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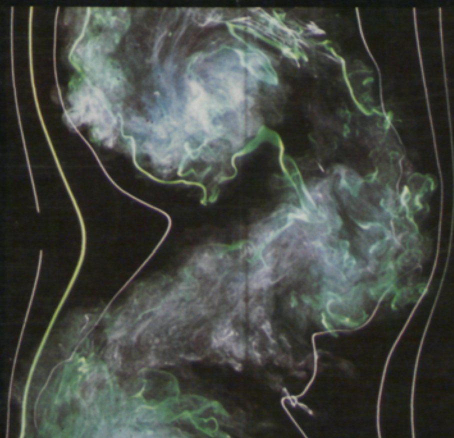

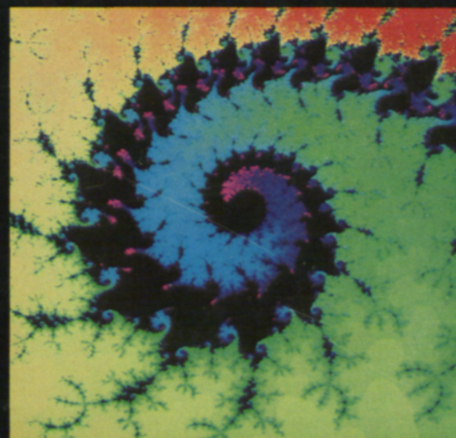

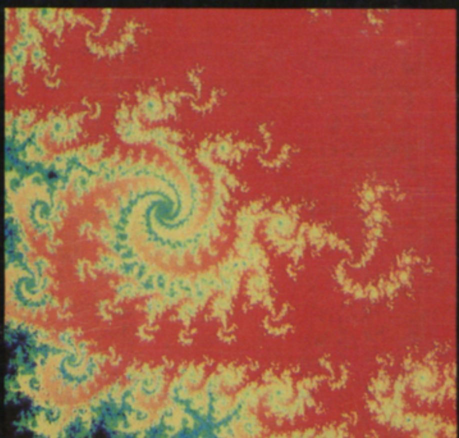

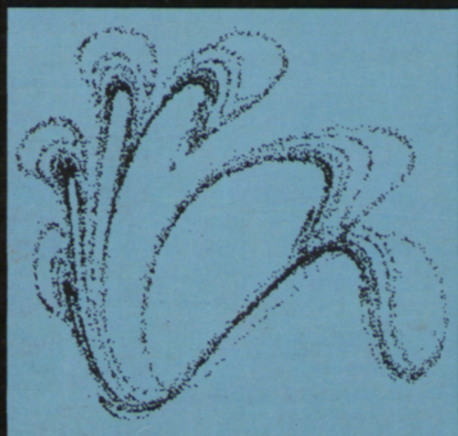

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CHAOS AND PHYSICAL SYSTEMS

IN THIS ISSUE

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The cover illustrations relate to the articles in this issue. Top center: a photograph of turbulent flow, discussed by Leibovich and Lumley. Top right and bottom left: Two pictures generated by a computer iterating complex polynomials; see the article by Hubbard. Bottom center: a Poincaré map, "fingerprint of chaos", provided by Moon.

The illustrations opposite are experimental "snapshots of chaos" taken at different places in a strange attractor. The system is the vibrating buckled beam discussed by both Moon and Holmes. Two basins of attraction are seen.

Moon

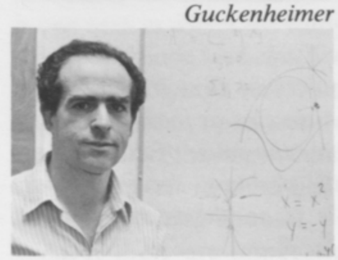
Holmes

Hubbard

Leibovich

Lumley

Guckenheimer



"But, you will ask, how could a uniform chaos coagulate at first irregularly in heterogeneous veins or masses to cause hills?... Tell me the cause of this, and the answer will perhaps serve for the chaos."—Sir Isaac Newton in a letter circa 1681

PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA.

Autore *J S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

This year is the three-hundredth anniversary of Newton's Principia. *The title page reproduced here is from the first edition.*

FRACTALS AND CHAOS: NEW CONCEPTS IN MECHANICS

by Francis C. Moon

Until very recently, a prediction that new discoveries in dynamics would be made three hundred years after publication of Newton's *Principia* would have been thought naive or foolish. Yet in the last decade new phenomena have been observed in all areas of nonlinear dynamics.

We were all taught in physics that Newtonian mechanics is deterministic—that given the initial position and velocity of each particle in a system, one could predict its history forever. But while this idea has worked well in planetary science, it does not apply to all problems in solid and fluid mechanics because there are many nonlinear phenomena in which the dynamic history is very sensitive to changes in the initial conditions. Foremost among these phenomena is chaotic oscillation, the emergence of random-like motions from completely deterministic systems.

Chaotic oscillations have long been recognized in fluid mechanics, but only recently have they been observed in low-order mechanical and electrical systems. Along with these discoveries has come the recognition that nonlinear difference and differential equations can admit nonperiodic solutions that appear to be random even though no random quantities appear in the equations. This has prompted the development of new mathematical ideas, new ways of looking at dynamical solutions, that are now making their way into the engineering laboratory.

SYSTEMS KNOWN TO EXHIBIT CHAOTIC VIBRATIONS

Buckled structures

Structural systems with play or backlash

Aeroelastic systems

Rail systems with wheel-rail dynamics

Magneto-mechanical actuators

Control systems with nonlinear elements

Rotating or gyroscopic systems

Robotic manipulators under periodic control

Nonlinear circuits

Lasers

WHY CHAOTIC VIBRATIONS ARE IMPORTANT TO STUDY

The importance of the study of chaotic vibrations lies not only in the observation of new phenomena in mechanics, but also because of the new mathematical ideas it introduces into the subject of vibrations. These new concepts include Cantor sets, Poincaré maps, fractal dimension, Lyapunov exponents, and strange attractors. Several decades ago mathematical tools such as the Laplace transform and the Fast Fourier transform made their way into the modern vibration-and-control laboratories. Today new concepts arising out of the study of chaos are changing the way engineers study and measure nonlinear vibrations.

Noise in mechanical systems—often attributed to "experimental demons"—is one area in which the study of chaotic vibrations may be helpful. The recognition that chaotic vibrations can arise from low-order, nonlinear deterministic systems offers the hope of understanding the source of random-like noise and doing something about it.

Also, there are implications for the

Figure 1a

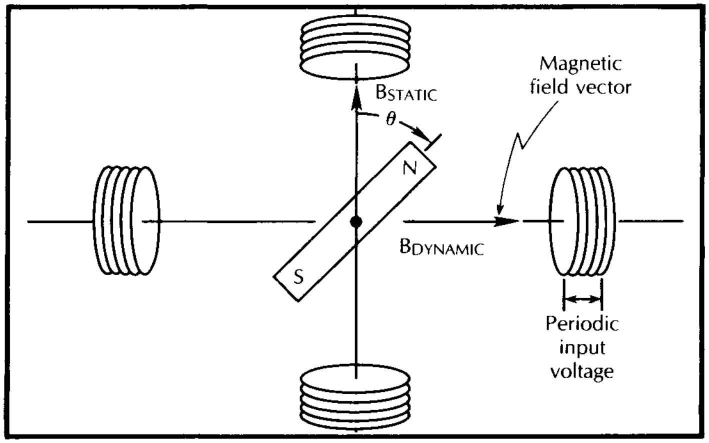
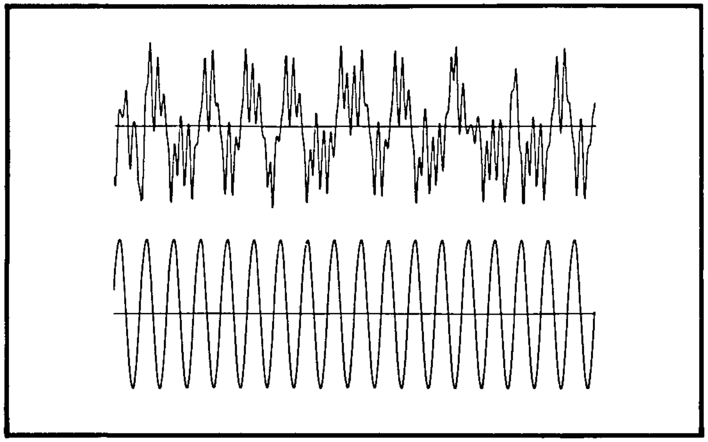


Figure 1b



general design procedure. Having recognized that chaos may exist in certain mechanical systems, engineers will want to know when they will encounter chaotic vibrations so as to design around them. Currently standard texts do not even mention such motions, let alone deal with criteria for their existence. Engineers would also like to characterize chaotic vibrations. The classical method of spectral analysis is one tool, but other indices—such as the fractal dimension and the Lyapunov exponent—that give some measure of how chaotic the motion really is have emerged.

Finally, the recognition that simple nonlinearities may lead to chaotic solutions brings into question the usefulness of numerical simulation of nonlinear systems. According to conventional wisdom, larger and faster supercomputers will allow engineers to make more precise predictions of system behavior. For nonlinear problems with chaotic dynamics, however, the time history is sensitive to initial conditions and therefore precise knowledge of the future is not possible.

Figure 1. Chaotic motion in a stepper motor.

The sketch in Figure 1a diagrams a motor with a rotating permanent magnet dipole and four magnetic field coils. When steady voltage is applied to the vertical coils and a sinusoidal voltage is applied to the horizontal pair, chaotic motion may occur instead of the regular motion expected according to classical mechanics.

The traces in Figure 1b show (at the top) a chaotic time history of rotor speed and (below) a time history of input voltage across horizontal field coils.

WHEN DO CHAOTIC VIBRATIONS OCCUR?

Chaotic vibrations occur in mechanical or electro-mechanical systems when there is some strong nonlinearity. For example, nonlinear elasticity or spring elements, nonlinear damping such as stick-slip friction, play or bilinear springs, backlash, limiters, or nonlinear electrical circuits and systems can be the cause. Fluid-related forces and nonlinear boundary conditions are known sources, and materials-based nonlinearities such as hysteretic magnetic or inelastic properties are suspected.

Dozens of physical systems that exhibit deterministic chaos are now known. The most familiar examples involve turbulence in fluids, but chaotic behavior has been observed also in structures, acoustic systems, magnetomechanical devices, and lasers.

EXPERIMENTS IN CHAOTIC VIBRATION

In our laboratory we have built and are experimenting with a number of simple devices that generate chaotic vibrations. Examples described here are a magnetic rotor, a vibrating elastic beam, an acoustically excited mechanical oscillator, and a mechanical positioning device with linear feedback control.

The magnetic rotor has only one degree of freedom. As shown in Figure 1*a*, four stationary magnetic poles excite rotary motion of a magnetic dipole (this configuration is used in stepper-motor technology). In our experiments we applied a steady voltage across the vertical pair of poles and a sinusoidally varying voltage across the horizontal

Figure 2

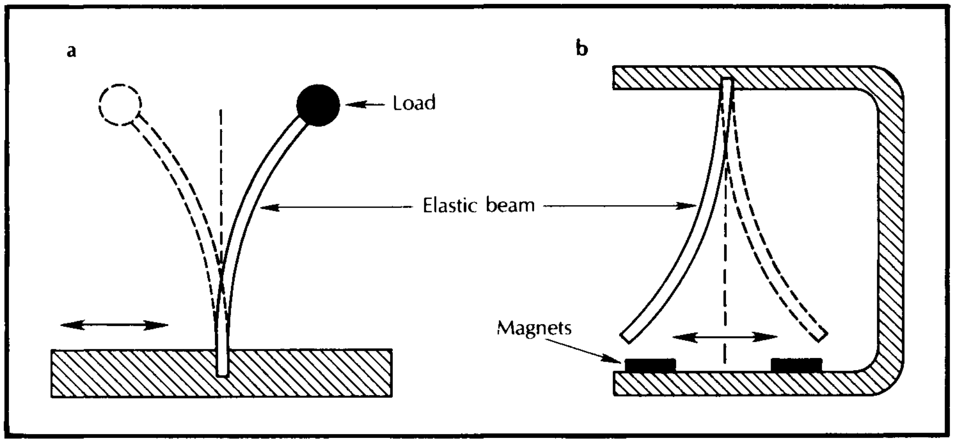


Figure 2. The vibrating buckled beam.

In Figure 2a a load is applied to a thin elastic steel beam, causing it to buckle. Under periodic excitation (applied as a sinusoidal force on the frame) the beam will oscillate, apparently at random, between two equilibrium positions.

In Figure 2b the beam is buckled by magnetic forces. (See also Figure 3, page 17, in the article by Philip Holmes.)

pair. According to classical mechanics, this combination of steady and periodic torques creates oscillatory or rotary motion, but at certain frequencies and forcing amplitudes we observed nonperiodic dynamics (Figure 1*b*).

