that Q satisfies the conditions of Proposition C.3, thus it has a polar decomposition. We can write $Q(dt, du) = Q(dt|u)\sigma(du)$ for some finite measure σ on $\mathbb{S}^{d-1}$ and some measurable family of Borel measures $\{Q(dt|u)\}_{u\in\mathbb{S}^{d-1}}$ on $(0, \infty)$.

For $A \in \mathfrak{B}(\mathbb{R}^d)$ we have

$$\begin{aligned}M(A) &= \int_{\mathbb{R}^d} \int_0^\infty 1_A(xt) t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &= \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx|x|^{-1-1/p}) t^{-1-\alpha} e^{-t^p} dt |x|^{\alpha/p} Q(dx) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(tur^{-1/p}) t^{-1-\alpha} e^{-t^p} dt r^{\alpha/p} Q(dr|u)\sigma(du) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(us) s^{-1-\alpha} e^{-s^p r} ds Q(dr|u)\sigma(du) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(us) q_p(s,u) s^{-1-\alpha} ds\sigma(du), \qquad (2.15)\end{aligned}$$

where $q_p(s,u) = \int_0^\infty e^{-s^p r} Q(dr|u)$, which is completely monotone in r^p by Bernstein's theorem.

Now to show the uniqueness of R. Assume that two measures R^1 and R^2 satisfy (2.11), (2.13), and (2.14). Define measures Q^i by (2.9). By Proposition C.3, there is a polar decomposition of Q^i. Moreover by (2.15) and the uniqueness part of Proposition C.3, we can represent Q^i in polar coordinates as

$$Q^i(ds, du) = Q^i(ds|u)\sigma(du), \quad i = 1, 2,$$

where $\{Q^i(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ are measurable families of Borel measures on $(0,\infty)$. By the discussion following Corollary 2.4, for σ almost every u and all $r > 0$ the Laplace transform

$$q_1^i(r,u) = \int_0^\infty e^{-rs} Q^i(ds|u)$$

is finite and by Bernstein's theorem, it defines completely monotone functions $q_1^i(r,u)$, $i = 1, 2$. By (2.15)

$$M(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(us) s^{-1-\alpha} q_p^i(s,u) ds\sigma(du), \quad A \in \mathfrak{B}(\mathbb{R}^d),\ i = 1, 2,$$

where $q_p^i(r,u) = q_1^i(r^p, u)$ for $i = 1, 2$ and $r > 0$. By Theorem 16.10 in [Bil95] and the continuity in r of $q_p^i(r,u)$, $i = 1, 2$ this implies that

$$q_p^1(r,u) = q_p^2(r,u), \quad r > 0$$

σ almost everywhere. Since Laplace transforms uniquely determine measures $Q^1(\cdot|u) = Q^2(\cdot|u)$ for σ almost all u. Hence $Q^1 = Q^2$, and by (2.8) $R^1 = R^2$.

Now for Part 3. Let Q be given by (2.9). As we have seen, Q has a polar decomposition of the form $Q(dr, du) = Q(dr|u)\sigma(du)$. We can take the measures $Q(\cdot|u)$ to be probability measures if and only if Q is finite. From here the result follows by (2.10). □

Definition 2.5. *We will refer to the R measure in* (2.11) *as the **Rosiński measure**.*

Definition 2.6. *Fix $\alpha < 2$, $p > 0$. The class of p-tempered α-stable distributions will be denoted by TS^p_α. If $\mu \in TS^p_\alpha$ then $\mu = ID(0_{d\times d}, M, b)$, where M is given by* (2.5). *We will use the notation $TS^p_\alpha(R, b)$ to denote this distribution.*

When we write $\mu = TS^p_\alpha(R, b)$, we will implicitly assume that R satisfies the assumptions in Theorem 2.3. The following proposition will be our basic result relating a p-tempered α-stable distribution to the α-stable distribution that is being tempered.

Proposition 2.7. *Fix $\alpha < 2$ and $p > 0$. Let M be the Lévy measure of a proper p-tempered α-stable distribution with Rosiński measure R. M can be represented by* (2.5) *with*

$$\sigma(B) = \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right)|x|^\alpha R(dx), \quad B \in \mathfrak{B}(\mathbb{S}^{d-1}).$$

If, in addition, $\alpha \in (0, 2)$ and M^α_σ is the Lévy measure of an α-stable distribution with spectral measure σ then

$$M^\alpha_\sigma(B) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(tx)\, t^{-\alpha-1} dt R(dx), \quad B \in \mathfrak{B}(\mathbb{R}^d).$$

Proof. For the first part we have

$$\begin{aligned}\int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right)|x|^\alpha R(dx) &= \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right) Q(dx)\\ &= \int_B\int_0^\infty Q(ds|u)\sigma(du) = \sigma(B).\end{aligned}$$

The second part follows from the first by

$$\begin{aligned}M^\alpha_\sigma(B) &= \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_B(su)s^{-\alpha-1}ds\sigma(du)\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_B(sx/|x|)s^{-1-\alpha}ds|x|^\alpha R(dx)\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_B(tx)t^{-\alpha-1}dtR(dx),\end{aligned}$$

where the third equality follows by the substitution $t = s/|x|$. □

Proposition 2.8. *Fix $\alpha < 2$ and $p > 0$. Let M and R be related by (2.11) and assume that R satisfies (2.13) and (2.14). Assume that $R(\mathbb{R}^d) > 0$. For $q \in (0,2)$*

$$\int_{|x|\le 1} |x|^q M(dx) < \infty \Longleftrightarrow \alpha < q \text{ and } \int_{|x|\le 1} |x|^q R(dx) < \infty. \tag{2.16}$$

The proof of this result is similar to that of Proposition 2.8 in [Ros07].

Proof. First assume that $\int_{|x|\le 1} |x|^q M(dx) < \infty$, and choose $r > 0$ such that $R(|x| \le r) > 0$. We have

$$\begin{aligned} \infty &> \int_{|x|\le 1} |x|^q M(dx) \ge \int_{|x|\le r} |x|^q \int_0^{|x|^{-1}} t^{q-1-\alpha} e^{-t^p} dt R(dx) \\ &\ge e^{-r^{-p}} \int_{|x|\le r} |x|^q R(dx) \int_0^{r^{-1}} t^{q-1-\alpha} dt. \end{aligned}$$

Now for the other direction. If $\alpha < q$ and $\int_{|x|\le 1} |x|^q R(dx) < \infty$, then

$$\begin{aligned} &\int_{|x|\le 1} |x|^q M(dx) = \int_{\mathbb{R}^d} |x|^q \int_0^{|x|^{-1}} t^{q-1-\alpha} e^{-t^p} dt R(dx) \\ &\le \int_{|x|\le 1} |x|^q R(dx) \int_0^\infty t^{q-1-\alpha} e^{-t^p} dt + \int_{|x|>1} |x|^q \int_0^{|x|^{-1}} t^{q-1-\alpha} dt R(dx) \\ &\le \int_{|x|\le 1} |x|^q R(dx) \int_0^\infty t^{q-1-\alpha} e^{-t^p} dt + (q-\alpha)^{-1} \int_{|x|>1} |x|^\alpha R(dx), \end{aligned}$$

which is finite. □

We will now give conditions for when p-tempered α-stable distributions are stable, when they are selfdecomposable, and when they are compound Poisson.

Proposition 2.9. *Fix $\alpha < 2$, $p > 0$, and $\beta \in (0 \vee \alpha, 2)$. Let $\mu = TS^p_\alpha(R, b)$ and let σ be a finite measure on $\mathbb{S}^{d-1}$. Then $\mu = S_\beta(\sigma, b)$ if and only if*

$$R(A) = K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi) \tag{2.17}$$

where $K = \int_0^\infty t^{\beta-\alpha-1} e^{-t^p} dt$.

Note that when $\sigma \neq 0$

$$\int_{\mathbb{R}^d} |x|^\alpha R(dx) = K^{-1}\sigma(\mathbb{S}^{d-1}) \int_0^\infty r^{-(\beta-\alpha)-1} dr = \infty.$$

Thus no stable distributions are proper p-tempered α-stable.

Proof. Note that $R(\{0_d\}) = 0$ and for any $\gamma \in [0, \beta)$

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\gamma\right) R(dx) = K^{-1}\sigma(\mathbb{S}^{d-1}) \int_0^\infty (r^{1-\beta} \wedge r^{\gamma-\beta-1}) dr < \infty.$$

Thus R satisfies the conditions given in Theorem 2.3. If R is of the given form and M is the Lévy measure of μ then

$$\begin{aligned} M(A) &= K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(rt\xi) t^{-1-\alpha} e^{-t^p} dt r^{-1-\beta} dr \sigma(d\xi) \\ &= K^{-1} \int_0^\infty t^{\beta-\alpha-1} e^{-t^p} dt \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi). \end{aligned}$$

The converse follows by the fact that R uniquely determines the Lévy measure of a p-tempered α-stable distribution. □

To characterize when tempered stable distribution are selfdecomposable, we first recall the following well known result (see e.g. Theorem 15.10 in [Sat99]).

Lemma 2.10. *A probability measure μ is selfdecomposable if and only if $\mu = ID(A, M, b)$ and we can write*

$$M(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(r\eta) k_\eta(r) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\eta), \quad B \in \mathfrak{B}(\mathbb{R}^d), \tag{2.18}$$

where σ is a finite Borel measure on $\mathbb{S}^{d-1}$ and $k_\eta(r)$ is a nonnegative function measurable in $\eta \in \mathbb{S}^{d-1}$, decreasing in $r > 0$, and with $\lim_{r\to\infty} k_\eta(r) = 0$.

Proposition 2.11. *Fix $p > 0$.*

1. For $\alpha \in [0, 2)$ the class TS^p_α is contained in the class of selfdecomposable distributions.

2. For $\alpha < 0$ the class TS^p_α contains distributions that are not selfdecomposable.

Proof. The first part follows immediately from Lemma 2.10 and (2.5). For the second part, note that by (2.5), for any $\alpha < 0$, $p > 0$ and finite Borel measure σ on $\mathbb{S}^{d-1}$ there is a measure $\mu \in TS^p_\alpha$ with a Lévy measure given by

$$M(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ut) t^{-1-\alpha} e^{-t^p} dt \sigma(du), \ A \in \mathfrak{B}(\mathbb{R}^d).$$

By Lemma 2.10 for μ to be a selfdecomposable distribution it is necessary that the function $f(t) = e^{-t^p} t^{-\alpha}$ is decreasing. However, using basic calculus, it is easy to see that the function is actually increasing when $t < (|\alpha|/p)^{1/p}$. □

For $\alpha \in [0, 2)$ p-tempered α-stable distributions inherit several important properties of selfdecomposable distributions. By Theorem 27.13 in [Sat99], any non-degenerate selfdecomposable distribution is absolutely continuous with respect to Lebesgue measure. Moreover, by Theorem 53.1 in [Sat99], when $d = 1$, they are unimodal.

Recall that a probability measure μ is called compound Poisson if its characteristic function can be written as

$$\hat{\mu}(z) = \exp\left\{ \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 \right) M(dx) \right\}, \tag{2.19}$$

where M is a finite Lévy measure. To classify when tempered stable distributions are compound Poisson we begin with a lemma.

Lemma 2.12. *Let M be given by (2.11). M is finite if and only if either $R = 0$ or $\alpha < 0$ and R is a finite measure.*

Proof. Observing that

$$\begin{aligned} R(\mathbb{R}^d) e^{-1} \int_0^1 t^{-1-\alpha} dt &\leq \int_{\mathbb{R}^d} \int_0^\infty e^{-t^p} t^{-1-\alpha} dt R(dx) \\ &\leq R(\mathbb{R}^d) \left(\int_0^1 t^{-1-\alpha} dt + \int_1^\infty e^{-t^p} t^{-1-\alpha} dt \right) \end{aligned}$$

gives the result. □

This implies the following.

Proposition 2.13. *If $\mu = TS^p_\alpha(R,b)$, then μ is compound Poisson if and only if either $R(\mathbb{R}^d)=0$ or $\alpha<0$, R is a finite measure, and $b=\int_{\mathbb{R}^d}\frac{x}{1+|x|^2}R(dx)$.*

2.1 Identifiability

Following the idea of Corollaries 2.5 and 2.6 in [Ros07], we will show that for a fixed $p>0$, in the subclass of proper p-tempered α-stable distribution, the parametrization (α,R) of the Lévy measure is identifiable. We will then show that within the class of all p-tempered α-stable distribution, the parametrization (p,α,R) is not identifiable. Finally, we will categorize the relationship between classes with different values of p and α.

Proposition 2.14. *Fix $\alpha<2$, $p>0$, and let M be given by* (2.11).

i) The map $s\mapsto s^\alpha M(|x|>s)$ is decreasing and $\lim_{s\to\infty}s^\alpha M(|x|>s)=0$.

ii) If $\alpha\in(0,2)$ then

$$\lim_{s\downarrow 0}s^\alpha M(|x|>s)=\frac{1}{\alpha}\int_{\mathbb{R}^d}|x|^\alpha R(\mathrm{d}x)$$

and if $\alpha\le 0$ then

$$\lim_{s\downarrow 0}s^\alpha M(|x|>s)=\infty.$$

iii) If $\alpha<0$ then

$$\lim_{s\downarrow 0}s^\alpha M(|x|<s)=\frac{1}{|\alpha|}\int_{\mathbb{R}^d}|x|^\alpha R(dx)$$

and if $\alpha\in[0,2)$ then for all $s>0$

$$M(|x|<s)=\infty.$$

Note that for a fixed $p>0$, this implies that in the class of proper p-tempered α-stable distributions both $\lim_{s\downarrow 0}s^\alpha M(|x|>s)=\infty$ and $M(|x|<s)=\infty$ if and only if $\alpha=0$.

Proof. We begin with the first two parts. We have

$$\begin{aligned} s^\alpha M(|x|>s) &= s^\alpha \int_{\mathbb{R}^d}\int_{s|x|^{-1}}^\infty t^{-1-\alpha}e^{-t^p}dtR(dx) \\ &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty u^{-1-\alpha}e^{-(su)^p}duR(dx). \end{aligned}$$

Thus the map $s \mapsto s^\alpha M(|x|>s)$ is decreasing. For large enough s, the integrand in the above is bounded by $u^{-1-\alpha}e^{-u^p}$, which is integrable since

$$\int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty u^{-1-\alpha}e^{-u^p}duR(dx) = \int_{|x|>1} M(dx) < \infty,$$

thus by dominated convergence $\lim_{s\to\infty} s^\alpha M(|x|>s) = 0$.

By monotone convergence we have

$$\begin{aligned} \lim_{s\downarrow 0} s^\alpha M(|x|>s) &= \lim_{s\downarrow 0}\int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty u^{-1-\alpha}e^{-(su)^p}duR(dx) \\ &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty u^{-1-\alpha}duR(dx). \end{aligned}$$

Thus if $\alpha \in (0,2)$ then

$$\lim_{s\downarrow 0} s^\alpha M(|x|>s) = \frac{1}{\alpha}\int_{\mathbb{R}^d}|x|^\alpha R(dx)$$

and if $\alpha \le 0$ then

$$\lim_{s\downarrow 0} s^\alpha M(|x|>s) = \infty.$$

Now for the third part. If $\alpha \in [0,2)$ then for all $s>0$

$$\begin{aligned} M(|x|<s) &= \int_{\mathbb{R}^d}\int_0^{s|x|^{-1}} t^{-1-\alpha}e^{-t^p}dtR(dx) \\ &\ge \int_{\mathbb{R}^d} e^{-(s/|x|)^p}\int_0^{s|x|^{-1}} t^{-1-\alpha}dtR(dx) = \infty. \end{aligned}$$

If $\alpha < 0$ then for all $s>0$

$$\begin{aligned} s^\alpha M(|x|<s) &= s^\alpha\int_{\mathbb{R}^d}\int_0^{s|x|^{-1}} t^{-1-\alpha}e^{-t^p}dtR(dx) \\ &= \int_{\mathbb{R}^d}\int_0^{|x|^{-1}} u^{-1-\alpha}e^{-(su)^p}duR(dx). \end{aligned}$$

Thus by monotone convergence

$$\begin{aligned}
\lim_{s\downarrow 0} s^\alpha M(|x|>s) &= \lim_{s\downarrow 0}\int_{\mathbb{R}^d}\int_0^{|x|^{-1}} u^{-1-\alpha}e^{-(su)^p}duR(dx)\\
&= \int_{\mathbb{R}^d}\int_0^{|x|^{-1}} u^{-1-\alpha}duR(dx)\\
&= \frac{1}{|\alpha|}\int_{\mathbb{R}^d}|x|^\alpha R(dx).
\end{aligned}$$

This completes the proof. □

By Theorem 2.3 we know that R is uniquely determined by M. Combining this fact with the previous result gives the following.

Proposition 2.15. *Fix $p>0$. In the subclass of proper p-tempered stable distributions, the parametrization (R,α) is identifiable.*

However, in general, the parameters α and p are not identifiable. This will become apparent from the following results.

Proposition 2.16. *Fix $\alpha<2$, $\beta\in(\alpha,2)$, and let $K=\int_0^\infty s^{\beta-\alpha-1}e^{-s^p}ds$. If $\mu=TS^p_\beta(R,b)$ and*

$$R'(A)=K^{-1}\int_{\mathbb{R}^d}\int_0^1 1_A(ux)u^{-\beta-1}\left(1-u^p\right)^{(\beta-\alpha)/p-1}duR(dx) \tag{2.20}$$

then R' is the Rosiński measure of a p-tempered α-stable distribution and $\mu=TS^p_\alpha(R',b)$.

Proof. First we will show that R' is the Rosiński measure of some p-tempered α-stable distribution. Let $C=\max_{u\in[0,.5]}\left(1-u^p\right)^{(\beta-\alpha)/p-1}$. Observe that

$$\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}du<\infty.$$

We have

$$
\begin{aligned}
K\int_{|x|\le 1}|x|^2R'(dx) &= \int_{\mathbb{R}^d}|x|^2\int_0^1 1_{|x|u\le 1}u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&= \int_{|x|\le 2}|x|^2\int_0^1 1_{|x|u\le 1}u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&\quad +\int_{|x|>2}|x|^2\int_0^{|x|^{-1}} u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&\le \int_{|x|\le 2}|x|^2R(dx)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}du\\
&\quad +C\int_{|x|>2}|x|^2\int_0^{|x|^{-1}} u^{1-\beta}duR(dx)\\
&= \int_{|x|\le 2}|x|^2R(dx)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}du\\
&\quad +\frac{C}{2-\beta}\int_{|x|\ge 2}|x|^\beta R(dx)<\infty.
\end{aligned}
$$

When $\alpha\in(0,2)$ we have

$$
\begin{aligned}
K\int_{|x|>1}|x|^\alpha R'(dx) &= \int_{|x|>1}|x|^\alpha\int_{|x|^{-1}}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&= \int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^{1/2} u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&\quad +\int_{|x|\ge 2}|x|^\alpha\int_{1/2}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&\quad +\int_{2>|x|>1}|x|^\alpha\int_{|x|^{-1}}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}duR(dx)\\
&\le C\int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^\infty u^{\alpha-1-\beta}duR(dx)\\
&\quad +\int_{|x|>1}|x|^\beta R(dx)\int_{1/2}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}du\\
&= \frac{C}{\beta-\alpha}\int_{|x|\ge 2}|x|^\beta R(dx)\\
&\quad +\int_{|x|>1}|x|^\beta R(dx)\int_{1/2}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}du<\infty.
\end{aligned}
$$

For $\delta>0$ let C_δ be a constant such that for all $u>0$, $\log(u)\le C_\delta u^\delta$ (exists by

4.1.37 in [AS72]). Now if $\alpha = 0$ and $\epsilon \in (0, \beta)$ we have

$$
\begin{aligned}
K \int_{|x|>1} \log|x| R'(dx) &\le K C_\epsilon \int_{|x|>1} |x|^\epsilon R'(dx) \\
&= C_\epsilon \int_{|x|>1} |x|^\epsilon \int_{|x|^{-1}}^1 u^{\epsilon-1-\beta}(1-u^p)^{\beta/p-1} du R(dx) \\
&\le C_\epsilon \int_{|x|>1} |x|^\epsilon \int_{|x|^{-1}}^1 u^{\epsilon-1-\beta}(1-u^p)^{(\beta-\epsilon)/p-1} du R(dx) \\
&= C_\epsilon \frac{C}{\beta-\epsilon} \int_{|x|\ge 2} |x|^\beta R(dx) \\
&\quad + C_\epsilon \int_{|x|>1} |x|^\beta R(dx) \int_{1/2}^1 u^{\epsilon-1-\beta}(1-u^p)^{(\beta-\epsilon)/p-1} du < \infty,
\end{aligned}
$$

where the last line follows as in the previous case.

When $\alpha < 0$ we have

$$
\begin{aligned}
K \int_{|x|>1} R'(dx) &= \int_{|x|>1} \int_{|x|^{-1}}^1 u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\
&\le \int_{|x|>2} \int_{|x|^{-1}}^{1/2} u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\
&\quad + \int_{|x|>1} R(dx) \int_{1/2}^1 u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1} du \\
&\le C \int_{|x|>2} \int_{|x|^{-1}}^1 u^{-1-\beta} du R(dx) \\
&\quad + \int_{|x|>1} R(dx) \int_{1/2}^1 u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1} du.
\end{aligned}
$$

Here the second integral is finite and the first equals

$$
\frac{C}{\beta} \int_{|x|>2} \left(|x|^\beta - 1\right) R(dx),
$$

when $\beta \neq 0$ and

$$
\int_{|x|>2} \log|x| R(dx),
$$

when $\beta = 0$. Thus the integral is finite.

Now to show the result. Let

$$\begin{aligned}
M(A) &= \int_{\mathbb{R}^d}\int_0^\infty 1_A(tx)t^{-1-\alpha}e^{-t^p}dtR'(dx) \\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^1 1_A(utx)t^{-1-\alpha}e^{-t^p}u^{-\beta-1}\left(1-u^p\right)^{(\beta-\alpha)/p-1}dudtR(dx) \\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^t 1_A(vx)t^{\beta-\alpha-1}e^{-t^p}v^{-\beta-1}\left(1-v^p/t^p\right)^{(\beta-\alpha)/p-1}dvdtR(dx) \\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_v^\infty 1_A(vx)t^{p-1}e^{-t^p}v^{-\beta-1}\left(t^p-v^p\right)^{(\beta-\alpha)/p-1}dtdvR(dx) \\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)e^{-v^p}v^{-\beta-1}dvR(dx)\int_0^\infty e^{-s^p}s^{\beta-\alpha-1}ds \\
&= \int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)e^{-v^p}v^{-\beta-1}dvR(dx),
\end{aligned}$$

where the third line follows by the substitution $v = ut$ and the fifth by the substitution $s^p = t^p - v^p$. □

To prove a similar result for the parameter p, we need to set up some notation. For $r \in (0,1)$, let f_r be the density of the r-stable distribution such that

$$\int_0^\infty e^{-tx}f_r(x)dx = e^{-t^r}. \tag{2.21}$$

Thus for $0 < p < q < \infty$

$$\int_0^\infty e^{-t^q x}f_{p/q}(x)dx = e^{-t^p}. \tag{2.22}$$

For all $r \in (0,1)$, such a density exists by Proposition 1.2.12 in [ST94]. However, the only case where an explicit formula is known is

$$f_{.5}(s) = (2\sqrt{\pi})^{-1}e^{-1/(4s)}s^{-3/2}1_{[s>0]} \tag{2.23}$$

(see 29.3.82 in [AS72]). By Theorem 5.4.1 in [UZ99]

$$f_r(x) \sim ax^{(r-2)/(2-2r)}\exp\{-bx^{-r/(1-r)}\} \text{ as } x \to 0 \tag{2.24}$$

where $a = (2\pi(1-r))^{-1/2}r^{1/(2-2r)}$ and $b = (1-r)r^{r/(1-r)}$. This implies that if $X \sim f_r$ and $\beta \geq 0$ then

$$E|X|^{-\beta} < \infty. \tag{2.25}$$

Proposition 2.17. *Fix $\alpha < 2$, $0 < p < q$. If $\mu = TS^p_\alpha(R, b)$ and*

$$R'(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(s^{-1/q}x)s^{\alpha/q} f_{p/q}(s)dsR(dx) \tag{2.26}$$

then R' is the Rosiński measure of a q-tempered α-stable distribution and $\mu = TS^q_\alpha(R', b)$. Moreover, μ is a proper p-tempered α-stable distribution if and only if it is a proper q-tempered α-stable distribution.

Proof. First we will show that R' is, in fact, the Rosiński measure of a q-tempered α-stable distribution. We have

$$\begin{aligned} \int_{|x|\le 1} |x|^2 R'(dx) &= \int_{\mathbb{R}^d} |x|^2 \int_{|x|^q}^\infty s^{-(2-\alpha)/q} f_{p/q}(s)dsR(dx) \\ &\le \int_{|x|\le 1} |x|^2 R(dx) \int_0^\infty s^{-(2-\alpha)/q} f_{p/q}(s)ds \\ &\quad + \int_{|x|>1} |x|^\alpha R(dx) \int_0^\infty f_{p/q}(s)ds < \infty. \end{aligned}$$

When $\alpha \in (0, 2)$

$$\begin{aligned} \int_{|x|>1} |x|^\alpha R'(dx) &= \int_{\mathbb{R}^d} |x|^\alpha \int_0^{|x|^q} f_{p/q}(s)dsR(dx) \\ &\le \int_{|x|\ge 1} |x|^\alpha R(dx) \int_0^\infty f_{p/q}(s)ds \\ &\quad + \int_{|x|<1} |x|^2 R(dx) \int_0^\infty s^{-(2-\alpha)/q} f_{p/q}(s)ds < \infty. \end{aligned}$$

When $\alpha = 0$

$$\begin{aligned} \int_{|x|>1} \log|x| R'(dx) &= \int_{\mathbb{R}^d} \int_0^{|x|^q} \log|xs^{-1/q}| f_{p/q}(s)dsR(dx) \\ &\le \int_{|x|\le 1} |x| \int_0^{|x|^q} s^{-1/q} f_{p/q}(s)dsR(dx) \\ &\quad + \int_{|x|>1} \int_0^{|x|^q} \left(\log|x| + s^{-1/q}\right) f_{p/q}(s)dsR(dx) \\ &\le \int_{|x|\le 1} |x|^2 R(dx) \int_0^\infty s^{-2/q} f_{p/q}(s)ds \\ &\quad + \int_{|x|>1} \log|x| R(dx) \int_0^\infty f_{p/q}(s)ds \\ &\quad + \int_{|x|>1} R(dx) \int_0^\infty s^{-1/q} f_{p/q}(s)ds < \infty, \end{aligned}$$

where the second line follows by 4.1.33 in [AS72]. When $\alpha < 0$

$$\begin{aligned}
\int_{|x|>1} R'(dx) &= \int_{\mathbb{R}^d} \int_0^{|x|^q} s^{\alpha/q} f_{p/q}(s) ds R(dx) \\
&= \int_{|x|<1} \int_0^{|x|^q} s^{-|\alpha|/q} f_{p/q}(s) ds R(dx) \\
&\quad + \int_{|x|\geq 1} \int_0^{|x|^q} s^{-|\alpha|/q} f_{p/q}(s) ds R(dx) \\
&\leq \int_{|x|<1} |x|^2 R(dx) \int_0^{\infty} s^{-(|\alpha|+2)/q} f_{p/q}(s) ds \\
&\quad + \int_{|x|\geq 1} R(dx) \int_0^{\infty} s^{-|\alpha|/q} f_{p/q}(s) ds < \infty.
\end{aligned}$$

Let M' be the Lévy measure of $TS^q_\alpha(R', b)$. We have

$$\begin{aligned}
M'(A) &= \int_{\mathbb{R}^d} \int_0^{\infty} 1_A(tx) t^{-1-\alpha} e^{-t^q} dt R'(dx) \\
&= \int_{\mathbb{R}^d} \int_0^{\infty} \int_0^{\infty} 1_A(s^{-1/q} tx) t^{-1-\alpha} e^{-t^q} dt s^{\alpha/q} f_{p/q}(s) ds R(dx) \\
&= \int_{\mathbb{R}^d} \int_0^{\infty} 1_A(vx) v^{-1-\alpha} \int_0^{\infty} e^{-v^q s} f_{p/q}(s) ds dv R(dx) \\
&= \int_{\mathbb{R}^d} \int_0^{\infty} 1_A(vx) v^{-1-\alpha} e^{-v^p} dv R(dx),
\end{aligned}$$

where the third line follows by the substitution $v = s^{-1/q} t$.

Now observe that

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^\alpha R'(dx) &= \int_{\mathbb{R}^d} |x|^\alpha \int_0^{\infty} s^{-\alpha/q} s^{\alpha/q} f_{p/q}(s) ds R(dx) \\
&= \int_{\mathbb{R}^d} |x|^\alpha R(dx).
\end{aligned}$$

Thus, $\int |x|^\alpha R(dx) < \infty$ if and only if $\int |x|^\alpha R'(dx) < \infty$ and the last statement of the proposition holds by Part 3 of Theorem 2.3. □

Corollary 2.18. *Fix $\alpha < 2$ and $p > 0$. Let $\mu \in TS^p_\alpha$.*

1. *For any $q \geq p$, $\mu \in TS^q_\alpha$.*
2. *For any $\beta \leq \alpha$, $\mu \in TS^p_\beta$.*

2.2 Moments

In this section we will give necessary and sufficient conditions for the finiteness of various moments. We will then give explicit formulas for the nth cumulants when they exist. For simplicity, throughout this section, we will use the notation M to refer to the Lévy measure of a p-tempered α-stable distribution.

Theorem 2.19. *Fix $\alpha \in (0,2)$ and $p > 0$. If $\mu = TS^p_\alpha(R,b)$ then*

(i) $\int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty$ when $0 \le q < \alpha$;

(ii) $\int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty \iff \int_{|x|>1} |x|^q R(dx) < \infty$ when $q > \alpha$;

(iii) $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \iff \int_{|x|>1} |x|^\alpha \log|x| R(dx) < \infty$;

(iv) if $p \in (0,1]$ and $\theta > 0$ then

$$\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty \iff R(\{|x| > \theta^{-1/p}\}) = 0.$$

This result and its proof are based on Proposition 2.7 in [Ros07].

Proof. By Proposition C.4, the moment conditions on μ are equivalent to the moment conditions on M restricted to the set $\{|x| > 1\}$. Using (2.12) we have

$$\begin{aligned}\int_{|x|>1} |x|^q M(dx) &= p^{-1} \int_{|x|\le 1} |x|^q \int_{|x|^{-p}}^\infty t^{(q-\alpha)/p-1} e^{-t} dt R(dx) \\ &\quad + p^{-1} \int_{|x|>1} |x|^q \int_{|x|^{-p}}^\infty t^{(q-\alpha)/p-1} e^{-t} dt R(dx) =: p^{-1}(I_1 + I_2).\end{aligned}$$

Let $C := \sup_{t\ge 1} t^{(2+q-\alpha)/p} e^{-t}$.

$$\begin{aligned} I_1 &\le C \int_{|x|\le 1} |x|^q \int_{|x|^{-p}}^\infty t^{-2/p-1} dt R(dx) = \frac{p}{2} C \int_{|x|\le 1} |x|^{q+2} R(dx) \\ &\le pC \int_{|x|\le 1} |x|^2 R(dx) < \infty. \end{aligned}$$

So the finiteness of $\int_{|x|>1} |x|^q M(dx)$ is decided by I_2.

If $q < \alpha$ then

$$I_2 \le \int_{|x|>1} |x|^q \int_{|x|^{-p}}^\infty t^{(q-\alpha)/p-1} dt R(dx) = \frac{p}{\alpha - q} \int_{|x|>1} |x|^\alpha R(dx) < \infty,$$

this proves (i).

If $q > \alpha$, then (ii) follows by

$$\left(\int_1^\infty t^{(q-\alpha)/p-1}e^{-t}dt\right)\int_{|x|>1}|x|^q R(dx) \leq I_2 \leq \Gamma\left(\frac{q-\alpha}{p}\right)\int_{|x|>1}|x|^q R(dx).$$

To show (iii) assume $q = \alpha$,

$$\begin{aligned} I_2 &\leq \int_{|x|>1}|x|^\alpha \int_{|x|^{-p}}^1 t^{-1}dtR(dx) + \int_{|x|>1}|x|^\alpha\int_1^\infty e^{-t}dtR(dx) \\ &= \int_{|x|>1}|x|^\alpha\left(p\log|x| + e^{-1}\right)R(dx) \end{aligned} \tag{2.27}$$

and

$$I_2 \geq \int_{|x|>1}|x|^\alpha\int_{|x|^{-p}}^1 t^{-1}e^{-1}dtR(dx) = pe^{-1}\int_{|x|>1}|x|^\alpha \log|x| R(dx).$$

Now to show (iv). Assume $R(\{|x| > \theta^{-1/p}\}) = 0$. By (2.11) we have

$$\begin{aligned} \int_{|x|>1} e^{\theta|x|^p}M(dx) &= \int_{|x|\leq\theta^{-1/p}}\int_{|x|^{-1}}^\infty e^{(\theta|x|^p-1)t^p}t^{-1-\alpha}dtR(dx) \\ &\leq \int_{|x|<(2\theta)^{-1/p}}\int_{|x|^{-1}}^\infty e^{-t^p/2}t^{2-\alpha}t^{-3}dtR(dx) \\ &\quad + \int_{(2\theta)^{-1/p}\leq|x|\leq\theta^{-1/p}}\int_{|x|^{-1}}^\infty t^{-1-\alpha}dtR(dx) \\ &\leq C\int_{|x|<(2\theta)^{-1/p}}\int_{|x|^{-1}}^\infty t^{-3}dtR(dx) + \frac{1}{\alpha}\int_{|x|\geq(2\theta)^{-1/p}}|x|^\alpha R(dx) \\ &\leq \frac{1}{2}C\int_{|x|<(2\theta)^{-1/p}}|x|^2R(dx) + \frac{1}{\alpha}\int_{|x|\geq(2\theta)^{-1/p}}|x|^\alpha R(dx) < \infty, \end{aligned}$$

where $C := \sup_{t\geq(2\theta)^{1/p}} t^{2-\alpha}e^{-t^p/2}$.

Conversely, if $R(\{|x| > \theta^{-1/p}\}) > 0$ then there is some $\epsilon > 0$ such that $R(\{|x|^p > \theta^{-1} + \epsilon\}) > 0$. So

$$\begin{aligned} \int_{|x|>1} e^{\theta|x|^p}M(dx) &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty e^{(\theta|x|^p-1)t^p}t^{-1-\alpha}dtR(dx) \\ &\geq \int_{|x|^p>\theta^{-1}+\epsilon}\int_{|x|^{-1}}^\infty e^{\theta\epsilon t^p}t^{-1-\alpha}dtR(dx) = \infty, \end{aligned}$$

as required. □

We will now consider the case when $\alpha \le 0$.

Theorem 2.20. *Fix $\alpha \le 0$ and $p > 0$. If $\mu = TS^p_\alpha(R, b)$ then*
(i) if $q \ge 0$, then $\int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty \iff \int_{|x|>1} |x|^q R(dx) < \infty$;
(ii) if $p \in (0, 1]$, $\alpha < 0$ and $\theta > 0$, then $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty \iff R(\{|x| \ge \theta^{-1/p}\}) = 0$ and $\int_{0<|x|^{-p}-\theta<1} (|x|^{-p} - \theta)^{\alpha/p} |x|^\alpha R(dx) < \infty$;
(iii) if $p \in (0, 1]$, $\alpha = 0$, and $\theta > 0$ then $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty \iff R(\{|x| \ge \theta^{-1/p}\}) = 0$ and $\int_{0<|x|^{-p}-\theta<1} |\log(1 - |x|^p\theta)| R(dx) < \infty$.

Note that in (ii) and (iii) we have the condition, $R(\{|x| \ge \theta^{-1/p}\}) = 0$, whereas in Theorem 2.19 part (iv) we have a similar condition, but with strict inequality. Note also that the set $\{0 < |x|^{-p} - \theta < 1\} = \{(1+\theta)^{-1/p} < |x| < \theta^{-1/p}\}$. The latter form is somewhat more appealing, but it loses the emphasis on why the integrals may diverge.

Proof. The proof of (i) is similar to the proof of Theorem 2.19 (ii). Let $\alpha \le 0$, we have

$$\begin{aligned} \int_{|x|>1} e^{\theta|x|^p} M(dx) &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty e^{(\theta|x|^p - 1)t^p} t^{-\alpha-1} dt R(dx) \\ &\ge \int_{|x|^p \ge \theta^{-1}} R(dx) \int_{\theta^{1/p}}^\infty t^{-\alpha-1} dt. \end{aligned}$$

This shows the necessity of $R(\{|x| \ge \theta^{-1/p}\}) = 0$ in both Parts (ii) and (iii). We will henceforth assume that this property holds both when showing necessity and sufficiency. We have

$$\begin{aligned} \int_{|x|>1} e^{\theta|x|^p} M(dx) &= \int_{|x|<\theta^{-1/p}} \int_{|x|^{-1}}^\infty e^{(\theta|x|^p - 1)t^p} t^{-1-\alpha} dt R(dx) \\ &= p^{-1} \int_{0<|x|^{-p}-\theta} (1 - \theta|x|^p)^{\alpha/p} \int_{|x|^{-p}-\theta}^\infty e^{-u} u^{-1-\alpha/p} du R(dx). \end{aligned}$$

This can be divided into two parts

$$p^{-1}\int_{1\le|x|^{-p}-\theta}(1-\theta|x|^p)^{\alpha/p}\int_{|x|^{-p}-\theta}^{\infty}e^{-u}u^{-1-\alpha/p}duR(dx)$$
$$+p^{-1}\int_{0<|x|^{-p}-\theta<1}(1-\theta|x|^p)^{\alpha/p}\int_{|x|^{-p}-\theta}^{\infty}e^{-u}u^{-1-\alpha/p}duR(dx)=:p^{-1}(I_1+I_2).$$

Let $C_1:=\sup_{u>1}e^{-u}u^{-1-\alpha/p}(u+\theta)^{(2-\alpha)/p+1}$. We have,

$$\begin{aligned}I_1 &\le \int_{1\le|x|^{-p}-\theta}|x|^{\alpha}\int_{|x|^{-p}-\theta}^{\infty}e^{-u}u^{-1-\alpha/p}duR(dx)\\ &\le C_1\int_{1\le|x|^{-p}-\theta}|x|^{\alpha}\int_{|x|^{-p}-\theta}^{\infty}(u+\theta)^{(\alpha-2)/p-1}duR(dx)\\ &= C_1\frac{p}{2-\alpha}\int_{|x|\le(1+\theta)^{-1/p}}|x|^2R(dx)<\infty.\end{aligned}$$

Thus finiteness is determined by I_2.

If $\alpha<0$ and $0<|x|^{-p}-\theta<1$, we have

$$\int_1^{\infty}e^{-u}u^{-1-\alpha/p}du\le\int_{|x|^{-p}-\theta}^{\infty}e^{-u}u^{-1-\alpha/p}du\le\Gamma(-\alpha/p).$$

Thus I_2 is finite if and only if

$$\int_{0<|x|^{-p}-\theta<1}(|x|^{-p}-\theta)^{\alpha/p}|x|^{\alpha}R(dx)=\int_{0<|x|^{-p}-\theta<1}(1-\theta|x|^p)^{\alpha/p}R(dx)<\infty.$$

If $\alpha=0$ then for $0<|x|^{-p}-\theta<1$, we have

$$\int_{|x|^{-p}-\theta}^{\infty}e^{-u}u^{-1}du = \int_1^{\infty}e^{-u}u^{-1}du+\int_{|x|^{-p}-\theta}^{1}e^{-u}u^{-1}du,$$

where the first integral is finite. For the second one, we have

$$\int_{|x|^{-p}-\theta}^{1}e^{-u}u^{-1}du \le \int_{|x|^{-p}-\theta}^{1}u^{-1}du=-\log(|x|^{-p}-\theta),$$

and

$$\int_{|x|^{-p}-\theta}^{1}e^{-u}u^{-1}du \ge e^{-1}\int_{|x|^{-p}-\theta}^{1}u^{-1}du=-e^{-1}\log(|x|^{-p}-\theta).$$

Thus, in this case, the finiteness of I_2 is equivalent to the finiteness of

$$-\int_{0<|x|^{-p}-\theta<1}\log(|x|^{-p}-\theta)R(dx)=\int_{0<|x|^{-p}-\theta<1}\left|\log(|x|^{-p}-\theta)\right|R(dx).$$

This completes the proof. □

Theorem 2.21. *Fix $\alpha < 2$ and $p > 0$. If $\mu = TS_\alpha^p(R, b)$ then*

(i) If $0 < q < p$, and $q \le 1$ then for any $\theta > 0$

$$\int_{\mathbb{R}^d} e^{\theta|x|^q}\mu(dx) < \infty \Longleftrightarrow \int_{|x|>1}\int_{|x|^{-1}}^{(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) < \infty. \quad (2.28)$$

A sufficient condition for the finiteness of $\int_{\mathbb{R}^d} e^{\theta|x|^q}\mu(dx)$ is

$$\int_{|x|>1} e^{(\theta|x|^q)^{p/(p-q)}} |x|^{-(1+\alpha)q/(p-q)} R(dx) < \infty. \quad (2.29)$$

(ii) If $R \neq 0$ then $\int_{\mathbb{R}^d} e^{\theta|x|\log|x|}\mu(dx) = \infty$ for every $\theta > 0$.

For the case where $\alpha \in (0, 2)$, $p = 2$, and $q = 1$, the sufficient condition in (2.29) is given in [BRKF11].

Note that the condition in (2.29) immediately implies the following. Assume that $0 < q < p$ and $q \le 1$. If $R(\{|x| > a\}) = 0$ for some $a \in [0, \infty)$ then $\int_{\mathbb{R}^d} e^{\theta|x|^q}\mu(dx) < \infty$ for every $\theta \in \mathbb{R}$. However by Theorem 2.19 and Theorem 2.20, for $R \neq 0$ it is impossible for $\int_{\mathbb{R}^d} e^{\theta|x|^p}\mu(dx)$ to be finite for every $\theta \in \mathbb{R}$. In particular, this means that for $\int_{\mathbb{R}^d} e^{\theta|x|}\mu(dx)$ to be finite for every $\theta \in \mathbb{R}$ it is necessary to have $p > 1$.

