DISTRIBUTED VECTOR GAUSSIAN SOURCE-CODING AND DISTRIBUTED HYPOTHESIS TESTING

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by
Md. Saifur Rahman
January 2012
Distributed compression is desired in applications in which data is collected in a distributed manner by several sensors and information about the data is sent to a processing center, which uses these information to meet an end goal. In this work, we focus on two such applications: (1) distributed source coding and (2) distributed hypothesis testing.

In distributed source coding, we determine the rate region of the vector Gaussian one-helper source-coding problem under a covariance matrix distortion constraint. The rate region is achieved by a simple scheme that separates the lossy vector quantization from the lossless spatial compression. We introduce a novel analysis technique, namely distortion projection. The converse is established by combining distortion projection with two other analysis techniques that have been employed in the past to obtain partial results for the problem. We also study an extension to a special case of the problem in which the primary source is a vector and the helper’s observation is a scalar and consider separate distortion constraints on both sources. We provide an outer bound to the rate region of this problem and show that it is partially tight in general and completely tight in some nontrivial cases.

In distributed hypothesis testing, we study a problem in which data is compressed distributively and sent to a detector that seeks to decide between two possible distributions for the data. The aim is to characterize all achievable en-
coding rates and exponents of the type 2 error probability when the type 1 error probability is at most a fixed value. For related problems in distributed source coding, schemes based on random binning perform well and are often optimal. For distributed hypothesis testing, however, the use of binning is hindered by the fact that the overall error probability may be dominated by errors in the binning process. We show that despite this complication, binning is optimal for a class of problems in which the goal is to “test against conditional independence.” We then use this optimality result to give an outer bound for a more general class of instances of the problem. We also extend the “test against independence” result of Ahlswede and Csiszár to the vector Gaussian case.
Md. Saifur Rahman was born and brought up in Patna, a city in the eastern part of India. He received his Master of Science degree in Electrical Engineering from Cornell University in 2010, and his Bachelor of Technology degree in Instrumentation Engineering from Indian Institute of Technology (IIT), Kharagpur in 2006 with a minor in Electronics and Electrical Communication Engineering. After his junior year at IIT, he spent the summer of 2005 at the University of New Mexico, Albuquerque as an intern and worked on the nonuniformity correction for infrared sensors. During 2006-07, he was a software engineer at Samsung India Software Operations, Bangalore. He joined the School of Electrical and Computer Engineering at Cornell University in the Fall of 2007 as a graduate student. At Cornell, he has been a member of the Foundations of Information Engineering group and has been undergoing his doctoral research under the supervision of Professor Aaron Wagner. He spent the summer of 2011 at Samsung Telecommunications America, Dallas working on the design of low-complexity quantizers for Hybrid ARQ systems. His main research interests include distributed compression, optimization, and networks. In particular, he has been working on distributed compression of vector sources and compression for hypothesis testing.
This document is dedicated to my family.
ACKNOWLEDGEMENTS

My special thanks go to my thesis advisor Aaron Wagner. I have been privileged to pursue research under him, work with him as a teaching assistant, and learn the fundamentals of Information Theory in the class he taught. I am indebted to him for being such an inspiration and for guiding me all the way in my research. I have particularly benefitted from his style of advising as he is understanding, systematic in his approach, and puts emphasis on details and thoroughness. All these have helped me immensely in carrying out this work.

I would also like to extend my thanks to the other committee members, Lang Tong, Adrian Lewis, and Salman Avestimehr for their valuable suggestions and feedbacks which have helped me conduct this work. Also, the classes they taught were extremely informative and have contributed to my research. I would like to acknowledge other professors at Cornell whose classes have benefitted me: Thomas Parks, Zygmunt Haas, C. Richard Johnson Jr., Stephen Wicker, Mark Lewis, David Williamson, Philip Protter, Iouli Iliachenko, and Eugene Dynkin. I am also thankful to Scott Coldren for helping me out at various points during this work both academically and non-academically. My thanks also extend to my friends and fellow graduate students at Cornell: Ebad Ahmed, Amine Laourine, Yücel Altuğ, Benjamin Kelly, Muhammad Adnan, Swarnavo Sarkar, Ajeet Kumar, Amandeep Singh, Prayut Bhamawat, Saba Khan, Meng Wang, Nithin Michael, Alireza Vahid, and Ilan Shomorony. I am particularly thankful to Ebad and Amine for several helpful discussions we had and continue to have about our research.

I am thankful to Jorge Pezoa and Hoang Nguyen for their guidance and help during my two internships. I would also like to thank Aurobinda Routray, my undergraduate thesis advisor at IIT Kharagpur, for always encouraging me
to do research. My thanks also go to my friends from IIT Kharagpur, Amit Prakash, Sambit Sahu, Bitihotra Routroy, Susmit Jha, Surya Mohan, Azim Iqbal, and Shad Kirmani for always inspiring and encouraging me to pursue graduate studies and also for all the memories of the wonderful days that we spent together. I also express my gratitude to Prof. M. Akhtar, Prof. Ram Iqbal Singh, Er. Randhir Kumar, and Anil Parmar who helped me pass the entrance examination to IITs. A few other friends who have been part of this accomplishment in some way are Dilshad Alam, Tauseef Fakhri, Maroof Hashmi, and Nooruzzuha.

I am extremely grateful to my parents Md. Shah Mahmood Alam and Rehana Khatoon, my wife Saba Khan, my daughter Aisha Saif Rahman, my siblings Hafiz Dr. Md. Obaidur Rahman, Kaniz Fatma, Saleha Perween, Salma Perween, Md. Abdur Rahman, Roquiyya Tarannum, Nahid Hena, Md. Ebadur Rahman, and Sufia Rahman, my in-laws Dr. Sayeed Ahmed Khan, Nafis Khan, and Shoeb Khan, and other relatives Suraiyya, Naushad Alam, and Dr. Mohd. Athar Ansari, for always being with me, and supporting and helping me in all my accomplishments.

Finally, I am indebted to Abdur Razzaque Sir for being my go-to person all my life.
# TABLE OF CONTENTS

Biographical Sketch ............................................................. iii
Dedication .............................................................................. iv
Acknowledgements ................................................................. v
Table of Contents ................................................................. vii
List of Figures ....................................................................... ix

1 Introduction
1.1 Distributed Source Coding ................................................. 1
1.2 Distributed Hypothesis Testing ........................................... 5
1.3 Organization of the Thesis ................................................. 12

2 Vector Gaussian One-Helper Source-Coding ....................... 14
2.1 Notation ........................................................................... 14
2.2 Problem Formulation ....................................................... 16
2.3 Gaussian Achievable Scheme .......................................... 17
2.4 Main Result ..................................................................... 19
2.5 Overview of the Converse Argument ............................... 20

3 Scalar-Help-Vector Solution ................................................. 24
3.1 Rate Region .................................................................... 25
3.2 Core Optimization Problem .......................................... 25
3.3 Converse Ingredients ..................................................... 37
  3.3.1 Distortion Projection ............................................... 40
  3.3.2 Oohama’s Approach ................................................ 41
3.4 Converse Proof of the Main Result ................................ 43
3.5 Extension to Two Constraints ........................................ 45
  3.5.1 An Outer Bound ..................................................... 47
  3.5.2 Tightness of the Outer Bound .................................. 51
  3.5.3 Numerical Example ................................................ 57

4 Vector-Help-Vector Solution ............................................... 58
4.1 Rate Region .................................................................... 58
4.2 Properties of the Optimal Gaussian Solution .................... 60
4.3 Converse Ingredients ..................................................... 71
  4.3.1 Distortion Projection ............................................... 75
  4.3.2 Source Enhancement ............................................... 78
  4.3.3 Oohama’s Approach ................................................ 82
4.4 Converse Proof of the Main Result ................................ 85
4.5 Solution for the General Case ........................................ 86
## LIST OF FIGURES

1.1 Vector Gaussian one-helper source-coding problem. ............... 3  
1.2 A Gaussian achievable scheme. ........................................ 4  
1.3 $L$-encoder general hypothesis testing. .......................... 6  
1.4 Shimokawa-Han-Amari achievable region for a fixed channel $P_{U_1|X_1}$. ................................................. 8  
1.5 Gel’fand and Pinsker hypothesis testing against independence. 11  
1.6 Gaussian many-help-one hypothesis testing against independence. 11  
3.1 Inner and outer bounds for the two-distortion extension of the scalar-help-vector case. ............................. 56  
5.1 Related source coding problem. ................................. 109  
5.2 Shimokawa-Han-Amari achievable region for a fixed $P_{U_1|X_1}$. 112  
5.3 Gaussian many-help-one source coding problem. .............. 124  
5.4 Regions of pair $(\rho_0, \rho_1)$ for which the outer bound is nontrivial. 134  
5.5 Outer and inner bounds for four examples. ...................... 141
CHAPTER 1
INTRODUCTION

Information Theory revolves around the seminal work of Shannon [1]. His work made it possible to develop technologies that have made profound impacts in our life. The developments in the areas of communication, networking, security, and control are a few examples where we have made a huge step-forward in past few decades. However, the future developments of these existing areas and new emerging ones are challenging because of several practical reasons. One of the challenges is that in reality the data that we are interested in is not available at one location. It is rather collected in a distributed manner at several locations and these locations are often constrained to communicate their data to a central location. They therefore communicate a summary of their data keeping in mind the end goal at the central location. This results in a classical distributed compression problem. Out of many applications of distributed compression, we focus on two in this thesis. The first is distributed source coding in which data is compressed distributively keeping in mind that a part or whole of it needs to be reproduced exactly or approximately at the central location. The second is distributed hypothesis testing in which data is compressed so that the central location can make a decision between two possible distributions for the data.

1.1 Distributed Source Coding

Consider a distributed surveillance system in which cameras are mounted at two locations to monitor an area under surveillance [2]. Both locations need to communicate their videos to a central location which uses them to make surveil-
lance decisions. However, due to practical constraints, these videos need to be compressed. Since two videos are of the same geographical area, they are correlated. So, the problem at hand is how to exploit this correlation and compress these videos so that the central location can recover them losslessly or with some loss. Similar problem arises in applications like cooperative relaying [3, 4] and distributed video coding [5]. More generally, these problems fall in the realm of distributed source coding.

The general area of distributed source coding has received considerable attention. The study began with the work by Slepian and Wolf [6]. They considered a lossless problem in which two correlated memoryless sources must be reproduced by the decoder with arbitrarily small error probability. Almost immediately, researchers began to extend Slepian and Wolf’s result to lossy problems in which the sources must be reconstructed with an average distortion of no more than a given amount. Wyner and Ziv [7] considered the lossy problem in which the decoder reconstructs a source to within an allowable distortion with the help of side information about the source. Since then, the problem has been extended in several directions by Berger [8], Tung [9], Berger et al. [10], Oohama [11], Viswanathan and Berger [12], Wagner et al. [13], Liu and Viswanath [14], and others.

In most of these extensions, compression is done on the blocks of source samples with the assumption that these samples are independent and identically distributed (i.i.d.). This assumption is not true in reality. The data samples are often dependent and have different distributions. Video frames and MIMO relaying are two examples of vector-valued sources. One way to handle such sources is to model them as vectors with non-i.i.d. components and consider
compressing the i.i.d. strings of these vectors. The challenge, however, is that the existing solutions to the scalar distributed source coding problems do not apply directly to their vector extensions. In this work, we study one such problem, namely vector Gaussian one-helper source-coding problem.

In vector Gaussian one-helper source-coding problem, as depicted in Fig. 1.1, there are two vector Gaussian sources $X$ and $Y$. Encoders 1 and 2 observe two i.i.d. strings distributed according to $X$ and $Y$, respectively, and separately send messages to the decoder at rates $R_1$ and $R_2$ bits per observation, respectively, using noiseless channels. The decoder uses both messages to estimate $X$ such that a given distortion constraint on the average error covariance matrix is satisfied. The goal is to determine the rate region of the problem, which is the set of all rate pairs $(R_1, R_2)$ that allow us to satisfy the distortion constraint for some design of the encoders and the decoder.

![Figure 1.1: Vector Gaussian one-helper source-coding problem.](image)

Oohama [11] studied the one-helper problem in which both sources are scalar and gave a complete characterization of the rate region. The achievability proof is a Gaussian scheme that is described in more detail below. The converse argument uses the entropy-maximizing property of the Gaussian distribution and the entropy power inequality (EPI), and it bears a certain resemblance to Bergmans’ earlier converse for the scalar Gaussian broadcast channel [15].
As such, one might hope that the channel enhancement technique introduced by Weingarten et al. [16] to solve the MIMO Gaussian broadcast channel would be sufficient to solve the problem considered here. This turns out not to be the case, however. Among other contributions, Liu and Viswanath [14] showed that channel enhancement yields an outer bound for the vector one-helper problem that is not tight in general. We later improved it slightly by showing that the Gaussian scheme achieves a portion of the boundary of the rate region [17]. The enhancement technique was applied in a different way by Zhang [18], who called it source enhancement, but this also yielded an outer bound that is not always tight.

![Figure 1.2: A Gaussian achievable scheme.](image)

In this work, we first solve a simpler version that cannot be solved using existing techniques, namely that in which $Y$ is a scalar and $X$ is a vector [19, 20]. The proof did not use enhancement, but it did require a novel technique that we call distortion projection. We then show that distortion projection, source enhancement, and Oohama’s converse technique together are sufficient to solve the general problem in which both $X$ and $Y$ are vectors [21, 22]. In particular, we determine the rate region exactly and show that a vector extension of the Gaussian scheme used by Oohama is optimal. In this scheme, as depicted in Fig. 1.2, encoder 1 vector quantizes (VQ) its observations using a Gaussian test channel as in point-to-point rate-distortion theory. It then compresses the quantized
values using Slepian-Wolf (SW) encoding [6]. Encoder 2 just vector quantizes its observations using another Gaussian test channel. The decoder decodes the quantized values and estimates the observations of encoder 1 using a minimum mean-squared error (MMSE) estimator.

We also consider an extension to the special case of the problem in which \( Y \) is a scalar and \( X \) is a vector by imposing separate distortion constraints on both sources. We provide an outer bound to the rate region of this extended problem. The outer bound is partially tight in general and completely tight in some nontrivial cases.

### 1.2 Distributed Hypothesis Testing

Consider the problem of measuring the traffic on two links in a communication network and inferring whether the two links are carrying any common traffic [23, 24]. Evidently, this inference cannot be made by inspecting the measurements from one of the links alone, except in the extreme situation in which that link carries no traffic at all. Thus it is necessary to transport the measurements from one of the links to the other, or to transport both measurements to a third location. The measured data is potentially high-rate, however, so this transportation may require that the data be compressed. This raises the question of how to compress data when the goal is not to reproduce it \textit{per se}, but rather to perform inference. A similar problem arises when inferring the speed of a moving vehicle from the times that it passes certain waypoints.

These problems can be modeled mathematically by the setup depicted in Fig. 1.3, which we call the \( L \)-encoder general hypothesis testing problem. A vector
source \((X_1, \ldots, X_L, Y)\) has different joint distributions \(P_{X_1, \ldots, X_L, Y}\) and \(Q_{X_1, \ldots, X_L, Y}\) under two hypotheses \(H_0\) and \(H_1\), respectively. Encoder \(l\) observes an i.i.d. string distributed according to \(X_i\) and sends a message to the detector at a finite rate of \(R_l\) bits per observation using a noiseless channel. The detector, which has access to an i.i.d. string distributed according to \(Y\), makes a decision between the hypotheses. The detector may make two types of error: the type 1 error (\(H_0\) is true but the detector decides otherwise) and the type 2 error (\(H_1\) is true but the detector decides otherwise). The type 1 error probability is upper bounded by a fixed value. The type 2 error probability decreases exponentially fast, say with an exponent \(E\), as the length of the i.i.d. strings increases. The goal is to characterize the rate-exponent region of the problem, which is the set of all achievable rate-exponent vectors \((R_1, \ldots, R_L, E)\), in the regime in which the type 1 error probability is small. This problem was first introduced by Berger [25] (see also [26]) and arises naturally in many applications. Yet despite these applications, the theoretical understanding of this problem is far from complete, especially when compared with its sibling, distributed source coding, where random binning has been shown to be a key ingredient in many optimal schemes.

![Diagram](image)

**Figure 1.3:** \(L\)-encoder general hypothesis testing.

Note that if one of the variables in the set \((X_1, \ldots, X_L, Y)\) has a different marginal distribution under \(P_{X_1, \ldots, X_L, Y}\) and \(Q_{X_1, \ldots, X_L, Y}\), then one of the termi-
nals can detect the underlying hypothesis with an exponentially-decaying type 2 error probability, even without receiving any information from the other terminals, and could communicate this decision to other terminals by broadcasting a single bit. Motivated by the applications mentioned above, we shall focus our attention on the case in which the variables $X_1, \ldots, X_L, Y$ have the same marginal distributions under both hypotheses.

Ahlswede and Csiszár [27] studied a special case of this problem in which $L = 1$. They presented a scheme in which the encoder sends a quantized value of $X_1$ to the detector which uses it to perform the test with the help of $Y$. Their scheme, although suboptimal in general, is optimal for a special case of the problem which they call “test against independence.” Their scheme was later improved by Han [28] and Shimokawa-Han-Amari [29]. In the latter improvement, the encoder first quantizes $X_1$, then bins the quantized value using a Slepian and Wolf encoder [6]. The detector first decodes the quantized value with the help of $Y$ and then performs a likelihood ratio test. In this scheme, type 2 errors can occur in two different ways: the binning can fail so that the receiver decodes the wrong codeword and therefore makes an incorrect decision, or the true codeword can be decoded correctly yet be atypically distributed with $Y$, again resulting in an incorrect decision. Moreover, there is a tension between these two forms of error. If the codeword is a high fidelity representation of $X_1$, then binning errors are likely, yet the detector is relatively unlikely to make an incorrect decision if it decodes the codeword correctly. If the codeword is a low fidelity representation, then binning errors are unlikely, but the detector is more likely to make an incorrect decision when it decodes correctly.

Fig. 1.4 illustrates this tradeoff for a fixed test channel $P_{U_1|X_1}$ used for quan-
Figure 1.4: Shimokawa-Han-Amari achievable region for a fixed channel $P_{U_1|X_1}$.

tization. All mutual information quantities are computed with respect to $P$. $\rho^*_2(U_1)$ and $\rho^*_1(U_1)$ are the exponents associated with type 2 errors due to binning errors and assuming correct decoding of the codeword, respectively. Formulas for each are available in [26]. For low rates, binning errors are common and $\rho^*_2(U_1)$ dominates the overall exponent. For high rates, binning errors are uncommon and $\rho^*_1(U_1)$ dominates the overall exponent. To achieve the overall performance, the test channel should be chosen so that these two exponents are equal; if they are not, then making the test channel slightly more or less noisy will yield better performance. A similar tradeoff arises in the analysis of error exponents of binning-based schemes for the Wyner-Ziv problem [30, 31, 32, 33] and in the design of short block-length codes for Wyner-Ziv or joint source-channel coding. Evidently the benefit accrued from binning is reduced when one considers error exponents, as opposed to when the design criterion is vanishing error probability or average distortion, because the error exponent associated with the binning process itself may dominate the overall performance.
The Shimokawa-Han-Amari scheme uses random, unstructured binning. It is known from the lossless source coding literature that structured binning schemes can strictly improve upon unstructured binning schemes in terms of the error exponents [34, 35, 36]. Thus, two questions naturally arise:

1. Is the tradeoff depicted in Fig. 1.4 fundamental to the problem or an artifact of a suboptimal scheme?
2. Can the scheme be improved by using structured binning?

We conclusively answer both questions and show that unstructured binning is optimal in several important cases. We begin by considering a special case of the problem that we call \( L \)-encoder hypothesis testing against conditional independence. Here \( Y \) is replaced by a three-source \((X_{L+1}, Y, Z)\) such that \( Z \) induces conditional independence between \((X_1, \ldots, X_L, X_{L+1})\) and \( Y \) under \( H_1 \). In addition, \((X_1, \ldots, X_L, X_{L+1}, Z)\) and \((Y, Z)\) have the same distributions under both hypotheses. This problem is a generalization of the single-encoder test against independence studied by Ahlswede and Csizár [27].

For this problem we provide an achievable region, based on a scheme we call Quantize-Bin-Test, that reduces to the Shimokawa-Han-Amari region for \( L = 1 \) yet is significantly simpler. We also introduce an outer bound similar to the outer bound for the distributed rate-distortion problem given by Wagner and Anantharam [37]. The idea is to introduce an auxiliary random variable that induces conditional independence between the sources. This technique of obtaining an outer bound has been used to prove results in many distributed source coding problems [13, 37, 38, 39, 40, 41].

The inner (achievable) and outer bounds are shown to match in three exam-
The first is the case in which there is only one encoder \((L = 1)\). Although this problem is simply the conditional version of the test against independence studied by Ahlswede and Csiszár [27], the conditional version is much more complicated due to the necessary introduction of binning. It follows that the Shimokawa-Han-Amari scheme is optimal for \(L = 1\), providing what appears to be the first nontrivial optimality result for this scheme. This problem arises in detecting network flows in the presence of common cross-traffic that is known to the detector. Here \(X_1\) represents the network traffic measured at a remote location, \(Y\) is the traffic measured at the detector, and \(Z\) represents the cross-traffic. The goal is to detect the presence of common traffic beyond \(Z\), i.e., to determine whether \(Z\) captures all of the dependence between \(X_1\) and \(Y\).

The second is a problem inspired by a result of Gel’fand and Pinsker [42]. We refer to this as the Gel’fand and Pinsker hypothesis testing against independence problem, the setup of which is shown in Fig. 1.5. Here \(X_{L+1}\) and \(Z\) are deterministic and there is a source \(X\) which under \(H_0\) is the minimum sufficient statistic for \(Y\) given \((X_1, \ldots, X_L)\) such that \(X_1, \ldots, X_L, Y\) are conditionally independent given \(X\). We characterize the set of rate vectors \((R_1, \ldots, R_L)\) that achieve the centralized exponent \(I(X;Y)\). We show that the Quantize-Bin-Test scheme is optimal for this problem.

The third is the Gaussian many-help-one hypothesis testing against independence problem, the setup of which is shown in Fig. 1.6. Here the sources are jointly Gaussian and there is another scalar Gaussian source \(X\) observed by the main encoder which sends a message to the detector at a rate \(R\). The encoder observing \(X_i\) is now referred to as the helper \(l\). We characterize the rate-exponent region of this problem in a special case when \(X_1, \ldots, X_L, Y\) are...
conditionally independent given \( X \). We use results on related source coding problem by Oohama [43] and Prabhakaran et al. [44] to obtain an outer bound, which we show is achieved by the Quantize-Bin-Test scheme.

Figure 1.6: Gaussian many-help-one hypothesis testing against independence.

For all three examples, we obtain the solution by observing that the relevant error exponent takes the form of a mutual information, and thereby relate the problem to a source-coding problem. This correspondence was first observed
by Ahlswede and Csiszár [27]. Tian and Chen later applied it in the context of successive refinement [45]. These three conclusive results enable us to answer both of the above questions. Because the Shimokawa-Han-Amari scheme is optimal for $L = 1$, the tradeoff that it entails, depicted in Fig. 1.4, must be fundamental to the problem. Moreover, as both the Shimokawa-Han-Amari and Quantize-Bin-Test schemes do not use structured binning, we conclude that it is not necessary for this problem, at least in the special case considered here.

As a byproduct of our results, we obtain an outer bound for a class of more general hypothesis testing problems. This is the first nontrivial outer bound for the problem, and numerical experiments show that it is quite close to the existing achievable regions in many cases. We also extend the test against independence result of Ahlswede and Csiszár to the vector Gaussian case. All of these results are also available in [17, 46, 47].

1.3 Organization of the Thesis

The organization of the thesis is as follows. In Chapter 2, we formulate the vector Gaussian one-helper source-coding problem, state the main result, and present the Gaussian achievable scheme. We end the chapter with an overview of the converse proof of the main result. Chapter 3 is devoted to the special case of the problem in which the main source is a vector and the helper’s observation is a scalar. This chapter also contains an outer bound for the more general problem in which there are separate distortion constraints on both sources. In Chapter 4, we present the proof for the most general case of the problem in which both sources are vectors. The distributed hypothesis testing problem is
studied in Chapter 5. We start with the $L$-encoder general hypothesis testing problem in Section 5.2. Section 5.3 is devoted to the $L$-encoder hypothesis testing against conditional independence problem. The next three sections are on three special cases of this problem. An outer bound for the $L$-encoder general hypothesis testing problem is given in Section 5.7. Section 5.8 concludes the chapter with the vector Gaussian extension of Ahlswede and Csiszár’s result on test against independence.
In this chapter, we formulate the vector Gaussian one-helper source-coding problem and present a scheme that is optimal for this problem. For convenience, we adopt the following terminology in the rest of the thesis. We refer to the case in which both sources are scalar as the scalar-help-scalar. The case in which the main source is a scalar and the helper’s observation is a vector is referred to as the vector-help-scalar. The remaining two cases are similarly referred to as the scalar-help-vector and vector-help-vector. The scalar-help-scalar case of the problem was solved by Oohama [11]. Oohama’s solution extends to the vector-help-scalar case. However, it on its own is not sufficient to solve the remaining two cases. The insufficiency of Oohama’s solution and the techniques needed to solve the remaining two cases are discussed in the end of this chapter. The complete solution to the scalar-help-vector case is presented in Chapter 3. The key technique needed for the solution is distortion projection. The vector-help-vector case is solved in Chapter 4. The solution for this case requires distortion projection and source enhancement.

2.1 Notation

We use uppercase to denote random variables and vectors. Boldface is used to distinguish vectors from scalars. Arbitrary realizations of random variables and vectors are denoted in lowercase. Auxiliary random variables associated with scalar sources are written as non-boldface, but vector values are allowed. Auxiliary random variables associated with vector sources are written as boldface.
For a random vector $X$, $X^n$ denotes an i.i.d. vector of length $n$, $X^n(i)$ denotes its $i$th component, and $X^n(i : j)$ denotes the $i$th through $j$th components. The superscript $T$ denotes matrix transpose. In order to be consistent with the notation used in practice, we use $| \cdot |$ to denote the cardinality of the range of a function, the cardinality of a set, and the determinant of a matrix. The meaning of the notation will be clear from its argument and also from the context. The notation $x^+$ denotes $\max(x, 0)$. $\mathbb{R}^m$ is used to denote the $m$-dimensional Euclidean space. The closure of a set $A$ is denoted by $\overline{A}$. We use $\sigma_Y^2$ and $\sigma_{Y|V}^2$ to denote the variance of $Y$ and the conditional variance of $Y$ given $V$, respectively. The covariance matrix of $X$ is denoted by $K_X$. The conditional covariance matrix of $X$ given $Y$ is denoted by $K_{X|Y}$ and is defined as

$$K_{X|Y} \triangleq E\left[ (X - E(X|Y)) (X - E(X|Y))^T \right].$$

All vectors are column vectors and are $m$-dimensional, unless otherwise stated. We use $I_m$ to denote an $m \times m$ identity matrix. With a little abuse of notation, $0$ is used to denote both zero vectors and zero matrices of appropriate dimensions. We use $\text{Diag}(d_1, d_2, \ldots, d_p)$ to denote a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_p$. The trace of a matrix $A$ is denoted by $\text{Tr}(A)$. For two real symmetric matrices $A$ and $B$, $A \succcurlyeq B$ ($A \succ B$) means that $A - B$ is positive semidefinite (definite). Similarly, $A \preccurlyeq B$ ($A \prec B$) means that $B - A$ is positive semidefinite (definite). All logarithms in this thesis are to the base 2. The notation $X \leftrightarrow Y \leftrightarrow Z$ means that $X, Y,$ and $Z$ form a Markov chain in this order. We use $\text{span}\{c_i\}_{i=1}^l$ to denote the subspace spanned by $\{c_i\}_{i=1}^l$. Likewise, $\text{span}\{A\}$ denotes the subspace spanned by the columns of $A$. 

15
2.2 Problem Formulation

Let $X$ and $Y$ be two generic zero-mean jointly Gaussian random vectors with covariance matrices $K_X$ and $K_Y$, respectively. The dimensions of $X$ and $Y$ are assumed to be $m$ and $k$, respectively. Let $\{(X^n(i), Y^n(i))\}_{i=1}^n$ be a sequence of i.i.d. random vectors with the distribution at a single stage being the same as that of the generic pair $(X,Y)$. As depicted in Fig. 1.1, encoder 1 observes $X^n$ and sends a message to the decoder using an encoding function

$$f_1^{(n)} : \mathbb{R}^{mn} \mapsto \{1, \ldots, M_1^{(n)}\}.$$  

Analogously, encoder 2 observes $Y^n$ and sends a message to the decoder using another encoding function

$$f_2^{(n)} : \mathbb{R}^{kn} \mapsto \{1, \ldots, M_2^{(n)}\}.$$  

The decoder uses both messages to estimate $X^n$ using a decoding function

$$g^{(n)} : \{1, \ldots, M_1^{(n)}\} \times \{1, \ldots, M_2^{(n)}\} \mapsto \mathbb{R}^{mn}.$$  

**Definition 1.** A rate-distortion vector $(R_1, R_2, D)$ is achievable for the vector Gaussian one-helper source-coding problem if there exist a block length $n$, encoding functions $f_1^{(n)}$ and $f_2^{(n)}$, and a decoding function $g^{(n)}$ such that

$$R_i \geq \frac{1}{n} \log M_i^{(n)} \text{ for all } i \in \{1, 2\}, \text{ and}$$

$$D \geq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \left( X^n(i) - \hat{X}^n(i) \right) \left( X^n(i) - \hat{X}^n(i) \right)^T \right],$$

where

$$\hat{X}^n \triangleq g^{(n)} \left( f_1^{(n)} (X^n), f_2^{(n)} (Y^n) \right).$$

Let $\mathcal{RD}$ be the set of all achievable rate-distortion vectors. Define

$$\mathcal{RD}(D) \triangleq \{(R_1, R_2) : (R_1, R_2, D) \in \mathcal{RD}\}.$$
We call \( \mathcal{R}(D) \) the rate region for the vector Gaussian one-helper source-coding problem.

Our goal is to characterize the rate region \( \mathcal{R}(D) \). Note that the matrix distortion constraint is more general in the sense that it subsumes other natural distortion constraints such as a finite number of upper bounds on the mean square error of reproductions of linear functions of the source. In particular, it subsumes the case in which the distortion constraint is on the mean square error of reproductions of the components of \( X \).

Since we are interested in a quadratic distortion constraint, without loss of generality we can restrict the decoding function to be the MMSE estimate of \( X^n \) based on the received messages. Therefore, \( \hat{X}^n \) can be written as

\[
\hat{X}^n = E \left[ X^n | f_1^n(X^n), f_2^n(Y^n) \right].
\]

The case in which \( K_X \not\preceq D \) has a trivial solution. In this case, the rate region is the entire nonnegative quadrant. So, we assume that \( K_X \not\preceq D \) does not hold in the rest of the thesis. This means that there exists a direction \( z \neq 0 \) such that

\[
z^T K_X z > z^T D z. \tag{2.1}
\]

### 2.3 Gaussian Achievable Scheme

In this section, we present a Gaussian achievable scheme, depicted in Fig. 1.2. The scheme is well-known and is often referred to as the Berger-Tung scheme [8, 9]. This scheme is known to be optimal for several Gaussian distributed source-coding problems [10, 11, 12, 13, 40, 43, 44]. However, it is not optimal in
some cases. For instance, a lattice-based scheme can outperform it if the goal is to reconstruct a hidden random vector that is jointly Gaussian with $X$ and $Y$ [39, 48]. The discrete memoryless version of the scheme is not optimal if the sources have common components [49]. For the problem under consideration however, the Berger-Tung scheme is indeed optimal. We present an overview of the scheme here. The details for similar problem setups can be found in [10, 11].

Let $S$ be the set of zero-mean jointly Gaussian random vectors $U$ and $V$ such that

(i) $U$, $X$, $Y$, and $V$ form a Markov chain $U \leftrightarrow X \leftrightarrow Y \leftrightarrow V$, and

(ii) $K_{X|U,V} \leq D$.

Consider any $(U, V) \in S$ and a large block length $n$. Let $R'_1 \triangleq I(X; U) + \epsilon$, where $\epsilon > 0$. To construct the codebook for encoder 1, first generate $2^{nR'_1}$ independent codewords $U^n$ randomly according to the marginal distribution of $U$, and then uniformly distribute them into $2^{nR'_1}$ bins. Encoder 2’s codebook is constructed by generating $2^{nR_2}$ independent codewords $V^n$ randomly according to the marginal distribution of $V$.

Given a source sequence $X^n$, encoder 1 looks for a codeword $U^n$ that is jointly typical with $X^n$, and sends the index $b$ of the bin to which $U^n$ belongs. Encoder 2, upon observing $Y^n$, sends the index of the codeword $V^n$ that is jointly typical with $Y^n$. The decoder receives the two indices, then looks into the bin $b$ for a codeword $U^n$ that is jointly typical with $V^n$. The decoder can recover $U^n$.
and $V^n$ with high probability as long as

$$R_1 \geq I(X;U|V) \quad \text{and}$$

$$R_2 \geq I(Y;V).$$

The decoder then computes the MMSE estimate of the source $X^n$ given the messages $U^n$ and $V^n$, and (ii) above guarantees that this estimate will satisfy the covariance matrix distortion constraint. Let

$$\mathcal{R}_G(D) \triangleq \{(R_1, R_2) : \text{there exists } (U, V) \in S \text{ such that}$$

$$R_1 \geq I(X;U|V) \quad \text{and}$$

$$R_2 \geq I(Y;V)\}. $$

It then follows that the Gaussian achievable scheme achieves $\mathcal{R}_G(D)$.

### 2.4 Main Result

**Theorem 1.** The Gaussian achievable scheme achieves the rate region for the vector Gaussian one-helper source-coding problem

$$\mathcal{R}(D) = \mathcal{R}_G(D).$$

It is immediate that

$$\mathcal{R}_G(D) \subseteq \mathcal{R}(D).$$

We prove the reverse inclusion (converse) in Chapters 3 and 4. The next section gives a nonrigorous overview of the argument.
2.5 Overview of the Converse Argument

The starting point of our proof is Oohama’s converse for the scalar-help-scalar case [11], which proceeds as follows. Let $f_1^{(n)}$ and $f_2^{(n)}$ be encoding functions and $g^{(n)}$ be a decoding function that achieve the rate-distortion vector $(R_1, R_2, D)$. Let $C_1 \triangleq f_1^{(n)}(X^n)$ and $C_2 \triangleq f_2^{(n)}(Y^n)$. By standard steps, we have

\[
R_2 \geq \log M_2^{(n)} \\
\geq H(C_2) \\
= I(Y^n; C_2), \text{ and}
\]

\[
R_1 \geq \log M_1^{(n)} \\
\geq H(C_1) \\
\geq H(C_1|C_2) \\
= I(X^n; C_1|C_2) \\
= I(X^n; C_1, C_2) - I(X^n; C_2).
\]

It follows that

\[
nR_1 \geq \inf_{C_1, C_2} I(X^n; C_1, C_2) - I(X^n; C_2) \\
\text{subject to } \sum_{i=1}^n E [ (X^n(i) - E[X^n(i)|C_1, C_2])^2 ] \leq nD, \quad (2.2)
\]

\[
I(Y^n; C_2) \leq nR_2, \text{ and}
\]

\[X^n \leftrightarrow Y^n \leftrightarrow C_2.
\]

Now this infimum can be lower bounded by separately optimizing each term

\[
nR_1 \geq \inf_{(C_1, C_2)} \frac{1}{\sum_{i=1}^n E [ (X^n(i) - E[X^n(i)|C_1, C_2])^2 ] \leq nD} I(X^n; C_1, C_2) \]

\[
- \sup_{C_2 : X^n \leftrightarrow Y^n \leftrightarrow C_2, I(Y^n; C_2) \leq nR_2} I(X^n; C_2). \quad (2.3)
\]
The first optimization problem,
\[
\inf_{(C_1, C_2)} I(X^n; C_1, C_2), \quad (2.4)
\]
which we call the *distortion problem*, can be solved using the entropy-maximizing property of the Gaussian distribution and the concavity of the logarithm. The second problem,
\[
\sup_{C_2 : X^n \leftrightarrow Y^n \leftrightarrow C_2, I(Y^n; C_2) \leq nR_2} I(X^n; C_2), \quad (2.5)
\]
which we call the *helper problem*, can be solved via the conditional version of the entropy power inequality [15]. Substituting these solutions into (2.3) yields exactly the $R_1$ achieved by the scheme from the previous section for the given $R_2$ and $D$. This completes Oohama’s converse proof for the scalar-help-scalar case.

The key to the proof is that separately minimizing the two terms in (2.2) does not decrease the objective. More precisely, for any pair $(C_1^*, C_2^*)$ that achieves the infimum in (2.2) we have
\[
I(X^n; C_1^*; C_2^*) = \inf_{(C_1, C_2)} I(X^n; C_1, C_2), \quad (2.6)
\]
and
\[
I(X^n; C_2^*) = \sup_{C_2 : X^n \leftrightarrow Y^n \leftrightarrow C_2, I(Y^n; C_2) \leq nR_2} I(X^n; C_2). \quad (2.7)
\]
Whenever (2.6) occurs, we shall say that the *distortion problem incurs no loss*. Whenever (2.7) occurs, we shall say that the *helper problem incurs no loss*.

It is not difficult to verify that this proof also works for the vector-help-scalar case, i.e., when $X$ is a scalar and $Y$ is a vector. In particular, both the distortion and helper problems incur no loss in this case. When both $X$ and $Y$ are vectors (vector-help-vector case), the proof breaks down in three places:
1. The distortion problem incurs a loss in general. For instance, if $D \preceq K_X$, then the distortion problem is solved by choosing $C_1$ and $C_2$ so that

$$\sum_{i=1}^{n} E \left[ (X^n(i) - E[X^n(i)|C_1, C_2]) (X^n(i) - E[X^n(i)|C_1, C_2])^T \right] = nD.$$ 

That is, the constraint is met with equality. For the original problem in (2.2), on the other hand, even if $D \preceq K_X$ we can only guarantee that

$$\sum_{i=1}^{n} E \left[ (X^n(i) - E[X^n(i)|C_1^*, C_2^*]) (X^n(i) - E[X^n(i)|C_1^*, C_2^*])^T \right] \preceq nD,$$

and equality may not hold. The lack of equality is easiest to see when $K_Y$ is poorly conditioned. If $K_Y$ has essentially one nonzero eigenvalue, then the helper will allocate all of its rate in the direction of the associated eigenvector. If $R_2$ is large, this could result in “overshooting” the distortion constraint in that direction.

2. The helper problem also incurs a loss in general. One way of seeing this is to note that if the goal is only to maximize the mutual information in (2.5), then one might choose $C_2$ to send information about a direction of $Y$ along which the distortion constraint $D$ is not active. This would necessarily deviate from the optimizer $C_2^*$ of the original problem.

3. The vector EPI does not solve the helper problem in general.

To address the first issue, observe that the distortion problem incurs no loss if the optimizers $C_1^*$ and $C_2^*$ for the original problem happen to meet the distortion constraint with equality, i.e., it holds that

$$\sum_{i=1}^{n} E \left[ (X^n(i) - E[X^n(i)|C_1^*, C_2^*]) (X^n(i) - E[X^n(i)|C_1^*, C_2^*])^T \right] = nD.$$ 

We show that it is possible to reduce the general case to this one by projecting the source and the distortion constraint in the directions in which the distortion constraint is met with equality for the candidate optimal scheme. We call
this process *distortion projection*. This addresses the first issue. For the scalar-help-vector case, the second and third issues do not arise, and hence *distortion projection* together with Oohama’s converse arguments is sufficient to solve the problem [20]. Chapter 3 contains the detailed solution for this case.

Liu and Viswanath [14] showed that the *channel enhancement* technique of Weingarten *et al.* [16] is sufficient to solve the helper problem in the vector case, thereby addressing the third issue. Their solution, however, is not sufficient to handle the second issue. Recently, Zhang [18] introduced a variation on *channel enhancement* called *source enhancement*. *Source enhancement* effectively replaces the original problem with a relaxation for which the helper problem incurs no loss and the vector EPI solves the helper problem, although Zhang does not describe it in this way. This addresses the second and third issues. Thus, it appears that *distortion projection*, *source enhancement*, and Oohama’s approach together should be sufficient to solve the vector-help-vector case of the problem. We show in Chapter 4 that this is indeed true.
CHAPTER 3
SCALAR-HELP-VECTOR SOLUTION

In this chapter, we complete the solution for the scalar-help-vector case of the problem. It requires a novel analysis technique *distortion projection*. In the end of this chapter, we study an extended problem in which there is a separate distortion constraint on each of the two sources. We provide an outer bound to the rate region of this extended problem.