This problem can be simulated on the computer using a differential equation similar to the one that describes the motion of a pendulum:

$$\frac{d^2\theta}{dt^2} + \delta\frac{d\theta}{dt} + \sin\theta = f\cos\theta\cos\omega t$$

where θ represents the rotation angle, δ is a damping coefficient, the third term on the left represents the steady magnetic torque, and the right-hand term represents the periodic magnetic torque. This is an ordinary differential equation for a nonlinear, periodically forced system. Although fairly simple, it has no known general solutions.

The vibrating buckled beam (Figure 2) is an example of a two-state system. Under compression loading, thin plates or long beams can suffer a buckling deformation, creating multiple equilibrium positions. Under periodic excitation, our buckled beam can jump between two equilibrium positions in an unpredictable manner. This example of chaotic vibration is described also in the article by Philip Holmes, whose collaboration provided the important theoretical framework for the experimental work.

The coupling between acoustic vibrations and mechanical oscillations has been studied with the experiment shown in Figure 3. Acoustic vibrations are set up in a long tube and a mechanical oscillator is placed near one end. Although both the acoustic tube and the mechanical oscillator are linear devices, the coupling between them is not, and the result is random-like vibration caused by nonlinear boundary

"Dozens of physical systems that exhibit deterministic chaos are now known."

Figure 3a

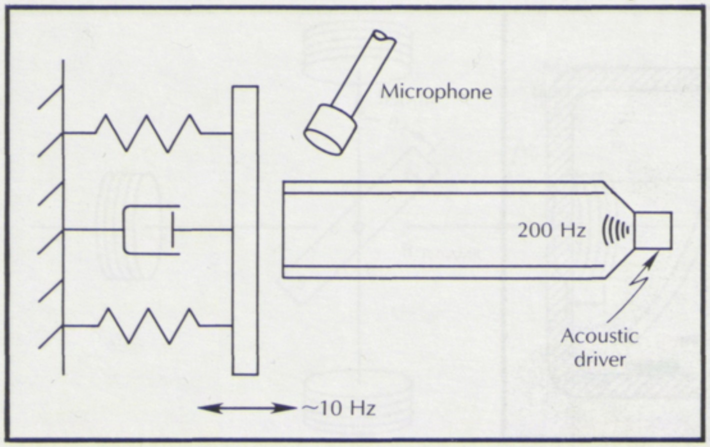


Figure 3b

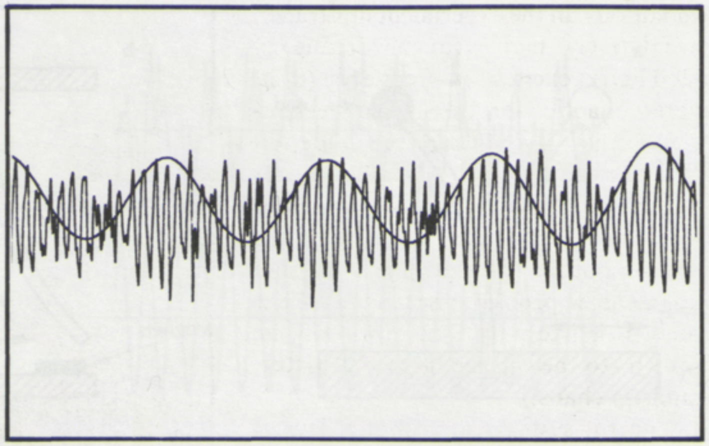


Figure 4

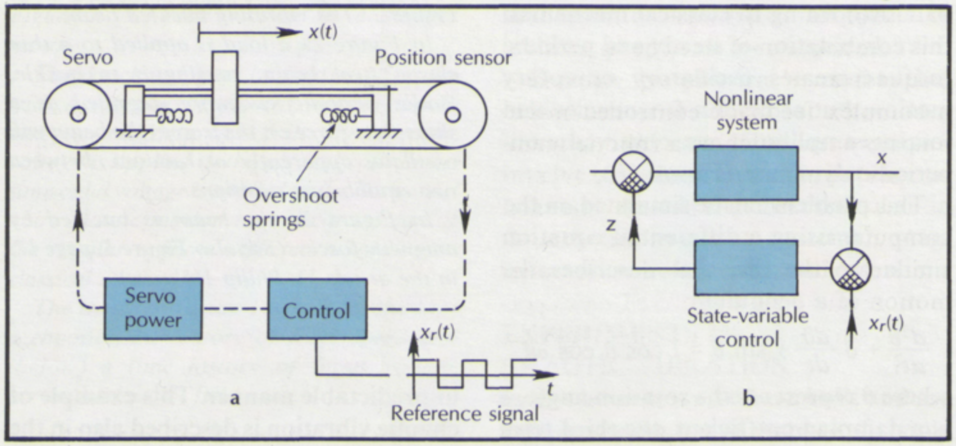


Figure 5

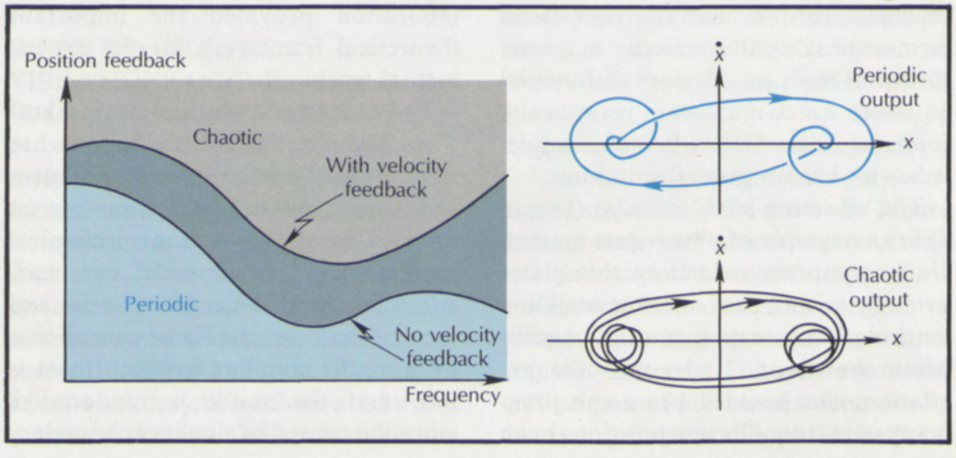


Figure 3. Organ-pipe chaos exhibited by an acoustically excited mechanical oscillator. The sketch in a *shows a resonant acoustic tube with one end partially blocked by a plate on elastic springs. A time history of acoustic ouput showing chaotic modulation of the carrier signal is reproduced in* b, *which also shows the induced low-frequency oscillation of the elastic end plug.*

Figure 4. A mechanical positioning device that can exhibit unwanted chaotic dynamics. The diagram in a *shows the linear servomotor feedback system. The overshoot springs introduce the possibility of nonlinearity, however. The block diagram in* b *shows the nonlinear physical plant and the linear control system.*

Figure 5. Avoiding chaos. Nonperiodic, chaotic dynamics exhibited by the Figure 4 device can be quenched by adding velocity feedback to the system. The two curves represent boundaries between periodic and chaotic behavior in the plane of position gain and frequency. Adding rate feedback has raised the boundary, giving the system a wider operating range in which the dynamics are regular.

conditions. In the experiment illustrated, a relatively high acoustic frequency (200 hertz) excites low-frequency (8–10 hertz) chaotic vibration in the mechanical oscillator, and chaotic modulation of the acoustic signal. (Wind-instrument musicians, of course, have known about mechanical chaos for centuries: if the lips are not pursed just right, a buzzing noise can be produced instead of a clear tone. It is likely that this noise will be shown to be an example of deterministic chaos.)

Control systems for robotic devices are studied to see whether they have the potential for chaotic response. Essentially we ask: Can a robot be "schizoid"? Stated in another way the question is: If a complex feedback controlled mechanical system (such as a robotic manipulator) is programmed to perform a sequence of tasks in a periodic manner, will the system always perform this task precisely the same way in each cycle? Our tentative answer, based on experiments with a simple mechanical device, is no.

The apparatus, illustrated in Figure 4, is a mechanical positioning device with a linear servomotor feedback system. The mass is programmed to shuttle back and forth between two positions in a periodic way, as might be required in some tool-positioning operation in a factory. We have observed, however, that as we increase the repetition frequency, the system can exhibit nonperiodic, chaotic dynamics (Figure 5). The nonlinearity in this system is a spring-like mechanical constraint at either end of the positioning track. We have found, though, that we can quench the chaos by adding rate feedback. The effect of adding feedback damping is to shift the chaos boundary, as shown in Figure 5; the system is given a wider operating range within which the dynamics are chaos-free.

"...it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible...."—Henri Poincaré in "Science and Method," circa 1900, an essay in *Mathematical Creation.*

Such problems are not all academic. Dr. Ferdinand Hendricks of IBM-Yorktown Heights has found that chaotic vibrations limit the upper frequency at which impact printers can operate in computer-output systems. IBM is experimenting with feedback control to quench such chaotic impacts and increase the speed of mechanical printers. This might become the first practical payoff of chaos research in engineering.

NEW EXPERIMENTAL METHODS IN VIBRATIONS

The experimental methods and diagnostic instruments in the field of engineering vibrations have always been closely coupled to the level of mathematical understanding. As new concepts have entered the theory of nonlinear vibration, new experimental methods based on mathematical ideas such as Poincaré mapping, fractal sets, and Lyapunov exponents have been developed in the research laboratory. It is likely that within the next decade these research tools will move from the laboratory into the engineering workplace in the form of new electronic vibration-diagnostic equipment.

One of the new tools is the Poincaré map, named for the great dynamicist and mathematician of the early twentieth century, Henri Poincaré. This provides a way of viewing the dynamics of a system at discrete times (represented by dots on an oscilloscope screen) rather than as a continuous time trace. The chaotic vibration of a buckled elastic beam, for example, looks random when it is viewed as a continous trace of tip displacement and tip velocity (Figure

Figure 6

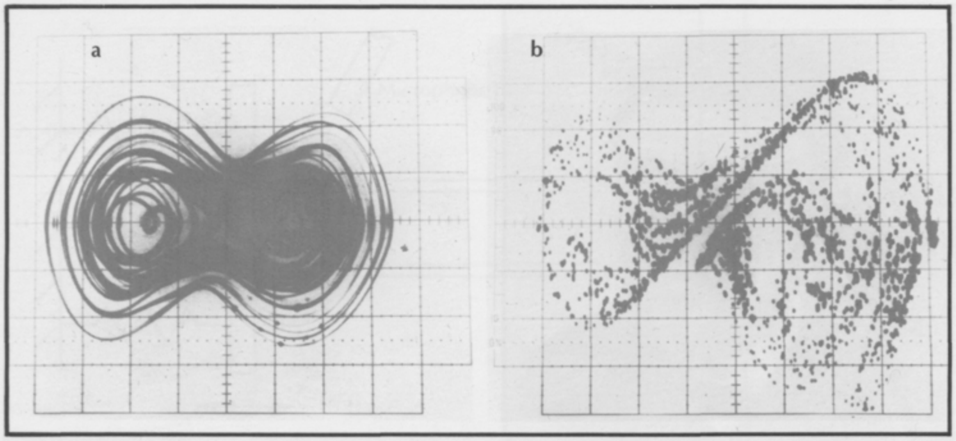


Figure 6. Chaotic vibration of the buckled beam under forced motion. A plot of position versus velocity (a) *shows apparently random oscillations. A pattern emerges, however, in a Poincaré map* (b).

6*a*). However, when it is viewed at certain discrete times that are synchronous with the driving-force frequency, the distribution of points or dots shows a stable (and sometimes beautiful) structure or pattern (Figure 6*b*). Instead of a uniform distribution of points, as one would get with complete randomness, the points tend toward a *strange attractor*. The pattern shows dense coverage in certain areas and open spaces in other areas. Mathematicians call such a distribution of points a *mapping* because if a point at time t is given, the physics of the problem tells one how to plot or map the next point, at time $t+\tau$.

Poincaré maps have become the "fingerprints of chaos." They give a qualitative picture of chaotic behavior, and are widely used to classify dynamical motion as periodic or chaotic. In a periodic motion, the number of points on the map is finite; for example, a subharmonic oscillation of period five has a Poincaré map of five points. A chaotic Poincaré map shows a distribution of many points.

FRACTAL DIMENSION IN STUDIES OF VIBRATION

Quantitative measures of chaos, including Lyapunov exponents and fractal dimension, have also emerged. We are interested in how these might be applied to problems of structural chaotic vibration, and have worked initially with fractal dimension.