Proof. We begin with Part (i). By Proposition C.4, the problem is equivalent to the finiteness of

$$\begin{aligned}
\int_{|x|>1} e^{\theta|x|^q} M(dx) &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\
&= \int_{|x|\le 1}\int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\
&\quad + \int_{|x|>1}\int_{|x|^{-1}}^{(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\
&\quad + \int_{|x|>1}\int_{(\theta|x|^q)^{1/(p-q)}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) =: I_1 + I_2 + I_3.
\end{aligned}$$

For the first integral, we have

$$\begin{aligned} I_1 &= \int_{|x|\le 1}\int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{1-\alpha} t^{-2} dt R(dx) \\ &\le \int_{|x|\le 1} |x|^2 \int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{1-\alpha} dt R(dx) \\ &\le \int_{|x|\le 1} |x|^2 R(dx) \int_1^{\infty} e^{(\theta t^{q-p}-1)t^p} t^{1-\alpha} dt, \end{aligned}$$

which is finite since $q < p$. For the third integral we have

$$\begin{aligned} I_3 &= \int_{|x|>1}\int_{(\theta|x|^q)^{1/(p-q)}}^{\infty} e^{\left(\theta|x|^q t^{q-p}-1\right)t^p} t^{-1-\alpha} dt R(dx) \\ &= \frac{1}{(p-q)}\int_{|x|>1} (\theta|x|^q)^{-\alpha/(p-q)} \int_1^{\infty} e^{-(1-1/u)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du R(dx), \end{aligned}$$

where the second line follows by the substitution $u = t^{p-q}/(\theta|x|^q)$. If $\alpha \in [0,2)$ then this implies that

$$I_3 \le \frac{\theta^{-\alpha/(p-q)}}{(p-q)} \int_{|x|>1} R(dx) \int_1^{\infty} e^{-(1-1/u)(u\theta)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du < \infty.$$

If $\alpha < 0$ let $C = \sup_{s\ge 0} e^{-s} s^{(1-\alpha)/p} < \infty$. We have

$$\begin{aligned} I_3 &\le \frac{C}{(p-q)} \int_{|x|>1} (\theta|x|^q)^{-\alpha/(p-q)} \times \\ &\quad \times \int_1^{\infty} \left[(1-1/u)(u\theta|x|^q)^{p/(p-q)}\right]^{(\alpha-1)/p} u^{-1-\alpha/(p-q)} du R(dx) \\ &\le \frac{C}{(p-q)} \int_{|x|>1} (\theta|x|^q)^{-\alpha/(p-q)} (\theta|x|^q)^{(\alpha-1)/(p-q)} R(dx) \int_1^{\infty} u^{-1-1/(p-q)} du \\ &= C \int_{|x|>1} (\theta|x|^q)^{-1/(p-q)} R(dx) \le C\theta^{-1/(p-q)} \int_{|x|>1} R(dx) < \infty. \end{aligned}$$

Thus everything is determined by I_2. For the sufficient condition we need to bound I_2. We have

$$\begin{aligned} &\int_{|x|^{-1}}^{.5(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{2-\alpha} t^{-3} dt \\ &\qquad \le e^{(.5)^q (\theta|x|^q)^{p/(p-q)}} (.5)^{2-\alpha} (\theta|x|^q)^{(2-\alpha)/(p-q)} \int_{|x|^{-1}}^{\infty} t^{-3} dt \\ &\qquad = \frac{\theta^{(2-\alpha)/(p-q)}}{2^{3-\alpha}} e^{(.5)^q (\theta|x|^q)^{p/(p-q)}} |x|^{q(2-\alpha)/(p-q)+2}, \end{aligned}$$

and

$$\int_{.5(\theta|x|^q)^{1/(p-q)}}^{(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt$$
$$\leq (.5)^{-(|\alpha|+1)} e^{(\theta|x|^q)^{p/(p-q)}} (\theta|x|^q)^{-(1+\alpha)/(p-q)} \int_0^\infty e^{-t^p} dt.$$

Since for large enough $|x|$

$$e^{(.5)^q(\theta|x|^q)^{p/(p-q)}} |x|^{q(2-\alpha)/(p-q)+2} \leq e^{(\theta|x|^q)^{p/(p-q)}} |x|^{-(1+\alpha)q/(p-q)}$$

the result follows.

Now for Part (ii). For any $b > 0$, let $T_b = \{|x| > b\}$. Since $R \neq 0$ and $R(\{0\}) = 0$, there exists an $\epsilon > 0$ such that $R(T_\epsilon) > 0$ hence for any $b > 0$

$$\begin{aligned} M(T_b) &= \int_{\mathbb{R}^d} \int_0^\infty 1_{t|x|>b} e^{-t^p} t^{-1-\alpha} dt R(dx) \\ &\geq \int_{|x|>\epsilon} \int_{b|x|^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt R(dx) \\ &\geq \int_{|x|>\epsilon} \int_{b\epsilon^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt R(dx) \\ &= R(T_\epsilon) \int_{b\epsilon^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt > 0. \end{aligned}$$

Thus by Theorem 26.1 in [Sat99], the result follows. □

Now that we know which moments are finite, we can find the form of the cumulants. For proper 1-tempered α-stable distributions with $\alpha \in (0, 2)$ this was given in [TW06]. However, our proof is different and we correct a mistake in their formula for c_1. We begin by first proving the result for general infinitely divisible distributions.

First we establish some notation. For $x \in \mathbb{R}^d$ and $n \geq 1$ define the d^n-dimensional vector $x^{\otimes n}$ recursively by $x^{\otimes 1} = x$ and $x^{\otimes n} = x \otimes x^{\otimes(n-1)}$, where $\otimes$ is the Kronecker product. Let A be an $m \times n$ matrix and a_j its jth column. Define

$$\mathrm{Vec}(A) := [a_1^T, \ldots, a_n^T]^T, \tag{2.30}$$

which is an element of $\mathbb{R}^{nm}$. For more information on Kronecker products and the Vec operator consult [MN88].

Let $\phi : \mathbb{R}^d \mapsto \mathbb{R}^m$. Let $J_\phi(u)$ be the Jacobian matrix of ϕ evaluated at $u \in \mathbb{R}^d$. When it exists, recursively define

$$D_u^{\otimes 1}\phi(u) = \text{Vec}(J_\phi^T(u))$$

and

$$D_u^{\otimes n}\phi = D_u^{\otimes 1}\left(D_u^{\otimes(n-1)}\phi\right).$$

Note that $D_u^{\otimes n}\phi$ is a function mapping $\mathbb{R}^d$ into $\mathbb{R}^{md^n}$. Let μ be a probability measure such that $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$. The k**th cumulant** is defined to be

$$c_k = (-i)^k D_z^{\otimes k} C_\mu \Big|_{z=0_d}, \tag{2.31}$$

when it exists, and it is undefined otherwise. This explains why C_μ is called the cumulant generating function. Note that if $X \sim \mu$ then $c_1 = \mathrm{E}X$ and $c_2 = \text{Vec}\,[\text{Cov}(X)]$. For more information consult [Ter03] or [TW06].

Let $X \sim ID(A, M, b)$ and assume that for some $N \geq 1$, $\mathrm{E}|X|^N < \infty$. By Proposition C.4, this means that

$$\int_{|x|>1} |x|^N M(dx) < \infty, \tag{2.32}$$

and the cumulant generating function can be written as

$$C_X(z) = -\frac{\langle z, Az\rangle}{2} + \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle\right) M(dx) + i\langle z, b_1\rangle, \tag{2.33}$$

where b_1 is given by (C.11). The mth cumulant is the vector of all mth order partial derivatives of this function with respect to the components of $z = (z_1, \ldots, z_d)^T$ evaluated at $z = 0_d$. We will first consider the integrand. Let

$$f(z, x) = e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle.$$

Let $j_1, j_2, ...$ be a sequence of elements in $\{1, 2, ..., d\}$. We have

$$\frac{\partial}{\partial z_{j_1}} f(z,x) = i x_{j_1} \left[e^{i\langle x,z\rangle} - 1 \right]$$

and by induction, for $n \geq 2$

$$\frac{\partial^n}{\partial z_{j_n} \cdots \partial z_{j_1}} f(z,x) = i^n e^{i\langle x,z\rangle} \prod_{k=1}^{n} x_{j_k}.$$

Note that we are only interested in z near 0_d, so assume that $|z| \leq 1$. Using Lemma C.2 and the Cauchy-Schwartz inequality, we have

$$\left| \frac{\partial}{\partial z_j} f(z,x) \right| \leq |x| \left(|\langle x,z\rangle| \wedge 2 \right) \leq |x| \left((|x||z|) \wedge 2 \right) \leq |x| \left(|x| \wedge 2 \right) = |x|^2 \wedge (2|x|),$$

which is integrable under assumption (2.32). Similarly, if $n \leq N$ then

$$\left| \frac{\partial^n}{\partial z_{j_n} \cdots \partial z_{j_1}} f(z,x) \right| \leq |x|^n,$$

which is integrable under assumption (2.32). Let $A = (a_{ij})$. By Theorem 16.8 in [Bil95],

$$\frac{\partial}{\partial z_{j_1}} C_X(z) = -\sum_{m=1}^{d} a_{j_1 m} z_m + i \int_{\mathbb{R}^d} x_{j_1} \left(e^{i\langle x,z\rangle} - 1 \right) M(dx) + i b_1^{j_1}$$

where $b_1^{j_1}$ is the j_1 element of b_1 and

$$\frac{\partial^2}{\partial z_{j_2} \partial z_{j_1}} C_X(z) = -a_{j_1 j_2} - \int_{\mathbb{R}^d} x_{j_1} x_{j_2} e^{i\langle x,z\rangle} M(dx)$$

and for $N \geq n \geq 3$

$$\frac{\partial^n}{\partial z_{j_n} \cdots \partial z_{j_1}} C_X(z) = i^n \int_{\mathbb{R}^d} \left(\prod_{m=1}^{n} x_{j_m} \right) e^{i\langle x,z\rangle} M(dx).$$

Evaluating the above at $z = 0_d$ and multiplying by $(-i)^n$ gives the following result.

Proposition 2.22. *Let $X \sim ID^1(A, M, b_1)$ with M satisfying (2.32) for some $N \geq 1$, then for any $1 \leq n \leq N$, the nth cumulant c_n exists. Moreover, $c_1 = b_1$, $c_2 = \mathrm{Vec}(A) + \int_{\mathbb{R}^d} x^{\otimes 2} M(dx)$, and for all $3 \leq n \leq N$ $c_n = \int_{\mathbb{R}^d} x^{\otimes n} M(dx)$.*

Now, by plugging in the form of the Lévy measure of a p-tempered α-stable distributions given in (2.11) we get the following result.

Proposition 2.23. *Fix $p > 0$ and $\alpha < 2$. If $X \sim TS^p_\alpha(R, b)$ and $\int_{|x|>1} |x|^N R(dx) < \infty$ for some $N \geq 1$ then for any $1 \leq n \leq N$ the nth cumulant c_n exists. Moreover,*

$$c_1 = b_1 = b + \int_{\mathbb{R}^d} \int_0^\infty x \frac{|x|^2}{1 + |x|^2 t^2} t^{2-\alpha} e^{-t} dt R(dx)$$

and for all $2 \leq n \leq N$, $c_n = p^{-1}\Gamma\left(\frac{n-\alpha}{p}\right) \int_{\mathbb{R}^d} x^{\otimes n} R(dx)$.

2.3 Characteristic Functions

In this section we will give more explicit forms for the characteristic functions of certain p-tempered α-stable distributions. In light of Theorem 2.19, whenever $\alpha \in (1, 2)$ we can use Parametrization 1 (see (C.10)). Similarly, in light of Proposition 2.8 and Part 3 of Theorem 2.3, when $\alpha < 1$ and the p-tempered α-stable distribution is proper, we can use Parametrization 0 (see (C.7)). We will now give a simpler form of the characteristic function in the case when $p = 1$. For the case $\alpha \in (0, 2)$ this is Theorem 2.9 in [Ros07].

Theorem 2.24. *Fix $\alpha < 2$, $p = 1$, and let $\mu = TS^1_\alpha(R, b)$.*

1. If $\int_{|x|>1} |x| R(dx) < \infty$, and

$$b_1 = b + \int_{\mathbb{R}^d} \int_0^\infty x \frac{|x|^2}{1 + |x|^2 t^2} t^{2-\alpha} e^{-t} dx R(dx)$$

then the characteristic function is given by

$$\hat{\mu}(z) = \exp\left\{ \int_{\mathbb{R}^d} \psi_\alpha(\langle z, x \rangle) R(dx) + i\langle z, b_1 \rangle \right\}, \tag{2.34}$$

where

$$\psi_\alpha(s) = \begin{cases} \Gamma(-\alpha)[(1 - is)^\alpha - 1 + i\alpha s] & \alpha \neq 0, 1 \\ -\log(1 - is) - is & \alpha = 0 \\ (1 - is)\log(1 - is) + is & \alpha = 1 \end{cases}. \tag{2.35}$$

In particular the characteristic function can be written in the form (2.34) *when* $1 < \alpha < 2$*, or*

$$\alpha = 1 \quad \textit{and} \quad \int_{|x|>1} |x| \log |x| R(dx) < \infty, \tag{2.36}$$

or

$$\alpha < 1 \quad \textit{and} \quad \int_{\mathbb{R}^d} |x| R(dx) < \infty. \tag{2.37}$$

2. If $\alpha < 1$*,* $\int_{|x|\le 1} |x| R(dx) < \infty$*, and*

$$b_0 = b - \int_{\mathbb{R}^d} \int_0^\infty \frac{x}{1+|x|^2 t^2} t^{-\alpha} e^{-t} dt R(dx)$$

then the characteristic function is given by

$$\hat{\mu}(z) = \exp\left\{\psi_\alpha^0(\langle z, x\rangle) R(dx) + i\langle z, b_0\rangle\right\}, \tag{2.38}$$

where

$$\psi_\alpha^0(s) = \begin{cases} \Gamma(-\alpha)[(1-is)^\alpha - 1] & \alpha \neq 0 \\ -\log(1-is) & \alpha = 0 \end{cases}. \tag{2.39}$$

In particular if μ *is a proper* TS_α^1 *distribution with* $\alpha < 1$*, its characteristic function can be written in the form* (2.38)*.*

By Theorem 2.19 and Theorem 2.20 the condition $\int_{|x|>1} |x| R(dx) < \infty$ is equivalent to the condition $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$ and in this case we have $b_1 = \int_{\mathbb{R}^d} x \mu(dx)$. In Case 2 b_0 is the drift vector. Another situation where a relatively nice form for the characteristic function is known is when $p = 2$ and $\alpha \in [0, 2)$. In this case [BRKF11] gives a form for the characteristic function in terms of confluent hypergeometric functions.

Before proceeding to the proof, we should say a few things about the complex functions that we are using. Throughout we will use the principle branch of the complex logarithm. In this case, if $z \in \mathbb{C}$ with $\Re z > 0$ then

$$\log(z) = \log |z| + i \arctan(\Im z / \Re z), \tag{2.40}$$

where arctan refers to the branch of the arctangent whose image is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In particular $\log(1-is) = \frac{1}{2}\log(1+s^2) - i\arctan(s)$. Likewise, if $z \in \mathbb{C}$ with $\Re z > 0$ and $\gamma \in \mathbb{R}$ then $z^\gamma = \exp\{\gamma \log z\} = \exp\{\gamma\left[\log|z| + i\arctan(\Im z/\Re z)\right]\}$. For details about branches see e.g. [Fis99].

Proof. When $\alpha \in (0,2)$ this is Theorem 2.9 in [Ros07]. Thus we will assume that $\alpha \leq 0$. We will need the fact that when $\alpha < 0$ and $w \in \mathbb{C}$ with $\Re(w) > 0$ then

$$\int_0^\infty e^{-wt} t^{-\alpha-1} dt = w^\alpha \Gamma(-\alpha). \tag{2.41}$$

(see 6.1.1 in [AS72]).

We start with the case $\int_{|x|>1} |x| R(dx) < \infty$, which by Theorem 2.20 is equivalent to $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$, which implies that we can use Parametrization 1. Thus the characteristic function can be written as

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle x,z\rangle t} - 1 - i\langle x,z\rangle t\right) t^{-1-\alpha} e^{-t} dt R(dx) + i\langle z, b_1\rangle\right\}.$$

For simplicity of notation, let $s = \langle x, z\rangle$. Using (2.41) for $\alpha < 0$ we have

$$\begin{aligned}\int_0^\infty (e^{ist} - 1 - ist) e^{-t} t^{-\alpha-1} dt &= \int_0^\infty (e^{-(1-is)t} - e^{-t} - ise^{-t}t) t^{-\alpha-1} dt \\ &= \Gamma(-\alpha)[(1-is)^\alpha - 1 + i\alpha s].\end{aligned}$$

Now for the case when $\alpha = 0$. Using L'Hospital's rule and the fact that $\Gamma(-x) = \frac{-\pi}{\Gamma(x+1)\sin(\pi x)}$ we get

$$\begin{aligned}\lim_{\alpha\to 0} \Gamma(-\alpha)[(1-is)^\alpha - 1 + i\alpha s] &= \lim_{\alpha\to 0} \frac{-\pi[(1-is)^\alpha - 1 + i\alpha s]}{\Gamma(\alpha+1)\sin(\pi\alpha)} \\ &= \lim_{\alpha\to 0} \frac{-[(1-is)^\alpha \log(1-is) + is]}{\cos(\pi\alpha)} \\ &= -\log(1-is) - is.\end{aligned}$$

Let $\alpha < 0$. If $t \in (0,1)$ then

$$\begin{aligned}|(e^{ist} - 1 - ist) t^{-\alpha-1} e^{-t}| &\leq \frac{|\cos(st) - 1 + i\sin(st) - ist|}{t} \\ &= \frac{\sqrt{2 - 2\cos(st) + s^2t^2 - 2st\sin(st)}}{t}.\end{aligned}$$

To see that this is integrable on $t \in (0,1)$, note that by L'Hospital's rule

$$\lim_{t\downarrow 0} \frac{2-2\cos(st)+s^2t^2-2st\sin(st)}{t^2} = \lim_{t\downarrow 0} \frac{s^2t-s^2t\cos(st)}{t} = 0.$$

Now, if $t \geq 1$, by Lemma C.2 we have

$$|(e^{ist}-1-ist)t^{-\alpha-1}e^{-t}| \leq s^2t^{1-\alpha}e^{-t},$$

which is integrable over $t \in [1,\infty)$. Now by dominated convergence

$$\begin{aligned}
\int_0^\infty \left(e^{it\langle x,z\rangle}-1-it\langle x,z\rangle\right)e^{-t}t^{-1}dt &= \int_0^\infty \lim_{\alpha\nearrow 0}\left(e^{it\langle x,z\rangle}-1-it\langle x,z\rangle\right)e^{-t}t^{-\alpha-1}dt \\
&= \lim_{\alpha\nearrow 0}\int_0^\infty \left(e^{it\langle x,z\rangle}-1-it\langle x,z\rangle\right)e^{-t}t^{-\alpha-1}dt \\
&= \lim_{\alpha\nearrow 0}\Gamma(-\alpha)[(1-i\langle x,z\rangle)^\alpha-1+it\langle x,z\rangle] \\
&= -\log(1-i\langle x,z\rangle)-i\langle x,z\rangle,
\end{aligned}$$

as required.

When $\int_{|x|\leq 1}|x|R(dx)<\infty$, by Proposition 2.8, we can use Parametrization 0 and the characteristic function can be written as

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d}\left(e^{i\langle x,z\rangle}-1\right)M(dx)+i\langle z,b_0\rangle\right\}.$$

Showing that $\int_{\mathbb{R}^d}\left(e^{i\langle x,z\rangle}-1\right)M(dx)$ has the required form is similar to the previous part. □

In the proof we can use L'Hospital's rule because the denominator is real. However, in general, L'hopital's rule may fail for complex valued functions of real numbers, see [Car58].

2.4 Regular Variation

In this section we will derive necessary and sufficient conditions for tempered stable distributions to have regularly varying tails. This is important both from an applied and a theoretical perspective. In applications there are many situation when we assume that the tails are regularly varying. For example there is some evidence to suggest that financial returns have regularly varying tails, see e.g. [CT04]. On the theoretical side, this will help us in Section 4.2, when we classify long and short time behavior of tempered stable Lévy processes.

Fix $\alpha < 2$ and $p > 0$. Let $\mu \in TS^p_\alpha$ and let $X \sim \mu$. When $\alpha \in (0, 2)$ Theorem 2.19 implies that $E|X|^\varrho < \infty$ for all $\varrho \in [0, \alpha)$. Thus, by Proposition B.7, μ cannot have regularly varying tails with index $\varrho \in (-\alpha, 0]$. However, depending on R, it can have regularly varying tails with index $\varrho \leq -\alpha$. For the case when $\alpha \leq 0$ a p-tempered α-stable distribution may have regularly varying tails with tail index ϱ for any $\varrho \leq 0$. In this section we will categorize when a p-tempered α-stable distribution have regularly varying tails with index $\varrho < (-\alpha) \wedge 0$. The case when $\alpha \in (0, 2)$ and $\varrho = -\alpha$ will be treated later (see Corollary 4.9). First we introduce some notation.

Let $k : (0, \infty) \mapsto \mathbb{R}$. We define the **Mellin transform** of k by

$$\hat{k}(z) = \int_0^\infty u^{z-1} k(1/u) du \tag{2.42}$$

for all $z \in \mathbb{C}$ for which the integral converges. We will need the following result.

Proposition 2.25. *Let $-\infty < \gamma < \varrho < \tau < \infty$, $\ell \in RV_0^\infty$, and $c \in \mathbb{R}$. Let k be a continuous and non-negative function on $(0, \infty)$ such that*

$$\sum_{n=-\infty}^{\infty} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) < \infty \tag{2.43}$$

and

$$\hat{k}(z) \neq 0 \quad \text{when } \Re z = \varrho. \tag{2.44}$$

Let U be a monotone, right continuous function on $(0, \infty)$ and

$$\limsup_{r \downarrow 0} \frac{|U(r)|}{r^\gamma} < \infty. \tag{2.45}$$

Then

$$\int_0^\infty k(x/t)dU(t) \sim c\varrho\hat{k}(\varrho)x^\varrho\ell(x) \quad \text{as } x \to \infty \tag{2.46}$$

if and only if

$$U(x) \sim cx^\varrho\ell(x) \quad \text{as } x \to \infty. \tag{2.47}$$

Proof. This combines Theorems 4.4.2 and 4.9.1 in [BGT87]. □

We will now apply this result to our situation. Fix $\alpha \in (0,2)$, $p > 0$, and let σ be a finite nonzero measure on $\mathbb{S}^{d-1}$. Let R be the Rosiński measure of a p-tempered α-stable distribution and let M be the corresponding Lévy measure. For all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ define for $r > 0$

$$M_D(r) = M(|x| > r, x/|x| \in D)$$

and

$$R_D(r) = R(|x| > r, x/|x| \in D).$$

Note that for any integrable function $f : \mathbb{R} \to \mathbb{R}$

$$\int_{x/|x| \in D} f(|x|)R(dx) = -\int_0^\infty f(x)dR_D(x).$$

Lemma 2.26. *If $\varrho < (-\alpha) \wedge 0$ and $\ell \in RV_0^\infty$ then*

$$M_D(r) \sim \sigma(D)r^\varrho\ell(r) \text{ as } r \to \infty$$

if and only if

$$R_D(r) \sim \sigma(D)\frac{p}{\Gamma\left(\frac{|\rho|-\alpha}{p}\right)}r^\varrho\ell(r) \text{ as } r \to \infty.$$

Proof. For simplicity, let $\beta = (-\alpha) \wedge 0$. Note that

$$\begin{aligned} M_D(r) &= \int_{x/|x| \in D} \int_{r|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &= -\int_0^{\infty} \int_{r/x}^{\infty} t^{-1-\alpha} e^{-t^p} dt dR_D(x) \\ &= -\int_0^{\infty} k(r/x) R_D(dx), \end{aligned}$$

where

$$k(s) = \int_s^{\infty} t^{-\alpha-1} e^{-t^p} dt = p^{-1} \int_{s^p}^{\infty} t^{-\alpha/p-1} e^{-t} dt.$$

Note that, for $\Re z < \beta$

$$\begin{aligned} \hat{k}(z) &= \int_0^{\infty} u^{z-1} k(1/u) du = \int_0^{\infty} u^{z-1} \int_{1/u}^{\infty} t^{-1-\alpha} e^{-t^p} dt du \\ &= \int_0^{\infty} u^{z+\alpha-1} \int_1^{\infty} t^{-1-\alpha} e^{-(t/u)^p} dt du \\ &= \int_0^{\infty} u^{-z-\alpha-1} e^{-u^p} du \int_1^{\infty} t^{z-1} dt = -\frac{1}{pz} \Gamma\left(\frac{-z-\alpha}{p}\right). \end{aligned}$$

It remains to show that the assumptions of Proposition 2.25 hold. It is easy to see that k is a continuous and non-negative function on $(0, \infty)$ and that $\hat{k}(z)$ has no zeros. Let $\tau \in (\varrho, \beta)$ and let $\gamma < \varrho \wedge (-2)$. Let $C = \sup_{t \geq 1} t^{-\alpha/p-1} e^{-t/2}$. We have

$$\begin{aligned} &p \sum_{n=0}^{\infty} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) = \sum_{n=0}^{\infty} e^{|\gamma| n} \int_{e^{np}}^{\infty} t^{-\alpha/p-1} e^{-t/2} e^{-t/2} dt \\ &\leq C \sum_{n=0}^{\infty} e^{|\gamma| n} \int_{e^{np}}^{\infty} e^{-t/2} dt = 2C \sum_{n=0}^{\infty} e^{|\gamma| n} e^{-e^{np}/2} < \infty. \end{aligned}$$

We also have

$$p \sum_{n=-\infty}^{-1} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) = \sum_{n=1}^{\infty} e^{-|\tau| n} \int_{e^{-np}}^{\infty} t^{-\alpha/p-1} e^{-t} dt.$$

If $\alpha < 0$ then this is bounded by

$$\int_0^{\infty} t^{-\alpha/p-1} e^{-t} dt \sum_{n=1}^{\infty} e^{-|\tau| n} < \infty.$$

If $\alpha = 0$ then it is bounded by

$$\sum_{n=1}^{\infty} e^{-|\tau|n} \left(\int_{e^{-np}}^{1} t^{-1} dt + \int_{1}^{\infty} e^{-t} dt \right) = \sum_{n=1}^{\infty} e^{-|\tau|n} \left(np + e^{-1} \right) < \infty.$$

If $\alpha \in (0, 2)$ then it is bounded by

$$\sum_{n=1}^{\infty} e^{-|\tau|n} \int_{e^{-np}}^{\infty} t^{-\alpha/p-1} dt = \frac{p}{\alpha} \sum_{n=1}^{\infty} e^{-|\tau|n+\alpha n} = \frac{p}{\alpha} \sum_{n=1}^{\infty} e^{-(|\tau|-\alpha)n} < \infty.$$

Recall that $\gamma < -2$. Note that $-R_D(r)$ is a right continuous, monotonely increasing function on $(0, \infty)$ with

$$\begin{aligned} \limsup_{r \downarrow 0} \frac{|-R_D(r)|}{r^\gamma} &\leq \limsup_{r \downarrow 0} r^2 \int_{|x|>r} R(dx) \\ &\leq \limsup_{r \downarrow 0} \int_{1>|x|>r} |x|^2 R(dr) + \limsup_{r \downarrow 0} r^2 R\left(|x| \geq 1\right) \\ &\leq \int_{|x|<1} |x|^2 R(dr) < \infty. \end{aligned}$$

This completes the proof. □

Theorem 2.27. *Fix $\alpha < 2$, $p > 0$. Let μ be $TS_\alpha^p(R, b)$. Let M be the Lévy measure of μ. If $\varrho < (-\alpha) \wedge 0$*

$$\mu \in RV_\varrho^\infty(\sigma) \Longleftrightarrow M \in RV_\varrho^\infty(\sigma) \Longleftrightarrow R \in RV_\varrho^\infty(\sigma). \tag{2.48}$$

Moreover, if $M \in RV_\varrho^\infty(\sigma)$ then for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $\sigma(D) > 0$

$$\lim_{r \to \infty} \frac{R\left(|x| > r, x/|x| \in D\right)}{M\left(|x| > r, x/|x| \in D\right)} = \frac{p}{\Gamma\left(\frac{|\varrho|-\alpha}{p}\right)}.$$

In the case when $\alpha \in (0, 2)$, necessary and sufficient conditions for regular variation with index $-\alpha$ of the tails of a p-tempered α-stable distribution will be given in Corollary 4.9.

Proof. The relationship between M and R follows from Lemma 2.26 and (B.15). The relationship between the regular variation of μ and M is well know, see for example [HL06a]. □

We will now show a related result, which will be useful for finding necessary and sufficient conditions for long time behavior of tempered stable distributions to be Gaussian. We begin with a lemma.

Lemma 2.28. *Fix $\beta \in \mathbb{R}$, $p > 0$, and let $k(t) = t^\beta e^{-t^p}$. Then for all z with $\Re z < \beta$*

$$\hat{k}(z) = p^{-1}\Gamma\left(\frac{\beta - z}{p}\right)$$

and for any $-\infty < \gamma < \varrho < \tau < \beta < \infty$, k satisfies (2.43) *and* (2.44)*.*

Proof. For z with $\Re z < \beta$

$$\begin{aligned}\hat{k}(z) &= \int_0^\infty u^{z-1}k(1/u)du = \int_0^\infty u^{z-1-\beta}e^{-1/u^p}du \\ &= p^{-1}\int_0^\infty v^{\frac{\beta-z}{p}-1}e^{-v}dv = p^{-1}\Gamma\left(\frac{\beta-z}{p}\right).\end{aligned}$$

Since this is never equal to zero (2.44) holds.

For any $\gamma < \varrho < \tau < \beta$ (2.43) holds because

$$\begin{aligned}&\sum_{n=1}^\infty \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \le t \le e^{n+1}} k(t) = \sum_{n=1}^\infty e^{-\gamma n} \sup_{e^n \le t \le e^{n+1}} t^\beta e^{-t^p} \\ &\le \sum_{n=1}^\infty e^{-\gamma n} e^{|\beta|(n+1)} e^{-e^{np}} = \sum_{n=1}^\infty e^{(|\beta|-\gamma)n+|\beta|} e^{-e^{np}} < \infty\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=-\infty}^{-1} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \le t \le e^{n+1}} k(x) \le \sum_{n=1}^\infty e^{\tau n} \sup_{e^{-n} \le t \le e^{-n+1}} t^\beta \\ &\le \sum_{n=1}^\infty e^{\tau n} e^{-n\beta+|\beta|} = \sum_{n=1}^\infty e^{-(\beta-\tau)n+|\beta|} < \infty.\end{aligned}$$

This completes the proof. □

Proposition 2.29. *Fix $\alpha < 2$, $p > 0$, and $C \in L(\mathbb{R}^d)$ with* tr$C > 0$*. Let R be the Rosiński measure of a p-tempered α-stable distributions. Let M be the Lévy measure of this distribution. Define*

$$A_t = \int_{|x| \le t} xx^T M(dx)$$

and

$$B_t = \int_{|x|\le t} xx^T R(dx).$$

Then $A \in LRV_0^\infty(C)$ if and only if $B \in LRV_0^\infty(C)$. Moreover, when this holds

$$\lim_{t\to\infty} \frac{\mathrm{tr}A_t}{\mathrm{tr}B_t} = p^{-1}\Gamma\left(\frac{2-\alpha}{p}\right).$$

Proof. For $z \in \mathbb{R}^d$, let

$$f^z(s) = \int_{|x|\le s} \langle x, z\rangle^2 M(dx), \tag{2.49}$$

$$g^z(s) = \int_{|x|\le s} \langle x, z\rangle^2 R(dx), \text{ and} \tag{2.50}$$

$$G^z(s) = \int_0^s g^z(t)dt. \tag{2.51}$$

Note that

$$\begin{aligned} f^z(s) &= \int_0^\infty g^z(s/t)t^{1-\alpha}e^{-t^p}dt = s^{-1}\int_0^\infty g^z(u)(s/u)^{3-\alpha}e^{-(s/u)^p}du \\ &= s^{-1}\int_0^\infty k(s/u)dG^z(u), \end{aligned}$$

where

$$k(s) = s^{3-\alpha}e^{-s^p}.$$

We will now use Proposition 2.25 to show that $f^z \in RV_0^\infty$ if and only if $G^z \in RV_1^\infty$. First we need to verify that the assumptions hold. Since $3-\alpha > 1$, Lemma 2.28 implies that k satisfies the required conditions, and

$$\lim_{s\downarrow 0} |G^z(s)| \le \lim_{s\downarrow 0} s \int_{|x|\le s} \langle x, z\rangle^2 R(dx) = 0,$$

implies that so does G^z. Thus there are constants c_z and a slowly varying function ℓ with

$$sf^z(s) \sim c_z s\ell(s) \quad \text{as} \quad s\to\infty \tag{2.52}$$

if and only if

$$G^z(s) \sim c_z s\ell(s)\frac{p}{\Gamma\left(\frac{2-\alpha}{p}\right)} \quad \text{as} \quad s \to \infty. \tag{2.53}$$

Since g^z is a monotonely increasing function, by Karamata's Theorem and the Monotone Density Theorem (Theorems 1.5.11 and 1.7.2 in [BGT87]) (2.53) holds if and only if

$$g^z(s) \sim c_z\ell(s)\frac{p}{\Gamma\left(\frac{2-\alpha}{p}\right)} \quad \text{as} \quad s \to \infty. \tag{2.54}$$

From here the result holds to Proposition B.13. When the above holds we have

$$\lim_{t\to\infty}\frac{\mathrm{tr}A_t}{\mathrm{tr}B_t} = p^{-1}\Gamma\left(\frac{2-\alpha}{p}\right),$$

as required. □

CHAPTER 3

LIMITS OF SEQUENCES OF TEMPERED STABLE DISTRIBUTIONS

In this chapter we will discuss the possible weak limits of p-tempered α-stable distributions for fixed $\alpha < 2$, $p > 0$. It turns out that the class TS^p_α is not closed under weak convergence. We will introduce the smallest class that contains TS^p_α and is closed under weak convergence. Moreover, this class is closed under taking finite convolutions. We will call this the class of extended p-tempered α-stable distributions (ETS^p_α). In Section 3.1 we will define this class for $\alpha \in (0, 2)$ and give a limit theorem for convergence. In Section 3.2 we will extend these results to $\alpha \leq 0$. In the remaining sections we will give some extensions and applications. Throughout this chapter it will be very important that we use the compactification of $\mathbb{R}^d$ defined in Appendix A.4.

3.1 Sequences with $\alpha \in (0, 2)$

First, recall that, by (2.5), the Lévy measure of a p-tempered α-stable distribution can be given by

$$M(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) q_p(r, u) r^{-1-\alpha} dr \sigma(du), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{3.1}$$

where σ is a finite measure on $\mathbb{S}^{d-1}$ and $q_p(r, u)$ is jointly measurable in r and u and

$$q_p(r, u) = \int_0^\infty e^{-r^p t} Q(dt|u), \tag{3.2}$$

for some measurable family $\{Q(\cdot|u)\}_{u \in \mathbb{S}^{d-1}}$ of Borel measures on $(0, \infty)$ satisfying certain integrability conditions. We will now extend this class slightly.

Definition 3.1. *Fix $\alpha \in (0, 2)$ and $p > 0$. The class of* ***extended p-tempered α-stable distributions*** *is the class of infinitely divisible distributions that may* *have a Gaussian part and where the Lévy measure is given by* (3.1) *with* $q_p(r,u)$ *defined by* (3.2) *as before, but now allowing the family of measures* $\{Q(\cdot|u)\}_{u\in\mathbb{S}^{d-1}}$ *to have a point mass at* 0*. We denote this class by* ETS_α^p*.*

We can write the Lévy measure as

$$\begin{aligned} M(A) &= \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_A(ru)\int_{[0,\infty)} e^{-r^p s}Q(ds|u)r^{-1-\alpha}dr\sigma(du) \\ &= \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_A(ru)r^{-1-\alpha}drQ(\{0\}|u)\sigma(du) \\ &\quad + \int_{\mathbb{R}^d}\int_0^\infty 1_A(rx)r^{-1-\alpha}e^{-r^p}drR(dx), \qquad A\in\mathfrak{B}(\mathbb{R}^d) \end{aligned}$$

where R is the Rosiński measure defined as before. Again, R must satisfy the conditions given in Theorem 2.3. Note that a distribution is in ETS_α^p if and only if it can be written as the convolution of a Gaussian distribution, an α-stable distribution, and an element of TS_α^p.

We will now put the Lévy measure into a form that is easier to work with. To do this we will define a measure on $\bar{\mathbb{R}}^d$. It is important to emphasize that, in this context, by $\bar{\mathbb{R}}^d$ we are referring to the construction given in Appendix A.4. Define a measure ν on $\bar{\mathbb{R}}^d$ by $\nu(\{0_d\})=0$,

$$\nu(dx) = \left(|x|^2\wedge|x|^\alpha\right)R(dx) \quad \text{for} \quad x\in\mathbb{R}^d_0, \tag{3.3}$$

and $\int_{\mathbb{I}^{d-1}} 1_A(x)\nu(dx) = \int_{\mathbb{S}^{d-1}} 1_A(u\infty)Q(\{0\}|u)\sigma(du)$. Thus, for $A\in\mathfrak{B}(\bar{\mathbb{R}}^d)$

$$\nu(A) = \int_{\mathbb{R}^d_0} 1_A(x)\left(|x|^2\wedge|x|^\alpha\right)R(dx) + \int_{\mathbb{S}^{d-1}} 1_A(x\infty)Q(\{0\}|x)\sigma(dx). \tag{3.4}$$

Note that this is a finite measure. We will call it the **extended Rosiński measure**. If $\mu\in ETS_\alpha^p$ and it has Lévy measure M, then for any measurable function

f, which is integrable with respect to M, we have

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)M(dx) &= \int_{\mathbb{S}^{d-1}}\int_0^\infty f(tx)t^{-1-\alpha}dtQ(\{0\}|x)\sigma(dx)\\ &\quad+\int_{\mathbb{R}^d}\int_0^\infty f(tx)t^{-1-\alpha}e^{-t^p}dtR(dx)\\ &= \int_{\mathbb{I}^{d-1}}\int_0^\infty f(t\xi(x))t^{-1-\alpha}e^{-(t/|x|)^p}dt\nu(dx)\\ &\quad+\int_{\mathbb{R}^d}\int_0^\infty f(t\xi(x))t^{-1-\alpha}e^{-(t/|x|)^p}dt|x|^\alpha R(dx)\\ &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty f(t\xi(x))t^{-1-\alpha}\frac{e^{-(t/|x|)^p}}{1\wedge|x|^{2-\alpha}}dt\nu(dx).\end{aligned}$$

In the above, and throughout, we adopt the convention that if $x \in \mathbb{I}^{d-1}$ then $|x|^{-1} = 0$. From the extended Rosiński measure ν we can get back the Rosiński measure R by

$$R(dx) = 1_{\mathbb{R}^d}(x)\frac{1}{|x|^\alpha \wedge |x|^2}\nu(dx).$$

Proposition 3.2. *For a fixed $\alpha \in (0,2)$ and $p > 0$, the extended Rosiński measure ν uniquely determines the Lévy measure of an extended p-tempered α-stable distribution.*

Proof. This follows from the fact that ν uniquely determines R and $Q(\{0\}|x)\sigma(dx)$. By Theorem 2.3 and Remark 14.4 in [Sat99], these measures uniquely determine the Lévy measures of the p-tempered α-stable and α-stable parts. □

We will denote a distribution in ETS^p_α by $ETS^p_\alpha(A,\nu,b)$, where A is the Gaussian part, ν is the extended Rosiński measure, and b is the shift. We can now state the main result of this section.

Theorem 3.3. *Fix $\alpha \in (0,2)$ and $p > 0$. Let $\mu_n = ETS^p_\alpha(A_n,\nu_n,b_n)$. If $\mu_n \xrightarrow{w} \mu$ for some probability measure μ, then $\mu = ETS^p_\alpha(A,\nu,b)$. Moreover, $\mu_n \xrightarrow{w} \mu$ if and only if the following conditions hold:*

1. $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d_0$, *2.* $b_n \to b$*, and*

3.

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}(A_n+H_n^\epsilon)=\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}(A_n+H_n^\epsilon)=A, \tag{3.5}$$

where

$$H_n^\epsilon=\int_{|x|<\sqrt{\epsilon}}\frac{xx^T}{|x|^2}\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n(dx). \tag{3.6}$$

To see that the extended Rosiński measure may contribute to the Gaussian part, see Proposition 3.12 below. On the other hand, the extended Rosiński measure does not contribute to the Gaussian part if and only if

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathrm{tr}H_n^\epsilon=0. \tag{3.7}$$

Note that

$$\mathrm{tr}H_n^\epsilon=\int_{|x|<\sqrt{\epsilon}}\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n(dx)$$

and for $\epsilon\in(0,1)$

$$\begin{aligned}\int_{|x|<\epsilon}\nu_n(dx)\int_0^1 t^{1-\alpha}e^{-t^p}dt &\le \int_{|x|<\sqrt{\epsilon}}\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n(dx)\\ &\le \int_{|x|<\sqrt{\epsilon}}\nu_n(dx)\int_0^\infty t^{1-\alpha}e^{-t^p}dt.\end{aligned}$$

Thus (3.7) holds if and only if

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}\nu_n(dx)=0. \tag{3.8}$$

Similarly the limit has no α-stable part if and only if

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|\ge N}\nu_n(dx)=0. \tag{3.9}$$

Before proceeding to the proof, we will give several examples to show that the conditions of Theorem 3.3 are independent of each other. First, we need some notation. Let $I_{d\times d}$ be the $d\times d$ identity matrix and let $e_1=(1,0,\ldots,0)^T\in\mathbb{R}^d$.

Example 1. Let $A_n = 0_{d\times d}$, $\nu_n = 0$, and $b_n = n_d$. This satisfies Condition 1 (with $\nu = \nu_1$) and Condition 3 (with $A = 0_{d\times d}$). However, Condition 2 does not hold and μ_n does not converge weakly. □

Example 2. Let $A_n = 0_{d\times d}$, $b_n = 0$, and $\nu_n = \delta_{e_1}$ when n is even and $\nu_n = 0$ when n is odd. This satisfies Condition 2 (with $b = 0$) and Condition 3 (with $A = 0_{d\times d}$). However, Condition 1 does not hold, and again μ_n does not converge weakly. □

Example 3. Let $A_n = nI_{d\times d}$, $b_n = 0_d$, and $\nu_n = 0$. This satisfies Condition 1 (with $\nu = \nu_1$) and Condition 2 (with $b = 0_d$), but not Condition 3. It is again easy to see that μ_n does not converge weakly. □

The next example will show that we do, in fact, need both parts of the third condition.

