The Collapse Times of Tall, Multi-story Buildings of Constant Cross-section

Bradford Howland[1], Frank M. Howland[2], Howard C. Howland[3]

[1] Madison Wisconsin
[2] Department of Economics, Wabash College, Crawfordsville, Indiana
[3] Department of Neurobiology and Behavior, Cornell University, Ithaca, N.Y.

Corresponding author: Howard C. Howland
Email: hch2@cornell.edu
Telephone: 607-255-4716
Fax: 607-254-1303

Keywords: World Trade Center Towers, Building demolition, Building collapse, collapse time. Computer model of collapse.

Version of 23X09

10/23/2009 3:52 PM

Abstract

We present a simple mathematical model of the collapse of tall multi-story buildings in general and of the World Trade Center (WTC) towers in particular with the object of predicting their collapse times. In constructing the model we first consider two modes of demolition, one in which the supports of the bottom floor are destroyed and a second where the supports of the topmost level are destroyed. In both modes it is assumed that the retardation of the brittle structure of the building is insignificant. In the first model the entire building collapses in free-fall, i.e. with one *g* acceleration. In the second mode of collapse we show that for very tall buildings the ratio of the time for collapse and the free fall times, as well as the reciprocal velocities of collapse, approach the square root of 3 as the number of floors is increased indefinitely. We then model the destruction of the WTC towers as a combination of these two modes of collapse. In this third mode of collapse, the destruction of the building results in an agglomeration of floors impacted from the top by free-falling floors and impacting the lower floors below it. It may be shown that the agglomeration has an acceleration of (3/5) *g*. A model constructed along these lines for the collapse of the WTC towers, which had fractures originating at different floors, results in collapse times that differ by 1.83 seconds. This difference accords well with the measured 2 second difference in collapse times derived from video and seismic records.

The Collapse Times of Tall, Multi-story Buildings of Constant Cross-section

Bradford Howland, Frank M. Howland, Howard C. Howland

I Introduction and statement of the problem

An important parameter that was measured during the collapse of each of the World Trade Center buildings was the duration of the collapse. These measurements were made by correlation of the video data, which indicated times of initiation of collapse, and seismic data from an observatory located 34 kilometers north of the event. These times, so determined, were nine and 11 seconds, for the buildings impacted at the 82nd and 98th floors (Anonymous 2008), respectively. Due to uncertainty about the delay in receiving the seismic signal of collapse, the two second difference between the collapse times is of greater accuracy than the measured absolute times of collapse. This paper models the collapse times of the buildings and thereby explains the difference in collapse times.

The mechanics of the collapse were previously considered in detail by Bazant and Verdure (2007). However, these authors assumed that the destruction occurs in two sequential phases, first a “crush down” phase and then, in their words, “After the lower crushing front hits the ground, the upper crushing front of the compacted zone can begin propagating into the falling upper part of the tower…”. In contrast, as will be seen below, we believe that both types of crushing occurred simultaneously in the fall of the twin towers. This is because the stories above the fracture are in free fall until they strike the compacted zone (which we term the “agglomeration”) and the agglomeration is falling with an acceleration less than that of gravity due to the reduction of velocity each time the agglomeration strikes an underlying stationary story.

It was pointed out to us as we prepared the final draft of this paper that a model formulated along the same lines as ours, but less complete, could be found on a website (Kuhn,2008). We note the differences between that model and ours at the appropriate point below.

II Two Contrasting demolition techniques

We consider first the dynamics of a commonly used demolition technique used for tall, reinforced concrete buildings. The support posts at the bottom floor are wired for explosives which are simultaneously detonated. The building accordingly collapses as a free-falling structure with the lowest floor impacting the ground first. The retardation caused by energy absorption of the brittle structure being negligible compared to the gravitational energy liberated, the time of collapse, T, is very nearly equal to the freefall time, T_{ff}, for any heavy object falling from the height, H, of the topmost floor. Thus:

$$T \approx T_{ff} = \sqrt{2H/g} \qquad (1)$$

where g is the gravitational constant of acceleration. We note that the seismic signature of such a collapse would be many small impacts of individual floors with the ground, unequally spaced in time.