Without loss of generality, we can write

\[ X = aY + N, \]

where \( a \) is a vector, and \( N \) is a zero-mean Gaussian random vector with the covariance matrix \( K_N \) and is independent of \( Y \). Therefore, we have

\[ K_X = aa^T \sigma_Y^2 + K_N. \]

If \( a = 0 \) or \( \sigma_Y^2 = 0 \), then the problem reduces to the point-to-point vector Gaussian rate-distortion problem, which can be solved using existing techniques. Therefore, we assume that \( a \neq 0 \) and \( \sigma_Y^2 > 0 \) in the rest of the chapter. We also assume that \( K_X \) and \( D \) are positive definite. The other cases of the problem can be reduced to this case by applying an invertible transformation. The reduction for the more general case of the problem (vector-help-vector case) is presented in Section 4.5.
3.1 Rate Region

Let us define the following set

$$
\mathcal{R}^*(D) \triangleq \left\{ (R_1, R_2) : R_1 \geq \min_{K} \frac{1}{2} \log \frac{|aa^T \sigma_Y^2 2^{-2R_2} + K_N|}{|K|} \right. \\
\text{s.t. } 0 \preceq K \preceq D \text{ and } \\
K \preceq aa^T \sigma_Y^2 2^{-2R_2} + K_N \right\}.
$$

Our main result in this case can be stated as

**Theorem 2.**

$$
\mathcal{R}(D) = \mathcal{R}_G(D) = \mathcal{R}^*(D).
$$

The second equality in Theorem 2 is proved later in Section 3.3 (problem \(P_G\) and Lemma 1). To prove the first equality, we have from the Gaussian achievable scheme that

$$
\mathcal{R}(D) \supseteq \mathcal{R}_G(D).
$$

So, it suffices to prove the reverse inclusion (converse), which is presented in Section 3.4. We next study an optimization problem that plays a crucial role in the proof.

3.2 Core Optimization Problem

Consider the point-to-point rate-distortion problem for a vector Gaussian source with a covariance matrix distortion constraint. In this setup, an i.i.d. zero-mean vector Gaussian source \(Z^n\) with the covariance matrix \(K_z\) is observed
by the encoder, which sends a message to the decoder over a rate-constrained channel using an encoding function

\[ f^{(n)} : \mathbb{R}^{mn} \mapsto \{1, \ldots, M^{(n)}\}. \]

The decoder uses the message to give an estimate \( \hat{Z}^n \) of the source \( Z^n \) such that

\[ \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( Z^n(i) - \hat{Z}^n(i) \right) \left( Z^n(i) - \hat{Z}^n(i) \right)^T \right] \precsim D_Z. \]

As mentioned in Section 2.2, we can assume without loss of generality that \( D_Z \) and \( K_Z \) are strictly positive definite and

\[ \hat{Z}^n = E [Z^n | f^{(n)}(Z^n)]. \]

In a single-letter form, the rate-distortion function of the source \( Z^n \) is given by the optimal value of the following optimization problem

\[
\min_{U} \quad I(Z; U)
\]

subject to \( K_{Z|U} \precsim D_Z. \)

Choosing \( U \) to be jointly Gaussian with \( Z \) is optimal for this problem due to the fact that Gaussian distribution maximizes differential entropy for a given covariance matrix [53, Theorem 8.6.5]. Thus this problem is equivalent to the following matrix optimization problem

\[
F(D_Z, K_Z) \triangleq \max \log |K_{Z|U}|
\]

subject to \( 0 \precsim K_{Z|U} \precsim D_Z \) and \( K_{Z|U} \precsim K_Z. \)

(3.1)

Note that we need to impose the constraint

\[ K_{Z|U} \precsim K_Z \]
because of the fact that the conditional covariance is no more than the unconditional covariance in a positive semidefinite sense. The remainder of this section is devoted to establishing certain properties that the optimal solution to this problem must satisfy. We shall establish these properties via the Karush-Kuhn-Tucker (KKT) conditions [50, p. 267]. One could also proceed by simultaneously diagonalizing $K_Z$ and $D_Z$, but we expect the KKT approach to be more useful for generalizations, as we will see in Chapter 4.

Because the objective of the optimization problem (3.1) is continuous and

$$\{0 \preceq K_{Z|U} \preceq D_Z \text{ and } K_{Z|U} \preceq K_Z\}$$

is a compact set, there exists an optimal solution $K_{Z|U}^*$ to (3.1). The Lagrangian associated with (3.1) is

$$\log |K_{Z|U}| + \text{Tr}\{K_{Z|U}\Lambda - (K_{Z|U} - D_Z)M_1 - (K_{Z|U} - K_Z)M_2\},$$

where $\Lambda$, $M_1$, and $M_2$ are positive semidefinite Lagrange multiplier matrices corresponding to the constraints $K_{Z|U} \succeq 0$, $K_{Z|U} \preceq D_Z$, and $K_{Z|U} \preceq K_Z$, respectively. Because (3.1) is convex and satisfies Slater’s condition [50, p. 265], there exist optimal dual matrices, $\Lambda^*$, $M_1^*$, and $M_2^*$, which together with $K_{Z|U}^*$ must satisfy the KKT conditions [50, p. 267]

$$K_{Z|U}^{*-1} + \Lambda^* - M_1^* - M_2^* = 0,$$  \hspace{1cm} (3.2)

$$K_{Z|U}^*\Lambda^* = 0,$$  \hspace{1cm} (3.3)

$$(D_Z - K_{Z|U}^*)M_1^* = 0,$$  \hspace{1cm} (3.4)

$$(K_Z - K_{Z|U}^*)M_2^* = 0,$$  \hspace{1cm} (3.5)

$$\Lambda^*, M_1^*, M_2^* \succeq 0.$$  \hspace{1cm} (3.6)

The condition (3.2) is obtained by setting the gradient of the Lagrangian with
respect to $K_{Z|U}$ equal to zero. Conditions (3.3) – (3.5) are complementary slackness conditions.

Observe that in the optimization problem (3.1), the constraint $K_{Z|U} \succeq 0$ is never active, so

$$\Lambda^* = 0.$$  \hspace{1cm} (3.7)

Since $\log |\cdot|$ is strictly convex over the domain of positive definite matrices, $K_{Z|U}^*$ is the unique maximizer of the problem. Let

$$Q \triangleq \{(M_1^*, M_2^*)\}$$

be the set of all pairs $(M_1^*, M_2^*)$ of Lagrange multiplier matrices that satisfy the KKT conditions. Consider any convergent sequence of pairs $(M_1^n, M_2^n)$ in $Q$. By the continuity of the KKT conditions, it follows that the limit of this sequence belongs to $Q$. Therefore, $Q$ is a closed set. Now (3.2) and (3.7) together imply that every pair $(M_1^*, M_2^*)$ in $Q$ is such that

$$M_1^* + M_2^* = K_{Z|U}^{*-1},$$

which is a fixed matrix. Hence, the set $Q$ is bounded. We thus conclude that $Q$ is a compact set. This along with the continuity of the $\text{Tr}(\cdot)$ function imply that there exists $(\bar{M}_1, \bar{M}_2)$ in $Q$ that solves the optimization problem

$$\min \text{ Tr}(M_1^*)$$

subject to \quad $(M_1^*, M_2^*) \in Q.$  \hspace{1cm} (3.8)

Since $\bar{M}_1$ and $\bar{M}_2$ are positive semidefinite, we can write their spectral decompositions as

$$\bar{M}_1 = \sum_{i=1}^{r} \lambda_i s_i s_i^T$$

and

$$\bar{M}_2 = \sum_{i=1}^{l} \gamma_i t_i t_i^T,$$  \hspace{1cm} (3.9) (3.10)
where

(a) $0 \leq r, l \leq m,$

(b) $\lambda_i, \gamma_j > 0,$ for all $i \in \{1, \ldots, r\}$ and for all $j \in \{1, \ldots, l\},$ and

(c) $\{s_i\}_{i=1}^r$ and $\{t_i\}_{i=1}^l$ are sets of orthonormal vectors.

Note that we allow $r$ and $l$ to be zero because $\bar{M}_1$ and $\bar{M}_2$ can be zero. We have from (3.4), (3.5), (3.9), and (3.10) that

\[(D_Z - K_{Z|U}^*) \sum_{i=1}^r \lambda_i s_i s_i^T = 0\]
\[(K_Z - K_{Z|U}^*) \sum_{i=1}^l \gamma_i t_i t_i^T = 0,\]

which imply that

\[(D_Z - K_{Z|U}^*) s_i = 0, \quad \forall i \in \{1, \ldots, r\}\]  \hspace{1cm} (3.11)
\[(K_Z - K_{Z|U}^*) t_i = 0, \quad \forall i \in \{1, \ldots, l\}.\]  \hspace{1cm} (3.12)

Define the matrices

\[S \triangleq \left[ \sqrt{\lambda_1} s_1, \sqrt{\lambda_2} s_2, \ldots, \sqrt{\lambda_r} s_r \right]\]
\[T \triangleq \left[ \sqrt{\gamma_1} t_1, \sqrt{\gamma_2} t_2, \ldots, \sqrt{\gamma_l} t_l \right].\]

Let $C$ be an $m \times m$ positive definite matrix.

**Definition 2.** A non-zero $m \times p$ matrix $E$ is $C$-orthogonal if $E^T CE$ is a diagonal matrix.

**Definition 3.** A non-zero $m \times p$ matrix $E$ and a non-zero $m \times q$ matrix $F$ are cross $C$-orthogonal if $E^T CF = 0.$
Definition 4. A non-zero vector \( w \) is in \( \text{span}\{c_i\}_{i=1}^l \) if there exist real numbers \( \{\gamma_i\}_{i=1}^l \) such that

\[
w = \sum_{i=1}^l \gamma_i c_i.
\]

We denote this as

\[
w \in \text{span}\{c_i\}_{i=1}^l.
\]

We have the following theorem about the optimal solution to the optimization problem (3.1).

Theorem 3. (a) If \( r > 0 \), then \( S^T(K_Z - K_Z^*|U)\) is strictly positive definite.

(b) \( [S, T] \) is square and invertible.

(c) \( [S, T] \) is \( K_Z^*|U \)-orthogonal with

\[
[S, T]^T K_Z^*|U [S, T] = I_m.
\]

(d) \( S \) is \( D_Z \)-orthogonal with

\[
S^T D_Z S = I_r.
\]

(e) \( T \) is \( K_Z \)-orthogonal with

\[
T^T K_Z T = I_l.
\]

(f) \( S \) and \( T \) are cross \( D_Z \)-orthogonal.

(g) \( S \) and \( T \) are cross \( K_Z \)-orthogonal.

Proof. For part (a), it suffices to show that \( S^T(K_Z - K_Z^*|U)S \) is non-singular. Suppose otherwise that it is singular. Then there exists \( 0 \neq e \in \mathbb{R}^r \) such that

\[
e^T S^T(K_Z - K_Z^*|U)Se = 0.
\]
Let \( w \triangleq Se \). We then have
\[
w = \sum_{i=1}^{r} e_i s_i, 
\]
where \( e_i \) is the \( i \)-th component of \( e \), and
\[
(K_Z - K_{Z|U}^* \right) w = 0. \tag{3.14}
\]
Let
\[
\lambda_{\min} \triangleq \min \{ \lambda_1, \lambda_2, \ldots, \lambda_r \} \tag{3.15}
\]
and pick any \( \epsilon \) such that
\[
0 < \epsilon \leq \frac{\lambda_{\min}}{\|w\|^2}. \tag{3.16}
\]
Consider any \( 0 \neq z \in \mathbb{R}^m \). Let \( z_S \) be the projection of \( z \) on \( \text{span}\{S\} \) and \( z_{S\perp} \) be the projection of \( z \) on the space orthogonal to \( \text{span}\{S\} \). We can then write
\[
z = z_S + z_{S\perp}.
\]
Using this, we obtain
\[
z^T \bar{M}_1 z = (z_S + z_{S\perp})^T \bar{M}_1 (z_S + z_{S\perp})
\]
\[
= z_S^T \bar{M}_1 z_S + z_S^T \bar{M}_1 z_{S\perp} + z_{S\perp}^T \bar{M}_1 z_S + z_{S\perp}^T \bar{M}_1 z_{S\perp}
\]
\[
= z_S^T \bar{M}_1 z_S, \tag{3.17}
\]
where (3.17) follows because \( \bar{M}_1 z_{S\perp} = 0 \). Similarly,
\[
w^T z = w^T z_S. \tag{3.18}
\]
We now have

\[ z^T (\bar{M}_1 - \epsilon w w^T) z = z^T \bar{M}_1 z - \epsilon (w^T z)^2 \]

\[ = z_S^T \bar{M}_1 z_S - \epsilon (w^T z_S)^2 \quad (3.19) \]

\[ \geq z_S^T \bar{M}_1 z_S - \epsilon \|w\|^2 \|z_S\|^2 \quad (3.20) \]

\[ = \sum_{i=1}^{r} \lambda_i (s_i^T z_S)^2 - \epsilon \|w\|^2 \|z_S\|^2 \quad (3.21) \]

\[ \geq \lambda_{\min} \sum_{i=1}^{r} (s_i^T z_S)^2 - \epsilon \|w\|^2 \|z_S\|^2 \quad (3.22) \]

\[ = \lambda_{\min} \|z_S\|^2 - \epsilon \|w\|^2 \|z_S\|^2 \]

\[ = \|z_S\|^2 (\lambda_{\min} - \epsilon \|w\|^2) \quad (3.23) \]

\[ \geq 0, \quad (3.24) \]

where

(3.19) follows from (3.17) and (3.18),

(3.20) follows from the Cauchy-Schwartz Inequality,

(3.21) follows from (3.9),

(3.22) follows from (3.15),

(3.23) follows because

\[ \|z_S\|^2 = \sum_{i=1}^{r} (s_i^T z_S)^2, \] and

(3.24) follows from (3.16).

This proves that \( \bar{M}_1 - \epsilon w w^T \) is a positive semidefinite matrix. Let us define the matrices

\[ \bar{M}_1 \triangleq \bar{M}_1 - \epsilon w w^T \] and

\[ \bar{M}_2 \triangleq \bar{M}_2 + \epsilon w w^T. \]
We then have

(i) \( \tilde{M}_1, \tilde{M}_2 \succ 0, \)

(ii) \( \tilde{M}_1 + \tilde{M}_2 = \bar{M}_1 + \bar{M}_2 = K_{Z|U}^* \)

(iii) \[
\begin{align*}
(D_z - K_{Z|U}^*) \tilde{M}_1 &= (D_z - K_{Z|U}^*) (\bar{M}_1 - \epsilon ww^T) \\
&= (D_z - K_{Z|U}^*) \tilde{M}_1 - (D_z - K_{Z|U}^*) \epsilon ww^T \\
&= 0 - (D_z - K_{Z|U}^*) \epsilon \sum_{i,j=1}^r e_ie_js_is_j^T \tag{3.25}
\end{align*}
\]

where

(3.25) follows from (3.4) and (3.13), and

(3.26) follows from (3.11), and

(iv) \[
\begin{align*}
(K_z - K_{Z|U}^*) \tilde{M}_2 &= (K_z - K_{Z|U}^*) (\bar{M}_2 + \epsilon ww^T) \\
&= (K_z - K_{Z|U}^*) \bar{M}_2 + (K_z - K_{Z|U}^*) \epsilon ww^T \\
&= 0, \tag{3.27}
\end{align*}
\]

where (3.27) follows from (3.5) and (3.14).

Now (i) through (iv) above imply that \((\tilde{M}_1, \tilde{M}_2)\) is feasible for the optimization problem (3.8). However,

\[
\text{Tr}(\tilde{M}_1) = \text{Tr}(\bar{M}_1) - \text{Tr}(\epsilon ww^T) < \text{Tr}(\bar{M}_1),
\]
which is a contradiction to the assumption that \((\bar{M}_1, \bar{M}_2)\) solves the problem (3.8). Therefore, \(S^T (K_Z - K_{Z|U}^*) S\) is non-singular.

The proof of part (b) is similar to that of part (a). We first show by contradiction that columns of \([S, T]\) are linearly independent. Suppose otherwise that they are linearly dependent. Since columns of both \(S\) and \(T\) are linearly independent, there exists

\[
0 \neq w = \sum_{i=1}^{r} a_i s_i = \sum_{i=1}^{l} b_i t_i,
\]

where \(a_i\)'s and \(b_i\)'s are real numbers. This means that

\[
w \in \text{span}\{S\}
\]

and

\[
w \in \text{span}\{T\}.
\]

Pick any \(\epsilon\) such that

\[
0 < \epsilon \leq \frac{\lambda_{\min}}{\|w\|^2}.
\]

Then as proved in part (a), \(\bar{M}_1 - \epsilon ww^T\) is a positive semidefinite matrix and the matrices \(\bar{M}_1\) and \(\bar{M}_2\) defined as before satisfy (i) through (iii) above. Moreover, we have

\[
(\bar{K}_Z - K_{Z|U}^*) \bar{M}_2 = (K_Z - K_{Z|U}^*) (M_2 + \epsilon ww^T)
\]

\[
= (K_Z - K_{Z|U}^*) \bar{M}_2 + (K_Z - K_{Z|U}^*) \epsilon \sum_{i,j=1}^{l} b_i b_j t_i t_j^T
\]

\[
= 0,
\]

where (3.28) follows from (3.5) and (3.12). We again conclude that \((\bar{M}_1, \bar{M}_2)\) is feasible for the problem (3.8), and hence as before we have arrived at a contradiction. Therefore, columns of \([S, T]\) are linearly independent, i.e.

\[
r + l \leq m.
\]
Next (3.2), (3.7), (3.9), and (3.10) together yield

\[ K_{\mathbf{Z}^*|\mathbf{U}}^{-1} = \mathbf{M}_1 + \mathbf{M}_2 = \sum_{i=1}^{r} \lambda_i \mathbf{s}_i \mathbf{s}_i^T + \sum_{i=1}^{l} \gamma_i \mathbf{t}_i \mathbf{t}_i^T, \]  

(3.30)

which implies that

\[ r + l \geq m, \]  

(3.31)

because otherwise \( K_{\mathbf{Z}^*|\mathbf{U}}^{-1} \) will be singular. Now (3.29) and (3.31) imply that

\[ r + l = m, \]

which means that \([\mathbf{S}, \mathbf{T}]\) is a square matrix, and is therefore invertible because it has linearly independent columns.

For part (c), on post-multiplying (3.30) by \( K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \), we obtain

\[ \mathbf{s}_1 = \sum_{i=1}^{r} \lambda_i \mathbf{s}_i \left( \mathbf{s}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \right) + \sum_{i=1}^{l} \gamma_i \mathbf{t}_i \left( \mathbf{t}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \right), \]

which can be re-written as

\[ \mathbf{s}_1 \left( 1 - \lambda_1 \left( \mathbf{s}_1^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \right) \right) - \sum_{i=2}^{r} \lambda_i \mathbf{s}_i \left( \mathbf{s}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \right) = \sum_{i=1}^{l} \gamma_i \mathbf{t}_i \left( \mathbf{t}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 \right). \]  

(3.32)

Since columns of \([\mathbf{S}, \mathbf{T}]\) are linearly independent from part (b), coefficients of all vectors in (3.32) must be zero, i.e.

\[ \lambda_1 \mathbf{s}_1^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 = 1, \]

\[ \mathbf{s}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 = 0, \; \forall i \in \{2, \ldots, r\}, \]  

and

\[ \mathbf{t}_i^T K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_1 = 0, \; \forall i \in \{1, \ldots, l\}. \]

Likewise, on post-multiplying (3.30) by \( K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_2, \ldots, K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{s}_r, \; K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{t}_1 \ldots, K_{\mathbf{Z}^*|\mathbf{U}}^* \mathbf{t}_l \) and then equating all coefficients to zero, we obtain similar equations. In sum-
mary,

\[ \lambda_i s_i^T K_{Z|U}^* s_i = 1, \quad \forall i \in \{1, \ldots, r\}, \]
\[ \gamma_i t_i^T K_{Z|U}^* t_i = 1, \quad \forall i \in \{1, \ldots, l\}, \]
\[ s_i^T K_{Z|U}^* s_j = 0, \quad \forall i, j \in \{1, \ldots, r\}, i \neq j, \]
\[ t_i^T K_{Z|U}^* t_j = 0, \quad \forall i, j \in \{1, \ldots, l\}, i \neq j, \quad \text{and} \]
\[ s_i^T K_{Z|U}^* t_j = 0, \quad \forall i \in \{1, \ldots, r\}, \forall j \in \{1, \ldots, l\}, \]

which imply that

\[ [S, T]^T K_{Z|U}^* [S, T] = I_m. \quad (3.33) \]

Hence, \([S, T]\) is \(K_{Z|U}^*\)-orthogonal.

For parts (d) through (g), we have from (3.11) and (3.12) that

\[ D_Z S = K_{Z|U}^* S \quad \text{and} \]
\[ K_Z T = K_{Z|U}^* T, \]

which along with (3.33) imply

\[ S^T D_Z S = S^T K_{Z|U}^* S = I_r, \]
\[ T^T K_Z T = T^T K_{Z|U}^* T = I_l, \]
\[ T^T D_Z S = T^T K_{Z|U}^* S = 0, \quad \text{and} \]
\[ S^T K_Z T = S^T K_{Z|U}^* T = 0. \]

This completes the proof of Theorem 3. \(\square\)

It is clear from Theorem 3 that \(\text{span}\{S\}\) is the set of directions in which the encoder sends information until the distortion constraint is met with equality. Furthermore, \(\text{span}\{T\}\) is the set of directions in which the encoder sends no
information and hence the $K_Z$ constraint is met with equality in such directions. Note that if $S$ is an empty matrix, then the rate-distortion function is zero. We shall now use these properties to solve an optimization problem, which is at the center of our main result in this case.

### 3.3 Converse Ingredients

Let us define the main optimization problem $(P)$ as

$$
(P) \triangleq \min_{U, V} I(X; U|V)
$$

subject to $R_2 \geq I(Y; V)$,

$$
D \succeq K_{X|U,V}, \text{ and }
$$

$$
X \leftrightarrow Y \leftrightarrow V.
$$

where $X$, $Y$, and $D$ are defined as before. We have the following theorem.

**Theorem 4.** A Gaussian $(U, V)$ is an optimal solution of the main optimization problem $(P)$.

We prove this theorem in the remainder of the section. Let us first restrict the solution space to Gaussian distributions. This results in an optimization problem $(P_G)$ over the conditional covariance matrix $K_{X|U,V}$ and the conditional
variance $\sigma^2_{Y|V}$. Formally, it can be defined as

$$(P_G) \triangleq \min_{K_{X|U,V}, \sigma^2_{Y|V}} \frac{1}{2} \log \frac{|K_{X|V}|}{|K_{X|U,V}|}$$

subject to $R_2 \geq \frac{1}{2} \log \frac{\sigma^2_Y}{\sigma^2_{Y|V}}$, $D \succeq K_{X|U,V} \succeq 0$, and $K_{X|V} \succeq K_{X|U,V}$,

where

$$K_{X|V} = aa^T \sigma^2_{Y|V} + K_N.$$ 

Since restricting the solution space to Gaussian distributions can only increase the optimal value of the main optimization problem $(P)$, we immediately have

$$v(P_G) \geq v(P),$$

where $v(P)$ is the optimal value of the optimization problem $(P)$. We use similar notation to denote optimization problems and their optimal values in the rest of the thesis. It is therefore sufficient to prove the reverse inequality

$$v(P_G) \leq v(P).$$

We can rewrite $(P_G)$ as

$$(P_G) \triangleq \min_{\sigma^2_{Y|V}} \frac{1}{2} \log |K_{X|V}| - \frac{1}{2} v(F(D, K_{X|V}))$$

subject to $R_2 \geq \frac{1}{2} \log \frac{\sigma^2_Y}{\sigma^2_{Y|V}}$, (3.34)

which is a double optimization problem. Note that for a fixed $\sigma^2_{Y|V}$, the inner optimization problem is exactly $F(D, K_{X|V})$, which was defined in (3.1).

Since $(P_G)$ has a continuous objective and a compact feasible set, there exists an optimal solution $(K_{X|U^*, V^*}, \sigma^2_{Y|V^*})$ to it, where $U^*$ and $V^*$ represent the
corresponding optimal Gaussian solution. The following lemma states that it is optimal for encoder 2 to use all $R_2$ bits for sending a message to the decoder.

**Lemma 1.** We can assume without loss of optimality that $(U^*, V^*) \in S$, and

$$
\sigma_{Y|V^*}^2 = \sigma_Y^2 2^{-2R_2}.
$$

(3.35)

**Proof.** See Appendix A.1. $\square$

Lemma 1 and (3.34) imply that the optimal value of $(P_G)$ is

$$
v(P_G) = \frac{1}{2} \log |K_{X|V^*}| - \frac{1}{2} v\left(F\left(D, K_{X|V^*}\right)\right),
$$

(3.36)

where

$$
K_{X|V^*} = aa^T \sigma_Y^2 2^{-2R_2} + K_N,
$$

(3.37)

and $K_{X|U^*, V^*}$ is optimal for the problem $F\left(D, K_{X|V^*}\right)$ with the optimal value

$$
v\left(F\left(D, K_{X|V^*}\right)\right) = \frac{1}{2} \log |K_{X|U^*, V^*}|.
$$

(3.38)

As discussed in Section 3.2, $K_{X|U^*, V^*}$ gives two sets of directions $S$ and $T$ which satisfy the properties in Theorem 3. On substituting (3.38) into (3.36), we obtain

$$
v(P_G) = \frac{1}{2} \log \begin{vmatrix} K_{X|V^*} \end{vmatrix} - \frac{1}{2} \log |K_{X|U^*, V^*}|
$$

(3.39)

$$
= \frac{1}{2} \log \left|\begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} K_{X|V^*} \begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} \right| - \frac{1}{2} \log |K_{X|U^*, V^*}|
$$

(3.40)

$$
= \frac{1}{2} \log \left|\begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} K_{X|V^*} \begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} \right| - \frac{1}{2} \log |I_m|
$$

(3.41)

$$
= \frac{1}{2} \log \left|\begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} K_{X|V^*} \begin{vmatrix} S \end{vmatrix} \begin{vmatrix} T \end{vmatrix} \right| - \frac{1}{2} \log |I_l|
$$

(3.42)
where

(3.39) follows because \([S, T]\) is invertible from Theorem 3(b),

(3.40) follows because \([S, T]\) is \(K_{X|U^* Y^*}\)-orthogonal from Theorem 3(c), and

(3.41) follows because \(T\) is \(K_{X|Y^*}\)-orthogonal, and \(S\) and \(T\) are cross \(K_{X|Y^*}\)-orthogonal from Theorem 3(e) and 3(g), respectively.

### 3.3.1 Distortion Projection

The optimal Gaussian solution to \((P_G)\) suggests a way to lower bound \((P)\) by projecting the main source \(X\) on \(S\) and imposing the distortion constraint on the subspace spanned by the columns of \(S\). Note that the distortion constraint is tight on this subspace for the optimal Gaussian solution. We refer to this method of lower bounding \((P)\) as distortion projection.

The projected optimization problem \((\tilde{P})\) is now defined as

\[
(\tilde{P}) \triangleq \min_{U, V} I(S^T X; U|V) \\
\text{subject to } R_2 \geq I(Y; V),
\]

\[
S^T D S \succeq S^T K_{X|U^* Y^*} S, \text{ and } \\
S^T X \leftrightarrow Y \leftrightarrow V.
\]

We next show that the main optimization problem \((P)\) is lower bounded by the projected optimization problem \((\tilde{P})\). Since \([S, T]\) is invertible from Theorem 3(b)
and the mutual information is non-negative, we have

\[ I(X; U|V) = I \left( [S, T]^T X; U|V \right) \]
\[ = I \left( S^T X, T^T X; U|V \right) \]
\[ = I \left( S^T X; U|V \right) + I \left( T^T X; U|V, S^T X \right) \]
\[ \geq I \left( S^T X; U|V \right) . \]  \hfill (3.43)

Now any \((U, V)\) satisfying

\[ D \succcurlyeq K_{X|U,V} \text{ and} \]
\[ X \leftrightarrow Y \leftrightarrow V \]

also satisfies

\[ S^T D S \succcurlyeq S^T K_{X|U,V} S \text{ and} \]
\[ S^T X \leftrightarrow Y \leftrightarrow V . \]

Therefore, the feasible set of \((P)\) is contained in that of \((\bar{P})\). Moreover, (3.43) above implies that the objective value of \((P)\) is no less than that of \((\bar{P})\). We hence conclude that the \textit{projected optimization problem} \((\bar{P})\) lower bounds the \textit{main optimization problem} \((P)\), i.e.

\[ v(P) \geq v(\bar{P}). \]  \hfill (3.44)

3.3.2 Oohama’s Approach

We now apply Oohama’s approach [11] on \((\bar{P})\). The objective of \((\bar{P})\) can be decomposed as

\[ I \left( S^T X; U|V \right) = I \left( S^T X; U, V \right) - I \left( S^T X; V \right) . \]  \hfill (3.45)
Using this, we define two subproblems that are used to lower bound the projected optimization problem ($\tilde{P}$). The first subproblem ($\tilde{P}_1$) minimizes the first mutual information in the right-hand-side of (3.45) subject to the distortion constraint in ($\tilde{P}$) and the second subproblem ($\tilde{P}_2$) maximizes the second mutual information in the right-hand-side of (3.45) subject to the rate constraint and the Markov condition in ($\tilde{P}$). In other words, ($\tilde{P}_1$) is defined as

$$
(\tilde{P}_1) \triangleq \min_{U,V} I(S^T X; U, V)
$$

subject to $S^T D S \succeq S^T K_{X|U,V} S$,

and ($\tilde{P}_2$) is defined as

$$
(\tilde{P}_2) \triangleq \max_V I(S^T X; V)
$$

subject to $R_2 \geq I(Y; V)$ and

$$
S^T X \leftrightarrow Y \leftrightarrow V.
$$

It is clear from the decomposition in (3.45) and from the definitions of ($\tilde{P}$), ($\tilde{P}_1$), and ($\tilde{P}_2$) that ($\tilde{P}_1$) and ($\tilde{P}_2$) lower bound ($\tilde{P}$), i.e.

$$v(\tilde{P}) \geq v(\tilde{P}_1) - v(\tilde{P}_2). \quad (3.46)$$

We now give two lemmas about the optimal solutions to subproblems ($\tilde{P}_1$) and ($\tilde{P}_2$).

**Lemma 2.** A Gaussian ($U, V$) with the conditional covariance matrix $K_{X|U,V}$ is optimal for the subproblem ($\tilde{P}_1$), and the optimal value is

$$v(\tilde{P}_1) = \frac{1}{2} \log |S^T K_{X|U,V} S|.$$

**(Proof.** See Appendix A.2. \qed

42
Lemma 3. A Gaussian $V$ with the conditional variance $\sigma_{Y|V}^2$ is optimal for the sub-problem $(\tilde{P}_2)$, and the optimal value is

$$v(\tilde{P}_2) = \frac{1}{2} \log \frac{|S^T K_X S|}{|S^T K_{X|V} S|}.$$  \hspace{1cm} (3.48)

Proof. See Appendix A.3. \hfill \square

On substituting (3.47) and (3.48) into (3.46), we obtain

$$v(\tilde{P}) \geq \frac{1}{2} \log |S^T K_{X|V} S|.$$  \hspace{1cm} (3.49)

Now (3.42), (3.44), and (3.49) together yield

$$v(P) \geq v(P_G),$$

which proves that a Gaussian $(U, V)$ is optimal for the main optimization problem $(P)$.

3.4 Converse Proof of the Main Result

Liu and Viswanath gave a single-letter outer bound to the rate region in [14]. We shall use a similar outer bound that is reminiscent of the Berger-Tung outer bound [8, 9].

Lemma 4. If $(R_1, R_2, D)$ is achievable, then there exist $U$ and $V$ such that

$$R_1 \geq I(X; U|V),$$

$$R_2 \geq I(Y; V),$$

$$D \succ K_{X|U,V}, \text{ and}$$

$$X \leftrightarrow Y \leftrightarrow V.$$
Proof. See Appendix A.4.

We are now ready to prove the converse of the first equality in Theorem 2. Suppose \((R_1, R_2, D)\) is achievable. Then

\[
R_1 \geq v(P) \quad (3.50)
\]

\[
= v(P_G) \quad (3.51)
\]

\[
= \frac{1}{2} \log |K_{X|V^*}| - \frac{1}{2} v \left( F(D, K_{X|V^*}) \right) \quad (3.52)
\]

\[
= \min_{K} \frac{1}{2} \log \left| \frac{aa^T \sigma_Y^2 2^{-2R_2} + K_N}{|K|} \right|
\]

\[
\text{s.t. } 0 \preceq K \preceq D \text{ and } \quad (3.53)
\]

\[
K \preceq aa^T \sigma_Y^2 2^{-2R_2} + K_N,
\]

where

(3.50) follows from Lemma 4,

(3.51) follows from Theorem 4,

(3.52) follows from (3.36), and

(3.53) follows from the definition of \( F(\cdot, \cdot) \) and (3.37).

And if \((R_1, R_2, D) \in \hat{\mathcal{R}}\), then (3.53) again holds because (3.52) is continuous in \((R_2, D)\). This completes the proof of the first equality in Theorem 2.

Remark 3.1: It follows from Theorem 2 and Lemma 1 that one can add the constraints

\[
U \leftrightarrow X \leftrightarrow Y \leftrightarrow V \quad \text{and}
\]

\((U, V, X, Y)\) are jointly Gaussian

to the main optimization problem \((P)\) without changing its optimal value.
3.5 Extension to Two Constraints

In this section, we extend the scalar-help-vector case of the problem by considering a separate distortion constraint on each of the two sources. The formulation of the problem is similar to that of Section 2.2. Now, however, the decoder uses the received messages from the encoders to estimate both $X^n$ and $Y^n$ using the decoding functions

$$g_1^{(n)} : \{1, \ldots, M_1^{(n)}\} \times \{1, \ldots, M_2^{(n)}\} \mapsto \mathbb{R}^{mn} \text{ and }$$
$$g_2^{(n)} : \{1, \ldots, M_1^{(n)}\} \times \{1, \ldots, M_2^{(n)}\} \mapsto \mathbb{R}^n,$$

respectively.

**Definition 5.** A rate-distortion vector $(R_1, R_2, D, d)$ is achievable for this extended source-coding problem if there exist a block length $n$, encoding functions $f_1^{(n)}$ and $f_2^{(n)}$, and decoding functions $g_1^{(n)}$ and $g_2^{(n)}$ such that

$$R_i \geq \frac{1}{n} \log M_i^{(n)} \text{ for all } i \in \{1, 2\},$$
$$D \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( X^n(i) - \hat{X}^n(i) \right) \left( X^n(i) - \hat{X}^n(i) \right)^T \right], \text{ and }$$
$$d \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ (Y^n(i) - \hat{Y}^n(i))^2 \right],$$

where

$$\hat{X}^n \triangleq g_1^{(n)} \left( f_1^{(n)}(X^n), f_2^{(n)}(Y^n) \right)$$
$$= E \left[ X^n | f_1^{(n)}(X^n), f_2^{(n)}(Y^n) \right], \text{ and }$$

$$\hat{Y}^n \triangleq g_2^{(n)} \left( f_1^{(n)}(X^n), f_2^{(n)}(Y^n) \right)$$
$$= E \left[ Y^n | f_1^{(n)}(X^n), f_2^{(n)}(Y^n) \right].$$

Let $\mathcal{RD}'$ be the set of all achievable rate-distortion vectors. Define

$$\mathcal{R}(D, d) \triangleq \left\{ (R_1, R_2) : (R_1, R_2, D, d) \in \mathcal{RD}' \right\}.$$
We call $\mathcal{R}(D, d)$ the rate region for this extended source-coding problem with two distortion constraints.

We can assume without loss of generality that the components $(X_1, \ldots, X_m)$ of $X$ and $Y$ are standard normal, and $(X_1, Y)$ is independent of $(X_2, \ldots, X_m)$. Starting from any problem, we can obtain an equivalent problem with this structure by applying an invertible transformation [51, 52]. This can be done as follows. Let

$$w_1, w_2, \ldots, w_m$$

be an orthonormal basis in $\mathbb{R}^m$ starting at

$$w_1 \triangleq \frac{1}{\rho} \left( \sigma_Y K_X^{-1/2} a \right),$$

where

$$\rho \triangleq \| \sigma_Y K_X^{-1/2} a \|$$

is the 2-norm of $\sigma_Y K_X^{-1/2} a$. Define the matrices

$$W \triangleq [w_1, w_2, \ldots, w_m] \quad \text{and} \quad T_X \triangleq W^T K_X^{-1/2}.$$

Then the transformation is given by

$$\bar{X} \triangleq T_X X \quad \text{and} \quad \bar{Y} \triangleq \frac{1}{\sigma_Y} Y.$$

The covariance matrix of $\bar{X}$ is

$$K_{\bar{X}} = T_X K_X T_X^T$$

$$= W^T K_X^{-1/2} K_X K_X^{-1/2} W$$

$$= W^T W$$

$$= I_m,$$
and the cross-covariance between $\tilde{X}$ and $\tilde{Y}$ is

$$K_{\tilde{X}\tilde{Y}} = \frac{1}{\sigma_Y} T_X K_{XY}$$

$$= \frac{1}{\sigma_Y} \left( W^T K_{X}^{-1/2} \right) \left( \sigma_Y^2 a \right)$$

$$= W^T \left( \sigma_Y K_{X}^{-1/2} a \right)$$

$$= W^T (\rho w_1)$$

$$= (\rho, 0, \ldots, 0)^T.$$

Since $T_X$ is invertible, the equivalent distortion constraints are

$$T_X D T_X^T \succeq 1_n n \sum_{i=1}^n E \left[ \left( \tilde{X}^n(i) - \hat{\tilde{X}}^n(i) \right) \left( \tilde{X}^n(i) - \hat{\tilde{X}}^n(i) \right)^T \right]$$

and

$$d \sigma_Y^2 \geq \frac{1}{n} \sum_{i=1}^n E \left[ \left( \tilde{Y}^n(i) - \hat{\tilde{Y}}^n(i) \right)^2 \right].$$

Since the above transformation is invertible, it does not incur any information loss. We therefore have an equivalent structured problem. So in the rest of the section, we will assume that our original problem has this structure with $\rho$ being the correlation coefficient between $X_1$ and $Y$.

### 3.5.1 An Outer Bound

First note that if there is no distortion constraint between $Y^n$ and $\hat{Y}^n$, then the problem reduces to the scalar-help-vector case, and hence we have

$$R(D, d) \subseteq R^*(D). \quad (3.54)$$

This bound is tight for large $R_2$ because the distortion constraint between $Y^n$ and $\hat{Y}^n$ is always satisfied for large $R_2$. One can consider the vector-help-scalar relaxation of the problem in which there is no distortion constraint between $X^n$
and \( \hat{X}^n \), and obtain another outer bound to the rate region. However, the outer bound thus obtained is not tight in general even for large \( R_1 \). This is because the optimal solution to this relaxed problem is such that encoder 1 sends information about \( X_1 \) only. The rest of the components of \( X \) are ignored and therefore the distortion constraint between \( X^n \) and \( \hat{X}^n \) is not satisfied in general. We obtain an improved outer bound by retaining some of the constraints imposed on \( X^n \). Specifically, we separately impose the constraints on \( X_1 \) and \((X_2, \ldots, X_m)\).

Due to the assumed independence of \( X_1, X_2, \ldots, X_m \), these two constraints decouple. The first can be handled using the results of [11] and [13] while the second can be handled using point-to-point rate-distortion theory.

Let us denote \((X_2, \ldots, X_m)^T\) by \( \bar{X} \) and write

\[
D = \begin{pmatrix}
D_1 & b^T \\
b & \bar{D}
\end{pmatrix},
\]

where \( D_1 \) is a positive number, \( b \) is a \((m - 1)\)-dimensional vector, and \( \bar{D} \) is a \((m - 1) \times (m - 1)\) positive definite matrix. Let \( \bar{R}_1(\bar{D}) \) be the point-to-point rate-distortion function of the source \( \bar{X} \) under a covariance matrix distortion constraint \( \bar{D} \). Then from the discussion in Section 3.2, we have

\[
\bar{R}_1(\bar{D}) = -\frac{1}{2} v\left(F(\bar{D}, I_{m-1})\right) = -\frac{1}{2} \log |\bar{D}^*|,
\]

(3.55)

where \( \bar{D}^* \) is the optimal solution to the problem \( F(\bar{D}, I_{m-1}) \) defined in (3.1).

Define the sets

\[
\mathcal{R}^*_2(d) \triangleq \left\{(R_1, R_2) : R_2 \geq \frac{1}{2} \log^+ \left[\frac{1}{d} \left(1 - \rho^2 + \rho^2 2^{-2(R_1 - \bar{R}_1(\bar{D}))}\right)\right]\right\},
\]

and

\[
\mathcal{R}^*_{\text{sum}}(D_1, d) \triangleq \left\{(R_1, R_2) : R_1 - \bar{R}_1(\bar{D}) + R_2 \geq \frac{1}{2} \log^+ \left[\frac{(1 - \rho^2) \beta(D_1, d)}{2D_1 d}\right]\right\},
\]

48
where
\[
\beta(D_1, d) \triangleq 1 + \sqrt{1 + \frac{4\rho^2 D_1 d}{(1 - \rho^2)^2}}.
\]

We have the following outer bound.