Example 4. Let $A_n = 0_{d\times d}$, $b_n = 0_d$, and $\nu_n = \delta_{e_1/n}$ when n is even and $\nu_n = 0$ when n is odd. This satisfies Condition 1 (with ν being the zero measure) and Condition 2 (with $b = 0_d$). However, when n is odd, $H_n^\epsilon = 0_{d\times d}$, whereas when n is even and $n^2 > \epsilon^{-1}$

$$H_n^\epsilon = e_1 e_1^T \int_0^{\epsilon n} t^{1-\alpha} e^{-t^p} dt \to e_1 e_1^T \int_0^{\infty} t^{1-\alpha} e^{-t^p} dt \neq 0_{d\times d},$$

where the limit is taken as $n \to \infty$ along the odd integers. □

To facilitate the proof of Theorem 3.3, we begin with several lemmas.

Lemma 3.4. *Fix $\alpha < 2$, $p > 0$. If $|s| \le 1$ then $\int_0^\infty (\cos(ts) - 1)\, t^{-1-\alpha} e^{-t^p} dt \le -\frac{11}{24} s^2 \int_0^1 t^{1-\alpha} e^{-t^p} dt$.*

Proof. We have

$$\begin{aligned}
&\int_0^\infty (\cos(ts) - 1)\, t^{-1-\alpha} e^{-t^p} dt \le \int_0^1 (\cos(ts) - 1)\, t^{-1-\alpha} e^{-t^p} dt \\
&\le \int_0^1 \left(\frac{s^4 t^4}{24} - \frac{s^2 t^2}{2} \right) t^{-1-\alpha} e^{-t^p} dt = \int_0^1 \left(\frac{s^4 t^2}{24} - \frac{s^2}{2} \right) t^{1-\alpha} e^{-t^p} dt \\
&\le -\frac{11}{24} s^2 \int_0^1 t^{1-\alpha} e^{-t^p} dt,
\end{aligned}$$

where the second inequality follows by the Taylor expansion of cosine and the remainder theorem for alternating series. □

Lemma 3.5. *Let the sequence* $\{\mu_n\}$ *be as in Theorem 3.3.*

1. If $\mu_n \xrightarrow{w} \mu$ *for some probability measure* μ *then* $\sup \nu_n(\bar{\mathbb{R}}^d) < \infty$.

2. If $\nu_n \xrightarrow{v} \nu$ *on* $\bar{\mathbb{R}}^d_0$ *for some finite measure* ν *then for any* $\delta > 0$ *we have* $\sup \nu_n(|x| \geq \delta) < \infty$.

3. If (3.5) *holds with some matrix* A *then for any* $\delta > 0$ $\sup \nu_n(|x| < \delta) < \infty$.

Proof. The second part follows immediately from Proposition A.8. Now for the first part. Assume that $\mu_n \xrightarrow{w} \mu$. By Lemma 3.4, for $|z| \leq 1$

$$\begin{aligned}
|\hat{\mu}_n(z)| &\leq \left|\exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{it\langle x,z\rangle} - 1 - i\frac{t\langle x,z\rangle}{1+|x|^2}\right) t^{-1-\alpha} e^{-t^p} dt R_n(dx)\right\}\right| \\
&\leq \exp\left\{\int_{|x|\leq 1}\int_0^\infty \left(\cos\left(t\langle x,z\rangle\right) - 1\right) t^{-1-\alpha} e^{-t^p} dt R_n(dx)\right\} \\
&\leq \exp\left\{-\frac{11}{24}\int_0^1 t^{1-\alpha} e^{-t^p} dt \int_{|x|\leq 1} \langle x,z\rangle^2 R_n(dx)\right\},
\end{aligned}$$

where the first inequality follows by the fact that we can write μ_n as a convolution of a Gaussian, an α-stable, and an element of TS^p_α. By Proposition 2.5 in [Sat99] $|\hat{\mu}_n(z)| \to |\hat{\mu}(z)|$ uniformly on compact sets, and for some $b > 0$, $|\hat{\mu}(z)| > b$ on a neighborhood of zero. Thus on this neighborhood we have

$$b < \exp\left\{-\frac{11}{24}\int_0^1 t^{1-\alpha} e^{-t^p} dt \int_{|x|\leq 1} \langle x,z\rangle^2 R_n(dx)\right\},$$

for large enough n. This implies that for every $z \in \mathbb{R}^d$

$$\sup_n \int_{|x|\leq 1} \langle x,z\rangle^2 R_n(dx) < \infty,$$

and hence $\sup \nu_n\left(|x| \leq 1\right) < \infty$.

By Proposition C.5, μ is infinitely divisible. Let M_n be the Lévy measure of μ_n and M the Lévy measure of μ. Let f_1 be a non-negative, continuous, bounded, real-valued function vanishing on a neighborhood of zero such that $f_1(y) = 1$ for $|y| \geq 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^d} f_1(x) M_n(dx) &= \int_{\bar{\mathbb{R}}^d} \int_0^\infty f_1(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \nu_n(dx) \\
&\geq \int_{|x|\geq 1} \int_1^\infty t^{-1-\alpha} e^{-(t/|x|)^p} dt \nu_n(dx) \\
&\geq \int_{|x|=\infty} \int_1^\infty t^{-1-\alpha} dt \nu_n(dx) \\
&\quad + \int_{\infty > |x| \geq 1} \int_1^2 t^{-1-\alpha} e^{-(t/|x|)^p} dt \nu_n(dx) \\
&\geq \alpha^{-1} \nu_n(|x| = \infty) + e^{-2^p} \frac{2^\alpha - 1}{\alpha 2^\alpha} \nu_n(\infty > |x| \geq 1) \\
&\geq e^{-2^p} \frac{2^\alpha - 1}{\alpha 2^\alpha} \nu_n(|x| \geq 1). \qquad (3.10)
\end{aligned}$$

By Proposition C.5, the left hand side converges to $\int_{\mathbb{R}^d} f_1(x) M(dx)$. Thus if $\limsup_{n\to\infty} \nu_n(|x| \geq 1) = \infty$ then $\int_{|x|>1} M(dx) = \infty$, contradicting the fact that M is a Lévy measure. Hence $\sup \nu_n \left(\bar{\mathbb{R}}^d\right) < \infty$.

For the third part, fix $\delta > 0$, let $\epsilon = \delta^2$, and assume that (3.5) holds. Note that for any $z \in \mathbb{R}^d$, $|\langle z, H_n^\epsilon z\rangle| < \infty$. Thus, since $\langle z, xx^T z\rangle = \langle x, z\rangle^2$, this means that for any $z \in \mathbb{R}^d$

$$\begin{aligned}
\infty &> \limsup_{n\to\infty} \int_{|x|<\sqrt{\epsilon}} \langle \xi(x), z\rangle^2 \int_0^{\epsilon|x|^{-1}} t^{1-\alpha} e^{-t^p} dt \nu_n(dx) \\
&\geq \limsup_{n\to\infty} \int_0^{\sqrt{\epsilon}} t^{1-\alpha} e^{-t^p} dt \int_{|x|<\sqrt{\epsilon}} \langle \xi(x), z\rangle^2 \nu_n(dx).
\end{aligned}$$

This implies that $\limsup_{n\to\infty} \int_{|x|\leq\sqrt{\epsilon}} \nu_n(dx) < \infty$, which completes the proof. □

Lemma 3.6. *Let the sequence $\{\mu_n\}$ be as in Theorem 3.3, and let M_n be the Lévy measure of μ_n. If $\sup_n \nu_n(\bar{\mathbb{R}}^d) < \infty$ then*

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \left(A_n + \int_{|x|\leq\epsilon} xx^T M_n(dx) \right) \\
&\qquad = \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \left(A_n + \int_{|x|<\sqrt{\epsilon}} \frac{xx^T}{|x|^2} \int_0^{\epsilon|x|^{-1}} t^{1-\alpha} e^{-t^p} dt \nu_n(dx) \right),
\end{aligned}$$

and the result remains true if we replace $\limsup$ *by* $\liminf$.

Proof. We have

$$\begin{aligned}
\int_{|x|\le\epsilon} xx^T M_n(dx) &= \int_{\mathbb{I}^{d-1}} \int_0^{\epsilon} \xi(x)[\xi(x)]^T t^{1-\alpha} dt \nu_n(dx) \\
&\quad + \int_{\infty>|x|\ge 1} \int_0^{\epsilon|x|^{-1}} xx^T t^{1-\alpha} e^{-t^p} dt |x|^{-\alpha} \nu_n(dx) \\
&\quad + \int_{1>|x|\ge\sqrt{\epsilon}} \int_0^{\epsilon|x|^{-1}} xx^T t^{1-\alpha} e^{-t^p} dt |x|^{-2} \nu_n(dx) \\
&\quad + \int_{|x|<\sqrt{\epsilon}} \int_0^{\epsilon|x|^{-1}} xx^T t^{1-\alpha} e^{-t^p} dt |x|^{-2} \nu_n(dx) \\
&=: I_1^{n,\epsilon} + I_2^{n,\epsilon} + I_3^{n,\epsilon} + I_4^{n,\epsilon}.
\end{aligned}$$

Since $\sup_n \nu_n(\bar{\mathbb{R}}^d) < \infty$, there is a C such that $\nu_n(\bar{\mathbb{R}}^d) < C$ for all n. Let $I_m^{n,\epsilon}(i,j)$ be the (i,j)th component of $I_m^{n,\epsilon}$ for $m = 1, 2, 3, 4$. Note that if $x = (x_1, \ldots, x_d)^T$ then $|x_i| \le |x|$ for $i = 1, \ldots, d$. We have

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} |I_1^{n,\epsilon}(i,j)| &\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \nu_n(\mathbb{I}^{d-1}) \frac{\epsilon^{2-\alpha}}{2-\alpha} \\
&\le \lim_{\epsilon\downarrow 0} C \frac{\epsilon^{2-\alpha}}{2-\alpha} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} |I_2^{n,\epsilon}(i,j)| &\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{\infty>|x|\ge 1} |x|^{2-\alpha} \int_0^{\epsilon|x|^{-1}} t^{1-\alpha} dt \nu_n(dx) \\
&\le \lim_{\epsilon\downarrow 0} C \frac{\epsilon^{2-\alpha}}{2-\alpha} = 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} |I_3^{n,\epsilon}(i,j)| &\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{1>|x|\ge\sqrt{\epsilon}} \int_0^{\epsilon|x|^{-1}} t^{1-\alpha} dt \nu_n(dx) \\
&\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{1>|x|\ge\sqrt{\epsilon}} \int_0^{\sqrt{\epsilon}} t^{1-\alpha} dt \nu_n(dx) \\
&\le \lim_{\epsilon\downarrow 0} C \frac{\epsilon^{1-\alpha/2}}{2-\alpha} = 0.
\end{aligned}$$

The fact that if $\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} |a_n^\epsilon| = 0$ then

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} (a_n^\epsilon + b_n^\epsilon) = \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} b_n^\epsilon \quad \text{and} \quad \lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} (a_n^\epsilon + b_n^\epsilon) = \lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} b_n^\epsilon$$

completes the proof. $\square$

Proof of Theorem 3.3. Let M_n be the Lévy measure of μ_n.

Assume that $\mu_n \xrightarrow{w} \mu$. By Proposition C.5 μ is infinitely divisible with some Lévy triplet (A, M, b) and $b_n \to b$, $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}^d_0$, and

$$\begin{aligned} A &= \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \left(A_n + \int_{|x|<\epsilon} xx^T M_n(dx) \right) \\ &= \lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} \left(A_n + \int_{|x|<\epsilon} xx^T M_n(dx) \right). \end{aligned}$$

Combining this with Lemma 3.5 and Lemma 3.6 gives (3.5).

It remains to show that there is a extended Rosiński measure ν such that $\mu = ETS^p_\alpha(A, \nu, b)$ and $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d_0$. By Lemma 3.5, $\sup \nu_n(\bar{\mathbb{R}}^d) < \infty$. Thus by Proposition A.10 there is a vaguely convergent on $\bar{\mathbb{R}}^d$ subsequence $\{\nu_{n_j}\}$ of $\{\nu_n\}$. Let $\tilde{\nu}$ be the vague limit of this subsequence. Let f be any non-negative, bounded, continuous function on $\bar{\mathbb{R}}^d$ such that, for some $\epsilon > 0$, $f(x) = 0$ when $|x| \le \epsilon$. For $x \in \bar{\mathbb{R}}^d$ let

$$g_\alpha(x) = \int_\epsilon^\infty f(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt. \tag{3.11}$$

Since $f(\xi(x)t)\frac{e^{-(t/|x|)^p}}{1\wedge|x|^{2-\alpha}}$ is uniformly bounded and $\int_\epsilon^\infty t^{-1-\alpha} dt < \infty$, g_α is bounded and by dominated convergence it is continuous on $\bar{\mathbb{R}}^d$. Thus

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) M(dx) &= \lim_{j\to\infty} \int_{\mathbb{R}^d} f(x) M_{n_j}(dx) \\ &= \lim_{j\to\infty} \int_{\bar{\mathbb{R}}^d} g_\alpha(x) \nu_{n_j}(dx) = \int_{\bar{\mathbb{R}}^d} g_\alpha(x) \tilde{\nu}(dx), \end{aligned}$$

where the first equality follows by the fact that $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}^d_0$. Thus for all $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned} M(A) &= \int_{\bar{\mathbb{R}}^d} \int_0^\infty 1_A(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \tilde{\nu}(dx) \\ &= \int_{\bar{\mathbb{R}}^d} \int_0^\infty 1_A(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \nu(dx), \end{aligned} \tag{3.12}$$

where $\nu_{|_{\bar{\mathbb{R}}^d_0}} = \tilde{\nu}_{|_{\bar{\mathbb{R}}^d_0}}$ and $\nu(\{0_d\}) = 0$. The second line follows by the fact that

$$\lim_{x \to 0} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} = 0.$$

Since Lévy measures are unique, this proves that the class ETS^p_α is closed under weak convergence. Moreover, since ν uniquely determines the Lévy measure, Proposition A.10 implies that $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d_0$.

Now for the other direction. Let M be the Lévy measure of μ. Assume that $b_n \to b$, (3.5) holds, and $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d_0$. Combining (3.5) with Lemmas 3.5 and 3.6 gives (C.16). It remains to show that $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}^d_0$. Let f be a bounded, continuous function of $\bar{\mathbb{R}}^d_0$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) M_n(dx) &= \int_{\bar{\mathbb{R}}^d} \int_0^\infty f(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \nu_n(dx) \\
&\to \int_{\bar{\mathbb{R}}^d} \int_0^\infty f(\xi(x)t) t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \nu(dx) \\
&= \int_{\mathbb{R}^d} f(x) M(dx),
\end{aligned}$$

where the convergence follows by arguments similar to the other direction. This completes the proof. □

We will now give a version of the theorem for convergence within the subclass TS^p_α. We begin with a lemma.

Lemma 3.7. *Let M_n be a sequence of Radon measures on $\bar{\mathbb{R}}^d_0$ such that $M_n(\mathbb{I}^{d-1}) = 0$. Then $M_n \xrightarrow{v} M_1$ on $\bar{\mathbb{R}}^d_0$ if and only if $M_n \xrightarrow{v} M_1$ on $\mathbb{R}^d_0$ and*

$$\lim_{N \to \infty} \limsup_{n \to \infty} M_n(|x| > N) = 0. \tag{3.13}$$

Proof. Let (a_m) be a sequence of positive real numbers monotonely increasing to infinity such that $M_1(|x| = a_m) = 0$. First assume that $M_n \xrightarrow{v} M_1$ on $\bar{\mathbb{R}}^d_0$. It follows that $M_n \xrightarrow{v} M_1$ on $\mathbb{R}^d_0$, and by Proposition A.8

$$\lim_{m \to \infty} \limsup_{n \to \infty} M_n(|x| \geq a_m) \leq \lim_{m \to \infty} M_1(|x| \geq a_m) = 0,$$

where the last equality follows by the fact that M_1 is a finite measure with $M_1(\mathbb{I}^{d-1}) = 0$. Thus (3.13) holds.

Now assume that $M_n \xrightarrow{v} M_1$ on $\mathbb{R}_0^d$ and (3.13) holds. Let B be a relatively compact Borel subset of $\bar{\mathbb{R}}_0^d$ and a continuity set of M_1. Using Proposition A.8 we have

$$\begin{aligned}\lim_{n\to\infty} M_n(B) &= \lim_{m\to\infty}\lim_{n\to\infty}\left[M_n(B\cap[|x|<a_m]) + M_n(B\cap[|x|\geq a_m])\right]\\ &= M_1(B\cap[|x|<\infty]) = M_1(B),\end{aligned}$$

where the second line follows by (3.13). □

Corollary 3.8. *Fix $\alpha \in (0,2)$, $p>0$, let $\mu_n = TS_\alpha^p(R_n, b_n)$, and let $\mu = TS_\alpha^p(R,b)$. Then $\mu_n \xrightarrow{w} \mu$ if and only if $R_n \xrightarrow{v} R$ on $\mathbb{R}_0^d$, $b_n \to b$,*

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|\geq N}|x|^\alpha R_n(dx) = 0, \tag{3.14}$$

and

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}|x|^2 R_n(dx) = 0. \tag{3.15}$$

Note that we only need $R_n \xrightarrow{v} R$ on $\mathbb{R}_0^d$ not $R_n \xrightarrow{v} R$ on $\bar{\mathbb{R}}_0^d$. Of course topologically, these two types of convergence are very different. However, it is easy to see that the second type always implies the first, but not the other way around.

Proof. By Theorem 3.3 $\mu_n \xrightarrow{w} \mu$ if and only if $b_n \to b$,

$$\left(|x|^2\wedge|x|^\alpha\right)R_n(dx) \xrightarrow{v} \left(|x|^2\wedge|x|^\alpha\right)R(dx) \text{ on } \bar{\mathbb{R}}_0^d,$$

and

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|\leq\sqrt{\epsilon}} xx^T\int_0^{\epsilon|x|^{-1}} t^{1-\alpha}e^{-t^p}dtR(dx) = 0_{d\times d}.$$

By (3.8) the last part is equivalent to (3.15). Thus it suffices to show that $(|x|^2\wedge|x|^\alpha)R_n(dx) \xrightarrow{v} (|x|^2\wedge|x|^\alpha)R(dx)$ on $\bar{\mathbb{R}}_0^d$ if and only if $R_n \xrightarrow{v} R$ on $\mathbb{R}_0^d$ and (3.14) holds. This follows by Lemma 3.7. □

3.2 Sequences with $\alpha \le 0$

We will now give a version of Theorem 3.3 for the case when $\alpha \le 0$. As before, we will need to extend the class slightly. However, in this case, we only need to add a Gaussian part.

Definition 3.9. *Fix $p > 0$ and $\alpha \le 0$. The class of **extended p-tempered α-stable distributions** is the class of infinitely divisible distributions that may have a Gaussian part and where the Lévy measure is given by* (3.1)*. We denote this class by ETS^p_α.*

Clearly, the class ETS^p_α is the class TS^p_α, but allowing for a Gaussian part. As with p-tempered α-stable distributions, the Lévy measure can be written in terms of the Rosiński measure R as in (2.11). However, for the purposes of this section it will be more useful to use a transformed version of this measure. For $\alpha \le 0$ we define the **extended Rosiński measure** ν as follows $\nu\left(\mathbb{I}^{d-1}\right) = \nu(\{0_d\}) = 0$ and for $x \in \mathbb{R}^d_0$

$$\nu(dx) = \begin{cases} \left(|x|^2 \wedge [1 + \log^+ |x|]\right) R(dx) & \text{if } \alpha = 0 \\ \left(|x|^2 \wedge 1\right) R(dx) & \text{if } \alpha < 0 \end{cases}. \tag{3.16}$$

Thus, for $A \in \mathfrak{B}(\bar{\mathbb{R}}^d)$

$$\nu(A) = \begin{cases} \int_{\mathbb{R}^d_0} 1_A(x) \left(|x|^2 \wedge [1 + \log^+ |x|]\right) R(dx) & \text{if } \alpha = 0 \\ \int_{\mathbb{R}^d_0} 1_A(x) \left(|x|^2 \wedge 1\right) R(dx) & \text{if } \alpha < 0 \end{cases}.$$

Note that this is a finite measure. Let μ is a extended p-tempered α-stable distribution with Lévy measure M, and let f be any measurable function, which is integrable with respect to M. If $\alpha = 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)M(dx) &= \int_{\mathbb{R}^d}\int_0^\infty f(tx)t^{-1}e^{-t^p}dtR(dx)\\ &= \int_{\mathbb{R}^d}\int_0^\infty f(t\xi(x))t^{-1}e^{-(t/|x|)^p}dtR(dx)\\ &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty f(t\xi(x))t^{-1}\frac{e^{-(t/|x|)^p}}{|x|^2\wedge[1+\log^+|x|]}dt\nu(dx)\end{aligned}$$

Similarly, if $\alpha < 0$ we have

$$\int_{\mathbb{R}^d} f(x)M(dx) = \int_{\bar{\mathbb{R}}^d}\int_0^\infty f(t\xi(x))t^{-1-\alpha}\frac{e^{-(t/|x|)^p}}{|x|^{2-\alpha}\wedge|x|^{-\alpha}}dt\nu(dx).$$

From the extended Rosiński measure ν we can get back the Rosiński measure R by

$$R(dx) = \begin{cases} \left(|x|^2\wedge[1+\log^+|x|]\right)^{-1}\nu(dx) & \text{if } \alpha = 0\\ \left(|x|^2\wedge 1\right)^{-1}\nu(dx) & \text{if } \alpha < 0\end{cases}. \tag{3.17}$$

Proposition 3.10. *For $p > 0$ and $\alpha \leq 0$. The extended Rosiński measure ν uniquely determines the Lévy measure of a extended p-tempered α-stable distribution.*

Proof. By Theorem 2.3 the Rosiński measure uniquely determines the Lévy measure of a p-tempered α-stable distribution. Thus, the result follows from the fact that there is a one-to-one relationship between R and ν. □

We will denote a particular element of ETS^p_α by $ETS^p_\alpha(A,\nu,b)$, where A is the Gaussian part, ν is the extended Rosiński measure, and b is the shift. We will now give a result analogous to Theorem 3.3.

Theorem 3.11. *Fix $p > 0$ and $\alpha \leq 0$ Let $\mu_n = ETS^p_\alpha(A_n,\nu_n,b_n)$. If $\mu_n \xrightarrow{w} \mu$ for some probability measure μ, then $\mu = ETS^p_\alpha(A,\nu,b)$. Moreover, $\mu_n \xrightarrow{w} \mu$ if and only if the following conditions hold:*

1. $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d_0$, *2.* $b_n \to b$*, and*

3.

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\left(A_n + H_n^\epsilon\right) = \lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\left(A_n + H_n^\epsilon\right) = A, \tag{3.18}$$

where

$$H_n^\epsilon = \int_{|x|<\sqrt{\epsilon}} \frac{xx^T}{|x|^2}\int_0^{\epsilon|x|^{-1}} t^{1-\alpha}e^{-t^p}dt\nu_n(dx). \tag{3.19}$$

Note that by Lemma 3.7, Part 1 in the above is equivalent to the following conditions

$$R_n \stackrel{v}{\to} R \text{ on } \mathbb{R}_0^d \tag{3.20}$$

$$\lim_{N\to\infty}\limsup_{n\to\infty} R_n(|x|>N) = 0 \text{ when } \alpha < 0 \tag{3.21}$$

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|>N}\log|x|R_n(dx) = 0 \text{ when } \alpha = 0, \tag{3.22}$$

where we get R from ν by (3.17).

The proof is similar to the proof of Theorem 3.3, we will only highlight the differences.

Proof of Theorem 3.11 when $\alpha = 0$. In Lemma 3.5, we can replace (3.10) with

$$\begin{aligned}
\int_{\mathbb{R}^d} f_1(x)M_n(dx) &= \int_{\mathbb{R}^d}\int_0^\infty f_1(xt)t^{-1}e^{-t^p}dtR_n(dx) \\
&\geq \int_{|x|\geq 1}\int_{|x|^{-1}}^e t^{-1}e^{-t^p}dtR_n(dx) \\
&\geq e^{-e^p}\int_{|x|\geq 1}\int_{|x|^{-1}}^e t^{-1}dtR_n(dx) \\
&= e^{-e^p}\int_{|x|\geq 1}\left(1+\log|x|\right)R_n(dx) = e^{-e^p}\nu_n\left(|x|\geq 1\right).
\end{aligned}$$

Then in Lemma 3.6 we have $I_1^{n,\epsilon} = 0_{d\times d}$ and

$$I_2^{n,\epsilon} = \int_{\infty>|x|\geq 1}\int_0^{\epsilon|x|^{-1}} xx^Tte^{-t^p}dt\left(1+\log|x|\right)^{-1}\nu_n(dx).$$

We have

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}|I_2^{n,\epsilon}(i,j)| &\leq \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{\infty>|x|\geq 1}|x|^2\int_0^{\epsilon|x|^{-1}} tdt\nu_n(dx)\\ &= \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}.5\epsilon^2\nu_n(|x|\geq 1)=0.\end{aligned}$$

Then in (3.11), we define

$$g_0(x)=\int_{\epsilon|x|^{-1}}^{\infty} f(xt)\,t^{-1}\frac{e^{-t^p}}{|x|^2\wedge\left[1+\log^+|x|\right]}dt.$$

By Proposition A.13 to guarantee that $\int_{\mathbb{R}^d} g_0(x)\nu_{n_j}(dx)\to\int_{\mathbb{R}^d} g_0(x)\tilde{\nu}(dx)$, it suffices to show that $g_0(x)$ is bounded and continuous on $\mathbb{R}^d$. Let N be an upper bound on f. If $|x|\leq 1$ then

$$1_{[t\geq\epsilon|x|^{-1}]}f(xt)t^{-1}e^{-t^p}|x|^{-2}\leq 1_{[t\geq 0]}N\epsilon^{-2}te^{-t^p},$$

which is integrable with respect to t. Now assume that $|x|\geq 1$ and fix $\delta\in(0,|x|)$. If x' is such that $|x'-x|<\delta$ then

$$1_{[t\geq\epsilon|x'|^{-1}]}f(x't)t^{-1}e^{-t^p}\left[1+\log|x'|\right]^{-1}\leq 1_{[t\geq\epsilon(|x|+\delta)^{-1}]}Nt^{-1}e^{-t^p},$$

which is integrable with respect to t. Thus, by dominated convergence (Theorem 16.8 in [Bil95]) g_0 is continuous on $\mathbb{R}^d$. Now to show that $g_0(x)$ is bounded. When $|x|\leq 1$ then, as before

$$g_0(x)\leq N\epsilon^{-2}\int_0^{\infty} te^{-t^p}dt<\infty.$$

If $|x|>1$

$$\begin{aligned}g_0(x) &\leq N\left[1+\log|x|\right]^{-1}\int_{\epsilon|x|^{-1}}^{\infty} t^{-1}e^{-t^p}dt\\ &\leq N\left[1+\log|x|\right]^{-1}\int_{\epsilon|x|^{-1}}^{\epsilon e} t^{-1}dt+N\int_{\epsilon e}^{\infty} t^{-1}e^{-t^p}dt\\ &= N+N\int_{\epsilon e}^{\infty} t^{-1}e^{-t^p}dt.\end{aligned}$$

Observing that

$$\lim_{x\to 0}\frac{e^{-(t/|x|)^p}}{|x|^2\wedge\left[1+\log^+|x|\right]}=0,$$

and performing a change of variables, we can replace (3.12) with

$$\begin{aligned}M(A) &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty 1_A(\xi(x)t)t^{-1}\frac{e^{-(t/|x|)^p}}{|x|^2\wedge\left[1+\log^+|x|\right]}dt\tilde{\nu}(dx)\\ &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty 1_A(\xi(x)t)t^{-1}\frac{e^{-(t/|x|)^p}}{|x|^2\wedge\left[1+\log^+|x|\right]}dt\nu(dx).\end{aligned}$$

This completes the changes. □

Proof of Theorem 3.11 when $\alpha<0$. In Lemma 3.5, we can replace (3.10) with

$$\begin{aligned}\int_{\mathbb{R}^d} f_1(x)M_n(dx) &= \int_{\mathbb{R}^d}\int_0^\infty f_1(xt)t^{-1-\alpha}e^{-t^p}dt\frac{1}{1\wedge|x|^2}\nu_n(dx)\\ &\geq \nu_n(|x|\geq 1)\int_1^\infty t^{-1-\alpha}e^{-t^p}dt.\end{aligned}$$

Then in Lemma 3.6 we have $I_1^{n,\epsilon}=0_{d\times d}$ and

$$I_2^{n,\epsilon}=\int_{|x|\geq 1}xx^T\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n(dx).$$

We have

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}|I_2^{n,\epsilon}(i,j)| &\leq \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|\geq 1}|x|^2\int_0^{\epsilon/|x|}t^{1-\alpha}dt\nu_n(dx)\\ &\leq \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{\epsilon^{2-\alpha}}{2-\alpha}\int_{|x|\geq 1}|x|^\alpha\nu_n(dx)\\ &= \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}C\frac{\epsilon^{2-\alpha}}{2-\alpha}=0.\end{aligned}$$

Then in (3.11), we define

$$g_\alpha(x)=\int_{\epsilon|x|^{-1}}^\infty f(xt)t^{-1-\alpha}\frac{e^{-t^p}}{|x|^2\wedge 1}dt.$$

To guarantee that $\int_{\mathbb{R}^d}g_0(x)\nu_{n_j}(dx)\to\int_{\mathbb{R}^d}g_0(x)\tilde{\nu}(dx)$ we need to show that $g_0(x)$ is bounded and continuous. Let N be the upper bound on f. Note that if $|x|\geq 1$ we have

$$f(xt)t^{-1-\alpha}\frac{e^{-t^p}}{|x|^2\wedge 1}1_{t>\epsilon|x|^{-1}}\leq Nt^{-1-\alpha}e^{-t^p},$$

which is integrable on $[0,\infty)$. If $|x|<1$ we have

$$f(xt)t^{-1-\alpha}\frac{e^{-t^p}}{|x|^2\wedge 1}1_{t>\epsilon|x|^{-1}}\le Nt^{-2}t^{1-\alpha}\frac{e^{-t^p}}{|x|^2}1_{t>\epsilon|x|^{-1}}\le M\epsilon^{-2}t^{1-\alpha}e^{-t^p},$$

which is integrable on $[0,\infty)$. Thus g_α is bounded, and by dominated convergence it is continuous on $\bar{\mathbb{R}}^d$.

Observing that

$$\lim_{x\to 0}\frac{e^{-(t/|x|)^p}}{|x|^{2-\alpha}\wedge|x|^{-\alpha}}=0,$$

and performing a change of variables, we can replace (3.12) with

$$\begin{aligned} M(A) &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty 1_A(\xi(x)t)t^{-1-\alpha}\frac{e^{-(t/|x|)^p}}{|x|^{2-\alpha}\wedge|x|^{-\alpha}}dt\tilde{\nu}(dx)\\ &= \int_{\bar{\mathbb{R}}^d}\int_0^\infty 1_A(\xi(x)t)t^{-1-\alpha}\frac{e^{-(t/|x|)^p}}{|x|^{2-\alpha}\wedge|x|^{-\alpha}}dt\nu(dx). \end{aligned}$$

This completes the changes. □

3.3 Closure Properties

In this section we will show that ETS^p_α is, in fact, the smallest class that contains TS^p_α and is closed under weak convergence.

Proposition 3.12. *Fix $\alpha<2$ and $p>0$.*

1. If $\mu=N(0_d,A)$ then there is a sequence $\{\mu_n\}$ in TS^p_α such that $\mu_n\xrightarrow{w}\mu$.

2. If $\alpha\in(0,2)$ and $\mu=S_\alpha(\sigma,0_d)$ then there is a sequence $\{\mu_n\}$ in TS^p_α such that $\mu_n\xrightarrow{w}\mu$.

3. The class ETS^p_α is the smallest class that contains TS^p_α and is closed under weak convergence. Moreover, this class is closed under finite convolution.

Proof. First observe that by using L'Hospital's rule twice we get

$$\lim_{s\to 0}\frac{e^{i\langle x,z\rangle rs}-1-\frac{i\langle x,z\rangle sr}{1+|xr|^2s^2}}{s^2}=-\frac{1}{2}\langle x,z\rangle^2r^2.$$

Now, let $R = N(0_d, cA)$, where $c = \left[\int_0^\infty r^{1-\alpha} e^{-r^p} dr\right]^{-1}$. Let $X = (X_1, \dots, X_d)^T \sim R$, and let $R_n(A) = n^2 \int_{\mathbb{R}^d} 1_A(xn^{-1}) R(dx)$. This satisfies the assumptions of a Rosiński measure. Let $\mu_n \sim TS_\alpha^p(R_n, 0_d)$. We have

$$\begin{aligned}
C_{\mu_n}(z) &= \int_{\mathbb{R}^d} \int_0^\infty \left(e^{i\langle x,z\rangle r} - 1 - \frac{i\langle x, z\rangle r}{1+|x|^2 r^2} \right) r^{-1-\alpha} e^{-r^p} dr R_n(dx) \\
&= n^2 \int_{\mathbb{R}^d} \int_0^\infty \left(e^{i\langle x,z\rangle r/n} - 1 - \frac{i\langle x, z\rangle r/n}{1+|x/n|^2 r^2} \right) r^{-1-\alpha} e^{-r^p} dr R(dx) \\
&\to -\frac{1}{2} \int_{\mathbb{R}^d} \langle x, z\rangle^2 R(dx) \int_0^\infty r^{1-\alpha} e^{-r^p} dr \\
&= -\frac{1}{2} \int_{\mathbb{R}^d} \langle x, z\rangle^2 R(dx) c^{-1} \\
&= -\frac{1}{2} \sum_{i=1}^d \sum_{j=2}^d z_i z_j E[X_i X_j] c^{-1} = -\frac{1}{2} \langle z, Az\rangle,
\end{aligned}$$

where the fourth line follows by dominated convergence and the fact that

$$\begin{aligned}
& n^2 \left| e^{i\langle x,z\rangle r/n} - 1 - \frac{i\langle x, z\rangle r/n}{1+|x/n|^2 r^2} \right| \\
&= \frac{n^2}{1+|x/n|^2 r^2} \left| e^{i\langle x,z\rangle r/n} - 1 - i\langle x, z\rangle r/n + |x/n|^2 r^2 \left(e^{i\langle x,z\rangle r/n} - 1 \right) \right| \\
&\le n^2 \left(.5\langle x, z\rangle^2 r^2/n^2 + 2|x/n|^2 r^2 \right) \\
&\le .5|x|^2|z|^2 r^2 + 2|x|^2 r^2 = (.5|z|^2 + 2)|x|^2 r^2
\end{aligned}$$

and

$$(.5|z|^2 + 2) \int_{\mathbb{R}^d} |x|^2 \int_0^\infty r^{1-\alpha} e^{-r^p} dr R(dx) < \infty.$$

Now for the second part. Let

$$R(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ut) e^{-t} t^{-\alpha} dt \sigma(du), \qquad A \in \mathfrak{B}(\mathbb{R}^d).$$

It is easy to show that R satisfies the assumptions of a Rosiński measure and that

$$\sigma(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|}\right) |x|^\alpha R(dx), \qquad A \in \mathfrak{B}(\mathbb{S}^{d-1}).$$

Let

$$R_n(A) = n^{-\alpha} \int_{\mathbb{R}^d} 1_A(xn) R(dx), \qquad A \in \mathfrak{B}(\mathbb{R}^d),$$

and let μ_n and M_n be the corresponding probability and Lévy measures. We have

$$\begin{aligned}
C_{\mu_n}(z) &= \int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle x,z\rangle r} - 1 - \frac{i\langle x,z\rangle r}{1+|x|^2r^2}\right) r^{-1-\alpha}e^{-r^p}drR_n(dx) \\
&= n^{-\alpha}\int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle x,z\rangle rn} - 1 - \frac{i\langle x,z\rangle rn}{1+|xn|^2r^2}\right) r^{-1-\alpha}e^{-r^p}drR(dx) \\
&= \int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle x,z\rangle t/|x|} - 1 - \frac{i\langle x,z\rangle t/|x|}{1+t^2}\right) t^{-1-\alpha}e^{-(t|x|^{-1}n^{-1})^p}dt|x|^\alpha R(dx) \\
&\to \int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle x,z\rangle t/|x|} - 1 - \frac{i\langle x,z\rangle t/|x|}{1+t^2}\right) t^{-1-\alpha}dt|x|^\alpha R(dx) \\
&= \int_{\mathbb{S}^{d-1}}\int_0^\infty \left(e^{i\langle u,z\rangle t} - 1 - \frac{i\langle u,z\rangle t}{1+t^2}\right) t^{-1-\alpha}dt\sigma(du),
\end{aligned}$$

where the fourth line follows by the substitution $t = rn|x|$ and the fifth line by dominated convergence.

The third part is an immediate consequence of the first two. □

Definition 3.13. *For $\alpha < 2$, $p > 0$. A random vector is called an* ***elementary p-tempered α-stable random vector*** *on $\mathbb{R}^d$ if it can be written as Ux, where $x \in \mathbb{R}^d_0$ is a nonrandom vector and $U \sim ID(0, M, b)$ is an infinitely divisible random variable on $\mathbb{R}$ with $b \in \mathbb{R}$ and $M(dt) = c1_{[t>0]}t^{-1-\alpha}e^{-t^p}dt$, for some $c > 0$.*

For $\lambda \in \mathbb{R}$, we have

$$\mathrm{E}e^{i\lambda U} = \exp\left\{c\int_0^\infty \left(e^{i\lambda t} - 1 - \frac{i\lambda t}{1+t^2}\right) t^{-1-\alpha}e^{-t^p}dt + i\lambda b\right\}$$

thus for $z \in \mathbb{R}^d$

$$\begin{aligned}
\mathrm{E}e^{i\langle z,Ux\rangle} &= \exp\left\{c\int_0^\infty \left(e^{i\langle z,x\rangle t} - 1 - \frac{i\langle z,x\rangle t}{1+t^2}\right) t^{-1-\alpha}e^{-t^p}dt + i\langle z, xb\rangle\right\} \\
&= \exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{i\langle y,z\rangle t} - 1 - \frac{i\langle y,z\rangle t}{1+t^2}\right) t^{-1-\alpha}e^{-t^p}dtR(dy) + i\langle z, xb\rangle\right\}
\end{aligned}$$

where $R(dy) = c\delta_x(dy)$. Clearly, a random vector is the finite sum of elementary p-tempered α-stable random vectors if and only if its distribution is an element of TS^p_α with Rosiński measure R having a finite support with no point mass at 0. Equivalently, this is true for the extended Rosiński measure ν.

Theorem 3.14. *Fix $\alpha < 2$, $p > 0$. The class ETS^p_α is the smallest class of distributions closed under convolution and weak convergence and containing all elementary p-tempered α-stable distributions. In fact, $\mu \in ETS^p_\alpha$ if and only if there are μ_n, $n = 1, 2, \ldots$ with $\mu_n \xrightarrow{w} \mu$ such that each μ_n is the distribution of the sum of a finite number of independent elementary p-tempered α-stable random vectors.*

For the case when $p = 1$ and $\alpha \in \{-1, 0\}$ this was shown in Theorem F of [BNMS06]. There the results was shown using certain integral representations. Similar representations for the case $\alpha < 2$ and $p > 0$ are given in [MN09]. However, in the case when $\alpha \in (0, 2)$ the properties of the representation are different. Thus it appears that a proof analogous to that of [BNMS06] can only be constructed when $\alpha \leq 0$. We use a different approach, which works for all $\alpha < 2$.

Proof. In light of Proposition 3.12, it suffices to show that we can approximate any distribution in TS^p_α. Let $\mu = TS^p_\alpha(R, b)$ and let ν be its extended Rosiński measure. By Proposition A.11 there is a sequence (ν_n) of measures on $\bar{\mathbb{R}}^d$ with a finite support such that $\nu_n(\{0_d\}) = 0$, $\nu_n(\mathbb{I}^{d-1}) = 0$, and $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d$. Let $\mu_n = ETS^p_\alpha(0_{d\times d}, \nu_n, b)$. Note that

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{|x|<\epsilon} |x|^2 R_n(dx) \leq \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \nu_n\left(|x| \leq \epsilon\right) \leq \lim_{\epsilon\downarrow 0} \nu\left(|x| \leq \epsilon\right) = 0,$$

where the second inequality follows by Proposition A.8. Thus, $\mu_n \xrightarrow{w} \mu$ by Theorem 3.3 when $\alpha \in (0, 2)$ and by Theorem 3.11 when $\alpha \leq 0$. □

Corollary 3.15. *ETS^p_α is the smallest class of distributions closed under convolution and convergence and containing all proper p-tempered α-stable distributions.*

Proof. This follows immediately from Theorem 3.14 and the fact that all elementary p-tempered α-stable distributions are proper. □

3.4 Duality

In this section we will introduce dual Rosiński measures. The idea is analogous to the dual Lévy measures defined in Appendix C.2. Fix $p > 0$ and $\alpha \in (-\infty, 2)\setminus\{0\}$. Throughout this section we will assume that $\alpha \neq 0$ because there is no natural definition of a dual Rosiński measure in this case. Let R be the Rosiński measure of a p-tempered α-stable distribution. In particular R is a Lévy measure and in fact $R \in \mathfrak{M}^\gamma$, where

$$\gamma = \alpha \vee 0. \tag{3.23}$$

Thus it has a γ-dual Lévy measure given by

$$R^\gamma(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^2}\right)|x|^{2+\gamma}R(dx), \qquad A \in \mathfrak{B}(\mathbb{R}^d). \tag{3.24}$$

Note that R^γ is itself the Rosiński measure of some p-tempered α-stable distribution. We will call R^γ the **dual Rosiński measure** of R, and, for simplicity, we will denote it by R^*. It is important to note that if M is the Lévy measure of $TS^p_\alpha(R, b)$ then the dual of M is *not* the Lévy measure of $TS^p_\alpha(R^*, b)$.