We next consider a contrasting demolition method, which we have invented for this problem. Here, the fracture is initiated by destroying all the supports for the topmost surface, i.e the roof of the building, simultaneously. In this hypothetical case, the mode of collapse differs greatly from the previous one, and the collapse time will be considerably greater due to inertia effects. We assume that the collisions between the concrete floors are inelastic. At the first instant, the top surface (assumed to have the mass of a floor), now unsupported, accelerates downward at 1g until it impacts the floor below. The two merged floor masses, retaining half the velocity that the first attained by conservation of momentum, now accelerate at 1g until the next collision, where the growing stack of the floors loses one-third of its velocity, etc. The termination of the process occurs when the stack of all the floors impacts the ground with a well-defined single seismic signature.

A computer simulation for a 110 story building of 416 meters in height revealed that the collapse time for the “top-down" collapse mode, T_c, is 1.629 times the fee freefall time of

9.24 seconds. Furthermore the velocity at impact of this stack is 1/ 1.720 times the freefall velocity, v_{ff}, where

$$v_{ff} = \sqrt{2gH} \tag{2}$$

Neither of these ratios appears to be especially significant; however the situation becomes clearer when the number of floors in the computer simulation are increased from an initial value of 100 floors by successive orders of magnitude. New collapse times and velocity ratios are given in Table 1.

Table 1 Times and Terminal Velocities of Collapsing Buildings of Various Numbers of Stories where Collapse starts with topmost floor*

No. of Stories	**Time of Collapse in Seconds**	**Ratio of $T_c^{\#}/T_{ff}$**	**Ratio of v_c^{+}/ v_{ff}**
110	15.009	1.629	1.720
100	14.269	1.624	1.719
10,000	151.172	1.721	1.732
1,000,000	1520.547	1.731	1.732
100,000,000	15214.327	1.732	1.732

*Note that the square root of 3 is 1.73205. The value for g in New York City used in these calculations is 32.161 ft/sec^2 (Hodgman 1952). $^{\#}T_c$ is the time for collapse. T_{ff} is the time for free fall through the building's height, 416 meters. v_c is the terminal velocity of the collapse v_{ff} is the terminal velocity of free fall.

We note that, as the number of floors is successively increased, both ratios, one more gradually, approach the value 1.73205, or the square root of three! The mathematical explanation for this curious phenomenon is given in Appendix I where it is assumed that the number of floors is infinite -- matter being distributed evenly between the topmost floor and the ground. We note that a collapse time of the square root of three times the freefall time corresponds to a downward acceleration of the stack of one-third g.

The exact time of collapse for any number of floors in the “top-down" mode can thus be obtained by a simple computer simulation.

Kuhn (2008) investigates this second demolition method and arrives at the same model as ours above for the special case of the top-down collapse, i.e. one initiated at the very top floor. Kuhn (2008) attempts to generalize the problem, considering demolitions in which the fracture is below the top of building, e.g. the 96th floor. His solution is to treat the entire structure above the fracture as a single mass which then successively impacts the floors below. We believe that the floors above the point of destruction should instead be modeled as a collection of individual masses separated by very frail structures of negligible strength; this is Kuhn’s implicit assumption in modeling the impacts on floors below.

Qualitatively, the situation is as follows: the portion of the building above the level of the initial fracture falls freely as an intact structure with the acceleration of gravity, collapsing into the stationary intact section of the building below the fracture. As the collapse proceeds, there accumulates between these sections of the building a plurality of floors which we term the “agglomeration". We shall show that the downward acceleration of the agglomeration, for the case of infinitely many floors, is exactly 3/5 that of gravity. We note that this value is at least reasonable, since it must be more than that of the top-down collapse and less than that of the freefall value, the agglomeration being impacted by collisions with floors both above and below it. The calculation of the acceleration of the agglomeration for an infinite number of floors uses an extension of the method used to calculate the one third g acceleration of the stack in a top-down collapse; it is given in appendix II.

III Consequences of the (3/5) *g* acceleration of the agglomeration

Appendix II demonstrates that the downward acceleration of the agglomeration approaches 3/5 *g*, as the number of floors approaches infinity. The assumption that this value suffices for approximately solving the collapse of a building with a finite number of floors leads to several interesting results. Consider first the building of height *H*, with a fracture initiating a lesser height *h*. The agglomeration begins to form at height *h* and falls with the acceleration of 3/5 *g*. We now ask: at what height h* will the agglomeration hit the ground at the same time as the top floor in freefall mode? We set these times to be equal as follows:

$$\sqrt{2H/g} \quad = \quad \sqrt{2h^*/(3/5)g} \qquad (3)$$

thus $$h^* = \quad (3/5)\,H. \qquad (4)$$

Let us now assume that we have a 110 story building with the fracture initiating at 3/5 of the height, i.e. the 66th floor level. The foregoing result implies that the top 44 floors, falling with the help of gravity, will succeed in demolishing the lower 66 floors, all in freefall time.