**Theorem 5.**

\[
\mathcal{R}(D, d) \subseteq \mathcal{R}^*(D) \cap \mathcal{R}^*_2(d) \cap \mathcal{R}^*_\text{sum}(D_1, d).
\] (3.56)

**Proof.** Consider \((R_1, R_2) \in \mathcal{R}(D, d)\). Let \(C_1 \triangleq f_1^{(n)}(X^n)\) and \(C_2 \triangleq f_2^{(n)}(Y^n)\). Then

\[
nR_2 \geq \log M_2^{(n)}
\geq H(C_2)
\geq H(C_2|C_1)
\geq I(Y^n; C_2|C_1)
= I(Y^n; C_2) - I(Y^n; C_1)
\geq I(Y^n; \hat{Y}^n) - I(Y^n; C_1), \quad \text{and} \quad (3.57)
\]

\[
nR_1 \geq \log M_1^{(n)}
\geq H(C_1)
\geq I(X^n; C_1)
= I(X^n_1; C_1) + I(\bar{X}^n; C_1|X^n_1). \quad (3.58)
\]

We can lower bound the second mutual information in (3.58) as follows

\[
I(\bar{X}^n; C_1|X^n_1) = I(\bar{X}^n; C_1, X^n_1)
= I(\bar{X}^n; C_1, X^n_1, Y^n)
= I(\bar{X}^n; C_1, C_2, X^n_1, Y^n)
\geq I(\bar{X}^n; C_1, C_2)
\geq I(\bar{X}^n, \hat{X}^n), \quad (3.60)
\]
where (3.59) follows because
\[ \bar{X}^n \leftrightarrow (C_1, X_1^n) \leftrightarrow Y^n. \]

Define the following optimization problem
\[
\min_{C_1} \frac{1}{n} I \left( X^n; \hat{X}^n \right)
\]
subject to
\[
\frac{1}{n} \sum_{i=1}^{n} E \left[ \left( \bar{X}^n(i) - \hat{X}^n(i) \right) \left( \bar{X}^n(i) - \hat{X}^n(i) \right)^T \right] \preceq \bar{D}. \tag{3.61}
\]

This is the multi-letter form of the point-to-point rate-distortion problem for the source \( \bar{X}^n \) under a covariance matrix distortion constraint \( \bar{D} \). Therefore, from the discussion in Section 3.2, a Gaussian \( C_1 \) is optimal for this problem and from (3.55), the optimal conditional covariance matrix is \( \bar{D}^* \) and the optimal value is \( \bar{R}_1(\bar{D}) \). From (3.57), (3.58), (3.60), and the definition of the optimization problem (3.61), we conclude that \((R_1, R_2)\) satisfies
\[
R_2 \geq \frac{1}{n} I(Y^n; \hat{Y}^n) - \frac{1}{n} I(Y^n; C_1),
\]
\[
R_1 - \bar{R}_1(\bar{D}) \geq \frac{1}{n} I(X^n_1; C_1), \quad \text{and}
\]
\[
d \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( Y^n(i) - \hat{Y}^n(i) \right)^2 \right].
\]

By invoking Oohama’s lower bounding technique [11] next, we obtain
\[
R_2 \geq \frac{1}{2} \log^+ \left[ \frac{1}{d} \left( 1 - \rho^2 + \rho^2 2^{-2(R_1 - \bar{R}_1(\bar{D}))} \right) \right],
\]
which implies that
\[
(R_1, R_2) \in \mathcal{R}_2^*(d). \tag{3.62}
\]
We now proceed to lower bound the sum-rate.

\[ n(R_1 + R_2) \geq H(C_1, C_2) \]

\[ \geq I(X^n, Y^n; C_1, C_2) \]

\[ = I(X_1^n, Y^n; C_1, C_2) + I(\bar{X}_1^n; C_1, C_2| X_1^n, Y^n) \]

\[ = I(X_1^n, Y^n; C_1, C_2) + I(\bar{X}_1^n; C_1| X_1^n) \]

\[ = I(X_1^n, Y^n; C_1, C_2) + I(\bar{X}_1^n; \hat{X}_n), \quad (3.63) \]

where (3.63) follows from (3.60). The sum-rate can be lower bounded further by minimizing two mutual informations in (3.63) separately subject to separate distortion constraints. Using the sum-rate lower bounding technique by Wagner et al. [13], the first mutual information is minimized subject to the distortion constraints \( D_1 \) and \( d \) on the sources \( X_1^n \) and \( Y^n \), respectively. We omit the details to avoid repetition. Minimizing the second mutual information subject to the covariance matrix distortion constraint \( \bar{D} \) is the optimization problem (3.61) again. We therefore have that

\[ R_1 - \bar{R}_1(D) + R_2 \geq \frac{1}{2} \log^+ \left[ \frac{(1 - \rho^2) \beta(D_1, d)}{2D_1 d} \right], \]

which implies that

\[ (R_1, R_2) \in \mathcal{R}_{\text{sum}}^*(D_1, d). \quad (3.64) \]

Now (3.54), (3.62), and (3.64) together establish (3.56).

\[ \square \]

### 3.5.2 Tightness of the Outer Bound

We will prove that the boundary of the rate region \( \mathcal{R}(D, d) \) partially coincides with the boundary of \( \mathcal{R}^*(D) \) in general, coincides with the outer bound (3.56)
completely if $b = 0$, and partially coincides with the boundary of $\mathcal{R}^*_2(d)$ if $b \neq 0$ and a condition holds. Let

$$R^*_1 \triangleq \inf \left\{ R_1 : R_1 > R_1(\hat{D}) + \frac{1}{2} \log \frac{1}{D_1} \quad \text{and} \quad \hat{D} - D^* \gtrsim \frac{bb^T}{D_1 - 2^{-2(R_1 - R_1(\hat{D}))}} \right\}.$$ 

We have the following lemma.

**Lemma 5.** (a) There exists $R^*_2 \geq 0$ such that

$$\mathcal{R}(D, d) \cap \{ R_2 \geq R^*_2 \} = \mathcal{R}^*(D) \cap \{ R_2 \geq R^*_2 \}. \quad (3.65)$$

(b) If $b = 0$, then

$$\mathcal{R}(D, d) = \mathcal{R}^*(D) \cap \mathcal{R}^*_2(d) \cap \mathcal{R}^*_{\text{sum}}(D_1, d). \quad (3.66)$$

(c) If $b \neq 0$ and $R^*_1 < \infty$, then

$$\mathcal{R}(D, d) \cap \{ R_1 \geq R^*_1 \} = \mathcal{R}^*_2(d) \cap \{ R_1 \geq R^*_1 \}. \quad (3.67)$$

**Proof.** As explained in Section 2.3, the optimal scheme depicted in Fig. 1.2 is such that encoder 2 vector quantizes its observation using a Gaussian test channel as in point-to-point rate-distortion theory. So, the average distortion between $Y^n$ and $\hat{Y}^n$ decreases as $R_2$ increases. Hence, there exists a nonnegative number $R^*_2$ such that for any $R_2 \geq R^*_2$, the distortion constraint between $Y^n$ and $\hat{Y}^n$ is satisfied and therefore the region

$$\mathcal{R}^*(D) \cap \{ R_2 \geq R^*_2 \}$$

is achievable for the problem. This along with the outer bound (3.54) prove the equality in (3.65).
For part (b), we will first prove that if $b = 0$, i.e. $D$ is block diagonal, then

$$\mathcal{R}^*(D) = \left\{(R_1, R_2) : R_1 - \bar{R}_1(\bar{D}) \geq \frac{1}{2} \log^+ \left[ \frac{1}{D_1} (1 - \rho^2 + \rho^2 2^{-2R_2}) \right] \right\}. \quad (3.68)$$

The optimization problem in the definition of $\mathcal{R}^*(D)$ is

$$\min_K \frac{1}{2} \log |\frac{aa^T \sigma_Y^2 2^{-2R_2} + K_N}{K}|$$

subject to $0 \preceq K \preceq D$ and

$$K \preceq aa^T \sigma_Y^2 2^{-2R_2} + K_N. \quad (3.69)$$

Since our problem has a special structure as explained above, we have

$$aa^T \sigma_Y^2 2^{-2R_2} + K_N = \text{Diag} \{1 - \rho^2 + \rho^2 2^{-2R_2}, 1, \ldots, 1\}.$$

Consider any feasible

$$K = \begin{pmatrix} K_1 & c^T \\ c & \bar{K} \end{pmatrix},$$

where $K_1$ is a positive number, $c$ is a $(m - 1)$-dimensional non-zero vector, and $\bar{K}$ is a $(m - 1) \times (m - 1)$ positive definite matrix. Let us define

$$\bar{K} \triangleq \begin{pmatrix} K_1 & 0 \\ 0 & \bar{K} \end{pmatrix}.$$

We then have

$$|K| = |\bar{K}| |(K_1 - c^T \bar{K}^{-1} c)| < |\bar{K}| K_1 = |\bar{K}|.$$

Therefore, without loss of optimality we can restrict the feasible solutions to be of the following form

$$\bar{K} = \begin{pmatrix} K_1 & 0 \\ 0 & \bar{K} \end{pmatrix}.$$
The restricted feasible set

\[
\{ 0 \preceq \tilde{K} \preceq D \text{ and } \tilde{K} \preceq \text{Diag}\{1 - \rho^2 + \rho^2 2^{-2R_2}, 1, \ldots, 1\}\}
\]

is equivalent to

\[
\{ 0 \preceq K_1 \preceq \min(D_1, 1 - \rho^2 + \rho^2 2^{-2R_2}), 0 \preceq \tilde{K} \preceq D, \text{ and } \tilde{K} \preceq I_{m-1}\}.
\]

The objective of (3.69) can be decomposed as

\[
\frac{1}{2} \log \frac{1 - \rho^2 + \rho^2 2^{-2R_2}}{|\tilde{K}|} = \frac{1}{2} \log \frac{1 - \rho^2 + \rho^2 2^{-2R_2}}{K_1} + \frac{1}{2} \log \frac{1}{\tilde{K}}.
\]

Therefore, its optimal value equals the sum of the optimal values of subproblems

\[
\min_{K} \frac{1}{2} \log \frac{1}{|K|},
\]

subject to \(0 \preceq K \preceq D\) and

\[
\tilde{K} \preceq I_{m-1},
\]

and

\[
\min_{K_1} \frac{1}{2} \log \frac{1 - \rho^2 + \rho^2 2^{-2R_2}}{K_1},
\]

subject to \(0 \leq K_1 \leq \min(D_1, 1 - \rho^2 + \rho^2 2^{-2R_2})\).

The first subproblem is the point-to-point rate-distortion problem for the source \(X\) under a distortion constraint \(D\), and hence its optimal value is \(\bar{R}_1(D)\). The optimal value of the second subproblem is

\[
\frac{1}{2} \log^+ \left[ \frac{1}{D_1} \left( 1 - \rho^2 + \rho^2 2^{-2R_2} \right) \right].
\]

Thus, the optimal value of the optimization problem (3.69) is

\[
\bar{R}_1(D) + \frac{1}{2} \log^+ \left[ \frac{1}{D_1} \left( 1 - \rho^2 + \rho^2 2^{-2R_2} \right) \right],
\]

54
which proves the inequality in (3.68). It is now easy to verify that the outer bound (3.56) coincides with the shifted boundary of the rate region of the scalar Gaussian two-encoder source-coding problem [13], where the shift is by the amount \( \bar{R}_1(D) \) in the direction of \( R_1 \) axis. So, by using the point-to-point rate-distortion optimal code for the source \( \bar{X} \) in conjunction with the separation-based optimal scheme for the sources \( X_1 \) and \( Y \) [13], we can achieve the outer bound. We therefore have the equality in (3.66).

For part (c), it suffices to show that if the conditions in Lemma 5(c) hold, then

\[
\begin{pmatrix}
2^{-2(R_1^* - \bar{R}_1(D))} & 0 \\
0 & \bar{D}^*
\end{pmatrix} \preceq \begin{pmatrix}
D_1 & b^T \\
b & \bar{D}
\end{pmatrix} = D. \tag{3.70}
\]

This will imply that the region

\[ \mathcal{R}_2^*(d) \cap \{R_1 \geq R_1^*\} \]

is achievable for the problem by using a scheme in which the source \( \bar{X} \) is encoded and decoded as in the point-to-point rate-distortion theory, and the sources \( X_1 \) and \( Y \) are encoded and decoded using the scalar version of the Gaussian scheme of Section 2.3, treating \( Y \) as the main source and \( X_1 \) as the helper. Consider any

\[ 0 \neq x \triangleq \begin{pmatrix}
y \\
z
\end{pmatrix} \in \mathbb{R}^m. \]
Then
\[
x^T \left[ D - \begin{pmatrix} 2^{-2(R_1^*-R_1(D))} & 0 \\ 0 & D^* \end{pmatrix} \right] x
= \begin{pmatrix} y \\ z^T \end{pmatrix} \begin{pmatrix} D_1 - 2^{-2(R_1^*-R_1(D))} & b^T \\ b & \bar{D} - \bar{D}^* \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}
= y^2 \left( D_1 - 2^{-2(R_1^*-R_1(D))} \right) + 2y(z^T b) + z^T (\bar{D} - \bar{D}^*) z
= \left( D_1 - 2^{-2(R_1^*-R_1(D))} \right) \left( y + \frac{z^T b}{D_1 - 2^{-2(R_1^*-R_1(D))}} \right)^2
+ z^T \left( D - D^* - \frac{bb^T}{D_1 - 2^{-2(R_1^*-R_1(D))}} \right) z
\geq 0,
\]
where (3.71) and 3.72) follow from the conditions in Lemma 5(c). This implies that (3.70) holds, and hence we have the equality in (3.67).
3.5.3 Numerical Example

To illustrate the outer bound in Theorem 5, we study a numerical example and compare the outer bound with the inner bound achieved by the Gaussian scheme. Let us consider a two-dimensional problem in which

\[ \rho = 0.7, \]
\[ d = 0.5, \quad \text{and} \]
\[ D = \begin{pmatrix} 0.5 & 0.05 \\ 0.05 & 0.5 \end{pmatrix}. \]

Fig. 3.1 shows the inner and outer bounds for this problem. The bold black plot is our outer bound, which is obtained by the intersection of three bounds in (3.56). It is evident that a portion of the outer bound obtained by the scalar-help-vector relaxation of the problem coincides with the inner bound.
CHAPTER 4
VECTOR-HELP-VECTOR SOLUTION

In this chapter, we present the solution for the vector-help-vector case of the problem. We can assume without loss of generality\(^1\) that

\[ X = Y + N, \]

where \( N \) is a zero-mean Gaussian random vector with the covariance matrix \( K_N \) and is independent of \( Y \). For now, we assume that \( K_X, K_Y, \) and \( D \) are positive definite. The general case of the problem will be addressed in Section 4.5.

4.1 Rate Region

The rate region \( R(D) \) is a closed convex set in the nonnegative quadrant. It is closed by definition and is convex because any convex combination of two points in the rate region is in the rate region as it can be achieved by time-sharing between the encoding and decoding strategies of the two points. Therefore, we can characterize it completely by its supporting hyperplane, which can be expressed as the following optimization problem

\[ R(D, \mu) \triangleq \inf_{(R_1, R_2) \in R(D)} \mu R_1 + R_2, \]

\(^1\)Since \( X \) and \( Y \) are jointly Gaussian, we can write

\[ X = AY + N, \]

where \( A \) is an \( m \times k \) matrix and \( N \) is an \( m \)-dimensional zero-mean Gaussian random vector that is independent of \( Y \). Since there is no distortion constraint on \( Y \), and \( AY \) is the sufficient statistic of \( X \) in \( Y \) (i.e., \( X \leftrightarrow Y \leftrightarrow AY \) and \( X \leftrightarrow AY \leftrightarrow Y \)), we can relabel \( AY \) with \( Y \) and write

\[ X = Y + N. \]
where $\mu$ is a nonnegative real number. Let us define

$$R^*(D, \mu) \triangleq \begin{cases} v(P_{pt}) & \text{if } 0 \leq \mu \leq 1 \\ v(P_{G1}) & \text{if } \mu > 1, \end{cases}$$

where $v(P_{pt})$ and $v(P_{G1})$ respectively are the optimal values of the optimization problems $(P_{pt})$ and $(P_{G1})$, which are defined as

$$(P_{pt}) \triangleq \min_{K_X|U} \frac{\mu}{2} \log \frac{|K_X|}{|K_X|U}$$

subject to $K_X \succeq K_X|U \succ 0$ and $D \succeq K_X|U$,

and

$$(P_{G1}) \triangleq \min_{K_Y|V, K_X|U, V} \frac{\mu}{2} \log \frac{|K_Y| + K_N}{|K_X|U, V} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y|V}$$

subject to $K_Y \succeq K_Y|V \succ 0$, $K_Y|V + K_N \succeq K_X|U, V \succ 0$, and $D \succeq K_X|U, V$.

Let us define the weighted sum rate that is achievable by the Gaussian achievable scheme as

$$R_G(D, \mu) \triangleq \min_{(R_1, R_2) \in \mathcal{R}_G(D)} \mu R_1 + R_2.$$ 

In terms of the weighted sum rate, our main result is as follows

**Theorem 6.**

$$R(D, \mu) = R_G(D, \mu) = R^*(D, \mu).$$

The second equality in the theorem is proved in Appendix B.1. To prove the first equality, we have from the Gaussian achievable scheme that

$$R(D, \mu) \leq R_G(D, \mu).$$
So, it suffices to prove the reverse inequality (converse). Since the proof of it rather is long, we divide it into sections. In the next section, we study the optimization problem \((P_{G_1})\) in the definition of \(R^*(D, \mu)\) and establish several properties that its optimal solution satisfies. We use these properties in Section 4.3 to prove the main result needed for the converse. We finally complete the proof of the first equality in Theorem 6 in Section 4.4.

### 4.2 Properties of the Optimal Gaussian Solution

Consider the optimization problem \((P_{G_1})\), and note first that the constraints

\[
\begin{align*}
K_{Y|V} \succeq 0 & \quad \text{and} \\
K_{X|U,V} \succeq 0
\end{align*}
\]

are never active because otherwise the objective value is infinite. Therefore we can ignore these constraints in the study of the problem. Now, instead of studying \((P_{G_1})\) directly as it is, we study an equivalent formulation. This formulation is also implicit in [18]. Note that if \(K_{Y|V}\) and \(K_{X|U,V}\) are feasible for \((P_{G_1})\), then there exist two positive semidefinite matrices \(B_1\) and \(B_2\) such that

\[
\begin{align*}
K_{Y|V} &= K_Y - B_2, \\
K_{X|U,V} &= K_{Y|V} + K_N - B_1 \\
&= K_Y - B_2 + K_N - B_1 \\
&= K_X - B_1 - B_2, \quad \text{and} \\
K_X - B_1 - B_2 &\preceq D.
\end{align*}
\]
Therefore, \((P_{G1})\) is equivalent to the following problem

\[
(P_{G2}) \triangleq \min_{B_1, B_2} \frac{\mu}{2} \log \frac{|K_X - B_2|}{|K_X - B_1 - B_2|} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y - B_2|}
\]

subject to \(B_i \geq 0\) for all \(i \in \{1, 2\}\), and

\[
D \geq K_X - B_1 - B_2.
\]

We next establish important properties that the optimal solution to \((P_{G2})\) satisfies.

Since \((P_{G2})\) has continuous objective and a compact feasible set, there exists an optimal solution \((B^*_1, B^*_2)\) to it. The Lagrangian of the problem is [50, Sec. 5.9.1]

\[
\frac{\mu}{2} \log \frac{|K_X - B_2|}{|K_X - B_1 - B_2|} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y - B_2|} - \text{Tr}(B_1 M_1 + B_2 M_2 - (K_X - B_2 - D) \Lambda),
\]

where \(M_1, M_2, \) and \(\Lambda\) are positive semidefinite Lagrange multiplier matrices corresponding to the constraints \(B_1 \geq 0, B_2 \geq 0, \) and \(D \geq K_X - B_1 - B_2,\) respectively. The KKT conditions for this problem are [50, Sec. 5.9.2]

\[
\frac{\mu}{2} (K_X - B_1^* - B_2^*)^{-1} = \Lambda^* + M_1^*, \quad (4.1)
\]

\[
\frac{\mu}{2} (K_X - B_1^* - B_2^*)^{-1} - \frac{\mu}{2} (K_X - B_2^*)^{-1} + \frac{1}{2} (K_Y - B_2^*)^{-1} = \Lambda^* + M_2^*, \quad (4.2)
\]

\[
B_i^* M_i^* = 0, \forall i \in \{1, 2\} \quad (4.3)
\]

\[
(K_X - B_1^* - B_2^* - D) \Lambda^* = 0, \quad \text{and} \quad (4.4)
\]

\[
M_1^*, M_2^*, \Lambda^* \geq 0, \quad (4.5)
\]

where \(M_1^*, M_2^*,\) and \(\Lambda^*\) are optimal Lagrange multiplier matrices. Conditions (4.1) and (4.2) respectively are obtained by setting gradients of the objective with respect to \(B_1\) and \(B_2\) to zero. Conditions (4.3) and (4.4) are the slackness con-
ditions on the Lagrange multiplier matrices. We next establish that these KKT conditions must hold at \((B_1^*, B_2^*)\).

**Lemma 6.** There exist matrices \(M_1^*, M_2^*,\) and \(\Lambda^*\) that satisfy the KKT conditions (4.1) – (4.5).

**Proof.** See Appendix B.2. \(\square\)

Let us define

\[
\Delta^* \triangleq \Lambda^* - \frac{\mu}{2} \left[ (K_X - B_1^* - B_2^*)^{-1} - (K_X - B_2^*)^{-1} \right].
\]

It follows from conditions (4.1) and (4.2) that

\[
\Delta^* = \frac{\mu}{2} (K_X - B_2^*)^{-1} - M_1^* = \frac{1}{2} (K_X - B_2^*)^{-1} - M_2^*.
\] \hspace{1cm} (4.6)

We have the following lemma.

**Lemma 7.** \(\Delta^*\) is a nonzero positive semidefinite matrix.

**Proof.** See Appendix B.3. \(\square\)

If \(\Delta^*\) happens to be positive definite, then *distortion projection* turns out to be unnecessary. To handle the case in which \(\Delta^*\) is singular, we shall use *distortion projection*. Since \(\Delta^*, M_1^*,\) and \(M_2^*\) are positive semidefinite, we can write their spectral decompositions as

\[
\Delta^* = \sum_{i=1}^{r} \lambda_i s_i s_i^T, \quad \text{(4.7)}
\]

\[
M_1^* = \sum_{i=1}^{p} \alpha_i a_i a_i^T, \quad \text{and} \hspace{1cm} \text{(4.8)}
\]

\[
M_2^* = \sum_{i=1}^{q} \beta_i b_i b_i^T, \quad \text{(4.9)}
\]

where
(i) \(0 < r \leq m\),
(ii) \(0 \leq p, q \leq m\),
(iii) \(\lambda_i > 0\) for all \(i \in \{1, \ldots, r\}\),
(iv) \(\alpha_i > 0\) for all \(i \in \{1, \ldots, p\}\),
(v) \(\beta_i > 0\) for all \(i \in \{1, \ldots, q\}\), and
(vi) \(\{s_i\}_{i=1}^r, \{a_i\}_{i=1}^p, \text{ and } \{b_i\}_{i=1}^q\) are sets of orthonormal vectors.

Note that we allow \(p\) and \(q\) to be zero because \(M_1^*\) and \(M_2^*\) can be zero. Since (4.6) implies that
\[
\Delta^* + M_1^* = \frac{\mu}{2}(K_X - B_2^*)^{-1} \succ 0 \quad \text{and} \quad \Delta^* + M_2^* = \frac{1}{2}(K_Y - B_2^*)^{-1} \succ 0,
\]
we must have that
\[
r + p \geq m \quad \text{and} \quad r + q \geq m.
\]
This means that if \(r + p = m\), then \(s_1, s_2, \ldots, s_r, a_1, a_2, \ldots, a_p\) must be linearly independent. Similarly, if \(r + q = m\), then \(s_1, s_2, \ldots, s_r, b_1, b_2, \ldots, b_q\) must be linearly independent.

Define the matrix
\[
S \triangleq \left[ \sqrt{\lambda_1}s_1, \sqrt{\lambda_2}s_2, \ldots, \sqrt{\lambda_r}s_r \right].
\]
It now follows from the definition of \(\Delta^*\) that
\[
\Lambda^* \succ \Delta^* = SS^T
\]
because
\[(K_X - B_1^* - B_2^*)^{-1} \succcurlyeq (K_X - B_2^*)^{-1}.\]

This and (4.4) imply that
\[(K_X - B_1^* - B_2^* - D)S = 0. \quad (4.10)\]

In addition to Definitions 2 through 4, we use the following definitions in this chapter. Let \(\{C_1, C_2, \ldots, C_t\}\) be a set of \(m \times m\) positive definite matrices.

**Definition 6.** A non-zero \(m \times p\) matrix \(E\) is \(\{C_1, C_2, \ldots, C_t\}\)-orthogonal if it is \(C_i\)-orthogonal for all \(i \in \{1, 2, \ldots, t\}\).

**Definition 7.** A non-zero \(m \times p\) matrix \(E\) and a non-zero \(m \times q\) matrix \(F\) are cross \(\{C_1, C_2, \ldots, C_t\}\)-orthogonal if they are cross \(C_i\)-orthogonal for all \(i \in \{1, 2, \ldots, t\}\).

We have the following theorem about the optimal solution to the optimization problem \((P_{G_2})\).

**Theorem 7.** There exist two matrices
\[
T \triangleq [t_1, t_2, \ldots, t_{m-r}]
\]
and
\[
W \triangleq [w_1, w_2, \ldots, w_{m-r}]
\]
such that \([S, T]\) and \([S, W]\) are invertible and if \(r < m\) then

(a) \(t_1, t_2, \ldots, t_{m-r} \in \text{span}\{a_i\}_{i=1}^p\),

(b) \(T\) is \(\{(K_X - B_2^*), (K_X - B_1^* - B_2^*)\}\)-orthogonal with
\[
T^T(K_X - B_2^*)T = T^T(K_X - B_1^* - B_2^*)T,
\]

64
(c) \( S \) and \( T \) are cross \( \{D, (K_X - B_2^*), (K_X - B_1^* - B_2^*)\} \)-orthogonal,

(d) \( w_1, w_2, \ldots, w_{m-r} \in \text{span}\{b_i\}_{i=1}^q\),

(e) \( W \) is \( \{K_Y, (K_Y - B_2^*)\} \)-orthogonal with

\[
W^T K_Y W = W^T (K_Y - B_2^*) W, \quad \text{and}
\]

(f) \( S \) and \( W \) are cross \( \{K_Y, (K_Y - B_2^*)\} \)-orthogonal.

Proof. It suffices to consider \( r < m \) case. Since \( \Delta^* = SS^T \) is rank deficient in this case, there exists \( z_1 \neq 0 \) such that

\[
S^T z_1 = 0.
\]

Let us define

\[
t_1 \triangleq (K_X - B_2^*)^{-1} z_1.
\]

Therefore

\[
S^T (K_X - B_2^*) t_1 = 0.
\]

We have from (4.6), (4.7), and (4.8) that

\[
\frac{\mu}{2} (K_X - B_2^*)^{-1} = \Delta^* + M_1^* = SS^T + \sum_{i=1}^p \alpha_i a_i a_i^T.
\]

On post-multiplying this by \( (K_X - B_2^*) t_1 \), we obtain

\[
\frac{\mu}{2} t_1 = SS^T (K_X - B_2^*) t_1 + \sum_{i=1}^p \alpha_i a_i (K_X - B_2^*) t_1
\]

\[
= \sum_{i=1}^p \alpha_i a_i (a_i^T (K_X - B_2^*) t_1).
\]

This proves that

\[
t_1 \in \text{span}\{a_i\}_{i=1}^p.
\]

We next show that

\[
t_1 \notin \text{span}\{s_i\}_{i=1}^r.
\]
Suppose otherwise that
\[ t_1 \in \text{span}\{s_i\}_{i=1}^r. \]
Then there exist real numbers \( \{c_i\}_{i=1}^r \) such that
\[ t_1 = \sum_{i=1}^r c_i s_i. \]
Since \( S^T(K_X - B_2^*)t_1 = 0 \), we have
\[ s_i^T(K_X - B_2^*)t_1 = 0 \quad \text{for all} \quad i \in \{1, 2, \ldots, r\}. \]
On multiplying this by \( c_i \) and then summing over all \( i \) in \( \{1, 2, \ldots, r\} \), we obtain
\[ t_1^T(K_X - B_2^*)t_1 = 0, \]
which is a contradiction because \( K_X - B_2^* \) is positive definite. We therefore have that
\[ t_1 \notin \text{span}\{s_i\}_{i=1}^r. \]
We have shown so far that there exists \( t_1 \in \text{span}\{a_i\}_{i=1}^p \) such that the rank of \( [S, t_1] \) is \( r + 1 \) and
\[ S^T(K_X - B_2^*)t_1 = 0. \]
Let us now assume that there exists
\[ T_j \triangleq [t_1, t_2, \ldots, t_j], \]
where
\[ t_1, t_2, \ldots, t_j \in \text{span}\{a_i\}_{i=1}^p \]
and \( 1 \leq j < m - r \) such that the rank of \( [S, T_j] \) is \( r + j \),
\[ S^T(K_X - B_2^*)T_j = 0, \]
and
\[ t_k^T(K_X - B_2^*)t_l = 0 \]
for all $k \neq l$ in $\{1, 2, \ldots, j\}$. Then there exists $z_{j+1} \neq 0$ such that

$$[S, T_j]^T z_{j+1} = 0.$$  

Let us define

$$t_{j+1} \triangleq (KX - B_2^*)^{-1} z_{j+1}.$$  

We therefore have that

$$[S, T_j]^T (KX - B_2^*) t_{j+1} = 0.$$  

It can be shown as before that

$$t_{j+1} \in \text{span}\{a_i\}_{i=1}^p$$  

and

$$t_{j+1} \notin \text{span}\{\{s_i\}_{i=1}^r, \{t_k\}_{k=1}^j\}.$$  

Hence, the rank of $[S, T_{j+1}]$, where

$$T_{j+1} \triangleq [T_j, t_{j+1}],$$  

is $r + j + 1$,

$$S^T (KX - B_2^*) T_{j+1} = 0,$$  

and

$$t_k^T (KX - B_2^*) t_l = 0,$$  

for all $k \neq l$ in $\{1, 2, \ldots, j + 1\}$. It now follows from the mathematical induction that there exist

$$t_1, t_2, \ldots, t_{m-r} \in \text{span}\{a_i\}_{i=1}^p$$  

such that if we define

$$T \triangleq [t_1, t_2, \ldots, t_{m-r}],$$
then \([S, T]\) is invertible,

\[ S^T(K_X - B_2^*)T = 0, \quad \text{and} \]
\[ T^T(K_X - B_2^*)T = G, \]

where

\[ G \triangleq \text{Diag}\left\{ (t_1^T(K_X - B_2^*)t_1), (t_2^T(K_X - B_2^*)t_2), \ldots, (t_{m-r}^T(K_X - B_2^*)t_{m-r}) \right\}. \]

Since \(B_1^*T = 0\) from (4.3) and \((K_X - B_1^* - B_2^*)S = DS\) from (4.10), we immediately have that

\[ S^T(K_X - B_2^*)T = S^T(K_X - B_1^* - B_2^*)T = S^TDT = 0 \quad \text{and} \]
\[ T^T(K_X - B_2^*)T = T^T(K_X - B_1^* - B_2^*)T = G. \]

This completes the proof of parts (a) through (c) of the theorem.

For parts (d) through (f), we have from (4.6), (4.7), and (4.9) that

\[ \frac{1}{2}(K_Y - B_2^*)^{-1} = \Delta^* + M_2^* = SS^T + \sum_{i=1}^{q} \beta_i b_i b_i^T. \]

Similar to the previous case, we can find

\[ w_1, w_2, \ldots, w_{m-r} \in \text{span}\{b_i\}_{i=1}^{q} \]

such that if we define

\[ W \triangleq [w_1, w_2, \ldots, w_{m-r}], \]

then \([S, W]\) is invertible,

\[ S^T(K_Y - B_2^*)W = 0, \quad \text{and} \]
\[ W^T(K_Y - B_2^*)W = H, \]

where

\[ H \triangleq \text{Diag}\left\{ (w_1^T(K_Y - B_2^*)w_1), (w_2^T(K_Y - B_2^*)w_2), \ldots, (w_{m-r}^T(K_Y - B_2^*)w_{m-r}) \right\}. \]
Since $B_2^*W = 0$ from (4.3), we conclude

$$S^T K_Y W = S^T (K_Y - B_2^*) W = 0 \quad \text{and} \quad W^T K_Y W = W^T (K_Y - B_2^*) W = H.$$  

This completes the proof of parts (d) through (f) of the theorem. \qed

We have the following corollary of Theorem 7.

**Corollary 1.** If $r < m = r + p$, then we can set

$$t_i = \sqrt{\alpha_i} a_i$$

for all $i$ in $\{1, 2, \ldots, p\}$. Similarly, if $r < m = r + q$, then we can set

$$w_i = \sqrt{\beta_i} b_i$$

for all $i$ in $\{1, 2, \ldots, q\}$.

**Proof.** Let $r < m = r + p$ and let us set

$$t_i = \sqrt{\alpha_i} a_i$$

for all $i$ in $\{1, 2, \ldots, p\}$ in the definition of $T$. We have from (4.6), (4.7), and (4.8) that

$$\frac{\mu}{2} (K_X - B_2^*)^{-1} = \sum_{i=1}^{p} \lambda_i s_i s_i^T + \sum_{i=1}^{p} \alpha_i a_i a_i^T. \quad (4.11)$$

Now, on post-multiplying (4.11) by $(K_X - B_2^*) s_1$, we obtain

$$\frac{\mu}{2} s_1 = \sum_{i=1}^{p} \lambda_i s_i (s_i^T (K_X - B_2^*) s_1) + \sum_{i=1}^{p} \alpha_i a_i (a_i^T (K_X - B_2^*) s_1).$$
which can be re-written as
\[
\begin{align*}
\mathbf{s}_1 \left( \frac{\mu}{2} - \lambda_1 \left( \mathbf{s}_1^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 \right) \right) - \sum_{i=2}^{r} \lambda_i \mathbf{s}_i \left( \mathbf{s}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 \right) \\
= \sum_{i=1}^{p} \alpha_i \mathbf{a}_i \left( \mathbf{a}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 \right). 
\end{align*}
\]
(4.12)

Since \([\mathbf{S}, \mathbf{T}]\) is invertible from (4.11), its columns are linearly independent. Hence, the coefficients of all vectors in (4.12) must be zero. Therefore,
\[
\lambda_1 \mathbf{s}_1^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 = \frac{\mu}{2},
\]
\[
\mathbf{s}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 = 0, \quad \forall i \in \{2, \ldots, r\}, \quad \text{and}
\]
\[
\mathbf{a}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_1 = 0, \quad \forall i \in \{1, \ldots, p\}.
\]

Likewise, on post-multiplying (4.11) by \((\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_2, \ldots, (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_r, (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{a}_1, \ldots, (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{a}_p\) and then equating all coefficients to zero, we obtain similar equations. In summary,
\[
\lambda_i \mathbf{s}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_i = \frac{\mu}{2}, \quad \forall i \in \{1, \ldots, r\},
\]
\[
\alpha_i \mathbf{a}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{a}_i = \frac{\mu}{2}, \quad \forall i \in \{1, \ldots, p\},
\]
\[
\mathbf{s}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{s}_j = 0, \quad \forall i, j \in \{1, \ldots, r\}, i \neq j,
\]
\[
\mathbf{a}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{a}_j = 0, \quad \forall i, j \in \{1, \ldots, p\}, i \neq j, \quad \text{and}
\]
\[
\mathbf{s}_i^T (\mathbf{K}_X - \mathbf{B}_2^*) \mathbf{a}_j = 0, \quad \forall i \in \{1, \ldots, r\}, \forall j \in \{1, \ldots, p\}.
\]

Hence,
\[
[\mathbf{S}, \mathbf{T}]^T (\mathbf{K}_X - \mathbf{B}_2^*) [\mathbf{S}, \mathbf{T}] = \frac{\mu}{2} \mathbf{I}_m.
\]
(4.13)

The parts (a) through (c) of Theorem 7 follow immediately from (4.3), (4.4), and (4.13) because \(\mathbf{M}_1^* = \mathbf{T} \mathbf{T}^T\) in this case.

The proof for the case when \(r < m = r + q\) is exactly similar. It starts with
the following from (4.6), (4.7), and (4.9)

\[
\frac{1}{2}(K_Y - B_2^*)^{-1} = \Delta^* + M_2^* = \sum_{i=1}^{r} \lambda_i s_i s_i^T + \sum_{i=1}^{q} \beta_i b_i b_i^T.
\]

\[\square\]

In summary, the key properties of the optimal Gaussian solution are as follows. If \(\Delta^*\) (and hence \(S\)) is not invertible, then there exist two matrices \(T\) and \(W\) such that their columns respectively are in \(\text{span}\{a_i\}_{i=1}^{p}\) and \(\text{span}\{b_i\}_{i=1}^{q}\), \([S, T]\) and \([S, W]\) are invertible, \(S\) and \(T\) are cross \((K_X - B_2^*)\)-orthogonal, and \(S\) and \(W\) are cross \((K_Y - B_2^*)\)-orthogonal. We shall exploit these properties in the next section to prove the optimality of an optimization problem, which is central to prove our main result.

### 4.3 Converse Ingredients

Let us define the main optimization problem as

\[
(P) \triangleq \min_{U, V} \mu I(X; U|V) + I(Y; V)
\]

subject to \(K_{X|U,V} \preceq D\) and

\[
X \leftrightarrow Y \leftrightarrow V,
\]

where \(X, Y, D,\) and \(\mu\) are defined as before. We have the following theorem.

**Theorem 8.** A Gaussian \((U, V)\) is an optimal solution of the main optimization problem \((P)\).

We prove this theorem in the remainder of the section. The proof for \(\mu\) in
[0, 1] is easy. In this case, the objective of \((P)\) can be lower bounded as

\[
\mu I(X; U|V) + I(Y; V)
\]

\[
= \mu I(X; U, V) - \mu I(X; V) + I(Y; V)
\]

\[
= \mu I(X; U) + \mu I(X; V|U) + \mu[I(Y; V) - I(X; V)] + (1 - \mu) I(Y; V)
\]

\[
\geq \mu I(X; U)
\]

\[
= \mu h(X) - \mu h(X|U)
\]

\[
\geq \frac{\mu}{2} \log \frac{|K_X|}{|K_{X|U}|}, \tag{4.15}
\]

where

(4.14) follows because of the facts that \(I(Y; V) \geq 0\) and \(I(X; V|U) \geq 0\), and we have

\[I(Y; V) - I(X; V) \geq 0\]

because of the data processing inequality [53, Theorem 2.8.1] and the Markov chain \(X \leftrightarrow Y \leftrightarrow V\), and

(4.15) follows because the Gaussian distribution maximizes the differential entropy for a given covariance matrix [53, Theorem 8.6.5], i.e.,

\[h(X|U) \leq \frac{1}{2} \log \left(\left(\frac{2\pi e}{m}\right)^n |K_X| \right).
\]

Inequalities (4.14) and (4.15) become equalities if we choose a Gaussian \((U, V)\) such that \(V\) is independent of \((X, Y, U)\). Because of the distortion constraint in \((P)\), the conditional covariance of \(X\) given \((U, V)\) should satisfy

\[0 \leq K_{X|U,V} = K_{X|U} \leq D.
\]

Since conditioning reduces covariance in a positive semidefinite sense, we also have

\[K_{X|U} \leq K_X.
\]
Hence, if $\mu$ is in $[0, 1]$, then a Gaussian $(U, V)$ is an optimal solution of the main optimization problem $(P)$ and the optimal value is

$$v(P) = \min_{K_{X|U}} \frac{\mu}{2} \log \frac{|K_X|}{|K_{X|U}|}$$

subject to $K_X \succeq K_{X|U} \succeq 0$ and $D \succeq K_{X|U}$

$$= v(P_{pt-pa}).$$

(4.16)

We therefore assume that $\mu > 1$ in the rest of the section.