Since R^* is the γ-dual Lévy measure of R, all of the properties from Section C.2 carry over. In particular

$$(R^*)^* = R \tag{3.25}$$

and for $\alpha \in (0, 2)$

$$\int_{\mathbb{R}^d} |x|^2 R(dx) < \infty \Longleftrightarrow \int_{\mathbb{R}^d} |x|^\alpha R^*(dx) < \infty. \tag{3.26}$$

Thus the dual of a finite variance p-tempered α-stable distribution with $\alpha \in (0, 2)$ is proper and vice versa. Moreover, in this case, the dual of a proper p-tempered α-stable distribution with a finite variance is a proper p-tempered α-stable distribution with a finite variance.

We will also need to define the dual of the extended Rosiński measure. Let $\mu = TS^p_\alpha(R, b)$ and define **the dual extended Rosiński measure** as

$$\nu^*(A) = \int_{\mathbb{R}^d} 1_A(x) \left(|x|^\gamma \wedge |x|^2\right) R^*(dx) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^2}\right) \nu(dx), \tag{3.27}$$

where $\nu(dx) = (|x|^\gamma \wedge |x|^2) R(dx)$. Note that we have not defined the dual extended Rosiński measure for all elements of ETS^p_α. For the others there is no natural definition. Our main use of duals will be for proving limit theorems. These will follow from the following proposition.

Proposition 3.16. *Fix $\alpha \in (-\infty, 2) \setminus \{0\}$, $p > 0$. If $X_n \sim TS^p_\alpha(R_n, 0_d)$ and $X^*_n \sim TS^p_\alpha(R^*_n, 0_d)$ then*

$$X_n \xrightarrow{d} X_1 \Longleftrightarrow X^*_n \xrightarrow{d} X^*_1.$$

Proof. By (3.25), it suffices to show only one direction. Assume that $X_n \xrightarrow{d} X_1$ holds. We have $R_n \xrightarrow{v} R_1$ on $\mathbb{R}^d_0$,

$$\lim_{N\to\infty} \limsup_{n\to\infty} \int_{|x|>N} |x|^\gamma R_n(dx) = 0, \tag{3.28}$$

and

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{|x|<\epsilon} |x|^2 R_n(dx) = 0. \tag{3.29}$$

For $\alpha \in (0, 2)$ this follows from Corollary 3.8, and for $\alpha < 0$ it follows from Theorem 3.11.

Let f be a continuous function mapping $\mathbb{R}^d$ into $\mathbb{R}$ such that f vanishes on a neighborhood of 0 and on a neighborhood of ∞. Note that this implies that $g(x) = f\left(\frac{x}{|x|^2}\right)|x|^{\gamma+2}$ is also a continuous function mapping $\mathbb{R}^d$ into $\mathbb{R}$ and vanishing on a neighborhood of 0 and on a neighborhood of ∞. We have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) R^*_n(dx) = \lim_{n\to\infty} \int_{\mathbb{R}^d} f\left(\frac{x}{|x|^2}\right)|x|^{\gamma+2} R_n(dx)$$
$$= \int_{\mathbb{R}^d} f\left(\frac{x}{|x|^2}\right)|x|^{\gamma+2} R_1(dx) = \int_{\mathbb{R}^d} f(x) R^*_1(dx).$$

By (3.29), we have

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|\geq N}|x|^\gamma R_n^*(dx) = \lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|\leq 1/N}|x|^2 R_n(dx) = 0$$

and by (3.28) we have

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}|x|^2 R_n^*(dx) = \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|>1/\epsilon}|x|^\gamma R_n(dx) = 0.$$

From here, the result follows by Corollary 3.8 for $\alpha\in(0,2)$ and by Theorem 3.11 for $\alpha<0$. □

Lemma 3.17. *Fix $\alpha\in(0,2)$, $p>0$. Let $\nu,\nu_1,\nu_2,\dots$ be a sequence of extended Rosiński measures of distributions in ETS^p_α such that $\nu_n(\mathbb{I}^{d-1})=0$ and $\nu(\mathbb{I}^{d-1})\geq 0$. If $\nu_n\xrightarrow{v}\nu$ on $\bar{\mathbb{R}}^d_0$ then*

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\int_{|x|<\sqrt{\epsilon}}\xi(x)\left[\xi(x)\right]^T\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n^*(dx)$$
$$= K\int_{\mathbb{I}^{d-1}}\xi(x)\left[\xi(x)\right]^T\nu(dx), \tag{3.30}$$

where $K=\int_0^\infty t^{1-\alpha}e^{-t^p}dt$.

Proof. It suffices to show that for all $z\in\mathbb{R}^d$

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\int_{|x|<\sqrt{\epsilon}}\langle\xi(x),z\rangle^2\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt\nu_n^*(dx) = K\int_{\mathbb{I}^{d-1}}\langle\xi(x),z\rangle^2\nu(dx).$$

Let

$$f_\epsilon(x) = \langle\xi(x),z\rangle^2\int_0^{\epsilon|x|^{-1}}t^{1-\alpha}e^{-t^p}dt.$$

Thus for $x\in\mathbb{R}^d_0$

$$f_\epsilon\left(\frac{x}{|x|^2}\right) = \langle\xi(x),z\rangle^2\int_0^{\epsilon|x|}t^{1-\alpha}e^{-t^p}dt.$$

Define

$$g_\epsilon(x) = \begin{cases} 0 & \text{if } x=0 \\ f_\epsilon\left(\frac{x}{|x|^2}\right) & \text{if } x\in\mathbb{R}^d_0 \\ \langle\xi(x),z\rangle^2 K & \text{if } x\in\mathbb{I}^{d-1}\end{cases}.$$

Note that g_ϵ is a bounded and continuous function. For $\epsilon > 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{|x|<\sqrt{\epsilon}} f_\epsilon(x)\nu_n^*(dx) &= \lim_{n\to\infty}\int_{|x|>1/\sqrt{\epsilon}} g_\epsilon\,(x)\,\nu_n(dx)\\ &= \int_{|x|>1/\sqrt{\epsilon}} g_\epsilon\,(x)\,\nu(dx),\end{aligned}$$

where the second equality follows by Proposition A.8 and the fact that g_ϵ is bounded and continuous. Now observe that

$$\begin{aligned}\limsup_{\epsilon\downarrow 0}\int_{|x|>1/\sqrt{\epsilon}} g_\epsilon\,(x)\,\nu(dx) &\leq \lim_{\epsilon\downarrow 0} K\int_{|x|>1/\sqrt{\epsilon}} \langle\xi(x),z\rangle^2\nu(dx)\\ &= K\int_{\mathbb{I}^{d-1}} \langle\xi(x),z\rangle^2\nu(dx)\end{aligned}$$

and

$$\begin{aligned}\liminf_{\epsilon\downarrow 0}\int_{|x|>1/\sqrt{\epsilon}} g_\epsilon\,(x)\,\nu(dx) &\geq \lim_{\epsilon\downarrow 0}\int_{|x|>1/\epsilon^2} g_\epsilon\,(x)\,\nu(dx) \qquad (3.31)\\ &\geq \lim_{\epsilon\downarrow 0}\int_0^{1/\epsilon} t^{1-\alpha}e^{-t^p}dt\int_{|x|>1/\epsilon^2}\langle\xi(x),z\rangle^2\nu(dx)\\ &= K\int_{\mathbb{I}^{d-1}}\langle\xi(x),z\rangle^2\nu(dx).\end{aligned}$$

This completes the proof. □

For $\beta\in(\gamma,2)$, the distribution $S_\beta(\sigma,b)\in TS_\alpha^p$ and it has Rosiński measure B given by (2.17). It is easy to see that

$$B^*(A) = K^{-1}\int_{\mathbb{S}^{d-1}}\int_0^\infty 1_A(ur)r^{-(\alpha+2-\beta)-1}dr\sigma(du),$$

where $K=\int_0^\infty t^{\beta-\gamma-1}e^{-t^p}dt$. Note that $(\gamma+2-\beta)\in(\gamma,2)$ and B^* is the Rosiński measure of $S_{\gamma+2-\beta}(\sigma,b)$. Thus Proposition 3.16 implies that if a sequence in TS_α^p converges weakly to $S_\beta(\sigma,b)$ then the sequence in TS_α^p with the dual Rosiński measures converges to $S_{\gamma+2-\beta}(\sigma,b)$.

Now focus on the case when $\alpha\in(0,2)$. The distribution $S_\alpha(\sigma,b)$ is not an element of TS_α^p. Nevertheless, by Lemma 3.17, if a sequence in TS_α^p converges to $S_\alpha(\sigma, b)$ then the sequence with the dual Rosiński measures converges to a Gaussian. We summarize these results below.

Corollary 3.18. *Fix $\alpha \in (-\infty, 2) \setminus \{0\}$, $p > 0$. Let $X_n \sim TS^p_\alpha(R_n, 0_d)$ and let $X^*_n \sim TS^p_\alpha(R^*_n, 0_d)$.*

1. If $\beta \in (\gamma, 2)$, $X \sim S_\beta(\sigma, 0_d)$, and $X^* \sim S_{2+\gamma-\beta}(\sigma, 0_d)$ then

$$X_n \xrightarrow{d} X \iff X^*_n \xrightarrow{d} X^*.$$

2. If $\alpha \in (0, 2)$ and $X \sim S_\alpha(\sigma, 0_d)$ then

$$X_n \xrightarrow{d} X \Longrightarrow X^*_n \xrightarrow{d} X^*,$$

where $X^* \sim N(0_d, A)$ with

$$A = \int_0^\infty t^{1-\alpha} e^{-t^p} dt \int_{\mathbb{S}^{d-1}} uu^T \sigma(du).$$

Note that, in the second part, the arrow is only in one direction. The following is a counter example to the reverse implication. For another counterexample see Example 6 in Section 4.2.

Example 5 Let $d = 1$, $\alpha \in (0, 2)$, $p > 0$, and $\nu_n = \delta_{(-1)^n/n}$. We have

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \int_{|x|<\sqrt{\epsilon}} \int_0^{\epsilon |x|^{-1}} t^{1-\alpha} e^{-t^p} dt \nu^*_n(dx) &= \lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \int_0^{\epsilon n} t^{1-\alpha} e^{-t^p} dt \\ &= \int_0^\infty t^{1-\alpha} e^{-t^p} dt. \end{aligned}$$

Thus by Theorem 3.3 the sequence $ETS^p_\alpha(0, \nu_n, 0)$ converges weakly to the distribution $N(0, \int_0^\infty t^{1-\alpha} e^{-t^p} dt)$. However, $\nu^*_n = \delta_{(-1)^n n}$. Thus $\nu^*_{2n}([1, \infty]) = 1$ and $\nu^*_{2n+1}([1, \infty]) = 0$. Hence, by Proposition A.8, ν^*_n cannot converge vaguely on $\bar{\mathbb{R}}_0$. Thus by Theorem 3.3 the sequence $ETS^p_\alpha(0, \nu^*_n, 0)$ does not converge weakly. □

CHAPTER 4

TEMPERED STABLE LÉVY PROCESSES

4.1 Path Properties

Fix $\alpha < 2$ and $p > 0$. We will call a Lévy process $\{X_t : t \geq 0\}$ in $\mathbb{R}^d$ where $X_1 \sim TS^p_\alpha(R, b)$ a **p-tempered α-stable Lévy process**. A **proper p-tempered α-stable Lévy process** will be defined analogously. In this section, we will discuss some properties of such processes. Many of these properties make use of the càdlàg property of Lévy processes. Thus, the results in this section are not for Lévy processes in law. See Definition D.1 and the discussion at the end of Appendix D for details.

Proposition 4.1. *Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, b)$. Assume that $R(\mathbb{R}^d) > 0$.*

1) The paths of $\{X_t : t \geq 0\}$ are discontinuous a. s.

2) The paths of $\{X_t : t \geq 0\}$ are piecewise constant a. s. if and only if $\alpha < 0$, R is a finite measure, and $b = \frac{x}{1+|x|^2}$.

3) If $\alpha < 0$ and R is a finite measure, then, almost surely, jumping times are infinitely many and countable in increasing order, and the first jumping time has an exponential distribution with mean $1/a$, where $a = R(\mathbb{R}^d) \int_0^\infty e^{-t^p} t^{-1-\alpha} dt$.

4). If $\alpha \geq 0$ or R is an infinite measure, then, almost surely, jumping times are countable and dense in $[0, \infty)$.

Proof. 1) follows by Theorem 21.1 in [Sat99]. 2) follows by Theorem 21.2 in [Sat99] and Proposition 2.13. 3) and 4) follow by Theorem 21.3 in [Sat99] and Lemma 2.12. □

We will now discuss when the variation of a p-tempered α-stable Lévy process is finite.

Definition 4.2. *Let $-\infty < a < b < \infty$. Let $f : [a, b] \to \mathbb{R}^d$ be a cádlág function. For every $q > 0$ we define*

$$V_q(f; a, b) = \sup \sum_{j=1}^{n} |f(t_i) - f(t_{i-1})|^q,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of the interval $[a, b]$.

*1. If $V_q(f; a, b) < \infty$ we say that f has **finite q-variation on** $[a, b]$.*

*2. If f is defined on all of $\mathbb{R}$, it is said to have **finite q-variation** if it has finite q-variation on each compact interval.*

*3. A stochastic process $\{X_t(\omega) : \ t \geq 0, \omega \in \Omega\}$ is said to be of **finite q-variation** if the paths are of finite q-variation for almost all $\omega \in \Omega$. When $q = 1$ we just say that the process has **finite variation**.*

Finiteness of q-variation gives some useful results about possible definitions of stochastic integrals. It is well known that if a function (or a stochastic process) has finite 1-variation, then one can define a Stieljes integral with respect to it for a large class of functions. When the 1-variation is infinite, under some assumptions about the finiteness of q-variation for some $q > 0$ of the integrand and integrator, we can define generalizations of Stieljes integrals, see [DN98] for details.

We now list some well known path properties of Lévy processes, and specialize them to the case of p-tempered α-stable Lévy processes.

Theorem 4.3. *Let $\{X_t : \ t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0_{d\times d}, M, b)$. When $\int_{|x|\leq 1} |x| M(dx) < \infty$ further assume that $b_0 = 0$ (where b_0 is defined by* (C.8)*). If*

$$\beta = \inf \left\{ \gamma > 0 : \int_{|x|\leq 1} |x|^\gamma M(dx) < \infty \right\}.$$

then $\beta \in [0, 2]$, and

1. if $\gamma < \beta$ then

i) $\limsup_{t\to 0} t^{-1/\gamma}|X_t| = \infty$ *with probability* 1 *and*

ii) $V_\gamma(X(\cdot);0,1) = \infty$ *with probability* 1*;*

2. if $\gamma > \beta$ *then*

i) $\lim_{t\to 0} t^{-1/\gamma} X_t \to 0$ *as with probability* 1 *and*

ii) $V_\gamma(X(\cdot);0,1) < \infty$ *with probability* 1*;*

3. for $\alpha < 2$ *and* $p > 0$*, if* $X_1 \sim TS^p_\alpha(R,b)$*, then* $\beta = (\alpha \vee r)$*, where* $r = \inf\left\{\gamma > 0 : \int_{|x|\le 1} |x|^\gamma R(dx) < \infty\right\}$.

In Parts 1 i) and 2 i) we can replace X_t by $\sup_{0\le s\le t}|X_s|$ see (3.4) in [Pru81].

Proof. The fact that $\beta \in [0,2]$ follows by the definition of a Lévy measure. 1 i) and ii) are Theorems 3.3 and 4.1 in [BG61]. 2 i) is Theorem 3.1 in [BG61]. 2 ii) follows from Theorem 4.2 in [BG61], Theorem 2 in [Mon72] and the discussion on page 1214 in [Mon72]. Part 3 follows immediately from Proposition 2.8. □

The theorem implies that, when $\alpha \in (0,2)$, for proper p-tempered α-stable distributions, $\beta = \alpha$. Note that according to page 494 of [BG61], $\beta = \alpha$ for α-stable Lévy processes as well. For the case of finite variation, there is more that can be said.

Proposition 4.4. *Let* $\{X_t : t \ge 0\}$ *be a Lévy process with* $X_1 \sim TS^p_\alpha(R,b)$*.*
1) If $\alpha \in [1,2)$ *or* $\int_{|x|\le 1}|x|R(dx) = \infty$*, then, almost surely,* X_t *has infinite variation on* $(0,t]$ *for any* $t \in (0,\infty)$*.*
2) If $\alpha < 1$ *and* $\int_{|x|\le 1}|x|R(dx) < \infty$*, then, almost surely,* X_t *has finite variation on* $(0,t]$ *for any* $t \in (0,\infty)$*. The variation function* $V_t(\omega)$ *of* $X_t(\omega)$ *is a subordinator with*

$$E[e^{-uV_t}] = \exp\left\{t\left(\int_{\mathbb{R}^d}\int_0^\infty \left(e^{-ur|x|}-1\right)e^{-r^p}r^{-1-\alpha}dr R(dx) - u|b_0|\right)\right\}, \quad u \ge 0,$$

where b_0 *is given by* (C.8)

Proof. For general Lévy processes with Lévy measure M, Theorem 21.9 in [Sat99] says that the variation is infinite if and only if $\int_{|x|\le 1}|x|M(dx)<\infty$. For p-tempered α-stable distributions this conditions is equivalent to $\alpha<1$ and $\int_{|x|\le 1}|x|R(dx)<\infty$ by Proposition 2.8. In the case of finite variation, for general Lévy processes the Laplace transform of the variation function is given by Theorem 21.9 in [Sat99]. The form here follows by (2.11). □

In particular this implies that a p-tempered α-stable Lévy process has finite variation if and only if $\alpha<1$ and $\int_{|x|\le 1}|x|R(\mathrm{d}x)<\infty$. In light of Theorem 2.3, proper p-tempered α-stable Lévy processes with $\alpha<1$ have finite variation.

By Lemma 2.2 for $\alpha\in(0,2)$ and $p>0$ proper p-tempered α-stable distributions are generalized tempered stable with tempering function q_p and limiting function $g\equiv 1$. Thus they inherit properties of those models.

Since, for proper p-tempered α-stable distributions $q_p(r,u)\le 1$ for all $r\ge 0$ and $u\in\mathbb{S}^{d-1}$, we can use Theorem 5.5 in [RS10] to get a series representation of the corresponding Lévy processes. This representation depends on two functions, which in this case will be $h(x)\equiv 1$ and $\rho^{-1}(x)=(\alpha x)^{-1/\alpha}$. We also get conditions for absolute continuity with respect to an α-stable Lévy process. This follows from Theorem 4.1 in [RS10]. First we need more details about the probability space. In this we follow [RS10].

Let $\Omega=D([0,\infty),\mathbb{R}^d)$ be the space of càdlàg mappings ω from $[0,\infty)$ into $\mathbb{R}^d$. A process $\{X_t : t\ge 0\}$ on $\mathbb{R}^d$ is said to be canonical if $X_t(\omega)=\omega(t)$, $t\ge 0$, $\omega\in\Omega$, and Ω is equipped with the σ-algebra $\mathcal{F}=\sigma\{X_s : s\ge 0\}$ and the right-continuous natural filtration $\mathcal{F}_t=\bigcap_{s>t}\sigma\{X_u : u\le s\}$, $t\ge 0$. The canonical process is completely determined by a probability measure P on $(\Omega,\mathcal{F})$. Let $\Delta X_t=X_t-X_{t-}$ and let $P_{|\mathcal{F}_t}$ denote the restriction of P to the σ-algebra $\mathcal{F}_t$.

Theorem 4.5. *Fix $\alpha\in(0,2)$, $p>0$. In the above setting, consider two probability* *measures* P_0 *and* P *on* $(\Omega, \mathcal{F})$. *Let* $\{X_t : t \geq 0\}$ *be the canonical process. Assume that under* P, $X_1 \sim TS^p_\alpha(R, b)$ *and under* P_0, $X_1 \sim S_\alpha(\sigma, a)$. *Assume that* R *is related to* σ *by Proposition 2.7. Then*

(i) $P_{0|\mathcal{F}_t}$ *and* $P_{|\mathcal{F}_t}$ *are mutually absolutely continuous for every* $t > 0$ *if and only if*

$$\int_{\mathbb{S}^{d-1}} \int_0^1 [1 - q_p(r, u)]^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) < \infty \tag{4.1}$$

and

$$b - a = \int_{\mathbb{R}^d} \int_0^\infty \frac{x}{1 + |x|^2 t^2} t^{-\alpha} (e^{-t^p} - 1) dt R(dx). \tag{4.2}$$

(ii) If $P_{0|\mathcal{F}_t}$ *and* $P_{|\mathcal{F}_t}$ *are not mutually absolutely continuous for some* $t > 0$, *then they are singular for all* $t > 0$.

(iii) If (4.1) *and* (4.2) *hold, then for every* $t > 0$

$$\frac{\mathrm{d}P_{|\mathcal{F}_t}}{\mathrm{d}P_{0|\mathcal{F}_t}} = e^{U_t}, \ P_0\text{-}a.s. \tag{4.3}$$

where $\{U_t : t \geq 0\}$ *is a Lévy process on* $\mathbb{R}^d$ *defined on the probability space* $(\Omega, \mathcal{F}, P_0)$. *It is given by*

$$\begin{aligned} U_t \;\; = \;\; \lim_{\epsilon \downarrow 0} \Bigg\{ & \sum_{\{s \in (0,t] : |\Delta X_s| > \epsilon} \log q_p \left(|\Delta X(s)|, \frac{\Delta X(s)}{|\Delta X(s)|} \right) \\ & + t \int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty [1 - q_p(r, u)] r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) \Bigg\}. \end{aligned}$$

Convergence in the above is uniform in t *on any bounded interval,* P_0*-a.s. The Lévy measure* ν *of the distribution of* U_1 *is concentrated on* $(-\infty, 0)$ *and is determined by*

$$\int_{-\infty}^0 F(s)\nu(ds) = \int_{\mathbb{S}^{d-1}} \int_0^\infty F(\log q_p(r, u)) r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)$$

for every Borel function F. *The characteristic function of* U_1 *is given by*

$$E^{P_0} e^{i\theta U_1} = \exp\left\{ i\theta\gamma_U + \int_{-\infty}^0 \left[e^{i\theta v} - 1 - i\theta v 1_{[-1,0)}(v) \right] \nu(\mathrm{d}v) \right\}, \tag{4.4}$$

where

$$\gamma_U = -\int_{-\infty}^{0} \left(e^v - 1 - v1_{[-1,0)}(v)\right) \nu(\mathrm{d}v)$$

We have $E^{P_0}[e^{U_t}] = E^P[e^{-U_t}] = 1$.

Proof. For generalized tempered α-stable distributions analogues of Parts (i) and (ii) are given in Theorem 4.1 of [RS10]. To specialize them to proper p-tempered α-stable distributions, it suffices to plug the tempering function q_p and limiting function $g = 1$ into the formula and perform a change of variables. In [RS10] the analogue of (4.1) is actually

$$\int_{\mathbb{S}^{d-1}} \int_0^1 (1 - [q_p(r,u)]^{1/2})^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) < \infty.$$

However, this is equivalent to (4.1) by the fact that for $x \in [0,1]$

$$.25(1-x)^2 \leq (1-\sqrt{x})^2 \leq (1-x)^2.$$

For any Lévy process, an analogue of Part (iii) is given in Theorem 33.2 of [Sat99]. To specialize it to p-tempered α-stable distributions we again need to plug in the correct form for the Lévy measure. □

We will show that (4.1) implies that the norm of the integral in (4.2) is finite. Observe that by Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \int_0^1 r[1 - q_p(r,u)] r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) \\
&\leq \left(\int_{\mathbb{S}^{d-1}} \int_0^1 r^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)\right)^{1/2} \left(\int_{\mathbb{S}^{d-1}} \int_0^1 [1 - q_p(r,u)]^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)\right)^{1/2} \\
&= \left(\frac{\sigma(\mathbb{S}^{d-1})}{2-\alpha}\right)^{1/2} \left(\int_{\mathbb{S}^{d-1}} \int_0^1 [1 - q_p(r,u)]^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)\right)^{1/2}.
\end{aligned}$$

Thus if (4.1) holds, then by (2.7) and (2.9)

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_0^{|x|^{-1}} (1 - e^{-t^p}) t^{-\alpha} dt |x| R(dx) \\
&\qquad = \int_{\mathbb{S}^{d-1}} \int_0^1 r[1 - q_p(r,u)] r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) < \infty. \qquad (4.5)
\end{aligned}$$

From here it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^\infty \frac{|x|}{1+|x|^2t^2}\left(1-e^{-t^p}\right)t^{-\alpha}dtR(dx)\\
&=\int_{\mathbb{R}^d}\int_0^{|x|^{-1}} \frac{|x|}{1+|x|^2t^2}\left(1-e^{-t^p}\right)t^{-\alpha}dtR(dx)\\
&\quad+\int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty \frac{|x|}{1+|x|^2t^2}\left(1-e^{-t^p}\right)t^{-\alpha}dtR(dx)\\
&\le \int_{\mathbb{R}^d}\int_0^{|x|^{-1}} \left(1-e^{-t^p}\right)t^{-\alpha}dt|x|R(dx)+\int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty \frac{|x|}{1+|x|^2t^2}t^{-\alpha}dtR(dx)\\
&=\int_{\mathbb{R}^d}\int_0^{|x|^{-1}} \left(1-e^{-t^p}\right)t^{-\alpha}dt|x|R(dx)+\int_{\mathbb{R}^d}|x|^\alpha R(dx)\int_1^\infty \frac{t^{-\alpha}}{1+t^2}dt<\infty.
\end{aligned}$$

Further, from (4.5), we can show that a necessary condition for (4.1) to hold is $\int_{|x|\le 1}|x|R(dx)<\infty$ and $p>\alpha-1$. Of course this always holds for $\alpha\in(0,1]$. By (4.5) we have

$$\infty>\int_{\mathbb{R}^d}\int_0^{|x|^{-1}}(1-e^{-t^p})u^{-\alpha}du|x|R(dx)\ge\int_{|x|\le 1}|x|R(dx)\int_0^1 t^{-\alpha}(1-e^{-t^p})dt,$$

and since $e^{-t^p}\le 1-t^p+t^{2p}/2$

$$\begin{aligned}
\int_0^1 t^{-\alpha}(1-e^{-t^p})dt &\ge \int_0^1 t^{-\alpha}(t^p-t^{2p}/2)dt\\
&= \int_0^1 t^{p-\alpha}(1-t^p/2)dt=.5\int_0^1 t^{p-\alpha}dt,
\end{aligned}$$

which is infinite when $p\le\alpha-1$.

4.2 Long and Short Time Behavior for TS^p_α Distributions

In Appendix D.2 we characterize the long and short time behavior of Lévy processes. In this section we will specialize those results to the p-tempered α-stable case. Thus for $c\in\{0,\infty\}$ we will give necessary and sufficient conditions for when there exists a random vector Y whose distributions is not concentrated at a point and a function $a_t>0$ such that $a_tX_t\stackrel{d}{\to}Y$ as $t\to c$. Note that the Rosiński measure of $a_t X_t$ is given by

$$R_t(A) = t \int_{\mathbb{R}^d} 1_A(a_t x) R(dx), \qquad A \in \mathfrak{B}\left(\mathbb{R}^d\right). \tag{4.6}$$

4.2.1 Long Time Behavior

We begin with the case where the limit is not Gaussian.

Theorem 4.6. *Fix $\alpha < 2$, $p > 0$, and $\beta \in (\alpha \vee 0, 2)$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, 0_d)$ and let $Y \sim S_\beta(\sigma, 0_d)$ with $\sigma \neq 0$. There exists a function a_t such that*

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to \infty \tag{4.7}$$

if and only if $R \in RV^\infty_{-\beta}(\sigma)$. Moreover, in this case $a \in RV^\infty_{-1/\beta}$ and

$$a_t \sim s^{1/\beta}/V^{\leftarrow}(t) \text{ as } t \to \infty, \tag{4.8}$$

where $s = \frac{p}{\alpha\Gamma\left(\frac{\beta-\alpha}{p}\right)} \sigma(\mathbb{S}^{d-1})$ and $V(t) = 1/R\left(|x| > t\right)$.

Proof. Let M be the Lévy measure of $TS^p_\alpha(R, 0_d)$. By Theorem D.6, (4.7) holds if and only if $M \in RV^\infty_{-\beta}(\sigma)$. By Theorem 2.27, this holds if and only if $R \in RV^\infty_{-\beta}(\sigma)$. The form of a_t follows from Theorem D.6, Theorem 2.27, and Proposition B.2. □

We will now consider the case where the limiting distribution is Gaussian.

Theorem 4.7. *Fix $\alpha < 2$, $p > 0$, and let B be a nonnegative definite matrix with $B \neq 0_{d\times d}$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, 0_d)$ and define*

$$A_t = \int_{|x|\leq t} xx^T R(dx). \tag{4.9}$$

Then

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to \infty \tag{4.10}$$

with $Y \sim N(0_d, B)$ *if and only if* $A \in LRV_0^\infty(B)$. *Moreover, in this case* $a \in RV_{-1/2}^\infty$ *and*

$$a_t \sim K^{-1/2}/g^{\leftarrow}(t) \text{ as } t \to \infty, \tag{4.11}$$

where $g(t) = t^2/\int_{|x|\le t}|x|^2R(dx)$ *and*

$$K = \int_0^\infty t^{1-\alpha}e^{-t^p}dt = p^{-1}\Gamma\left(\frac{2-\alpha}{p}\right). \tag{4.12}$$

Proof. Let M be the Lévy measure of X_1 and let

$$C_t = \int_{|x|\le t} xx^T M(dx).$$

By Theorem D.11, (4.10) holds if and only if $C \in LRV_0^\infty(B)$. When it holds, $a_t \in RV_{-1/2}^\infty$ with $a_t \sim 1/h^{\leftarrow}(t)$ where $h(t) = t^2/\int_{|x|\le t}|x|^2M(dx) = t^2/\mathrm{tr}C_t$. By Proposition 2.29, $C \in LRV_0^\infty(B)$ if and only if $B \in LRV_0^\infty(B)$. Moreover, in this case $\mathrm{tr}C_t \sim K\mathrm{tr}A_t$ as $t \to \infty$. Thus $h(t) \sim K^{-1}g(t)$ as $t \to \infty$ and $h^{\leftarrow}(t) \sim K^{1/2}g^{\leftarrow}(t)$ as $t \to \infty$. Hence (4.11) holds. □

We now turn to the case when $\alpha \in (0,2)$ and the limiting stable distribution has the same index of stability as the one being tempered. Note that an α-stable distribution with spectral measure σ is an element of ETS_α^p with extended Rosiński measure given by

$$\nu(A) = \int_{\mathbb{S}^{d-1}} 1_A(\infty x)\sigma(dx), \qquad A \in \mathfrak{B}(\bar{\mathbb{R}}^d).$$

In particular, this means that $\nu(\mathbb{R}^d) = 0$.

Proposition 4.8. *Fix* $\alpha \in (0,2)$. *Let* $\{X_t : t \ge 0\}$ *be a Lévy process with* $X_1 \sim ETS_\alpha^p(0_{d\times d}, \nu, 0_d)$ *where* $\nu(\mathbb{I}^{d-1}) = 0$ *and let* $Y \sim S_\alpha(\sigma, 0_d)$ *with* $\sigma \ne 0$. *Then*

$$a_tX_t \xrightarrow{d} Y \text{ as } t \to \infty \tag{4.13}$$

if and only if $\nu \in RV_0^\infty(\sigma)$. *Moreover, when this holds,* $a \in RV_{-1/\alpha}^\infty$ *with*

$$a_t \sim 1/V^{\leftarrow}(t), \tag{4.14}$$

where $V(t) = \sigma(\mathbb{S}^{d-1})t^\alpha/\nu(|x| > t)$.

Proof. By Proposition B.8 $\nu \in RV_0^\infty(\sigma)$ if and only if there is a function a_t with

$$\lim_{t\to\infty} t a_t^\alpha \nu\left(|x| > s/a_t, \xi(x) \in D\right) = \sigma(D). \tag{4.15}$$

When this holds $a \in RV_{-1/\alpha}^\infty$ and a_t is as in (4.14). Thus, it suffices to show that (4.13) holds if and only if (4.15) holds. Let R be the Rosiński measure of the distribution of X_1, let ν' be the extended Rosiński measure of Y, and let ν_t is the extended Rosiński measure of $a_t X_t$.

First assume that (4.13) holds. Lemma D.4 implies that $\lim_{t\to\infty} a_t = 0$. By Theorem 3.3 $\nu_t \xrightarrow{v} \nu'$ on $\bar{\mathbb{R}}_0^d$. Let $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ such that $\sigma(\partial D) = 0$. For $s \in (0,\infty)$, let $A_D^s = \{|x| > s, \xi(x) \in D\}$. We have $\nu'(\partial A_D^s) = 0$. Thus by Proposition A.8

$$\lim_{t\to\infty} \nu_t(A_D^s) = \nu'(A_D^s) = \nu'(\infty D) = \sigma(D). \tag{4.16}$$

When $s \geq 1$ and t is large enough

$$\nu_t(A_D^s) = t a_t^\alpha \int_{|x|>s/a_t} 1_D(\xi(x))|x|^\alpha R(dx) = t a_t^\alpha \nu\left(|x| > s/a_t, \xi(x) \in D\right).$$

Thus, by (4.16) for $s \geq 1$ (4.15) holds. When $s \in (0,1)$ and t is large we have

$$\begin{aligned} t a_t^\alpha \nu\left(|x| > s/a_t, \xi(x) \in D\right) &= t a_t^\alpha \nu\left(|x| > 1/a_t, \xi(x) \in D\right) \\ &\quad + t a_t^\alpha \nu\left(1/a_t \geq |x| > s/a_t, \xi(x) \in D\right). \end{aligned}$$

We will show that the second part goes to 0 when t gets large. Once we show this, by the case when $s = 1$, we will have

$$\lim_{t\to\infty} t a_t^\alpha \nu\left(|x| > s/a_t, \xi(x) \in D\right) = \lim_{t\to\infty} t a_t^\alpha \nu\left(|x| > 1/a_t, \xi(x) \in D\right) = \sigma(D).$$

To show that the other part goes to 0 note that for large enough t, $s/a_t > 1$

$$\begin{aligned}
& ta_t^\alpha \nu\left(1/a_t \geq |x| > s/a_t, \xi(x) \in D\right) \leq ta_t^\alpha \nu\left(1/a_t \geq |x| > s/a_t\right) \\
&= ta_t^\alpha \int_{1/a_t \geq |x| > s/a_t} |x|^\alpha R(dx) \leq ts^{\alpha-2} \int_{1 \geq |x| a_t > s} (a_t|x|)^2 R(dx) \\
&= s^{\alpha-2} \int_{1 \geq |x| > s} \nu_t(dx) = s^{\alpha-2} \nu_t(1 \geq |x| > s) \underset{t\to\infty}{\to} s^{\alpha-2}\nu'(1 \geq |x| > s) = 0.
\end{aligned}$$

Thus (4.15) holds.

Now assume (4.15) holds. Recall that this means that $\nu \in RV_0^\infty(\sigma)$. By Lemma A.12, this means that $\nu_t \stackrel{v}{\to} \nu'$ on $\bar{\mathbb{R}}_0^d$ as $t \to \infty$. By Theorem 3.3 it remains to check that the limit has no Gaussian part. By (3.8), it suffices to show that

$$\limsup_{t\to\infty} \nu_t(|x| < 1) = 0.$$

Since $\lim_{t\to\infty} a_t = 0$, for large enough t, $1/a_t > 1$. Thus

$$\begin{aligned}
\lim_{t\to\infty} \nu_t(|x| < 1) &= \lim_{t\to\infty} ta_t^2 \int_{|x|<1/a_t} |x|^2 R(dx) \\
&= \sigma(\mathbb{S}^{d-1}) \lim_{t\to\infty} \frac{a_t^2 \int_{|x|<1/a_t} |x|^2 R(dx)}{a_t^\alpha \nu(|x| > 1/a_t)} \\
&= \sigma(\mathbb{S}^{d-1}) \lim_{t\to\infty} \frac{\int_{|x|<1/a_t} |x|^2 R(dx)}{(1/a_t)^{2-\alpha} \int_{|x|>1/a_t} |x|^\alpha R(dx)} = 0.
\end{aligned}$$

To see that the last equality holds we use the fact that (B.14) implies that the function $\nu(|x| > \cdot) \in RV_0^\infty$ thus the equality holds by Corollary B.5. From here the result follows. □

By Theorem D.6, (4.13) holds for some random vector $Y \sim S_\alpha(\sigma, 0_d)$ with $\sigma(\mathbb{S}^{d-1}) > 0$ if and only if the Lévy measure of X_1 is regularly varying with index $-\alpha$. Thus Proposition 4.8 implies the following result.

Corollary 4.9. *Fix $\alpha \in (0,2)$, $p > 0$. Let $\mu = TS_\alpha^p(R,b)$. Let M be the Lévy measure of μ. If $\nu(dx) = (|x|^\alpha \wedge |x|^2)\, R(dx)$ then*

$$\mu \in RV_{-\alpha}^\infty(\sigma) \iff M \in RV_{-\alpha}^\infty(\sigma) \iff \nu \in RV_0^\infty(\sigma). \tag{4.17}$$

4.2.2 Short Time Behavior

In this section we will categorize the short time behavior of p-tempered α-stable Lévy processes. Our approach will be similar to how we showed short time behavior for general Lévy processes in Appendix D.2.1. However, now instead of using the dual Lévy measure, we will use the dual Rosiński measure.

Theorem 4.10. *Fix $p > 0$, $\alpha \in (-\infty, 2) \setminus \{0\}$ and define $\gamma = \alpha \vee 0$. Let $\sigma \neq 0$ be a finite measure on $\mathbb{S}^{d-1}$, let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, 0_d)$, and let $\{X^*_t : t \geq 0\}$ be a Lévy process with $X^*_1 \sim TS^p_\alpha(R^*, 0_d)$.*
1. Fix $\beta \in (\gamma, 2)$. Let $Y \sim S_\beta(\sigma, 0_d)$. There is a function a_t such that

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to \infty \tag{4.18}$$

if and only if

$$b_t X^*_t \xrightarrow{d} Y^* \text{ as } t \downarrow 0 \tag{4.19}$$

where $Y^* \sim S_{2+\gamma-\beta}(\sigma, 0_d)$, $b_t \sim 1/a_{h^{-1}(1/t)}$, and h is some strictly monotonely increasing function with $h(t) \sim t^{-1} a_t^{-2-\gamma}$ as $t \to \infty$.
2. If $\alpha \in (0, 2)$ let $Y \sim S_\alpha(\sigma, 0_d)$. If there is a function a_t such that (4.18) holds then (4.19) holds with $Y^* \sim N(0_d, A)$ where $A = K \int_{\mathbb{S}^{d-1}} xx^T \sigma(dx)$, $b_t \sim 1/a_{h^{-1}(1/t)}$, K is as in (4.12), and $h(t)$ is some strictly monotonely increasing function with $h(t) \sim t^{-1} a_t^{-2-\alpha}$ as $t \to \infty$.

Theorem 4.6 implies that $a \in RV^\infty_{-1/\beta}$ thus, by Proposition B.2 a function h of the required form exists. Note that $h \in RV^\infty_{(2+\gamma-\beta)/\beta}$ thus $h^{-1} \in RV^\infty_{\beta/(2+\gamma-\beta)}$ and $a_{h^{-1}(t)} \in RV^\infty_{-1/(2+\gamma-\beta)}$. Thus $b_t \sim 1/a_{h^{-1}(1/t)} \in RV^0_{-1/(2+\gamma-\beta)}$.

Proof. Assume that $\beta \in [\alpha, 2)$ when $\alpha \in (0, 2)$ or $\beta \in (0, 2)$ when $\alpha < 0$. Note that, by Slutzky's Theorem, it suffices to consider that case when $a_t = [th(t)]^{-1/(2+\gamma)}$

and $b_t = 1/a_{h^{-1}(1/t)}$. For any $B \in \mathfrak{B}(\mathbb{R}^d)$ let

$$R_t^1(B) = t\int_{\mathbb{R}^d} 1_B(xa_t)R(dx) \quad \text{and} \quad R_t^2(B) = t\int_{\mathbb{R}^d} 1_B(xb_t)R^*(dx)$$

be the Rosiński measures of a_tX_t and $b_tX_t^*$ respectively.

Assume that (4.18) holds. We have

$$\begin{aligned}
\left(R_t^1\right)^*(B) &= \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|^2}\right)|x|^{2+\gamma}R_t^1(dx) \\
&= ta_t^{2+\gamma}\int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|^2}a_t^{-1}\right)|x|^{2+\gamma}R(dx) \\
&= ta_t^{2+\gamma}\int_{\mathbb{R}^d} 1_B\left(xa_t^{-1}\right)R^*(dx).
\end{aligned}$$

Note that this is the Rosiński measure of $a_t^{-1}X^*_{ta_t^{2+\gamma}}$. Thus, by Corollary 3.18 this implies that

$$a_t^{-1}X^*_{ta_t^{2+\gamma}} \xrightarrow{d} Y^* \quad \text{as } t \to \infty.$$

We have

$$\begin{aligned}
\lim_{t\to\infty} a_t^{-1}X^*_{ta_t^{2+\gamma}} &= \lim_{t\downarrow 0} a_{1/t}^{-1}X^*_{t^{-1}a_{1/t}^{2+\gamma}} = \lim_{t\downarrow 0} a_{1/t}^{-1}X^*_{1/h(1/t)} \\
&= \lim_{u\downarrow 0} a_{h^{-1}(1/u)}^{-1}X^*_u = \lim_{u\downarrow 0} b_uX^*_u,
\end{aligned}$$

where the second line follows by the substitution $u = 1/h(1/t)$.

Now assume that $\beta \neq \alpha$ and (4.19) holds. As before we have

$$\left(R_t^2\right)^*(B) = tb_t^{2+\gamma}\int_{\mathbb{R}^d} 1_B\left(xb_t^{-1}\right)R(dx),$$

and by Corollary 3.18 this implies that

$$b_t^{-1}X_{tb_t^{2+\gamma}} \xrightarrow{d} Y \quad \text{as } t \downarrow 0.$$

We have

$$\begin{aligned}
\lim_{t\downarrow 0} b_t^{-1}X_{tb_t^{2+\gamma}} &= \lim_{t\to\infty} b_{1/t}^{-1}X_{t^{-1}b_{1/t}^{2+\gamma}} = \lim_{t\to\infty} a_{h^{-1}(t)}X_{t^{-1}a_{h^{-1}(t)}^{-2-\gamma}} \\
&= \lim_{t\to\infty} a_{h^{-1}(t)}X_{h^{-1}(t)} = \lim_{u\to\infty} a_uX_u,
\end{aligned}$$

where the third equality follows by the fact that $t = h\left(h^{-1}(t)\right) = \frac{1}{h^{-1}(t)a^{2+\gamma}_{h^{-1}(t)}}$ and the fourth by the substitution $u = h^{-1}(t)$. $\square$

We will now give an example to show that the converse of Part 2 in Theorem 4.10 does not hold, in the sense that Gaussian long time behavior does not imply α-stable short time behavior. This also provides another counterexample to the converse of the second part of Corollary 3.18.