More generally, if the fracture begins *K* floors below the top, the agglomeration will cease to fall with acceleration of (3/5) *g* when it has fallen (3/2) *K* floors, since it will then have used up all the floors above the agglomeration. From this point on, the pile of floors already accumulated will fall as a stack with reduced acceleration of approximately (1/3) *g*, in the “top down" model of collapse described before. Note that the above is true only for fractures initiated above the 66th floor, as was the case with both of the World Trade Center buildings.

The general computational solution to the problem of determining the collapse times is now at hand: a) for fractures originating above the (3/5) *H* level, the time will be found by adding the times for two successive modes of collapse, first the agglomeration with downward acceleration of (3/5) *g*, then the top-down mode of collapse with the stack accelerating at roughly (1/3) *g*. For this case the joining of the two solutions ignores one subtle error in the calculation of the total collapse time. We show in Appendix III that the velocity of the agglomeration exceeds that of a stack for an equivalent height building, by the factor $\sqrt{27/25}$ or approximately 1.04. Thus, at the transition point between the two phases of the collapse, the computer simulation must increase the velocity of the stack by 4% before proceeding with the remainder of the calculation.

For fractures originating below (3/5) of the height of the building, the agglomeration will hit the ground before the freefall time for the top of the building with reduced force of impact. This will occur at the time $\sqrt{(2h^*/g)(F/66)}$ where F is the number of the floor where the fracture initiates. This, the major seismic impact, will be followed by smaller irregularly spaced impacts of individual floors, the collapse sequence terminating at the freefall time $\sqrt{2H/g}$.

For a specific example of how the solutions for the agglomeration and the stack are joined together, consider a 110 story building with a fracture initiated at the 98th floor, 12 floors below the top, the case of the World Trade Tower. In the agglomeration mode of collapse, the top 12 floors, falling with an acceleration of (3/5) g into the lower structure will demolish 1.5 times as many floors, i.e. 18, or all above the 80th floor, 30 floors below the top. At this time there remain no more floors above the agglomeration, therefore the collapse proceeds as a “top-down" collapse with the lower value of acceleration, g/3. The corresponding stack velocity must then be increased by 1.04, or $\sqrt{27/25}$, and then the remaining time of the collapse will be computed. The times of the two modes are then added. The building collapse times are shown, as a function of the floor number of initial fracture, in fig. 1.

We note from this curve that the collapse times of the two World Trade Center buildings with fractures originating at the 82nd and 98th floors are, respectively 10.49 seconds and 12.32 seconds. It is the difference between these two times which should be compared to the difference of the measured seismic and video derived times of nine and 11 seconds. The agreement is therefore between 1.83 and 2 seconds. This near equality is the most important substantiation of the calculation presented here. It should be noted that the difference between the measured elapsed times is independent of any assumption as to the exact delay time in the seismic wave, or, more unlikely, to a timing discrepancy with the video records. The results also imply that the seismic delay times are in error by 1.49 seconds, on average.

IV Summary

We have attempted to calculate the collapse times of the buildings of the World Trade Center as a function of the height of the floor at which fracture began. It is generally agreed that the collision between reinforced concrete floors will be inelastic, and further that the strength of the supporting structure, once the collapse is initiated, will have minimal effect on the rate of progression of the collapse. Instead, inertial factors, for example collision with stationary floors below, dominate. For the special case of "top – down" collapse, initiated at the 110^{th}, or topmost floor, a computer simulation indicates that the collapse will require 1.629 times the freefall time from the top, or 15.0 seconds.

Out of curiosity, we extended this computer calculation to buildings of several orders of magnitude more floors, i.e. 1000, 10,000 etc. and found this interesting result: Both the time for collapse divided by the freefall time and the reciprocal velocity as compared to the freefall velocity approached the square root of three, as the number of floors was increased indefinitely. The theoretical justification of this result is easily proven using algebra, under the assumption that matter is evenly distributed between the top and bottom floors. The proof is given in appendix I.

The more complex case, wherein the fracture is initiated at some level below the top floor can also be treated by a simple extension of this algebraic argument. The resultant gathering “agglomeration" of floors, impacted from both the top free-falling structure, and the stationary intact section below is shown in Appendix II to have a downward acceleration of (3/5)g. Both modes of collapse played a part in the World Trade Center building collapses. The resulting collapse times are shown to be consistent with the data obtained from seismic and video observations.