Let us first restrict the solution space of $(P)$ to Gaussian distributions. This results in an optimization problem $(P_{G1})$, or equivalently $(P_{G2})$, defined in Section 4.2. For convenience, we shall work with the $(P_{G2})$ formulation in this section. First note that since restricting the solution space to Gaussian distributions can only increase the optimal value of the main optimization problem $(P)$, we immediately have

$$v(P_{G1}) = v(P_{G2}) \geq v(P).$$

(4.17)

So, it suffices to prove the reverse inequality. Let $(B_1^*, B_2^*)$ be an optimal solution to $(P_{G2})$. As discussed in Section 4.2, $(B_1^*, B_2^*)$ gives three matrices $S$, $T$, and $W$ which satisfy the properties in Theorem 7. Using these properties, the optimal
value of \((P_{G2})\) can be expressed as

\[
v(P_{G2}) = \mu \log \frac{|K_X - B_2^*|}{|K_X - B_1^* - B_2^*|} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y - B_2^*|} = \mu \log \left| \frac{|S, T^T (K_X - B_2^*) S |}{|S, T^T (K_X - B_1^* - B_2^*) S|} \right| + \frac{1}{2} \log \left| \frac{|S, W^T K_Y S, W|}{|S, W^T (K_Y - B_2^*) S, W|} \right|
\]

(4.18)

\[
= \mu \log \left| \frac{S^T (K_X - B_2^*) S}{0} \begin{pmatrix} 0 & T^T (K_X - B_2^*) T \\ T^T (K_X - B_2^*) S & 0 \end{pmatrix} \right| + \frac{1}{2} \log \left| \frac{S^T K_Y S}{0} \begin{pmatrix} 0 & W^T K_Y W \\ W^T (K_Y - B_2^*) W & 0 \end{pmatrix} \right|
\]

(4.19)

\[
= \mu \log \left| \frac{S^T (K_X - B_2^*) S}{0} \right| + \frac{1}{2} \log \left| \frac{S^T K_Y S}{W^T K_Y W} \right|
\]

\[
= \mu \log \left| \frac{S^T (K_X - B_2^*) S}{|S^T DS|} \right| + \frac{1}{2} \log \left| \frac{S^T K_Y S}{|S^T (K_Y - B_2^*) S|} \right|
\]

(4.20)

where

(4.18) follows because \([S, T]\) and \([S, W]\) are invertible,

(4.19) follows because \(S\) and \(T\) are cross \((K_X - B_2^*), (K_X - B_1^* - B_2^*)\)-orthogonal, and \(S\) and \(W\) are cross \((K_Y, (K_Y - B_2^*))\)-orthogonal, and
(4.20) follows from (4.10) and the facts that
\[
T^T (K_X - B_2^*) T = T^T (K_X - B_1^* - B_2^*) T \quad \text{and}
\]
\[
W^T K_Y W = W^T (K_Y - B_2^*) W.
\]

### 4.3.1 Distortion Projection

The special structure to the optimal Gaussian solution of \((P_{G2})\) suggests the use of distortion projection to lower bound \((P)\) by projecting the sources \(X\) and \(Y\) on \(S\) and imposing the distortion constraint on the subspace spanned by the columns of \(S\). Let us define

\[
\tilde{X} \equiv S^T X,
\]
\[
\tilde{Y} \equiv S^T Y,
\]
\[
\tilde{D} \equiv S^T DS,
\]
\[
\tilde{B}_1^* \equiv S^T B_1^* S,
\]
\[
\tilde{B}_2^* \equiv S^T B_2^* S,
\]
\[
\tilde{M}_1^* \equiv (S^T (K_X - B_1^*) S)^{-1} S^T (K_X - B_1^*) M_1^* (K_X - B_1^*) S (S^T (K_X - B_1^*) S)^{-1},
\]

and

\[
\tilde{M}_2^* \equiv (S^T (K_Y - B_2^*) S)^{-1} S^T (K_Y - B_2^*) M_2^* (K_Y - B_2^*) S (S^T (K_Y - B_2^*) S)^{-1}.
\]

Since \(S\) has full column rank, we immediately have that

\[
K_X, K_Y, \tilde{D} \succ 0,
\]
\[
\tilde{B}_1^*, \tilde{B}_2^* \succ 0, \quad \text{and}
\]
\[
\tilde{M}_1^*, \tilde{M}_2^* \succ 0.
\]
The *projected optimization problem* \( (\tilde{P}) \) is now defined as

\[
(\tilde{P}) \triangleq \min_{\tilde{U}, \tilde{V}} \mu I(\tilde{X}; \tilde{U}|\tilde{V}) + I(\tilde{Y}; \tilde{V})
\]

subject to

\[
K_{\tilde{X}|U,V} \preceq \tilde{D} \quad \text{and} \quad \tilde{X} \leftrightarrow \tilde{Y} \leftrightarrow \tilde{V}.
\]

We next show that the *main optimization problem* \( (P) \) is lower bounded by the *projected optimization problem* \( (\tilde{P}) \). Since \([S, T]\) and \([S, W]\) are invertible and mutual information is nonnegative, we obtain

\[
\mu I(X; U|V) + I(Y; V)
\]

\[
= \mu I(S^TX, T^TX; U|V) + I(S^TY, W^TY; V)
\]

\[
= \mu I(S^TX; U|V) + \mu I(T^TX; U|V, S^TX) + I(S^TY; V) + I(W^TY; V|S^TY)
\]

\[
\geq \mu I(\tilde{X}; U|V) + I(\tilde{Y}; V). \tag{4.21}
\]

Consider any \((U, V)\) feasible for \((P)\). Then

\[
D \succ K_{X|U,V} \quad \text{and} \quad X \leftrightarrow Y \leftrightarrow V \tag{4.22}
\]

Now (4.22) implies

\[
\tilde{D} = S^TDS \succ S^TK_{X|U,V}S = K_{\tilde{X}|U,V}. \tag{4.24}
\]
and (4.23) yields

\[
0 = I(X; V|Y)
= I(S^T X; V|Y) + I(T^T X; V|Y, S^T X)
\geq I(S^T X; V|Y) \quad \text{(4.25)}
= I(S^T X; V|S^T Y, W^T Y) \quad \text{(4.26)}
= h(S^T X|S^T Y, W^T Y) - h(S^T X|V, S^T Y, W^T Y)
\geq h(S^T X|S^T Y) - h(S^T X|V, S^T Y) \quad \text{(4.27)}
= I(S^T X; V|S^T Y)
= I(\tilde{X}; V|\tilde{Y})
\geq 0, \quad \text{(4.28)}
\]

where

(4.25) and (4.28) follows because mutual information is nonnegative,

(4.26) follows because \([S, W]\) is invertible, and

(4.27) follows because conditioning reduces entropy and we have from Theorem 7 that \(W^T Y\) is independent of \(S^T Y\), which implies that \(W^T Y\) is also independent of \(S^T X\) because \(X = Y + N\).

Now (4.28) is equivalent to

\[\tilde{X} \leftrightarrow \tilde{Y} \leftrightarrow V,\]

which together with (4.24) implies that \((U, V)\) is feasible for \((\tilde{P})\). Hence, the feasible set of \((P)\) is contained in that of \((\tilde{P})\). Moreover, (4.21) above implies that the objective of \((P)\) is no less than that of \((\tilde{P})\). We therefore have that the
projected optimization problem \( (\tilde{P}) \) lower bounds the main optimization problem \( (P) \), i.e.,

\[
v(P) \geq v(\tilde{P}).
\]  

(4.29)

By restricting the solution space of \( (\tilde{P}) \) to Gaussian distributions, we obtain its Gaussian version

\[
(\tilde{P}_{G2}) \triangleq \min_{\tilde{B}_1, \tilde{B}_2} \frac{\mu}{2} \log \frac{|K_X - \tilde{B}_2|}{|K_X - B_1 - B_2|} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y - B_2|}
\]

subject to \( \tilde{B}_i \succ 0 \) for all \( i \in \{1, 2\} \), and \( \tilde{D} \succ K_X - \tilde{B}_1 - \tilde{B}_2 \).

It is easy to verify that the projected optimal Gaussian solution \( (\tilde{B}_1^*, \tilde{B}_2^*) \) is feasible for \( (\tilde{P}_{G2}) \) and it meets the projected distortion constraint \( \tilde{D} \) with equality from (4.10). We next show that \( (\tilde{B}_1^*, \tilde{B}_2^*) \) is in fact optimal for \( (\tilde{P}) \).

Remark 4.1: If \( r = m \), then there is no need for distortion projection because \( S \) is invertible, and hence so is \( \Delta^* \).

4.3.2 Source Enhancement

In this subsection, we use the KKT conditions (4.1) through (4.5) satisfied by \( (B_1^*, B_2^*) \) to derive conditions that must be satisfied by \( (\tilde{B}_1^*, \tilde{B}_2^*) \). These conditions are then used to define the enhanced optimization problem, which lower bounds \( (\tilde{P}) \). We show that the optimal solution to the enhanced optimization problem is Gaussian, in particular \( (\tilde{B}_1^*, \tilde{B}_2^*) \) is optimal for the problem. This will in turn prove that \( (\tilde{B}_1^*, \tilde{B}_2^*) \) is optimal for \( (\tilde{P}) \). This approach of lower bounding is referred to as the source enhancement \[18\] and is similar to the channel enhancement idea of Weingarten et al. \[16\].
We start with the following key lemma.

**Lemma 8.** For $K_X, K_Y, \tilde{D}, \tilde{B}_i^*,$ and $\tilde{M}_i^*$, where $i \in \{1, 2\}$, defined as above, the following hold

\[
I_r = \frac{\mu}{2}(K_X - \tilde{B}_2^*)^{-1} - \tilde{M}_1^* = \frac{1}{2}(K_Y - \tilde{B}_2^*)^{-1} - \tilde{M}_2^*,
\]

(4.30)

\[
\tilde{B}_i^*\tilde{M}_i^* = 0 \text{ for all } i \in \{1, 2\}, \text{ and }
\]

(4.31)

\[
K_X - \tilde{B}_1^* - \tilde{B}_2^* = \tilde{D}.
\]

(4.32)

**Proof.** See Appendix B.4.

---

Let $K_X$ and $K_Y$ be two real symmetric matrices satisfying

\[
\frac{\mu}{2}(K_X - \tilde{B}_2^*)^{-1} - \tilde{M}_1^* = \frac{\mu}{2}(K_X - \tilde{B}_2^*)^{-1} \quad \text{and}
\]

(4.33)

\[
\frac{1}{2}(K_Y - \tilde{B}_2^*)^{-1} - \tilde{M}_2^* = \frac{1}{2}(K_Y - \tilde{B}_2^*)^{-1}.
\]

(4.34)

We now have the following lemma, which is similar to [16, Lemmas 11, 12].

**Lemma 9.** For $K_X, K_Y, K_X, K_Y, \tilde{B}_i^*, \tilde{M}_i^*$, $i \in \{1, 2\}$, defined as above, and $\mu > 1$, the following hold

\[
K_X - \tilde{B}_2^* = \frac{\mu}{2}I_r,
\]

(4.35)

\[
K_Y - \tilde{B}_2^* = \frac{1}{2}I_r,
\]

(4.36)

\[
K_X \succ K_Y \succ K_Y \succ 0,
\]

(4.37)

\[
K_X \succ K_X \succ 0,
\]

(4.38)

\[
\frac{|K_Y|}{|K_Y - B_2^*|} = \frac{|K_Y|}{|K_Y - B_2^*|}, \quad \text{and}
\]

(4.39)

\[
\frac{|K_X - B_2^*|}{|K_X - B_2^* - B_2^*|} = \frac{|K_X - B_2^*|}{|K_X - B_2^* - B_2^*|}.
\]

(4.40)

**Proof.** See Appendix B.5.

---

79
Let $\hat{X}$ and $\hat{Y}$ be two zero-mean Gaussian random vectors with covariance matrices $K_{\hat{X}}$ and $K_{\hat{Y}}$, respectively. Since $K_{\hat{X}} \succ K_{\hat{Y}}$ from (4.37), we can write
\[
\hat{X} = \hat{Y} + \hat{N},
\]
where $\hat{N}$ is a zero-mean Gaussian random vector with the covariance matrix
\[
K_{\hat{N}} = K_{\hat{X}} - K_{\hat{Y}} = \frac{\mu - 1}{2} I_r,
\]
and is independent of $\hat{Y}$. Similarly, we can use (4.37) and (4.38) to relate $\hat{X}$ and $\hat{Y}$ with $\tilde{X}$ and $\tilde{Y}$, respectively, and write
\[
\hat{X} = \tilde{X} + N_1 \quad \text{and} \quad \hat{Y} = \tilde{Y} + N_2,
\]
where $N_1$ and $N_2$ are two zero-mean Gaussian random vectors with covariance matrices
\[
K_{N_1} = K_{\hat{X}} - K_{\tilde{X}} \quad \text{and} \quad K_{N_2} = K_{\hat{Y}} - K_{\tilde{Y}},
\]
respectively, and they are independent of $\tilde{X}$ and $\tilde{Y}$. Using (4.32), we define
\[
\hat{D} \triangleq \tilde{D} + K_{N_1} = K_{\hat{X}} - B_1^* - B_2^*. \quad (4.41)
\]
The enhanced optimization problem $(\hat{P})$ is now defined as
\[
(\hat{P}) \triangleq \min_{U, V} \mu I(X; U|V) + I(Y; V)
\]
subject to $K_{X|U,V} \preceq \hat{D}$ and
\[
\hat{X} \leftrightarrow \hat{Y} \leftrightarrow V.
\]
We next show that $(\hat{P})$ lower bounds $(\tilde{P})$. Consider any $(U, V)$ feasible for $(\hat{P})$. Without loss of optimality, we can assume that the joint distribution between $\hat{X}, \hat{Y}, U,$ and $V$ is
\[
\hat{p} \triangleq p_{\hat{X}, \hat{Y}} p_{U|\hat{X}, \hat{Y}} p_{V|\hat{Y}}.
\]
Now, \( \tilde{p} \) induces two conditional distributions as follows

\[
p_{V|\hat{Y}} = \int_{\hat{Y}} p_{V|\hat{Y}} p_{\hat{Y}|\hat{Y}}
\]

\[
p_{U|\hat{X},V} = \int_{\hat{X}} p_{U|\hat{X},V} p_{\hat{X}|\hat{X},V},
\]

where

\[
p_{\hat{X}|X,V} = \frac{p_{}\hat{x},x p_{V|\hat{x}}}{\int_{\hat{x}} p_{}\hat{x},x p_{V|\hat{x}}}.\]

Then

\[
\hat{p} \triangleq p_{\hat{X},\hat{Y}} p_{U|\hat{X},V} p_{V|\hat{Y}}
\]

is a joint distribution between \( \hat{X}, \hat{Y}, U, \) and \( V \). It is clear that \( \hat{p} \) satisfies the Markov condition

\[
\hat{X} \leftrightarrow \hat{Y} \leftrightarrow V. \tag{4.42}
\]

Moreover, (4.41) and the distortion constraint in the definition of \( \hat{P} \) yield

\[
K_{\hat{X}|U,V} = K_{\hat{X}|U,V} + K_{N_1} \leq \hat{D} + K_{N_1} = \hat{D}. \tag{4.43}
\]

We next use the chain rule of mutual information to obtain

\[
I(\hat{X}, \hat{X}; U|V) = I(\hat{X}; U|V) + I(\hat{X}; U|V, \hat{X})
= I(\hat{X}; U|V) + I(\hat{X}; U|V, \hat{X})
= I(\hat{X}; U|V)
\]

and

\[
I(\hat{Y}, \hat{Y}; V) = I(\hat{Y}; V) + I(\hat{Y}; V|\hat{Y})
= I(\hat{Y}; V) + I(\hat{Y}; V|\hat{Y})
= I(\hat{Y}; V).
\]
Since mutual information is nonnegative, these imply that

\[ I(\tilde{X}; U|V) \geq I(\hat{X}; U|V) \]  \hspace{1cm} (4.44)

and

\[ I(\tilde{Y}; V) \geq I(\hat{Y}; V) \]  \hspace{1cm} (4.45)

Now (4.42) and (4.43) together imply that the distribution \( \hat{p} \), and hence \((U, V)\), is feasible for \((\hat{P})\). Therefore, the feasible set of \((\hat{P})\) is contained in that of \((\bar{P})\). Moreover, (4.44) and (4.45) assert that the objective value of \((\bar{P})\) is no more than that of \((\hat{P})\). We therefore conclude that the enhanced optimization problem \((\hat{P})\) lower bounds the projected optimization problem \((\bar{P})\), i.e.,

\[ v(\bar{P}) \geq v(\hat{P}). \]  \hspace{1cm} (4.46)

**Remark 4.2:** If \( r < m = r + p \), then there is no need to enhance the source \( \tilde{X} \) and the distortion \( \tilde{D} \) because \( M_1^* = TT^T \) from Corollary 1, and hence \( \tilde{M}_1^* = 0 \). Similarly, if \( r < m = r + q \), then there is no need to enhance the source \( \tilde{Y} \) because \( M_2^* = WW^T \) from Corollary 1 again, and hence \( \tilde{M}_2^* = 0 \). Finally, if \( r < m = r + p = r + q \), then there is no need for source enhancement.

### 4.3.3 Oohama’s Approach

We now apply Oohama’s approach [11] to prove that \((\tilde{B}_1^*, \tilde{B}_2^*)\) is optimal for \((\hat{P})\). The objective of \((\bar{P})\) can be decomposed as

\[ \mu I(\tilde{X}; U|V) + I(\tilde{Y}; V) = \mu I(\hat{X}; U, V) - [\mu I(\tilde{X}; V) - I(\hat{Y}; V)]. \]  \hspace{1cm} (4.47)

We next define two subproblems that are used to lower bound the enhanced optimization problem \((\bar{P})\). The first subproblem \((\bar{P}_1)\) minimizes the first mutual
information in the right-hand-side of (4.47) subject to the distortion constraint in \((\hat{P})\) and the second subproblem \((\hat{P}_2)\) maximizes the expression within the parenthesis in the right-hand-side of (4.47) subject to the Markov condition in \((\hat{P})\). In other words, \((\hat{P}_1)\) is defined as

\[
(\hat{P}_1) \triangleq \min_{\mathbf{U}, \mathbf{V}} \mu I(\hat{X}; \mathbf{U}, \mathbf{V})
\]

subject to \(K_{\hat{X}|\mathbf{U}, \mathbf{V}} \preceq \hat{D},\)

and \((\hat{P}_2)\) is defined as

\[
(\hat{P}_2) \triangleq \max_{\mathbf{V}} \mu I(\hat{X}; \mathbf{V}) - I(\hat{Y}; \mathbf{V})
\]

subject to \(\hat{X} \leftrightarrow \hat{Y} \leftrightarrow \mathbf{V}^83\).

It is clear from the decomposition in (4.47) and from the definitions of \((\hat{P}), (\hat{P}_1),\) and \((\hat{P}_2)\) that \((\hat{P}_1)\) and \((\hat{P}_2)\) lower bound \((\hat{P}), i.e.,\)

\[
v(\hat{P}) \geq v(\hat{P}_1) - v(\hat{P}_2). \tag{4.48}\]

We now give two lemmas about the optimal solutions to subproblems \((\hat{P}_1)\) and \((\hat{P}_2)\).

**Lemma 10.** A Gaussian \((\mathbf{U}, \mathbf{V})\) with the conditional covariance matrix

\[
K_{\hat{X}|\mathbf{U}, \mathbf{V}} = K_{\hat{X}} - \hat{B}_1^* - \hat{B}_2^* = \hat{D}
\]

is optimal for the subproblem \((\hat{P}_1)\), and the optimal value is

\[
v(\hat{P}_1) = \frac{\mu}{2} \log \frac{|K_{\hat{X}}|}{|\hat{D}|}. \tag{4.49}\]

**Proof.** See Appendix B.6. \(\square\)

**Lemma 11.** A Gaussian \(\mathbf{V}\) with the conditional covariance matrix

\[
K_{\hat{Y}|\mathbf{V}} = K_{\hat{Y}} - \hat{B}_2^*
\]
is optimal for the subproblem $(\hat{P}_2)$, and the optimal value is

$$v(\hat{P}_2) = \frac{\mu}{2} \log \left| \begin{array}{c} K_X \\ K_X - B_2^* \end{array} \right| - \frac{1}{2} \log \left| \begin{array}{c} K_Y \\ K_Y - B_2^* \end{array} \right|. \tag{4.50}$$

**Proof.** See Appendix B.7. \hfill \Box

Substituting (4.49) and (4.50) into (4.48), we obtain

$$v(\hat{P}) \geq \frac{\mu}{2} \log \left| \begin{array}{c} K_X - B_2^* \\ D \end{array} \right| + \frac{1}{2} \log \left| \begin{array}{c} K_Y \\ K_Y - B_2^* \end{array} \right| = \frac{\mu}{2} \log \left| \begin{array}{c} K_X - B_2^* \\ D \end{array} \right| + \frac{1}{2} \log \left| \begin{array}{c} K_Y \\ K_Y - B_2^* \end{array} \right| \tag{4.51}$$

$$= v(P_{G2}), \tag{4.52}$$

where

(4.51) follows from (4.32), (4.39), (4.40), and (4.41), and

(4.52) follows from (4.20).

We conclude from (4.29), (4.46), and (4.52) that

$$v(P) \geq v(P_{G2}).$$

It now follows from this and (4.17) that

$$v(P) = v(P_{G1}) = v(P_{G2}), \tag{4.53}$$

which proves that a Gaussian $(U, V)$ is optimal for the main optimization problem $(P)$. This completes the proof of Theorem 8.
4.4 Converse Proof of the Main Result

We start with the following outer bound.

Lemma 12. If the rate-distortion vector \((R_1, R_2, D)\) is achievable then there exist random vectors \(U\) and \(V\) such that

\[
R_1 \geq I(X; U|V),
\]
\[
R_2 \geq I(Y; V),
\]
\[
D \geq K_{X|U,V}, \text{ and}
\]
\[
X \leftrightarrow Y \leftrightarrow V.
\]

The proof of the lemma is similar to Lemma 4 and is omitted. We are now ready to prove the converse of the first equality in Theorem 6. If \((R_1, R_2, D)\) is achievable, then

\[
\mu R_1 + R_2 \geq v(P) \tag{4.54}
\]
\[
= \begin{cases} 
  v(P_{pt-p2}) & \text{if } 0 \leq \mu \leq 1 \\
  v(P_{G1}) & \text{if } \mu > 1 
\end{cases} \tag{4.55}
\]
\[
= \mathcal{R}^*(D, \mu), \tag{4.56}
\]

where

(4.54) follows from Lemma 12, and

(4.55) follows from (4.16) and (4.53).

And if \((R_1, R_2, D) \in \mathcal{RD}\), then (4.56) again holds because \(\mathcal{R}^*(D, \mu)\) is continuous.
in D. So, (4.56) is a lower bound for any \( (R_1, R_2) \) in the rate region \( \mathcal{R}(D) \). Hence,

\[
\mathcal{R}(D, \mu) = \inf_{(R_1, R_2) \in \mathcal{R}(D)} \mu R_1 + R_2 \\
\geq \mathcal{R}^*(D, \mu).
\]

This completes the proof of the first equality in Theorem 6.

Remark 4.3: It follows from Theorem 6 that one can add the constraints

\[
U \leftrightarrow X \leftrightarrow Y \leftrightarrow V \quad \text{and} \quad (U, V, X, Y) \text{ are jointly Gaussian}
\]

to the optimization problem

\[
(P) \triangleq \min_{U,V} \mu I(X; U|V) + I(Y; V)
\]

subject to \( K_{X|U,V} \preceq D \) and

\[
X \leftrightarrow Y \leftrightarrow V,
\]

without changing its optimal value.

4.5 Solution for the General Case

In this section, we lift the assumptions on \( K_X, K_Y \), and D and allow them to be any positive semidefinite matrices. We shall show that the Gaussian achievable scheme is optimal for this general problem. For this section, we denote the rate region of the problem by \( \mathcal{R}(K_X, K_Y, D) \). Note that \( K_X \) and \( K_Y \) completely specify the joint distribution of \( X \) and \( Y \) because we continue to assume that \( X = Y + N \). Similarly, \( \mathcal{R}_G(K_X, K_Y, D) \) is used to denote the rate region achieved by the Gaussian achievable scheme. We use \( \mathcal{R}(K_X, K_Y, D, \mu) \) and
\( R_G(K_X, K_Y, D, \mu) \) to denote the two minimum weighted sum-rates. Likewise, we denote the set \( S \) defined in Section 2.3 by \( S(K_X, K_Y, D) \). We use similar notation later in the section. We start with the following extension.

**Theorem 9.** If \( K_X \) and \( D \) are positive definite, and \( K_Y \) is positive semidefinite, then

\[
R(K_X, K_Y, D, \mu) = R_G(K_X, K_Y, D, \mu).
\]

**Proof.** It suffices to prove that

\[
R(K_X, K_Y, D, \mu) \geq R_G(K_X, K_Y, D, \mu).
\]

If \( K_Y \) is positive definite (hence nonsingular), then the result follows from Theorem 6. We therefore assume that \( K_Y \) is singular and has a rank \( p < m \). The eigen decomposition of \( K_Y \) is

\[
K_Y = Q\Sigma Q^T,
\]

where \( Q \) is an orthogonal matrix and

\[
\Sigma = \text{Diag}(\alpha_1, \ldots, \alpha_p, 0, \ldots, 0).
\]

Let us partition \( Q \) as

\[
Q = [Q_1, Q_2],
\]

where \( Q_1 \) is an \( m \times p \) matrix. Let us define

\[
Q^T K_N Q \triangleq \begin{pmatrix} E & F^T \\ F & G \end{pmatrix},
\]

where \( E, F, \) and \( G \) are submatrices of dimensions \( p \times p, (m - p) \times p, \) and \( (m - p) \times (m - p) \), respectively. Since \( Q_2^T K_Y Q_2 = 0 \) and \( X = Y + N \), we have that

\[
G = Q_2^T K_N Q_2 = Q_2^T K_X Q_2 \succ 0,
\]

87
i.e., \( G \) is positive definite. Using this, we define

\[
A \triangleq \begin{pmatrix}
I_p & -F^T G^{-1} \\
0 & I_{m-p}
\end{pmatrix} Q^T.
\]

\( A \) defines a transformed problem in which the transformed sources are

\[
\hat{X} \triangleq AX \quad \text{and} \quad \hat{Y} \triangleq AY,
\]

which satisfy

\[
\hat{X} = \hat{Y} + \hat{N},
\]

where \( \hat{N} \triangleq AN \), and the transformed distortion matrix is

\[
D \triangleq ADA^T.
\]

The covariance matrix of the transformed source \( \hat{Y} \) is

\[
K_Y = AK_Y A^T = \Sigma = \begin{pmatrix}
\Sigma_1 & 0 \\
0 & 0
\end{pmatrix},
\]

where

\[
\Sigma_1 \triangleq \text{Diag}(\alpha_1, \ldots, \alpha_p),
\]

and the covariance matrix of \( \hat{N} \) is

\[
K_N = AK_N A^T
\]

\[
= \begin{pmatrix}
I_p & -F^T G^{-1} \\
0 & I_{m-p}
\end{pmatrix} \begin{pmatrix}
E & F^T \\
F & G
\end{pmatrix} \begin{pmatrix}
I_p & 0 \\
-G^{-1}F & I_{m-p}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
E - F^T G^{-1} F & 0 \\
0 & G
\end{pmatrix}.
\]
Using these, the covariance matrix of the transformed source $\bar{X}$ can be expressed as

$$K_{\bar{X}} = K_Y + K_N = \begin{pmatrix} \Sigma_1 + E - F^T G^{-1} F & 0 \\ 0 & G \end{pmatrix}.$$ 

Since $A$ is invertible, the above transformation is information lossless, and hence the transformed problem is equivalent to the original problem. Therefore,

$$R(K_X, K_Y, D, \mu) = R(K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu) \text{ and } R_G(K_X, K_Y, D, \mu) = R_G(K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu).$$

So, it is sufficient to prove that

$$R(K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu) \geq R_G(K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu).$$

Let us define the following matrices

$$K_{N_1^{(n)}} \triangleq \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{n} G \end{pmatrix} \text{ and } K_{N_2^{(n)}} \triangleq \begin{pmatrix} E - F^T G^{-1} F & 0 \\ 0 & \left(1 - \frac{1}{n}\right) G \end{pmatrix},$$

where $n$ is a positive integer. It is clear that these matrices are positive semidefinite and they satisfy

$$K_N = K_{N_1^{(n)}} + K_{N_2^{(n)}}.$$

Let $\bar{N}_1^{(n)}$ and $\bar{N}_2^{(n)}$ be zero-mean vector Gaussian sources with covariance matrices $K_{\bar{N}_1^{(n)}}$ and $K_{\bar{N}_2^{(n)}}$, respectively. In addition, suppose they are independent of
each other and all other vector Gaussian sources. We can then write

$$\bar{X} = \bar{Y} + \bar{N}_1^{(n)} + \bar{N}_2^{(n)}.$$ 

Let us consider a new problem in which encoder 1 has access to $\bar{X}$, encoder 2 has access to $(\bar{Y}, \bar{N}_1^{(n)})$, and the distortion constraint on $\bar{X}$ is $\bar{D}$. This problem is clearly a relaxation to the original problem because encoder 2 has access to more information about $\bar{X}$ than the original problem. In other words, any feasible scheme for the original problem is also feasible for this new problem. Now since there is no distortion constraint on $\bar{Y}$ and the sufficient statistic of $\bar{X}$ in $(\bar{Y}, \bar{N}_1^{(n)})$ is $\bar{Y} + \bar{N}_1^{(n)}$, this new problem is equivalent to the problem in which encoder 2, instead of $(\bar{Y}, \bar{N}_1^{(n)})$, has access to the sum $\bar{Y} + \bar{N}_1^{(n)}$. Let us denote this sum by $\bar{Y}^{(n)}$, i.e.,

$$\bar{Y}^{(n)} \triangleq \bar{Y} + \bar{N}_1^{(n)},$$

which has a positive definite covariance matrix

$$K_{\bar{Y}^{(n)}} = K_{\bar{Y}} + K_{\bar{N}_1^{(n)}} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \frac{1}{n} G \end{pmatrix}.$$ 

It follows that

$$\mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}^{(n)}}, \bar{D}, \mu \right) \leq \mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu \right).$$

Since this is true for all $n$ and $\mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}^{(n)}}, \bar{D}, \mu \right)$ is monotonically increasing in $n$, we obtain

$$\lim_{n \to \infty} \mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}^{(n)}}, \bar{D}, \mu \right) \leq \mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}}, \bar{D}, \mu \right). \quad (4.57)$$

Since $K_{\bar{X}}, K_{\bar{Y}^{(n)}},$ and $\bar{D}$ are positive definite, the conclusion of Theorem 6 holds for this sequence of relaxed problems, i.e., for each $n$

$$\mathcal{R} \left( K_{\bar{X}}, K_{\bar{Y}^{(n)}}, \bar{D}, \mu \right) = \mathcal{R}_G \left( K_{\bar{X}}, K_{\bar{Y}^{(n)}}, \bar{D}, \mu \right).$$
This and (4.57) together imply that
\[ \lim_{n \to \infty} R_G \left( K_X, K_Y^{(n)}, D, \mu \right) \leq R \left( K_X, K_Y, D, \mu \right). \] (4.58)

Now for each \( n \), there exists \( (U^{(n)}, V^{(n)}) \) in \( S \left( K_X, K_Y^{(n)}, D \right) \) such that
\[ R_G \left( K_X, K_Y^{(n)}, D, \mu \right) = \mu I \left( \bar{X}; U^{(n)} | V^{(n)} \right) + I \left( \bar{Y}^{(n)}; V^{(n)} \right). \] (4.59)

Since \( \bar{X}, \bar{Y}^{(n)}, U^{(n)}, \) and \( V^{(n)} \) are jointly Gaussian, we can without loss of generality parameterize them by positive semidefinite matrices \( B_1 \) and \( B_2 \) as in the definition \((P_{G2})\). These matrices lie in a compact set because they satisfy the KKT conditions that are continuous, and they are bounded as \( B_1 + B_2 \prec K_X \). Therefore, there exists a subsequence of \( K_Y^{(n)} \) along which \( (U^{(n)}, V^{(n)}) \) converges to \( (U, V) \) in \( S \left( K_X, K_Y, D \right) \). Since the right-hand-side of (4.59) is continuous in \( (\bar{Y}^{(n)}, U^{(n)}, V^{(n)}) \), this implies
\[ \lim_{n \to \infty} R_G \left( K_X, K_Y^{(n)}, D, \mu \right) = \mu I \left( \bar{X}; U | V \right) + I \left( \bar{Y}; V \right) \geq R_G \left( K_X, K_Y, D, \mu \right). \] (4.60)

It now follows from (4.58) and (4.60) that
\[ R \left( K_X, K_Y, D, \mu \right) \geq R_G \left( K_X, K_Y, D, \mu \right). \]

This proves Theorem 9. \qed

We next use Theorem 9 to prove our result for the most general case of the problem.

**Theorem 10.** For any positive semidefinite \( K_X, K_Y, \) and \( D \), we have
\[ R \left( K_X, K_Y, D, \mu \right) = R_G \left( K_X, K_Y, D, \mu \right). \]
Proof. Let us suppose that the rank of $K_X$ is $p \leq m$. Since $K_X$ is positive semidefinite, its eigen decomposition is

$$K_X = Q \Sigma Q^T,$$

where $Q$ is an orthogonal matrix and

$$\Sigma = \text{Diag}(\alpha_1, \ldots, \alpha_p, 0, \ldots, 0).$$

Let us partition $Q$ as

$$Q \triangleq [Q_1, Q_2],$$

where $Q_1$ is an $m \times p$ matrix. Since $Q_2^T K_X Q_2 = 0$ and $X = Y + N$, we have

$$Q_2^T K_Y Q_2 = Q_2^T K_N Q_2 = 0,$$

which implies that

$$Q_2^T K_Y Q_2 = \begin{pmatrix} Q_1^T K_Y Q_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_2^T K_N Q_2 = \begin{pmatrix} Q_1^T K_N Q_1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Let us define

$$Q^T D Q \triangleq \begin{pmatrix} E & F^T \\ F & G \end{pmatrix},$$

where $E$, $F$, and $G$ are submatrices of dimensions $p \times p$, $(m - p) \times p$, and $(m - p) \times (m - p)$, respectively. We need the following lemma.

**Lemma 13.** [50, Appendix A.5.5, p. 651] $Q^T D Q \succ 0$ if and only if

$$G \succ 0,$$

$$E - F^T G^+ F \succ 0, \quad \text{and}$$

$$(I_{m-p} - GG^+) F = 0,$$

92
where \( G^+ \) is the pseudo-inverse or Moore-Penrose inverse of \( G \) [50, Appendix A.5.4, p. 649].

Let
\[
T \triangleq \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \triangleq \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} Q^T,
\]
where \( T_1 \) is a \( p \times m \) matrix. Using this, we obtain a transformed problem in which the transformed sources are
\[
\bar{X} \triangleq \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \triangleq \begin{pmatrix} T_1 X \\ T_2 X \end{pmatrix} = TX \quad \text{and}
\]
\[
\bar{Y} \triangleq \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \triangleq \begin{pmatrix} T_1 Y \\ T_2 Y \end{pmatrix} = TY.
\]
Using Lemma 13, we obtain the transformed distortion matrix
\[
\bar{D} \triangleq TDT^T
\]
\[
= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} Q^T D Q \begin{pmatrix} I_p & 0 \\ -G^+F & I_{m-p} \end{pmatrix}
\]
\[
= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} \begin{pmatrix} E & F^T \\ F & G \end{pmatrix} \begin{pmatrix} I_p & 0 \\ -G^+F & I_{m-p} \end{pmatrix}
\]
\[
= \begin{pmatrix} E - F^TG^+F & 0 \\ 0 & G \end{pmatrix}
\]
\[
= \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{4.61}
\]
where
\[
D_1 \triangleq E - F^TG^+F \quad \text{and}
\]
\[
D_2 \triangleq G.
\]
The covariance matrix of the transformed source $\bar{X}$ is

$$K_{\bar{X}} = TK_xT^T$$

$$= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} Q^T K_x Q \begin{pmatrix} I_p & 0 \\ -G+F & I_{m-p} \end{pmatrix}$$

$$= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_p & 0 \\ -G+F & I_{m-p} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\Sigma_1 \triangleq \text{Diag}(\alpha_1, \ldots, \alpha_p),$$

and the covariance matrix of the transformed source $\bar{Y}$ is

$$K_{\bar{Y}} = TK_yT^T$$

$$= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} Q^T K_y Q \begin{pmatrix} I_p & 0 \\ -G+F & I_{m-p} \end{pmatrix}$$

$$= \begin{pmatrix} I_p & -F^TG^+ \\ 0 & I_{m-p} \end{pmatrix} \begin{pmatrix} Q_1^T K_y Q_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_p & 0 \\ -G+F & I_{m-p} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1^T K_y Q_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $X_2$ and $Y_2$ are deterministic. So, we can set

$$X_2 = Y_2 = 0.$$
Since $T$ is invertible, the distortion constraint is equivalent to

$$TDT^T \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( \bar{X}_n(i) - \hat{X}_n(i) \right) \left( \bar{X}_n(i) - \hat{X}_n(i) \right)^T \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E \left[ \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix} \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix}^T \right]$$

$$= \begin{pmatrix} \frac{1}{n} \sum_{i=1}^{n} E \left[ \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix} \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix}^T \right] & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.62)$$

Since $D_1$ and $D_2$ are positive semidefinite from Lemma 13, (4.61) and (4.62) imply that the equivalent distortion constraint is

$$D_1 \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix} \begin{pmatrix} X_1^n(i) - \hat{X}_1^n(i) \\ 0 \end{pmatrix}^T \right].$$

Since $T$ is invertible, the above transformation is information lossless, and hence the transformed problem is equivalent to the original problem. Moreover, the transformed problem is essentially $p$-dimensional with the sources $X_1$ and $Y_1$, and the distortion matrix $D_1$ such that

$$K_{X_1} = \Sigma_1 \succ 0 \quad \text{and} \quad X_1 = Y_1 + N_1,$$

where $N_1 \triangleq T_1 N$. We therefore have that

$$\mathcal{R}(K_X, K_Y, D, \mu) = \mathcal{R}(K_{X_1}, K_{Y_1}, D_1, \mu) \quad \text{and} \quad (4.63)$$

$$\mathcal{R}_G(K_X, K_Y, D, \mu) = \mathcal{R}_G(K_{X_1}, K_{Y_1}, D_1, \mu). \quad (4.64)$$

Since $K_{X_1}$ is positive definite, if $D_1$ is singular, then the right-hand side of (4.63) and (4.64) are both infinite, so the conclusion trivially holds. Otherwise, we have that $K_{X_1}$ and $D_1$ are positive definite and $K_{Y_1}$ is positive semidefinite. In
that case Theorem 9 implies that
\[
\mathcal{R}(K_{X_1}, K_{Y_1}, D_1, \mu) = \mathcal{R}_G(K_{X_1}, K_{Y_1}, D_1, \mu).
\]
This together with (4.63) and (4.64) establishes the desired equality
\[
\mathcal{R}(K_X, K_Y, D, \mu) = \mathcal{R}_G(K_X, K_Y, D, \mu).
\]
Theorem 10 is thus proved. \qed
In this chapter, we formulate the distributed hypothesis testing problem, and state and prove all results. We start with the $L$-encoder general hypothesis testing problem. We then give the inner and outer bounds for a class of problems, namely $L$-encoder hypothesis testing against conditional independence. We prove that the inner and outer bounds coincide for three special instances of the problems. We also give an outer bound for a class of the general problem. The numerical examples show that the outer bound is quite close to known inner bounds in some cases. We end this chapter by extending the test against independence result of Ahlswede and Csiszár to vector Gaussian case.

5.1 Notation

We continue to use the notation defined in Section 2.1. In addition, we use the following notation in this chapter. For a random variable $X$, $X^n(i^c)$ denotes all but the $i$th component of $X^n$. We use $\mathcal{L}$ to denote the set $\{1, \ldots, L\}$. For $S \subseteq \mathcal{L}$, $S^c$ denotes the complement set $\mathcal{L} \setminus S$ and $X^n_S(i)$ denotes $(X^n_l(i))_{l \in S}$. When $S = \mathcal{L}$, we simply write $X^n_\mathcal{L}(i)$ as $X^n(i)$. Likewise when $S = \{l\}$, we write $X^n_{\{l\}}(i)$ and $X^n_{\{l\}}^c(i)$ as $X^n_l(i)$ and $X^n_{\bar{l}}(i)$, respectively. $1_A$ denotes the indicator function of an event $A$. $\mathbb{R}_+^L$ is used to denote the positive orthant in $L$-dimensional Euclidean space. For $0 \leq p \leq 1$, $H_b(p)$ denotes the binary entropy function defined as

$$H_b(p) \triangleq -p \log p - (1 - p) \log(1 - p).$$

We use $\mathcal{N}(m, K)$ to denote the p.d.f. of a Gaussian random vector with the mean $m$ and the covariance matrix $K$. All entropy and mutual information quantities
are under the null hypothesis, \( H_0 \), unless otherwise stated.

### 5.2 \( L \)-Encoder General Hypothesis Testing

Let \( (X_1, \ldots, X_L, Y) \) be a generic source taking values in \( \prod_{i=1}^{L} X_i \times \mathcal{Y} \), where \( X_1, \ldots, X_L, \) and \( \mathcal{Y} \) are finite sets. The distribution of the source is \( P_{X_1 \ldots X_L Y} \) under the null hypothesis \( H_0 \) and is \( Q_{X_1 \ldots X_L Y} \) under the alternate hypothesis \( H_1 \), i.e.,

\[
H_0 : P_{X_1 \ldots X_L Y} \\
H_1 : Q_{X_1 \ldots X_L Y}.
\]

Let \( \{(X_n^i(i), \ldots, X_L^n(i), Y^n(i))\}_{i=1}^{n} \) be an i.i.d. sequence of random vectors with the distribution at a single stage being the same as that of \( (X_1, \ldots, X_L, Y) \).