Example 6. Let $d = 1$. Fix $\alpha \in (0, 2)$ and $p > 0$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, 0)$ and let $\{X^*_t : t \geq 0\}$ be a Lévy process with $X^*_1 \sim TS^p_\alpha(R^*, 0)$. Assume that $R \neq 0$ and define

$$F(t) = \int_{|x|\leq t} x^2 R(dx).$$

Assume that $F \in RV^\infty_0$. By Theorem 4.7 this means that there is a function $a_t > 0$ such that $a_t X_t \to Y$ for some Gaussian random vector Y not concentrated at a point. For large enough t define

$$G(t) = \frac{\int_{0\leq x\leq t} x^2 R(dx)}{\int_{|x|\leq t} x^2 R(dx)},$$

and assume that there exist two sequences (b_n) and (c_n) increasing to infinity such that there are $b, c \in (0, \infty)$, $b \neq c$ and

$$\lim_{n\to\infty} G(b_n) = b \quad \text{and} \quad \lim_{n\to\infty} G(c_n) = c. \tag{4.20}$$

This holds for instance when R is given by

$$\begin{aligned} R(A) &= \int_{e^{e^\pi}}^\infty 1_A(-t)\frac{2 - \cos(\log(\log t)) - \sin(\log(\log t))}{t^3}dt \\ &\quad + \int_{e^{e^\pi}}^\infty 1_A(t)\frac{2 + \cos(\log(\log t)) + \sin(\log(\log t))}{t^3}dt, \quad A \in \mathfrak{B}(\mathbb{R}). \end{aligned}$$

In this case

$$F(t) = \begin{cases} 0 & \text{if } t \leq e^{e^\pi} \\ 4(\log t - e^\pi) & \text{if } t > e^{e^\pi} \end{cases}$$

and

$$G(t)=\frac{[\log t][2+\sin(\log(\log t))]-2e^{\pi}}{4(\log t-e^{\pi})},\quad t>e^{e^{\pi}}.$$

Note that

$$G(t)\sim\frac{2+\sin(\log(\log t))}{4}\quad\text{as } t\to\infty.$$

With the sequences $(b_n)=(e^{e^{\pi n}})$ and $(c_n)=(e^{e^{(2n+.5)\pi}})$ we have

$$\lim_{n\to\infty}G(b_n)=1/2\quad\text{and}\quad\lim_{n\to\infty}G(c_n)=3/4.$$

We will now show that the short time behavior is not α-stable. Assume for the sake of contradiction that it is. By Proposition 4.13 (below) this means that there is a finite Borel measure $\sigma\neq 0$ on $\mathbb{S}^0$ such that $\nu^1\in RV_0^0(\sigma)$ where $\nu^1(dx)=|x|^\alpha R^*(dx)$. This means that

$$\begin{aligned}\frac{\sigma(\{1\})}{\sigma(\mathbb{S}^0)}=\lim_{t\downarrow 0}\frac{\nu^1(x>t)}{\nu^1(|x|>t)} &= \lim_{t\downarrow 0}\frac{\int_{x\geq t}x^\alpha R(dx)}{\int_{|x|\geq t}x^\alpha R(dx)}\\ &= \lim_{t\downarrow 0}\frac{\int_{0\leq x\leq 1/t}x^2 R(dx)}{\int_{|x|\leq 1/t}x^2 R(dx)}=\lim_{t\downarrow 0}G(1/t).\end{aligned}$$

The contradiction follows from (4.20), which shows that this limit does not actually exist. □

From Theorem 4.10 we easily get the following.

Corollary 4.11. *Fix $\alpha\in(-\infty,2)\setminus\{0\}$, $p>0$, $\beta\in(\alpha,2)$, and let σ be a finite, nonzero Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t:t\geq 0\}$ be a Lévy Process with $X_1\sim TS_\alpha^p(R,0_d)$ and let Y be a β-stable random vector with spectral measure σ. There exists a function $a_t>0$ such that*

$$a_tX_t\xrightarrow{d}Y\text{ as }t\downarrow 0 \tag{4.21}$$

if and only if $R\in RV_{-\beta}^0(\sigma)$. Moreover, in this case, $a\in RV_{-1/\beta}^0$ and

$$a_t\sim s^{-1/\beta}V^{\leftarrow}(1/t)\text{ as }t\downarrow 0, \tag{4.22}$$

where $s=\alpha/\sigma(\mathbb{S}^{d-1})$ and $V(t)=R(|x|>1/t)$.

Proof. By Theorem 4.10 short time behavior of $\{X_t : t \geq 0\}$ equivalent to long time behavior of the process $\{X_t^* : t \geq 0\}$ where $X_1^* \sim TS_\alpha^p(R^*, 0_d)$. By Theorem 4.6 this is equivalent to regular variation of the tails of R^*. This is equivalent to the regular variation at 0 of R by Proposition C.13. The form of a_t follows from Proposition B.10. □

We will now give necessary and sufficient conditions for the short time behavior to be Gaussian.

Theorem 4.12. *Fix $\alpha \in (-\infty, 2) \setminus \{0\}$, $p > 0$ and let $B \neq 0_{d\times d}$ be a nonnegative definite matrix. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS_\alpha^p(R, 0_d)$. Define*

$$A_t = \int_{|x|\leq t} xx^T R(dx). \tag{4.23}$$

There exists a function a_t such that

$$a_t X_t \xrightarrow{d} Y \quad \text{as } t \downarrow 0 \tag{4.24}$$

where $Y \sim N(0_d, B)$ if and only if $A \in LRV_0^0(B)$. Moreover, when this holds, $a \in RV_{-1/2}^0$ and

$$a_t \sim 1/h^{\leftarrow}(t) \text{ as } t \downarrow 0, \tag{4.25}$$

where

$$h(t) = t^2/\mathrm{tr}A_t. \tag{4.26}$$

Proof. Let R_t be the Rosiński measure of $a_t X_t$. First assume that $A \in LRV_0^0(B)$. Let h be defined by (4.26) and a_t by (4.25). This implies that $h \in RV_2^0$ and

$a \in RV^0_{-1/2}$. We have

$$\begin{aligned}
\lim_{t\downarrow 0} \int_{|x|\le\epsilon} xx^T R_t(dx) &= \lim_{t\downarrow 0} ta_t^2 \int_{|x|\le\epsilon/a_t} xx^T R(dx) \\
&= \lim_{t\downarrow 0} a_t^2 \int_{|x|\le\epsilon/a_t} xx^T R(dx) h(1/a_t) \\
&= \lim_{t\downarrow 0} a_t^2 \int_{|x|\le\epsilon/a_t} xx^T R(dx) h(\epsilon/a_t) \frac{h(1/a_t)}{h(\epsilon/a_t)} \\
&= \lim_{t\downarrow 0} \frac{a_t^2 \int_{|x|\le\epsilon/a_t} xx^T R(dx)}{\epsilon^{-2} a_t^2 \int_{|x|\le\epsilon/a_t} |x|^2 R(dx)} \epsilon^{-2} \\
&= \lim_{t\downarrow 0} \frac{\int_{|x|\le\epsilon/a_t} xx^T R(dx)}{\int_{|x|\le\epsilon/a_t} |x|^2 R(dx)} = B.
\end{aligned}$$

Lemma D.10 guarantees that $R_t \xrightarrow{v} 0$ on $\bar{\mathbb{R}}^d$ and

$$\lim_{s\to\infty} \lim_{t\to 0} \int_{|x|>s} |x|^\gamma R_t(dx) = 0,$$

where $\gamma = \alpha \vee 0$. From here the result holds by Theorem 3.3 and (3.14) when $\alpha \in (0,2)$ and from Theorem 3.11 and (3.21) when $\alpha < 0$.

Now assume that (4.24) holds. Lemma D.4 implies that $\lim_{t\downarrow 0} a_t = \infty$, and Theorem 3.3 and Theorem 3.11 imply that $R_t \xrightarrow{v} 0$ on $\bar{\mathbb{R}}^d_0$ as $t \downarrow 0$. Thus by Lemma D.9, for any $\epsilon > 0$

$$\limsup_{t\downarrow 0} ta_t^2 \int_{|x|\le\epsilon/a_t} xx^T R(dx) = \limsup_{t\downarrow 0} ta_t^2 \int_{|x|\le 1/a_t} xx^T R(dx)$$

and similarly for the liminf. Theorem 3.3 and Theorem 3.11 imply that for all $s > 0$

$$\lim_{t\downarrow 0} ta_t^2 \int_{|x|\le s/a_t} xx^T R(dx) = B$$

and thus

$$\lim_{t\downarrow 0} ta_t^2 \int_{|x|\le s/a_t} |x|^2 R(dx) = \mathrm{tr}B.$$

Let $U(t) = \int_{|x|\le t} |x|^2 R(dx)$. By Lemma D.4 we can use Proposition B.2 to get $U \in RV^0_0$. This implies that

$$\lim_{t\downarrow 0} \frac{\int_{|x|\le t} xx^T R(dx)}{\int_{|x|\le t} |x|^2 R(dx)} = \lim_{t\downarrow 0} \frac{ta_t^2 \int_{|x|\le 1/a_t} xx^T R(dx)}{ta_t^2 \int_{|x|\le 1/a_t} |x|^2 R(dx)} = \frac{B}{\mathrm{tr}B},$$

and hence $A \in LRV_0^0(B)$. $\square$

We will now derive necessary and sufficient conditions for short time behavior in the case when $\alpha \in (0, 2)$ and the limiting distribution is α-stable.

Proposition 4.13. *Fix $\alpha \in (0, 2)$, $p > 0$, and let σ be a finite Borel measure on $\mathbb{S}^{d-1}$ with $\sigma \neq 0$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS_\alpha^p(R, 0_d)$. Let $Y \sim S_\alpha(\sigma, 0_d)$. There exists a positive function a_t such that*

$$a_t X_t \xrightarrow{d} Y \quad \text{as } t \downarrow 0 \tag{4.27}$$

if and only if $\nu^1 \in RV_0^0(\sigma)$, where $\nu^1(dx) = |x|^\alpha R(dx)$. Moreover, when this holds,

$$a_t \sim V^{\leftarrow}(1/t) \text{ as } t \downarrow 0, \tag{4.28}$$

where $V(t) = \frac{t^\alpha}{\sigma(\mathbb{S}^{d-1})}\nu^1(|x| > 1/t)$ and $a \in RV_{-1/\alpha}^0$.

Note that by Proposition B.10 this implies that the spectral measure of Y is given by

$$\sigma(D) = \lim_{t \downarrow 0} t a^\alpha \nu^1(|x| > 1/a_t, \xi(x) \in D), \quad B \in \mathfrak{B}(\mathbb{S}^{d-1}). \tag{4.29}$$

In particular, if ν^1 is a finite measure (which, by Theorem 2.3, holds if and only if X_1 has a proper p-tempered α-stable distribution) then

$$V(t) \sim t^\alpha \frac{\nu^1(\mathbb{R}^d)}{\sigma(\mathbb{S}^{d-1})} \text{ as } t \to \infty$$

thus by Proposition B.2

$$a_t \sim t^{-1/\alpha} \left(\frac{\nu^1(\mathbb{R}^d)}{\sigma(\mathbb{S}^{d-1})}\right)^{-1/\alpha} \quad \text{as } t \to \infty$$

and hence

$$\sigma(D) = \frac{\sigma(\mathbb{S}^{d-1})}{\nu^1(\mathbb{R}^d)}\nu^1(\xi(x) \in D), \quad D \in \mathfrak{B}(\mathbb{S}^{d-1}). \tag{4.30}$$

This implies that if X_1 has a proper p-tempered α-stable distribution and $Y \sim S_\alpha(\sigma_1, 0)$ where

$$\sigma_1(D) = \nu^1(\xi(x) \in D) = \int_{\mathbb{R}^d} 1_D\left(\frac{x}{|x|}\right)|x|^\alpha R(dx), \quad D \in \mathfrak{B}(\mathbb{S}^{d-1}) \tag{4.31}$$

then

$$\lim_{t\downarrow 0} t^{-1/\alpha} X_t \xrightarrow{d} Y. \tag{4.32}$$

By Proposition 2.7, the measure in (4.31) is the spectral measure of the stable distribution that we are tempering. This is not surprising, in fact, a result analogous to (4.32) is true for all generalized tempered stable distributions, as shown in Theorem 3.1 of [RS10].

Proof. Let ν be the extended Rosiński measure of X_1, let ν' be the extended Rosiński measure of Y, and let R_t and ν_t be, respectively,the Rosiński measure and the extended Rosiński measure of $a_t X_t$.

First assume that (4.27) holds. By Lemma D.4, $\lim_{t\to\infty} a_t = \infty$, and by Theorem 3.3, $\nu_t \xrightarrow{v} \nu'$ on $\bar{\mathbb{R}}^d_0$ as $t \downarrow 0$. Let $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ such that $\sigma(\partial D) = 0$. For $s \in (0, \infty)$, let $A^s_D = \{|x| > s, \xi(x) \in D\}$. We have $\nu'(\partial A^s_D) = 0$. Thus by Proposition A.8

$$\lim_{t\downarrow 0} \nu_t(A^s_D) = \nu'(A^s_D) = \nu'(\infty D) = \sigma(D).$$

When $s \geq 1$

$$\nu_t(A^s_D) = t a_t^\alpha \int_{|x|>s/a_t} 1_D(\xi(x))|x|^\alpha R(dx) = t a_t^\alpha \nu^1\left(|x| > s/a_t, \xi(x) \in D\right).$$

When $s \in (0, 1)$

$$\nu_t(A^s_D) = t a_t^\alpha \nu^1\left(|x| > 1/a_t, \xi(x) \in D\right) + \nu_t(1 \geq |x| > s, \xi(x) \in D).$$

We have

$$\lim_{t\downarrow 0} \nu_t(1 \geq |x| > s, \xi(x) \in D) = \nu'(1 \geq |x| > t, \xi(x) \in D) = 0.$$

Thus for all $s > 0$.

$$\lim_{t\downarrow 0} ta_t^\alpha \nu^1(|x| > s/a_t, \xi(x) \in D) = \lim_{t\downarrow 0} \nu_t(A_D^s) = \sigma(D).$$

Hence (B.20) holds and by Proposition B.10 the function a_t is as given in (4.28) and $\nu^1 \in RV_0^0(\sigma)$.

Now assume that $\nu^1(dx) \in RV_0^0(\sigma)$. By Proposition B.10, the function a as given in (4.28) satisfies $a \in RV_{-1/\alpha}^0$ and for any $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and any $s > 0$

$$\lim_{t\downarrow 0} ta_t^\alpha \nu^1(|x| > s/a_t, \xi(x) \in D) = \sigma(D).$$

Thus, for $s \in (0, 1)$

$$\lim_{t\downarrow 0} ta_t^\alpha \nu^1(1/a_t \geq |x| > s/a_t, \xi(x) \in D) = 0$$

and hence

$$\begin{aligned}
&\lim_{t\downarrow 0} ta_t^2 \nu(1/a_t \geq |x| > s/a_t, \xi(x) \in D) \\
&\qquad\qquad \leq \lim_{t\downarrow 0} ta_t^\alpha \nu^1(1/a_t \geq |x| > s/a_t, \xi(x) \in D) = 0.
\end{aligned}$$

This implies that for all $s > 0$

$$\lim_{t\downarrow 0} \nu_t(A_D^s) = \lim_{t\downarrow 0} ta^\alpha \nu^1(|x| > s/a_t, \xi(x) \in D) = \sigma(D),$$

and by Lemma A.12 it follows that $\nu_t \xrightarrow{v} \nu'$ on $\bar{\mathbb{R}}_0^d$ as $t \downarrow 0$. It remains to show that the limit has no Gaussian part. By (3.8), it suffices to show that

$$\lim_{t\downarrow 0} \nu_t\{|x| < 1\} = 0.$$

By the asymptotic form of a_t we have

$$\begin{aligned}
\lim_{t\downarrow 0} \nu_t\{|x| < 1\} &= \lim_{t\downarrow 0} ta_t^2 \int_{|x|<1/a_t} |x|^2 R(dx) = \sigma(\mathbb{S}^{d-1}) \lim_{t\downarrow 0} \frac{a_t^2 \int_{|x|<1/a_t} |x|^2 R(dx)}{a_t^\alpha \nu^1(|x| > 1/a_t)} \\
&= \sigma(\mathbb{S}^{d-1}) \lim_{t\downarrow 0} \frac{\int_{|x|<1/a_t} |x|^2 R(dx)}{(1/a_t)^{2-\alpha} \int_{|x|>1/a_t} |x|^\alpha R(dx)} = 0,
\end{aligned}$$

where the last equality follows by Corollary B.5. From here the result follows by Theorem 3.3. □

Fix $\alpha \in (0,2)$ and $p > 0$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, 0_d)$. Let $\nu(dx) = (|x|^2 \wedge |x|^\alpha)\, R(dx)$ and $\nu^1(dx) = |x|^\alpha R(dx)$. If X_1 has a proper p-tempered α-stable distribution (or equivalently if ν^1 is a finite measure) then necessarily $\nu^1 \in RV_0^0(\sigma_1)$ where σ_1 is given by (4.31) and in a short time frame the Lévy process behaves like a random vector with the distribution $S_\alpha(\sigma_1, 0_d)$. On the other hand, by Theorem 4.8 if $\nu \in RV_0^\infty(\sigma_2)$ then in a long time frame the Lévy process behaves like an α-stable random vector with spectral measure

$$\sigma_2(B) = \lim_{r\to\infty} \frac{\int_{|x|>r} 1_B\left(\frac{x}{|x|}\right) |x|^\alpha R(dx)}{\int_{|x|>r} |x|^\alpha R(dx)}. \tag{4.33}$$

It is easy to see that σ_2 is absolutely continuous with respect to σ_1. However, σ_1 need not be absolutely continuous with respect to σ_2. To see this, consider the measure

$$R(B) = \int_{-\infty}^{-1} 1_B(x) \frac{1}{|x|^{2+\alpha}} dx + \int_2^\infty 1_B(x) \frac{1}{x^{1+\alpha} (\log x)^2} dx,\ \ B \in \mathfrak{B}(\mathbb{R}^d).$$

We have

$$\int_{\mathbb{R}} |x|^\alpha R(dx) = \int_{-\infty}^{-1} \frac{1}{x^2} dx + \int_2^\infty \frac{1}{x(\log x)^2} dx = 1 + \frac{1}{\log 2} < \infty.$$

To see that the measure $(|x|^2 \wedge |x|^\alpha)\, R(dx)$ is slowly varying observe that

$$\lim_{r\to\infty} \frac{\int_{x>r} x^\alpha R(dx)}{\int_{|x|>r} |x|^\alpha R(dx)} = \lim_{r\to\infty} \frac{1/\log r}{1/r + 1/\log r} = 1$$

and

$$\lim_{r\to\infty} \frac{\int_{x<-r} |x|^\alpha R(dx)}{\int_{|x|>r} |x|^\alpha R(dx)} = \lim_{r\to\infty} \frac{1/r}{1/r + 1/\log r} = 0.$$

Thus, we have $\sigma_2(\{-1\}) = 0$ but

$$\sigma_1(\{-1\}) = \int_{x<-1} |x|^\alpha R(dx) = 1.$$

CHAPTER 5

MULTIVARIATE PRELIMIT THEOREMS

In Theorem C.7 we state the (generalized) central limit theorem for iid random vectors. In applications, this theorem is important because it implies the following approximation. If $X_1, X_2, \ldots$ are iid random vectors in the domain of attraction of a stable random vector Y, then, for large enough n, we have

$$S_n = X_1 + \cdots + X_n \overset{d}{\approx} a_n^{-1} Y + b_n, \tag{5.1}$$

where $a_n > 0$ and $b_n \in \mathbb{R}^d$ are the normalizing constants. However, unless we know a lot about the distribution of X_1, we have no way of knowing when n is large enough or what the distribution of Y is. In fact, we cannot even know its index of stability. By the CLT, to know the index of stability we need to know a lot about the tails of the distribution of X_1. But, as Klebanov, Rachev, and Székely point out in [KRS99], "[f]initely many empirical observations can never justify any tail behavior...." To remedy this, they introduce a central "prelimit" theorem, which does not depend on the tails but on the central part of the distribution. Since it does not depend on the tails, it cannot tell what the limiting stable distribution will be. However, they show that the distribution of S_n may be well approximated by a particular stable distribution for large (but not too large) values of n before ultimately converging to a potentially different stable distribution. Their results are given in [KRS99] and [KRS00]. In this chapter, we will generalize their results to d-dimensions.

It is important to note that, since the prelimit theorem does not consider the tail behavior of X_1, the stable distribution approximating the distribution of S_n may have very different tail behavior from that of S_n. In particular, if S_n is well approximated by a stable distribution with heavier tails than those of S_n, then a tempered stable distribution may do an even better job as it would not only be a good approximation of the distribution in some central region, but it would also provide a better fit to the tails. In this way, the prelimit theorem helps to explain why tempered stable distributions appear to work well in a variety of applications.

5.1 Theoretical Results

In this section we will state our theoretical results. The proofs will be postponed until Section 5.3. We begin by setting up the notation. Let X be a d-dimensional random vector. We will denote its characteristic function by $\hat{\mu}_X(z)$ and its distribution function by $F_X(x)$.

The convolution of two measurable functions f and g is defined by

$$f * g(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy \tag{5.2}$$

at any point $x \in \mathbb{R}^d$ where the integral exists. Similarly, when F is a (signed) measure and g is a measurable function, then the convolution of F and g is defined by

$$F \star g(x) = \int_{\mathbb{R}^d} g(x-y)F(dy) \tag{5.3}$$

at every point $x \in \mathbb{R}^d$ where the integral exists. If we choose $g = H$, a cdf, then $F \star H$ can also be viewed as a convolution of two measures. Note that if F has density f with respect to the d-dimensional Lebesgue measure, then $F \star g = f * g$.

A function f on $\mathbb{R}^d$ is said to satisfy the **Lipschitz condition** with coefficient M if

$$|f(x) - f(y)| \leq M|x-y| \tag{5.4}$$

for every $x, y \in \mathbb{R}^d$. We will use the notation $f \in Lip_M$ to denote this. Note that if h is differentiable and if $M := \sup_{x\in\mathbb{R}^d} |\nabla h(x)| < \infty$, then by the multivariate analogue of the mean value theorem (Theorem 12.9 in [Apo74]) we have

$$|h(x) - h(y)| = |\langle \nabla h(\theta), x-y\rangle| \leq M|x-y|,$$

where θ is some point on the line segment connecting x and y. Thus $h \in Lip_M$.

A useful distance on the space of probability measures on $\mathbb{R}^d$ can be defined as follows. Let $c, \gamma \geq 0$. For d-dimensional random vectors X and Y we set

$$d_{c,\gamma}(X,Y) = \sup_{|z|\geq c} \frac{|\hat{\mu}_X(z) - \hat{\mu}_Y(z)|}{|z|^\gamma}. \tag{5.5}$$

Note that this measures the distance between two probability laws and not two random variables, thus the notation $d_{c,\gamma}(F_X, F_Y)$ would be more precise. However, we will use, as is common, the notation given in (5.5).

If Y is a strictly α-stable random vector then by (C.19) for every n

$$Y \stackrel{d}{=} \frac{Y_1 + Y_2 + \cdots + Y_n}{n^{1/\alpha}},$$

where $Y_1, \ldots, Y_n$ are iid copies of Y. Let $X_1, X_2, \ldots$ be iid random variables, and let

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n^{1/\alpha}}.$$

Following [KRS99], we observe that

$$\begin{aligned} d_{0,\gamma}(S_n, Y) &= \sup_{z\in\mathbb{R}^d} \frac{\left|\hat{\mu}^n_{X_1}(z/n^{1/\alpha}) - \hat{\mu}^n_Y(z/n^{1/\alpha})\right|}{|z|^\gamma} \\ &\leq n \sup_{z\in\mathbb{R}^d} \frac{\left|\hat{\mu}_{X_1}(z/n^{1/\alpha}) - \hat{\mu}_Y(z/n^{1/\alpha})\right|}{|z|^\gamma} = \frac{1}{n^{\gamma/\alpha-1}} d_{0,\gamma}(X_1, Y), \end{aligned}$$

where the inequality in the second line follows by the fact that $(z^n - y^n) = (z - y)\sum_{i=1}^n y^{i-1} z^{n-i}$, and the equality in that line follows by the substitution $u = z/n^{1/\alpha}$. Thus, if $d_{0,\gamma}(X_1, Y) < \infty$ for some $\gamma > \alpha$ then X_1 is in the domain of normal attraction of Y.

Let h be a probability density on $\mathbb{R}^d$. We define another distance on the space of probability laws on $\mathbb{R}^d$ by setting, for two d-dimensional random vectors X and Y,

$$K_h(X,Y) = \sup_{x\in\mathbb{R}^d} |F_X \star h(x) - F_Y \star h(x)|. \tag{5.6}$$

We will now state conditions under which this distance metrizes weak convergence. The proof will be given in Section 5.3.

Lemma 5.1. *If h satisfies a Lipschitz condition and the corresponding characteristic function does not vanish then K_h metrizes weak convergence on $\mathbb{R}^d$.*

We now state our main theorem.

Theorem 5.2. *Fix $\alpha \in (0, 2]$. Let h be a probability density on $\mathbb{R}^d$. Assume that $h \in Lip_{M_h}$ for some $M_h > 0$. Let $X_1, X_2, \dots$ be iid d-dimensional random vectors, and let $S_n = n^{-1/\alpha} \sum_{j=1}^{n} X_j$. Let Y be a strictly α-stable d-dimensional random vector. For any $\gamma > \alpha$, we have*

$$K_h(S_n, Y) \leq \inf_{a,\Delta>0} \left\{ \frac{d_{\Delta n^{-1/\alpha},\gamma}(X_1, Y)}{n^{\gamma/\alpha - 1}} \frac{2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)} + \frac{2}{\pi^d}[\Delta \wedge (2a)]^d + M_h \frac{12d}{\pi a} \right\}. \tag{5.7}$$

We can simplify this bound by applying the inequality $[\Delta \wedge (2a)] \leq \Delta$ and analytically optimizing over a. Thus, under the conditions of Theorem 5.2, we have

$$K_h(S_n, Y) \leq \inf_{\Delta>0} \left\{ \frac{2}{\pi^d}\Delta^d + \frac{\gamma+d+1}{\gamma+d} \left(\frac{(12M_h)^{\gamma+d} 2^{\gamma+1} d^{3(\gamma+d)/2}}{\Gamma(d/2)\pi^{3d/2+\gamma}} \frac{d_{\Delta n^{-1/\alpha},\gamma}(X_1, Y)}{n^{\gamma/\alpha - 1}} \right)^{1/(\gamma+d+1)} \right\}. \tag{5.8}$$

We can think of the quantitative bounds presented in the theorem in the following way. Suppose that the random vector X is such that, for some $\gamma > \alpha$, the distance $d_{c,\gamma}(X, Y)$ remains "not too big" even for certain reasonably small values of $c > 0$. In that case one can choose $a > 0$ large, $\Delta > 0$ small, and have the upper bound on the distance between the distribution of S_n and that of the α-stable random vector small for fairly large values of n.

5.2 Extensions

We will consider two extensions. The first is to random summation and the second to the case when the random vectors are only conditionally iid.

5.2.1 Random Summation

Our results can be formulated in the somewhat more general framework of random summation. The papers [KRS99] and [KRS00] give a version of the prelimit theorem for random sums. Following their approach, we give a version of the prelimit theorem for random summation in d-dimensions.

Let $\Theta \subset (0,1)$ such that there is a sequence $\{\theta_n\}$ in Θ with $\theta_n \to 0$. Let $\mathcal{N} = \{\nu_\theta : \theta \in \Theta\}$ be a collection of probability measures on $\mathbb{N}$. Throughout assume that $N_\theta \sim \nu_\theta$ and

$$\mathrm{E}N_\theta = \frac{1}{\theta}.$$

Definition 5.3. *Fix $\alpha \in (0,2]$ and let $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \mu$. If for every $\theta \in \Theta$*

$$X_1 \overset{d}{=} \theta^{1/\alpha} \sum_{i=1}^{N_\theta} X_i \tag{5.9}$$

*then we say that μ has an **$\mathcal{N}$-strictly α-stable distribution**.*

Note that when $\Theta = \{1/n : n \in \mathbb{N}\}$ and $\nu_\theta = \delta_{1/\theta}$, the definition reduces to the usual definition of strictly α-stable distributions. Another important example is when $\Theta = (0,1)$ and for each $\theta \in \Theta$, ν_θ is a geometric distribution with mean $1/\theta$. In this case $\mathcal{N}$-strictly α-stable distributions are called strictly geo-stable. See [KMRV00] and the references therein for more details. $\mathcal{N}$-strictly α-stable distribution are usually studied within the more general context of random infinite divisibility. This is developed in [Bun96], [GK96], and [KR96].

Sufficient conditions on $\mathcal{N}$ to guarantee the existence of $\mathcal{N}$-strictly α-stable distributions are given in [KMM87], [Bun96], [GK96], and [KR96]. For completeness, we give a sufficient condition that combines results from [KMM87] and [KR96].

Let P_θ be the generating function of ν_θ. By this we mean that

$$P_\theta(s) = \mathrm{E}s^{N_\theta}, \quad s \in [0,1].$$

Let $\mathcal{P}$ be the smallest class of functions that is closed under composition and is generated by the family $\{P_\theta : \theta \in \Theta\}$. Note that $\mathcal{P}$ with the operation of composition forms a semigroup. In [KR96] it is shown that if $\mathcal{P}$ commutes then there exists a function ϕ such that for every $\theta \in \Theta$ and every $s \in [0,1]$

$$P_\theta(s) = \phi\left(\frac{1}{\theta}\phi^{-1}(s)\right). \tag{5.10}$$

If $\mathcal{P}$ commutes and

$$\{1/n : n \in \mathbb{N}\} \subset \Theta \tag{5.11}$$

then a probability measure μ is $\mathcal{N}$-strictly α-stable if and only if its characteristic function is of the form

$$\phi\left(-C_\alpha(z)\right), \tag{5.12}$$

where C_α is the cumulant generating function of a strictly α-stable distribution. In this case, we call the strictly α-stable distribution with characteristic function $e^{C_\alpha(z)}$, the **corresponding strictly α-stable distribution**.

By analogy with domains of attraction, we define the domain of $\mathcal{N}$-attraction as follows. The **domain of $\mathcal{N}$-attraction** of a probability measure μ is the class of all probability measures μ' such that if $Y \sim \mu$ and $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \mu'$ then for each $\theta \in \Theta$ there are $a_\theta > 0$ and $b_\theta \in \mathbb{R}^d$ such that

$$a_\theta \sum_{n=1}^{N_\theta} X_n - b_\theta \xrightarrow{d} Y. \tag{5.13}$$

The domain of $\mathcal{N}$-attraction of an $\mathcal{N}$-strictly α-stable distribution is nonempty, and when $\mathcal{P}$ commutes and (5.11) holds, it is the same as the domain of attraction of the corresponding strictly α-stable distribution.

We now formulate a version of Theorem 5.2 for the case of random summation.

Proposition 5.4. *Let $X_1, X_2, \dots$ be iid random vectors independent of N_θ and set*

$$S_\theta = \theta^{1/\alpha} \sum_{i=1}^{N_\theta} X_i.$$

Fix $\alpha \in (0, 2]$ and let Y be a $\mathcal{N}$-strictly α-stable distributions. If h is a probability density on $\mathbb{R}^d$ such that $h \in Lip_{M_h}$ for some $M_h > 0$ then for any $\gamma > \alpha$

$$\begin{aligned} K_h(S_\theta, Y) \quad \leq \quad \inf_{a,\Delta>0} \Bigg\{ & \theta^{\gamma/\alpha-1} d_{\Delta\theta^{-1/\alpha},\gamma}(X_1, Y) \frac{2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)} \\ & + \frac{2}{\pi^d}[\Delta \wedge (2a)]^d + M_h \frac{12d}{\pi a} \Bigg\}. \end{aligned} \tag{5.14}$$

5.2.2 Conditional Independence

Fix $n \in \mathbb{N}$. Let $(\Theta, \mathfrak{T}, \rho)$ be a probability space. Let $\{\mu_\theta : \theta \in \Theta\}$ be a family of probability measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ such that $\mu_\theta(B)$ is a measurable function in θ for every $B \in \mathfrak{B}(\mathbb{R}^d)$. Let $\mu_\theta^{\times n}$ be the product measure of μ_θ with itself n times. By a standard application of Dynkin's π-λ Theorem (Theorem 3.2 in [Bil95]) $\mu_\theta^{\times n}(B)$ is a measurable function in θ for every $B \in \mathfrak{B}(\mathbb{R}^d)$. Define

$$\mu(B) = \int_\Theta \mu_\theta^{\times n}(B)\rho(d\theta), \qquad B \in \mathfrak{B}(\mathbb{R}^{dn}). \tag{5.15}$$

Thus for any $B_1, \dots, B_n \in \mathfrak{B}(\mathbb{R}^d)$ we have

$$\mu(B_1 \times \cdots \times B_n) = \int_\Theta \Pi_{i=1}^n \mu_\theta(B_i)\rho(d\theta). \tag{5.16}$$

Note that if $(X_1^T, \dots, X_n^T)^T \sim \mu$ and $T_n = X_1 + \cdots + X_n$ then

$$\hat{\mu}_{T_n}(z) = \int_\Theta \hat{\mu}_\theta^n(z)\rho(dx), \quad z \in \mathbb{R}^d. \tag{5.17}$$

The random vectors $X_1, \ldots, X_n$ are called **conditionally iid** or **mixed iid**. For more about such random vectors and their relation to exchangeable and contractable sequences see Chapter 11 in [Kal02].

Let ν be an arbitrary probability measure on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ and define, for $\gamma > 0$ and $c \geq 0$, the distance

$$d_{c,\gamma}(\mu, \nu; \rho) = \sup_{|z|>c} \frac{\int |\hat{\mu}_\theta(z) - \hat{\nu}(z)| \, \rho(d\theta)}{|z|^\gamma}.$$

Proposition 5.5. *Let $X_1, \ldots, X_n$ be random vectors on $\mathbb{R}^d$ with $(X_1^T, \ldots, X_n^T)^T \sim \mu$. Fix $\alpha \in (0, 2]$, $\gamma > \alpha$ and let*

$$S_n = n^{-1/\alpha} \sum_{k=1}^n X_k.$$

Let Y be a strictly α-stable distributions. If h is a probability density on $\mathbb{R}^d$ such that $h \in Lip_{M_h}$ for some $M_h > 0$ then for any $\gamma > \alpha$ we have

$$\begin{aligned} K_h(S_n, Y) \quad \leq \quad \inf_{a,\Delta>0} \Bigg\{ & \frac{d_{\Delta n^{-1/\alpha},\gamma}(\mu, \mu_Y; \rho)}{n^{\gamma/\alpha - 1}} \frac{2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma + d)} \\ & + \frac{2}{\pi^d} \left[\Delta \wedge (2a)\right]^d + M_h \frac{12d}{\pi a} \Bigg\}. \end{aligned} \tag{5.18}$$

5.3 Proofs

We start by listing, for ease of reference, several well know properties of convolutions and Fourier transforms. To simplify the notation, we will write L^p for $L^p(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d), \lambda_d)$, where λ_d is Lebesgue measure on $\mathbb{R}^d$. By $\|\cdot\|$ we will denote the norm in L^p. The **Fourier transform** of a function $f \in L^2$ is denoted by $\tilde{f}$, and is defined by

$$\tilde{f}(z) = \underset{N\to\infty}{\text{l.i.m.}} \int_{|x|\leq N} e^{i\langle x,z\rangle} f(x)dx, \tag{5.19}$$

where l.i.m. is understood to be the limit in L^2. In the case where $f \in L^1 \cap L^2$ then (by e.g. Theorem 7.2 in [Bar95]) $\tilde{f}(z) = \int_{\mathbb{R}^d} e^{i\langle x,z\rangle} f(x)dx$ Lebesgue almost everywhere.

Theorem 5.6. *1. Let $1 \le p, q, r \le \infty$ such that $1 + 1/r = 1/p + 1/q$. If $f \in L^p$ and $g \in L^q$, then $f * g$ exists for Lebesgue almost every x, it is an element of L^r, and it satisfies Young's Inequality: $\|f * g\|_r \le \|f\|_p \|g\|_q$. If $r = \infty$ then $f * g$ exists for all x.*

2. Let $p \ge 1$ and $g \in L^p$. If F is a finite signed measure, then $F \star g$ is defined for Lebesgue almost every x and $F \star g \in L^p$.

3. Let $f, g \in L^2$. Then $\widetilde{f * g} = \tilde{f}\tilde{g}$.

4. Let $f, g \in L^2$. Then $f * g(x) = (2\pi)^{-d}\widetilde{(\tilde{f}\tilde{g})}(-x)$ for Lebesgue almost every x.

5. Let $f \in L^2$. If $\tilde{f} \in L^1 \cap L^2$ then $f \in L^\infty \cap L^2$ and $\|f\|_\infty \le (2\pi)^{-d}\|\tilde{f}\|_1$.

Proof. See Propositions 8.6-8.9 in [Fol99] for Part 1. Part 2 is in Proposition 3.9.9 in [Bog07]. The rest of the statements are in Proposition 6.8.1 and Theorem 6.8.1 in [Sta05]. □

Proposition 5.7. *Let h be a probability density on $\mathbb{R}^d$. Assume that $h \in Lip_{M_h}$ for some $M_h > 0$. Let X, Y be d-dimensional random vectors. For any $\gamma > 0$, we have*

$$K_h(X,Y) \le \inf_{a,\Delta>0} \left\{ d_{\Delta,\gamma}(X,Y) \frac{2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)} + \frac{2}{\pi^d}[\Delta \wedge (2a)]^d + M_h \frac{12d}{\pi a} \right\}. \tag{5.20}$$

Proof. Suppose that, for $a > 0$, V_a is a measurable function on $\mathbb{R}^d$ with the following properties. Define $B_a(x) := |x|V_a(x)$, $x \in \mathbb{R}^d$. Assume that $V_a, \tilde{V}_a, B_a \in L^1$, $|\tilde{V}_a| \le M$, $\tilde{V}_a(0) = 1$, and $\tilde{V}_a(x) = 0$ for $x \notin [-2a, 2a]^d$. We will show that we have a bound

$$K_h(X,Y) \le \inf_{a,\Delta>0} \left\{ d_{\Delta,\gamma}(X,Y) \frac{M 2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)} + \frac{2M}{\pi^d}[\Delta \wedge (2a)]^d + 2M_h \int_{\mathbb{R}^d} |t||V_a(t)|dt \right\}. \tag{5.21}$$

The proof will then be completed by choosing an appropriate function V_a.

Notice that for every $x \in \mathbb{R}^d$,

$$\begin{aligned}
|F_X \star h(x) - F_Y \star h(x)| &\leq |F_X \star h(x) - (F_X \star h) * V_a(x)| \\
&+ |F_Y \star h(x) - (F_Y \star h) * V_a(x)| \\
&+ \Big| \Big((F_X \star h) * V_a(x) - (F_X \star h) * I * V_a(x) \Big) \\
&\quad - \Big((F_Y \star h) * V_a(x) - (F_Y \star h) * I * V_a(x) \Big) \Big| \\
&+ \Big| (F_X \star h) * I * V_a(x) - (F_Y \star h) * I * V_a(x) \Big) \Big| := \sum_{j=1}^{4} T_j(x), \quad (5.22)
\end{aligned}$$

where for $\Delta > 0$,

$$I(x) = \prod_{j=1}^{d} \frac{\sin(\Delta x_j)}{\pi x_j}, \; x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d.$$

Note that I is an L^2 function and its Fourier transform is given by $\tilde{I}(z) = 1_{[-\Delta,\Delta]^d}(z) = \prod_{j=1}^{d} 1_{[-\Delta,\Delta]}(z_j)$. All the convolutions in (5.22) are well defined by parts 1 and 2 of Theorem 5.6.

Note that $\int_{\mathbb{R}^d} V_a(x)dx = \tilde{V}_a(0) = 1$. If $G \in Lip_M$ then

$$\begin{aligned}
|G(x) - G * V_a(x)| &\leq \int_{\mathbb{R}^d} |G(x) - G(x-t)| \, |V_a(t)| dt \\
&\leq M \int_{\mathbb{R}^d} |t||V_a(t)| dt. \quad (5.23)
\end{aligned}$$

Since $h \in Lip_{M_h}$, so are $F_X \star h$ and $F_Y \star h$. We conclude that

$$T_j(x) \leq M_h \int_{\mathbb{R}^d} |t||V_a(t)| dt, \quad j = 1, 2\,. \quad (5.24)$$

Further, by Part 5 of Theorem 5.6, $V_a \in L^p$ for all $1 \leq p \leq \infty$, and, clearly, so is the function $(F_X - F_Y) \star h$. By Part 1 of Theorem 5.6, the same is true for the convolution $[(F_X - F_Y) \star h] * V_a$. Denote by Z is a random vector with density h, independent, where appropriate, of X and Y. By Parts 3 and 5 of Theorem 5.6

we obtain

$$\begin{aligned} T_4(x) \le \|[(F_X - F_Y)\star h] * V_a * I\|_\infty &\le (2\pi)^{-d}\|[\hat{\mu}_{X+Z} - \hat{\mu}_{Y+Z}]\tilde{V}_a\tilde{I}\|_1 \\ &\le \frac{2M}{\pi^d}[\Delta \wedge (2a)]^d. \end{aligned} \tag{5.25}$$

This leaves only one term to consider in (5.22). By parts (3) and (4) of Theorem 5.6 we have

$$T_3(x) = (2\pi)^{-d}\left|\int_{\mathbb{R}^d}(1-\tilde{I}(z))\left(\hat{\mu}_X(z)-\hat{\mu}_Y(z)\right)\tilde{h}(z)\tilde{V}_a(z)e^{-i\langle z,x\rangle}dz\right|.$$

Note that for every $z \in \mathbb{R}^d$ and $\Delta > 0$

$$\begin{aligned} \left|(1-\tilde{I}(z))\hat{\mu}_X(z) - (1-\tilde{I}(z))\hat{\mu}_Y(z)\right| &\le |z|^\gamma \sup_{|t|>\Delta}\frac{|\hat{\mu}_X(t)-\hat{\mu}_Y(t)|}{|t|^\gamma} \\ &= |z|^\gamma d_{\Delta,\gamma}(X,Y). \end{aligned}$$

Therefore,

$$\begin{aligned} T_3(x) &\le (2\pi)^{-d}\int_{\mathbb{R}^d}\left|(1-\tilde{I}(z))[\hat{\mu}_X(z)-\hat{\mu}_Y(z)]\tilde{h}(z)\tilde{V}_a(z)\right|dz \\ &\le (2\pi)^{-d}d_{\Delta,\gamma}(X,Y)\int_{\mathbb{R}}|z|^\gamma|\tilde{V}_a(z)|dz \\ &\le M(2\pi)^{-d}d_{\Delta,\gamma}(X,Y)\int_{[-2a,2a]^d}|z|^\gamma dz \\ &\le M(2\pi)^{-d}d_{\Delta,\gamma}(X,Y)\int_{|z|\le 2a\sqrt{d}}|z|^\gamma dz \\ &= M(2\pi)^{-d}d_{\Delta,\gamma}(X,Y)\int_{\mathbb{S}^{d-1}}\int_0^{2a\sqrt{d}} r^\gamma dr\Lambda_{\mathbb{S}^{d-1}}(d\xi) \\ &= Md_{\Delta,\gamma}(X,Y)\frac{2^{1+\gamma}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)}, \end{aligned} \tag{5.26}$$

where $\Lambda_{\mathbb{S}^{d-1}}$ is Lebesgue measure on $\mathbb{S}^{d-1}$. The last line follows by the fact that $\Lambda_{\mathbb{S}^{d-1}}(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ (see Section 5.2 in [Str99]). Now (5.21) follows from (5.22), (5.24), (5.25) and (5.26).