The calculation of a fractal dimension is an attempt to measure the extent to which a strange attractor fills up the integer-dimension phase space. In the Poincaré maps of Figure 7, for instance, the strange attractor does not lie on a one-dimensional line, nor does it fill up a region of the plane. A property that is characterized mathematically as a *Cantor set* leaves holes in the spaces, and a mathematical measure of the extent to which the points fill in the space would ascribe a dimension between one and two to the attractor.

A geometric definition of dimension called the *capacity* is based on the number of cubes N of sides ϵ that are needed to cover the set of points. The fractal dimension is then defined as

$$d_c = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln (1/\epsilon)}.$$

Thus the set of points in the Poincaré map of Figure 7*a* is found to have a fractal dimension of $d_c = 1.5$. With higher damping (Figure 7*b*) the fractal dimension is $d_c = 1.1$, closer to the integer 1, and the map looks more like a linear geometric object. We have used this idea to calculate fractal dimensions for various experimental chaotic vibrations of our buckled-beam apparatus, and plan to perform similar calculations for other chaotic systems.

A direct optical method of making computations of this kind was developed recently by C.-K. Lee, a graduate student working in our laboratory. Essentially, the method incorporates the parallel-processing features of optical sampling. In future work we intend to use Cornell's new supercomputer, with its parallel-processing capability, to calculate fractal dimensions from experimental data on chaotic vibration.

PREDICTABILITY AND CHAOS IN MECHANICS

Another application of fractals in nonlinear dynamics has been in the study of basins of attraction and predictability. An example is illustrated in Figures 8 and 9, which pertain to the buckled-beam problem of Figure 2.

The problem can be modeled by two differential equations:

$$\frac{dx}{dt} = y; \; \frac{dy}{dt} = -\gamma y + x(1-x^2) + f \cos \omega t.$$

Figure 7

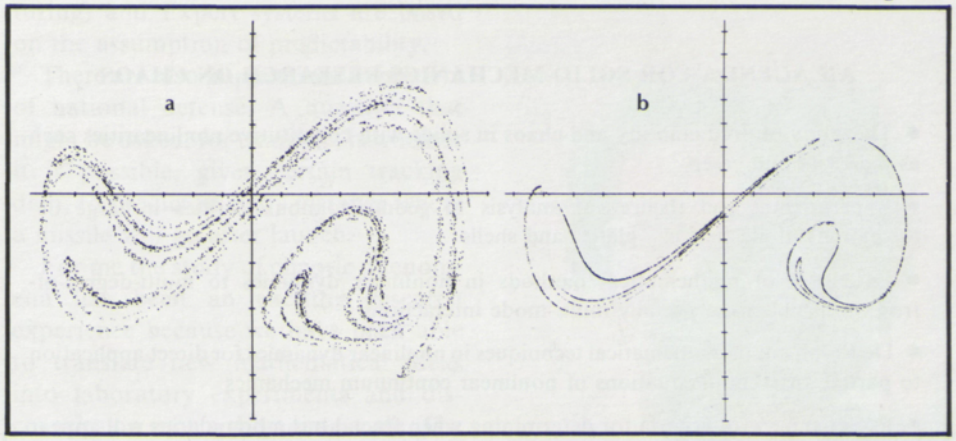


Figure 7. Poincaré maps of chaotic oscillations with different fractal dimensions. Lower damping results in a spreading of the dots, as in a*, which has a fractal dimension of 1.5. Higher damping, as in* b*, produces an almost linear array of dots and a fractal dimension close to unity.*

Figure 8

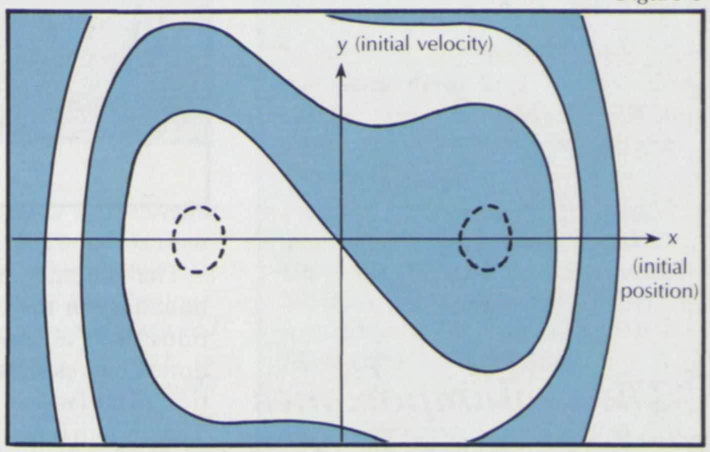


Figure 8. Basins of attraction. In the buckled-beam problem of Figure 2 there are two periodic orbits, shown here by the dotted lines. The smooth curves represent boundaries separating the basins of attraction: depending upon where they are located in the plane, the initial conditions will lead to periodic vibrations about either the left or the right equilibrium position of the buckled beam.

Figure 9. A fractal basin boundary. At a critical value of the forcing amplitude in the buckled-beam experiment, the boundary of the basins of attraction take on a fractal structure. The computer-generated pattern in a *shows points corresponding to 160,000 initial conditions in the position and velocity plane. The scale is the same as in Figure 8. The region in the tinted box is expanded in* b*, which is also generated from 160,000 initial conditions and also shows fractal structure.*

Figure 9a

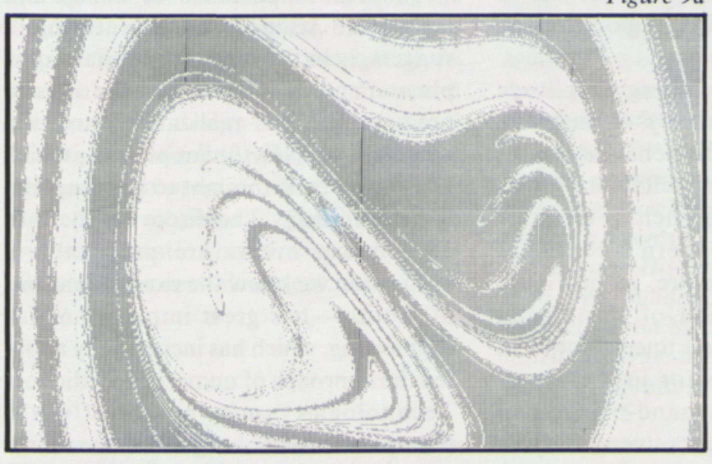
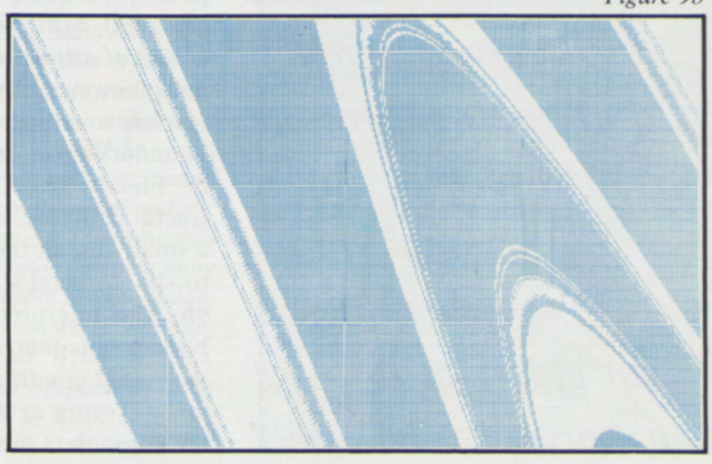


Figure 9b



AN AGENDA FOR SOLID-MECHANICS RESEARCH ON CHAOS

- The study of predictability and chaos in solids with constitutive nonlinearities such as plasticity and creep
- Experimental and theoretical analysis of geometric nonlinearities in large deformations of elastic rods, plates, and shells
- Extension of mathematical methods in nonlinear dynamics to multi-degree-of-freedom problems, especially multi-mode interactions
- Development of mathematical techniques in nonlinear dynamics for direct application to partial differential equations of nonlinear continuum mechanics
- Establishment of criteria for determining when fractal basin boundaries will appear in nonlinear systems with many degrees of freedom
- Determination of the statistical nature of chaotic vibrations in solid systems and the effects of outside noise
- Investigation of possible connections between chaotic dynamics and microphysical failure mechanisms such as fracture, fatigue, friction, and wear

"The real importance of chaos and fractals to science and engineering...is in the recognition that many phenomena in classical physics are not predictable."

The diagram in Figure 8 shows a boundary in the plane of initial conditions (x,y) at time equal to zero. The dotted curves show the long-time vibration orbit (which may not be chaotic). The smooth boundary separates starting points, which lead to either the left or the right orbit in the regions called *basins of attraction.*

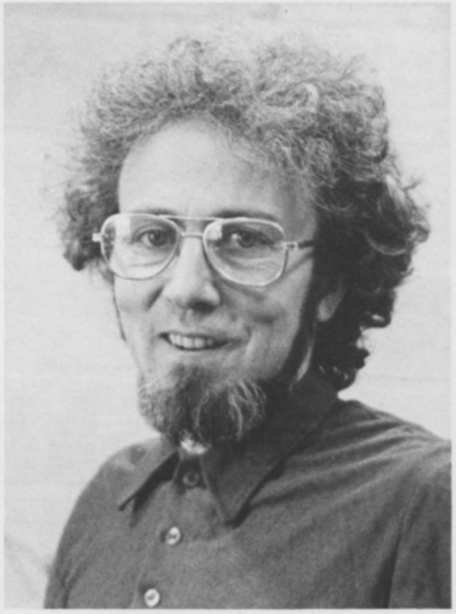
If, however, the forcing amplitude reaches a critical value $f=f_c$, then the boundary takes on the behavior shown in Figure 9*a*. This behavior shows fractal structure, and when we examine a small area in the pattern it continues to show fractal structure, as in Figure 9*b*. The interpretation of this fractal boundary is that small uncertainties in the initial conditions (or in the experimental setup or the round-off error in the computer) produces uncertainty as to whether the beam vibrates to the left or to the right. This kind of behavior is not confined to mechanical vibration; other examples are being discovered in systems that follow the laws of classical physics.

The real importance of chaos and fractals to science and engineering, I suggest, is in the recognition that many phenomena in classical physics are *not* predictable. This realization throws a new light on Newtonian physics, which has always been thought to be completely deterministic. The discovery that not all physical events are predictable—even when we know the exact equations of motion—has great implications for engineering, which has increasingly adopted the approach of numerical prediction. Technologies such as CAD/CAM (computer-aided design and manufacturing) and Expert systems are based on the assumption of predictability.

There are also implications in matters of national defense. A question that might be asked, for example, is whether it is possible, given certain tracking data, to predict in every instance where a missile will go after launch.

For me the study of chaotic phenomena has been an exciting research experience because we have been able to translate new mathematical ideas into laboratory experiments and discoveries on such a short time scale. The range of new discoveries has spread rapidly from mathematics to physics to many areas of applied science and engineering. I am not sure, though, what the residue of this period of research on chaos will be in ten or twenty years. Possibly the work on chaos will follow a progression like that of catastrophe theory, which had a mercurial rise and ebb and has not left as large an imprint as early enthusiasts predicted. We shall see.

Certainly there is much work to accomplish. In the box on the facing page is a list of topics that should be on the agenda for research on chaos in the field of solid mechanics. From an engineering perspective, the questions that should be asked are: (1) How can an engineer design so as to avoid chaotic motions? (2) How can chaotic motions be suppressed when they occur? And (3) If all else fails, how does one live with chaotic phenomena and unpredictability in technical devices?



Francis C. Moon, professor of theoretical and applied mechanics at Cornell, is chairman of his department. After receiving his undergraduate degree at the Pratt Institute, he came to Cornell for graduate study and earned the Ph.D. in 1966. He taught at Princeton University in the Department of Aerospace and Mechanical Sciences before returning to Cornell as a member of the faculty.

Chaotic dynamics, his subject in this article, is his major current research interest, but his work has covered a wide spectrum of problems in the dynamics and solids and structures. For example, in 1981 he was a visiting engineer with the magnetic fusion division of Lawrence Livermore Laboratory, and he is a consultant to Argonne National Laboratory. His publications include more than one hundred papers and a book, Magneto-Solid Mechanics, *which was published in 1984 by John Wiley & Sons.* Chaotic Vibrations *will be published this year by Wiley. Moon is an associate editor of the* Journal of Applied Mechanics.

SOME BOOKS ON CHAOS, FRACTALS, AND TURBULENCE

See also books listed by Philip Holmes at the end of his article.

Leslie, D. C. 1973. *Developments in the theory of turbulence.* Oxford: Clarendon Press. (A summary of the pre-chaos mathematical theory of fluid turbulence.)

Mandelbrot, B. B. 1982. *The fractal geometry of nature.* San Francisco: Freeman. (A discussion of the many applications of fractals; excellent pictures.)