As depicted in Fig. 1.3, encoder \( l \) observes \( X_i^n \), then sends a message to the detector using an encoding function

\[
f_i^{(n)} : X_i^n \mapsto \{1, \ldots, M_i^{(n)}\}.
\]

\( Y^n \) is available at the detector which uses it and the messages from the encoders to make a decision between the hypotheses based on a decision rule

\[
g^{(n)}(m_1, \ldots, m_L, y^n) = \begin{cases} 
H_0 & \text{if } (m_1, \ldots, m_L, y^n) \text{ is in } A \\
H_1 & \text{otherwise},
\end{cases}
\]

where

\[
A \subseteq \prod_{i=1}^{L} \{1, \ldots, M_i^{(n)}\} \times \mathcal{Y}^n
\]

is the acceptance region for \( H_0 \). The encoding functions \( f_i^{(n)} \) and the detector \( g^{(n)} \) are such that the type 1 error probability does not exceed a fixed \( \epsilon \) in \( (0, 1) \),
i.e.,
\[ P_{f_l^n (X^n_l)} (Y_n(A^c)) \leq \epsilon, \]
and the type 2 error probability does not exceed \( \eta \), i.e.,
\[ Q_{f_l^n (X^n_l)} (Y_n(A)) \leq \eta. \]

**Definition 8.** A rate-exponent vector
\[(R, E) = (R_1, \ldots, R_L, E)\]
is achievable for a fixed \( \epsilon \) if for any positive \( \delta \) and sufficiently large \( n \), there exist encoding functions \( f_l^n \) and a detector \( g^n \) such that
\[ \frac{1}{n} \log M_l^n \leq R_l + \delta \text{ for all } l \text{ in } \mathcal{L}, \text{ and} \]
\[ -\frac{1}{n} \log \eta \geq E - \delta. \]

Let \( \mathcal{RE}_\epsilon \) be the set of all achievable rate-exponent vectors for a fixed \( \epsilon \). The rate-exponent region \( \mathcal{RE} \) is defined as
\[ \mathcal{RE} \triangleq \bigcap_{\epsilon > 0} \mathcal{RE}_\epsilon. \]

Our goal is to characterize the region \( \mathcal{RE} \).

**Remark 5.1:** This formulation has an obvious asymmetry between the type 1 and type 2 error probabilities; the type 2 error probability is required to decrease to zero exponentially, but the type 1 error probability is only required to decrease to zero at any rate. This is akin to Stein’s lemma [53, Theorem 11.8.3]. One could also consider the symmetric problem in which it is required that both type 1 and type 2 error probabilities tend to zero exponentially with exponents \( E_1 \) and \( E_2 \), respectively, and the goal is to characterize all achievable rate-exponent vectors \((R_1, \ldots, R_L, E_1, E_2)\). This formulation has been studied
previously and some results have been obtained for two special cases of the problem, namely zero-rate and one-bit compression [26]. The general problem, however, is difficult even for the test against independence with $L = 1$. Schemes for the asymmetric problem are applicable here, but the resultant achievable regions will have Chernoff-type exponents [53, p. 384]. These exponents are difficult to analyze, and proving their optimality, if they are in fact optimal, seems outside the reach of existing techniques.

5.2.1 Entropy Characterization of the Rate-Exponent Region

We start with the entropy characterization of the rate-exponent region. We shall use it later in the chapter to obtain inner and outer bounds. Define the set

$$
\mathcal{RE}_* \triangleq \bigcup_n \bigcup_{(f_l^{(n)})_{l \in \mathcal{L}}} \mathcal{RE}_* \left( n, \left( f_l^{(n)} \right)_{l \in \mathcal{L}} \right),
$$

where

$$
\mathcal{RE}_* \left( n, \left( f_l^{(n)} \right)_{l \in \mathcal{L}} \right) \triangleq \left\{ (R,E) : R_l \geq \frac{1}{n} \log \left| f_l^{(n)}(X_l^n) \right| \text{ for all } l \in \mathcal{L}, \text{ and } \right. \\
E \leq \frac{1}{n} D \left( P \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}} Y^n \left| Q \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}} Y^n \right) \right) \right\}.
$$

(5.1)

We have the following Proposition.

**Proposition 1.**

$$
\mathcal{RE} = \overline{\mathcal{RE}_*}.
$$

The proof of Proposition 1 is a straightforward generalization of that of Theorem 1 in [27] and is hence omitted. Ahlswede and Csiszár [27] showed that for
$L = 1$, the strong converse holds, i.e., $\mathcal{RE}_\epsilon$ is independent of $\epsilon$. Thus, $\overline{\mathcal{RE}_\epsilon}$ is essentially a characterization for both $\mathcal{RE}$ and $\mathcal{RE}_\epsilon$. One can expect the same to hold for the problem under consideration. It however remains to be investigated. We next study a class of instances of the problem before returning to the general problem in Section 5.7.

5.3 $L$-Encoder Hypothesis Testing against Conditional Independence

We consider a class of instances of the general problem, referred to as the $L$-encoder hypothesis testing against conditional independence, and obtain inner and outer bounds to the rate-exponent region. These bounds coincide and characterize the region completely in some cases. Moreover, the outer bound for this problem can be used to give an outer bound for a more general class of problems, as we shall see later.

Let $X_{L+1}$ and $Z$ be discrete memoryless sources taking values in finite sets $\mathcal{X}_{L+1}$ and $\mathcal{Z}$, respectively such that $(X, X_{L+1})$ and $Y$ are conditionally independent given $Z$ under $H_1$, and the distributions of $(X, X_{L+1}, Z)$ and $(Y, Z)$ are the same under both hypotheses, i.e.,

$$H_0 : P_{XX_{L+1}Y|Z}P_Z$$
$$H_1 : P_{XX_{L+1}|Z}P_{Y|Z}P_Z.$$  

The problem formulation is the same as before with $Y$ replaced by $(X_{L+1}, Z, Y)$ in it. The reason for focusing on this special case is that the relative entropy in (5.1) becomes a mutual information, which simplifies the analysis. Let $\mathcal{RE}^{CI}$
be the rate-exponent region of this problem. Here “CI” stands for conditional independence. Let

\[ \mathcal{RE}_{CI}^* \triangleq \bigcup_{n} \bigcup_{(f_l^{(n)})_{l \in L}} \mathcal{RE}_{CI}^* \left( n, \left( f_l^{(n)} \right)_{l \in L} \right), \]

where

\[ \mathcal{RE}_{CI}^* \left( n, \left( f_l^{(n)} \right)_{l \in L} \right) \triangleq \left\{ \left( R, E \right) : R_l \geq \frac{1}{n} \log \left| f_l^{(n)}(X_l^n) \right| \text{ for all } l \in \mathcal{L}, \text{ and } \right. \]

\[ \left. E \leq \frac{1}{n} I \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in L}, X_{L+1}^n; Y^n \big| Z^n \right) \right\}. \]

We have the following corollary as a consequence of Proposition 1.

**Corollary 2.**

\[ \mathcal{RE}_{CI} = \overline{\mathcal{RE}_{CI}^*}. \]

With mutual information replacing relative entropy, the problem can be analyzed using techniques from distributed rate-distortion. In particular, both inner and outer bounds for that problem can be applied here.

### 5.3.1 Quantize-Bin-Test Inner Bound

Our inner bound is based on a simple scheme which we call the Quantize-Bin-Test scheme. In this scheme, encoders, as in the Shimokawa-Han-Amari scheme, quantize and then bin their observations, but the detector now performs the test directly using the bins. The inner bound obtained is similar to the generalized Berger-Tung inner bound for distributed source coding [8, 9, 54]. Let \( \Lambda_i \) be the set of finite-alphabet random variables \( \lambda_i = (U_1, \ldots, U_L, T) \) satisfying
(C1) $T$ is independent of $(X, X_{L+1}, Y, Z)$, and

(C2) $U_l \leftrightarrow (X_l, T) \leftrightarrow (U_l, X_l, X_{L+1}, Y, Z)$ for all $l$ in $L$.

Define the set

$$\mathcal{RE}_i^{CI}(\lambda_i) \triangleq \left\{ (R, E) : \sum_{l \in S} R_l \geq I(X_S; U_S|U_{S^c}, X_{L+1}, Z, T) \text{ for all } S \subseteq L, \text{ and } E \leq I(Y; U, X_{L+1}|Z, T) \right\}$$

and let

$$\mathcal{RE}_i^{CI} \triangleq \bigcup_{\lambda_i \in \Lambda_i} \mathcal{RE}_i^{CI}(\lambda_i).$$

The following lemma asserts that $\mathcal{RE}_i^{CI}$ is computable and closed.

**Lemma 14.** (a) $\mathcal{RE}_i^{CI}$ remains unchanged if we impose the following cardinality bound on $(U, T)$ in $\Lambda_i$

$$|U_l| \leq |X_l| + 2^L - 1 \text{ for all } l \text{ in } L, \text{ and }$$

$$|T| \leq 2^L.$$

(b) $\mathcal{RE}_i^{CI}$ is closed.

**Proof.** See Appendix C.1.

Although the cardinality bound is exponential in the number of encoders, one can obtain an improved bound by exploiting the contra-polymatroid structure of $\mathcal{RE}_i^{CI}$ [55, 56]. We do not do so here because it is technically involved and we just want to prove that $\mathcal{RE}_i^{CI}$ is computable. The following theorem gives an inner bound to the rate-exponent region.
Theorem 11. \[ \mathcal{RE}_i^{CI} \subseteq \mathcal{RE}^{CI}. \]

Proof. See Appendix C.2. \qed

Remark 5.2: Although our inner bound is for the special case of the test against conditional independence, it can be generalized for the general case. But, the inner bound thus obtained will be quite complicated with competing exponents, and it is not needed in this work.

It is worth pointing out that the Quantize-Bin-Test scheme is in general suboptimal for problems in which encoders’ observations have common randomness, i.e., there exist deterministic functions of encoders’ observations that is common to encoders. However, it is straightforward to generalize this scheme by using the idea from the common-component scheme for distributed source coding problems [49].

5.3.2 Outer Bound

Let \( \Lambda_o \) be the set of finite-alphabet random variables \( \lambda_o = (U, W, T) \) satisfying

(C3) \( (W, T) \) is independent of \( (X, X_{L+1}, Y, Z) \), and

(C4) \( U_l \leftrightarrow (X_l, W, T) \leftrightarrow (U_{l'}, X_{l'}, X_{L+1}, Y, Z) \) for all \( l \) in \( \mathcal{L} \),

and let \( \chi \) be the set of finite-alphabet random variable \( X \) such that \( X_1, \ldots, X_{L}, X_{L+1}, Y \) are conditionally independent given \( (X, Z) \). Note that \( \chi \)
is nonempty because it contains \((X, X_{L+1})\). For a given \(X\) in \(\chi\) and \(\lambda_o\) in \(\Lambda_o\), the joint distribution of \(X, (X, X_{L+1}, Y, Z)\), and \(\lambda_o\) satisfy the Markov condition

\[
X \leftrightarrow (X, X_{L+1}, Y, Z) \leftrightarrow \lambda_o.
\]

Define the set

\[
\mathcal{RE}^{CI}_o(X, \lambda_o) \triangleq \{(R, E) : \sum_{l \in S} R_l \geq I(X; U_S | U_{S^c}, X_{L+1}, Z, T) + \sum_{l \in S} I(X_l; U_l | X, W, X_{L+1}, Z, T) \forall S \subseteq \mathcal{L}, \text{ and } E \leq I(Y; U, X_{L+1} | Z, T)\}.
\]

Also let

\[
\mathcal{RE}^{CI}_o \triangleq \bigcap_{X \in \chi} \bigcup_{\lambda_o \in \Lambda_o} \mathcal{RE}^{CI}_o(X, \lambda_o).
\]

We have the following outer bound to the rate-exponent region.

**Theorem 12.**

\[
\mathcal{RE}^{CI}_* \subseteq \mathcal{RE}^{CI}_o.
\]

And therefore

\[
\mathcal{RE}^{CI} \subseteq \overline{\mathcal{RE}^{CI}_o}.
\]

**Proof.** The proof of the first inclusion is presented in Appendix C.3. The first inclusion and Corollary 2 imply the second inclusion. \(\square\)

This outer bound is similar to an outer bound for multiterminal source coding obtained by Wagner and Anantharam [37]. As noted there, the key step is the introduction of the auxiliary random variable \(X\), which, unlike most auxiliary random variables, does not represent a component of the code. Rather, it is
used to induce conditional independence among the observations. Conditional independence is a useful simplifying assumption in distributed detection [57] and multiterminal source coding [13, 38, 39, 41, 42, 43, 44]. This paper will show that it is also useful here. The utility of this bound is that it allows us to handle problems that lack an intrinsic conditional independence. The bound tends to be tightest when the problem already contains the right conditional independence structure. The next three sections provide examples. In Section 5.7, we will see how to extend the outer bound to a more general setting.

5.4 1-Encoder Hypothesis Testing against Conditional Independence

In this section, we study a special case in which $L = 1$. This problem is the conditional version of the test against independence studied by Ahlswede and Csiszár [27]. The conditional version however is complicated because of the binning process. We prove that the inner and outer bounds coincide for this problem, which in turn proves that the Quantize-Bin-Test scheme is optimal. We also prove that in this case the Shimokawa-Han-Amari inner bound simplifies to the Quantize-Bin-Test inner bound, establishing that the Shimokawa-Han-Amari scheme is also optimal.
5.4.1 Rate-Exponent Region

Theorem 13.

\[ \mathcal{RE}^{CI} = \mathcal{RE}_o^{CI} = \mathcal{RE}_i^{CI} = \mathcal{RE}^{CI} \triangleq \left\{ (R_1, E) : \text{there exists } U_1 \text{ such that} \right. \]
\[ R_1 \geq I(X_1; U_1|X_2, Z), \]
\[ E \leq I(Y; U_1, X_2|Z), \]
\[ |U_1| \leq |X_1| + 1, \text{ and} \]
\[ U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z) \right\}. \] (5.2)

Proof. To show (5.2), it suffices to show that

\[ \mathcal{RE}_o^{CI} \subseteq \mathcal{RE}_i^{CI}, \]

because \( \mathcal{RE}_i^{CI} \) is closed from Lemma 14(b). Consider \((R_1, E)\) in \( \mathcal{RE}_o^{CI} \). Take \( X = X_2 \). It is evident that \( X_2 \) is in \( \chi \). Then there exists \( \lambda_o = (U_1, W, T) \) in \( \Lambda_o \) such that \((R_1, E)\) is in \( \mathcal{RE}_o^{CI}(X_2, \lambda_o) \), i.e.,

\[ R_1 \geq I(X_2; U_1|X_2, Z, T) + I(X_1; U_1|X_2, Z, W, T) \]
\[ = I(X_1; U_1|X_2, Z, W, T), \]

and

\[ E \leq I(Y; U_1, X_2|Z, T) \]
\[ = H(Y|Z, T) - H(Y|U_1, X_2, Z, T) \]
\[ \leq H(Y|Z, W, T) - H(Y|U_1, X_2, Z, W, T) \] (5.4)
\[ = I(Y; U_1, X_2|Z, W, T), \]
where (5.4) follows from conditioning reduces entropy and the fact that \((Y, Z)\) is independent of \((W, T)\). If we set \(\tilde{T} = (W, T)\), then it is easy to verify that 
\[ \lambda_i = (U_1, \tilde{T}) \] is in \(\Lambda_i\) and we have

\[
R_1 \geq I(X_1; U_1|X_2, Z, \tilde{T}) \quad \text{and} \\
E \leq I(Y; U_1, X_2|Z, \tilde{T}).
\] (5.5) (5.6)

Therefore, \((R_1, E)\) is in \(\mathcal{RE}^{CI}_i(\lambda_i)\), which implies that \((R_1, E)\) is in \(\mathcal{RE}^{CI}_i\). This completes the proof of (5.2).

To prove (5.3), it suffices to show that 
\[
\mathcal{RE}^{CI}_i \subseteq \mathcal{RE}^{CI}.
\]

The reverse containment immediately follows if we restrict \(T\) to be deterministic in the definition of \(\mathcal{RE}^{CI}_i\). Continuing from the proof of (5.2), let \(\tilde{U}_1 = (U_1, \tilde{T})\). Since \((U_1, \tilde{T})\) is in \(\Lambda_i\), we have that \(\tilde{T}\) is independent of \((X_1, X_2, Y, Z)\) and that 
\[ U_1 \leftrightarrow (\tilde{T}, X_1) \leftrightarrow (X_2, Y, Z). \]

Both together imply that 
\[ \tilde{U} \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z). \]

We next have from (5.5) that
\[
R_1 \geq I(X_1; U_1|X_2, Z, \tilde{T}) \quad \text{and} \\
= I(X_1; U_1|X_2, Z, \tilde{T}) + I(X_1; \tilde{T}|X_2, Z) \quad \text{(5.7)} \\
= I(X_1; U_1, \tilde{T}|X_2, Z) \\
= I(X_1; \tilde{U}_1|X_2, Z),
\]
where (5.7) follows because \(\tilde{T}\) is independent of \((X_1, X_2, Y, Z)\). And (5.6) similarly yields
\[
E \leq I(Y; \tilde{U}_1, X_2|Z).
\]
Using the support lemma [58, Lemma 3.4, pp. 310] as in the proof of Lemma 14(a), we can obtain the cardinality bound

$$|\tilde{U}_1| \leq |X_1| + 1.$$ 

We thus conclude that $(R_1, E)$ is in $\tilde{RE}^{CI}$. \hfill \Box

### 5.4.2 Related Source Coding Problem

![Related source coding problem](image)

Figure 5.1: Related source coding problem.

The conclusion of Theorem 13 can be used to relate the problem to a source coding problem which is depicted in Fig. 5.1. Here encoders 0 and 1 respectively compress i.i.d. strings distributed according to $Y$ and $X_1$ and send messages to the decoder at rates $R_0$ and $R_1$, respectively. The decoder losslessly reproduces the $Y$ string using the two messages and the side information $(X_2, Z)$. We are interested in characterizing the rate region of this problem. This problem is a generalization of the source coding problem studied by Ahlswede and Körner [59]. It follows from the generalization of their result that the rate region of this
problem is

\[ \mathcal{R}^{SC} \triangleq \{(R_0, R_1) : \text{there exists } U_1 \text{ such that} \]
\[ R_1 \geq I(X_1; U_1 | X_2, Z), \]
\[ R_0 \geq H(Y | U_1, X_2, Z), \]
\[ |U_1| \leq |X_1| + 1, \text{ and} \]
\[ U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z) \}. \]

On comparing the rate region \( \mathcal{R}^{SC} \) with the rate-exponent region \( \mathcal{RE}^{CI} \) in Theorem 13, we can conclude that there is a one-to-one correspondence between the two if we replace \( R_0 \) with \( H(Y | Z) - E \) and vice versa. Hence, the converse proof of the source coding problem applies to the hypothesis testing problem at hand. This is the key behind all converse results in this chapter. Ahlswede and Csiszár first observed this relation in [27]. Tian and Chen [45] later used it under the successive refinement setting.

### 5.4.3 Optimality of Shimokawa-Han-Amari Scheme

The Shimokawa-Han-Amari scheme operates as follows. Consider a test channel \( P_{U_1|X_1} \), a sufficiently large block length \( n \), and \( \alpha > 0 \). Let \( \tilde{R}_1 = I(X_1; U_1) + \alpha \).

To construct the codebook, we first generate \( 2^{n\tilde{R}_1} \) independent codewords \( U_1^n \), each according to \( \prod_{i=1}^n P_{U_1}(u_{1i}) \), and then distribute them uniformly into \( 2^{nR_1} \) bins. The codebook and the bin assignment are revealed to the encoder and the detector. The encoder first quantizes \( X_1^n \) by selecting a codeword \( U_1^n \) that is jointly typical with it. With high probability, there will be at least one such codeword. The encoder then sends to the detector the index of the bin to which the codeword \( U_1^n \) belongs. The joint type of \( (X_1^n, U_1^n) \) is also sent to the detector,
which requires zero additional rate asymptotically. The detector finds a codeword \( \hat{U}_1^n \) in the bin that minimizes the empirical entropy \( H(U_1^n, Y^n) \). It then performs the test and declares \( H_0 \) if and only if both \((X_1^n, U_1^n)\) and \((Y^n, \hat{U}_1^n)\) are jointly typical under \( H_0 \). The inner bound thus obtained is as follows. Define

\[
A(R_1) \triangleq \left\{ U_1 : R_1 \geq I(X_1; U_1|X_2, Y, Z), \ U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z), \ \text{and} \ |U_1| \leq |X_1| + 1 \right\}
\]

\[
B(U_1) \triangleq \left\{ P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} : P_{\hat{U}_1, \hat{X}_1} = P_{U_1, X_1} \ \text{and} \ P_{\hat{U}_1, \hat{X}_2, \hat{Y}, \hat{Z}} = P_{U_1, X_2, Y, Z} \right\}, \ \text{and}
\]

\[
C(U_1) \triangleq \left\{ P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} : P_{\hat{U}_1, \hat{X}_1} = P_{U_1, X_1}, \ P_{\hat{X}_2, \hat{Y}, \hat{Z}} = P_{X_2, Y, Z}, \ \text{and} \ H(\hat{U}_1|\hat{X}_2, \hat{Y}, \hat{Z}) \geq H(U_1|X_2, Y, Z) \right\}.
\]

In addition, define the exponents

\[
\rho^*_1(U_1) \triangleq \min_{P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} \in B(U_1)} D\left( P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} \parallel P_{U_1, X_1, X_2, Y, Z} \right)
\]

\[
\rho^*_2(U_1) \triangleq \begin{cases} +\infty & \text{if } R_1 \geq I(U_1; X_1) \\ \rho_2(U_1) & \text{otherwise} \end{cases}
\]

\[
\rho_2(U_1) \triangleq \left[ R_1 - I(X_1; U_1|X_2, Y, Z) \right]^+
\]

\[
+ \min_{P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} \in C(U_1)} D\left( P_{\hat{U}_1, \hat{X}_1, \hat{X}_2, \hat{Y}, \hat{Z}} \parallel P_{U_1, X_1, X_2, Y, Z} \right).
\]

Finally, define

\[
E_{SHA}(R_1) \triangleq \max_{U_1 \in A(R_1)} \min (\rho^*_1(U_1), \rho^*_2(U_1)).
\]

Recall that \( \rho^*_2(U_1) \) and \( \rho^*_1(U_1) \) are the exponents associated with type 2 errors due to binning errors and assuming correct decoding of the codeword, respectively.

**Theorem 14.** \([29]\) \((R_1, E)\) is in the rate-exponent region if

\[
E \geq E_{SHA}(R_1).
\]
Fig. 5.2 shows the Shimokawa-Han-Amari achievable exponent as a function of the rate assuming a fixed channel $P_{U_1|X_1}$ is used for quantization. This is simply Fig. 1.4 particularized to the 1-encoder hypothesis testing against conditional independence problem. For rates $R_1 \geq I(X_1, U_1|X_2, Z)$, $\rho_1^*(U_1)$ dominates $\rho_2^*(U_1)$ and there is no penalty for binning at these rates as the exponent stays the same. Therefore, we can bin all the way down to the rate $R_1 = I(X_1, U_1|X_2, Z)$ without any loss in the exponent. However, if we bin further at rates $R_1$ in $[I(X_1, U_1|X_2, Y, Z), I(X_1, U_1|X_2, Z))$, then $\rho_2^*(U_1)$ dominates $\rho_1^*(U_1)$, the exponent decreases linearly with $R_1$, and the performance deteriorates all the way down to a point at which the message from the encoder is useless. At this point, the binning rate $R_1$ equals $I(X_1, U_1|X_2, Y, Z)$ and the exponent equals $I(Y; X_2|Z)$, which is the exponent when the detector ignores the encoder’s message. This competition between the exponents makes the optimality of the Shimokawa-Han-Amari scheme unclear. We prove that it is indeed optimal by showing that the Shimokawa-Han-Amari inner bound simplifies to the Quantize-Bin-Test inner bound.
ner bound, which by Theorem 13 is tight. Let us define

$$\mathcal{A}^*(R_1) \triangleq \left\{ U_1 : R_1 \geq I(X_1; U_1|X_2, Z), \ U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z), \ \text{and} \ |U_1| \leq |X_1|+1 \right\}$$

and

$$E_{QBT}(R_1) \triangleq \max_{U_1 \in \mathcal{A}^*(R_1)} I(Y; U_1, X_2|Z).$$

We have the following theorem.

**Theorem 15.** If \((R_1, E)\) is in the rate-exponent region, then

$$E \leq E_{QBT}(R_1) = E_{SHA}(R_1).$$

**Proof.** The inequality follows from Theorem 13. To prove the equality, it is sufficient to show that

$$E_{SHA}(R_1) \geq E_{QBT}(R_1).$$

The reverse inequality follows from Theorems 13 and 14. Since conditioning reduces entropy and any \(U_1\) in \(\mathcal{A}^*(R_1)\) satisfies the Markov chain

$$U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y, Z),$$

we have

$$R_1 \geq I(X_1; U_1|X_2, Z)$$

$$= H(U_1|X_2, Z) - H(U_1|X_1X_2, Z)$$

$$\geq H(U_1|X_2, Y, Z) - H(U_1|X_1X_2, Y, Z)$$

$$= I(X_1; U_1|X_2, Y, Z),$$

which means that \(U_1\) is in \(\mathcal{A}(R_1)\). Hence, \(\mathcal{A}^*(R_1) \subseteq \mathcal{A}(R_1)\). This implies that

$$E_{SHA}(R_1) \triangleq \max_{U_1 \in \mathcal{A}(R_1)} \min \left( \rho_1^*(U_1), \rho_2^*(U_1) \right)$$

$$\geq \max_{U_1 \in \mathcal{A}^*(R_1)} \min \left( \rho_1^*(U_1), \rho_2^*(U_1) \right). \quad (5.8)$$
Now the objective of the optimization problem in the definition of $\rho_1^*(U_1)$ can be lower bounded as

$$D(P_{\tilde{U}_1 \tilde{X}_1 \tilde{X}_2 \tilde{Y} \tilde{Z}} \| P_{U_1|X_1} P_{X_1|X_2} P_{Y|Z} P_Z) \geq D(P_{\tilde{U}_1 \tilde{X}_2 \tilde{Y} \tilde{Z}} \| P_{U_1} P_{Y|Z} P_Z)$$

$$= D(P_{U_1 X_2 Y Z} \| P_{U_1} P_{Y|Z} P_Z)$$

$$= I(Y; U_1, X_2|Z).$$

The lower bound is achieved by the distribution $P_{U_1 X_2 Y Z} P_{X_1|U_1 X_2 Z}$ in $B(U_1)$. Therefore,

$$\rho_1^*(U_1) = I(Y; U_1, X_2|Z).$$

Similarly, we can lower bound the optimization problem in the definition of $\rho_2(U_1)$ as

$$D(P_{\tilde{U}_1 \tilde{X}_1 \tilde{X}_2 \tilde{Y} \tilde{Z}} \| P_{U_1|X_1} P_{X_1|X_2} P_{Y|Z} P_Z) \geq D(P_{\tilde{U}_1 \tilde{X}_2 \tilde{Y} \tilde{Z}} \| P_{X_2} P_{Y|Z} P_Z)$$

$$= D(P_{X_2 Y Z} \| P_{X_2} P_{Y|Z} P_Z)$$

$$= I(Y; X_2|Z),$$

and the lower bound is achieved by the distribution $P_{X_2 Y Z} P_{U_1 X_1|X_2 Z}$ in $C(U_1)$. Therefore,

$$\rho_2(U_1) = [R_1 - I(X_1; U_1|X_2, Y, Z)]^+ + I(Y; X_2|Z).$$

Consider any $U_1$ in $A^*(R_1)$. If $R_1 \geq I(X_1; U_1)$, then

$$\min (\rho_1^*(U_1), \rho_2^*(U_1)) = \rho_1^*(U_1)$$

$$= I(Y; U_1, X_2|Z).$$

(5.9)
And if \( I(X_1; U_1) > R_1 \geq I(X_1; U_1|X_2, Z) \), then

\[
\min \left( \rho_1^*(U_1), \rho_2^*(U_1) \right) \\
= \min \left( I(Y; U_1, X_2|Z), R_1 - I(X_1; U_1|X_2, Y, Z) + I(Y; X_2|Z) \right) \\
\geq \min \left( I(Y; U_1, X_2|Z), I(X_1; U_1|X_2, Z) - I(X_1; U_1|X_2, Y, Z) + I(Y; X_2|Z) \right) \\
= \min \left( I(Y; U_1, X_2|Z), I(Y; U_1|X_2, Z) + I(Y; X_2|Z) \right) \\
= \min \left( I(Y; U_1, X_2|Z), I(Y; U_1, X_2|Z) \right) \\
= I(Y; U_1, X_2|Z).
\]

(5.10)

Now (5.8) through (5.10) imply

\[
E_{SHA}(R_1) \geq \max_{U_1 \in A^*(R_1)} I(Y; U_1, X_2|Z) \\
= E_{QBT}(R_1).
\]

Theorem 15 is thus proved. \(\Box\)

### 5.5 Gel’fand and Pinsker Hypothesis Testing against Independence

In some cases, it is possible to achieve the centralized performance, which is obviously the best that we can hope for, even with data that is compressed in a decentralized manner. For such problems, we would like to characterize the rates for which we can compress the data and achieve the centralized performance. We study one such problem in this section. We call this the Gel’fand and Pinsker hypothesis testing against independence problem, because it is related to the source coding problem studied by Gel’fand and Pinsker [42]. This problem is a special case of the \(L\)-encoder hypothesis testing against conditional
independence problem. For this problem, we characterize the set of all rate vectors that achieve the centralized type 2 error exponent, which is the exponent attained by using the uncompressed data and is given by Stein’s lemma [53, Theorem 11.8.3]. In particular, we show that the inner and outer bounds of Section 5.3 associated with the centralized type 2 error exponent coincide.

We focus on a class of problems in which $X_{L+1}$ and $Z$ are deterministic and there exists a function of $X_1, \ldots, X_L$, say $X$, such that under $H_0$,

(C5) $X_1, \ldots, X_L, Y$ are conditionally independent given $X$, and

(C6) for any finite-alphabet random variable $U$ such that $Y \leftrightarrow X \leftrightarrow U$ and $Y \leftrightarrow U \leftrightarrow X$, we have $H(X|U) = 0$.

The condition (C5) is the usual conditional independence condition that has been studied extensively in distributed source coding [13, 37, 38, 39, 40, 41]. Note that this condition also implies that $X$ is a sufficient statistic for $Y$ given $X$. The condition (C6) imposes an additional constraint on $X$. It is required that under $H_0$, $X$ be a minimal sufficient statistic for $Y$ given $X$, i.e., $X$ is a function of every other sufficient statistic.

Certainly not every $X$ that satisfies (C6) satisfies (C5). And not every $X$ satisfying (C5) satisfies (C6). As an example, let $W$ be a probability transition matrix with distinct rows. Suppose that under $H_0$, we have that $X_2, \ldots, X_L$ and $Y$ are conditionally i.i.d. given $X_1$ with conditional marginal $W$. Then choosing $X = X_1$ clearly satisfies (C5) and can be shown to satisfy (C6) (see Appendix C.4 for the proof of similar result). Choose $X$ to be the vector $(X_1, \ldots, X_L)$, on the other hand, satisfies (C5) but not (C6).

We shall now characterize the centralized rate region, the set of rate vectors
that achieve the centralized type 2 error exponent \( I(X; Y) = I(X; Y) \), for this class of problems. More precisely, we shall characterize the set

\[
\{ R : (R, I(X; Y)) \in \mathcal{R}^C I \},
\]

denoted by \( \mathcal{R}^C I(I(X; Y)) \). Let us similarly define

\[
\mathcal{R}^C I_i(I(X; Y)) \triangleq \{ R : (R, I(X; Y)) \in \mathcal{R}^C I_i \}
\]

and

\[
\overline{\mathcal{R}}^C O(I(X; Y)) \triangleq \left\{ R : (R, I(X; Y)) \in \overline{\mathcal{R}}^C O \right\}.
\]

We need the following lemma.

**Lemma 15.** Condition (C6) is equivalent to

\[(C7) \text{ For any positive } \epsilon, \text{ there exists a positive } \delta \text{ such that for all finite-alphabet random variable } U \text{ such that } Y \leftrightarrow X \leftrightarrow U \text{ and } I(X; Y|U) \leq \delta, \text{ we have } H(X|U) \leq \epsilon.\]

**Proof.** See Appendix C.4. \(\square\)

Let us define a function

\[
\phi(\delta) \triangleq \inf \left\{ \epsilon : \text{ for all finite-alphabet } U \text{ such that } Y \leftrightarrow X \leftrightarrow U \text{ and } I(X; Y|U) \leq \delta, \text{ we have } H(X|U) \leq \epsilon \right\}.
\]

It is clear that \( \phi \) is continuous at zero with the value \( \phi(0) = 0 \). We have the following theorem.

**Theorem 16.**

\[
\mathcal{R}^C I(I(X; Y)) = \mathcal{R}^C I_i(I(X; Y)) = \overline{\mathcal{R}}^C O(I(X; Y)).
\]
Proof. It suffices to show that
\[ \overline{R}^C I_o(I(X; Y)) \subseteq R^C I_i(I(X; Y)). \]

Consider any \( R \) in \( \overline{R}^C I_o(I(X; Y)) \), any positive \( \delta \), and \( X \) defined as above. Then there exists \( \lambda_o = (U, W, T) \) in \( \Lambda_o \) such that \( (R_1 + \delta, \ldots, R_L + \delta, I(X; Y) - \delta) \) is in \( \mathcal{RE}_o^C I(X, \lambda_o) \), i.e.,
\[
\sum_{l \in S} (R_l + \delta) \geq I(X; U_S|U_{S^c}, T) + \sum_{l \in S} I(X_l; U_l|X, W, T) \quad \forall \ S \subseteq \mathcal{L}, \quad \text{and} \quad (5.11)
\]
\[
I(X; Y) - \delta \leq I(Y; U|T). \tag{5.12}
\]

We have the Markov chain
\[ Y \leftrightarrow X \leftrightarrow (U, T), \]
which implies
\[
I(X; Y|U, T) = H(Y|U, T) - H(Y|X, U, T)
\]
\[
= H(Y|U, T) - H(Y|X)
\]
\[
= I(X; Y) - I(Y; U|T)
\]
\[
\leq \delta,
\]
where the last inequality follows from (5.12). Therefore, by the definition of \( \phi \) function
\[
H(X|U, T) \leq \phi(\delta). \tag{5.13}
\]

Now
\[
I(X; U_S|U_{S^c}, T) = H(X|U_{S^c}, T) - H(X|U, T)
\]
\[
\geq H(X|U_{S^c}, W, T) - \phi(\delta) \tag{5.14}
\]
\[
\geq I(X; U_S|U_{S^c}, W, T) - \phi(\delta),
\]
118
where (5.14) follows from (5.13) and the fact that conditioning reduces entropy. This together with (5.11) implies
\[
\sum_{l \in S} (R_l + \delta + \phi(\delta)) \geq I(X; U_S|U_{S^c}, W, T) + \sum_{l \in S} I(X_l; U_l|X, W, T)
\]
\[
= I(X; U_S|U_{S^c}, W, T) + I(X_S; U_{S^c}, X, W, T)
\]
\[
= I(X, X_S; U_S|U_{S^c}, W, T)
\]
\[
\geq I(X_S; U_S|U_{S^c}, W, T).
\]

Again since conditioning reduces entropy and \(Y\) is independent of \((W, T)\), we obtain from (5.12) that
\[
I(X; Y) - \delta \leq I(Y; U|T)
\]
\[
= H(Y|T) - H(Y|U, T)
\]
\[
\leq H(Y|W, T) - H(Y|U, W, T)
\]
\[
= I(Y; U|W, T).
\]

Define \(\tilde{T} = (W, T)\). It is then clear that \(\lambda_i = (U, \tilde{T})\) is in \(\Lambda_i\),
\[
\sum_{l \in S} (R_l + \delta + \phi(\delta)) \geq I(X_S; U_S|U_{S^c}, \tilde{T}) \text{ for all } S \subseteq \mathcal{L}, \text{ and}
\]
\[
I(X; Y) - \delta \leq I(Y; U|\tilde{T}).
\]

Hence, \((R_1 + \delta + \phi(\delta), \ldots, R_L + \delta + \phi(\delta), I(X; Y) - \delta)\) is in \(\mathcal{R}\mathcal{E}_i^{CI}(\lambda_i)\), which implies that \((R, I(X; Y))\) is in \(\mathcal{R}\mathcal{E}_i^{CI}\) because \(\mathcal{R}\mathcal{E}_i^{CI}\) is closed from Lemma 14(b). Therefore, \(R\) is in \(\mathcal{R}_i^{CI}(I(X; Y))\). \(\Box\)
5.6 Gaussian Many-Help-One Hypothesis Testing against Independence

We now turn to a continuous example of the problem studied in Section 5.3. This problem is related to the quadratic Gaussian many-help-one source coding problem [13, 43, 44]. We first obtain an outer bound similar to the one in Theorem 12, and then show that it is achieved by the Quantize-Bin-Test scheme.

Let \((X,Y,X_1,\ldots,X_L)\) be a zero-mean Gaussian random vector such that under both hypotheses
\[X_l = X + N_l\]
for each \(l\) in \(L\). \(X\) and \(Y\) are correlated under null hypothesis \(H_0\) and are independent under alternate hypothesis \(H_1\), i.e.,
\[H_0 : Y = X + N\]
\[H_1 : Y \perp X.\]

We assume that \(X, N, N_1, N_2, \ldots, N_L\) are mutually independent, and that \(\sigma_N^2\) and \(\sigma_{N_l}^2\) are positive. The setup of the problem is shown in Fig. 1.6. Unlike the previous problem, we now allow \(X\) to be observed by an encoder, which sends a message to the detector at a finite rate \(R\). We use \(f^{(n)}\) to denote the corresponding encoding function. In order to be consistent with the source coding terminology, we call this the main encoder. The encoder observing \(X_l\) is now called helper \(l\). We assume that \(X_{L+1}\) and \(Z\) are deterministic. The rest of the problem formulation is similar to the one in Section 5.2. Let \(\mathcal{RE}^{MHO}\) be the rate-exponent region of this problem. We need the entropy characterization of \(\mathcal{RE}^{MHO}\). For
that, define
\[
\mathcal{RE}_{*}^{MHO} \triangleq \bigcup_{n} \bigcup_{f^{(n)}, f^{(n)}_{l} \in \mathcal{L}} \mathcal{RE}_{*}^{MHO} (n, (f^{(n)}_{l})),
\]
where
\[
\mathcal{RE}_{*}^{MHO} (n, (f^{(n)}_{l} \in \mathcal{L})) \triangleq \left\{ (R, R_{l}, E) : R \geq \frac{1}{n} \log \left| f^{(n)}(X_{n}) \right|, \right.
\]
\[
R_{l} \geq \frac{1}{n} \log \left| f^{(n)}_{l}(X_{n}) \right| \text{ for all } l \text{ in } \mathcal{L}, \text{ and}
\]
\[
E \leq \frac{1}{n} I \left( Y^{n}; f^{(n)}(X^{n}), \left( f^{(n)}_{l}(X^{n}) \right)_{l \in \mathcal{L}} \right).
\]

Corollary 3.
\[
\mathcal{RE}^{MHO} = \mathcal{RE}_{*}^{MHO}.
\]

The corollary follows as an extension of Proposition 1. Define the set
\[
\tilde{\mathcal{RE}}^{MHO} \triangleq \left\{ (R, R_{1}, \ldots, R_{L}, E) : \text{there exists } (r_{1}, \ldots, r_{L}) \in \mathbb{R}_{+}^{L} \text{ such that}
\right.
\]
\[
R_{l} \geq r_{l} \text{ for all } l \text{ in } \mathcal{L}, \text{ and}
\]
\[
R + \sum_{l \in S} R_{l} \geq \frac{1}{2} \log^{+} \left[ \frac{1}{D} \left( \frac{1}{\sigma_{X}^{2}} + \sum_{l \in S^{c}} \frac{1 - 2^{-2r_{l}}}{\sigma_{N_{l}}^{2}} \right)^{-1} \right] + \sum_{l \in S} r_{l}, \forall S \subseteq \mathcal{L},
\]
where
\[
D = (\sigma_{X}^{2} + \sigma_{N}^{2}) 2^{-2E} - \sigma_{N}^{2}.
\]

Theorem 17.
\[
\mathcal{RE}_{*}^{MHO} = \tilde{\mathcal{RE}}^{MHO}.
\]

Proof. The proof of inclusion \( \mathcal{RE}_{*}^{MHO} \subseteq \tilde{\mathcal{RE}}^{MHO} \) is similar to the converse proof of the Gaussian many-help-one source coding problem by Oohama [43] and Vinod et al. [44] (see also [37]). Their proofs continue to work if we replace the
original mean square error distortion constraint with the mutual information constraint that we have here. It is noteworthy though that Wang et al.’s [60] approach does not work here because it relies on the distortion constraint.

We start with the continuous extension of Theorem 12. Let \( \Lambda_o \) be the set of random variables \( \lambda_o = (U, U, W, T) \) such that each take values in a finite-dimensional Euclidean space and collectively they satisfy

(C8) \( (W, T) \) is independent of \( (X, X, Y) \),

(C9) \( U \leftrightarrow (X, W, T) \leftrightarrow (U, X, Y) \),

(C10) \( U_l \leftrightarrow (X_l, W, T) \leftrightarrow (U, U_l, X, X_l, Y) \) for all \( l \) in \( L \), and

(C11) the conditional distribution of \( U_l \) given \( (W, T) \) is discrete for each \( l \).

Define the set

\[
\mathcal{RE}^{MHO}_o(\lambda_o) \triangleq \left\{ (R, R, E) : R_l \geq I(X_l; U_l|X, W, T) \text{ for all } l \text{ in } L, \right. \]

\[
R + \sum_{l \in S} R_l \geq I(X; U, U_{S^c}|U_{S^c}, T) + \sum_{l \in S} I(X_l; U_l|X, W, T), \forall S \subseteq L, \]  \hspace{1cm} (5.15)

\[
E \leq I(Y; U, U|T) \}
\]

Finally, let

\[
\mathcal{RE}^{MHO}_o \triangleq \bigcup_{\lambda_o \in \Lambda_o} \mathcal{RE}^{CI}_o(\lambda_o).
\]

We have the following lemma.