Let $W(x) = \frac{12\sin^4(x/2)}{\pi x^4}$, $x \in \mathbb{R}$. This is called the Jackson-de la Vallée-Poussin kernel. Its Fourier transform is given by

$$\tilde{W}(x) = \begin{cases} 1 - \frac{3x^2}{2} + \frac{3|x|^3}{4} & |x| \leq 1 \\ \frac{1}{4}(2-|x|)^3 & 1 \leq |x| \leq 2 \\ 0 & |x| \geq 2 \end{cases},$$

see page 119 in [Ach92]. For $a > 0$, let $W_a(x) = aW(xa)$, so that $\tilde{W}_a(x) = \tilde{W}(x/a)$, and define for $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$,

$$V_a(x) = \prod_{j=1}^{d} W_a(x_j).$$

Then also

$$\tilde{V}_a(z) = \prod_{j=1}^{d} \tilde{W}_a(z_j)$$

for $z = (z_1, \ldots, z_d)^T \in \mathbb{R}^d$.

Let $B_a(x) = |x|V_a(x)$. Note that $\tilde{V}_a(z) \leq 1$, V_a, $\tilde{V}_a$, $B_a \in L^1$, and $\tilde{V}(0) = 1$. Therefore, the function V_a satisfies the assumptions imposed in the beginning of the proof. Further, we have

$$\int_{\mathbb{R}^d} |x| \, |V_a(x)| \, dx = \left(\frac{12}{\pi}\right)^d \frac{2^{1-3d}}{a} \int_{\mathbb{R}^d} |x| \prod_{j=1}^{d} \frac{\sin^4(x_i)}{x_i^4} dx \leq \frac{6d}{\pi a}. \quad (5.27)$$

This follows easily from the facts that $|x| \leq \sum_{i=1}^{d} |x_i|$, $\int_0^\infty \frac{\sin^4 x}{x^4} dx = \pi/3$ (see [GR00] 3.821), and

$$\int_0^\infty \frac{\sin^4 v}{v^3} dv = \int_0^1 \frac{\sin^4 v}{v^3} dv + \int_1^\infty \frac{\sin^4 v}{v^3} dv \leq \int_0^1 v dv + \int_1^\infty \frac{1}{v^3} dv = 1.$$

This completes the proof. □

Now, to prove the Theorem and the various extensions it suffices to give bounds on $d_{\Delta,\gamma}(X,Y)$, when X is replaced by S_n or S_θ and Y follows a strictly stable or strictly $\mathcal{N}$-stable distribution.

In the context of Theorem 5.2, by the strict stability of Y, we have

$$\begin{aligned} d_{\Delta,\gamma}(S_n, Y) &= \sup_{|z|\geq\Delta} \frac{|\hat{\mu}^n_{X_1}(z/n^{1/\alpha}) - \hat{\mu}^n_Y(z/n^{1/\alpha})|}{|z|^\gamma} \\ &\leq n \sup_{|z|\geq\Delta} \frac{|\hat{\mu}_{X_1}(z/n^{1/\alpha}) - \hat{\mu}_Y(z/n^{1/\alpha})|}{|z|^\gamma} \\ &= n^{-(\gamma/\alpha-1)} \sup_{|z|\geq\Delta n^{-1/\alpha}} \frac{|\hat{\mu}_{X_1}(z) - \hat{\mu}_Y(z)|}{|z|^\gamma} \\ &= n^{-(\gamma/\alpha-1)} d_{\Delta n^{-1/\alpha},\gamma}(X_1, Y) \end{aligned}$$

From here, (5.7) follows by Proposition 5.7.

In the context of Section 5.2.1, we have

$$\begin{aligned} |\hat{\mu}_{S_\theta}(z) - \hat{\mu}_Y(z)| &= \left| \mathrm{E}\left[\exp\left\{ i\theta^{1/\alpha}\langle z, \sum_{i=1}^{N_\theta} X_i\rangle \right\} - \exp\left\{ i\theta^{1/\alpha}\langle z, \sum_{i=1}^{N_\theta} Y_i\rangle \right\} \right] \right| \\ &= \sum_{n=1}^\infty |\hat{\mu}^n_{X_1}(\theta^{1/\alpha}z) - \hat{\mu}^n_Y(\theta^{1/\alpha}z)| P(N_\theta = n) \\ &\leq |\hat{\mu}_{X_1}(\theta^{1/\alpha}z) - \hat{\mu}_Y(\theta^{1/\alpha}z)| \sum_{n=1}^\infty nP(N_\theta = n) \\ &\leq |\hat{\mu}_{X_1}(\theta^{1/\alpha}z) - \hat{\mu}_Y(\theta^{1/\alpha}z)| EN_\theta. \end{aligned}$$

Thus

$$\begin{aligned} d_{\Delta,\gamma}(S_\theta, Y) &\leq \theta^{-1} \sup_{|x|>\Delta} \frac{|\hat{\mu}_{X_1}(\theta^{-1/\alpha}z) - \hat{\mu}_Y(\theta^{-1/\alpha}z)|}{|z|^\gamma} \\ &= \theta^{\gamma/\alpha-1} \sup_{|x|>\Delta\theta^{-1/\alpha}} \frac{|\hat{\mu}_{X_1}(z) - \hat{\mu}_Y(z)|}{|z|^\gamma} \\ &= \theta^{\gamma/\alpha-1} d_{\Delta\theta^{-1/\alpha},\gamma}(X_1, Y). \end{aligned}$$

From here, (5.14) follows by Proposition 5.7.

In the context of Section 5.2.2, we have

$$\begin{aligned} d_{a,\Delta}(S_n, Y) &= \sup_{|z|>\Delta} \frac{|\hat{\mu}_{S_n}(z) - \hat{\mu}_Y(z)|}{|z|^\gamma} \\ &= \sup_{|z|>\Delta} \frac{\left|\int \hat{\mu}_\theta^n(zn^{-1/\alpha})\rho(d\theta) - \hat{\mu}_Y^n(zn^{-1/\alpha})\right|}{|z|^\gamma} \\ &= n^{-\gamma/\alpha} \sup_{|z|>\Delta n^{-1/\alpha}} \frac{\left|\int \hat{\mu}_\theta^n(z)\rho(dW) - \hat{\mu}_Y^n(z)\right|}{|z|^\gamma} \\ &\leq n^{-\gamma/\alpha} \sup_{|z|>\Delta n^{-1/\alpha}} \frac{\int |\hat{\mu}_\theta^n(z) - \hat{\mu}_Y^n(z)|\,\rho(dW)}{|z|^\gamma} \\ &\leq n^{-(\gamma/\alpha-1)} \sup_{|z|>\Delta n^{-1/\alpha}} \frac{\int |\hat{\mu}_\theta(z) - \hat{\mu}_Y(z)|\,\rho(dW)}{|z|^\gamma} \\ &= n^{-(\gamma/\alpha-1)} d_{\Delta n^{-1/\alpha},\gamma}(\mu, \mu_Y; \mu). \end{aligned}$$

From here, (5.18) follows by Proposition 5.7.

We will now show that, under the required conditions, K_h does, in fact, metrize weak convergence.

Proof of Lemma 5.1. Let h be the pdf of a probability distribution, and assume that $h \in Lip_{M_h}$ and $\tilde{h}(z) \neq 0$ for all $z \in \mathbb{R}^d$. Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of probability measures on $\mathbb{R}^d$.

First assume that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} h(x-y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x-y)\mu(dy) \right| = 0.$$

Note that $\int_{\mathbb{R}^d} h(x-y)\mu_n(dy)$ and $\int_{\mathbb{R}^d} h(x-y)\mu(dy)$ are densities with respect to Lebesgue measure. By Scheffé's theorem (Theorem 16.12 in [Bil95]) and Proposition 2.5 in [Sat99] for all $z \in \mathbb{R}^d$, $\hat{\mu}_n(z)\tilde{h}(z) \to \hat{\mu}(z)\tilde{h}(z)$, and so $\hat{\mu}_n(z) \to \hat{\mu}(z)$ for all $z \in \mathbb{R}^d$. This implies that $\mu_n \xrightarrow{w} \mu$.

Conversely, assume that $\mu_n \xrightarrow{w} \mu$. Note that since h is a continuous density, it is bounded by some constant B. Fix $\epsilon > 0$. There is a $T > 0$ such that $\mu(|x| = T) = 0$ and $\mu(|x| > T) < \epsilon/(4B)$. Thus, by the Portmanteau Theorem, for all n large enough $\mu_n(|x| > T) < \epsilon/(4B)$. Note also that there is an $R > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x| > R$ and $|y| \leq T$ we have $h(x-y) < \epsilon/4$. This implies that for n large enough

$$
\begin{aligned}
&\sup_{|x|>R} \left| \int_{\mathbb{R}^d} h(x-y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x-y)\mu(dy) \right| \\
&\leq \sup_{|x|>R} \left[\int_{|y|\leq T} h(x-y)\mu_n(dy) + \int_{|y|>T} h(x-y)\mu_n(dy) \right. \\
&\qquad \left. + \int_{|x|\leq T} h(x-y)\mu(dy) + \int_{|x|>T} h(x-y)\mu(dy) \right] \\
&\leq \epsilon/4 + B\mu_n(|y| > T) + \epsilon/4 + B\mu(|y| > T) < \epsilon.
\end{aligned}
$$

Now for every x with $|x| \leq R$ let $U_x = \{y \in \mathbb{R}^d : |x - y| < \epsilon/(3M_h)\}$. Since the set $\{U_x\}_{|x|\leq R}$ is an open cover of the compact set $|x| \leq R$ there is a finite subcover. Let this be $\{U_{x_i} : i = 1, \ldots, m\}$. Since h is a bounded, continuous function, $\int_{\mathbb{R}^d} h(x-y)\mu_n(dy) \to \int_{\mathbb{R}^d} h(x-y)\mu(dy)$ for every $x \in \mathbb{R}^d$. Thus there is an n large enough such that for each i

$$
\left| \int_{\mathbb{R}^d} h(x_i - y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x_i - y)\mu(dy) \right| < \epsilon/3.
$$

For such n we have

$$
\begin{aligned}
&\sup_{|x|\leq R} \left| \int_{\mathbb{R}^d} h(x-y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x-y)\mu(dy) \right| \\
&\leq \max_{i=1,\ldots,m} \sup_{x\in U_{x_i}} \left\{ \left| \int_{\mathbb{R}^d} h(x-y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x_i-y)\mu_n(dy) \right| \right. \\
&\qquad + \left| \int_{\mathbb{R}^d} h(x_i-y)\mu_n(dy) - \int_{\mathbb{R}^d} h(x_i-y)\mu(dy) \right| \\
&\qquad \left. + \left| \int_{\mathbb{R}^d} h(x_i-y)\mu(dy) - \int_{\mathbb{R}^d} h(x-y)\mu(dy) \right| \right\} \\
&\leq \max_{i=1,\ldots,m} \sup_{x\in U_{x_i}} \{M_h|x - x_i| + \epsilon/3 + M_h|x - x_i|\} < \epsilon.
\end{aligned}
$$

This completes the proof. □

APPENDIX A

CONVERGENCE OF MEASURES

In this appendix we state, for ease of reference, some basic results about weak convergence of probability measures on $\mathbb{R}^d$ and vague convergence of Radon measures on certain topological spaces. For this last part we will need some topology, thus we give some basic definitions in Appendix A.2. We begin with weak convergence.

A.1 Weak Convergence of Probability Measures on $\mathbb{R}^d$

In this section we will list some important properties of weak convergence on $\mathbb{R}^d$. A standard reference is [Bil99]. We begin with the definition.

Definition A.1. *Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of probability measures on $\mathbb{R}^d$. If for all continuous and bounded real valued functions f*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\mu_n(dx) = \int_{\mathbb{R}^d} f(x)\mu(dx)$$

*then we say that the sequence $\{\mu_n\}$ **converges weakly on** $\mathbb{R}^d$ to μ and write $\mu_n \stackrel{w}{\to} \mu$. Equivalently, if $X_n \sim \mu_n$ and $X \sim \mu$ then, when this holds, we say that the sequence (X_n) **converges in distribution** to X and write $X_n \stackrel{d}{\to} X$ or $\lim_{n\to\infty} X_n \stackrel{d}{=} X$.*

It is well known that if (X_n) is a sequence of random vectors and $a \in \mathbb{R}^d$ is a constant then $X_n \stackrel{d}{\to} a$ if and only if $X_n \stackrel{p}{\to} a$ (see for example Theorem 25.3 in [Bil95]). We will now state a number of useful results about weak convergence on $\mathbb{R}^d$. We will consider these results to be so fundamental to the theory that we will often use them without reference.

Proposition A.2. *1.(Portmanteau Theorem) Let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$. The following are equivalent:*

i) $\mu_n \xrightarrow{w} \mu$*;*

ii) $\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\mu_n(dx) = \int_{\mathbb{R}^d} f(x)\mu(dx)$ *for all bounded, uniformly continuous functions* f*;*

iii) $\mu_n(B) \to \mu(B)$ *for all* $B \in \mathfrak{B}(\mathbb{R}^d)$ *with* $\mu(\partial B) = 0$*;*

iv) $\limsup_{n\to\infty} \mu_n(K) \le \mu(K)$ *for all closed sets* K*;*

v) $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$ *for all open sets* G*.*

2.(Cramér-Wold device) Let $X, X_1, X_2, \dots$ *be random vectors. Then* $X_n \xrightarrow{d} X$ *if and only if* $\langle z, X_n\rangle \xrightarrow{d} \langle z, X\rangle$ *for all* $z \in \mathbb{R}^d$*.*

3.(Continuous Mapping Theorem) Let $X, X_1, X_2, \dots$ *be random vectors and let* μ *be the law of* X*. Let* g *be a Borel function mapping* $\mathbb{R}^d$ *into* $\mathbb{R}^k$ *and let* D *be the set of discontinuities of* g*. If* $X_n \xrightarrow{d} X$ *and* $\mu(D) = 0$ *then*

$$g(X_n) \xrightarrow{d} g(X).$$

4. Let (X_n) *and* (Y_n) *be independent sequences of random vectors. If there exist random vectors* X *and* Y *independent of each other such that* $X_n \xrightarrow{d} X$ *and* $Y_n \xrightarrow{d} Y$ *then*

$$X_n + Y_n \xrightarrow{d} X + Y.$$

5. (Slutzky's Theorem) Fix $b \in \mathbb{R}^d$ *and* $c \in \mathbb{R}$*. Let* (X_n) *be a sequence of random vector such that* $X_n \xrightarrow{d} X$ *for some random vector* X*. If* (Y_n) *is a sequence of random vectors with* $Y_n \xrightarrow{p} b$ *then*

$$X_n + Y_n \xrightarrow{d} X + b.$$

If (Z_n) *is a sequence of random variables with* $Z_n \xrightarrow{p} c$ *then*

$$Z_n X_n \xrightarrow{d} cX$$

and if $c \neq 0$

$$X_n/Z_n \xrightarrow{d} X/c.$$

6. (Convergence of Types) Let Z and W be non-constant random vectors. Let (Z_n) be a sequence of random vectors, $b_n > 0$, and $c_n \in \mathbb{R}^d$. If $Z_n \xrightarrow{d} Z$ and $b_n Z_n + c_n \xrightarrow{d} W$ then $b_n \to b$, $c_n \to c$ for some $b \in (0, \infty)$ and $c \in \mathbb{R}^d$ and $bZ + c \stackrel{d}{=} W$.

Proof. Part 1 is Theorem 2.1 in [Bil99]. Part 2 is Corollary 5.5 in [Kal02]. Part 3 is Theorem 1.10 in [Sha03]. Part 4 is an immediate consequence of Part 3. Part 5 is Theorem 1.11 in [Sha03]. Finally, Part 6 is Lemma 13.10 in [Sat99]. □

A.2 Definitions From Topology

In this section we will define some basic concepts from topology, which will be useful for discussing vague convergence in Appendix A.3. We begin by defining a topological space.

Definition A.3. *Let E be a set. If τ is a collection of subsets of E such that*

1. $\emptyset, E \in \tau$,

2. τ is closed under finite intersections,

3. τ is closed under arbitrary unions,

*then τ is called a **topology**, (E, τ) is called a **topological space**, and the sets in τ are called **open sets**. The complement of an open set is called a **closed set**. If, in addition, for any $a, b \in E$ with $a \neq b$ there are $A, B \in \tau$ with $a \in A$, $b \in B$, and $A \cap B = \emptyset$ then we say that the space is **Hausdorff**.*

When we fix a particular topology, we will just write E instead of (E, τ). If a set E is equipped with a metric d, then the metric induces a topology on E. This is the smallest topology containing sets of the form

$$\{y \in E : d(x, y) < r\}, \ r > 0, \ x \in E.$$

For any topological space (E, τ), the class of **Borel sets** on E is the smallest σ-algebra generated by τ. We will use the notation $\mathfrak{B}(E)$ to denote this class. If $A \subset E$ then the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A. We write A° for the interior of A and $\bar{A}$ for the closure of A. Note that $A^\circ \subset A \subset \bar{A}$. We write $\partial A = \bar{A} \setminus A^\circ$ to denote the **boundary** of A. Now, we will define compact sets.

Definition A.4. *Let (E, τ) be a Hausdorff space and let $A \subset E$. If for any collection $\tau_0 \subset \tau$ with $A \subset \bigcup \tau_0$ there is a finite subcollection $\tau_1 \subset \tau_0$ such that $A \subset \bigcup \tau_1$ then A is called a **compact set**. If A is such that its closure is compact then A is called **relatively compact**.*

For more on topological spaces see [Mun00] or Chapter 7 in [Bau81].

A.3 Vague Convergence

Let (E, τ) be a locally compact Hausdorff space with a countable basis. By **locally compact** we mean that every $x \in E$ is contained in a relatively compact open set, and by **countable basis** we mean that there exists a countable collection of open sets $\{G_n\}$ such that every open set G can be written as a finite or countable union of elements of $\{G_n\}$. By Theorem 7.6.1 in [Bau81] this implies that (E, τ) is a Polish space. Thus there exists a metric ρ which generates τ and, in this metric, E is a complete and separable metric space. A measure on $(E, \mathfrak{B}(E))$ is called a **Radon measure** if it is finite on any compact subset of E. Let $M(E)$ denote the set of Radon measures on $(E, \mathfrak{B}(E))$.

Definition A.5. *Let $\mu, \mu_1, \mu_2, \dots$ be a sequence in $M(E)$. We say that $\{\mu_n\}$ **converges vaguely** to μ and write $\mu_n \xrightarrow{v} \mu$ if for all continuous, real valued* *functions f on E with compact support*

$$\lim_{n\to\infty}\int_E f(x)\mu_n(dx) = \int_E f(x)\mu(dx).$$

Proposition A.6. *Vague convergence on $M(E)$ is metrizable as a complete, separable metric space.*

Proof. This is Proposition 3.17 in [Res87]. □

In discussing vague convergence, the following definition will be useful.

Definition A.7. *A set $B \in \mathfrak{B}(E)$ is a* ***continuity set*** *of a measure μ if $\mu(\partial B) = 0$.*

We will now give some equivalent formulations of vague convergence. The equivalence of the first three parts in the next result, is called **the Portmanteau Theorem for vague convergence**.

Proposition A.8. *Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence in $M(E)$. The following are equivalent:*

1. $\mu_n \xrightarrow{v} \mu$;

2. $\mu_n(B) \to \mu(B)$ for all relatively compact continuity sets B of μ;

3. $\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$ and $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ for all compact sets K and all open, relatively compact sets G;

4. for all relatively compact continuity sets B of μ and all measurable functions f, which are continuous and bounded on B, $\int_B f(x)\mu_n(dx) \to \int_B f(x)\mu(dx)$.

Proof. The equivalence of Parts 1, 2, and 3 is given in Proposition 3.12 in [Res87]. The equivalence between Parts 1 and 2 implies that $\mu_n \xrightarrow{v} \mu$ if and only if for any relatively compact set B with $\mu(\partial B) = 0$, $\mu_{|B} \xrightarrow{v} \mu_{|B}$. Thus, Part 4 is equivalent to Part 1. □

Part 2 of the above proposition shows that vague convergence is equivalent to convergence of the measures on relatively compact continuity sets of the limiting measure. The following result shows that it suffices to show convergence on certain smaller classes of sets. We call these **convergence determining sets**.

Proposition A.9. *Let $\mu, \mu_1, \mu_2, \ldots$ be finite measures and let $\mathcal{A} \subset \mathfrak{B}(E)$ be a class of relatively compact open sets satisfying:*

1) $\mathcal{A}$ is closed under finite intersections, and

2) any relatively compact open set is a countable union of elements of $\mathcal{A}$.

If

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)$$

for every $A \in \mathcal{A}$, then $\mu_n \xrightarrow{v} \mu$.

Proof. If $\mathcal{A}$ satisfies the additional assumption:

3) For any compact set V, there exist $A_1, \ldots, A_m \in \mathcal{A}$ such that $V \subset \bigcup_{i=1}^m A_i$

then, for the case where $E = \mathbb{R}^d$, this is Lemma 2.6.2 in [Cup75]. Little in the proof changes for our more general situation.

However, 3) is not needed since it is implied by 2). Let V be a compact set in $\mathfrak{B}(E)$. Since E is locally compact, V can be covered by relatively compact open sets. Thus, by 2) there is an open cover of V made up of elements of $\mathcal{A}$, and since V is compact there is a finite subcover. □

We have seen several ways to show vague convergence of a sequence of measures to a particular measure. We will now give a result on how to show that a sequence has a vague limit.

Proposition A.10. *Let $\{\mu_n\}$ be a sequence in $M(E)$ such that $\sup \mu_n(B) < \infty$ for every relatively compact $B \in \mathfrak{B}(E)$. The following hold:*

1. The set $\{\mu_n : n \in \mathbb{N}\}$ is relatively compact.

2. *There exists a subsequence* $\{\mu_{n_k}\}$, *which converges vaguely to some measure* μ.

3. *If there is a measure* μ *such that for every vaguely convergent subsequence* $\{\mu_{n_k}\}$, *we have* $\mu_{n_k} \xrightarrow{v} \mu$ *as* $k \to \infty$, *then* $\mu_n \xrightarrow{v} \mu$.

A version of this result for the case where $E = \mathbb{R}^d$ is given in Theorem 2.6.1 and Corollary 2.6.1 of [Cup75].

Proof. The first part follows immediately from Proposition 3.16 in [Res87]. Theorem 28.2 in [Mun00] says that Part 1 is equivalent to Part 2 in a metrizable space. Since, by Proposition A.6, $M(E)$ is metrizable, Part 2 holds. Once we know that there is a vaguely convergent subsequence, Part 3 is immediate. □

Proposition A.11. *If* $\Lambda \in M(E)$ *then there is a sequence* $\{\Lambda_n\}$ *in* $M(E)$ *such that* Λ_n *has a finite support for each* n *and* $\Lambda_n \xrightarrow{v} \Lambda$.

Proof. This is a version of Theorem 7.7.3 in [Bau81]. □

A.4 Vague Convergence in $\bar{\mathbb{R}}^d_0$

Most of the time, we will not need the generality of the previous section. Instead we will confine ourselves to vague convergence of Radon measures on either $\mathbb{R}^d$, $\mathbb{R}^d_0$, or certain compactifications at infinity of these sets. These compactifications will be denoted $\bar{\mathbb{R}}^d$ and $\bar{\mathbb{R}}^d_0$. There are many nonequivalent ways to define such compactifications. Often, when we deal with measures that place no mass on $\bar{\mathbb{R}}^d \setminus \mathbb{R}^d$, the precise definition of $\bar{\mathbb{R}}^d$ will not matter. However, in other cases, we will need to know exactly what is meant by $\bar{\mathbb{R}}^d$. In this section we will define $\bar{\mathbb{R}}^d$ and $\bar{\mathbb{R}}^d_0$ in a way that will be needed in Chapter 3. We will then discuss vague convergence on $\bar{\mathbb{R}}^d_0$.

Recall that if $x \in \mathbb{R}^d_0$ then $x = |x|\frac{x}{|x|}$, thus we can identify every element of $\mathbb{R}^d_0$ with an element of $(0,\infty) \times \mathbb{S}^{d-1}$. Let $\bar{\mathbb{R}}^d_0 = (0,\infty] \times \mathbb{S}^{d-1}$ and $\bar{\mathbb{R}}^d = \bar{\mathbb{R}}^d_0 \cup \{0_d\}$.

We will write $\mathbb{I}^{d-1} = \{\infty\} \times \mathbb{S}^{d-1}$. Since we can identify any element of $\mathbb{R}^d$ with an element in $\bar{\mathbb{R}}^d \setminus \mathbb{I}^{d-1}$, we will generally write one to refer to the other. Also, for simplicity, we will sometimes write ∞u for the tuple (∞, u), and when $B \subset \mathbb{S}^{d-1}$ we will write ∞B for $\{\infty\} \times B$. With this notation, we have $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \mathbb{I}^{d-1}$. We will also need the following functions. Let $\xi : \bar{\mathbb{R}}^d \mapsto \mathbb{S}^{d-1} \cup \{0_d\}$ and $\vartheta : \bar{\mathbb{R}}^d \mapsto [0, \infty]$ be defined as follows. Let $\xi(0_d) = \vartheta(0_d) = 0$ and for $x \neq 0_d$ we have $x = (r, u)$ for $r \in (0, \infty]$ and $u \in \mathbb{S}^{d-1}$, let $\xi(x) = u$ and $\vartheta(x) = r$. Sometimes we will write $|x| := \vartheta(x)$.

Let $\stackrel{\bar{\mathbb{R}}_+}{\to}$ and $\stackrel{\mathbb{R}^d}{\to}$ denote, respectively, the usual convergence on $[0, \infty]$ and on $\mathbb{R}^d$. Let $x, x_1, x_2, \cdots \in \bar{\mathbb{R}}^d_0$. We will write

$$\lim_{n\to\infty} x_n = x$$

when $\vartheta(x_n) \stackrel{\bar{\mathbb{R}}_+}{\to} \vartheta(x)$ and $\xi(x_n) \stackrel{\mathbb{R}^d}{\to} \xi(x)$. Let τ_0 be the set of subsets on $\bar{\mathbb{R}}^d_0$ such that $A \in \tau_0$ if and only if for any $x \in A$ and any sequence (x_n) in $\bar{\mathbb{R}}^d_0$ with $\lim_{n\to\infty} x_n = x$ there is an N large enough such that for all $n \geq N$, $x_n \in A$. It is straightforward to show that τ_0 is a topology. Moreover, it is not difficult to see that the topological space $\left(\bar{\mathbb{R}}^d_0, \tau_0\right)$ is a locally compact Hausdorff space with a countable basis. The compact sets are closed sets that are bounded away from 0_d. Note that if $x, x_1, x_2, \cdots \in \bar{\mathbb{R}}^d_0 \setminus \mathbb{I}^{d-1}$ then $\lim_{n\to\infty} x_n = x$ if and only if $x_n \stackrel{\mathbb{R}^d}{\to} x$.

In a similar way we can define a topology on $\bar{\mathbb{R}}^d$. We define convergence to a point in $\bar{\mathbb{R}}^d \setminus \{0_d\}$ as before. For convergence to 0_d we write

$$\lim_{n\to\infty} x_n = 0_d$$

when $\vartheta(x_n) \stackrel{\bar{\mathbb{R}}_+}{\to} 0$. From here we define a topology τ in an analogous way to the previous case. The topological space $\left(\bar{\mathbb{R}}^d, \tau\right)$ is also a locally compact Hausdorff space with a countable basis. Here, the compact sets are closed sets.

We will now give some results about vague convergence on $\bar{\mathbb{R}}^d_0$. We begin with an important example of convergence determining sets.

Lemma A.12. *Let (μ_n) be a sequence of Radon measures on $\bar{\mathbb{R}}^d_0$. Let μ be a measure on $\bar{\mathbb{R}}^d_0$ such that $\mu(|x| = a) = 0$ for every $0 < a < \infty$ and*

$$\mu(A) = \int_{\mathbb{S}^{d-1}} \int_{(0,\infty]} 1_A(xu)\nu(dx)\sigma(du)$$

for some finite measure σ and some measure ν finite outside any neighborhood of 0. Then $\mu_n \xrightarrow{v} \mu$ on $\bar{\mathbb{R}}^d_0$ if and only if

$$\mu_n\left(|x| > t, \xi(x) \in D\right) \to \mu\left(|x| > t, \xi(x) \in D\right) \tag{A.1}$$

for every $t > 0$ and every $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$.

Proof. Clearly, if $\mu_n \xrightarrow{v} \mu$ on $\bar{\mathbb{R}}^d_0$ then (A.1) holds. Now assume that (A.1) holds. Let $\mathcal{A}$ be the class of measurable sets such that $A \in \mathcal{A}$ if and only if A is bounded away from 0_d and $\mu_n(A) \to \mu(A)$. If $A, B \in \mathcal{A}$ and $A \subset B$ then

$$\mu_n(B \setminus A) = \mu_n(B) - \mu_n(A) \to \mu(B) - \mu(A) = \mu(B \setminus A),$$

thus $B \setminus A \in \mathcal{A}$. By assumption sets of the form

$$\left\{x \in \bar{\mathbb{R}}^d : |x| > t, \xi(x) \in D\right\} \tag{A.2}$$

for $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $t > 0$ are elements of $\mathcal{A}$. Thus so are sets of the form

$$\left\{x \in \bar{\mathbb{R}}^d : a \geq |x| > b, \xi(x) \in D\right\},$$

where $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $0 < b < a \leq \infty$. Moreover, by continuity from above

$$\begin{aligned} \limsup_{n\to\infty} \mu_n\left(|x| = t, \xi(x) \in D\right) &\leq \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \mu_n\left(t \geq |x| > t - \epsilon, \xi(x) \in D\right) \\ &= \lim_{\epsilon\downarrow 0} \mu\left(t \geq |x| > t - \epsilon, \xi(x) \in D\right) = 0. \end{aligned}$$

This means that all sets of the form

$$\left\{x \in \bar{\mathbb{R}}^d : a > |x| > b, \xi(x) \in D\right\}, \tag{A.3}$$

where $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $0 < b < a \leq \infty$ are elements of $\mathcal{A}$.

Let $\mathcal{A}'$ be the class of sets of the form (A.2) and (A.3) where D is an open set in $\mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$. We will show that $\mathcal{A}'$ satisfies the assumptions of Proposition A.9. It is immediate that $\mathcal{A}'$ is a collection of open sets which satisfies Assumption 1). Assumption 2) follows from the fact that the space is separable and into any neighborhood we can fit sets from $\mathcal{A}'$. Thus, since for any $A \in \mathcal{A}'$ $\mu_n(A) \to \mu(A)$ the result holds by Proposition A.9. □

Next, we will show a useful characterization of vague convergence on $\bar{\mathbb{R}}^d_0$ for the special case when none of the measures place mass on $\bar{\mathbb{R}}^d \setminus \mathbb{R}^d$. First we need some notation. Let $C^\#$ be the class of bounded continuous Borel functions mapping $\mathbb{R}^d$ into $\mathbb{R}$, and let $C^\#_0$ be the subclass of $C^\#$ containing function that vanish on a neighborhood of zero.

Proposition A.13. *Let M_n for $n = 0, 1, 2, \ldots$ be a sequence of Borel measures on $\bar{\mathbb{R}}^d$ such that $M_n(\bar{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ for all n.*

1. If, for all $\epsilon > 0$, $M_n(|x| > \epsilon) < \infty$ then $M_n \xrightarrow{v} M_0$ on $\bar{\mathbb{R}}^d_0$ if and only if $\int_{\mathbb{R}^d} f(x)M_n(dx) \to \int_{\mathbb{R}^d} f(x)M_0(dx)$ for all $f \in C^\#_0$.

2. If M_n are finite measures, then $M_n \xrightarrow{v} M_0$ on $\bar{\mathbb{R}}^d$ if and only if $\int_{\mathbb{R}^d} f(x)M_n(dx) \to \int_{\mathbb{R}^d} f(x)M_0(dx)$ for all $f \in C^\#$.

Note that both in Parts 1 and 2 above, the assumptions on the measures M_n imply that they are Radon measures on $\bar{\mathbb{R}}^d_0$. This result implies that we do not need to worry about continuity of f at infinity, in fact we do not even need the limit to exist there. Note that for the proof we will only assumes that $\bar{\mathbb{R}}^d$ and $\bar{\mathbb{R}}^d_0$

are compactifications of $\mathbb{R}^d$ and $\mathbb{R}_0^d$ and not the precise forms that we previously defined.

Proof. We will only prove the first part since the proof of the second part is similar. First, assume that $M_n \xrightarrow{v} M_0$ on $\bar{\mathbb{R}}_0^d$. Let $H = \{T \in (0,\infty) : M_0(|x| = T) = 0\}$ and fix $f \in C_0^{\#}$. This means that there is a K such that $|f(x)| \leq K$ for all x and there is an $\epsilon \in H$ such that $f(x) = 0$ if $|x| < \epsilon$.

First assume that $f(x) \geq 0$ for all x. By Part 4 of Proposition A.8, if $T \in H$

$$\lim_{n\to\infty} \int_{\epsilon\leq|x|\leq T} f(x)M_n(dx) = \int_{\epsilon\leq|x|\leq T} f(x)M_0(dx).$$

Thus

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^d} f(x)M_n(dx) &\geq \lim_{H\ni T\uparrow\infty} \lim_{n\to\infty} \int_{\epsilon\leq|x|\leq T} f(x)M_n(dx) \\
&= \lim_{H\ni T\uparrow\infty} \int_{\epsilon\leq|x|\leq T} f(x)M_0(dx) \\
&= \int_{\mathbb{R}^d} f(x)M_0(dx),
\end{aligned}$$

where the last equality follows by dominated convergence. Since $M_0(\bar{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$, for any $\delta > 0$ there is a $T_\delta \in H$ such that $M_0(|x| \geq T_\delta) \leq \delta/K$. Thus

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)M_n(dx) &\leq \int_{\epsilon\leq|x|\leq T_\delta} f(x)M_n(dx) + KM_n(|x| > T_\delta) \\
&\to \int_{\epsilon\leq|x|\leq T_\delta} f(x)M_0(dx) + KM_0(|x| > T_\delta) \\
&\leq \int_{\mathbb{R}^d} f(x)M_0(dx) + \delta. \qquad \text{(A.4)}
\end{aligned}$$

Since this holds for all $\delta > 0$, $\limsup_{n\to\infty} \int_{\mathbb{R}^d} f(x)M_n(dx) \leq \int_{\mathbb{R}^d} f(x)M_0(dx)$. Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)M_n(dx) = \int_{\mathbb{R}^d} f(x)M_0(dx).$$

Extension to the case when f may be negative is immediate. The other direction follows from the definition of vague convergence. □

APPENDIX B

REGULAR VARIATION

Regularly varying functions are functions that have power-like behavior. Standard references are [Fel71], [BGT87], and [Res87]. First we will define regularly varying functions and state some of their properties. Next we will consider generalizations to regularly varying measures and regularly varying matrix valued functions.

B.1 Regularly Varying Functions

Definition B.1. *Fix $a \in \{0, \infty\}$. A positive Borel function f defined on a neighborhood of a is called* ***regularly varying at*** **a** *if there exists some $\rho \in \mathbb{R}$ such that*

$$\lim_{x \to a} \frac{f(tx)}{f(x)} = t^{\rho}.$$

The parameter ρ is called the ***index of regular variation****. We will write $f \in RV_{\rho}^{a}$.*

If $f \in RV_{\rho}^{\infty}$ then there is an $\ell \in RV_{0}^{\infty}$ such that $f(x) = x^{\rho}\ell(x)$. If $g(x) = f(1/x)$ then

$$f \in RV_{\rho}^{\infty} \Longleftrightarrow g \in RV_{-\rho}^{0}. \tag{B.1}$$

If $f \in RV_{\rho}^{\infty}$ then f is defined and locally bounded on $[X, \infty)$ for some $X > 0$. If $\rho > 0$ define

$$f^{\leftarrow}(x) = \inf \{y > X : f(y) > x\}. \tag{B.2}$$

By page 28 in [BGT87],

$$f^{\leftarrow} \in RV_{1/\rho}^{\infty} \tag{B.3}$$

and $f^{\leftarrow}$ is an asymptotic inverse of f in the sense that

$$f(f^{\leftarrow}(x)) \sim f^{\leftarrow}(f(x)) \sim x \text{ as } x \to \infty. \tag{B.4}$$

In fact, a function g is an asymptotic inverse of f if and only if $g(x) \sim f^{\leftarrow}(x)$ as $x \to \infty$. We will also need an asymptotic inverse for regular variation at 0. Let $f \in RV_\rho^0$ with $\rho > 0$ and $h(x) = 1/f(1/x)$. In this case $h \in RV_\rho^\infty$ and we can define

$$f^{\leftarrow}(x) = 1/h^{\leftarrow}(1/x). \tag{B.5}$$

Note that

$$f^{\leftarrow} \in RV_{1/\rho}^0, \tag{B.6}$$

and it is easy to see that $f^{\leftarrow}$ is the asymptotic inverse of f in the sense that

$$f(f^{\leftarrow}(x)) \sim f^{\leftarrow}(f(x)) \sim x \text{ as } x \downarrow 0. \tag{B.7}$$

The following lemma summarizes some important properties of regularly varying functions.

Proposition B.2. *Fix $\rho \in \mathbb{R}$ and $X \geq 0$. Let $f, g : [X, \infty) \mapsto [0, \infty)$.*

1. If $f \in RV_\rho^\infty$ then

$$\lim_{t\to\infty} f(t) = \begin{cases} 0 & \text{if } \rho < 0 \\ \infty & \text{if } \rho > 0 \end{cases}.$$

2. If $f \in RV_{\rho_1}^\infty$, $g \in RV_{\rho_2}^\infty$, and $\lim_{x\to\infty} g(x) = \infty$, then

$$f \circ g \in RV_{\rho_1\rho_2}^\infty.$$

3. If $\rho \neq 0$ and $f \in RV_\rho^\infty$ then there is a differentiable, strictly monotone function h such that $f(x) \sim h(x)$ as $x \to \infty$.

4. Let $f \in RV_\rho^\infty$ and let h, g be functions increasing to ∞. If $c > 0$ and $g(x) \sim ch(x)$ as $x \to \infty$ then $f(g(x)) \sim c^\rho f(h(x))$ as $x \to \infty$.

5. If $c > 0$, $\rho > 0$, and $f, g \in RV_\rho^\infty$ then

$$f(t) \sim cg(t) \text{ as } t \to \infty$$

if and only if

$$f^{\leftarrow}(t) \sim c^{-1/\rho} g^{\leftarrow}(t) \text{ as } t \to \infty.$$

6. If f is a monotone function and there are sequences of positive numbers λ_n and b_n such that $b_n \to \infty$, $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$, and if for all $x > 0$

$$\lim_{n\to\infty} \lambda_n f(b_n x) =: \chi(x) \tag{B.8}$$

exists and is positive and finite then there is a $\rho \in \mathbb{R}$ such that $\chi(x)/\chi(1) = x^\rho$ and $f \in RV_\rho^\infty$.

By (B.1), it follows that if (B.8) holds with $b_n \to 0$ then there is a $\rho \in \mathbb{R}$ such that $\chi(x)/\chi(1) = x^\rho$ and $f \in RV_\rho^0$.

Proof. See Propositions 2.3 and 2.6 in [Res07]. □

Proposition B.3. *Let $f \in RV_\rho^\infty$ for some $\rho \in \mathbb{R}$.*

1. If $\rho > 0$ then there exists a function a_s increasing to ∞ such that $\lim_{s\to\infty} \frac{f(ta_s)}{s} = t^\rho$. Moreover, this holds for a function a_s if and only if $a_s \sim f^{\leftarrow}(s)$. In this case $a \in RV_{1/\rho}^\infty$.

2. If $\rho < 0$ then there exists a function a_s increasing to ∞ such that $\lim_{s\to\infty} sf(ta_s) = t^\rho$. Moreover, this holds for a function a_s if and only if $a_s \sim g^{\leftarrow}(s)$, where $g(s) = 1/f(s)$. In this case $a \in RV_{1/|\rho|}^\infty$.