Appendix I

The (1/3) g Acceleration of the Stack

Let the vertical distances be represented as increasing downward, so that velocity and acceleration are positive values. A building is assumed to have height, H, and a total mass M, with uniform cross-section. The quotient M/H. equals m_0, the mass per unit height. We assume here that the fracture is initiated at the top floor, and that time, t, is measured there from. We shall use Newton's second law of motion, namely that force is equal to the time derivative of momentum. We further assume that the stack falls with a constant acceleration α; therefore the velocity of the stack at time t equals $\alpha\, t$.

The mass of the stack is equal to the height of fall, $\alpha\, t^2/2$, times its mass per unit height and therefore

$$M_{\mathrm{stack}} = m_0\, \alpha\, t^2/2 \qquad \text{(AI 1)}$$

The momentum of the stack is the velocity times its mass or $m_o\, \alpha^2\, t^3/2$. By Newton’s law the time derivative of this momentum, $(3/2)\, m_o\, \alpha^2\, t^2$, must equal the force of gravity acting on the stack or $g\, M_{stack}$. Thus:

$$(3/2)\, m_o\, \alpha^2\, t^2 = g\, m_o\, \alpha\, t^2/2 \qquad \text{(AI 2)}$$

This reduces to :

$$\alpha = g/3 \qquad \text{(AI 3)}$$

This is the acceleration of the stack in the “top-down” collapse. We note the curious fact that the collision of lower floors with the stack does not enter into the momentum calculation, since, being stationary, they carry no momentum.

Appendix II

The (3/5) *g* Acceleration of the Agglomeration

Assume, as in Appendix I, that time, *t*, is measured from the onset of the fracture which occurs a significant distance below the top of the building. The agglomeration is that of the floors which accumulate between the freefalling top section of the structure and the collapsing, stationary portion below. The downward acceleration of the agglomeration will be termed β and assumed to be constant. As before the mass per unit height of the building is m_0. The time, *t*, is measured from the onset of fracture. The mass of the agglomeration increases at the same rate as the height of the building, with the top section and freefall decreases. Thus:

$$M_{\mathrm{agg}} = m_o\,(1/2)\,g\,t^2 \qquad \text{(AII 1)}$$

The velocity of the agglomeration is equal to β t. The momentum of the agglomeration is accordingly equal to its mass times its velocity or:

$$MOM_{agg} = (1/2)\,m_0\,\beta\,g\,t^3 \qquad \text{(AII 2)}$$

Computing the time derivative of the momentum which has the dimension of force according to Newton's laws we obtain:

$$(d/dt)[(1/2)\,m_0\,\beta\,g\,t^3] = (3/2)\,m_0\,\beta\,g\,t^2 \qquad \text{(AII 3)}$$

Balanced against this rate of change of momentum are two terms: a) the force of gravity acting on the agglomeration and b) the rate of increase of momentum caused by the impact of the structure falling faster from above. We note that the impacts of the agglomeration with stationary floors below add no momentum to the agglomeration after each inelastic collision. (It is assumed here that the all inter-floor collisions are inelastic.)

We compute terms a) and b) as follows

a) The force of gravity, F, is equal to g times the mass of the agglomeration or:

$$F = m_o (1/2) g^2 t^2 \qquad \text{(AII 4)}$$

b) The rate of addition of momentum from floors impacted from above is equal to the product of the momentum per unit mass, which is its downward velocity, $g\,t$, times the rate of collision of mass above the agglomeration with the structure. This rate equals ($g – \beta$) t, their velocity difference. Thus term b) is given by:

$$d/dt\,(MOM_b) = m_o (g - \beta)\, t\, (g\, t) = m_o (g - \beta)\, g\, t^2 \qquad \text{(AII 5)}$$

Next, using Newton's second law, by setting the previously computed rate of change of momentum of the agglomeration equal to the gravitational force plus the rate of momentum transfer from the floors impacting from above, we have:

$$(3/2)\, m_0\, \beta\, g\, t^2 = m_o (1/2) g^2 t^2 + m_o(g - \beta)\, g\, t^2 \qquad \text{(AII 6)}$$

Dividing each of these terms by $m_o\,(1/2)\, g\, t^2$, we have:

$$3\beta = g + 2g - 2\beta \quad or \quad 5\beta = 3g \quad or \quad \beta = (3/5)g \qquad \text{(AII 7)}$$

Thus the acceleration of the agglomeration is three-fifths that of gravity.