Moon, F. C. 1986 (in press). *Chaotic vibrations.* New York: Wiley-Interscience. (An introductory book on chaos for engineers and applied scientists.)

Peitgen, H.-O., and P. H. Richter. *Frontiers of chaos.* Bremen: MAPART, Universtät Bremen. (A collection of beautiful photographs and essays on fractal sets related to the work of B. B. Mandelbrot and J. Hubbard.)

Schuster, H. G. 1984. *Deterministic chaos.* Weinheim: Physik-Verlag. (A readable introduction to ideas of chaos, starting with maps and ending with speculation on chaos in quantum systems.)

Sparrow, C. 1982. *The Lorenz equations: Bifurcations, chaos and strange attractors.* New York: Springer-Verlag. (A mathematical discussion of the famous convection turbulence equations.)

Tatsumi, T., ed. 1984. *Turbulence and chaotic phenomena in fluids.* 1984. Amsterdam: North-Holland. (A collection of papers on turbulence and chaos.)

CLEAR AIR TURBULENCE

The Dakotas and then Wyoming wrinkle
under us as the air wraps about us—
only the scale differs: those fine grains
and peaks are the land's flow, where years
extend to millennia. But cliffs bring up
the plateau's stretch with a leap

and up here seconds count as the wingtips dip
and bounce, breaking sight of the wrinkled face
below, the snow blown southward off ridges.
This air we're turned and bucked in sweeps
and fills huge cells over those ranges
which now shrug again and pull straight.

Unseen, the patterns stagger and break up;
what we would impose on them breaks up.
How can the air's heated, turning chaos be seen
as a fit end to its local order? And even granting this,
I still know that, in flight, volumes and pressures
far less properly described keep us alive.

Why wish to explain them if we can rely on
what's not understood? We can't. The plane drops
an instant. We're forced again to look
past the surface, the hills and knotted air,
to the blank place, always just ahead, where
if only for a moment, the heart stops.
—Philip Holmes

CHAOTIC DYNAMICS

by Philip Holmes

"If the doors of perception were cleansed every thing would appear to man as it is: Infinite."—William Blake, *The Marriage of Heaven and Hell*

Many physical systems exhibit complex dynamical behavior. The apparently random motion of a turbulent fluid is perhaps the best-known example, but vibrating structures, electrical circuits, and magnetic devices can also behave in complicated ways.

In many of these situations, we have what are generally felt to be good mathematical models of the basic physical phenomena. These models are in the form of deterministic nonlinear differential equations, and they, too, can behave in surprising ways. When the motion described by such an equation is effectively random, there is an apparent paradox: deterministic chaos.

In this article I will describe a deterministic mathematical process—the iteration of a function—that yields chaotic behavior, and then show how the same features occur among the solutions of a differential equation describing a physical situation—the vibrations of a buckled beam. While not as spectacular as hydrodynamic turbulence, this example is perhaps easier to understand. In the course of the article, I will also outline some of the newer analytical methods that are coming into use in the study of differential equations.

PHASE SPACES AND PARAMETER SPACES

For simplicity I will consider only those mathematical models that are in the form of *ordinary differential equations,* or ODEs. (I have come to feel that some ODEs are almost as poetic as their literary namesakes.) An ODE specifies a relationship between the rate of change of certain *dependent* variables and their instantaneous present values. The variables usually depend upon time or space, the *independent* variable. Newton's second law applied to the motion of a particle suspended by an elastic spring provides an example (see Figures 1 and 2 and the box on the following page).

The ODE provides a mathematical model of a physical system, but to determine the behavior of that system we must *solve* the ODE. Alas, we can only do this in general if the functional dependence on the state variables is *linear* ($\underset{\sim}{f}$ is a linear function of $\underset{\sim}{x}$ in Equation 6 in the box). Many models of important problems in engineering and the physical sciences are *nonlinear* ODEs and a complete solution—that is, a list of all possible functions $\underset{\sim}{x}(t)$ that satisfy the relationship of Equation 6 for all initial data $\underset{\sim}{x}(t_0)$ and parameter values $\underset{\sim}{\mu}$— cannot be found. One therefore often relies on numerical integration by computer, but this is expensive and time-consuming, and provides only piecemeal answers if the state or parameter spaces have more than two or three dimensions. To compound the problem, a description of the kinetics of a complicated system—a complex chemical reaction, say, or the dynamics of a particle system—may require many-dimensional space. For example, to fully describe the motion of two planets orbiting a star we must specify the positions and velocities of all these bodies in three-dimensional space: the vector $\underset{\sim}{x}$ in Equation 6 is

ORDINARY DIFFERENTIAL EQUATIONS
A Discussion in Terms of a Familiar Law

Newton's second law applied to the motion of a particle suspended by an elastic spring states that the force F on the particle is equal to its mass m multiplied by its acceleration a, or as an algebraic equation:

$$F = ma. \tag{1}$$

To turn this into a differential equation we do two things. First we realize that acceleration is the rate of change of velocity, and velocity is the rate of change of position. Thus, if we denote the particle's position at time t as $y(t)$, we have:

a = rate of change of (rate of change of $(y(t))$

or, using the notation of differential calculus (which Newton invented so that he could apply his law):

$$a = \frac{d^2y}{dt^2}. \tag{2}$$

Second, we must specify a function relating the force F felt by the particle to its position. A common assumption is that the spring is a linear elastic body and obeys Hooke's law (see Figure 1); thus F is proportional to the particle's displacement $y(t)-y_0$ from the position y_0 in which the spring is relaxed:

$$F = -k(y(t)-y_0). \tag{3}$$

Here k is the spring constant or stiffness and the minus sign reflects the fact that the force acts to oppose the displacement. Finally, substituting (2) and (3) into (1), we have our ordinary differential equation (ODE):

$$m\frac{d^2y}{dt^2} = -k(y-y_0), \quad \text{or} \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = \frac{k}{m}y_0. \tag{4}$$

If we denote the particle's velocity dy/dt by v, then (4) can be rewritten as a system of first-order ODEs:

$$\frac{dy}{dt} = v; \qquad \frac{dv}{dt} = -\frac{k}{m}y + \frac{k}{m}y_0. \tag{5}$$

Even a simple ODE like (4) or (5), with a linear force relationship, illustrates some important notions. The instantaneous *state* of the particle is specified by its position $y(t)$ and velocity $v(t)$; thus the *state space* is the set of all pairs (y,v); it is a Euclidean plane. In this context, *solutions* of the ODE are paths or curves in the state space. Solution of the equation involves the determination of the functions $y(t)$ and $v(t)$ given k,m,y_0 and the *initial conditions* $y(t_0)$, $v(t_0)$ at some starting time $t = t_0$ (see Figure 2). The system's behavior depends upon the parameters m and k—the mass of the particle and stiffness of the spring—and so the *parameter space* is the set of pairs (m,k), also a plane in this case.

In general, a system or set of ODEs can be written in the form

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}(\underset{\sim}{x},t,\underset{\sim}{\mu}), \tag{6}$$

where $\underset{\sim}{x} = (x_1,x_2,\ldots,x_n)$ is a vector of states and the vector-valued function $\underset{\sim}{f}$ describes the effects felt by the system due to its state $(\underset{\sim}{x})$ at the present time t with the particular parameters $(\underset{\sim}{\mu} = (\mu_1,\mu_2,\ldots,\mu_m))$ specified.

eighteen-dimensional (although this number can be reduced by using conservation laws). The large computer simulation programs that describe the dynamics of aircraft have state spaces with thousands of dimensions. Similarly, the dimensions of the parameter spaces (the set of parameters such as particle mass) can be very large in models of complicated systems.

The modern theory of *dynamical systems,* which stems from the work of the French mathematician Henri Poincaré (1854–1912) provides another approach to such problems. In this theory one asks not for detailed "exact" solutions, but for *qualitative* information on such aspects as stability, or whether a solution approaches a steady state or is periodic, almost periodic, or less regular. One tries, above all, to obtain a geometric picture of the set of all solutions in the n-dimensional phase space as the initial data vary, and to describe the qualitative changes or *bifurcations* the solutions undergo as the parameters are varied.

The methods used in dynamical systems theory are drawn from almost all the branches of pure mathematics: analysis and topology play major roles and even algebra and number theory make important appearances. However, this is not the appropriate place to introduce all these actors, and in the rest of this article I shall content myself with outlining one important technique and showing how it enables us to characterize "chaotic" solutions. This technique is *symbolic dynamics*, a way of relating solutions of ODEs to discrete sequences of symbols.

Figure 1

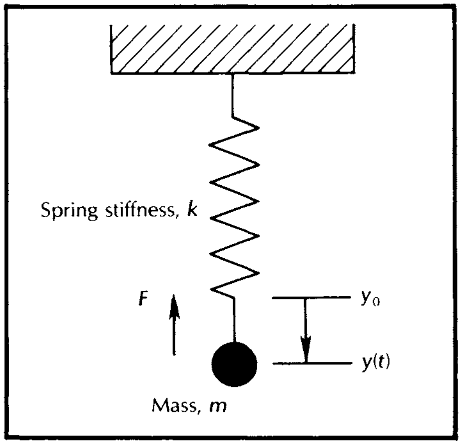


Figure 1. An oscillating particle, an example of Newton's second law. This law states that the force on a particle is equal to its mass multiplied by its acceleration: F = ma. This algebraic equation can be written as an ordinary differential equation (ODE) expressing the instantaneous state of the particle. (See the box on the opposite page).

Figure 2

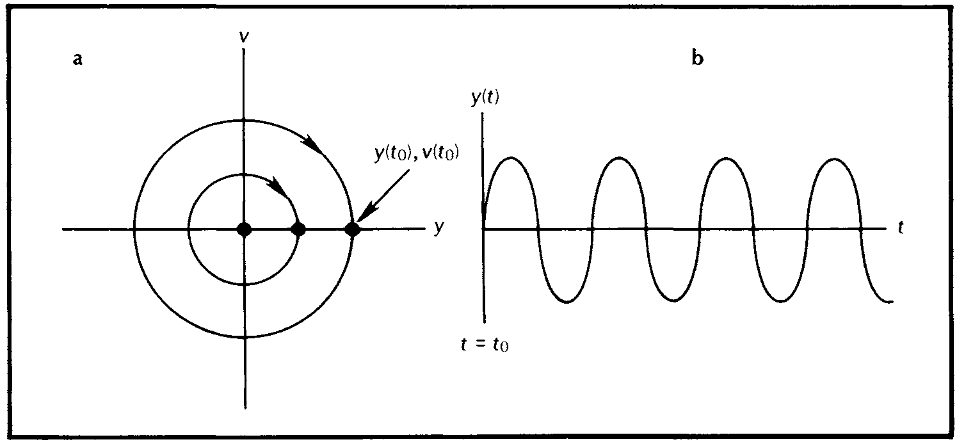


Figure 2. Phase space (a) and time history (b) for the simple linear ODE of Figure 1.

The instantaneous state of the particle is specified by its position y(t) and velocity v(t). In general, solutions of the ODE are paths or curves in the state space. Solving the ODE involves determining y(t) and v(t) given k, m, y_0, and the initial conditions.

For more complex systems, solutions are difficult or impossible. One alternative is computer integration. Another approach, the modern theory of dynamical systems, leads to a qualitative geometric picture showing such characteristics as stability, periodicity, or chaos.

But before introducing symbolic dynamics, I want to make an important observation. Systems of ODEs like Equation 6 are completely *deterministic*: if one knows the parameter values and the initial condition of the system at some starting time, then in principle one can "solve" the equation and so predict the state at *any* future time. A fundamental theorem guarantees that a unique solution to such a problem does exist. Unfortunately, as we shall see, existence does not preclude chaotic behavior. In this article we shall meet solutions that behave so badly they might as well be nondeterministic. Symbolic dynamics will capture these chaotic solutions and reveal their hidden order.

A PSEUDO RANDOM NUMBER GENERATOR

In this section I will describe a simple deterministic process that yields chaotic solutions. In the next section we will see how this rather abstract mathematical process is secretly present in an ODE that models the vibration of a beam.

We start with a little arithmetic involving base two numbers. Take any number between 0 and 1 (call it x), double it, subtract the integer part, double the result, subtract the integer part, and continue ad infinitum. Call the doubling-and-integer-subtracting operation f. Thus we perform

$$x \rightarrow f(x) \rightarrow f(f(x)) \rightarrow f(f(f(x))) \ldots;$$

for example:

$$\begin{array}{cccc} & (1.5668) & (1.1336) & \\ 0.7834 \rightarrow & 0.5668 \rightarrow & 0.1336 \rightarrow & 0.2672 \ldots. \end{array}$$

For compactness we write $f(f(x))$ as $f^{\circ 2}(x)$, etc. Each number x_0 generates a sequence or *orbit* $x_0, x_1, \ldots, x_n, \ldots$ with $x_1 = f(x_0)$ and $x_n = f^{\circ n}(x_0)$. This iterative mechanism is similar to that used in the numerical solution of differential equations. We shall interpret the sequence $\{x_n\}$ as a dynamical process, with x_n being the n^{th} "state" of our system (think of x as a measurement of position or velocity). As in our ODE, the process is clearly deterministic, since any state x_n is completely determined by the preceding state x_{n-1} and hence by the initial state x_0. Nonetheless, I shall show that lack of exact knowledge of x_0 (due to finite precision arithmetic, for example) implies that most orbits behave in an essentially random manner, and prediction of the (distant) future is impossible.