Proof. We begin with Part 1. Let $a_s = f^{\leftarrow}(s)$. By (B.3) $a \in RV_{1/\rho}^\infty$ and by Proposition B.2 it increases to ∞. Observe that

$$t^\rho = \lim_{s\to\infty} \frac{f(ta_s)}{f(a_s)} = \lim_{s\to\infty} \frac{f(ta_s)}{s} \frac{s}{f(a_s)} = \lim_{s\to\infty} \frac{f(ta_s)}{s}.$$

Now let b_s be a function increasing to ∞ such that $\lim_{s\to\infty} \frac{f(tb_s)}{s} = t^\rho$. Since $f^{\leftarrow}$ is regularly varying, by Part 4 of Proposition B.2 we have

$$\lim_{s\to\infty} \frac{b_s}{f^{\leftarrow}(s)} = \lim_{s\to\infty} \frac{f^{\leftarrow}(f(b_s))}{f^{\leftarrow}(s)} = \lim_{s\to\infty} \frac{f^{\leftarrow}(s)}{f^{\leftarrow}(s)} = 1.$$

For Part 2, note that $g \in RV_{|\rho|}^{\infty}$. From here, the result follows immediately from Part 1. □

The following result will be useful in the sequel.

Proposition B.4. *Fix $\gamma > \eta$. Suppose that μ is a Borel measure on $[0,\infty]$ such that $0 < \int_0^\infty (y^\gamma \wedge y^\eta)\mu(dy) < \infty$. Let*

$$U_\gamma(x) = \int_0^x y^\gamma \mu(dy)$$

and

$$V_\eta(x) = \int_x^\infty y^\eta \mu(dy).$$

1. If $U_\gamma \in RV_\rho^\infty$ for some $\rho \in \mathbb{R}$, then $\rho = \gamma - \alpha$ for some $\alpha \in [\eta, \gamma]$ and

$$\lim_{x\to\infty} \frac{x^{\gamma-\eta} V_\eta(x)}{U_\gamma(x)} = \frac{\gamma - \alpha}{\alpha - \eta}, \tag{B.9}$$

where we interpret the right side as infinity when $\alpha = \eta$.

2. If $V_\eta \in RV_\rho^\infty$ for some $\rho > \eta - \gamma$, then $\rho = \eta - \alpha$ for some $\alpha \in [\eta, \gamma)$ and (B.9) *holds.*

3. If (B.9) *holds with some $\alpha \in (\eta, \gamma)$, then there exists an $L \in RV_0^\infty$ such that*

$$U_\gamma(x) \sim (\alpha - \eta) x^{\gamma-\alpha} L(x) \quad \text{and} \quad V_\eta(x) \sim (\gamma - \alpha) x^{\eta-\alpha} L(x),$$

as $x \to \infty$.

4. If (B.9) *holds with $\alpha = \gamma$ then $U_\gamma \in RV_0^\infty$. If* (B.9) *holds with $\alpha = \eta$ then $V_\eta \in RV_0^\infty$.*

5. If $V_\eta \in RV_\rho^0$ for some $\rho \in \mathbb{R}$, then $\rho = \eta - \alpha$ for some $\alpha \in [\eta, \gamma]$ and

$$\lim_{x\downarrow 0} \frac{x^{\gamma-\eta} V_\eta(x)}{U_\gamma(x)} = \frac{\gamma - \alpha}{\alpha - \eta}, \tag{B.10}$$

where we interpret the right side as infinity when $\alpha = \eta$.

6. If $U_\gamma \in RV_\rho^0$ *for some* $\rho < \gamma - \eta$*, then* $\rho = \gamma - \alpha$ *for some* $\alpha \in (\eta, \gamma]$ *and* (B.10) *holds.*

7. If (B.10) *holds with some* $\alpha \in (\eta, \gamma)$*, then there exists an* $L \in RV_0^0$ *such that*

$$V_\eta(x) \sim (\gamma - \alpha)x^{\eta-\alpha}L(x) \quad \text{and} \quad U_\gamma(x) \sim (\alpha - \eta)x^{\gamma-\alpha}L(x),$$

as $x \downarrow 0$.

8. If (B.10) *holds with* $\alpha = \gamma$ *then* $U_\gamma \in RV_0^0$*. If* (B.10) *holds with* $\alpha = \eta$ *then* $V_\eta \in RV_0^0$*.*

Proof. We begin by considering Parts 1-4. When μ is a finite measure this result is given in Theorem 5.3.11 in [MS01] and Theorem 2 in Section VIII.9 of [Fel71].

Now assume that μ is not finite, but it satisfies the required integrability conditions. Define the measure

$$\mu'(A) = \int_1^\infty 1_A(y) y^\eta \mu(dy) + \delta_1(A) \int_0^1 y^\gamma \mu(dy).$$

For $x > 1$ we have

$$U_\gamma(x) = \int_0^x y^{\gamma-\eta} \mu'(dy)$$

and

$$V_\eta(x) = \int_x^\infty \mu'(dy).$$

Since μ' is a finite measure, the result follows from the finite case.

Now for parts 5-8. Let $T(y) = 1/y$ and define the measure μT^{-1} by

$$(\mu T^{-1})(A) = \mu(T^{-1}A).$$

Let $\tilde{V}_{-\gamma}(x) = U_\gamma(1/x)$ and let $\tilde{U}_{-\eta}(x) = V_\eta(1/x)$. By the change of variables formula (Theorem 16.13 in [Bil95]) we have

$$\tilde{V}_{-\gamma}(x) = U_\gamma(1/x) = \int_0^{1/x} y^\gamma \mu(dy) = \int_x^\infty u^{-\gamma} (\mu T^{-1})(du)$$

and

$$\tilde{U}_{-\eta}(x) = V_\eta(1/x) = \int_{1/x}^{\infty} y^\eta \mu(dy) = \int_0^x u^{-\eta}(\mu T^{-1})(du).$$

Similarly, by the change of variables formula, $0 < \int_0^\infty (y^{-\eta} \wedge y^{-\gamma})(\mu T^{-1})(dy) < \infty$. From here the result follows from Parts 1-4. □

By a change of variables we get the following extension.

Corollary B.5. *Fix $\gamma > \eta$. Let F map $\mathbb{R}^d$ into $\mathbb{R}$. Let M be a Borel measure on $\bar{\mathbb{R}}^d$ such that $0 < \int (|F(y)|^\gamma \wedge |F(y)|^\eta)\, M(dy) < \infty$. Define*

$$U_\gamma(x) = \int_{|F(y)| \le x} |F(y)|^\gamma M(dy)$$

and

$$V_\eta(x) = \int_{|F(y)| \ge x} |F(y)|^\eta M(dy).$$

The results of Proposition B.4 hold when U_γ and V_η are defined in this way.

B.2 Regularly Varying Measures

We will now define regularly varying measures. General information can be found in [MS01], [HL06b], and [Res07]. Assume that R is a measure on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ such that for some $X > 0$

$$R(\{x \in \mathbb{R}^d : |x| > X\}) < \infty \tag{B.11}$$

and for all $r > 0$

$$R(\{x \in \mathbb{R}^d : |x| > r\}) > 0. \tag{B.12}$$

In particular, these conditions hold for any Lévy measure with an unbounded support.

Definition B.6. *Fix $\rho \leq 0$. A measure Borel R on $\mathbb{R}^d$ satisfying* (B.11) *and* (B.12) *is said to be* ***regularly varying at ∞ with index*** *ρ if there is a finite, non-zero measure σ on $(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}))$ such that for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$*

$$\lim_{r\to\infty} \frac{R\left(|x| > rt, \frac{x}{|x|} \in D\right)}{R(|x| > r)} = t^\rho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}.$$

When this holds we will write $R \in RV_\rho^\infty(\sigma)$.

This is often called **regular variation of the tails** of R. Clearly, the measure σ is unique only up to a multiplicative constant. For $D \in \mathfrak{B}(\mathbb{S}^d)$ define

$$U_D(t) = R(|x| > t, x/|x| \in D). \tag{B.13}$$

When $\sigma(D) > 0$, $\sigma(\partial D) = 0$, and $R \in RV_\rho^\infty(\sigma)$

$$\lim_{r\to\infty} \frac{U_D(rt)}{U_D(r)} = \lim_{r\to\infty} \frac{U_D(rt)}{U_{\mathbb{S}^{d-1}}(r)} \frac{U_{\mathbb{S}^{d-1}}(r)}{U_D(r)} = t^\rho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})} \frac{\sigma(\mathbb{S}^{d-1})}{\sigma(D)} = t^\rho.$$

Thus, under the given conditions

$$U_D \in RV_\rho^\infty. \tag{B.14}$$

Moreover, $R \in RV_\rho^\infty(\sigma)$ if and only if there is an $\ell \in RV_0^\infty$ such that for all $D \in \mathfrak{B}(\mathbb{S}^d)$ with $\sigma(\partial D) = 0$

$$U_D(t) \sim \sigma(D) t^\rho \ell(t) \text{ as } t \to \infty. \tag{B.15}$$

Proposition B.7. *Let R be a measure satisfying* (B.11) *with some $X > 0$ and let $R \in RV_\rho^\infty(\sigma)$ for $\rho \leq 0$ and some finite measure σ. Then*

$$\int_{|x|\geq X} |x|^\gamma R(dx) \begin{cases} < \infty & \text{if } \gamma < |\rho| \\ = \infty & \text{if } \gamma > |\rho| \end{cases}.$$

The proof of this result is based on the proof of Lemma 2.1 in [BDM02].

Proof. When $\gamma \leq 0$ the result is immediate. Assume that $\gamma > 0$. By Fubini's Theorem

$$\begin{aligned}
\int_{|x|\geq X} |x|^\gamma R(dx) &= \int_{|x|\geq X} \int_0^{|x|} \gamma u^{\gamma-1} du R(dx) \\
&= \int_0^X \gamma u^{\gamma-1} du \int_{|x|\geq X} R(dx) + \int_X^\infty \gamma u^{\gamma-1} \int_{|x|\geq u} R(dx) du \\
&= X^\gamma R(|x| \geq X) + \int_X^\infty \gamma u^{\gamma-1} R(|x| \geq u) du = I_1 + I_2.
\end{aligned}$$

Clearly, $I_1 < \infty$. By (B.14) $R(|x| \geq u) = u^\rho \ell(u)$, where ℓ is some slowly varying function. By Proposition 1.5.10 in [BGT87]), if $\gamma < |\rho|$ then $I_2 < \infty$. Now assume that $\gamma > |\rho|$. Fix $\epsilon \in (0, \gamma - |\rho|)$. By Proposition 1.3.6 in [BGT87], there is an $X' > X$ such that for all $x > X'$, $x^\epsilon \ell(x) > 1$. Thus,

$$I_2 \geq \int_{X'}^\infty \gamma u^{\gamma-1-|\rho|-\epsilon} u^\epsilon \ell(u) du \geq \int_{X'}^\infty \gamma u^{\gamma-1-|\rho|-\epsilon} du = \infty.$$

This completes the proof. □

Proposition B.8. *Let σ be a finite, non-zero measure on $(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}))$. Fix $\varrho \leq 0$, $c > 0$, and let R be a measure on $\mathbb{R}^d$ satisfying* (B.11) *and* (B.12)*.*

1. If $R \in RV_\varrho^\infty(\sigma)$, $p \geq 0$, and $\varrho < p$ then there exists is a function a_s with $\lim_{s\to\infty} a_s = \infty$ *such that*

$$\lim_{s\to\infty} s a_s^{-p} R\left(|x| > a_s t, \frac{x}{|x|} \in D\right) = c\sigma(D) t^\varrho \tag{B.16}$$

for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$.

2. If there is a function a_s with $\lim_{s\to\infty} a_s = \infty$ *such that* (B.16) *holds for some $p \geq 0$ then $R \in RV_\varrho^\infty(\sigma)$.*

3. If $R \in RV_\varrho^\infty(\sigma)$, $p \geq 0$, and $\varrho < p$ then (B.16) *holds for some function a_s if and only if $a_s \sim V^{\leftarrow}(s)$ where $V(t) = c\sigma(\mathbb{S}^{d-1}) t^p / R(|x| > t)$. Moreover, in this case, $a \in RV_{1/(p+|\varrho|)}$.*

Let $r = c\sigma(\mathbb{S}^{d-1})$ and $U(t) = t^p/R(|x| > t)$. In this case $V(t) = rU(t)$. If $R \in RV_\varrho^\infty(\sigma)$ for some $\varrho \leq 0$ and $p + |\varrho| > 0$ then by Proposition B.2 $V^{\leftarrow}(t) = r^{-1/(p+|\varrho|)}U^{\leftarrow}(t)$. Thus, in the third part, we can write $a_s \sim r^{-1/(p+|\varrho|)}U^{\leftarrow}(s)$.

Proof. Let $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ such that $\sigma(\partial D) = 0$ and let $r = c\sigma(\mathbb{S}^{d-1})$.

First assume that $R \in RV_\varrho^\infty(\sigma)$. For $t > X$, let $V(t) = rt^p/R(|x| > t)$, and let $a_s = V^{\leftarrow}(s)$. Note that $\lim_{s\to\infty} a_s = \infty$. We have

$$\begin{aligned}
t^\varrho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})} &= \lim_{q\to\infty} \frac{R\left(|x| > qt, \frac{x}{|x|} \in D\right)}{R(|x| > q)} \\
&= \lim_{s\to\infty} \frac{r^{-1}a_s^{-p}R\left(|x| > a_s t, \frac{x}{|x|} \in D\right)}{r^{-1}a_s^{-p}R(|x| > a_s)} \\
&= \lim_{s\to\infty} V(a_s) r^{-1} a_s^{-p} R\left(|x| > a_s t, \frac{x}{|x|} \in D\right) \\
&= \lim_{s\to\infty} \frac{s}{c\sigma(\mathbb{S}^{d-1})} a_s^{-p} R\left(|x| > a_s t, \frac{x}{|x|} \in D\right).
\end{aligned}$$

Now assume that (B.16) holds for some $p \geq 0$ and some function a_s satisfying $\lim_{s\to\infty} a_s = \infty$. We have

$$\lim_{s\to\infty} \frac{R\left(|x| > st, \frac{x}{|x|} \in D\right)}{R(|x| > s)} = \lim_{s\to\infty} \frac{s a_s^{-p} R\left(|x| > a_s t, \frac{x}{|x|} \in D\right)}{s a_s^{-p} R(|x| > a_s)} = \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})} t^\varrho.$$

The third part follows by Proposition B.3. □

We can also define regularly varying measures at 0. We will need slightly different assumptions. Assume that R is a measure on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ such that for all $s > 0$

$$R(\{x \in \mathbb{R}^d : |x| > s\}) < \infty \tag{B.17}$$

and for some $r > 0$

$$R(\{x \in \mathbb{R}^d : |x| > r\}) > 0. \tag{B.18}$$

In particular, these conditions hold for nonzero Lévy measures.

Definition B.9. *Fix $\rho \le 0$. A Borel measure R on $\mathbb{R}^d$ satisfying* (B.17) *and* (B.18) *is said to be **regularly varying at 0 with index** ρ if there is a finite, non-zero measure σ on $(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}))$ such that for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$*

$$\lim_{r\downarrow 0} \frac{R\left(|x| > rt, \frac{x}{|x|} \in D\right)}{R(|x| > r)} = t^{\rho} \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}.$$

When this holds we will write $R \in RV^0_\rho(\sigma)$.

For $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ define $U_D(t) = R(|x| > t, x/|x| \in D)$. As before, we can show that if $\sigma(D) > 0$ and $R \in RV^0_\rho(\sigma)$ then $U_D \in RV^0_\rho$. Moreover, $R \in RV^0_\rho(\sigma)$ if and only if there is an $\ell \in RV^0_0$ such that for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$U_D(t) \sim \sigma(D) t^{\rho} \ell(t) \quad \text{as } t \downarrow 0. \tag{B.19}$$

We now give a version of Proposition B.8 for measures that are regularly varying at 0.

Proposition B.10. *Let σ be a finite, non-zero measure on $(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}))$. Fix $\varrho \le 0$, $c > 0$, and let R be a measure on $\mathbb{R}^d$ satisfying* (B.17) *and* (B.18).

1. If $R \in RV^0_\varrho(\sigma)$, $p \ge 0$, and $\varrho < p$ then there exists is a function a_t with $\lim_{t\downarrow 0} a_t = \infty$ such that

$$\lim_{t\downarrow 0} t a_t^p R\left(|x| > u/a_t, \frac{x}{|x|} \in D\right) = c\sigma(D) u^{\varrho} \tag{B.20}$$

for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$.

2. If there is a function a_t with $\lim_{t\downarrow 0} a_t = \infty$ such that (B.20) *holds for some $p \ge 0$ then $R \in RV^0_\varrho(\sigma)$.*

3. If $R \in RV^0_\varrho(\sigma)$, $p \ge 0$, and $\varrho < p$ then (B.20) *holds for some function a_t if and only if $a_t \sim V^{\leftarrow}(1/t)$ as $t \downarrow 0$ where $V(t) = \frac{t^p}{c\sigma(\mathbb{S}^{d-1})} R(|x| > 1/t)$. Moreover, in this case, $a_t \in RV^0_{-1/(p+|\varrho|)}$.*

Proof. Let $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ such that $\sigma(\partial D) = 0$ and let $s = c\sigma(\mathbb{S}^{d-1})$.

First assume that $R \in RV^0_\varrho(\sigma)$. For $t > r$ (where r is as in (B.18)), let $V(t) = s^{-1}t^p R(|x| > 1/t)$, and let $a_t = V^{\leftarrow}(1/t)$. Note that $\lim_{t\downarrow 0} a_t = \infty$. We have

$$\begin{aligned} t^\varrho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})} &= \lim_{v\downarrow 0} \frac{R\left(|x| > vt, \frac{x}{|x|} \in D\right)}{R(|x| > v)} \\ &= \lim_{u\downarrow 0} \frac{s^{-1}a_u^p R\left(|x| > t/a_u, \frac{x}{|x|} \in D\right)}{s^{-1}a_u^p R(|x| > 1/a_u)} \\ &= \lim_{u\downarrow 0} \frac{s^{-1}a_u^p R\left(|x| > t/a_u, \frac{x}{|x|} \in D\right)}{V(a_u)} \\ &= \lim_{u\downarrow 0} \frac{u}{c\sigma(\mathbb{S}^{d-1})} a_u^p R\left(|x| > t/a_u, \frac{x}{|x|} \in D\right). \end{aligned}$$

Now assume that (B.20) holds for some $p \geq 0$ and some function a_t with $\lim_{t\downarrow 0} a_t = \infty$. We have

$$\lim_{v\downarrow 0} \frac{R(|x| > uv, x/|x| \in D)}{R(|x| > v)} = \lim_{t\downarrow 0} \frac{ta_t^p R(|x| > u/a_t, x/|x| \in D)}{ta_t^p R(|x| > 1/a_t)} = u^\varrho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}.$$

The third part follows by Proposition B.3. □

B.3 Regularly Varying Matrix Valued Functions

Let $L(\mathbb{R}^d)$ be the set of $d \times d$ matrices. We define convergence on this set as pointwise convergence of the components. This is equivalent to convergence in the operator norm (see Proposition 2.1.8 in [MS01])

Definition B.11. *Fix $c \in \{0, \infty\}$ and $\rho \in \mathbb{R}$. Let $A : [0, \infty) \mapsto L(\mathbb{R}^d)$. We say that A_t is a* ***regularly varying at c matrix valued function with index of regular variation*** *ρ if there is a positive function $f \in RV^c_\rho$ and a matrix $B \in L(\mathbb{R}^d)$ with $B \neq 0_{d\times d}$ such that*

$$\lim_{t\to c} \frac{A_t}{f(t)} = B. \tag{B.21}$$

When $\mathrm{tr}B > 0$ then for t close enough to c we have $\mathrm{tr}A_t > 0$ and we can take $f(t) = \mathrm{tr}A_t$. Similarly, if B is nonnegative-definite then there is a $z \in \mathbb{R}^d$ with $\langle z, Bz\rangle > 0$. This implies that for t close enough to c $\langle z, A_t z\rangle > 0$ and we can take $f(t) = \langle z, A_t z\rangle$.

Definition B.12. *Fix $c \in \{0, \infty\}$ and $\rho \in \mathbb{R}$. Let $A : [0, \infty) \mapsto L(\mathbb{R}^d)$, let B be a matrix with $\mathrm{tr}B > 0$, and let $f(t) = \mathrm{tr}A_t$. If $f \in RV^c_\rho$ and for some $k > 0$*

$$\lim_{t\to c} \frac{A_t}{\mathrm{tr}A_t} = kB \tag{B.22}$$

we will write $A \in LRV^c_\rho(B)$.

In particular we will be interested in matrix-valued function of the form

$$A_t = \int_{|x|\le t} xx^T M(dx)$$

for some Borel measure M on $\mathbb{R}^d$. In this case

$$\mathrm{tr}A_t = \int_{|x|\le t} |x|^2 M(dx).$$

The following is a useful criterion for this situation.

Proposition B.13. *Fix $c \in \{0, \infty\}$ and $\rho \in \mathbb{R}$. Let μ be a Borel measure on $\mathbb{R}^d$ such that $\mu \neq 0$ and for every $t > 0$ $\int_{|x|\le t} |x|^2\mu(dx) < \infty$. Define $A_t = \int_{|x|\le t} xx^T\mu(dx)$. There exists a $B \in L(\mathbb{R}^d)$ such that $A_t \in LRV^c_\rho(B)$ if and only if there exists an $\ell \in RV^c_0$ such that for all $z \in \mathbb{R}^d$ there is a $\kappa_z \in \mathbb{R}$ with*

$$\langle z, A_t z\rangle \sim \kappa_z t^\rho \ell(t) \text{ as } t \to c, \tag{B.23}$$

and there is a $z^* \in \mathbb{R}^d$ such that $\kappa_{z^*} \neq 0$.

Proof. First assume that $A_t \in LRV^c_\rho(B)$. If $z \in \mathbb{R}^d$ then

$$\lim_{t\to c} \frac{\langle z, A_t z\rangle}{\mathrm{tr}A_t} = \langle z, Bz\rangle.$$

Hence (B.23) holds with $\kappa_z = \langle z, Bz \rangle$ and $\ell(t) = \mathrm{tr}(A_t)/t^\rho$. The fact that $\mathrm{tr}B > 0$ guarantees the existence of a z^* of the given form.

Now assume that there exists an $\ell \in RV_0^c$ such that for all $z \in \mathbb{R}^d$ there is a $\kappa_z \in \mathbb{R}$ for which (B.23) holds, and there is a $z^* \in \mathbb{R}^d$ with $\kappa_{z^*} \neq 0$. Let $A_t' = A_t / [t^\rho \ell(t)]$. By (B.23) for every $z \in \mathbb{R}^d$ we have $\lim_{t\to c}\langle z, A_t' z\rangle = k_z$. Thus by Corollary 2.1.9 of [MS01] there is a $B \in L(\mathbb{R}^d)$ such that $\lim_{t\to c} A_t' = B$ and $\kappa_z = \langle z, Bz \rangle$. This implies that

$$\mathrm{tr}A_t \sim \mathrm{tr}(B) t^\rho \ell(t) \text{ as } t \to c.$$

The fact that $\langle z^*, Bz^* \rangle \neq 0$ implies that $B \neq 0_{d\times d}$. Let $A_t = (a_t^{ij})$ and $B = (b^{ij})$. We have

$$|b^{ij}| = \lim_{t\to c} \frac{|a_t^{ij}|}{t^\rho \ell(t)} \leq \lim_{t\to c} \frac{\mathrm{tr}A_t}{t^\rho \ell(t)} = \mathrm{tr}B.$$

Thus $\mathrm{tr}B > 0$ and $A \in LRV_\rho^c(B)$. □

APPENDIX C

INFINITELY DIVISIBLE DISTRIBUTIONS

In this appendix we give some basic properties of infinitely divisible distributions on $\mathbb{R}^d$. A general reference is [Sat99].

Definition C.1. *A probability measure μ is called* ***infinitely divisible*** *if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that if $X \sim \mu$ and $Y_1^{(n)}, \dots, Y_n^{(n)} \overset{\text{iid}}{\sim} \mu_n$ then*

$$X \overset{d}{=} \sum_{i=1}^{n} Y_i^{(n)}.$$

We will denote the class of infinitely divisible distributions by ID.

The characteristic function of an infinitely divisible distribution μ on $\mathbb{R}^d$ never vanishes (see Lemma 7.5 in [Sat99]) and is given by $\hat{\mu}(z) = \exp\{C_\mu(z)\}$ where

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\frac{\langle z, x\rangle}{1+|x|^2} \right) M(dx), \tag{C.1}$$

A is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and M satisfies

$$M(\{0_d\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) M(dx) < \infty. \tag{C.2}$$

We call A the **Gaussian part** and M the **Lévy measure**. In fact, we will call any measure that satisfies (C.2) a Lévy measure. According to Theorem 8.1 in [Sat99], the measure μ is uniquely identified by the **Lévy triplet** (A, M, b). We will write

$$\mu = ID(A, M, b). \tag{C.3}$$

More generally, if $f : \mathbb{R}^d \mapsto \mathbb{R}$ is a measurable function such that for all $z \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \left| e^{i\langle z,x\rangle} - 1 - i\langle x, z\rangle f(x) \right| M(dx) < \infty, \tag{C.4}$$

we can write

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b^f, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle f(x) \right) M(dx), \tag{C.5}$$

where

$$b^f = b + \int_{\mathbb{R}^d} x \left(f(x) - \frac{1}{1+|x|^2} \right) M(dx). \tag{C.6}$$

Although we will usually assume the parametrization given in (C.1), there are cases where other parametrizations will be useful. When $\int_{|x|\leq 1} |x| M(dx) < \infty$, we can use **Parametrization 0** with

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b_0, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 \right) M(dx), \tag{C.7}$$

where

$$b_0 = b - \int_{\mathbb{R}^d} \frac{x}{1+|x|^2} M(dx). \tag{C.8}$$

In this case we will write

$$\mu = ID^0(A, M, b_0). \tag{C.9}$$

When $\int_{|x|>1} |x| M(dx) < \infty$ (which by Proposition C.4 below is equivalent to $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$), we can use **Parametrization 1** with

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b_1, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \right) M(dx), \tag{C.10}$$

where

$$b_1 = b + \int_{\mathbb{R}^d} x \frac{|x|^2}{1+|x|^2} M(dx). \tag{C.11}$$

In this case we will write

$$\mu = ID^1(A, M, b_1). \tag{C.12}$$

When dealing with characteristic functions of infinitely divisible distributions, the following well known (see for example (26.4) in [Bil95]) technical result is often useful.

Lemma C.2. *If $x \in \mathbb{R}$ then for $n \geq 0$,*

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \min\left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}.$$

It is often convenient to work with the polar decomposition of the Lévy measure. Its existence is given by the following result. We will, in fact, need a polar decomposition of measures that satisfy somewhat more complicated integrability conditions, thus we give the result in a slightly more general form. For Lévy measures, in the following proposition take $f_1 \equiv 1$ and $f_2(x) = x^2$.

Proposition C.3. *Let M be a Borel measure on $\mathbb{R}^d$ with $M \neq 0$ and $M(\{0_d\}) = 0$. Assume that there are continuous, nonnegative valued functions f_1, f_2 on $\mathbb{R}$ with $f_1(x) \wedge f_2(x) > 0$ for all $x \neq 0$ such that $\int_{\mathbb{R}^d} [f_1(|x|) \wedge f_2(|x|)]\, M(dx) < \infty$. There exists a Borel probability measure σ and a family $\{M_\xi : \xi \in \mathbb{S}^{d-1}\}$ of Borel measures on $(0, \infty)$ such that*

$$M_\xi(B) \text{ is measurable in } \xi \text{ for } B \in \mathfrak{B}((0,\infty)), \tag{C.13}$$

$$0 < M_\xi((0,\infty)) \leq \infty \text{ for each } \xi \in \mathbb{S}^{d-1}, \tag{C.14}$$

and

$$\nu(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(r\xi) M_\xi(dr) \sigma(d\xi) \text{ for all } B \in \mathfrak{B}(\mathbb{R}^d). \tag{C.15}$$

Here σ and $\{M_\xi\}$ are uniquely determined by M in the following sense: if σ, $\{M_\xi\}$ and σ', $\{M'_\xi\}$ both have the above properties then there is a Borel function $c(\xi)$ on $\mathbb{S}^{d-1}$ such that

$$0 < c(\xi) < \infty,$$

$$\sigma'(d\xi) = c(\xi)\sigma(d\xi),$$

$$c(\xi)M'_\xi(dr) = M_\xi(dr), \quad \text{for } \sigma \text{ almost every } \xi \in \mathbb{S}^{d-1}.$$

Proof. For Lévy measures, this is Lemma 2.1 in [BNMS06]. The proof remains essentially unchanged in our slightly more general situation. □

We will now categorize when the moments of infinitely divisible distributions are finite.

Proposition C.4. *Let $X \sim ID(A, M, b)$. For $\alpha, \beta \geq 0$ and $p \in [0, 1]$*

$$\mathrm{E}\left[|X|^\alpha e^{\beta|X|^p}\right] < \infty \iff \int_{|x|>1} |x|^\alpha e^{\beta|x|^p} M(dx) < \infty.$$

Proof. This is Corollary 25.8 in [Sat99]. □

Next, we categorize the possible weak limits of sequences of infinitely divisible distributions.

Proposition C.5. *Let $\{\mu_n\}$ be a sequence of infinitely divisible distributions such that $\mu_n = ID(A_n, M_n, b_n)$. If $\mu_n \xrightarrow{w} \mu$ then $\mu = ID(A, M, b)$. Moreover, $\mu_n \xrightarrow{w} \mu$ if and only if $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}_0^d$, $b_n \to b$, and*

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} (A_n + A_n^\epsilon) = \lim_{\epsilon \downarrow 0} \liminf_{n\to\infty} (A_n + A_n^\epsilon) = A, \tag{C.16}$$

where

$$A_n^\epsilon = \int_{|x|\leq\epsilon} x x^T M_n(dx). \tag{C.17}$$

Proof. This follows from Theorem 8.7 in [Sat99] and Proposition A.13. □

Note that the Lévy measure does not contribute to A if and only if

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \int_{|x|<\epsilon} |x|^2 M_n(dx) = \lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \mathrm{tr} A_n^\epsilon = 0. \tag{C.18}$$

C.1 Stable Distributions

In this section we will discuss the class of stable distributions. This is an important subclass of infinitely divisible distributions. A standard reference is [ST94].

Definition C.6. *Let μ be a probability measure on $\mathbb{R}^d$ and let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mu$. μ is called **stable** if for any n there are $a_n > 0$ and $b_n \in \mathbb{R}^d$ such that*

$$X_1 + \cdots + X_n \overset{d}{=} a_n X_1 + d_n, \tag{C.19}$$

*If for every n $d_n = 0_d$ then the distribution is called **strictly stable**.*

It turns out that $a_n = n^{-1/\alpha}$ for some $\alpha \in (0, 2]$. We call this parameter the **index of stability**, and we refer to any stable distribution with index α as **α-stable**. Note that for all $x \in \mathbb{R}^d_0$ the distribution δ_x is strictly 1-stable. The distribution δ_{0_d} is strictly stable for every α.

Fix $\alpha \in (0, 2]$. Let μ be an α-stable distribution. If $\alpha = 2$ then, by Theorem 14.1 in [Sat99], $\mu = ID(A, 0, b)$. In this case, we will sometimes write $\mu = N(b, A)$. If $\alpha \in (0, 2)$ then, by Theorems 14.1 in [Sat99], $\mu = ID(0_{d\times d}, M^\alpha_\sigma, b)$ where

$$M^\alpha_\sigma(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ur) r^{-1-\alpha} dr \sigma(du), \qquad A \in \mathfrak{B}\left(\mathbb{R}^d\right), \tag{C.20}$$

for some finite Borel measure σ on $\mathbb{S}^{d-1}$. In this case we will sometimes write $\mu = S_\alpha(\sigma, b)$. We call σ the **spectral measure** of the stable distribution. By Remark 14.4 in [Sat99], the parameters α and σ uniquely determine M^α_σ.

One reason for the importance of stable distributions is that they are the only possible limits of scaled and shifted sums of iid random vectors. More specifically, let $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \mu$ and define

$$S_n = \sum_{i=1}^n X_i. \tag{C.21}$$

If there exists a probability measure ν and sequences $a_n > 0$ and $b_n \in \mathbb{R}^d$ such that when $Y \sim \nu$

$$a_n (S_n - b_n) \xrightarrow{d} Y \tag{C.22}$$

then ν is a stable distribution. When this holds we say that μ (or equivalently X_1) is in **the domain of attraction** of ν (or equivalently of Y). If we can take $a_n = n^{-1/\alpha}$ (where α is the index of stability of ν) then we say that μ (or X_1) is in **the domain of normal attraction** of ν (or Y). The problem of categorizing the domains of attraction of stable distributions is solved by the so called (generalized) central limit theorem (CLT). When ν is full this is given in [Fel71] for the case $d = 1$ and in [Rva62] and [MS01] for other d. We now state the result.

Theorem C.7. *Let μ be a probability measure on $\mathbb{R}^d$.*
1. Let $A_t = \int_{|x|\le t} xx^T \mu(dx)$. μ is in the domain of attraction of a nondegenerate Gaussian distribution with covariance matrix A if and only if $A_t \in RV_0^\infty(A)$.
2. Fix $\alpha \in (0, 2)$. μ is in the domain of attraction of a nondegenerate α-stable law with spectral measure σ if and only if $\mu \in RV_{-\alpha}^\infty(\sigma)$.

Proof. In a slightly different form, Part 1 is given in Theorem 4.1 in [Rva62]. Equivalence with this form follows by Corollary B.5 and Proposition B.13. Part 2 is Theorem 8.2.18 in [MS01]. □

C.2 Duality

In this section we will define dual Lévy measures and discuss some of their properties. The dual Lévy measure, in a sense, "inverts" the Lévy measure, interchanging its behavior near zero and near infinity. The definition was first introduced in the context of integration with respect to additive processes in [Sat07]. Here we refer to the dual as the 0-dual, and extend the definition to more general β-duals. We will then use them to prove various limiting results. Most of the results in this section appear to be new.

We begin by setting up some notation. For $\beta \geq 0$, let $\mathfrak{M}^\beta$ be the class of Lévy measures M such that $\int_{|x|>1} |x|^\beta M(dx) < \infty$. In particular, $\mathfrak{M}^0$ is the class of all Lévy measures.

Definition C.8. *Fix $\beta \in [0,2]$. If $M \in \mathfrak{M}^\beta$ we will call the measure M^β its **β-dual Lévy measure** if $M^\beta(\{0_d\}) = 0$ and for any Borel subset $A \in \mathfrak{B}(\mathbb{R}^d)$*

$$M^\beta(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M(dx).$$

We will sometimes refer to the 0-dual of a Lévy measure simply as the **dual Lévy measure**. When discussing β-duals, we will implicitly assume that $M \in \mathfrak{M}^\beta$. To see that $M^\beta \in \mathfrak{M}^\beta$ observe that

$$\begin{aligned}\int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\beta\right) M^\beta(dx) &= \int_{\mathbb{R}^d} \left(|x|^{-2} \wedge |x|^{-\beta}\right) |x|^{2+\beta} M(dx) \\ &= \int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\beta\right) M(dx) < \infty.\end{aligned}$$

Moreover it is easy to see that

$$(M^\beta)^\beta = M. \tag{C.23}$$

We have

$$\int_{\mathbb{R}^d} |x|^2 M(dx) < \infty \Longleftrightarrow \int_{\mathbb{R}^d} |x|^\beta M^\beta(dx) < \infty, \tag{C.24}$$

and in particular

$$\int_{\mathbb{R}^d} |x|^2 M(dx) < \infty \Longleftrightarrow M^0(\mathbb{R}^d) < \infty. \tag{C.25}$$

We will now relate convergence of a sequence of Lévy measures to convergence of the sequence of their duals. We will then extend this to convergence of the corresponding distributions.

Lemma C.9. *Fix $\beta \in [0,2]$. Let $M, M_1, M_2, \dots \in \mathfrak{M}^\beta$. We have:*

1. $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}_0^d$ if and only if $M_n^\beta \xrightarrow{v} M^\beta$ on $\mathbb{R}^d$.

2. If $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}_0^d$ and

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \int_{|x|\le \epsilon} |x|^{2+\beta} M_n(dx) = 0$$

then $M_n^\beta \xrightarrow{v} M^\beta$ on $\bar{\mathbb{R}}^d$.

Proof. We begin with Part 1. Let $f : \bar{\mathbb{R}}^d \mapsto \mathbb{R}$ be a continuous function vanishing on a neighborhood of zero. The function $g(x) = f\left(\frac{x}{|x|^2}\right)|x|^{2+\beta}$ is a continuous and bounded mapping of $\bar{\mathbb{R}}^d$ into $\mathbb{R}$, vanishing on a neighborhood of infinity. Thus, if $M_n^\beta \xrightarrow{v} M^\beta$ on $\mathbb{R}^d$ then

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) M_n(dx) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} g(x) M_n^\beta(dx) \\ &= \int_{\mathbb{R}^d} g(x) M^\beta(dx) = \int_{\mathbb{R}^d} f(x) M(dx).\end{aligned}$$

Thus $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}_0^d$. The other direction of Part 1 is similar.

Now for Part 2. Let $f : \bar{\mathbb{R}}^d \mapsto \mathbb{R}$ be a continuous function. Since f is continuous on a compact set, it is bounded and thus

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \int_{|x|<\epsilon} f\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M_n(dx) = 0.$$

Let (ϵ_m) is a sequence decreasing to 0 such that $M(|x| = \epsilon_m) = 0$ for all m. We have

$$\begin{aligned}&\limsup_{n\to\infty} \int_{\mathbb{R}^d} f(x) M_n^\beta(dx) = \limsup_{n\to\infty} \int_{\mathbb{R}^d} f\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M_n(dx) \\ &= \lim_{m\to\infty} \limsup_{n\to\infty} \left(\int_{|x|\ge\epsilon_m} f\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M_n(dx) \right. \\ &\qquad\qquad \left. + \int_{|x|<\epsilon_m} f\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M_n(dx) \right) \\ &\le \lim_{m\to\infty} \int_{|x|\ge\epsilon_m} f\left(\frac{x}{|x|^2}\right) |x|^{2+\beta} M(dx) = \int_{\mathbb{R}^d} f(x) M^\beta(dx),\end{aligned}$$

where the last line follows by the fourth part of Proposition A.8 and the final equality follows by dominated convergence. By a similar argument we can show that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^d} f(x)M_n^\beta(dx) \geq \int_{\mathbb{R}^d} f(x)M^\beta(dx).$$

This completes the proof. □

This can be readily extended to convergence of infinitely divisible distributions.

Proposition C.10. *Fix $\beta \in [0,2]$. Let $M, M_1, M_2, \cdots \in \mathfrak{M}^\beta$. Assume that for every n $X_n \sim ID(0_{d\times d}, M_n, 0_d)$, $X_n^\beta \sim ID(0_{d\times d}, M_n^\beta, 0_d)$, $Y \sim ID(0_{d\times d}, M, 0_d)$, and $Y^\beta \sim ID(0_{d\times d}, M^\beta, 0_d)$ then*

$$X_n \xrightarrow{d} Y \textit{ and } \lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|>N}|x|^\beta M_n(dx) = 0$$

if and only if

$$X_n^\beta \xrightarrow{d} Y^\beta \textit{ and } \lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|>N}|x|^\beta M_n^\beta(dx) = 0.$$

Proof. By (C.23), it suffices to show only one direction. Assume that $X_n \xrightarrow{d} Y$. By Proposition C.5, this implies that $M_n \xrightarrow{v} M$ on $\bar{\mathbb{R}}_0^d$ and by (C.18)

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|>N}|x|^\beta M_n^\beta(dx) = \lim_{N\to\infty}\limsup_{n\to\infty}\int_{|x|<1/N}|x|^2 M_n(dx) = 0$$

and

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}|x|^{2+\beta}M_n(dx) \leq \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}|x|^2 M_n(dx) = 0.$$

Thus, by Lemma C.9 $M_n^\beta \xrightarrow{v} M^\beta$ on $\bar{\mathbb{R}}_0^d$. Note also that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\epsilon}|x|^2 M_n^\beta(dx) \quad = \quad \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|>1/\epsilon}|x|^\beta M_n(dx) = 0.$$

From here the result follows by Proposition C.5. □

Note that when $\beta = 0$, $X_n \xrightarrow{d} Y$ implies that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \int_{|x|>N} |x|^\beta M_n(dx) = \lim_{N\to\infty} \limsup_{n\to\infty} M_n\left(|x| > N\right) = 0.$$

Thus, we immediately get the following.

Corollary C.11. *If for every n $X_n \sim ID(0_{d\times d}, M_n, 0_d)$, $X_n^0 \sim ID(0_{d\times d}, M_n^0, 0_d)$, $Y \sim ID(0_{d\times d}, M, 0_d)$, and $Y^0 \sim ID(0_{d\times d}, M^0, 0_d)$ then*

$$X_n \xrightarrow{d} Y \iff X_n^0 \xrightarrow{d} Y^0.$$

For $\alpha \in (0,2)$, the Lévy measure of an α-stable distribution with spectral measure σ is M_σ^α and given by (C.20). We have

$$\begin{aligned} \left(M_\sigma^\alpha\right)^0(A) &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ur^{-1}) r^{-1-\alpha} r^2 dr \sigma(du) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ur) r^{-1-(2-\alpha)} dr \sigma(du). \end{aligned}$$

Note that $\alpha \in (0,2)$ when $(2-\alpha) \in (0,2)$ and $\left(M_\sigma^\alpha\right)^0$ is the Lévy measure of a $(2-\alpha)$-stable distribution. This fact, combined with Corollary C.11 gives the following result.

Corollary C.12. *Fix $\alpha \in (0,2)$. For every n let $X_n \sim ID(0_{d\times d}, M_n, 0_d)$ and $X_n^0 \sim ID(0_{d\times d}, M_n^0, 0_d)$. If $X_n \xrightarrow{d} X$, where X is an α-stable random vector with spectral measure σ, then $X_n^0 \xrightarrow{d} X^0$, where X^0 is a $(2-\alpha)$-stable random vector with spectral measure σ.*

We conclude our discussion of dual Lévy measures with a duality result for regular variation.

Proposition C.13. *Fix $\beta \in [0,2]$. Let $M \in \mathfrak{M}^\beta$ and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. If $\rho \in (-2-\beta, 0)$ then*

$$M^\beta \in RV_\rho^\infty(\sigma) \iff M \in RV_{-(\rho+2+\beta)}^0(\sigma). \tag{C.26}$$

Moreover if $\ell \in RV_0^\infty$ *and* $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial D) = 0$ *then*

$$M^\beta(|x| > t, x/|x| \in D) \sim \sigma(D) t^\rho \ell(t) \text{ as } t \to \infty \tag{C.27}$$

if and only if

$$M(|x| > t, x/|x| \in D) \sim \frac{|\rho|}{\rho + 2 + \beta} \sigma(D) t^{-\rho-2-\beta} \ell(1/t) \text{ as } t \downarrow 0. \tag{C.28}$$

Note that when $\rho \in (-2-\beta, 0)$ then $-\rho - 2 - \beta \in (-2-\beta, 0)$.