Appendix III

Derivation of the $\sqrt{27/25}$ Velocity Correction

It is instructive to compare the impact velocity of two modes of building collapse: that of the agglomeration, where the fracture is initiated at the 3/5 height level, e.g., the 66th floor of a World Trade building, with the velocity of impact into top-down collapse mode for a building of equal height. In each case all the floors hit the ground together.

The velocity of fall of the agglomeration, v_{agglom}, falling from 3/5 H at an acceleration of 3/5 g will be given by:

$$v_{agglom} = \sqrt{2(3/5)g(3/5)H} = \sqrt{(18/25)gH} \qquad \text{(AIII 1)}$$

where H is the height of the building.

Furthermore the velocity of the fall in the “top-down” collapse mode is:

$$v_{top\text{-}down} = \sqrt{(2/3)gH} \qquad \text{(AIII 2)}$$

since the acceleration is $g/3$.

The ratio of these two impact velocities equals $\sqrt{27/25}$, with the agglomeration falling approximately 4% faster than the building which collapses from the top down. This correction is easily made, at the transition point between the end of the agglomeration phase, and the beginning of "top-down" collapse in the computational program. The resulting curve of collapse times, versus height of initial fracture, incorporating the correction is given in the curve of fig. 1.

Appendix IV

Energy considerations

We first investigated the “top-down" model of collapse , and showed that the velocity of the stack which comprises all the floors above ground level, impacts the ground with a velocity close to $1/\sqrt{3}$ times the freefall velocity from the height of the building. The total gravitational energy available is equal to the weight of all the floors of the building times the height of the center of gravity which is *H*/2. If all of this gravitational energy were converted to kinetic energy, the stack would fall at a velocity equal to the freefall velocity divided by the square root of two, since energy is proportional to velocity squared. The kinetic energy of the stack is thus equal to the gravitational energy times $\left(1/\sqrt{3}\right)^2/\left(1/\sqrt{2}\right)^2$, or 2/3. The missing one third of the total energy is evidently dissipated in the totality of inelastic inter-floor collisions during the collapse.

One third of the total gravitational energy is still an enormous amount of energy, and will account for a good part of the fragmentation of the reinforced concrete parts of the structure. The same calculation for the maximum energy agglomeration -- due to impact at (3/5)*H*, or the 66th floor indicates that 72% of the available energy is expended when the agglomeration hits the ground, leaving 28% to be dissipated by inter-floor collisions.

Knowing the total energy of the falling stack or agglomeration, as the case may be, we could theoretically compute the Richter number for the impact; however we have no present knowledge of the efficiency with which the energy of impact is coupled to the seismic waves which are so generated. One can show that, since the Richter number is a logarithmic measure, the seismic impact for fractures initiated between the top floor and the 66th floor would differ by less than 0.023 Richter units.

Acknowledgments We thank Prof. Terrence Millar of the University of Wisconsin, Madison, Mathematics Department, Prof. emeritus Alex Andrew of Reading University, U.K, and Professor Richard Rand of the Departments of Mechanical and Aerospace Engineering and Mathematics at Cornell University for helpful comments. We wish to dedicate this paper to the memory of our father, Warren E. Howland of Purdue University who was a life-long member of the American Society of Civil Engineers.

References

Anonymous. (2008). “National Institute of Standards and Technology (NIST) Federal Building and Fire Safety Investigation of the World Trade Center Disaster Answers to Frequently Asked Questions (August 30, 2006)." http://wtc.nist.gov/pubs/factsheets/faqs_8_2006.htm (July 16, 2009).

Bazant, Z. P., Verdure, M. . (2007). "Mechanics of progressive collapse: learning from World Trade Center and building demolitions." *Journal of Engineering Mechanics*, 133(3), 308-319.

Hodgman, C. D. (1952). Handbook of Chemistry and Physics, Chemical Rubber Publishing Co., Cleveland, OH. .

Kuhn, K. (2008) “A Mathematical Analysis of the World Trade Center” (Nov 8, 2008) http://www.kennethkuhn.com/writings/commentaries/wtc_collapse_mathematics.pdf (Oct 21, 2009)

Figure Caption.

Fig. 1. Collapse times for a building of 110 stories. Segment **a** gives the time for complete collapse when the destruction begins above the 66th floor. Segment **b** gives the total collapse time when the fracture is at the 66th floor or below. Segment **c** gives the time it takes for the agglomeration of collapsed floors to hit the ground. During the time interval between segments **b** and **c**, the upper stories in free fall are individually hitting the agglomeration on the ground.