"...computer simulations that were discarded as 'unreliable' are now being reexamined for evidence of deterministic chaos."

To see this, we assign an infinite *symbol sequence* $a_0a_1...a_n...$ of 0's and 1's to each number x_0 using the rule

$$a_n = 0 \text{ if } 0 \leq f^{\circ n}(x_0) < ½,$$
$$a_n = 1 \text{ if } ½ \leq f^{\circ n}(x_0) \leq 1.$$

A little thought will show that $a_0a_1,...$ is simply the *binary expansion* of x_0:

$$x_0 = a_0/2 + a_1/2^2 +...+ a_k/2^{k+1} +...$$

Thus, the first few symbols for the example 0.7834 are 110010001.... The function f acts on the sequences $\{a_k\}$ as follows:

$$\begin{aligned} f(x_0) &= 2\,(a_0/2 + a_1/2^2 +... \\ &\quad + a_k/2^{k+1} +...) - \text{(integer part)} \\ &= a_0 + a_1/2 +... \\ &\quad + a_k/2^k +... - \text{(integer part)} \\ &= a_1/2 +...+ a_k/2^k, \\ &\quad \text{since } a_0 = 0 \text{ or } 1. \end{aligned}$$

Thus the sequence $a_0a_1a_2...$ becomes $a_1a_2...$; the leading symbol is removed and *information is lost.* If we knew x_0 to infinite accuracy (that is, if we had all the a_k's for k between 0 and ∞) this would not matter, for at each stage we would still know the state x_n exactly, but alas, our minds and our computers are finite and we can only store, say, the first N binary places $a_0a_1a_2...a_{N-1}$. The result is that, after N iterations, we cannot even say whether the state x_n lies above or below ½. The effect is known as *sensitive dependence on initial conditions:* the system amplifies small errors.

Worse is yet to come. The binary number construction shows that to each sequence of 0's or 1's there corresponds a number between zero and one, and vice versa. Thus, given *any* "random" sequence (generated by tossing a coin and assigning Heads = 0, Tails = 1, for example), there is an initial state x_0 such that the iteration $f^{\circ n}(x_0)$ realizes that sequence. Hence our system has infinitely many "random" or chaotic orbits. There are also infinitely many periodic orbits (periodic sequences—for example, 001001001...), but since the irrational numbers form a set of full measure, the "typical" behavior is chaotic rather than periodic.

Here all the complication comes from the *expansive* properties of f, the doubling operation, coupled with the fact that solutions cannot escape, due to removal of the integer part at each step. Small errors are magnified: no matter how small ϵ, for some N, $2^N\epsilon$ is large. One might object that this expansion and reinjection mechanism is probably rare, "real" systems tend to damp out errors, and "most" orbits should tend toward simple steady or periodic behaviors. And indeed, in spite of the mathematical work of Henri Poincaré (starting about 1880), G. D. Birkhoff, A. Andronov, and others, most applied scientists and engineers sought only such regular behaviors until about ten years ago. However, it is now recognized that many important systems and their mathematical models do exhibit chaos, and computer simulations that were discarded as "unreliable" are now being reexamined for evidence of deterministic chaos.

Before I turn to a mechanical example, I want to note that solutions of a "simple" model used in population studies, the logistic difference equation

$$x_{k+1} = \lambda x_k(1-x_k),\ 0 \leq x_k \leq 1,$$

behave in precisely the same fashion as those of the doubling map when $\lambda = 4$. It was during studies of this equation for

$\lambda < 4$ that Cornell physics professor Mitchell Feigenbaum first developed the theory of period-doubling cascades and the attendant notions of universal scaling behavior. Such period-doubling cascades are just one of infinitely many bifurcation sequences leading to the chaotic behavior as λ increases toward 4.

Iterated functions of complex variables (in the technical sense, a complex number is one of the form $a + ib$, where i is $\sqrt{-1}$) are also undergoing intensive study, as the article by John Hubbard in this issue indicates. In fact, the study of iteration in the complex plane has helped answer many questions raised earlier in the study of "real" iterated functions such as the logistic equation shown above.

A REAL EXAMPLE: A CHAOTIC VIBRATING BEAM

I will now discuss a device, constructed by my colleague Francis C. Moon, that provides a simple example of a chaotic mechanical system.

Figure 3 shows a slender steel beam, fixed at the upper end to a rigid frame. The beam is buckled to the right or the left by magnetic forces. The frame is subject to horizontal sinusoidal oscillation. Figure 4*a* shows a typical time trace of the position *y(t)* of the beam's tip, measured with a strain gauge glued near the root. One can see that the beam apparently oscillates "at random" about the left and right magnets.

Figure 4*b* is a theoretical time trace obtained from the simplest mathematical model of this system. Although the model differential equation that leads to this trace is a considerable simplification (it includes only a single mode of vibration—the fundamental mode) it

Figure 3. The magnetoelastic beam, a simple example of a chaotic mechanical system.

Figure 4. Displacement of the beam in Figure 3. The trace in a *is experimental and the trace in* b *is theoretical.*

The simplest mathematical model for this system is the differential equation

$$\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} - y + y^3 = \gamma\sin(t) \qquad (1)$$

for the tip displacement $y(t)$. Here $\delta\, dy/dt$ represents damping due to air resistance, proportional to velocity dy/dt, and γ is the amplitude of the applied sinusoidal force. This equation simply expresses Newton's second law, force = mass × acceleration:

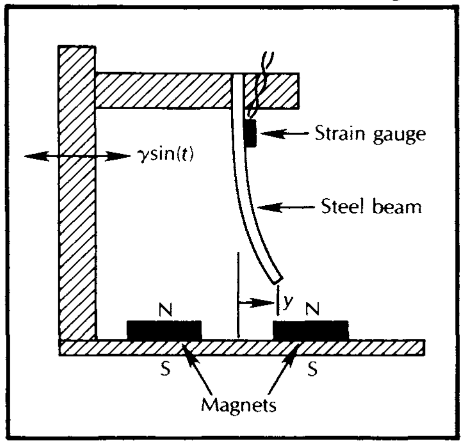
$$y - y^3 - \delta\frac{dy}{dt} + \gamma\sin(t) = \text{mass} \times \frac{d^2y}{dt^2} \qquad (2)$$

or, in terms of velocity v

$$dy/dt = v; \quad dv/dt = y - y^3 - \delta v + \gamma\sin(t), \qquad (3)$$

where the mass has been normalized to 1. The nonlinear force dependence on displacement reflects the two stable equilibrium positions over the magnets and the unstable central position.

Figure 3



Actually, our model differential equation represents a considerable simplification, since it includes only a single mode of vibration (the fundamental mode). It does, however, capture important qualitative features of the physical system, as will now be indicated. Numerical integration of this equation for certain parameter values δ,γ yields the time trace in b. *Clearly, the computer solution "looks like" the physically observed chaos of* a, *but we can actually* prove *that the equation really has chaotic solutions—that the complexity is not due to computational errors or truncation effects (see Figure 5).*

Figure 4

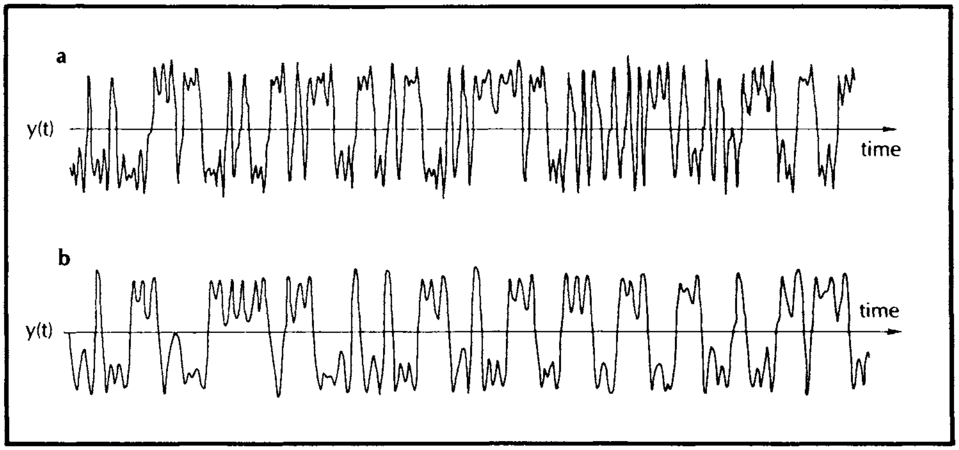
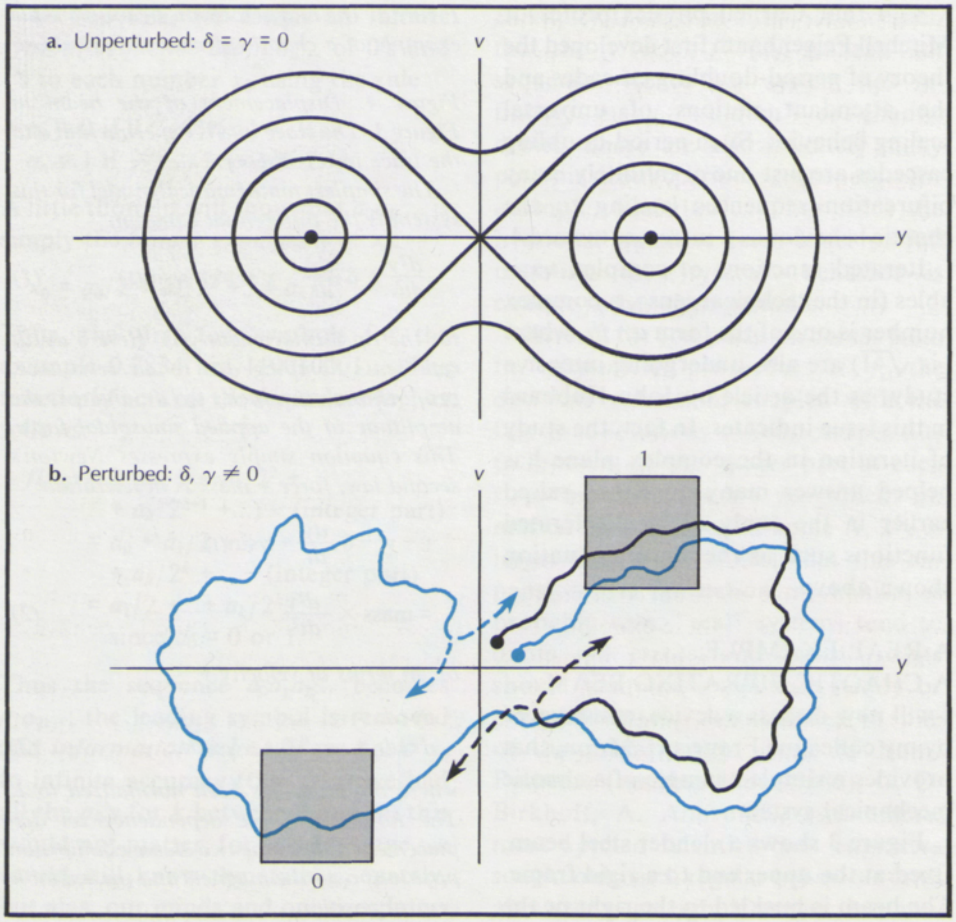


Figure 5. Proof that the vibrating-beam equation of Figure 4 has chaotic solutions. We proceed by perturbation from the exact analytical solution for $\delta = \gamma = 0$. A combination of analytical and topological techniques permits us to prove that, provided the force γ exceeds a critical fraction of the damping δ (specifically, if $\gamma > 4\cosh(\pi/2) / 3\sqrt{2\pi}$), then for any "random" or regular sequence of 0's and 1's there exist initial displacement and velocity conditions for the beam that lead to solutions visiting the left (0) and right (1) magnets in the order specified. In the proof we construct "boxes" near $y = -1$ and $y = +1$ in the (y,v)-phase space of the equation, label them 0 and 1, and show that solutions can be found that pass successively through the boxes in any predetermined order. Thus, as in the doubling function, there are deterministic motions of the beam that are effectively indistinguishable from random processes such as coin tossing.