**Lemma 16.**

\[
\mathcal{RE}^{MHO}_* \subseteq \mathcal{RE}^{MHO}_o.
\]
The inequalities (5.16) and (5.17) can be established as in the proof of Theorem 12. In particular, we obtain (5.16) by considering only those constraints on the sum of rate combinations that include $R$. The inequality (5.15) is not present in Theorem 12. However, it can be derived easily. We need the following lemma.

**Lemma 17.** [37, Lemma 9] If $\lambda_o$ is in $\Lambda_o$, then for all $S \subseteq L$,

$$2^{2I(X;U_S|W,T)} \leq 1 + \sum_{l \in S} \frac{1 - 2^{-2I(X_l;U_l|X,W,T)}}{\sigma_N^2/\sigma_X^2}. \tag{5.15}$$

Consider any $(R, R, E)$ in $RE_o^{MHO}$. Then there exists $\lambda_o$ in $\Lambda_o$ such that for all $S \subseteq L$,

$$R + \sum_{l \in S} R_l \geq I(X; U, U_S|U_{S^c}, T) + \sum_{l \in S} I(X_l; U_l|X, W, T)$$

$$= I(X; U, U|T) - I(X; U_{S^c}|T) + \sum_{l \in S} I(X_l; U_l|X, W, T), \tag{5.18}$$

and

$$E \leq I(Y; U, U|T). \tag{5.19}$$

We can lower bound the first term in (5.18) by applying the entropy power inequality [53] and obtain

$$2^{2h(Y|U, U, T)} = 2^{2h(X+N|U, U, T)}$$

$$\geq 2^{2h(X|U, U, T)} + 2^{2h(N)}$$

$$= 2^{2h(X|U, U, T)} + 2\pi e\sigma_N^2,$$

which simplifies to

$$h(Y|U, U, T) \geq \frac{1}{2} \log \left(2^{2h(X|U, U, T)} + 2\pi e\sigma_N^2\right). \tag{5.20}$$

Now (5.19) and (5.20) together imply

$$I(X; U, U|T) \geq \frac{1}{2} \log \frac{\sigma_X^2}{(\sigma_X^2 + \sigma_N^2)2^{-2E} - \sigma_N^2}. \tag{5.21}$$
We next upper bound the second term in (5.18). Since conditioning reduces entropy and \( X \) is independent of \( (W, T) \), we have

\[
I(X; U_{S^c}|T) = h(X|T) - h(X|U_{S^c}, T)
\leq h(X|W, T) - h(X|U_{S^c}, W, T)
= I(X; U_{S^c}|W, T).
\] (5.22)

Define

\[
r_i \triangleq I(X_i; U_i|X, W, T).
\]

Then we have from (5.18), (5.21), (5.22), and Lemma 17 that

\[
R + \sum_{l \in S} R_l \geq \frac{1}{2} \log \left[ \left( \frac{\sigma_X^2}{\sigma_N^2} + 2^{-2E} - 2 \right) \left( \frac{1}{\sigma_X^2} + \sum_{l \in S^c} \frac{1 - 2^{-2r_l}}{\sigma_N^2} \right)^{-1} \right] + \sum_{l \in S} r_l.
\]

On applying Lemma 16 and Corollary 3, we obtain \( \mathcal{RE}_{MHO} \subseteq \tilde{\mathcal{RE}}_{MHO} \).

\begin{figure}[h]
  \centering
  \includegraphics[width=0.8\textwidth]{fig53.png}
  \caption{Gaussian many-help-one source coding problem.}
\end{figure}

We use the Quantize-Bin-Test scheme to prove the reverse inclusion. Con-
sider \((R,R,E)\) in \(\mathcal{RE}^{MHO}\). Then there exists \(r \in \mathbb{R}_+^L\) such that

\[
R_l \geq r_l \text{ for all } l \text{ in } \mathcal{L}, \text{ and }
\]

\[
R + \sum_{l \in S} R_l \geq \frac{1}{2} \log^+ \left[ \frac{1}{D} \left( \frac{1}{\sigma_X^2} + \sum_{l \in S^c} \frac{1 - 2^{-2r_l}}{\sigma_{X_l}^2} \right)^{-1} \right] + \sum_{l \in S} r_l \text{ for all } S \subseteq \mathcal{L}.
\]

We therefore have from Oohama’s result [43] that \((R,R,D)\) is achievable for the quadratic Gaussian many-help-one source coding problem, the setup of which is shown in Fig. 5.3. In this problem, the main encoder and helpers operate as before. The decoder however uses all available information to estimate \(X\) such that the mean square error of the estimate is no more than a fixed positive number \(D\). Since \((R,R,D)\) is achievable, it follows from Oohama’s achievability proof that for any positive \(\delta\) and sufficiently large \(n\), there exist quantize and bin encoders \(f^{(n)}, f_1^{(n)}, \ldots, f_L^{(n)}\), and a decoder \(\psi^{(n)}\) such that

\[
R + \delta \geq \frac{1}{n} \log |f^{(n)}(X^n)|,
\]

\[
R_l + \delta \geq \frac{1}{n} \log |f_l^{(n)}(X_l^n)| \text{ for all } l \text{ in } \mathcal{L}, \text{ and }
\]

\[
D + \delta \geq \frac{1}{n} \sum_{i=1}^n E \left[ (X^n(i) - \hat{X}^n(i))^2 \right],
\]

where

\[
\hat{X}^n = \psi^{(n)}\left( f^{(n)}(X^n), \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}} \right).
\]

For each \(i\), we have

\[
E \left[ \left( Y^n(i) - \hat{X}^n(i) \right)^2 \right] = E \left[ \left( Y^n(i) - X^n(i) + X^n(i) - \hat{X}^n(i) \right)^2 \right]
\]

\[
= E \left[ \left( N^n(i) + X^n(i) - \hat{X}^n(i) \right)^2 \right]
\]

\[
= \sigma_N^2 + E \left[ \left( X^n(i) - \hat{X}^n(i) \right)^2 \right],
\]

where the last equality follows because

\[
Y^n(i) \leftrightarrow X^n(i) \leftrightarrow \hat{X}^n(i).
\]
By averaging over time, we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} E \left[ \left( Y^n(i) - \hat{X}^n(i) \right)^2 \right] = \sigma_N^2 + \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( X^n(i) - \hat{X}^n(i) \right)^2 \right] \leq \sigma_N^2 + D + \delta,
\]
where the last inequality follows from (5.25). Therefore, the code achieves a distortion \( \sigma_N^2 + D + \delta \) in \( Y \). Hence,
\[
\frac{1}{n} I \left( Y^n; \psi^{(n)} \left( f^{(n)}(X^n), \left( f_l^{(n)}(X^n) \right)_{l \in L} \right) \right) \geq \frac{1}{2} \log \frac{\sigma_X^2 + \sigma_N^2}{\sigma_N^2 + D + \delta},
\]
where the right-hand-side of the inequality is the rate-distortion function of \( Y \) at a distortion \( \sigma_N^2 + D + \delta \). Using this and the data processing inequality [53, Theorem 2.8.1], we obtain
\[
\frac{1}{n} I \left( Y^n; f^{(n)}(X^n), \left( f_l^{(n)}(X^n) \right)_{l \in L} \right) \geq \frac{1}{n} I \left( Y^n; \psi^{(n)} \left( f^{(n)}(X^n), \left( f_l^{(n)}(X^n) \right)_{l \in L} \right) \right) \geq \frac{1}{2} \log \frac{\sigma_X^2 + \sigma_N^2}{(\sigma_X^2 + \sigma_N^2)^2 - 2E} + \bar{\delta}
\]
(5.26)
\[
= E - \bar{\delta},
\]
(5.27)
where (5.26) follows for a positive \( \bar{\delta} \) such that \( \bar{\delta} \to 0 \) as \( \delta \to 0 \). We now have from (5.23), (5.24), and (5.27) that \((R, R, E)\) is in \( \mathcal{RE}^{MHO} \). Hence by Corollary 3, \( \mathcal{RE}^{MHO} \subseteq \mathcal{RE}^{MHO} \).

5.6.1 Special Cases

Consider the following special cases. We continue to use the terminology from the source coding literature.
1. **Gaussian CEO hypothesis testing against independence:** When $R = 0$, the problem reduces to the Gaussian CEO hypothesis testing against independence. Let $\mathcal{R}^\text{CEO}$ be the rate-exponent region of this problem. Define the set

$$\mathcal{R}^{\text{CEO}} \triangleq \left\{ (R_1, \ldots, R_L, E) : \text{there exists } r \in \mathbb{R}_+^L \text{ such that} \right. \sum_{l \in S} R_l \geq \frac{1}{2} \log^+ \left[ \frac{1}{D} \left( \frac{1}{\sigma_X^2} + \sum_{l \in S^c} \frac{1 - 2^{-2r_l}}{\sigma_{N_l}^2} \right)^{-1} \right] + \sum_{l \in S} r_l \forall S \subseteq L \right\}.$$ 

We immediately have the following corollary as a consequence of Theorem 17.

**Corollary 4.**

$$\mathcal{R}^{\text{CEO}} = \mathcal{R}^{\text{CEO}}.$$ 

2. **Gaussian one-helper hypothesis testing against independence:** When $L = 1$, the problem reduces to the Gaussian one-helper hypothesis testing against independence. Let $\mathcal{R}^\text{OH}$ be the rate-exponent region of this problem. Define the sets

$$\mathcal{R}^{\text{OH}} \triangleq \left\{ (R, R_1, E) : \text{there exists } r_1 \in \mathbb{R}_+ \text{ such that} \right. \begin{align*}
R_1 &\geq r_1, \\
R + R_1 &\geq \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{D} \right] + r_1, \text{ and} \\
R &\geq \frac{1}{2} \log^+ \left[ \frac{1}{D} \left( \frac{1}{\sigma_X^2} + \frac{1 - 2^{-2r_1}}{\sigma_{N_1}^2} \right)^{-1} \right] \right\},
\end{align*}$$

and

$$\mathcal{R}^{\text{OH}} \triangleq \left\{ (R, R_1, E) : R \geq \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{D} \right. (1 - \rho^2 + \rho^22^{-2R_1}) \left. \right] \right\},$$

where

$$\rho^2 = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_{N_1}^2}.$$
Corollary 5.

\[
\mathcal{RE}^{OH} = \widetilde{\mathcal{E}}^{OH} = \widehat{\mathcal{E}}^{OH}.
\]

Proof. The first equality follows from Theorem 17. Consider any \((R, R_1, E)\) in \(\mathcal{RE}^{OH}\). It must satisfy

\[
R \geq \min_{0 \leq r_1 \leq R_1} \max \left\{ \frac{1}{2} \log^+ \left[ \frac{1}{D} \left( \frac{1}{\sigma_X^2} + \frac{1 - 2^{-2r_1}}{\sigma_{N_1}^2} \right)^{-1} \right], \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{D} \right] + r_1 - R_1 \right\}
\]

\[
= \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{D} \left( 1 - \rho^2 + \rho^2 2^{-2R_1} \right) \right],
\]

where the equality is achieved by

\[
r_1 = R_1 + \frac{1}{2} \log \left( 1 - \rho^2 + \rho^2 2^{-2R_1} \right).
\] (5.28)

We therefore have that \((R, R_1, E)\) is in \(\widetilde{\mathcal{E}}^{OH}\), and hence \(\mathcal{RE}^{OH} \subseteq \widetilde{\mathcal{E}}^{OH}\). The proof of the reverse containment is trivial by noticing that for any \((R, R_1, E)\) in \(\mathcal{RE}^{OH}\), there exists \(r_1\) satisfying (5.28) such that all inequalities in the definition of \(\widetilde{\mathcal{E}}^{OH}\) are satisfied. \(\square\)

5.7 A More General Outer Bound

We return to the general problem formulated in Section 5.2. The problem remains open till date. Several inner bounds are known for \(L = 1\) [26, 27, 28, 29]. But even for \(L = 1\), there is no nontrivial outer bound with which to compare the inner bounds. We give an outer bound for a class of instances of the general problem.

Consider the class of instances such that \(P_X = Q_X\), i.e., the marginal distributions of \(X\) are the same under both hypotheses. Stein’s lemma [53, Theorem
11.8.3] asserts that the centralized type 2 error exponent for this class of problems is

\[ E_C \triangleq D(P_{XY} \| Q_{XY}), \]

which is achieved when X and Y both are available at the detector. Let

\[ \mathcal{RE}_C \triangleq \{ (R, E) : E \leq E_C \}. \]

We have the following trivial outer bound.

**Lemma 18.**

\[ \mathcal{RE} \subseteq \mathcal{RE}_C. \]

Let \( \Xi \) be the set of finite-alphabet random variable Z such that there exists two joint distributions \( P_{XYZ} \) and \( Q_{XYZ} \) satisfying

(C12) \( \sum_Z P_{XYZ} = P_{XY} \), the distribution under \( H_0 \),

(C13) \( \sum_Z Q_{XYZ} = Q_{XY} \), the distribution under \( H_1 \),

(C14) \( Q_{XYZ} = Q_{X|Z} Q_{Y|Z} Q_Z \), i.e., X and Y are conditionally independent given Z under the Q distribution, and

(C15) \( P_{XZ} = Q_{XZ} \), i.e., the joint distributions of \( (X, Z) \) are the same under both distributions.

Note that the joint distributions of \( (Y, Z) \) need not be the same under the two distributions. If \( P_{XYZ} \) and \( Q_{XYZ} \) are the joint distributions of X, Y, and Z under \( H_0 \) and \( H_1 \), respectively and Z is available to the detector, then the problem can be related to the L-encoder hypothesis testing against conditional independence. Now Z is not present in the original problem, but we can augment the sample space by introducing Z and supplying it to the decoder. The outer
bound for this new problem is then an outer bound for the original problem. Moreover, we can then optimize over $Z$ to obtain the best possible bound.

Let $\chi$ and $\Lambda_o$ be defined as in Section 5.3.2 with $X_{L+1}$ restricted to be deterministic. If $\Xi$ is nonempty, then for any $(Z, X, \lambda_o)$ in $\Xi \times \chi \times \lambda_o$, define the set

$$\mathcal{RE}_o(Z, X, \lambda_o) \triangleq \left\{ (R, E) : \sum_{l \in S} R_l \geq I(X; U_S|U_{S^c}, Z, T) + \sum_{l \in S} I(X_l; U_l|X, W, Z, T) \forall S \subseteq \mathcal{L}, \text{ and} \right.$$  

$$E \leq I(Y; U|Z, T) + D(\mathcal{P}_{Y|Z}\|\mathcal{Q}_{Y|Z}) \right\}.$$  

Finally, let

$$\mathcal{RE}_o \triangleq \begin{cases} \bigcap_{Z \in \Xi} \bigcap_{X \in \chi} \bigcup_{\lambda_o \in \Lambda_o} \mathcal{RE}_o(Z, X, \lambda_o) & \text{if } \Xi \text{ is nonempty} \\ \mathbb{R}^{L+1} & \text{otherwise.} \end{cases}$$  

We have the following outer bound to the rate-exponent region of this class of problems.

**Theorem 18.**

$$\mathcal{RE} \subseteq \overline{\mathcal{RE}_o} \cap \mathcal{RE}_c.$$  

**Proof.** In light of Proposition 1 and Lemma 18, it suffices to show that

$$\mathcal{RE}_* \subseteq \mathcal{RE}_o.$$  

Consider $(R, E)$ in $\mathcal{RE}_*$. Then there exists a block length $n$ and encoders $f^{(n)}_l$ such that

$$R_l \geq \frac{1}{n} \log |f^{(n)}_l(X^n_i)| \text{ for all } l \in \mathcal{L}, \text{ and}$$  

$$E \leq \frac{1}{n} D\left(\mathcal{P}_{(f^{(n)}_l(X^n_i))_{i \in \mathcal{L}}} \| \mathcal{Q}_{(f^{(n)}_l(X^n_i))_{i \in \mathcal{L}}} \right).$$  

130
Consider any $Z$ in $\Xi$. Then
\[
D \left( P_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n \| Q_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n \\
\leq D \left( P_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n Z^n \| Q_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n Z^n \\
= D \left( P_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n \| P_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Z^n Q Y^n | Z^n \\
= D \left( P_1 \left( f^{(n)}(X^n) \right) \right)_{l \in \mathcal{L}} Y^n \| Z^n P Y^n | Z^n \\
+ D \left( P Y^n \| Q Y^n | Z^n \right) \\
= I \left( f^{(n)}(X^n) \right)_{l \in \mathcal{L}} ; Y^n | Z^n \\
+ n D \left( P Y^n \| Q Y^n | Z^n \right),
\]
which together with (5.30) implies
\[
E \leq \frac{1}{n} I \left( f^{(n)}(X^n) \right)_{l \in \mathcal{L}} ; Y^n | Z^n \\
+ D \left( P Y^n \| Q Y^n | Z^n \right). \tag{5.31}
\]

It now follows from (5.29), (5.31), and Corollary 2 that
\[
\left( R, (E - D \left( P Y^n \| Q Y^n | Z^n \right))^{+} \right)
\]
is in $\mathcal{R}E_{s}^{CI}$. Therefore from Theorem 12, it must also be in $\mathcal{R}E_{o}^{CI}$. Hence for any $X$ in $\chi$, there exists $\lambda_0$ in $\Lambda_o$ such that $\left( R, (E - D \left( P Y^n \| Q Y^n | Z^n \right))^{+} \right)$ is in $\mathcal{R}E_{o}^{CI} (X, \lambda_0)$, i.e.,
\[
\sum_{i \in S} R_i \geq I(X; U_S | U_{S^c}, Z, T) + \sum_{i \in S} I(X_i; U_i | X, W, Z, T) \text{ for all } S \subseteq \mathcal{L},
\]
and
\[
(E - D \left( P Y^n \| Q Y^n | Z^n \right))^{+} \leq I(Y; U | Z, T).
\]
This means that $(R, E)$ is in $\mathcal{R}E_o(Z, X, \lambda_0)$, and hence $\mathcal{R}E_s \subseteq \mathcal{R}E_o.$

Although the outer bound above is not computable in general, it simplifies
to the following computable form for the special case in which \( L = 1 \). Let

\[ \tilde{\mathcal{R}}E \triangleq \bigcap_{Z \in \Xi} \{ (R_1, E) : \text{there exists } U_1 \text{ such that} \]

\[ R_1 \geq I(X_1; U_1|Z), \]
\[ E \leq I(Y; U_1|Z) + D(P_{Y|Z}\|Q_{Y|Z}|Z), \]
\[ |U_1| \leq |X_1| + 1, \text{ and} \]
\[ U_1 \leftrightarrow X_1 \leftrightarrow (Y, Z) \}. \]

**Corollary 6.** For 1-encoder general hypothesis testing,

\[ \overline{\mathcal{RE}_o} = \tilde{\mathcal{RE}} \]

and hence

\[ \mathcal{RE} \subseteq \tilde{\mathcal{RE}} \cap \mathcal{RE}_C. \]

**Proof.** It suffices to show that \( \overline{\mathcal{RE}_o} = \tilde{\mathcal{RE}} \). This immediately follows by noticing that given any \( Z \) in \( \Xi \), the outer bound can be related to the rate-exponent region of the 1-encoder hypothesis testing against conditional independence. The result then follows from Theorem 13. \( \square \)

It is easy to see that the outer bound is tight for the test against independence.

**Corollary 7.** (Test against independence, [27]) If \( Q_{X_1Y} = P_{X_1} P_Y \), then

\[ \mathcal{RE} = \tilde{\mathcal{RE}}. \]

**Proof.** This follows by choosing \( Z \) to be deterministic in the outer bound and then invoking the result of Ahlswede and Csiszár [27]. \( \square \)
Remark 5.3: The outer bound is not always better than the centralized outer bound. In particular, if

\[ D \left( P_{Y|Z} || Q_{Y|Z} | Z \right) \geq E_C \]

for all \( Z \) in \( \Xi \), then the outer bound is no better than the centralized outer bound.

5.7.1 Gaussian Case

To illustrate this bound, let us consider a Gaussian example in which \( X_1 \) and \( Y \) are zero-mean unit-variance jointly Gaussian sources with the correlation coefficients \( \rho_0 \) and \( \rho_1 \) under \( H_0 \) and \( H_1 \), respectively, where \( \rho_0 \neq \rho_1 \), \( \rho_0^2 < 1 \), and \( \rho_1^2 < 1 \). We can assume without loss of generality that \( 0 \leq \rho_1 \leq 1 \) because the case \( -1 \leq \rho_1 < 0 \) can be handled by multiplying \( Y \) by \(-1\). We use lowercase \( p \) and \( q \) to denote appropriate Gaussian densities under hypotheses \( H_0 \) and \( H_1 \), respectively. Let \( \mathcal{RE}^G \) be the rate-exponent region of this problem. We focus on the following three regions (Fig. 5.4) for which the outer bound is nontrivial.

\[
\mathcal{D}_1 \triangleq \{ (\rho_0, \rho_1) : 0 \leq \rho_1 < \rho_0 \leq 1 \}, \\
\mathcal{D}_2 \triangleq \{ (\rho_0, \rho_1) : 0 \leq \rho_1 \text{ and } 2\rho_1 - 1 \leq \rho_0 < \rho_1 \}, \text{ and} \\
\mathcal{D}_3 \triangleq \left\{ (\rho_0, \rho_1) : -1 \leq \rho_0 \leq 2\rho_1 - 1 \text{ and} \right. \\
\left. \frac{2(\log e)\rho_1}{1-\rho_1} \leq \frac{1}{2} \log \left( \frac{1 - \rho_1^2}{1 - \rho_0^2} \right) - \frac{\log e}{2} \left( \rho_1 - \rho_0 \right) \right\}.
\]
Figure 5.4: Regions of pair $(\rho_0, \rho_1)$ for which the outer bound is nontrivial.

**Outer Bound**

Let us define

\[
\rho \triangleq \begin{cases} 
\frac{\rho_0 - \rho_1}{1 - \rho_1} & \text{if } (\rho_0, \rho_1) \text{ is in } \mathcal{D}_1 \cup \mathcal{D}_2 \\
\frac{\rho_0 + \rho_1}{1 - \rho_1} & \text{if } (\rho_0, \rho_1) \text{ is in } \mathcal{D}_3,
\end{cases}
\]

and

\[
C \triangleq \begin{cases} 
0 & \text{if } (\rho_0, \rho_1) \text{ is in } \mathcal{D}_1 \cup \mathcal{D}_2 \\
\frac{2(\log e)\rho_1}{1 - \rho_1} & \text{if } (\rho_0, \rho_1) \text{ is in } \mathcal{D}_3.
\end{cases}
\]

The centralized type 2 error exponent is

\[
E^G_C \triangleq D(p_{X_1 Y} \| q_{X_1 Y})
= \frac{1}{2} \log \left( \frac{1 - \rho_1^2}{1 - \rho_0^2} \right) - \frac{(\log e)\rho_1(\rho_0 - \rho_1)}{1 - \rho_1^2}.
\]

Define the sets

\[
\mathcal{RE}^G_o \triangleq \left\{ (R_1, E) : E \leq \frac{1}{2} \log \left( \frac{1}{1 - \rho_1^2 + \rho_1^2 R_1^2} \right) + C \right\}
\]

and

\[
\mathcal{RE}^G_C \triangleq \left\{ (R_1, E) : E \leq E^G_C \right\}.
\]

We have the following outer bound.
Theorem 19. If \((\rho_0, \rho_1)\) is in \(D_1 \cup D_2 \cup D_3\), then

\[
\mathcal{RE}^G \subseteq \mathcal{RE}_o^G \cap \mathcal{RE}_C^G.
\]

Proof. The proof is in two steps. We first obtain a single letter outer bound similar to the one in Corollary 6 and then use it to obtain the desired outer bound.

Consider \((\rho_0, \rho_1)\) in \(D_1\). Let \(Z, Z', W,\) and \(V\) be standard normal random variables independent of each other. \(X_1\) and \(Y\) can be expressed as

\[
X_1 = \sqrt{\rho_1}Z + \sqrt{\rho_0 - \rho_1}Z' + \sqrt{1 - \rho_0}W \\
Y = \sqrt{\rho_1}Z + \sqrt{\rho_0 - \rho_1}Z' + \sqrt{1 - \rho_0}V
\]

under \(H_0\) and as

\[
X_1 = \sqrt{\rho_1}Z + \sqrt{1 - \rho_1}W \\
Y = \sqrt{\rho_1}Z + \sqrt{1 - \rho_1}V
\]

under \(H_1\). It is easy to verify that conditions (C12) through (C15) are satisfied if we replace the distributions by the corresponding Gaussian densities. Therefore, \(Z\) is in \(\Xi\). Define the set

\[
\tilde{\mathcal{RE}}_G^G \triangleq \left\{ (R_1, E) : \text{there exists } U_1 \text{ such that} \right. \\
R_1 \geq I(X_1; U_1|Z), \\
E \leq I(Y; U_1|Z) + D(p_{Y|Z}\|q_{Y|Z}|Z), \text{ and} \\
(Y, Z) \leftrightarrow X_1 \leftrightarrow U_1 \left\}
\]

Corollary 8.

\[
\mathcal{RE}^G \subseteq \tilde{\mathcal{RE}}_G^G \cap \mathcal{RE}_C^G.
\]

The proof is immediate as a continuous extension of Corollary 6. From Corollary 8, it suffices to show that

\[
\tilde{\mathcal{RE}}_G^G \subseteq \mathcal{RE}_o^G.
\]
Note first that

\[ D(p_{Y|Z} || q_{Y|Z}|Z) = 0 \]

here because the joint densities of \((Y, Z)\) are the same under both hypotheses.

Consider any \((R_1, E)\) in \(\widetilde{RE}^G\). Then there exists a random variable \(U_1\) such that \((Y, Z) \leftrightarrow X_1 \leftrightarrow U_1,\)

\[ R_1 \geq I(X_1; U_1|Z), \quad \text{and} \quad (5.32) \]

\[ E \leq I(Y; U_1|Z). \quad (5.33) \]

Since \(X_1, Y,\) and \(Z\) are jointly Gaussian under \(H_0,\) we can write by the linear estimation calculation that

\[ Y = \rho X_1 + \sqrt{\rho_1}(1 - \rho)Z + B, \]

where \(B\) is a zero-mean Gaussian random variable with the variance

\[ \sigma_{Y|X_1Z}^2 = (1 - \rho_1)(1 - \rho^2), \]

and is independent of \(X_1\) and \(Z.\) We now have

\[ h(Y|U_1, Z) = h(\rho X_1 + \sqrt{\rho_1}(1 - \rho)Z + B|U_1, Z) \]

\[ = h(\rho X_1 + B|U_1, Z) \]

\[ \geq \frac{1}{2} \log \left(2^{2h(\rho X_1|U_1, Z)} + 2^{2h(B)}\right) \]

\[ = \frac{1}{2} \log \left(\rho^2 2^{2h(X_1|U_1, Z)} + 2^{2h(B)}\right) \]

\[ = \frac{1}{2} \log \left(\rho^2 2^{2(h(X_1|Z) - I(X_1; U_1|Z))} + 2^{2h(B)}\right) \]

\[ = \frac{1}{2} \log \left(\rho^2 (1 - \rho_1) 2^{-2I(X_1; U_1|Z)} + (1 - \rho_1)(1 - \rho^2)\right) + \frac{1}{2} \log(2\pi e) \]

\[ \geq \frac{1}{2} \log \left(\rho^2 (1 - \rho_1) 2^{-2R_1} + (1 - \rho_1)(1 - \rho^2)\right) + \frac{1}{2} \log(2\pi e), \quad (5.34) \]

where
(5.34) follows from the entropy power inequality [53] because $X_1$ and $B$ are independent given $(U_1, Z)$, and

(5.35) follows because function

$$f(x) = \frac{1}{2} \log (p2^{-2x} + q)$$

is monotonically decreasing in $x$ for $p > 0$, and we have the rate constraint in (5.32).

Now (5.33) and (5.35) imply that

$$E \leq \frac{1}{2} \log \left( \frac{\sigma^2_{Y|Z}}{\rho^2(1 - \rho_1)2^{-2R_1} + (1 - \rho_1)(1 - \rho^2)} \right)$$

$$= \frac{1}{2} \log \left( \frac{1}{1 - \rho^2 + \rho^22^{-2R_1}} \right),$$

which proves that $(R_1, E)$ is in $\mathcal{RE}_o^G$. This completes the proof for the region $D_1$.

The proof is analogous for $(\rho_0, \rho_1)$ in the region $D_2$. The only difference is that under $H_0$, $X_1$ and $Y$ can now be expressed as

$$X_1 = \sqrt{\rho_1}Z + \sqrt{\rho_1 - \rho_0}Z' + \sqrt{1 - 2\rho_1 + \rho_0}W$$

$$Y = \sqrt{\rho_1}Z - \sqrt{\rho_1 - \rho_0}Z' + \sqrt{1 - 2\rho_1 + \rho_0}V.$$

Suppose now that $(\rho_0, \rho_1)$ is in $D_3$. One can verify that $-\rho_0 - \rho_1 > 0$ here. Hence, $X_1$ and $Y$ can be expressed as

$$X_1 = \sqrt{\rho_1}Z + \sqrt{-\rho_0 - \rho_1}Z' + \sqrt{1 + \rho_0}W$$

$$Y = -\sqrt{\rho_1}Z - \sqrt{-\rho_0 - \rho_1}Z' + \sqrt{1 + \rho_0}V$$

under $H_0$. Their expressions under $H_1$ are the same as before. It is evident that $Z$ is in $\Xi$. Therefore, the outer bound in Corollary 8 is valid for this case, which
implies that it suffices to show that
\[ \widetilde{\mathcal{RE}}^G \subseteq \mathcal{RE}_G^G. \]

Under \( H_0 \), the conditional distribution of \( Y \) given \( Z = z \) is Gaussian with the mean \(-\sqrt{\rho_1}z\) and the variance \( 1 - \rho_1 \). Similarly under \( H_1 \), it is Gaussian with the mean \( \sqrt{\rho_1}z \) and the variance \( 1 - \rho_1 \). We therefore obtain

\[
D(p_Y|Z\|q_Y|Z|Z) = \int_{z \in \mathbb{R}} p_Z(z)dz \int_{y \in \mathbb{R}} p_Y|Z(y|z) \log \frac{p_Y|Z(y|z)}{q_Y|Z(y|z)} dy
\]

\[
= \int_{z \in \mathbb{R}} p_Z(z)dz \int_{y \in \mathbb{R}} p_Y|Z(y|z) \log \left[ \exp \left(\frac{(y - \sqrt{\rho_1}z)^2}{2(1 - \rho_1)} - \frac{(y + \sqrt{\rho_1}z)^2}{2(1 - \rho_1)}\right) \right] dy
\]

\[
= \int_{z \in \mathbb{R}} p_Z(z)dz \int_{y \in \mathbb{R}} p_Y|Z(y|z) \left[-\frac{2(\log e)\sqrt{\rho_1}yz}{1 - \rho_1}\right] dy
\]

\[
= \frac{2(\log e)\rho_1}{1 - \rho_1} \int_{z \in \mathbb{R}} z^2 p_Z(z)dz
\]

Again, since \( X_1, Y, \) and \( Z \) are jointly Gaussian under \( H_0 \), we can write

\[ Y = \rho X_1 - \sqrt{\rho_1}(1 + \rho)Z + B, \]

where \( B \) is defined as before. The rest of the proof is identical to the region \( D_1 \) case.

Ahlswede and Csiszár's Inner Bound

We next compare the outer bound with Ahlswede and Csiszár’s inner bound, which is obtained by using a Gaussian test channel to quantize \( X_1 \). One can use
better inner bounds [28, 29], but they are quite complicated and for the Gaussian case considered here, Ahlswede and Csiszár’s bound itself is quite close to our outer bound in some cases. Let

\[
\mathcal{RE}_G^i \triangleq \left\{(R_1, E) : E \leq \frac{1}{2} \log \left( \frac{1 - \rho_1^2 (1 - 2^{-2R_1})}{1 - \rho_0^2 (1 - 2^{-2R_1})} \right) - \frac{(\log e)\rho_1 (\rho_0 - \rho_1) (1 - 2^{-2R_1})}{1 - \rho_1^2 (1 - 2^{-2R_1})} \right\}.
\]

**Proposition 2.** [27]

\[
\mathcal{RE}_i^G \subseteq \mathcal{RE}^G.
\]

**Proof.** Fix any \((R_1, E)\) in \(\mathcal{RE}_i^G\). Let \(U_1 = X_1 + P\), where \(P\) is a zero-mean Gaussian random variable independent of \((X_1, Y)\) such that

\[
I(X_1; U_1) = R_1,
\]

which implies that the variance of \(P\)

\[
\sigma_P^2 = \frac{1}{2^{2R_1} - 1}.
\]

The covariance matrix of \((U_1, Y)\) is

\[
K_0 = \begin{bmatrix}
1 + \sigma_P^2 & \rho_0 \\
\rho_0 & 1
\end{bmatrix}
\]

under \(H_0\) and is

\[
K_1 = \begin{bmatrix}
1 + \sigma_P^2 & \rho_1 \\
\rho_1 & 1
\end{bmatrix}
\]

under \(H_1\). It now follows from Ahlswede and Csiszár’s scheme [27, Theorem 5]
that the achievable exponent is

\[ E_{AC} = D(p_{U_1Y} || q_{U_1Y}) \]

\[ = \int_{\mathbb{R}^2} p_{U_1Y}(z) \log \frac{p_{U_1Y}(z)}{q_{U_1Y}(z)} \, dz \]

\[ = -\frac{1}{2} \log \left( (2\pi e)^2 |K_0| \right) - \int_{\mathbb{R}^2} p_{U_1Y}(z) \log q_{U_1Y}(z) \, dz \]

\[ = -\frac{1}{2} \log \left( (2\pi e)^2 |K_0| \right) - \int_{\mathbb{R}^2} p_{U_1Y}(z) \left[ -\frac{(\log e)}{2} z^T K_1^{-1} z - \frac{1}{2} \log ((2\pi)^2 |K_1|) \right] \, dz \]

\[ = \frac{1}{2} \log \frac{|K_1|}{|K_0|} - (\log e) + \frac{(\log e)}{2} \int_{\mathbb{R}^2} p_{U_1Y}(z) \left( z^T K_1^{-1} z \right) \, dz \]

\[ = \frac{1}{2} \log \left( \frac{1 + \sigma^2}{1 + \sigma^2_0} - \rho_0 \rho_1 \right) - \log e + \frac{(\log e)(1 + \sigma^2 - \rho_0 \rho_1)}{(1 + \sigma^2 - \rho^2_1)} \]

\[ = \frac{1}{2} \log \left( 1 - \rho_1^2 \frac{(1 - 2^{-2R_1})}{1 - \rho_0^2 (1 - 2^{-2R_1})} \right) - \frac{(\log e)(\rho_1 - \rho_1)(1 - 2^{-2R_1})}{1 - \rho_1^2 (1 - 2^{-2R_1})}. \]

This proves that \((R_1, E)\) is in \(\mathcal{RE}^G\). \(\square\)

The inner and outer bounds coincide for the test against independence.

**Corollary 9.** (Test against independence, [11, 27]) If \(X_1\) and \(Y\) are independent under \(H_1\), i.e., \(\rho_1 = 0\), then

\[ \mathcal{RE}^G = \mathcal{RE}^G_o = \mathcal{RE}^G_i = \left\{ (R_1, E) : E \leq \frac{1}{2} \log \left( \frac{1}{1 - \rho_0^2 + \rho_0^2 2^{-2R_1}} \right) \right\}. \]

**Numerical Results**

Fig. 5.5 shows the inner and outer bounds for four examples. Fig. 5.5(a)-(c) are the examples when \((\rho_0, \rho_1)\) is in \(D_1 \cup D_2\). Observe that the two bounds are quite close near zero and at all large rates. Fig. 5.5(d) is an example when \((\rho_0, \rho_1)\) is
in $D_3$. For this example, there is a gap between the inner and outer bounds at zero rate. This is due to the fact that in our outer bound, the joint densities of $(Y, Z)$ are different under the two hypotheses. Numerical results suggest that for a fixed $\rho_0$, the maximum gap between the inner and outer bounds decreases as we decrease $\rho_1$ and finally becomes zero at $\rho_1 = 0$, which is the test against independence.

5.8 1-Encoder Vector Gaussian Hypothesis Testing against Independence

In this section, we consider a vector Gaussian extension of Ahlswede and Csiszár’s test against independence result. Let $X$ and $Y$ be two zero-mean vector Gaussian sources with positive definite covariance matrices $K_X$ and $K_Y$, 

Figure 5.5: Outer and inner bounds for four examples.
respectively. We assume that under $H_0$
\[
Y = AX + N,
\]
where $A$ is an invertible matrix and $N$ is a zero-mean Gaussian random vector with the covariance matrix $K_N$. Assume further that $N$ is independent of $X$. $X$ and $Y$ are correlated under null hypothesis $H_0$ and are independent under alternate hypothesis $H_1$. More precisely, we have the following two hypotheses
\[
H_0 : p_{XY} = \mathcal{N}(0, K) \\
H_1 : p_{X} p_{Y} = \mathcal{N}(0, K_X) \mathcal{N}(0, K_Y),
\]
where $K$ is a block covariance matrix given by
\[
K \triangleq \begin{bmatrix}
K_X & K_XA^T \\
AK_X & K_Y
\end{bmatrix}.
\]
$X^n$ is observed by an encoder whereas $Y^n$ is available at the detector. The problem formulation is similar to Section 5.2. Let $\mathcal{RE}_{G,\text{vec}}$ be the rate-exponent region of this problem.

Let $(\lambda_i)_{1 \leq i \leq m}$ be the eigenvalues of $K_X^{-1/2} A^T K_Y^{-1} A K_X^{1/2}$. For $\nu \geq 0$ and for all $i \in \{1, \ldots, m\}$, define
\[
r_i(\nu) \triangleq -\frac{1}{2} \log \left( \frac{1 - \lambda_i}{\nu \lambda_i} \right),
\]
where $(x)^- = \min(x, 1)$. Using these, define
\[
\mathcal{RE}_{s, \text{vec}} \triangleq \left\{(R, E) : E \leq \sum_{i=1}^{m} \frac{1}{2} \log \left( \frac{1}{1 - \lambda_i + \lambda_i 2^{-r_i(\nu)}} \right) \text{ for } \nu \geq 0 \text{ s.t. } \sum_{i=1}^{m} r_i(\nu) = R \right\}.
\]
We have the following theorem.

**Theorem 20.**
\[
\mathcal{RE}_{G,\text{vec}} = \mathcal{RE}_{s, \text{vec}}.
\]
Proof. The problem at hand is related to the remote vector Gaussian source coding under mutual information constraint [51]. Both problems have similar converse proofs. We present the proof in brief here. The details can be found in [51, Theorem 3]. The main idea is to decompose the problem into component-wise scalar problems by applying an invertible transformation on sources $X$ and $Y$ [51, 52], and then apply the scalar converse solution on each component. This results in a water-pouring outer bound which can be achieved by applying Ahlswede and Csiszár’s scheme on each component [11, 27].

We start with the single letter outer bound similar to the one in Corollary 8. Define the set

$$\widetilde{\mathcal{RE}}_{G,vec} \triangleq \{(R, E) : \text{there exists } U \text{ such that } R \geq I(X; U), E \leq I(Y; U), \text{ and } Y \leftrightarrow X \leftrightarrow U\}.$$

Corollary 10.

$$\mathcal{RE}^{G,vec} \subseteq \widetilde{\mathcal{RE}}^{G,vec}.$$ 

This corollary is a vector extension of Corollary 8. Consider any $(R, E)$ in $\mathcal{RE}^{G,vec}$. It follows from Corollary 10 then that

$$E \leq \max_U I(Y; U)$$

subject to $R \geq I(X; U)$ (5.36)

$$U \leftrightarrow X \leftrightarrow Y.$$ 

We present the transformation next. Let $P^T \Lambda Q$ be the singular value decomposition of $K_Y^{-1/2} AK_X^{-1/2}$, where $P$ and $Q$ are two orthogonal matrices and $\Lambda$ is a
diagonal matrix with singular values at the diagonal. Define the matrices

\[ T_X \triangleq Q K_X^{-1/2}, \]
\[ T_Y \triangleq P K_Y^{-1/2}. \]

Then the transformation is given by

\[ \bar{X} \triangleq T_X X, \]
\[ \bar{Y} \triangleq T_Y Y, \]

respectively. Under this transformation, we have

\[ K_X = T_X K_X T_X^T = Q K_X^{-1/2} K_X K_X^{-1/2} Q^T = I_m, \]
\[ K_Y = T_Y K_Y T_Y^T = P K_Y^{-1/2} K_Y K_Y^{-1/2} P^T = I_m, \]

and

\[ K_{XY} = T_Y A K_X T_X^T = P K_Y^{-1/2} A K_X^{-1/2} Q^T = \Lambda. \]

So, \( \bar{X} \) and \( \bar{Y} \) have i.i.d. standard normal components and are only componentwise correlated. Moreover, since \( \Lambda \) is the cross covariance between \( X \) and \( Y \), its diagonal entries are all between 0 and 1. Since the transformation \( T_X \) and \( T_Y \) are full rank, they are information lossless. Therefore,

\[ I(X; U) = I(\bar{X}; U) \]
\[ I(Y; U) = I(\bar{Y}; U). \]

Using this in (5.36), we can obtain the following relaxation.

\[ R \geq I(\bar{X}; U) \]
\[ = \sum_{i=1}^{m} \left[ h(\bar{X}_i) - h(\bar{X}_i|U, \bar{X}_i^{-1}) \right] \]
\[ = \sum_{i=1}^{m} I(\bar{X}_i; U, \bar{X}_i^{-1}) \]
\[ = \sum_{i=1}^{n} I(\bar{X}_i; U_i), \quad (5.37) \]
where (5.37) follows by letting \( U_i \triangleq (U, \bar{X}^{i-1}) \), and
\[
I(\bar{Y}; U) = \sum_{i=1}^{m} \left[ h(\bar{Y}_i) - h(\bar{Y}_i|U, \bar{Y}^{i-1}) \right]
\leq \sum_{i=1}^{n} \left[ h(\bar{Y}_i) - h(\bar{Y}_i|U, \bar{Y}^{i-1}, \bar{X}^{i-1}) \right] \tag{5.38}
\]
\[
= \sum_{i=1}^{n} \left[ h(\bar{Y}_i) - h(\bar{Y}_i|U, \bar{X}^{i-1}) \right] \tag{5.39}
\]
\[
= \sum_{i=1}^{n} I(\bar{Y}_i; U_i),
\]
where

(5.38) follows because conditioning reduces differential entropy, and

(5.39) follows from the Markov condition \( \bar{Y}_i \leftrightarrow (U, \bar{X}^{i-1}) \leftrightarrow \bar{Y}^{i-1} \).