Proof. Fix $\rho \in (-2-\beta, 0)$ and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. Define

$$M^1(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^2}\right) M(dx).$$

We have

$$M(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^2}\right) M^1(dx) \quad \text{and} \quad M^\beta(A) = \int_{\mathbb{R}^d} 1_A(x) |x|^{-2-\beta} M^1(dx).$$

Note that

$$M^1(|x| < 1) = M(|x| > 1) < \infty.$$

For every $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$, let $V_D(t) = M\left(|x| > 1/t, \frac{x}{|x|} \in D\right)$ and $V_D^\beta(t) = M^\beta\left(|x| > t, \frac{x}{|x|} \in D\right)$. Note that the equivalence of (C.27) and (C.28) implies the equivalence in (C.26). Thus, it suffices to only show the first equivalence.

Assume that (C.27) holds. This means that there is an $\ell \in RV_0^\infty$ such that for $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$V_D^\beta(t) \sim \sigma(D) t^\rho \ell(t) \quad \text{as } t \to \infty.$$

If $\sigma(D) > 0$ then $V_D^\beta \in RV_\rho^\infty$ and

$$V_D^\beta(t) = M^\beta\left(|x| > t, \frac{x}{|x|} \in D\right) = \int_{\left[|x|>t, \frac{x}{|x|} \in D\right]} |x|^{-2-\beta} M^1(dx).$$

Thus, by Corollary B.5 since

$$V_D(t) = M\left(|x| > 1/t, \frac{x}{|x|} \in D\right) = \int_{\left[|x|<t, \frac{x}{|x|}\in D\right]} M^1(dx),$$

$V_D \in RV^{\infty}_{\rho+2+\beta}$ and

$$V_D(t) \sim \frac{|\rho|}{2+\beta+\rho}\sigma(D)t^{\rho+2+\rho}\ell(t) \quad \text{as } t \to \infty.$$

If $\sigma(D) = 0$ then for $A \in \mathfrak{B}(\mathbb{R}^d)$ define $M^1_D(A) = \int_A 1_D\left(\frac{x}{|x|}\right) M^1(dx)$ and $M^1_\epsilon(A) = \epsilon M^1(A)$ for any $\epsilon > 0$. We have

$$\int_{|x|>t} |x|^{-2-\beta}\left(M^1_D + M^1_\epsilon\right)(dx) \sim \epsilon\sigma(\mathbb{S}^{d-1})t^\rho \ell(t) \text{ as } t \to \infty.$$

As before, by Corollary B.5, this implies that

$$V_D(t) + \epsilon V_{\mathbb{S}^{d-1}}(t) \sim \epsilon\sigma(\mathbb{S}^{d-1})\frac{|\rho|}{2+\beta+\rho}t^{\rho+2+\beta}\ell(t) \text{ as } t \to \infty.$$

Thus

$$\lim_{t\to\infty} \frac{V_D(t)}{t^{\rho+2+\beta}\ell(t)} \leq \lim_{\epsilon\downarrow 0}\lim_{t\to\infty} \frac{V_D(t) + \epsilon V_{\mathbb{S}^{d-1}}(t)}{t^{\rho+2+\beta}\ell(t)} = \lim_{\epsilon\downarrow 0} \epsilon\sigma(\mathbb{S}^{d-1})\frac{|\rho|}{2+\beta+\rho} = 0.$$

Hence, for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$, (C.28) holds. The other direction is similar. □

APPENDIX D

LÉVY PROCESSES

D.1 Definition and Basic Results

Lévy processes are stochastic processes that generalize Brownian motion to allow for other infinitely divisible marginal distributions. A standard reference is [Sat99].

Definition D.1. *A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$ is called a* ***Lévy Process*** *if the following conditions are satisfied.*

1. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments property).

2. $X_0 = 0$ a.s.

3. The distribution of $X_{s+t} - X_s$ does not depend on s (stationary increments property).

4. For every $t \geq 0$ and $\epsilon > 0$ $\lim_{s \to t} P(|X_s - X_t| > \epsilon) = 0$ (stochastic continuity).

5. There is $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$ (càdlàg paths property).

A process that only satisfies conditions 1-4 is called a **Lévy process in law**.

Proposition D.2. *1. If $\{X_t : t \geq 0\}$ is a Lévy processes in law on $\mathbb{R}^d$ then for any $t \geq 0$ the distribution μ_t of X_t is infinitely divisible. Moreover $\hat{\mu}_t(z) = [\hat{\mu}_1(z)]^t$.*

2. Conversely, if μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a Lévy process in law $\{X_t : t \geq 0\}$ where $X_1 \sim \mu$.

3. If $\{X_t : t \geq 0\}$ and $\{X'_t : t \geq 0\}$ are Lévy processes in law on $\mathbb{R}^d$ such that $X_1 \overset{d}{=} X'_1$ then $\{X_t : t \geq 0\}$ and $\{X'_t : t \geq 0\}$ have the same finite dimensional distributions.

4. If $\{X_t : t \geq 0\}$ is a Lévy process in law on $\mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$ then *there exists a Lévy process* $\{X'_t : t \geq 0\}$ *on* $\mathbb{R}^d$ *defined on* $(\Omega, \mathcal{F}, P)$ *such that* $P(X_t = X'_t) = 1$ *for all* $t \geq 0$.

Proof. Parts 1-3 are Theorem 7.10 and Part 4 is Theorem 11.5 in [Sat99]. □

Throughout, we will focus on Lévy processes and not Lévy processes in law. However, we will rarely need the càdlàg paths property. Thus, in many cases one can read "Lévy process in law" for "Lévy process." An important exception is Section 4.1, where the assumption of càdlàg paths is necessary for several of the results.

D.2 Long and Short Time Behavior of Lévy Processes

In this section we will categorize the long and short time behavior of Lévy processes (see (1.6) for the definitions). In Proposition D.5 we will show that long time behavior is equivalent to finding the limits of scaled partial sums of iid copies of X_1. Thus a characterization of long time behavior is supplied by the central limit theorem. This is stated in Theorem C.7 for the case when the limiting distribution is nondegenerate. The general version seems to be well known but we could not track down a reference. For completeness and because of the importance of this result, we will give selfcontained proofs. Short time behavior in 1-dimension is given in [MM08]. For generalized tempered stable distributions in d-dimensions it is given in [RS10]. We will give this result in general. Before proceeding, we give a few preliminary results.

Lemma D.3. *Fix* $c \in \{0, \infty\}$. *Let* $\{X_t : t \geq 0\}$ *be a Lévy process and let* Y *be a random vector whose distribution is not concentrated at a point. If there exists a function* $a_t > 0$ *such that*

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to c$$

then Y has a stable distribution.

Proof. Let N be a fixed integer. Let $Y^1, Y^2, \ldots, Y^N$ be iid copies of Y and let $\{X_t^n : t \geq 0\}$ be independent Lévy processes with $X_1^n \stackrel{d}{=} X_1$. We have

$$\lim_{t \to c} a_{Nt} X_{Nt} \stackrel{d}{=} Y$$

and

$$\lim_{t \to c} a_t X_{Nt} = \lim_{t \to c} a_t \sum_{n=1}^{N} \left(X_{nt} - X_{(n-1)t} \right) \stackrel{d}{=} \lim_{t \to c} \sum_{n=1}^{N} a_t X_t^n \stackrel{d}{=} \sum_{n=1}^{N} Y^n.$$

Since Y is not concentrated at a point, neither is $\sum_{n=1}^{N} Y_i^n$, and by the Convergence of Types Theorem (Proposition A.2) there is a constant $a_N > 0$ such that

$$\sum_{n=1}^{N} Y^n \stackrel{d}{=} a_N Y.$$

From here the result follows by Definition C.6. □

Now we will give some properties of the scaling function a_t.

Lemma D.4. *Let $\{X_t : t \geq 0\}$ be a Lévy process, let Y a random vector such that $P(Y = 0_d) < 1$, and let a_t be a positive function.*

1. If $\lim_{t \downarrow 0} a_t X_t \stackrel{d}{=} Y$ then $\lim_{t \downarrow 0} a_t = \infty$ and $a_{1/t} \sim a_{1/(t+1)}$ as $t \to \infty$.

2. If $\lim_{t \to \infty} a_t X_t \stackrel{d}{=} Y$ then $\lim_{t \to \infty} a_t = 0$ and $a_t \sim a_{t+1}$ as $t \to \infty$.

Proof. We will prove the first part by contradiction. Let $\ell := \liminf_{t \downarrow 0} a_t$. Assume that $\ell < \infty$. This means that there is a sequence of real numbers (t_n) converging to 0 such that $\lim_{n \to \infty} a_{t_n} = \ell$. Thus, since $X_t \xrightarrow{p} 0_d$ as $t \downarrow 0$, by Slutzky's Theorem

$$\lim_{n \to \infty} a_{t_n} X_{t_n} \stackrel{d}{=} \ell 0_d = 0_d.$$

This contradicts the assumption that $P(Y = 0_d) < 1$. For $z \in \mathbb{R}^d$ let $C_{X_1}(z)$ be the cumulant generating function of X_1. The characteristic function of $a_{1/t} X_{1/t}$ is $\exp\left(\frac{1}{t} C_{X_1}(a_{1/t} z)\right)$. If $\hat{\mu}_Y(z)$ is the characteristic function of Y then

$$\hat{\mu}_Y(z) = \lim_{t \to \infty} \exp\left(\frac{1}{t} C_{X_1}(a_{1/t} z)\right) = \lim_{t \to \infty} \exp\left(\frac{1}{t+1} C_{X_1}(a_{1/t} z)\right)$$

where $\exp\left(\frac{1}{t+1}C_{X_1}(a_{1/t}z)\right)$ is the characteristic function of $a_{1/t}X_{1/(t+1)}$. From here the result follows from Slutzky's Theorem.

Now for the second part. Let $X_1 \sim ID(A, M, b)$ and let $Y \sim ID(A', M', b')$. The Lévy measure of $a_t X_t$ is $M_t(\cdot) = tM(\cdot/a_t)$. By Proposition C.5 for any $s > 0$ we have

$$\lim_{t\to\infty} tM(|x| > s/a_t) = \lim_{t\to\infty} M_t(|x| > s) = M'(|x| > s).$$

By Lemma D.3 Y is stable thus, if $M' \neq 0$ then $M'(|x| > s) > 0$ for every $s > 0$. This implies that $\lim_{t\to\infty} a_t = 0$. If $M' = 0$ then, again by Proposition C.5,

$$\lim_{t\to\infty} ta_t^2 A = A' \quad \text{and} \quad \lim_{t\to\infty} ta_t b = b'.$$

In this case, we have either $A' \neq 0_{d\times d}$ or $b' \neq 0_d$, and the above implies that $a_t \to 0$. Now let $X' \stackrel{d}{=} X_1$ be independent of $\{X_t : t \geq 0\}$. By Slutzky's Theorem, we have

$$Y \stackrel{d}{=} \lim_{t\to\infty} a_{t+1}X_{t+1} \stackrel{d}{=} \lim_{t\to\infty} (a_{t+1}X_t + a_{t+1}X') \stackrel{d}{=} \lim_{t\to\infty} a_{t+1}X_t.$$

From here the result follows by another application of Slutzky's Theorem. □

Now we will show that long time behavior is equivalent to only considering limits along the natural numbers.

Proposition D.5. *Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$. Let Y be an α-stable random vector with $\alpha \in (0, 2]$, whose distribution is not concentrated at a point. There exists a sequence (b_n) with*

$$b_n X_n \stackrel{d}{\to} Y \tag{D.1}$$

if and only if there exists a function a_t such that

$$a_t X_t \stackrel{d}{\to} Y \text{ as } t \to \infty \tag{D.2}$$

holds. Moreover this holds if and only if $a_t \sim b_{\lfloor t \rfloor}$ as $t \to \infty$.

Note that since Lévy processes have independent and stationary increments, $X_n \stackrel{d}{=} \sum_{i=1}^n X^i$, where $X^1, X^2, \ldots$ are iid copies of X_1. Thus, categorizing the long time behavior of Lévy processes is equivalent to finding the limits of scaled sums of independent random vectors.

Proof. It is immediate that (D.2) implies that (D.1) holds with $a_t \sim b_{\lfloor t \rfloor}$ as $t \to \infty$. Now assume that (D.1) holds for some sequence (b_n). Let $a_t \sim b_{\lfloor t \rfloor}$ as $t \to \infty$. We have

$$a_t X_t \quad = \quad b_{\lfloor t \rfloor} X_{\lfloor t \rfloor} + a_t \left(X_t - X_{\lfloor t \rfloor} \right) + \left(a_t / b_{\lfloor t \rfloor} - 1 \right) b_{\lfloor t \rfloor} X_{\lfloor t \rfloor}.$$

Since $a_t \sim b_{\lfloor t \rfloor}$ as $t \to \infty$ and $b_{\lfloor t \rfloor} X_{\lfloor t \rfloor} \stackrel{d}{\to} Y$ as $t \to \infty$, by Slutzky's Theorem

$$\lim_{t \to \infty} \left(a_t / b_{\lfloor t \rfloor} - 1 \right) b_{\lfloor t \rfloor} X_{\lfloor t \rfloor} = 0_d.$$

Now observe that

$$a_t \left| X_t - X_{\lfloor t \rfloor} \right| \stackrel{d}{=} a_t \left| X_{t - \lfloor t \rfloor} - X_0 \right| \stackrel{a.s.}{=} a_t |X_{t - \lfloor t \rfloor}| \le a_t \sup_{s \in [0,1]} |X_s|.$$

Since $\lim_{t \to \infty} a_t = 0$, by Slutzky's Theorem $a_t \sup_{s \in [0,1]} |X_s| \stackrel{d}{\to} 0$ and thus so does $a_t \left(X_t - X_{\lfloor t \rfloor} \right)$. Now, since $b_{\lfloor t \rfloor} X_{\lfloor t \rfloor} \stackrel{d}{\to} Y$, the result holds by using Slutzky's Theorem once again. □

In the next following section we will characterize the long and short time behavior of a Lévy processes $\{X_t : t \ge 0\}$ with $X_1 \sim ID(0_{d \times d}, M, 0_d)$. In Section D.2.2 we will consider the case when the Gaussian part may be nonzero.

D.2.1 For Lévy Processes With No Gaussian Part

Let $\{X_t : t \ge 0\}$ be a Lévy processes with $X_1 \sim ID(0_{d \times d}, M, 0_d)$ and let $a_t > 0$. Note that the Lévy measure of $a_t X_t$ is given by

$$M_t(A) = t \int_{\mathbb{R}^d} 1_A(a_t x) M(dx), \qquad A \in \mathfrak{B}\left(\mathbb{R}^d\right). \tag{D.3}$$

We begin with the case when the limiting distribution is infinite variance α-stable for some $\alpha \in (0, 2)$. The Lévy measure of an α-stable distribution with spectral measure $\sigma \neq 0$ is given by M_σ^α as defined in (C.20). Note that, by Proposition C.5, for the long (or short) time behavior of μ to be α-stable it is necessary that

$$M_t \xrightarrow{v} M_\sigma^\alpha \text{ on } \bar{\mathbb{R}}_0^d \text{ as } t \to c$$

where $c = \infty$ (or $c = 0$). We will show that this is also sufficient and that it is equivalent to regular variation of M.

Theorem D.6. *Fix $\alpha \in (0, 2)$ and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t : t \geq 0\}$ be a Lévy Process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$ and let $Y \sim S_\alpha(\sigma, 0_d)$. There exists a function $a_t > 0$ such that*

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to \infty \tag{D.4}$$

if and only if $M \in RV_{-\alpha}^\infty(\sigma)$. Moreover, in this case $a \in RV_{-1/\alpha}^\infty$ and

$$a_t \sim s^{1/\alpha}/V^{\leftarrow}(t), \tag{D.5}$$

where $s = \alpha^{-1}\sigma(\mathbb{S}^{d-1})$ and $V(t) = 1/M\left(|x| > t\right)$.

Proof. Let M_t be defined by (D.3). If (D.4) holds then by Lemma D.4 $\lim_{t\to\infty} a_t = 0$ and by Proposition C.5 $M_t \xrightarrow{v} M_\sigma^\alpha$ on $\bar{\mathbb{R}}_0^d$ as $t \to \infty$. Note that for all $b \geq 0$ $M_\sigma^\alpha(|x| = b) = 0$. Thus for any $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\begin{aligned} \lim_{t\to\infty} tM\left(|x| > b/a_t, \frac{x}{|x|} \in D\right) &= \lim_{t\to\infty} M_t\left(|x| > b, \frac{x}{|x|} \in D\right) \\ = M_\sigma^\alpha\left(|x| > b, \frac{x}{|x|} \in D\right) &= \int_D \int_b^\infty r^{-1-\alpha} dr\sigma(du) = \alpha^{-1}\sigma(D)b^{-\alpha}. \end{aligned}$$

Thus, by Proposition B.8, $M \in RV_{-\alpha}^\infty(\sigma)$ and a_t is in $RV_{-1/\alpha}^\infty$ and it satisfies (D.5).

Conversely, assume that $M \in RV^\infty_{-\alpha}(\sigma)$. Let M_t be as in (D.3) and a_t as in (D.5). By Proposition B.8, for any $b > 0$ and $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\lim_{t\to\infty} M_t\left(|x| > b, \frac{x}{|x|} \in D\right) = \lim_{t\to\infty} tM\left(|x| > b/a_t, \frac{x}{|x|} \in D\right)$$
$$= \alpha^{-1}\sigma(D)b^{-\alpha} = \int_D\int_b^\infty r^{-1-\alpha}dr\sigma(du) = M^\alpha_\sigma\left(|x| > b, \frac{x}{|x|} \in D\right).$$

Since for all $b \geq 0$, $M^\alpha_\sigma(|x| = b) = 0$ we can use Lemma A.12 to get $M_t \xrightarrow{v} M^\alpha_\sigma$ on $\bar{\mathbb{R}}^d_0$ as $t \to \infty$. From here it suffices to show that (C.18) holds.

Set $b_t = a_t^{-1}s^{1/\alpha}$. Note $a_t \in RV^\infty_{-1/\alpha}$ and thus $\lim_{t\to\infty} b_t = \infty$. Since M is regularly varying with index $-\alpha$ by Corollary B.5 if $f(t) = \int_{|x|\leq t}|x|^2M(dx)$ then $f \in RV^\infty_{2-\alpha}$ and

$$\limsup_{t\to\infty} ta_t^2\int_{|x|\leq\epsilon/a_t}|x|^2M(dx) = \limsup_{t\to\infty}\frac{f(\epsilon/a_t)}{a_t^{-2}M(|x| > b_t)}$$
$$= \limsup_{t\to\infty}\frac{f(\epsilon s^{-1/\alpha}b_t)}{s^{-2/\alpha}b_t^2M(|x| > b_t)}$$
$$= \limsup_{t\to\infty}\frac{f(\epsilon s^{-1/\alpha}b_t)}{f(b_t)}s^{2/\alpha}\frac{f(b_t)}{b_t^2M(|x| > b_t)} = \epsilon^{2-\alpha}s\frac{\alpha}{2-\alpha}.$$

Taking $\epsilon \downarrow 0$ shows that (C.18) holds. □

We will now use duality to derive short time behavior.

Theorem D.7. *Fix $\alpha \in (0, 2)$. Let $\{X_t : t > 0\}$ be a Lévy Process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$ and let $\{X^0_t : t > 0\}$ be a Lévy Process with $X^0_1 \sim ID(0_{d\times d}, M^0, 0_d)$. Let $X \sim S_\alpha(\sigma, 0_d)$. There is a function a_t such that*

$$a_tX_t \xrightarrow{d} X \text{ as } t \to \infty \tag{D.6}$$

if and only if there exists a function b_t with

$$b_tX^0_t \xrightarrow{d} X^0 \text{ as } t \downarrow 0 \tag{D.7}$$

where $X^0 \sim S_{2-\alpha}(\sigma, 0_d)$. Moreover, this holds if and only if $b_t \sim 1/a_{h^{-1}(1/t)}$ as $t \downarrow 0$ where $h(t)$ is any strictly monotonely increasing function with $h(t) \sim t^{-1}a_t^{-2}$ as $t \to \infty$.

From Theorem D.6 it follows that $a \in RV^{\infty}_{-1/\alpha}$ and thus by Proposition B.2 a function h of the required form exists. Note that $h \in RV^{\infty}_{(2-\alpha)/\alpha}$ thus $h^{-1} \in RV^{\infty}_{\alpha/(2-\alpha)}$ and $a_{h^{-1}(t)} \in RV^{\infty}_{-1/(2-\alpha)}$. Thus since $b_t \sim 1/a_{h^{-1}(1/t)}$ as $t \downarrow 0$ we have $b \in RV^{0}_{-1/(2-\alpha)}$.

Proof. Note that by Slutzky's Theorem, it suffices to show that the result holds when $a_t = [th(t)]^{-1/2}$ and $b_t = 1/a_{h^{-1}(1/t)}$. For any $B \in \mathfrak{B}(\mathbb{R}^d)$ define

$$M_t^1(B) = t\int_{\mathbb{R}^d} 1_B(xa_t)M(dx) \quad \text{and} \quad M_t^2(B) = t\int_{\mathbb{R}^d} 1_B(xb_t)M^0(dx).$$

These are, respectively, the Lévy measures of a_tX_t and $b_tX_t^0$.

Assume that (D.6) holds. We have

$$\begin{aligned}\left(M_t^1\right)^0(B) &= \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|^2}\right)|x|^2 M_t^1(dx) \\ &= ta_t^2\int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|^2}a_t^{-1}\right)|x|^2 M(dx) \\ &= ta_t^2\int_{\mathbb{R}^d} 1_B\left(xa_t^{-1}\right)M^0(dx).\end{aligned}$$

Thus, by Corollary C.12 this implies that

$$a_t^{-1}X^0_{ta_t^2} \xrightarrow{d} X^0 \quad \text{as } t\to\infty.$$

The result holds since

$$\begin{aligned}\lim_{t\to\infty} a_t^{-1}X^0_{ta_t^2} &= \lim_{t\downarrow 0} a_{1/t}^{-1}X^0_{t^{-1}a^2_{1/t}} = \lim_{t\downarrow 0} a_{1/t}^{-1}X^0_{1/h(1/t)} \\ &= \lim_{u\downarrow 0} a^{-1}_{h^{-1}(1/u)}X^0_u = \lim_{u\downarrow 0} b_uX^0_u,\end{aligned}$$

where the second line follows by the substitution $u = 1/h(1/t)$.

Conversely, assume that (D.7) holds. As before we have

$$\left(M_t^2\right)^0(B) = tb_t^2\int_{\mathbb{R}^d} 1_B\left(xb_t^{-1}\right)M(dx)$$

and by Corollary C.12 this implies that

$$b_t^{-1}X_{tb_t^2} \xrightarrow{d} X \quad \text{as } t\downarrow 0.$$

The result follows from the fact that

$$\begin{aligned}\lim_{t\downarrow 0} b_t^{-1} X_{tb_t^2} &= \lim_{t\to\infty} b_{1/t}^{-1} X_{t^{-1}b_{1/t}^2} = \lim_{t\to\infty} a_{h^{-1}(t)} X_{t^{-1}a_{h^{-1}(t)}^{-2}} \\ &= \lim_{t\to\infty} a_{h^{-1}(t)} X_{h^{-1}(t)} = \lim_{u\to\infty} a_u X_u,\end{aligned}$$

where the third equality follows by the fact that $t = h\left(h^{-1}(t)\right) = \frac{1}{h^{-1}(t)a_{h^{-1}(t)}^2}$ and the fourth by the substitution $u = h^{-1}(t)$. □

Corollary D.8. *Fix $\alpha \in (0,2)$ and let σ be a finite, nonzero Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t : t \geq 0\}$ be a Lévy Process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$ and let $Y \sim S_\alpha(\sigma, 0_d)$. There exists a function $a_t > 0$ such that*

$$a_t X_t \xrightarrow{d} Y \text{ as } t \downarrow 0 \tag{D.8}$$

if and only if $M \in RV_{-\alpha}^0(\sigma)$. Moreover, in this case, $a \in RV_{-1/\alpha}^0$ and

$$a_t \sim s^{-1/\alpha} V^{\leftarrow}(1/t) \text{ as } t \downarrow 0, \tag{D.9}$$

where $s = \alpha/\sigma(\mathbb{S}^{d-1})$ and $V(t) = M\left(|x| > 1/t\right)$.

Proof. This is an immediate consequence of Theorem D.7, Theorem D.6, and Proposition C.13. The form of a_t follows from Proposition B.10. □

We will now consider the case when the limiting distribution is Gaussian. We begin with two lemmas.

Lemma D.9. *Fix $\delta, \delta', \epsilon \in (0,\infty)$ and let M_n be a sequence of Lévy measures on $\mathbb{R}^d$. If $M_n \xrightarrow{v} 0$ on $\mathbb{R}_0^d$ then*

$$\lim_{n\to\infty}\left(\int_{|x|<\delta} xx^T M_n(dx) - \int_{|x|<\delta'} xx^T M_n(dx)\right) = 0_{d\times d}. \tag{D.10}$$

Note that, in the above, we only need convergence of M_n on $\mathbb{R}_0^d$ not $\bar{\mathbb{R}}_0^d$. In other words, in Definition A.5 it suffices to consider only functions that vanish both on a neighborhood of zero and on a neighborhood of infinity.

Proof. Without loss of generality assume that $\delta < \delta'$. For all $z \in \mathbb{R}^d$ we have

$$\left| \int_{|x|<\delta} \langle x, z\rangle^2 M_n(dx) - \int_{|x|<\delta'} \langle x, z\rangle^2 M_n(dx) \right| = \int_{\delta \le |x| < \delta'} \langle x, z\rangle^2 M_n(dx)$$
$$\le |z|^2 \int_{\delta \le |x| < \delta'} |x|^2 M_n(dx) \le |z|^2 (\delta')^2 M_n\left(\delta \le |x| < \delta'\right) \to 0,$$

where the convergence follows by Proposition A.8. □

Lemma D.10. *Fix $c \in \{0, \infty\}$. Let M be a Lévy measure on $\mathbb{R}^d$ and let $\alpha \in [0, 2)$ be such that $\int_{|x|>1} |x|^\alpha M(dx) < \infty$. Let B be a matrix with $\mathrm{tr}B > 0$, and*

$$A_t = \int_{|x|\le t} x x^T M(dx).$$

Assume that $A \in LRV_0^c(B)$. Let $a_t \sim 1/h^{\leftarrow}(t)$ as $t \to c$, where $h(t) = 1/g(t)$ and $g(t) = t^{-2} \int_{|x|\le t} |x|^2 M(dx)$. If

$$M_t(D) = t \int_{\mathbb{R}^d} 1_D(a_t x) M(dx), \qquad D \in \mathfrak{B}(\mathbb{R}^d),$$

then for all $s > 0$

$$\lim_{t\to c} \int_{|x|>s} |x|^\alpha M_t(dx) = 0.$$

In particular, $M_t \xrightarrow{v} 0$ on $\bar{\mathbb{R}}_0^d$ as $t \to c$.

Note that $A \in LRV_0^c(B)$ implies that $\int_{|x|\le t} |x|^2 M(dx) = \mathrm{tr}A_t$ is an element of RV_0^c. Thus $g \in RV_{-2}^c$ and $h \in RV_2^c$. The proof of this lemma is similar to that of Lemma 8.1.2 in [MS01].

Proof. Note that $A \in LRV_0^c(B)$ implies that $a \in RV_{-1/2}^c$. Define

$$V_\alpha(s) = \int_{|x|>s} |x|^\alpha M(dx) \text{ and } U_2(s) = \int_{|x|\le s} |x|^2 M(dx) = \mathrm{tr}A(s).$$

Note that $U_2 \in RV_0^c$ and $\lim_{t\to c} a_t = 1/c$. Thus, by Corollary B.5, for all $s \ge 0$

$$\begin{aligned} \lim_{t\to c} \frac{\int_{|x|>s} |x|^\alpha M_t(dx)}{\int_{|x|\le s} |x|^2 M_t(dx)} &= \lim_{t\to c} \frac{a_t^{-(2-\alpha)} \int_{|x|>s/a_t} |x|^\alpha M(dx)}{\int_{|x|\le s/a_t} |x|^2 M(dx)} \\ &= \lim_{t\to c} \frac{a_t^{-(2-\alpha)} V_\alpha(s/a_t)}{U_2(s/a_t)} = 0. \end{aligned}$$

Since $g \in RV^c_{-2}$ we have

$$\lim_{t\to c} tg(s/a_t)s^2 = \lim_{t\to c} \frac{g(s/a_t)}{g(1/a_t)} s^2 = 1$$

and thus

$$\lim_{t\to c} \int_{|x|>s} |x|^\alpha M_t(dx) = \lim_{t\to c} ta_t^\alpha \int_{|x|>s/a_t} |x|^\alpha M(dx) = \lim_{t\to c} ta_t^\alpha V_\alpha(s/a_t)$$
$$= \lim_{t\to c} \frac{a_t^{-(2-\alpha)} V_\alpha(s/a_t)}{U_2(s/a_t)} ta_t^2 U_2(s/a_t) = \lim_{t\to c} \frac{a_t^{-(2-\alpha)} V_\alpha(s/a_t)}{U_2(s/a_t)} tg(s/a_t)s^2 = 0.$$

This completes the proof. □

Theorem D.11. *Fix $c \in \{0, \infty\}$. Let B be a symmetric nonnegative-definite matrix with $B \neq 0_{d\times d}$. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$ such that $M \neq 0$. Define*

$$A_t = \int_{|x|\leq t} xx^T M(dx). \tag{D.11}$$

There exists a function a_t and a random vector $Y \sim N(B, 0_d)$ such that

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to c \tag{D.12}$$

if and only if $A \in LRV_0^c(B)$. Moreover, when this holds $a \in RV^c_{-1/2}$ and

$$a_t \sim 1/h^{\leftarrow}(t) \text{ as } t \to c, \tag{D.13}$$

where $h(t) = 1/g(t)$ and

$$g(t) = t^{-2} \int_{|x|\leq t} |x|^2 M(dx). \tag{D.14}$$

Remark 1: When (D.12) holds with a_t as in (D.13) then

$$\lim_{t\to c} \frac{\int_{|x|\leq t} xx^T M(dx)}{\int_{|x|\leq t} |x|^2 M(dx)} = B.$$

Remark 2: When $c = \infty$ and $\int_{\mathbb{R}^d} |x|^2 M(dx) < \infty$ then (D.12) holds with $a_t = t^{-1/2}$ and, in this case, the limiting covariance matrix is $\int_{\mathbb{R}^d} xx^T M(dx)$.

Proof. Let M_t be the Lévy measure of a_tX_t, as given by (D.3). First assume that $A \in LRV_0^c(B)$. Let g be defined by (D.14) and a_t by (D.13). This implies that $g \in RV_{-2}^c$ and $a \in RV_{-1/2}^c$. We have

$$\begin{aligned}
\lim_{t\to c}\int_{|x|\le\epsilon} xx^T M_t(dx) &= \lim_{t\to c} ta_t^2 \int_{|x|\le\epsilon/a_t} xx^T M(dx) \\
&= \lim_{t\to c}\frac{a_t^2\int_{|x|\le\epsilon/a_t} xx^T M(dx)}{g(1/a_t)} \\
&= \lim_{t\to c}\frac{a_t^2\int_{|x|\le\epsilon/a_t} xx^T M(dx)}{g(\epsilon/a_t)}\frac{g(\epsilon/a_t)}{g(1/a_t)} \\
&= \lim_{t\to c}\frac{a_t^2\int_{|x|\le\epsilon/a_t} xx^T M(dx)}{\epsilon^{-2}a_t^2\int_{|x|\le\epsilon/a_t}|x|^2M(dx)}\frac{g(\epsilon/a_t)}{g(1/a_t)} \\
&= \lim_{t\to c}\frac{\int_{|x|\le\epsilon/a_t} xx^T M(dx)}{\epsilon^{-2}\int_{|x|\le\epsilon/a_t}|x|^2M(dx)}\epsilon^{-2} = B.
\end{aligned}$$

From here the result follows by Lemma D.10 and Proposition C.5.

Now assume that (D.12) holds. Lemma D.4 implies that $\lim_{t\to c} a_t = 1/c$, and Proposition C.5 implies that $M_t \overset{v}{\to} 0$ on $\bar{\mathbb{R}}_0^d$ as $t \to c$. Thus by Lemma D.9, for any $\epsilon > 0$

$$\limsup_{t\to c} ta_t^2 \int_{|x|\le\epsilon/a_t} xx^T M(dx) = \limsup_{t\to c} ta_t^2 \int_{|x|\le 1/a_t} xx^T M(dx)$$

and similarly for the liminf. Proposition C.5 implies that for all $s > 0$

$$\lim_{t\to c} ta_t^2 \int_{|x|\le s/a_t} xx^T M(dx) = B$$

and thus

$$\lim_{t\to c} ta_t^2 \int_{|x|\le s/a_t} |x|^2 M(dx) = \mathrm{tr}B.$$

Let $U(t) = \int_{|x|\le t}|x|^2M(dx)$. By Lemma D.4 we can use Proposition B.2 to get $U \in RV_0^c$. This implies that

$$\lim_{t\to c}\frac{\int_{|x|\le t} xx^T M(dx)}{\int_{|x|\le t}|x|^2M(dx)} = \lim_{t\to c}\frac{ta_t^2\int_{|x|\le 1/a_t} xx^T M(dx)}{ta_t^2\int_{|x|\le 1/a_t}|x|^2M(dx)} = \frac{B}{\mathrm{tr}B},$$

and hence $A \in LRV_0^c(B)$. □

Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0_{d\times d}, M, 0_d)$, and for $\alpha \in (0,2)$ let $Y \sim S_\alpha(\sigma, 0_d)$. We have seen that there exists a function a_t such that $a_t X_t \xrightarrow{d} Y$ as $t \to \infty$ if and only if $M^0 \in RV^0_{2-\alpha}(\sigma)$. This leads to the question whether some form of slow variation of M^0 will be equivalent to the long time behavior being Gaussian. We have

$$A_t = \int_{|x|\leq t} xx^T M(dx) = \int_{1/t\leq |x|} \left(\frac{x}{|x|}\right)\left(\frac{x}{|x|}\right)^T M^0(dx) =: A^0_{1/t}.$$

Thus, there exists a nonnegative definite matrix B with $A \in LRV^\infty_0(B)$ if and only if $A^0 \in LRV^0_0(B)$. Moreover, taking traces we get

$$f(t) := \int_{|x|\leq t} |x|^2 M(dx) = \int_{1/t\leq|x|} M^0(dx).$$

The left side is in RV^∞_0 if and only if the right side is, which happens if and only if the measure on $[0,\infty)$

$$M_{abs}(A) = \int_{\mathbb{R}^d} 1_A(|x|) M^0(dx), \qquad A \in \mathfrak{B}([0,\infty))$$

is slowly varying at 0. For the long time behavior to be Gaussian it is necessary for $f \in RV^\infty_0$. Moreover, in one dimension it is also sufficient. Thus, in the one dimensional case, for the long time behavior to be Gaussian it is necessary and sufficient for M_{abs} to be slowly varying at 0. However, this is not true in higher dimensions.

D.2.2 For General Lévy Processes

In this section we will characterize long and short time behavior for the case when the Lévy process has a Gaussian part. We begin with a two lemmas.

Lemma D.12. *Fix $c \in \{0,\infty\}$. Let $A \neq 0_{d\times d}$ be a symmetric nonnegative definite matrix. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(A, 0, 0_d)$. Let Y be a* *stable random vector not concentrated at a point. There exists a function* $a_t > 0$ *such that*

$$a_t X_t \xrightarrow{d} Y \text{ as } t \to c \tag{D.15}$$

if and only if

$$a_t \sim k t^{-1/2} \text{ as } t \to c \tag{D.16}$$

for some $k \in (0, \infty)$. *In this case* $Y \sim ID(k^2 A, 0, 0_d)$.

Proof. Let a_t be as in (D.16). By the self-similarity of Brownian motion (see Section 13 in [Sat99]) we have

$$\lim_{t\to\infty} a_t X_t \stackrel{d}{=} \lim_{t\to\infty} k t^{-1/2} t^{1/2} X_1 \stackrel{d}{=} k X_1.$$

Now let a_t be any function such that (D.15) holds. We have

$$Y \stackrel{d}{=} \lim_{t\to c} a_t X_t = \lim_{t\to c} \frac{a_t}{t^{-1/2}} t^{-1/2} X_t \stackrel{d}{=} \lim_{t\to c} \frac{a_t}{t^{-1/2}} X_1.$$

Since convergence in distribution implies convergence of cumulant generating functions (see Lemma 7.7 in [Sat99]), there is a $k \in \mathbb{R}$ such that $a_t \sim k t^{-1/2}$ as $t \to c$. Since Y is not concentrated at a point $k \neq 0$, and the result holds. $\square$

Lemma D.13. *If* $\{X_t : t \geq 0\}$ *is a Lévy process with* $X_1 \sim ID(0_{d\times d}, M, 0_d)$ *then* $t^{-1/2} X_t \xrightarrow{p} 0_d$ *as* $t \downarrow 0$.

Proof. Let M_t be the Lévy measure of $t^{-1/2} X_t$. For any $\ell > 0$

$$\begin{aligned}
\lim_{t\downarrow 0} M_t(|x| > \ell) &= \lim_{t\downarrow 0} t M(|x| > \ell t^{1/2}) \\
&= \lim_{\epsilon\downarrow 0} \lim_{t\downarrow 0} \left[t \int_{\epsilon > |x| > \ell t^{1/2}} M(dx) + t \int_{|x| \geq \epsilon} M(dx) \right] \\
&\leq \lim_{\epsilon\downarrow 0} \lim_{t\downarrow 0} \left[\ell^{-2} \int_{\epsilon > |x|} |x|^2 M(dx) + t \int_{\epsilon \leq |x|} M(dx) \right] \\
&= \lim_{\epsilon\downarrow 0} \ell^{-2} \int_{\epsilon > |x|} |x|^2 M(dx) = 0,
\end{aligned}$$

and

$$\lim_{\epsilon\downarrow 0}\lim_{t\downarrow 0} t\int_{|x|\le\epsilon t^{1/2}} |x|^2 M(dx) = 0.$$

Thus the result follows by Proposition C.5. □

Theorem D.14. *Let $\{X_t : t \ge 0\}$ be a Lévy process with $P(X_1 = 0_d) < 1$ and $X_1 \sim ID(A, M, 0_d)$, let $\{Y_t : t \ge 0\}$ is a Lévy process with $Y_1 \sim ID(0_{d\times d}, M, 0_d)$, and let Z be a random vector not concentrated at a point. If $c \in \{0, \infty\}$ then there is $a_t > 0$ with*

$$a_t X_t \stackrel{d}{\to} Z \text{ as } t \to c \tag{D.17}$$

if and only if any of the following hold.

1. $c = \infty$, $\mathrm{E}|X_1|^2 < \infty$, and for some $k \ne 0$ $a_t \sim kt^{-1/2}$ as $t \to \infty$. Moreover, in this case $Z \sim N(0_d, B)$ with

$$B = k^2\left(A + \int_{\mathbb{R}^s} xx^T M(dx)\right).$$

2. $c = \infty$, $\mathrm{E}|X_1|^2 = \infty$, and $a_t Y_t \stackrel{d}{\to} Z$ as $t \to \infty$.

3. $c = 0$, $A \ne 0_{d\times d}$ and $a_t \sim kt^{-1/2}$ as $t \downarrow 0$, moreover, in this case $Z \sim N(0_d, k^2 A)$.

4. $c = 0$, $A = 0_{d\times d}$ and $a_t Y_t \stackrel{d}{\to} Z$ as $t \downarrow 0$.

Note that this implies that if $\{X_t : t \ge 0\}$ is a Lévy process with $X_1 \sim ID(A, M, b)$, then the results holds for the Lévy process $\{(X_t - tb) : t \ge 0\}$.

Proof. Part 1 follows immediately from Lemma D.12, Theorem D.11, and Remark 2 following that theorem. Part 3 follows from Lemma D.12 and Part 1 of Lemma D.13. Part 4 is immediate. It remains to show Part 2.

Let $\{W_t : t \ge 0\}$ be a Lévy process independent of $\{Y_t : t \ge 0\}$ with $W_1 \sim ID(A, 0, 0_d)$. Let a_t be such that $a_t Y_t \stackrel{d}{\to} Z$ as $t \to \infty$. We will first consider the case when Z is α-stable for $\alpha \in (0,2)$. In this case, by Theorem D.6 $a \in RV^{\infty}_{-1/\alpha}$ and thus $\lim_{t\to\infty} \frac{a_t}{t^{-1/2}} = 0$. We have

$$\lim_{t\to\infty} a_t X_t \stackrel{d}{=} \lim_{t\to\infty} (a_t Y_t + a_t W_t) \stackrel{d}{=} \lim_{t\to\infty} \left(a_t Y_t + \frac{a_t}{t^{-1/2}} t^{-1/2} W_t \right) \stackrel{d}{=} Z + 0W_1 = Z.$$

If $\alpha = 2$, we can use essentially the same argument. Let

$$\ell(t) = \left[\int_{|x|<\sqrt{t}} |x|^2 M(dx) \right]^{-1}.$$

By Theorem D.11 $\ell \in RV^{\infty}_0$ and for some $k \neq 0$ we have $a_t \sim k/h^{\leftarrow}(t)$, where $h(t) = t^2\ell(t^2)$. By Theorem 1.5.13 and Proposition 1.5.15 in [BGT87], there is a function $\ell^{\#}$ such that

$$h^{\leftarrow}(t) \sim t^{1/2}\ell^{\#}(t)$$

and $\lim_{t\to\infty} \ell(t) \left[\ell^{\#}(t\ell(t))\right]^2 = 1$. Since ℓ is slowly varying and $\lim_{t\to\infty} \ell(t) = 0$ this implies that $\lim_{t\to\infty} \ell^{\#}(t) = \infty$ and thus

$$\lim_{t\to\infty} \frac{a_t}{t^{-1/2}} = \lim_{t\to\infty} \frac{t^{-1/2}/\ell^{\#}(t)}{t^{-1/2}} = \lim_{t\to\infty} \frac{1}{\ell^{\#}(t)} = 0.$$

From here the result follows as before. $\square$

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