Figure 5



does capture important qualitative features of the physical system. Clearly, the computer solution "looks like" the physically observed chaos. And we can actually *prove* that the equation really does have chaotic solutions—that the complexity is not due to computational errors or truncation effects (see Figure 5). Our proof involves demonstrating that solutions can be found that visit the left ($y<0$) and right ($y>0$) halves of the phase space at will. Assigning the symbols 0 and 1 to these regions, we find all the regular and chaotic symbol sequences of the doubling function.

The expansive properties of the ODE, on which our proof depends, reflect the fact that the beam continually returns to the vicinity of the unstable equilibrium position midway between the magnets, where it is very sensitive to small variations in the force: it can flip from left to right or vice versa. The nonlinear term (y^3) in the equation models the fact that when the beam lies to the right or left of *both* magnets, a strong restoring force acts on it, and this reinjection mechanism keeps orbits trapped and causes the recurrent behavior. Just as in the doubling function, we have deterministic chaos, but here it occurs in a "simple" Newtonian mechanical system.

One might question the validity of numerical simulations of such extremely sensitive systems. All computers work with finite precision arithmetic and so, in view of the doubling example, can we hope for anything but nonsense? The answer must be qualified, but we can prove that many chaotic dynamical systems have a nice property (hyperbolicity) which implies that computer-generated solutions do correctly reflect the properties of typical orbits of the system. Although exact predictions arbitrarily far into the future cannot be made, we can calculate general features with good accuracy.

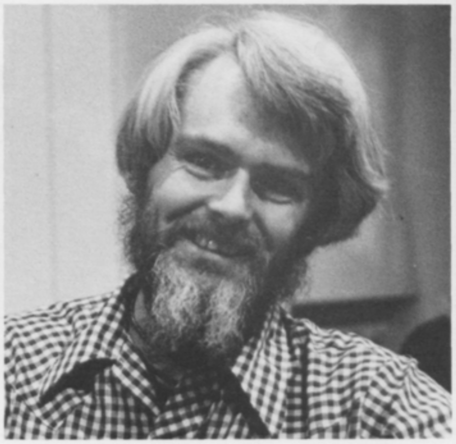
The analytical methods developed for this single-degree-of-freedom example have been extended to systems with many degrees of freedom and even to certain *partial* differential equations of continuum mechanics. Much the same picture of chaotic symbol sequences and sensitive dependence on initial conditions occurs in those cases also. One can, for example, include infinitely many modes of vibration in a generalization of the simple vibrating beam. Other applications include stability studies of electric-power-generating networks, of proton accelerators (atom smashers), of natural and artificial satellites, and the dynamics of superconducting junctions that have been proposed for use in advanced computer architectures. In all these examples the combination of unstable (expansive) *linear* behavior and *nonlinear* reinjection leads to chaotic motions. When almost all initial conditions lead to chaotic motions, we say that the system has a *strange attractor*.

NATURAL COMPLEXITY: A CONCLUDING MORAL

In this brief article I have tried to indicate how chaotic motions arise in deterministic systems in the absence of random inputs. One might modify Einstein's famous remark by suggesting that, while God does not play dice with the Universe, His (or Her) memory of the initial conditions is perhaps imperfect. Of course, I do not wish to suggest that random processes can always be replaced by deterministic ones as models for natural phenomena, merely that deterministic models can also behave in surprising ways. In fact, probabilistic methods are often appropriate for the study of deterministic chaos, and there is some interest in the effects of random perturbations on deterministic chaotic systems: in some cases the random forces "smooth" the chaotic behavior and promote regularity.

My main aim has been to illustrate that very simple mathematical models can exhibit extremely rich behaviors and thus can serve as models for aspects of Nature's complexity as well as her (or his) order. My examples also suggest that we be cautious in interpreting large-scale computer simulations of dynamical models.

It is perhaps fitting to end by remarking that Poincaré invented the "modern" theory of dynamical systems in the course of studying problems in celestial mechanics. Once thought the most orderly of all systems, even the Newtonian models of planetary and stellar motions can exhibit chaotic dynamics. Astronomical time scales are long, however: terrestrial social and political dynamics will probably have transformed our lives before the next astronomical catastrophe.



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Holmes received his undergraduate education at the University of Oxford, England, and earned the Ph.D. at Southampton University, where he held a research assistantship and a fellowship at the Institute of Sound and Vibration Research. In addition to many professional papers and a book written with John Guckenheimer (a contributor to this issue), Holmes has published several volumes of poetry.

FURTHER READING

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ORDER IN CHAOS

by John H. Hubbard

Most of the recent attention to the physical state called *chaos* has concentrated on the "disordered" side of the phenomenon. Researchers have spoken of such things as *strange attractors* and *Lyapunov exponents,* which are essentially statistical measurements describing average behavior. I want to describe instead the fine detail.

Not that statistical measurements are without interest. The application of dynamical systems theory to turbulent flows, for instance, seems to require such methods; it appears unlikely that for such extraordinarily complex problems we will ever have the sort of precise understanding that I propose. In any case, such information might be of little physical interest: the initial state of a system might never be known with enough precision to make more than average predictions.

ITERATING QUADRATIC POLYNOMIALS

Still, the problem I will examine is one that is usually thought of as chaotic: iteration of quadratic polynomials. (*Iteration* of a transformation is the process in which one applies the transformation over and over again, each time using the output of the previous transformation as the input for the next. The following example should explain what this means.)

The original motivation for iterating quadratic polynomials came from population dynamics. Suppose we have a species that goes through generations, and we denote the population of the nth generation as P_n. You might think (this is particularly relevant to life in Ithaca) of the population of gypsy moths in a given year. The plan is first to try to write down more or less realistic laws describing how the population in a given year depends on the population in the previous year, and then to see what this law predicts in the long run.

If we assume that there is a stable population P_S that the environment will support, then we might expect that the fertility rate would be proportional to how far the population is from this equilibrium—that is, something like $\lambda(P_S-P)$. This leads to the rule

$$P_{n+1} = \lambda(P_S-P_n)P_n.$$

If we scale the population so that $P_S = 1$, then the transformation being iterated is:

$$x_{n+1} = \lambda(1-x_n)x_n,$$

which is known as the *logistic model.*

An enormous amount of work has gone into understanding the behavior of sequences x_n defined by the logistic model, for various values of x_0 (the initial condition) and λ (the parameter). In particular, there are values of λ for which almost any x_0 leads to x_n converging to $-\infty$, and others for which x_n quite likely will cycle among some finite number of different values (the number can be anything), and still others for which it is hard to discern any pattern at all.

Another important problem is to see how robust the model is. It is absurd to imagine that any real system is described *exactly* by iteration of a quadratic polynomial. One might imagine that if the logistic model were just a rough model, then only its roughest behavior would be reflected in the real system, and any detailed information would be irrelevant. This is false. The behavior of quadratic iteration is *universal*: there

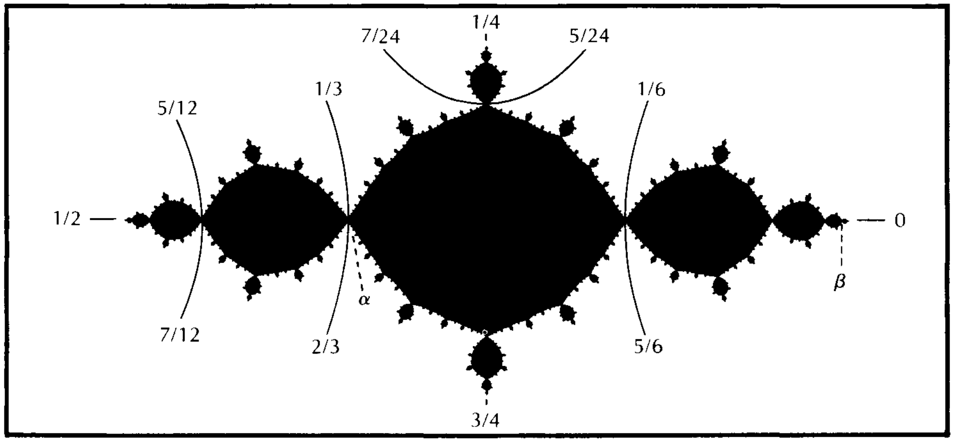


Figure 1. The set K_{-1}. The black locus K_{-1} is the set of complex numbers z such that the sequence z, z^2-1, $(z^2-1)^2-1$, ...does not tend to infinity. The curves drawn in color on the outside represent field lines one would get if the set were charged.

are whole classes of systems that exhibit exactly the same behavior under iteration as quadratic polynomials.

We now have quite a complete picture of how quadratic polynomials behave under iteration. Although the problem appears to concern only real numbers, this understanding arose from studying the iteration of polynomials on the complex plane rather than remaining restricted to the real line. Developing our understanding of universality also involved the use of complex numbers in essential ways.

ORDER IN AN INFINITELY COMPLEX SYSTEM

Let us consider the iteration of the polynomial z^2-1. In the complex plane there is a locus K such that if you pick z_0 in K, then the sequence (z_n) will remain in K forever, whereas if z_0 is outside K, the sequence will grow without bound.

This set K is represented in Figure 1. You can see that it is fairly complicated. In fact, if you were to perform appropriate blowups, you would find that it has detailed structure on all scales; and if you were to blow it up at any point near its boundary, you would always see much the same pattern. This property is called *self-similarity* and is closely related to *fractal dimension* (see the box on the following page). In the figure there are two points of special interest, marked α and β: they are the fixed points of the transformation. If you start at one of these points, you will stay there forever.

The best way to understand K in detail is to understand its outside. Suppose we built a metal model of K and charged it with electrons. Then the outside would carry an electric field, with field lines and equipotentials. Together these form a system of polar coordinates, and we assign angle 0 to the field line going to β. It turns out (this is not quite obvious) that the transformation sends field lines to field lines, simply doubling the corresponding angle. Since α is a fixed point, the field lines landing on α must be permuted among themselves; in fact, they must be exchanged. There are only two angles, θ_1 and θ_2, having the property that $\theta_2 = 2\theta_1$ and $\theta_1 = 2\theta_2$—namely, $\theta_1 = 120°$ and $\theta_2 = 240°$ or, better, $\theta_1 = 1/3$ and $\theta_2 = 2/3$ if you count in turns (the right way to count for this problem).

Continuing in this way, you can understand all the "pinching" required to make K.

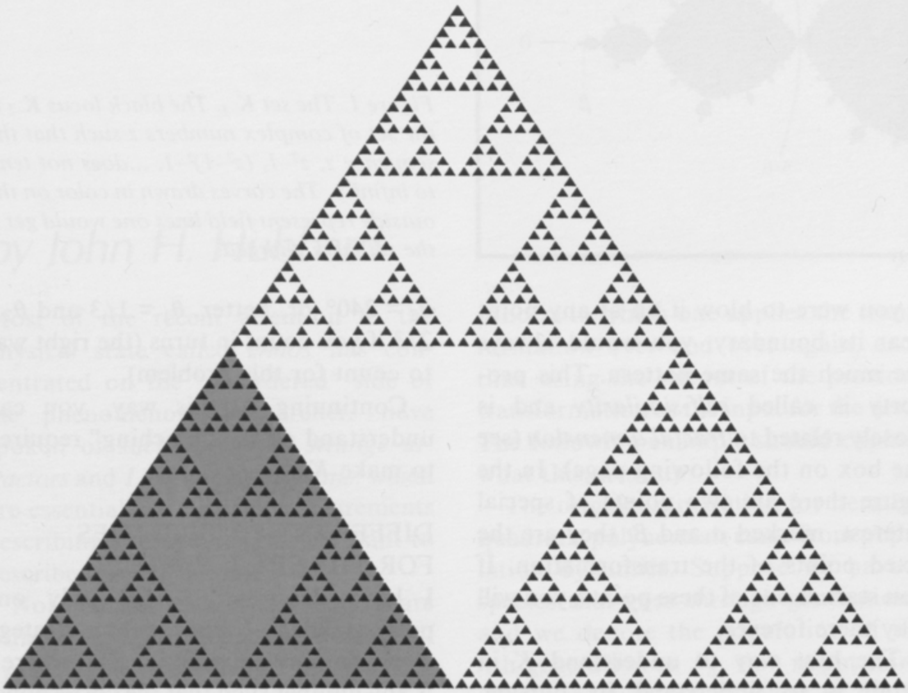
DIFFERENT POSSIBILITIES FOR THE SET K

I have described K for only one polynomial, z^2-1, but the same strategy works for any polynomial z^2+c, where c is any number such that the corresponding K (denoted K_c—the one in Figure 1 is K_{-1}) is *connected*. The sets may be far more complicated, as in Figure 2, but they can still be understood as "pinched discs", however complex the pattern of the pinching.