In addition, it is clear that for all \( i \) in \( \{1, \ldots, m\} \), we have the Markov condition

\[
U_i \leftrightarrow \bar{X}_i \leftrightarrow \bar{Y}_i.
\]

We therefore have the following relaxation
\[
E \leq \max_{U_i} \sum_{i=1}^{m} I(\bar{Y}_i; U_i)
\]
subject to \( R \geq \sum_{i=1}^{m} I(\bar{X}_i; U_i) \) \( \tag{5.40} \)
\( U_i \leftrightarrow \bar{X}_i \leftrightarrow \bar{Y}_i \ \forall i \in \{1, \ldots, m\}. \)

The converse proof of the theorem now follows by applying the scalar converse solution (see also [11]) on each component, and then obtaining a water-pouring solution by using standard arguments. We therefore conclude that \((R, E)\) in \( \mathcal{RE}^{G,vec}_* \), and hence
\[
\mathcal{RE}^{G,vec} \subseteq \mathcal{RE}^{G,vec}_*.
\]
As mentioned above, the reverse inclusion (achievability) follows by first applying the above transformation on $X$ and $Y$ and then using Ahlswede and Csiszár’s scheme on each component.
A.1 Proof of Lemma 1

Consider the case in which encoder 2’s message alone is sufficient to meet the distortion constraint. In this case, \( R_1 = 0 \), i.e. \( U^* \) is independent of \( (X, Y, V^*) \) and encoder 2 can use all available rate \( R_2 \) to transmit a message to the decoder. Therefore, without loss of optimality we can assume that \( (U^*, V^*) \) is in \( \mathcal{S} \) and \( \sigma_{Y|V^*}^2 = \sigma_Y^2 2^{-2R_2} \). Consider now the other case in which both encoders need to send messages to the decoder. We will first show that if we restrict the solution space of \( (P) \) to Gaussian distributions, then we can assume without loss of generality that we have the long Markov chain

\[
U \leftrightarrow X \leftrightarrow Y \leftrightarrow V.
\]

It suffices to show the same for Gaussian \( \bar{U} \) and \( \bar{V} \) such that \( \sigma_{Y|\bar{V}}^2 \) is feasible and \( K_{X|\bar{U},\bar{V}} \) is the corresponding optimal solution to the problem (3.1). Then from (3.34), \( (\sigma_{Y|\bar{V}}^2, K_{X|\bar{U},\bar{V}}) \) is a candidate to be an optimal solution to \( (P_G) \). From Section 3.2, \( K_{X|\bar{U},\bar{V}} \) gives two sets of directions \( S \) and \( T \) which satisfy the properties in Theorem 3. Let us define

\[
\bar{U} \triangleq S^T X + W,
\]

where \( W \) is a zero-mean Gaussian random vector independent of \( X \) and has a covariance matrix \( K_W \) that satisfies

\[
(S^T K_{X|\bar{U},\bar{V}} S)^{-1} = (S^T K_{X|\bar{V}} S)^{-1} + K_W^{-1}.
\]
Note that $K_W$ is strictly positive definite because

$$S^T K_{X|\hat{U},\bar{V}} S \prec S^T K_{X|\bar{V}} S$$

from Theorem 3(a). The conditional covariance of $X$ given $(\hat{U}, \bar{V})$ can be expressed as

$$K_{X|\hat{U},\bar{V}} = K_{X|\bar{V}} - E \left( X\hat{U}^T | \bar{V} \right) K_{\hat{U}|\bar{V}}^{-1} E \left( \hat{U}X^T | \bar{V} \right)$$

$$= K_{X|\bar{V}} - K_{X|\bar{V}} S (S^T K_{X|\bar{V}} S + K_W)^{-1} S^T K_{X|\bar{V}}.$$

Using this, we obtain

$$S^T K_{X|\hat{U},\bar{V}} S = S^T K_{X|\bar{V}} S - S^T K_{X|\bar{V}} S (S^T K_{X|\bar{V}} S + K_W)^{-1} S^T K_{X|\bar{V}} S$$

$$= \left( (S^T K_{X|\bar{V}} S)^{-1} + K_W^{-1} \right)^{-1}$$

$$= S^T K_{X|\hat{U},\bar{V}} S, \quad (A.2)$$

$$T^T K_{X|\hat{U},\bar{V}} T = T^T K_{X|\bar{V}} T - T^T K_{X|\bar{V}} S (S^T K_{X|\bar{V}} S + K_W)^{-1} S^T K_{X|\bar{V}} T$$

$$= T^T K_{X|\bar{V}} T$$

$$= I_l \quad (A.3)$$

$$= T^T K_{X|\hat{U},\bar{V}} T, \quad (A.4)$$

and

$$T^T K_{X|\hat{U},\bar{V}} S = T^T K_{X|\bar{V}} S - T^T K_{X|\bar{V}} S (S^T K_{X|\bar{V}} S + K_W)^{-1} S^T K_{X|\bar{V}} S$$

$$= 0 \quad (A.6)$$

$$= T^T K_{X|\hat{U},\bar{V}} S, \quad (A.7)$$

where

(A.2) follows from (A.1),

148
(A.3) and (A.6) follow because $S$ and $T$ are cross $K_{X|\bar{V}}$-orthogonal from Theorem 3(g),

(A.4) follows because $T$ is $K_{X|\bar{V}}$-orthogonal from Theorem 3(e),

(A.5) follows because $T$ is $K_{X|\bar{U},\bar{V}}$-orthogonal from Theorem 3(c), and

(A.7) follows because $S$ and $T$ are cross $K_{X|\bar{U},\bar{V}}$-orthogonal from Theorem 3(c).

In summary, we have

$$[S, T]^T K_{X|\bar{U},\bar{V}} [S, T] = [S, T]^T K_{X|\bar{U},\bar{V}} [S, T],$$

and hence

$$K_{X|\bar{U},\bar{V}} = K_{X|\bar{U},\bar{V}},$$

because $[S, T]$ is invertible from Theorem 3(b). This proves that we can assume that $\bar{U}$ is of the following form

$$\bar{U} = S^T X + W,$$

and therefore we have the following long Markov chain

$$\bar{U} \leftrightarrow X \leftrightarrow Y \leftrightarrow \bar{V}.$$

Hence, without loss of optimality we can assume that any feasible solution to $P_G$, in particular $(U^*, V^*)$, satisfies the long Markov chain. This proves that we can assume that $(U^*, V^*)$ is in $S$.

Encoder 2’s rate constraint implies

$$\sigma_{Y|V^*}^2 \geq \sigma_Y^2 2^{-2R_2}. \quad (A.8)$$

We want to prove equality in (A.8). Suppose otherwise that

$$\sigma_{Y|V^*}^2 > \sigma_Y^2 2^{-2R_2}. \quad (A.9)$$

149
Then there exists a zero-mean Gaussian random variable $\tilde{V}$ such that for some $\epsilon > 0$, the conditional variance of $Y$ given $\tilde{V}$ is

$$\sigma^2_{Y|\tilde{V}} = \sigma^2_{Y|V^*} - \epsilon > \sigma^2_{Y|V^*} - 2^{2R_2},$$

and $U^*, X, Y, \tilde{V}$, and $V^*$ form a Markov chain

$$U^* \leftrightarrow X \leftrightarrow Y \leftrightarrow \tilde{V} \leftrightarrow V^*.$$  \hfill (A.10)

We therefore have that

$$K_{X|U^*, \tilde{V}} = K_{X|U^*, \tilde{V}, V^*} \preceq K_{X|U^*, V^*} \preceq D,$$

and

$$K_{X|U^*, \tilde{V}} \preceq K_{X|\tilde{V}} = aa^T \sigma^2_{Y|\tilde{V}} + K_N,$$

which means that $(K_{X|U^*, \tilde{V}}, \sigma^2_{Y|\tilde{V}})$ is feasible for the problem $(P_G)$. We now have the following chain of inequalities

$$I(U^*; X|\tilde{V}) = I(U^*; X|\tilde{V}, V^*)$$

$$= I(U^*; X, \tilde{V}|V^*) - I(U^*; \tilde{V}|V^*)$$

$$< I(U^*; X, \tilde{V}|V^*)$$

$$= I(U^*; X|V^*) + I(U^*; \tilde{V}|V^*, X)$$

$$= I(U^*; X|V^*),$$  \hfill (A.13)

where

(A.11) and (A.13) follow from the Markov condition in (A.10), and
(A.12) follows because

\[ I(U^*; \tilde{V}|V^*) = \frac{1}{2} \log \frac{K_{U^*|V^*}}{|K_{U^*|V^*,\tilde{V}}|} \]

\[ = \frac{1}{2} \log \frac{K_{U^*|V^*}}{|K_{U^*|\tilde{V}}|} \]

\[ = \frac{1}{2} \log \frac{S^T K_{X|V^*} S + K_W}{S^T K_{X|\tilde{V}} S + K_W} \]

\[ > 0. \]

We have arrived at a contradiction to the assumption that \( U^* \) and \( V^* \) are the optimal Gaussian random variables. Therefore, the supposition (A.9) is wrong, and hence (A.8) holds with equality.

### A.2 Proof of Lemma 2

We have the following upper bound

\[ h \left( \frac{S^TX|U,V} \right) \leq \frac{1}{2} \log \left( (2\pi e)^r \left| S^Tk_{X|U,V}S \right| \right) \]

\[ \leq \frac{1}{2} \log \left( (2\pi e)^r \left| S^TDS \right| \right), \]

where

(A.14) follows because Gaussian distribution maximizes differential entropy for a given covariance matrix [53, Theorem 8.6.5], and

(A.15) follows from the distortion constraint and the concavity of the \( \log |·| \) function.
Inequalities (A.14) and (A.15) are equalities if \( X, U, \) and \( V \) are jointly Gaussian with the conditional covariance matrix \( K_{X|U,V} \) being such that
\[
S^{T}K_{X|U,V}S = S^{T}DS. \tag{A.16}
\]
Since \( K_{X|U^*,V^*} \) satisfies (A.16), we conclude that a Gaussian \((U, V)\) with the conditional covariance matrix \( K_{X|U^*,V^*} \) is optimal for the subproblem \((\tilde{P}_1)\), and the optimal value is
\[
v(\tilde{P}_1) = h(S^{T}X) - \frac{1}{2} \log ((2\pi e)^r |S^{T}DS|)
= \frac{1}{2} \log ((2\pi e)^r |S^{T}K_{X}S|) - \frac{1}{2} \log ((2\pi e)^r |I_r|) \tag{A.17}
= \frac{1}{2} \log |S^{T}K_{X}S|,
\]
where (A.17) follows because \( S \) is \( D \)-orthogonal from Theorem 3(d).

### A.3 Proof of Lemma 3

First note that if \( S^{T}a = 0 \), then
\[
S^{T}X = S^{T}(aY + N) = S^{T}N,
\]
which means that
\[
v(\tilde{P}_2) = 0
\]
because \( Y \) is independent of \( N \) and we have a Markov condition \( S^{T}X \leftrightarrow Y \leftrightarrow V \). This implies that any \( V \), in particular a Gaussian \( V \) with the conditional variance \( \sigma_{Y|V^*}^2 \), is optimal for the subproblem \((\tilde{P}_2)\). Therefore, Lemma 3 is trivially true in this case. Let us assume now that \( S^{T}a \neq 0 \), and let
\[
w_1, w_2, \ldots, w_r
\]
be an orthonormal basis in $\mathbb{R}^r$ starting at

$$w_1 \triangleq \frac{1}{c} (S^T K_X S)^{-1/2} S^T a,$$

where

$$c \triangleq \| (S^T K_X S)^{-1/2} S^T a \|.$$

Define the matrices

$$W \triangleq [w_1, w_2, \ldots, w_r]$$

and

$$T_X \triangleq W^T (S^T K_X S)^{-1/2},$$

and the transformation

$$\tilde{X} \triangleq T_X (S^T X).$$

Then the covariance matrix of $\tilde{X}$ is

$$K_{\tilde{X}} = T_X (S^T K_X S) T_X^T$$

$$= W^T (S^T K_X S)^{-1/2} (S^T K_X S)^{-1/2} W$$

$$= W^T W$$

$$= I_r,$$

and the cross-covariance matrix between $\tilde{X}$ and $Y$ is

$$K_{\tilde{X}Y} = T_X S^T K_{XY}$$

$$= W^T (S^T K_X S)^{-1/2} S^T a \sigma_Y^2$$

$$= W^T w_1 \| (S^T K_X S)^{-1/2} S^T a \|^2 \sigma_Y^2$$

$$= (c \sigma_Y^2, 0, \ldots, 0)^T.$$

This means that under this transformation, $\tilde{X}$ has i.i.d standard normal components, and $Y$ is correlated with $\tilde{X}_1$ only and is uncorrelated with the rest of the
components of $\tilde{X}$. Since the transformation matrix $T_X$ is full rank, we have

$$I(S^T X; V) = I(\tilde{X}; V)$$

$$= I(\tilde{X}_1; V) + I(\tilde{X}_2, \ldots, \tilde{X}_r; V|\tilde{X}_1)$$

$$= I(\tilde{X}_1; V),$$

(A.18)

where (A.18) follows because $(\tilde{X}_2, \ldots, \tilde{X}_r)$ is independent of $(V, \tilde{X}_1)$. It is also clear that $\tilde{X}_1, Y,$ and $V$ form a Markov chain

$$\tilde{X}_1 \leftrightarrow Y \leftrightarrow V.$$

Therefore, $(\tilde{P}_2)$ is equivalent to the following problem

$$\max_V I(\tilde{X}_1; V)$$

subject to $R_2 \geq I(Y; V)$ and

$$\tilde{X}_1 \leftrightarrow Y \leftrightarrow V,$$

Oohama [11] showed that a Gaussian $V$ with the conditional variance $\sigma_{V|X}^2$ is optimal for this problem. Hence, the same solution is optimal for $(\tilde{P}_2)$ too, and the optimal value is

$$v(\tilde{P}_2) = \frac{1}{2} \log \frac{|S^T K_X S|}{|S^T (aa^T \sigma_{V|X}^2 + K_N) S|}$$

$$= \frac{1}{2} \log \frac{|S^T K_X S|}{|S^T K_{X|V^*} S|},$$

(A.19)

where (A.19) follows from (3.37).
A.4 Proof of Lemma 4

Assume \((R_1, R_2, D)\) is achievable. Let \(C_1 \triangleq f_1^{(n)}(X^n)\) and \(C_2 \triangleq f_2^{(n)}(Y^n)\). Then using standard information-theoretic inequalities, we obtain

\[
nR_2 \geq I(Y^n; C_2) \tag{A.20}
\]

\[
= \sum_{i=1}^{n} [h(Y^n(i)|Y^n(1:i-1)) - h(Y^n(i)|Y^n(1:i-1), C_2)]
\]

\[
= \sum_{i=1}^{n} [h(Y^n(i)) - h(Y^n(i)|X^n(1:i-1), Y^n(1:i-1), C_2)]
\]

\[
\geq \sum_{i=1}^{n} [h(Y^n(i)) - h(Y^n(i)|X^n(1:i-1), C_2)] \tag{A.21}
\]

\[
= \sum_{i=1}^{n} I(Y^n(i); V(i)), \text{ and} \tag{A.22}
\]

\[
nR_1 \geq I(X^n; C_1|C_2)
\]

\[
= \sum_{i=1}^{n} I(X^n(i); C_1|X^n(1:i-1), C_2)
\]

\[
= \sum_{i=1}^{n} I(X^n(i); C_1|V(i)),
\]

where

(A.20) follows because \(Y^n(i) \leftrightarrow (Y^n(1:i-1), C_2) \leftrightarrow X^n(1:i-1),\)

(A.21) follows because conditioning reduces differential entropy, and

(A.22) follows by letting \(V(i) \triangleq (X^n(1:i-1), C_2).\)
Using the distortion constraint, we obtain

\[
D \geq \frac{1}{n} \sum_{i=1}^{n} E \left[ (X^n(i) - \hat{X}^n(i))(X^n(i) - \hat{X}^n(i))^T \right]
\]

\[= \frac{1}{n} \sum_{i=1}^{n} E \left[ (X^n(i) - E(X^n(i)|C_1, C_2))(X^n(i) - E(X^n(i)|C_1, C_2))^T \right]
\]

\[\geq \frac{1}{n} \sum_{i=1}^{n} E \left[ (X^n(i) - E(X^n(i)|C_1, C_2, X^n(1 : i-1)))(X^n(i) - E(X^n(i)|C_1, C_2, X^n(1 : i-1)))^T \right]
\]

\[= \frac{1}{n} \sum_{i=1}^{n} E \left[ (X^n(i) - E(X^n(i)|C_1, V(i)))(X^n(i) - E(X^n(i)|C_1, V(i)))^T \right],
\]

where (A.23) follows because conditioning reduces the covariance of the error in a positive semidefinite sense. Finally, for each \(i \in \{1, \ldots, n\}\), we have a Markov chain

\[X^n(i) = aY^n(i) + N^n(i) \leftrightarrow Y^n(i) \leftrightarrow V(i) = (X^n(1 : i-1), C_2)\]

because \(N^n(i)\) is independent of \((X^n(1 : i-1), C_2)\). Lemma 4 now follows by using standard time sharing arguments.
B.1 Proof of Second Equality in Theorem 6

Suppose $\mu$ is in $[0, 1]$. Then for any $(U, V)$ in $S$, we have

$$
\mu I(X; U|V) + I(Y; V)
= \mu I(X; U, V) - \mu I(X, V) + I(Y; V)
= \mu I(X; U) + \mu I(X; V|U) + \mu [I(Y, V) - I(X; V)] + (1 - \mu)I(Y; V)
\geq \mu I(X; U) \tag{B.1}
= \frac{\mu}{2} \log \frac{|K_X|}{|K_{X|U}|},
$$

where (B.1) follows because of the facts that $I(Y; V) \geq 0$ and $I(X; V|U) \geq 0$, and we have

$$I(Y, V) - I(X; V) \geq 0$$

because of the data processing inequality [53, Theorem 2.8.1] and the Markov chain $X \leftrightarrow Y \leftrightarrow V$. The inequality (B.1) is achieved by any $(U, V)$ in $S$ such that $V$ is independent of $(X, Y, U)$, and the conditional covariance of $X$ given $(U, V)$ satisfies

$$0 \preceq K_{X|U,V} = K_{X|U} \preceq D.$$

Since conditioning reduces covariance in a positive semidefinite sense, we have an additional constraint

$$K_{X|U} \preceq K_X.$$
We therefore have the following

\[
R_G(D, \mu) = \min_{(R_1, R_2) \in R_G(D)} \mu R_1 + R_2
\]

\[
= \min_{(U, V) \in S} \mu I(X; U|V) + I(Y; V)
\]

\[
= \min_{K_{X|U}} \frac{\mu}{2} \log \frac{|K_X|}{|K_{X|U}|}
\]

subject to \( K_X \succeq K_{X|U} \succ 0 \) and

\[
D \succeq K_{X|U}
\]

\[
= v(P_{pt-pd}).
\]

Suppose now that \( \mu > 1 \). Then any \((U, V)\) in \( S\) can be characterized by positive semidefinite conditional covariance matrices \( K_{Y|V} \) and \( K_{X|U, V} \) such that

\[
K_Y \succeq K_{Y|V} \succeq 0,
\]

\[
K_{Y|V} + K_N \succeq K_{X|U, V} \succeq 0,
\]

\[
D \succ K_{X|U, V},
\]

and

\[
I(X; U|V) = \frac{1}{2} \log \frac{|K_{Y|V} + K_N|}{|K_{X|U, V}|},
\]

\[
I(Y; V) = \frac{1}{2} \log \frac{|K_Y|}{|K_{Y|V}|}.
\]
In this case, we have

\[ R_G(D, \mu) = \min_{(R_1, R_2) \in R_G(D)} \mu R_1 + R_2 \]

\[ = \min_{(U, V) \in S} \mu I(X; U|V) + I(Y; V) \]

\[ = \min_{K_{Y|V}, K_{X|U,V}} \frac{\mu}{2} \log \frac{K_{Y|V} + K_N}{|K_{X|U,V}|} + \frac{1}{2} \log \frac{|K_Y|}{|K_{Y|V}|} \]

subject to \[ K_Y \succ K_{Y|V} \succ 0, \]

\[ K_{Y|V} + K_N \succ K_{X|U,V} \succ 0, \] and

\[ D \succ K_{X|U,V} \]

\[ = v(P_{G1}). \]

**B.2 Proof of Lemma 6**

We will be using several results and terms from Bertsekas et al. [61]. The book contains all of the background that these results need. The proof of the lemma is similar to the same of Lemma 5 in [16]. Let us first introduce a few notation used in the proof. We use \( \text{vec}(A_1, A_2) \) to denote the column vector created by the concatenation of the columns of \( m \times m \) matrices \( A_1 \) and \( A_2 \). If \( a = \text{vec}(A_1, A_2) \), then we use the notation \( \text{mat}(a) \) to denote the inverse operation to get back the pair \( (A_1, A_2) \), i.e.,

\[ \text{mat}(a) = (A_1, A_2). \]

The set of all column vectors created by the concatenation of the columns of \( m \times m \) symmetric matrices \( A_1 \) and \( A_2 \) is denoted by \( \mathcal{A} \), i.e.,

\[ \mathcal{A} \triangleq \{ \text{vec}(A_1, A_2) : A_i = A_i^T \text{ for all } i \in \{1, 2\} \}. \]
ri($B$) is used to denote the relative interior of the set $B$. The sum of the two vector sets $V_1$ and $V_2$ is denoted by $V_1 + V_2$ and is defined as

$$V_1 + V_2 \triangleq \{ v_1 + v_2 : v_i \in V_i \text{ for all } i \in \{1, 2\} \}.$$ 

We also need the following facts from linear algebra.

**Lemma 19.** (a) If $E$ is an $m \times n$ matrix and $F$ is an $n \times m$ matrix, then $\text{Tr}(EF) = \text{Tr}(FE)$.

(b) If $E$ and $F$ are positive semidefinite, then $EF = 0$ if and only if $\text{Tr}(EF) = 0$.

**Proof.** Part (a) immediately follows from the definition of $\text{Tr}(\cdot)$ function. Part (b) can be proved using the eigen decompositions of $E$ and $F$. □

We can re-write the problem ($P_{G2}$) as

$$\min_b h(b)$$

subject to $b \in B$,

where $b \triangleq \text{vec}(B_1, B_2)$ and

$$h(b) \triangleq \frac{\mu}{2} \log \frac{|K_X - B_2|}{|K_X - B_1 - B_2|} + \frac{1}{2} \log \frac{|K_Y|}{|K_Y - B_2|},$$

and the feasible set $B$ is written as

$$B \triangleq B_1 \cap B_2 \cap B_{12},$$

where for $i \in \{1, 2\}$

$$B_i \triangleq \{ \text{vec}(B_1, B_2) : B_i \succ 0 \} \cap A$$

and

$$B_{12} \triangleq \{ \text{vec}(B_1, B_2) : B_1 + B_2 \succeq K_X - D \} \cap A.$$
Since $h(\cdot)$ is continuously differentiable, it follows from [61, Proposition 4.7.1, p. 255] that the local minima $b^*$ must satisfy
\[ -\nabla h(b^*) \in T_B(b^*), \]
where $\nabla h(b^*)$ is the gradient of $h(\cdot)$ at $b^*$, and $T_B(b^*)^*$ is the polar cone of the tangent cone $T_B(b^*)$ of $B$ at $b^*$. Now since $B_i$ for all $i \in \{1, 2\}$ and $B_{12}$ are nonempty convex sets and $\text{ri}(B_1) \cap \text{ri}(B_2) \cap \text{ri}(B_{12})$ is nonempty, it follows from [61, Problem 4.23, p. 267] and [61, Proposition 4.6.3, p. 254] that
\[ T_B(b^*)^* = T_{B_1}(b^*)^* + T_{B_2}(b^*)^* + T_{B_{12}}(b^*)^*. \]
We next show that
\[ -\nabla h(b^*) \in T_{B_1}(b^*)^* \cap A + T_{B_2}(b^*)^* \cap A + T_{B_{12}}(b^*)^* \cap A. \]
Note that $-\nabla h(b^*)$ is a column concatenation of two $m \times m$ symmetric matrices. This together with (B.2) and (B.3) yields
\[ -\nabla h(b^*) = z_1 + z_2 + z_{12} \in A, \]
where for $i \in \{1, 2\}$
\[ z_i \in T_{B_i}(b^*)^* \text{ and } \]
\[ z_{12} \in T_{B_{12}}(b^*)^*. \]
Let us now define
\[ (K_i, L_i) \triangleq \text{mat}(z_i), \forall i \in \{1, 2\} \text{ and } \]
\[ (K_{12}, L_{12}) \triangleq \text{mat}(z_{12}). \]
Using this, we define
\[ \bar{z}_i \triangleq \text{vec}\left(\frac{1}{2} (K_i + K_i^T), \frac{1}{2} (L_i + L_i^T)\right), \forall i \in \{1, 2\} \text{ and } \]
\[ \bar{z}_{12} \triangleq \text{vec}\left(\frac{1}{2} (K_{12} + K_{12}^T), \frac{1}{2} (L_{12} + L_{12}^T)\right). \]
Since $B_1$ is a nonempty convex set, it follows from [61, Proposition 4.6.3, p. 254] that

$$z_1^T (b - b^*) \leq 0, \quad \forall b \in B_1. \quad (B.6)$$

Consider any $b \in B_1$. Let

$$(E_1, F_1) \triangleq \text{mat}(b - b^*).$$

We now obtain

$$\bar{z}_1^T (b - b^*) = \frac{1}{2} \text{Tr} \left( (K_1 + K_1^T) E_1 \right) + \frac{1}{2} \text{Tr} \left( (L_1 + L_1^T) F_1 \right)$$

$$= \text{Tr} \left( K_1 E_1 \right) + \text{Tr} \left( L_1 F_1 \right) \quad (B.7)$$

$$= z_1^T (b - b^*)$$

$$\leq 0, \quad (B.8)$$

where

(B.7) follows because $E_1$ and $F_1$ are symmetric, and

(B.8) follows from (B.6).

We conclude from (B.8) that

$$\bar{z}_1 \in T_{B_1}(b^*)^* \cap A. \quad (B.9)$$

We can similarly show that

$$z_2 \in T_{B_2}(b^*)^* \cap A \quad \text{and} \quad (B.10)$$

$$\bar{z}_{12} \in T_{B_{12}}(b^*)^* \cap A. \quad (B.11)$$
Now
\[ \bar{z}_1 + \bar{z}_2 + \bar{z}_{12} \]
\[ = \text{vec} \left( \frac{1}{2} \left( K_1 + K_2 + K_{12} + K_1^T + K_2^T + K_{12}^T \right) \right), \]
\[ = \text{vec} \left( (K_1 + K_2 + K_{12}), (L_1 + L_2 + L_{12}) \right) \]
\[ = z_1 + z_2 + z_{12} \]
\[ = -\nabla h(b^*), \] (B.13)

where

(B.12) follows because \( K_1 + K_2 + K_{12} \) and \( L_1 + L_2 + L_{12} \) are symmetric from (B.5), and

(B.13) follows from the equality in (B.5).

This together with (B.9) – (B.11) implies (B.4).

We now proceed to characterize the right-hand side of (B.4). Consider any \( z \in T_{B_1}(b^*) \cap A \). It again follows from [61, Proposition 4.6.3, p. 254] that
\[ z^T (b - b^*) \leq 0, \quad \forall b \in B_1. \] (B.14)

Let us define
\[ (M_1, M_2) \triangleq \text{mat}(z), \]
\[ (B_1, B_2) \triangleq \text{mat}(b), \] and
\[ (B_1^*, B_2^*) \triangleq \text{mat}(b^*). \]

Then (B.14) can be re-written as
\[ \sum_{i=1}^{2} \text{Tr}(M_i (B_i - B_i^*)) \leq 0, \quad \forall \text{vec}(B_1, B_2) \in B_1. \] (B.15)
We first show that $M_2 = 0$. Let us pick $(B_1, B_2) = (B_1^*, B_2^* + M_2)$. This means that

$$\text{Tr}(M_2 M_2) \leq 0,$$

which implies that $M_2 = 0$ because $M_2$ is symmetric. We next prove that $M_1$ is negative semidefinite. Suppose there exists $w \neq 0$ such that $w^T M_1 w > 0$. We then have

$$0 < w^T M_1 w = \text{Tr}(w^T M_1 w) = \text{Tr}(M_1 ww^T),$$

where the last equality follows from Lemma 19(a). But this contradicts (B.15) because $\text{vec}(B_1^* + ww^T, B_2^*) \in B_1$, and hence $M_1 \preceq 0$. We finally show that $M_1 B_1^* = 0$. Let $(B_1, B_2) = (\alpha B_1^*, B_2^*)$, where $\alpha > 1$. (B.15) then implies that

$$\text{Tr}(M_1 B_1^*) \leq 0.$$

Likewise, on picking $0 < \alpha < 1$, we obtain

$$\text{Tr}(M_1 B_1^*) \geq 0.$$

Both together establish

$$\text{Tr}(M_1 B_1^*) = 0,$$

which together with Lemma 19(b) implies that

$$M_1 B_1^* = 0$$

because $-M_1$ and $B_1^*$ are positive semidefinite. We therefore have that

$$T_{B_1}(b^*)^\ast \cap A \subseteq \{\text{vec}(M_1, 0) | M_1 \preceq 0 \text{ and } M_1 B_1^* = 0\}. \quad (B.16)$$

Similarly, we can show that

$$T_{B_2}(b^*)^\ast \cap A \subseteq \{\text{vec}(0, M_2) | M_2 \preceq 0 \text{ and } M_2 B_2^* = 0\}. \quad (B.17)$$
Consider any $z \in T_{B_{12}}(b^*)^* \cap A$. As before, we obtain

$$\sum_{i=1}^{2} \text{Tr}(\Lambda_i(B_i - B_i^*)) \leq 0, \quad \forall \text{vec}(B_1, B_2) \in B_{12},$$

(B.18)

where

$$(\Lambda_1, \Lambda_2) \triangleq \text{mat}(z).$$

On picking $(B_1, B_2) = (B_1^* + \Lambda_1, B_2^* - \Lambda_1)$, (B.18) yields

$$\text{Tr}(\Lambda_1\Lambda_1) - \text{Tr}(\Lambda_2\Lambda_1) \leq 0.$$

Similarly, picking $(B_1, B_2) = (B_1^* - \Lambda_2, B_2^* + \Lambda_2)$ gives

$$\text{Tr}(\Lambda_2\Lambda_2) - \text{Tr}(\Lambda_1\Lambda_2) \leq 0.$$

Both together imply that

$$\text{Tr}((\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_2)) \leq 0,$$

and therefore

$$\Lambda_1 - \Lambda_2 = 0,$$

because $\Lambda_1$ and $\Lambda_2$ are symmetric. Let us denote $\Lambda_1$ and $\Lambda_2$ by $\Lambda$. As before, we can show that $\Lambda \prec 0$. We next prove that

$$\text{Tr}(\Lambda(B_1^* + B_2^* - K_X - D)) = 0.$$

Observe that $(B_1, B_2) = (\alpha(B_1^* + B_2^* - K_X + D) + K_X - D - B_2^* B_2^*)$, where $\alpha > 0$, is a valid choice of $(B_1, B_2)$ in (B.18). For $\alpha > 1$, this implies

$$\text{Tr}(\Lambda(B_1^* + B_2^* - K_X - D)) \leq 0,$$

and for $0 < \alpha < 1$, it gives

$$\text{Tr}(\Lambda(B_1^* + B_2^* - K_X - D)) \geq 0.$$
Therefore,
\[
\text{Tr}(\Lambda(B_1^* + B_2^* - K_X - D)) = 0.
\]
This and Lemma 19(b) imply that
\[
\Lambda(B_1^* + B_2^* - K_X) = 0.
\]
We thus have that
\[
T_{B_{12}}(b^*) \cap A \subseteq \{ \text{vec}(\Lambda, \Lambda) | \Lambda \leq 0 \text{ and } \Lambda(B_1^* + B_2^* - K_X - D) = 0 \}. \quad (B.19)
\]
It now follows from (B.4), (B.16), (B.17), and (B.19) that
\[
\nabla h(b^*) = \text{vec}(M_1 + \Lambda, M_2 + \Lambda)
\]
for some \(M_1, M_2,\) and \(\Lambda\) such that
\[
M_i B_i^* = 0, \quad \text{for all } i \in \{1, 2\}\]
\[
\Lambda(B_1^* + B_2^* - K_X + D) = 0, \quad \text{and}
\]
\[
M_1, M_2, \Lambda \succcurlyeq 0.
\]
Lemma 6 now follows because
\[
\nabla h(b^*) = \text{vec}\left(\mu \left(\frac{1}{2}(K_X - B_1^* - B_2^*)^{-1},
\frac{1}{2}(K_X - B_1^* - B_2^*)^{-1} - \frac{\mu}{2}(K_X - B_1^*)^{-1} + \frac{1}{2}(K_Y - B_2^*)^{-1}\right)\right).
\]

### B.3 Proof of Lemma 7

Using (4.6), we obtain
\[
\Delta^* = \frac{\mu}{2}(K_X - B_2^*)^{-1} - M_1^*
\]
\[
= (K_X - B_2^*)^{-1} \left[ \frac{\mu}{2}(K_X - B_2^*) - (K_X - B_2^*) M_1^* (K_X - B_2^*) \right] (K_X - B_2^*)^{-1}.
\]
It is hence sufficient to show that

\[ \frac{\mu}{2} (K_X - B_2^*) - (K_X - B_2^*) M_1^*(K_X - B_2^*) \]

is positive semidefinite. On pre- and post-multiplying (4.1) by $K_X - B_1^* - B_2^*$, we obtain

\[ \frac{\mu}{2} (K_X - B_1^* - B_2^*) - (K_X - B_1^* - B_2^*) (M_1^* + \Lambda^*) (K_X - B_1^* - B_2^*) = 0. \]  \hspace{1cm} (B.20)

Using (4.3) and (4.4), we have

\[ (K_X - B_1^* - B_2^*) M_1^*(K_X - B_1^* - B_2^*) = (K_X - B_2^*) M_1^*(K_X - B_2^*) \]  \hspace{1cm} and  \hspace{1cm} (B.21)

\[ (K_X - B_1^* - B_2^*) \Lambda^* (K_X - B_1^* - B_2^*) = D \Lambda^* D. \]  \hspace{1cm} (B.22)

Now (B.20) through (B.22) together imply that

\[ \frac{\mu}{2} (K_X - B_2^*) - (K_X - B_2^*) M_1^*(K_X - B_2^*) = \frac{\mu}{2} B_1^* + D \Lambda^* D, \]

which is a positive semidefinite matrix.

We next show that $\Delta^*$ is nonzero. Suppose otherwise that

\[ \Delta^* = 0. \]

This together with (4.6) implies that

\[ M_1^* = \frac{\mu}{2} (K_X - B_2^*)^{-1} > 0 \]  \hspace{1cm} and  \hspace{1cm} \[ M_2^* = \frac{1}{2} (K_Y - B_2^*)^{-1} > 0, \]

i.e., $M_1^*$ and $M_2^*$ are positive definite. It now follows from (4.3) that

\[ B_1^* = B_2^* = 0, \]

which is a contradiction because $(0, 0)$ is not feasible for the optimization problem $(P_{G2})$ by (2.1).
B.4 Proof of Lemma 8

It is clear by definition that \(\tilde{B}_1^*, \tilde{B}_2^*, \tilde{M}_1^*, \text{and} \tilde{M}_2^*\) are positive semidefinite matrices. To prove (4.30), we use the first equality in (4.6) and obtain

\[
SS^T = \Delta^*
\]

\[
= \frac{\mu}{2} (K_X - B_2^*)^{-1} - M_1^*
\]

(B.23)

\[
= \frac{\mu}{2}[S, T] \begin{pmatrix}
S^T (K_X - B_2^*) S & 0 \\
0 & T^T (K_X - B_2^*) T
\end{pmatrix}^{-1} [S, T]^T - M_1^*
\]

(B.24)

\[
= \frac{\mu}{2}[S, T] \begin{pmatrix}
(S^T (K_X - B_2^*) S)^{-1} & 0 \\
0 & (T^T (K_X - B_2^*) T)^{-1}
\end{pmatrix} [S, T]^T - M_1^*
\]

(B.25)

where

(B.23) follows because \([S, T]\) is invertible, and

(B.24) follows because \(S\) and \(T\) are cross \((K_X - B_2^*)\)-orthogonal.

On pre- and post-multiplying (B.25) by \(S^T (K_X - B_2^*)\) and \((K_X - B_2^*) S\), respectively, and again using the fact that \(S\) and \(T\) are cross \((K_X - B_2^*)\)-orthogonal, we obtain

\[
(S^T (K_X - B_2^*) S) (S^T (K_X - B_2^*) S)
\]

\[
= \frac{\mu}{2} (S^T (K_X - B_2^*) S) - S^T (K_X - B_2^*) M_1^* (K_X - B_2^*) S,
\]
which is equivalent to

\[
I_r = \frac{\mu}{2} (S^T (K_X - B_2^*) S)^{-1} \\
- \left( S^T (K_X - B_2^*) S \right)^{-1} S^T (K_X - B_2^*) M_1^* (K_X - B_2^*) S \left( S^T (K_X - B_2^*) S \right)^{-1}.
\]  
(B.26)

Similarly, using the second equality in (4.6) together with the facts that \([S, W]\) is invertible and \(S\) and \(W\) are cross \((K_Y - B_2^*)\)-orthogonal, we obtain

\[
I_r = \frac{1}{2} \left( S^T (K_Y - B_2^*) S \right)^{-1} \\
- \left( S^T (K_Y - B_2^*) S \right)^{-1} S^T (K_Y - B_2^*) M_2^* (K_Y - B_2^*) S \left( S^T (K_Y - B_2^*) S \right)^{-1}.
\]  
(B.27)

Now (B.26) and (B.27) together can be written as

\[
I_r = \frac{\mu}{2} (K_X - B_2^*)^{-1} - M_1^* = \frac{1}{2} (K_Y - B_2^*)^{-1} - M_2^*.
\]  
(B.28)

This proves (4.30).

To prove (4.31), we have from (4.3) and (4.8) that

\[
B_1^* a_i = 0,
\]

for all \(i\) in \(\{1, 2, \ldots, p\}\). Since the columns of \(T\) are in \(\text{span}\{a_i\}_{i=1}^p\), we have

\[
B_1^* T = 0.
\]

This and (4.3) together imply

\[
B_1^* \left( M_1^* - \frac{\mu}{2} T \left( T^T (K_X - B_2^*) T \right)^{-1} T^T \right) = 0.
\]

We now use (B.25) and obtain

\[
B_1^* \left( \frac{\mu}{2} S \left( S^T (K_X - B_2^*) S \right)^{-1} S^T - SS^T \right) = 0,
\]
which can be re-written as

\[ B_1^* S \left( \frac{\mu}{2} (K_X - \hat{B}_2^*)^{-1} - I_r \right) S^T = 0. \]

Using the first equality in (B.28) yields

\[ B_1^* S \tilde{M}_1^* S^T = 0. \]

We next invoke Lemma 19(b) to obtain

\[ \text{Tr}(B_1^* S \tilde{M}_1^* S^T) = 0. \]

Using Lemma 19(a) gives

\[ \text{Tr}(S^T B_1^* S \tilde{M}_1^*) = 0, \]

which is equivalent to

\[ \text{Tr}(\tilde{B}_1^* \tilde{M}_1^*) = 0. \]

Since \( \tilde{B}_1^* \) and \( \tilde{M}_1^* \) are positive semidefinite, by invoking Lemma 19(b) again, we obtain

\[ \tilde{B}_1^* \tilde{M}_1^* = 0. \]

The proof of

\[ \tilde{B}_2^* \tilde{M}_2^* = 0. \]

is exactly similar. This proves (4.31). The proof of (4.32) is immediate from (4.10).