If K is not connected, the strategy outlined above does not work. In terms of the electrical analogy, K cannot be "charged" like a metal object because the electrons could not flow from one piece to another. Figure 3 represents what such a disconnected K looks like: a dust of points that form two globs, each of which forms two globs, and so on ad infinitum. Actually, all the

SELF-SIMILARITY AND FRACTAL DIMENSION

The relationship between self-similarity and fractal dimension is explained by Hubbard in terms of the set known as the Sierpinski gasket:



The Sierpinski gasket is obtained by taking a triangle; then removing the middle triangle, leaving three small triangles; then removing the middle triangles of those; etc. Hubbard claims that the set left over has dimension log3/log2.

This amazing fact can be seen as follows: If the part A (shaded in color) is blown up by a factor of 2, it becomes the whole set B. Now if the set has a fractal dimension d, then (recalling that area goes as the square of linear dimensions, volume as the cube,...) we should have:

(measure in dimension d of B) = $2^d \times$ (measure in dimension d of A).

On the other hand, since B consists of three copies of A, we must have:

(measure in dimension d of B) = $3 \times$ (measure in dimension d of A).

This leads to $2^d = 3$, which is $d = \log 3 / \log 2$, as promised.

This computation should make it clear in which sense self-similarity is related to fractal dimension.

Figure 2. The locus K_c (shown in black). K_c is the set of complex numbers z such that the sequence z, z^2+c, $(z^2+c)^2+c$, ...does not tend to infinity. The colors represent rates at which the same sequence does tend to infinity.

Figure 3. A disconnected K_c. In this case the set is a dust. It appears to form two globs, each of which under magnification forms two globs, ...

Figure 4. The entire Mandelbrot set (shown in black). The cusp of the cardioid is at 1/4; the end of the tail is at –2. The coloring on the outside is explained in the box labeled Computing the Mandelbrot Set.

Figure 5. A blowup of a section of Figure 4.

Figure 6. Another blowup from Figure 4. Notice the tracery of filaments leading to a small copy of M.

disconnected K_c look like that; unlike the connected K_c, which are all different, they are all alike.

THE MANDELBROT SET: INFINITE COMPLEXITY

This leads to the definition of the Mandelbrot set: *M is the set of c such that K_c is connected.* The set M is in the parameter space: for each c there is a corresponding K_c like those represented in Figures 1, 2, and 3. Somehow the Mandelbrot set is saying something about all polynomials at once. Figure 4 is a representation of the Mandelbrot set, and Figures 5 and 6 are blowups. (See the box on page 24 for an explanation of how these are computed.)

The Mandelbrot set is a source of unending wonder. For one thing, it has detailed structure on all scales. But

Figure 2

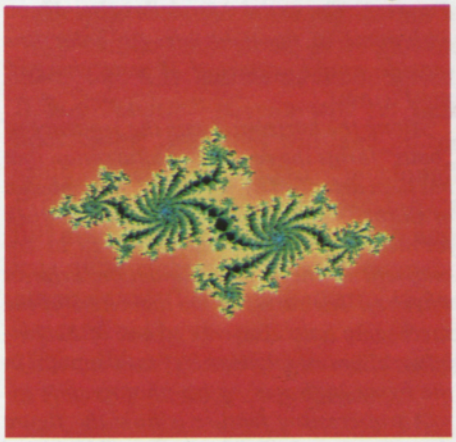


Figure 3

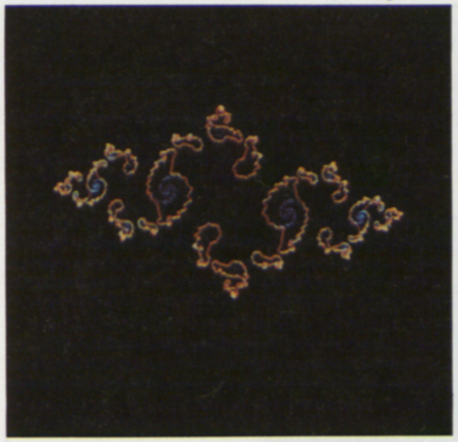


Figure 4

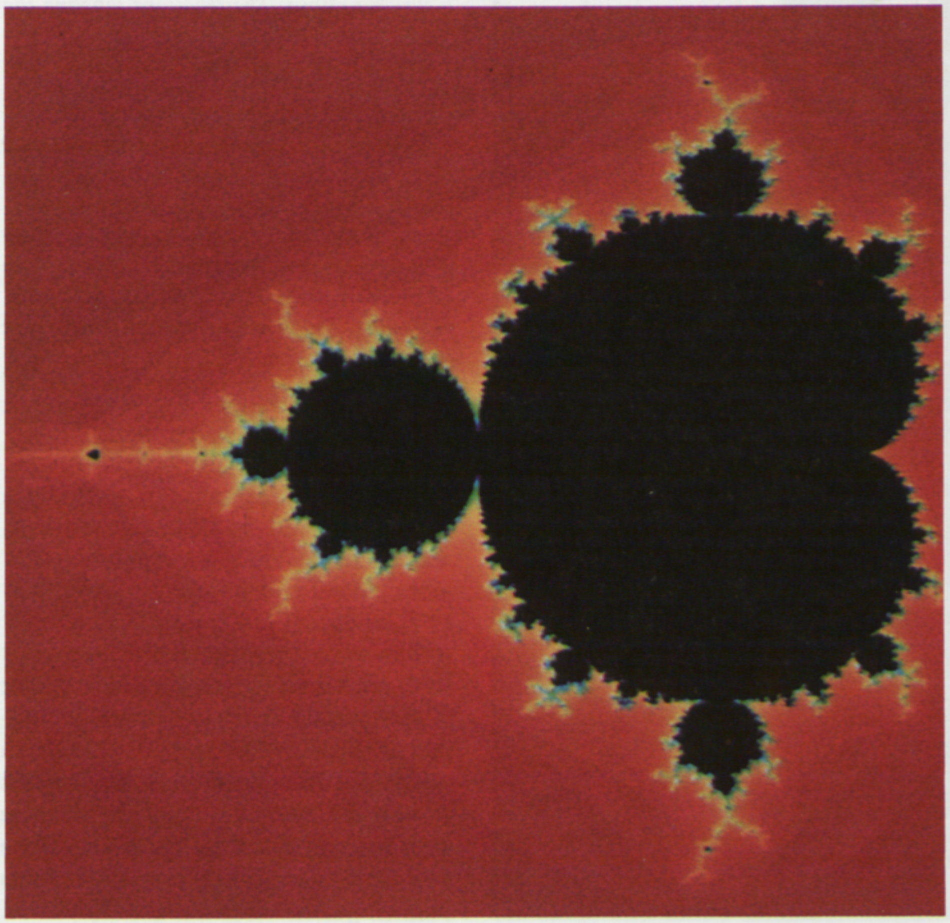


Figure 5

Figure 6

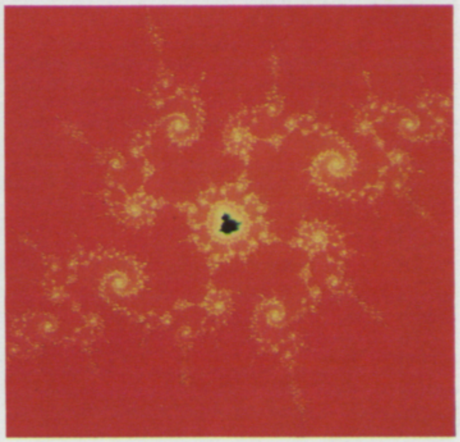


Figure 7

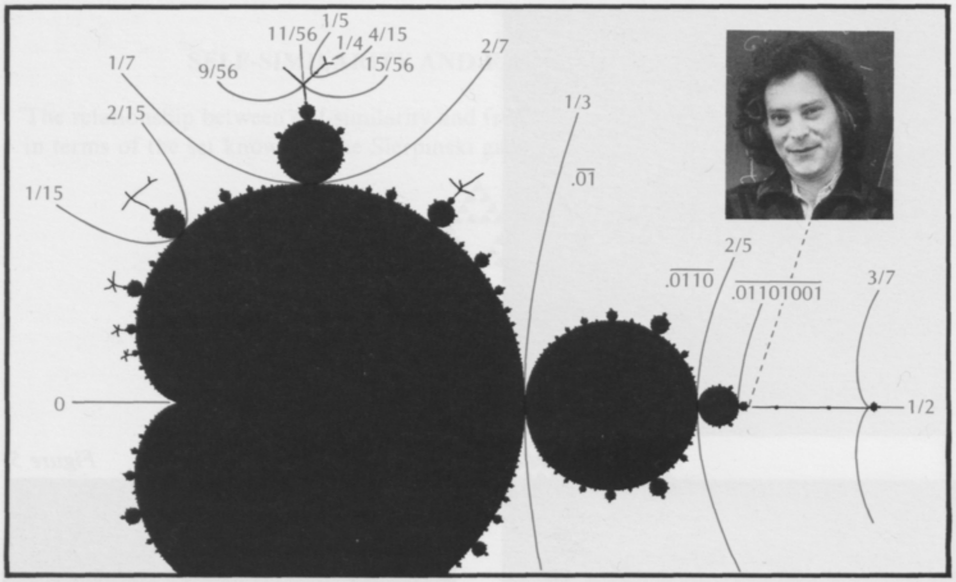


Figure 7. The entire Mandelbrot set again, with some of the field lines (in color) the outside would acquire if M were charged; see Figure 1.

Along the horizontal axis the sequence of "balls" forms the cascade of bifurcation that has been described by Mitchell J. Feigenbaum of Cornell's Department of Physics (in work for which he was recently honored as co-recipient of the Wolf Foundation Prize in Physics). The ratios of the diameters of the succeeding balls converge to one of Feigenbaum's "universal numbers" (at the point in the figure indicated by his photograph).

An interesting feature is how the figure demonstrates the "order in chaos" Hubbard is writing about. The fractions labeling the "field lines" indicate angles in a polar-coordinate system and can be written alternatively as base two numbers; for example, the angles at the tips of each ball can be written .01 (equivalent to 1/3), and .0110, equivalent to 2/5. The base two number at Feigenbaum's point is .0110100110010110.... The orderly sequence of these numbers, first discovered in 1922 as an abstract observation, was recently, independently, and simultaneously recognized as pertinent to chaotic theory by two researchers. One of them, Hubbard, showed that the sequence can be obtained according to an amazing rule: Begin by writing a digit, either 0 or 1; repeat this; repeat it again in reversed order; and continue in this way indefinitely. Thus one goes from .0 to .01 to .0110 to .01101001 to .0110100110010110....

COMPUTING THE MANDELBROT SET

It would be tremendously difficult to draw the Mandelbrot set (M) from its definition in terms of the connectivity of K_c. It turns out that there is a criterion that allows M to be drawn by a very simple program: c is an element of M if and only if 0 is in K_c. This leads to the following algorithm:

The Mandelbrot set consists of those complex numbers c such that the sequence

$$c_0 = 0,\ c_1 = c,\ c_2 = c^2+c,\ c_3 = (c^2+c)^2+c, \ldots$$

never satisfies $|c_n| > 2$.

For instance:

-1 is in M	: the sequence above is	$0, -1, 0, -1, \ldots$
i is in M	: the sequence above is	$0, i, -1+i, -i, -1+i, \ldots$
1 is not in M	: the sequence is	$0, 1, 2, 5, \ldots$

In practice, c ranges over points of some grid in the complex plane, and for each of these points the sequence above is computed until it gets larger than 2 or until some large number of terms like 1,000 have been computed.

The algorithm given here would lead to black-and-white pictures. These can be quite misleading, however: The Mandelbrot set is connected, made of up a spiderweb of thin filaments, all decorated with droplets with the same shape as the whole; and any grid, however fine, tends to step over the filaments. A better algorithm is to color c according to the first n for which $|c_n| > 2$; this is the program that generated Figures 4, 5, and 6.

unlike what happened for K_{-1} above, it does not repeat at finer and finer scales; it gets strictly more and more complicated. Each point contains its own address in each neighborhood of itself: If you are shown a blowup of some region of M, you can figure out where you are by counting how many strands come together at various nearby points.

Despite the amazing complication of M, it can be understood as a "pinched disc," like K_{-1}. The set M is connected. If you "charge" it, you will get field lines. Some field lines will land on the same point, leading to "pinching". Figure 7 shows some of this pinching; a full description can be given, although it is quite elaborate. For the moment, observe that -1 is in the ball attached to the cardioid by the pinching of $1/3$ and $2/3$, and that $1/3$ and $2/3$ were identified in K_{-1}; this is not a coincidence.

THE UNIVERSALITY OF QUADRATIC POLYNOMIALS

The description above gives a hint of what I mean by a detailed explanation of a complicated system. One perfectly reasonable question is: Who cares? If you have to work for years just to understand quadratic polynomials, you will presumably have to work much longer to understand anything else. Is it worth the effort?