**B.5 Proof of Lemma 9**

The proofs of (4.35) and (4.36) are easy. They follow from (4.30), (4.33), and (4.34). Since \( \mu > 1 \), (4.35) and (4.36) imply that

\[ K_X \succ K_Y. \]
$K_X$ and $K_Y$ are positive definite by definition. Since $\tilde{M}_1^*$ and $\tilde{M}_2^*$ are positive semidefinite,

$$K_X \succeq K_X \text{ and } K_Y \succeq K_Y$$

follow from (4.33) and (4.34), respectively. This proves (4.37) and (4.38). To prove (4.39), we have

$$\begin{align*}
\frac{|K_Y|}{|K_Y - \tilde{B}_2^*|} &= \frac{|K_Y - \tilde{B}_2^* + \tilde{B}_2^*|}{|K_Y - \tilde{B}_2^*|} \\
&= \frac{|I_r + \tilde{B}_2^*(K_Y - \tilde{B}_2^*)^{-1}|}{|I_r|} \\
&= \frac{|I_r + \tilde{B}_2^*[ (K_Y - \tilde{B}_2^*)^{-1} - 2\tilde{M}_2^*] |}{|I_r|} \\
&= \frac{|K_Y|}{|K_Y - \tilde{B}_2^*|},
\end{align*}$$

(B.29)

where

(B.29) follows from (4.31), and

(B.30) follows from (4.34).

To prove (4.40), we proceed similarly and obtain

$$\begin{align*}
\frac{|K_X - \tilde{B}_2^*|}{|K_X - \tilde{B}_1^* - \tilde{B}_2^*|} &= \frac{|I_r|}{|I_r - \tilde{B}_1^*(K_X - \tilde{B}_2^*)^{-1}|} \\
&= \frac{|I_r|}{|I_r - \tilde{B}_1^*[ (K_X - \tilde{B}_2^*)^{-1} - \frac{2}{\mu} \tilde{M}_1^*] |} \\
&= \frac{|I_r|}{|I_r - \tilde{B}_1^*(K_X - \tilde{B}_2^*)^{-1}|} \\
&= \frac{|K_X - \tilde{B}_2^*|}{|K_X - \tilde{B}_1^* - \tilde{B}_2^*|},
\end{align*}$$

(B.31)
where

(B.31) follows from (4.31), and

(B.32) follows from (4.33).

B.6 Proof of Lemma 10

We have

\[ h(\hat{X}|U, V) \leq \frac{1}{2} \log((2\pi e)^r |K_{\hat{X}|U, V}|) \tag{B.33} \]
\[ \leq \frac{1}{2} \log((2\pi e)^r |\hat{D}|), \tag{B.34} \]

where

(B.33) follows from the fact the Gaussian distribution maximizes the differential entropy for a given covariance matrix [53, Theorem 8.6.5], and

(B.34) follows from the distortion constraint in the definition of \( \hat{P}_1 \) and the concavity of \( \log \cdot \cdot \cdot \) function.

Inequalities (B.33) and (B.34) are equalities if \( \hat{X}, U, \) and \( V \) are jointly Gaussian with the conditional covariance matrix \( K_{\hat{X}|U, V} \) such that

\[ K_{\hat{X}|U, V} = \hat{D} = K_\hat{X} - \hat{B}_1^* - \hat{B}_1^*, \tag{B.35} \]

where the last equality follows from (4.41). We thus conclude that a Gaussian \( (U, V) \) with the conditional covariance matrix satisfying (B.35) is optimal for the
subproblem \(\tilde{P}_1\), and the optimal value is
\[
v(\tilde{P}_1) = \mu h(\tilde{X}) - \frac{\mu}{2} \log((2\pi e)^r |\tilde{D}|)
\]
\[
= \frac{\mu}{2} \log((2\pi e)^r |K_{\tilde{X}}|) - \frac{\mu}{2} \log((2\pi e)^r |\tilde{D}|)
\]
\[
= \frac{\mu}{2} \log \frac{|K_{\tilde{X}}|}{|\tilde{D}|}.
\]

B.7 Proof of Lemma 11

Since conditioned on \(V\), \(\tilde{Y}\) and \(\tilde{N}\) are independent, we use the vector EPI [53, Theorem 17.7.3] to obtain
\[
h(\tilde{Y}|V) - \mu h(\tilde{X}|V) = h(\tilde{Y}|V) - \mu h(\tilde{Y} + \tilde{N}|V)
\]
\[
\leq h(\tilde{Y}|V) - \frac{\mu r}{2} \log(2^{\frac{r}{2} h(\tilde{Y}|V)} + 2^{\frac{r}{2} h(\tilde{N})}). \tag{B.36}
\]

The inequality (B.36) is equality if \(\tilde{Y}\) and \(V\) are jointly Gaussian and the conditioned covariance matrix
\[
K_{\tilde{Y}|V} = aK_{\tilde{N}},
\]
for some constant \(a > 0\). By following standard calculus arguments, we can show that for \(\mu > 1\) the right-hand side of (B.36) is concave in \(h(\tilde{Y}|V)\) and has a global maximum at
\[
h(\tilde{Y}|V) = h(\tilde{N}) - \frac{r}{2} \log(\mu - 1). \tag{B.37}
\]

Let \(V_G\) and \(\tilde{Y}\) be jointly Gaussian such that the conditional covariance matrix of \(\tilde{Y}\) given \(V_G\) is
\[
K_{\tilde{Y}|V_G} = K_{\tilde{Y}} - \tilde{B}_2^*.
\]
We next show that this $V_G$ achieves equality in (B.36) and satisfies (B.37) simultaneously. We have from (4.35) and (4.36) that

$$K_Y - \tilde{B}_2^* = (\mu - 1)^{-1}K_N,$$  \hspace{1cm} (B.38)

i.e., the conditional covariance matrix $K_Y|V_G$ is proportional to $K_N$. Hence, (B.36) is satisfied with equality. Moreover, for this $V_G$, (B.37) and (B.38) are equivalent. Therefore,

$$h(\hat{Y}|V) - \mu h(\hat{X}|V) \leq \frac{1}{2} \log( (2\pi e)^r |K_Y - \tilde{B}_2^*| ) - \frac{\mu}{2} \log( (2\pi e)^r |K_X - \tilde{B}_2^*| ).$$

We thus conclude that $V_G$ is optimal for $(\hat{P}_2)$ and the optimal value is

$$v(\hat{P}_2) = \mu h(\hat{X}) - h(\hat{Y}) + \frac{1}{2} \log( (2\pi e)^r |K_Y - \tilde{B}_2^*| ) - \frac{\mu}{2} \log( (2\pi e)^r |K_X - \tilde{B}_2^*| )$$

$$= \frac{\mu}{2} \log( (2\pi e)^r |K_X| ) - \frac{1}{2} \log( (2\pi e)^r |K_Y| )$$

$$+ \frac{1}{2} \log( (2\pi e)^r |K_Y - \tilde{B}_2^*| ) - \frac{\mu}{2} \log( (2\pi e)^r |K_X - \tilde{B}_2^*| )$$

$$= \frac{\mu}{2} \log \frac{|K_X|}{|K_X - \tilde{B}_2^*|} - \frac{1}{2} \log \frac{|K_Y|}{|K_Y - \tilde{B}_2^*|}.$$
C.1 Proof of Lemma 14

The proof is rather well known and appears in source coding literature quite often. For instance, the similar proof can be found in [37]. Let us define

$$\hat{\Lambda}_i \triangleq \left\{ \lambda_i = (U, T) \in \Lambda_i : |U_i| \leq |X_i| + 2^L - 1 \text{ for all } l \in \mathcal{L}, \text{ and } |T| \leq 2^L \right\},$$

and

$$\hat{\mathcal{RE}}_i^{CI} \triangleq \bigcup_{\lambda_i \in \hat{\Lambda}_i} \mathcal{RE}_i^{CI}(\lambda_i).$$

We want to show that $\mathcal{RE}_i^{CI} = \hat{\mathcal{RE}}_i^{CI}$. We start with the deterministic $T$ case. Consider $\lambda_i = (U, T)$ in $\Lambda_i$, where $T$ is deterministic. For any $S \subseteq \mathcal{L}$ containing 1, we have

$$I(X_S; U_S|U_{S^c}, X_{L+1}, Z) = H(X_S|U_{S^c}, X_{L+1}, Z) - H(X_S|U_1^c, U_1, X_{L+1}, Z),$$

and for any nonempty $S$ not containing 1, we have

$$I(X_S; U_S|U_{S^c}, X_{L+1}, Z) = I(X_S; U_S|U_{S^c}\{1\}, U_1, X_{L+1}, Z).$$

Moreover,

$$I(Y; U, X_{L+1}|Z) = H(Y|X_{L+1}, Z) - H(Y|U_1^c, U_1, X_{L+1}, Z).$$

It follows from the support lemma [58, Lemma 3.4, pp. 310] that there exists $\hat{U}_1$ with $\hat{U}_1 \subseteq U_1$ such that

$$|\hat{U}_1| \leq |X_1| + 2^L - 1,$$
\[
\sum_{u_1 \in U_1} \Pr(X_1 = x_1|U_1 = u_1) \Pr(\hat{U}_1 = u_1) = \Pr(X_1 = x_1) \text{ for all } x_1 \text{ in } \mathcal{X}_1 \text{ but one},
\]

\[
H(X_S|U_1^c, U_1, X_{L+1}, Z) = H(X_S|U_1^c, \hat{U}_1, X_{L+1}, Z) \text{ for all } S \text{ containing } 1,
\]

\[
I(X_S; U_S|U_1^c \{1\}, U_1, X_{L+1}, Z) = I(X_S; U_S|U_1^c \{1\}, \hat{U}_1, X_{L+1}, Z)
\]

for all nonempty \( S \) not containing 1, and

\[
H(Y|U_1^c, U_1, X_{L+1}, Z) = H(Y|U_1^c, \hat{U}_1, X_{L+1}, Z).
\]

Since \( U_1 \leftrightarrow X_1 \leftrightarrow (U_1^c, X_1^c, X_{L+1}, Y, Z) \), if we replace \( U_1 \) by \( \hat{U}_1 \) then the resulting \( \lambda_i \) is in \( \Lambda_i \) and \( \mathcal{RE}^{CI}_i(\lambda_i) \) remains unchanged. By repeating this procedure for \( U_2, \ldots, U_L \), we conclude that there exists \( \hat{\lambda}_i = (\hat{U}, \hat{T}) \) in \( \hat{\Lambda}_i \) such that \( \hat{T} \) is deterministic and \( \mathcal{RE}^{CI}_i(\lambda_i) = \mathcal{RE}^{CI}_i(\hat{\lambda}_i) \).

We now turn to general \( T \). Consider \( \lambda_i = (U, T) \) in \( \Lambda_i \). Let \( (U, t) \) denote the joint distribution of \((U, T)\) conditioned on \( \{T = t\} \). It follows from the deterministic \( T \) case that for each \( t \) in \( T \), there exists \( \hat{U} \) such that \((\hat{U}, t)\) is in \( \hat{\Lambda}_i \) and \( \mathcal{RE}^{CI}_i(U, t) = \mathcal{RE}^{CI}_i(\hat{U}, t) \). Hence, on replacing \( U \) by \( \hat{U} \) for each \( t \) in \( T \), we obtain \((\hat{U}, T)\) in \( \Lambda_i \) such that \(|\hat{U}_l| \leq |\mathcal{X}_l| + 2^L - 1 \) for all \( l \) in \( \mathcal{L} \) and \( \mathcal{RE}^{CI}_i(U, T) = \mathcal{RE}^{CI}_i(\hat{U}, T) \). Now \( \mathcal{RE}^{CI}_i(\hat{U}, T) \) is the set of vectors \((R, E)\) such that

\[
\sum_{t \in S} R_t \geq I(X_S; U_S|U_1^c, X_{L+1}, Z, T) \text{ for all } S, \text{ and}
\]

\[
E \leq I(Y; U, X_{L+1}|Z, T).
\]

It again follows from the support lemma that there exists \( \hat{T} \) with \( \hat{T} \subseteq T \) such
that

\[ |\hat{T}| \leq 2^L, \]

\[ I(X_S; U_S| U_{Sc}, X_{L+1}, Z, T) = I(X_S; U_S| U_{Sc}, X_{L+1}, Z, \hat{T}), \]

and

\[ I(Y; U, X_{L+1}| Z, T) = I(Y; U, X_{L+1}| Z, \hat{T}). \]

We therefore have that \( \hat{\lambda}_i = (\hat{U}, \hat{T}) \) is in \( \hat{\Lambda}_i \) and \( \mathcal{RE}_i^{CI}(\lambda_i) = \mathcal{RE}_i^{CI}(\hat{\lambda}_i) \). This proves \( \mathcal{RE}_i^{CI} \subseteq \mathcal{RE}_i^{CI}, \) and hence \( \mathcal{RE}_i^{CI} = \mathcal{RE}_i^{CI} \) because the reverse containment trivially holds.

For part (b), it suffices to show that \( \mathcal{RE}_i^{CI} \) is closed. Consider any sequence \((R(n), E(n))\) in \( \mathcal{RE}_i^{CI} \) that converges to \((R, E)\). Since conditional mutual information is a continuous function, \( \hat{\Lambda}_i \) is a compact set. Hence, there exists a sequence \( \lambda_i^{(n)} = (U^{(n)}, T^{(n)}) \) in \( \hat{\Lambda}_i \) that converges to \( \lambda_i = (U, T) \) in \( \hat{\Lambda}_i \) such that \((R^{(n)}, E^{(n)})\) is in \( \mathcal{RE}_i^{CI}(\lambda_i^{(n)}) \), i.e.,

\[ \sum_{l \in S} R_l^{(n)} \geq I(X_S; U_S^{(n)}| U_{Sc}^{(n)}, X_{L+1}, Z, T^{(n)}) \quad \text{for all } S, \] and

\[ E^{(n)} \leq I(Y; U^{(n)}, X_{L+1}| Z, T^{(n)}). \]

Again, by the continuity of conditional mutual information, this implies that

\[ \sum_{l \in S} R_l \geq I(X_S; U_S| U_{Sc}, X_{L+1}, Z, T) \quad \text{for all } S, \] and

\[ E \leq I(Y; U, X_{L+1}| Z, T). \]

We thus have that \((R, E)\) is in \( \mathcal{RE}_i^{CI} \).

C.2 Proof of Theorem 11

We prove the deterministic \( T \) case. The general case follows by time sharing. Consider any \( \lambda_i = (U, T) \) in \( \Lambda_i \) with \( T \) being deterministic. Consider \((R, E)\)

177
such that

\[ \sum_{l \in S} R_l \geq I(X_S; U_S|U_{S^c}, X_{L+1}, Z) \quad \text{for all } S \subseteq \mathcal{L}, \quad \text{and} \]

(C.1)

\[ E \leq I(Y; U, X_{L+1}|Z). \quad \text{(C.2)} \]

It suffices to show that \((R, E)\) belongs to the rate-exponent region \(\mathcal{RE}^{CI}\).

Consider a sufficiently large block length \(n, \epsilon > 0, \mu > 0\). For each \(l\) in \(\mathcal{L}\), let \(\bar{R}_l \triangleq I(X_l; U_l) + \alpha\), where \(\alpha > 0\). To construct the codebook of encoder \(l\), we first generate \(2^{n\bar{R}_l}\) independent codewords \(U^n_l\), each according to \(\prod_{i=1}^n P_{U_i}(u_{ii})\), and then distribute them uniformly into \(2^{n(R_l+\epsilon)}\) bins. The codebooks and the bin assignments are revealed to the encoders and the detector. The encoding is done in two steps: quantization and binning. Encoder \(l\) first quantizes \(X^n_l\) by selecting a codeword \(U^n_l\) that is jointly \(\mu\)-typical with it. We adopt the typicality notion of Han [28]. If there is more than one such codeword, then encoder \(l\) selects one of them arbitrarily. If there is no such codeword, it selects an arbitrary codeword. The encoder then sends to the detector the index of the bin to which the codeword \(U^n_l\) belongs. In order to be consistent with our earlier notation, we denote this encoding function by \(f_l^{(n)}\). It is clear that the rate constraints are satisfied, i.e.,

\[ \frac{1}{n} \log \left| f_l^{(n)}(X^n_l) \right| = R_l + \epsilon \quad \text{for all } l \text{ in } \mathcal{L}. \quad \text{(C.3)} \]

The next lemma is a standard achievability result in distributed source coding.

**Lemma 20.** For any \(\delta > 0, \epsilon > 0, \mu > 0\), and all sufficiently large \(n\), there exists a function

\[ \varphi^{(n)} : \prod_{l=1}^L \left\{ 1, \ldots, 2^{n(R_l+\epsilon)} \right\} \times \mathcal{X}^n_{L+1} \times \mathcal{Z}^n \mapsto \prod_{l=1}^L U^n_l \]
such that (a) if

\[ V \triangleq \{ U^n, X^n_{L+1}, Y^n, Z^n \ \text{are jointly } \mu\text{-typical under } H_0 \}, \]

then \( P(V) \geq 1 - \delta \); and (b)

\[ p_e \triangleq P \left( \varphi(n) \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n_{L+1}, Z^n \right) \neq U^n \right) \leq \delta. \]

One can prove this lemma using standard random coding arguments. See [8, 9, 54] for proofs of similar results. Applying this lemma to the hypothesis testing problem at hand, we have

\[
\frac{1}{n} I \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n_{L+1} \biggm| Y^n, Z^n \right) \\
= \frac{1}{n} H(Y^n|Z^n) - \frac{1}{n} H \left( Y^n \biggm| \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n_{L+1}, Z^n \right) \\
= H(Y|Z) + \frac{1}{n} H \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}} \biggm| X^n_{L+1}, Z^n \right) \\
- \frac{1}{n} H \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, Y^n \biggm| X^n_{L+1}, Z^n \right). \tag{C.4}
\]

We can lower bound the second term in (C.4) as

\[
\frac{1}{n} H \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}} \biggm| X^n_{L+1}, Z^n \right) \\
= \frac{1}{n} I \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, U^n \biggm| X^n_{L+1}, Z^n \right) \\
= \frac{1}{n} H \left( U^n \biggm| X^n_{L+1}, Z^n \right) - \frac{1}{n} H \left( U^n \biggm| \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n_{L+1}, Z^n \right) \\
\geq \frac{1}{n} H \left( U^n \biggm| X^n_{L+1}, Z^n \right) - \frac{1}{n} H \left( U^n \biggm| \varphi(n) \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n_{L+1}, Z^n \right) \right) \tag{C.5} \\
\geq \frac{1}{n} H \left( U^n \biggm| X^n_{L+1}, Z^n \right) - \frac{1}{n} H_b(p_e) - p_e \sum_{l=1}^{L} \log |U_l| \tag{C.6} \\
\geq \frac{1}{n} H \left( U^n \biggm| X^n_{L+1}, Z^n \right) - \frac{1}{n} - \delta \sum_{l=1}^{L} \log |U_l|, \tag{C.7}
\]

where

(C.5) follows from data processing inequality [53, Theorem 2.8.1],
(C.6) follows from Fano’s inequality [53, Theorem 2.10.1], and

(C.7) follows Lemma 20(b) and the fact that \( H_b(p_e) \leq 1 \).

The third term in (C.4) can be upper bounded as

\[
\frac{1}{n} H \left( \left( f_i^{(n)} (X_i^n) \right)_{i \in \mathcal{L}}, Y^n | X_{L+1}^n, Z^n \right) \leq \frac{1}{n} H \left( U^n, \left( f_i^{(n)} (X_i^n) \right)_{i \in \mathcal{L}}, Y^n | X_{L+1}^n, Z^n \right) \\
= \frac{1}{n} H \left( U^n, Y^n | X_{L+1}^n, Z^n \right). \tag{C.8}
\]

On applying bounds (C.7) and (C.8) into (C.4), we obtain

\[
\frac{1}{n} I \left( \left( f_i^{(n)} (X_i^n) \right)_{i \in \mathcal{L}}, X_{L+1}^n, Y^n | Z^n \right) \\
\geq H (Y | Z) + \frac{1}{n} H (Y^n | X_{L+1}^n, Z^n) - \frac{1}{n} H \left( U^n, Y^n | X_{L+1}^n, Z^n \right) - \frac{1}{n} - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \\
= H (Y | Z) - \frac{1}{n} H (Y^n | U^n, X_{L+1}^n, Z^n) - \frac{1}{n} - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \\
\geq H (Y | Z) - \frac{1}{n} H \left( Y^n, 1_V | U^n, X_{L+1}^n, Z^n \right) - \frac{1}{n} - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \\
= H (Y | Z) - \frac{1}{n} H \left( 1_V | U^n, X_{L+1}^n, Z^n \right) - \frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V \right) \\
- \frac{1}{n} - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \\
\geq H (Y | Z) - \frac{1}{n} - \frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 1 \right) P(V) \\
- \frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 0 \right) P(V^c) - \frac{1}{n} - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \tag{C.9}
\]

\[
\geq H (Y | Z) - \frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 1 \right) - \frac{2}{n} - \delta \log |\mathcal{Y}| - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l|, \tag{C.10}
\]

where

(C.9) follows from the fact that \( H \left( 1_V | U^n, X_{L+1}^n, Z^n \right) \leq 1 \), and
(C.10) follows from Lemma 20(a) and the facts that

\[
\frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 0 \right) \leq \log |Y|, \quad \text{and} \quad P(V) \leq 1.
\]

We now proceed to upper bound the second term in (C.10). Let \( T^n_{\mu}(UX_{L+1}YZ) \) be the set of all jointly \( \mu \)-typical \((u^n, x^n_{L+1}, y^n, z^n)\) sequences. We need the following lemma.

**Lemma 21.** [28, Lemma 1(d)] If \( n \) is sufficiently large, then for any \((u^n, x^n_{L+1}, y^n, z^n)\) in \( T^n_{\mu}(UX_{L+1}YZ) \), we have

\[
P_{Y^n | U^n, X_{L+1}^n, Z^n}(y^n | u^n, x^n_{L+1}, z^n) \geq \exp \left[ -n \left( H(Y | U, X_{L+1}, Z) + 2\mu \right) \right].
\]

Using this lemma, we obtain

\[
\frac{1}{n} H \left( Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 1 \right) \\
= -\frac{1}{n} \sum_{T^n_{\mu}(UX_{L+1}YZ)} P_{U^n, X_{L+1}^n, Y^n, Z^n | 1_V = 1} \log P_{Y^n | U^n, X_{L+1}^n, Z^n, 1_V = 1} \\
= -\frac{1}{n} \sum_{T^n_{\mu}(UX_{L+1}YZ)} P_{U^n, X_{L+1}^n, Y^n, Z^n | 1_V = 1} \log \frac{P_{Y^n | U^n, X_{L+1}^n, Z^n}}{P_{1_V = 1 | U^n, X_{L+1}^n, Z^n}} \\
\leq \sum_{T^n_{\mu}(UX_{L+1}YZ)} P_{U^n, X_{L+1}^n, Y^n, Z^n | 1_V = 1} \left( H(Y | U, X_{L+1}, Z) + 2\mu \right) \\
= H(Y | U, X_{L+1}, Z) + 2\mu. \tag{C.11}
\]

Substituting (C.11) into (C.10) gives

\[
\frac{1}{n} I \left( \left( f_t^{(n)}(X_t^n) \right)_{t \in L} | X_{L+1}^n, Y^n | Z^n \right) \\
\geq I(Y; U, X_{L+1} | Z) - 2n - 2\mu - \delta \log |Y| - \delta \sum_{i=1}^L \log |\mathcal{U}_i| \\
\geq E - 3\mu - \delta \log |Y| - \delta \sum_{i=1}^L \log |\mathcal{U}_i|, \tag{C.12}
\]

181
where (C.12) follows from (C.2) and the fact that $n$ can be made arbitrarily large.

We conclude from (C.3) and (C.12) that

$$
\left( R_1 + \epsilon, \ldots, R_L + \epsilon, E - 3\mu - \delta \log |\mathcal{Y}| - \delta \sum_{l=1}^{L} \log |\mathcal{U}_l| \right)
$$

is in $\mathcal{R} \mathcal{E}^{CI}_*$. Since this is true for any $\delta > 0$, $\epsilon > 0$, and $\mu > 0$, we have that $(R, E)$ is in $\overline{\mathcal{R} \mathcal{E}^{CI}_*}$. This together with Corollary 2 implies that $(R, E)$ is in $\mathcal{R} \mathcal{E}^{CI}_*$.

### C.3 Proof of Theorem 12

Suppose $(R, E)$ is in $\mathcal{R} \mathcal{E}^{CI}_*$. Then there exists a block length $n$ and encoders $f_l^{(n)}$ such that

\begin{align*}
R_l &\geq \frac{1}{n} \log \left| f_l^{(n)}(X_l^n) \right| \quad \text{for all } l \in \mathcal{L}, \text{ and } \quad (C.13) \\
E &\leq \frac{1}{n} I\left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X_{L+1}^n; Y^n \right| Z^n \right). \quad (C.14)
\end{align*}

Consider any $X$ in $\chi$. Let $T$ be a time sharing random variable uniformly distributed over $\{1, \ldots, n\}$ and independent of $(X^n, X_{L+1}^n, X^n, Y^n, Z^n)$. Define

$$
X_l = X_l^n(T) \text{ for each } l \in \mathcal{L} \cup \{L + 1\},
$$

$$
X = X^n(T),
$$

$$
Y = Y^n(T),
$$

$$
Z = Z^n(T),
$$

$$
U_l = \left( f_l^{(n)}(X_l^n), X_{L+1}^n(1 : T - 1), X_L^n(T^c), Z^n(T^c) \right) \text{ for each } l \in \mathcal{L}, \text{ and }
$$

$$
W = \left( X^n(T^c), X_{L+1}^n(T^c), Z^n(T^c) \right).
$$

It is easy to verify that $\lambda_o = (U, W, T)$ is in $\Lambda_o$ and

$$
X \leftrightarrow (X, X_{L+1}, Y, Z) \leftrightarrow \lambda_o.
$$
It suffices to show that \((R, E)\) is in \(\mathcal{RE}_o^{CL}(X, \lambda_o)\). We obtain the following from (C.14)

\[
E \leq \frac{1}{n} I \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X_{L+1}^n; Y^n | Z^n \right) \\
= \frac{1}{n} \left[ H(Y^n | Z^n) - H \left( Y^n \left| \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X_{L+1}^n, Z^n \right) \right) \right] \\
= \frac{1}{n} \sum_{i=1}^{n} \left[ H(Y^n(i) | Z^n(i)) - H \left( Y^n(i) \left| \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, Y^n(1:i-1), X_{L+1}^n, Z^n \right) \right) \right] \tag{C.15} \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left[ H(Y^n(i) | Z^n(i)) - H \left( Y^n(i) \left| \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, Y^n(1:i-1), X_{L+1}^n, Z^n \right) \right) \right] \tag{C.16} \\
= \frac{1}{n} \sum_{i=1}^{n} I \left( Y^n(i); \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X^n(1:i-1), X_{L+1}^n(i^c), Z^n(i^c), X_{L+1}^n(i) \right) | Z^n(i) \right) \\
= I \left( Y^n(T); U, X_{L+1}^n(T) | Z^n(T), T \right) \\
= I \left( Y; U, X_{L+1}| Z, T \right),
\]

where

(C.15) follows from conditioning reduces entropy, and

(C.16) follows because of the Markov chain

\[
Y^n(1:i-1) \leftrightarrow (X^n(1:i-1), Z^n(1:i-1)) \\
\leftrightarrow \left( \left( f_l^{(n)}(X_l^n) \right)_{l \in \mathcal{L}}, X_{L+1}^n, Y^n(i), Z^n(i:n) \right).
\]
Now let $S \subseteq \mathcal{L}$. Then (C.13) implies

$$\sum_{l \in S} n R_l \geq \sum_{l \in S} \log |f_l^{(n)}(X^n_l)|$$

$$\geq \sum_{l \in S} H(f_l^{(n)}(X^n_l))$$

$$\geq H\left(\left(\left|f_l^{(n)}(X^n_l)\right|_{l \in S}\right)_{l \in S}\right)$$

$$\geq H\left(\left(f_l^{(n)}(X^n_l)_{l \in S}, X^n_{L+1}, Z^n\right)\right)$$

(C.17)

$$= I\left(X^n, X^n_S, \left(f_l^{(n)}(X^n_l)\right)_{l \in S}, \left(f_l^{(n)}(X^n_l)\right)_{l \in S^c}, X^n_{L+1}, Z^n\right)$$

$$= I\left(X^n, \left(f_l^{(n)}(X^n_l)\right)_{l \in S}, \left(f_l^{(n)}(X^n_l)\right)_{l \in S^c}, X^n_{L+1}, Z^n\right)$$

$$+ I\left(X^n_S, \left(f_l^{(n)}(X^n_l)\right)_{l \in S}, \left(f_l^{(n)}(X^n_l)\right)_{l \in S^c}, X^n, X^n_{L+1}, Z^n\right)$$

$$= \sum_{i=1}^n I\left(X^n(i), \left(f_l^{(n)}(X^n_l)\right)_{l \in S}, \left(f_l^{(n)}(X^n_l)\right)_{l \in S^c}, X^n(1 : i - 1), X^n_{L+1}, Z^n\right)$$

$$+ \sum_{l \in S} I\left(X^n_l, \left(f_l^{(n)}(X^n_l)\right)_{l \in S}, X^n, X^n_{L+1}, Z^n\right),$$

(C.18)

where

(C.17) follows from conditioning reduces entropy, and

(C.18) follows because $X$ is in $\chi$. 

184
We next lower bound the second sum in (C.18).

\[ I \left( X^n_l; f_l^{(n)} (X^n_l) \mid X^n, X^n_{L+1}, Z^n \right) \]
\[ = \sum_{i=1}^{n} I \left( X^n_l(i); f_l^{(n)} (X^n_l) \mid X^n, X^n_{L+1}^{(1 : i - 1)}, Z^n \right) \]
\[ = \sum_{i=1}^{n} \left[ H \left( X^n_l(i) \mid X^n, X^n_{L+1}^{(1 : i - 1)}, Z^n \right) - H \left( X^n_l(i) \mid f_l^{(n)} (X^n_l), X^n, X^n_{L+1}^{(1 : i - 1)}, Z^n \right) \right] \]
\[ \geq \sum_{i=1}^{n} \left[ H \left( X^n_l(i) \mid X^n, X^n_{L+1}, Z^n \right) - H \left( X^n_l(i) \mid f_l^{(n)} (X^n_l), X^n, X^n_{L+1}, Z^n \right) \right] \]
\[
\geq \sum_{i=1}^{n} I \left( X^n_l(i); f_l^{(n)} (X^n_l) \mid X^n, X^n_{L+1}, Z^n \right),
\]

(C.19)

where (C.19) again follows from conditioning reduces entropy. On applying (C.20) in (C.18), we obtain

\[
\sum_{l \in S} R_l \geq \frac{1}{n} \sum_{i=1}^{n} \left[ I \left( X^n_l(i); \left( f_l^{(n)} (X^n_l) \right) \right) \mid \left( f_l^{(n)} (X^n_l) \right) \right]_{l \in S^c}, X^n_{1 : i - 1}, X^n_{L+1}, Z^n \]
\[ + \sum_{l \in S} I \left( X^n_l(i); f_l^{(n)} (X^n_l) \mid X^n, X^n_{L+1}, Z^n \right). \]

(C.21)

If \( S^c \) is nonempty, then continuing from (C.21) gives

\[
\sum_{l \in S} R_l \geq I \left( X^n(T); U_S \mid U_{S^c}, X^n_{L+1}, Z^n(T), T \right)
\]
\[ + \sum_{l \in S} I \left( X^n(T); U_l \mid X^n(T), X^n_{L+1}(T), Z^n(T), X^n(T^c), X^n_{L+1}(T^c), Z^n(T^c), T \right) \]
\[ = I \left( X; U_S \mid U_{S^c}, X_{L+1}, Z, T \right) + \sum_{l \in S} I \left( X_l; U_l \mid X, W, X_{L+1}, Z, T \right). \]
Finally if $S = \mathcal{L}$, then

$$
I \left( X^n(i); \left( f^{(n)}_l (X^n_l) \right)_{l \in S} \bigg| \left( f^{(n)}_l (X^n_l) \right)_{l \in S^c}, X^n(1 : i - 1), X^n_{L+1}, Z^n \right)
= I \left( X^n(i); \left( f^{(n)}_l (X^n_l) \right)_{l \in S} \bigg| X^n(1 : i - 1), X^n_{L+1}, Z^n \right)
= I \left( X^n(i); \left( f^{(n)}_l (X^n_l) \right)_{l \in S}, X^n(1 : i - 1), X^n_{L+1}(i^c), Z^n(i^c) \bigg| X^n_{L+1}(i), Z^n(i) \right).
$$

(C.22)

Substituting (C.22) into (C.21) yields

$$
\sum_{l \in \mathcal{L}} R_l \geq I \left( X; U \big| X_{L+1}, Z, T \right) + \sum_{l \in \mathcal{L}} I \left( X_l; U_l \big| X, W, X_{L+1}, Z, T \right).
$$

This completes the proof of Theorem 12.

C.4 Proof of Lemma 15

It suffices to show that (C6) implies (C7). The other direction immediately follows by letting $\epsilon \to 0$. We can assume without loss of generality that $|\mathcal{X}| \geq 2$ because the lemma trivially holds otherwise. Let $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$ be the alphabet set of $X$. Let $P_i$ be the $i$th row of the stochastic matrix $P_{Y|X}$ corresponding to $X = i$. We need the following lemma.

**Lemma 22.** If (C6) holds, then rows $P_i$ corresponding to positive $P_X(i)$ are distinct.

**Proof.** The proof is by contradiction. Suppose that $P_X(1)$ and $P_X(2)$ are positive and $P_1 = P_2$. Let us define a random variable $U$ as

$$
U \triangleq \begin{cases} 
2 & \text{if } X = 1, 2 \\
X & \text{otherwise.}
\end{cases}
$$

186
The stochastic matrix $P_{X|U}$ has

$$P_{X|U}(1|2) = \frac{P_X(1)}{P_X(1) + P_X(2)},$$

$$P_{X|U}(2|2) = \frac{P_X(2)}{P_X(1) + P_X(2)},$$

and

$$P_{X|U}(i|i) = 1 \text{ for all } i \in \{3, 4, \ldots, |X|\}.$$ 

It is easy to see that $Y, X,$ and $U$ form a Markov chain

$$Y \leftrightarrow X \leftrightarrow U. \quad (C.23)$$

We now have

$$H(Y|U) = \sum_{i=2}^{|X|} H(Y|U = i)P_U(i)$$

$$= H(Y|U = 2)P_U(2) + \sum_{i=3}^{|X|} H(Y|U = i)P_U(i)$$

$$= H \left( \sum_{j=1}^{|X|} P_j P_{X|U}(j|2) \right) P_U(2) + \sum_{i=3}^{|X|} H \left( \sum_{j=1}^{|X|} P_j P_{X|U}(j|i) \right) P_U(i)$$

$$= H(P_2) P_U(2) + \sum_{i=3}^{|X|} H(P_i) P_U(i)$$

$$= \sum_{i=2}^{|X|} H(P_i) P_U(i), \quad (C.24)$$

187
and

\[
H(Y|X) = \sum_{j=1}^{[X]} H(P_j)P_X(j)
\]

\[
= \sum_{j=1}^{[X]} H(P_j) \left( \sum_{i=2}^{[X]} P_{X|U}(j|i)P_U(i) \right)
\]

\[
= \sum_{i=2}^{[X]} P_U(i) \sum_{j=1}^{[X]} P_{X|U}(j|i)H(P_j)
\]

\[
= P_U(2) \sum_{j=1}^{[X]} P_{X|U}(j|2)H(P_2) + \sum_{i=3}^{[X]} P_U(i) \sum_{j=1}^{[X]} P_{X|U}(j|i)H(P_j)
\]

\[
= P_U(2)H(P_2) + \sum_{i=3}^{[X]} P_U(i)H(P_i)
\]

\[
= \sum_{i=2}^{[X]} P_U(i)H(P_i).
\]

(C.25)

Now (C.23) through (C.25) together imply that \(I(X; Y|U) = 0\), and hence \(Y \leftrightarrow U \leftrightarrow X\). However,

\[
H(X|U) = \sum_{i=2}^{[X]} H(X|U = i)P_U(i)
\]

\[
= H(X|U = 2)P_U(2)
\]

\[
= H_b \left( \frac{P_X(1)}{P_X(1) + P_X(2)} \right) (P_X(1) + P_X(2))
\]

\[
> 0,
\]

which contradicts our assumption that (C6) holds. \(\Box\)

Consider any \(U\) that satisfies the Markov chain

\[U \leftrightarrow X \leftrightarrow Y.\]

We can assume without loss of generality that \(P_U(u)\) is positive for all \(u\) in \(\mathcal{U}\) because only positive \(P_U(u)\) contributes to \(H(X|U)\) and \(I(X; Y|U)\) in conditions
(C6) and (C7). Then

\[ I(X; Y|U) = H(Y|U) - H(Y|X) \]

\[ = \sum_{u \in \mathcal{U}} H(Y|U = u)P_U(u) - \sum_{i=1}^{|X|} P_X(i)H(P_i) \]

\[ = \sum_{u \in \mathcal{U}} H\left( \sum_{i=1}^{|X|} P_iP_{X|U}(i|u) \right)P_U(u) - \sum_{i=1}^{|X|} \left( \sum_{u \in \mathcal{U}} P_{X|U}(i|u)P_U(u) \right)H(P_i) \]

\[ = \sum_{u \in \mathcal{U}} P_U(u) \left[ H\left( \sum_{i=1}^{|X|} P_iP_{X|U}(i|u) \right) - \sum_{i=1}^{|X|} P_{X|U}(i|u)H(P_i) \right] \]

\[ = \sum_{u \in \mathcal{U}} P_U(u)T\left( P_{X|U}(\cdot|u) \right), \quad (C.26) \]

where (C.26) follows by setting

\[ T\left( P_{X|U}(\cdot|u) \right) \triangleq H\left( \sum_{i=1}^{|X|} P_iP_{X|U}(i|u) \right) - \sum_{i=1}^{|X|} P_{X|U}(i|u)H(P_i). \]

Since entropy is a strictly concave and continuous function, \( T \) is a nonnegative continuous function of \( P_{X|U}(\cdot|u) \). Moreover, for any \( u \) in \( \mathcal{U} \), \( P_{X|U}(i|u) = 0 \) for all \( i \) in \( X \) such that \( P_X(i) = 0 \). Let \( \mathcal{P} \) denote the set of all such \( P_{X|U}(\cdot|u) \). Define

\[ \gamma(\delta) \triangleq \sup_{P \in \mathcal{P}} \{ H(P) : T(P) \leq \delta \}. \]

It now follows from Lemma 22 that if \( T(P) = 0 \) for some \( P \) in \( \mathcal{P} \), then \( P \) must be a point mass and hence \( H(P) = 0 \). Therefore, \( \gamma(0) = 0 \). We next show that \( \gamma \) is continuous at 0. Consider a nonnegative sequence \( \delta_n \to 0 \). Then there exists a sequence of distributions \( P_n \) in \( \mathcal{P} \) such that

\[ T(P_n) \leq \delta_n \quad (C.27) \]

\[ H(P_n) \geq \frac{\gamma(\delta_n)}{2}. \quad (C.28) \]

Now, since the set of all distributions on \( X \) is a compact set, by considering a subsequence, we can assume without loss of generality that \( P_n \) converges to \( P \)
in $\mathcal{P}$. By letting $n \to \infty$ in (C.27), we obtain that $T(P) = 0$, i.e., $P$ is a point mass. Therefore, $H(P) = 0$. It now follows from (C.28) that $\gamma(\delta_n) \to 0 = \gamma(0)$ as $n \to \infty$. Hence, $\gamma$ is continuous at 0.

Fix $0 < \epsilon < \log |\mathcal{X}|$ (condition (C7) is always true for $\epsilon \geq \log |\mathcal{X}|$). Choose $\epsilon_1 > 0$ such that $\gamma(\epsilon_1/\log |\mathcal{X}|) + \epsilon_1 = \epsilon$. Set $\delta = (\epsilon_1/\log |\mathcal{X}|)^2$. Let $I(X; Y|U) \leq \delta$. Define the sets

$$U_1 \triangleq \{ u \in U : T(u) \leq \sqrt{\delta} \} \text{ and } U_2 \triangleq U \setminus U_1.$$  

Note that $U_1$ is nonempty because $\delta < 1$. We now have

$$\delta \geq I(X; Y|U) = \sum_{U} P_U(u) T(u) \geq \sum_{U_2} P_U(u) T(u) > \sqrt{\delta} \sum_{U_2} P_U(u),$$

which implies

$$\sum_{U_2} P_U(u) < \sqrt{\delta}.$$  

Hence,

$$H(X|U) = \sum_{U_1} H(X|U = u) P_U(u) + \sum_{U_2} H(X|U = u) P_U(u) < \gamma(\sqrt{\delta}) + \sqrt{\delta} \log |\mathcal{X}|$$

$$= \gamma(\epsilon_1/\log |\mathcal{X}|) + \epsilon_1$$

$$= \epsilon.$$