The justification lies in the *universality* of the Mandelbrot set. The *precise* dynamics of quadratic polynomials occurs in all sorts of transformations: Figures 8 and 9, for example, show pictures that arise from the iteration of Newton's method for cubic polynomials. Figure 8 is a z-plane picture, and shows unmistakable resemblance to Figure 2, whereas Figure 9 is a parameter-space picture, and clearly shows a Mandelbrot set.

If you look in the outside areas of these pictures, you will see dyadic trees. Follow a branch toward the sets in the middle, recording with 0's and 1's whether you turn left or right each time you have a choice. If you use the 0's and 1's to make a number written in base

Figure 8

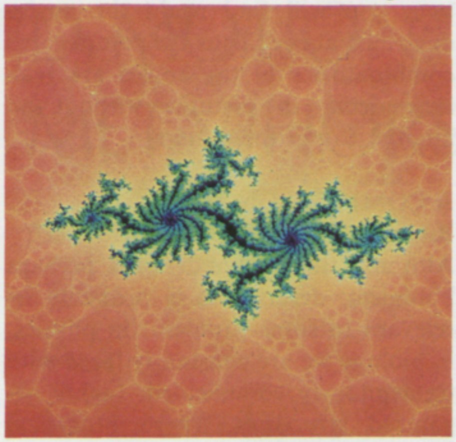
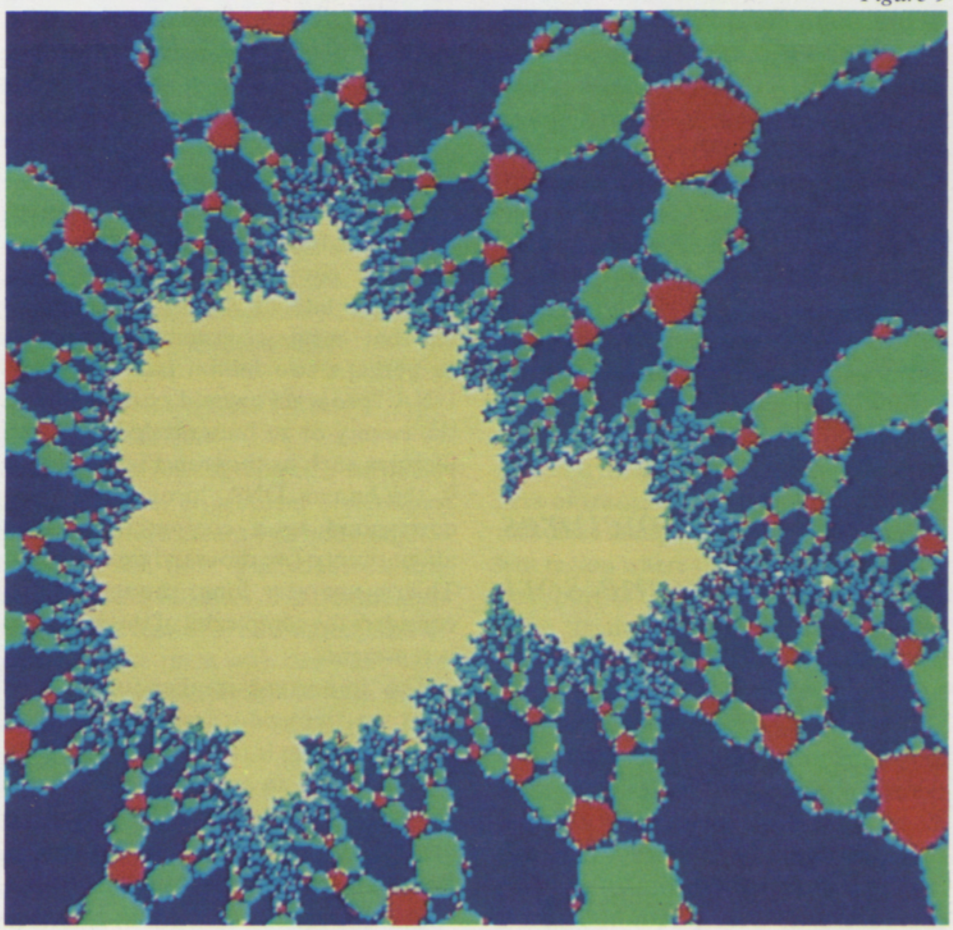


Figure 8. A region of nonconvergence, representing a portion of the set of initial conditions for Newton's method applied to a particular cubic polynomial. The black region is the set of initial values that do not lead to any root. Note the resemblance to Figure 2.

Figure 9. A drawing in the parameter space for Newton's method applied to cubic polynomials. The central shape is unmistakably a Mandelbrot set. If you follow branches of the dyadic tree on the outside, and encode lefts and rights as 0's and 1's, you find in base two the same numbers as in Figure 7.

Figure 9



two, you will find again the angles described above. Thus the complicated combinatorics of the angles are real: they occur in pictures without an attempt to put them there.

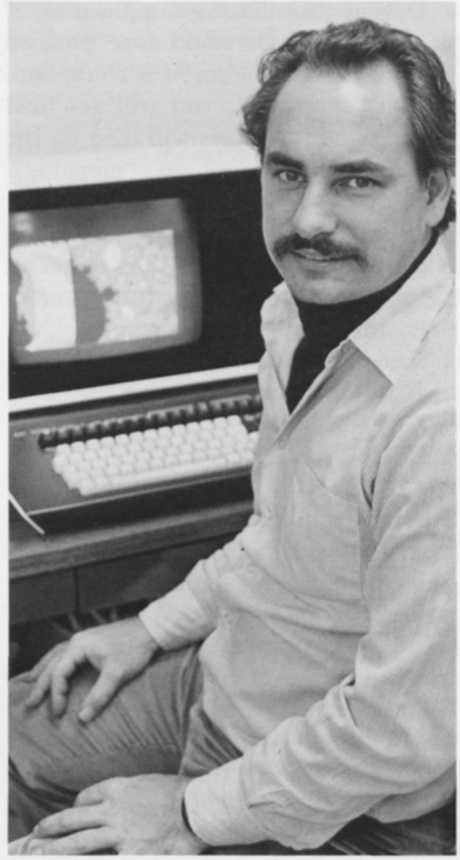
A COMPARISON TO BIOLOGICAL "PROGRAMMING"

Although this kind of detailed, perhaps botanical, description is probably irrelevant to some problems in physics, in biology we are constantly faced with tremendously complicated, highly organized structures. There, statistical measurements will not do; in living creatures, disorder is death.

I hope my pictures may be some sort of metaphor for biological objects: highly organized but extremely complicated. In that case, the computer programs that create the pictures would correspond to DNA. One can estimate that the information encoded in viral DNA is equivalent to perhaps fifty thousand bits of digital information, and that the human genome is equivalent to perhaps two billion bits. The viral DNA "program" would correspond to the twenty or so lines needed to create pictures such as those in Figures 8 and 9; the human DNA "program" would correspond to a computer program about twenty-five thousand times longer. This is not very long, though, if one considers the complexity of the organism it represents.

The important implication is that even the tremendous complexity of a biological form is represented by a code that is simple in comparison. Perhaps the sort of analysis that goes into understanding my pictures can contribute to actually understanding living creatures.

"I hope my pictures may be some sort of metaphor for biological objects: highly organized but extremely complicated."



John H. Hubbard, a professor in Cornell's Department of Mathematics, is an authority on the Mandelbrot set. He did much of the early work in making the detailed and beautiful computer-generated images—such as those reproduced here—that represent this intriguing and complex mathematical phenomenon.

Hubbard received the A.B. degree from Harvard University in 1967 and the Doctorat d'Etat from the University of Paris XI in 1973. He came to Cornell in 1977 after teaching at Harvard and at Paris, and in 1981–82 he returned to Paris during a year's leave.

COMPLEX FLUID MOTION: MODELS AND METAPHORS

by Sidney Leibovich and John L. Lumley

We live in a fluid environment. We need not be specialists in fluid physics to be fascinated by the motion of the gases and liquids all around us: our commonplace observations are sufficient for that. Our fascination stems in part from the remarkable visual patterns nature provides, from intricate cloud formations, to the vortices that were a favorite subject of da Vinci's, to the grand and seemingly abstract patterns seen in photographs of Earth and Jupiter taken from spacecraft. Just as commonly, our everyday impressions of fluid motion are tactile; we are struck with the irregularity of the fluctuations of the wind or the surface elevation of moving water.

The perception of order or disorder is often a matter of perspective. If what is being observed is large enough in scale, patterns may be apparent; on a range of smaller scales, no pattern may be manifest. A smooth, well-ordered fluid motion, called a *laminar* flow, is common neither in nature nor in the majority of flows of technological interest. More often, flows are either *turbulent*, involving contributions over a very wide range of length scales and shifting in a seemingly random way as time passes, or *transitional*, being laminar-like some of the time and chaotic or turbulent-like the rest. Coherent spatial patterns appear even in fully-developed turbulent flows, but they lack the persistence seen in laminar flows, and constantly form and dissolve at irregular times and at shifting locations. There is, in most fluid flows, a ceaseless competition between chaos and order.

ORDER AND DISORDER IN FLUID FLOW

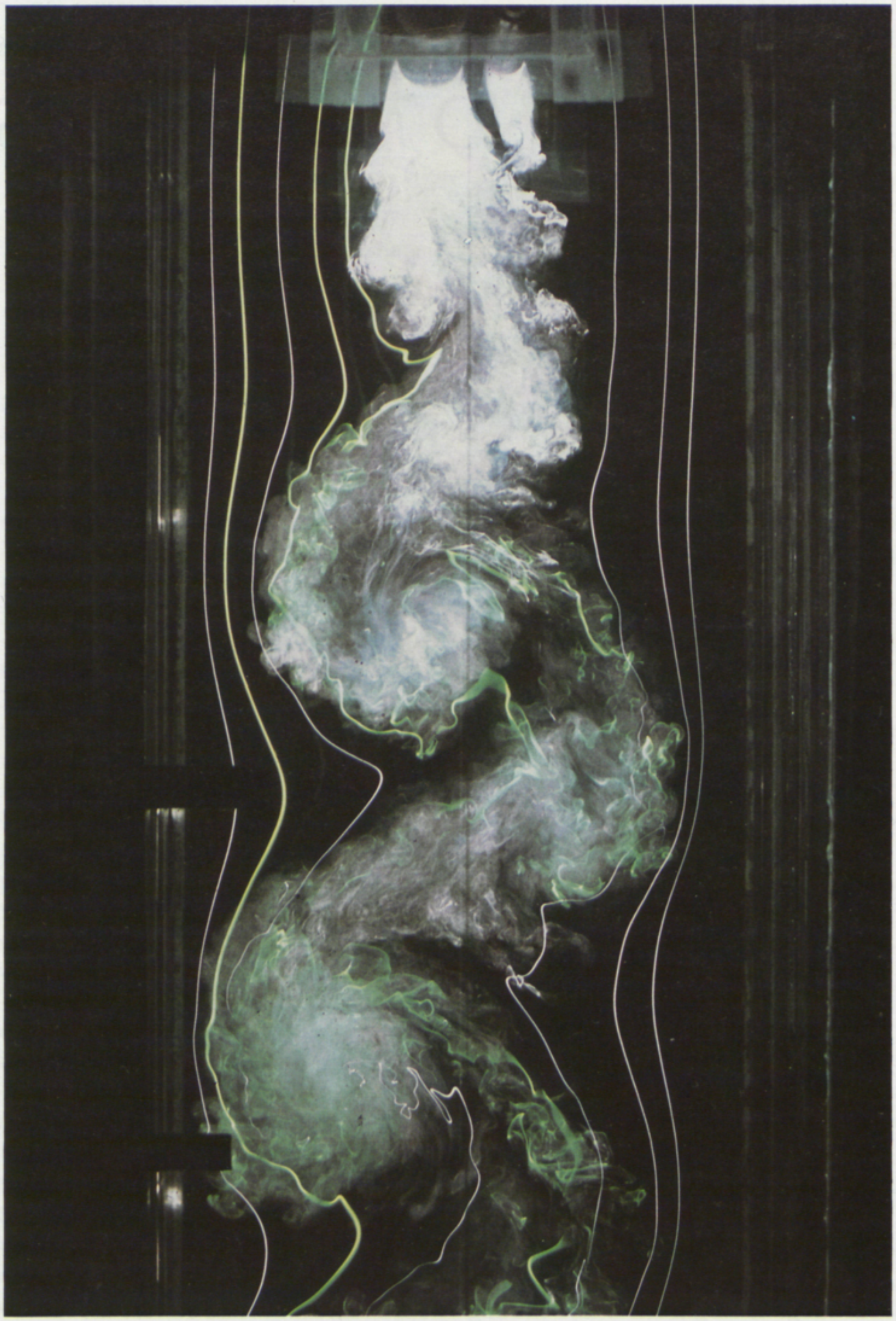
Figure 1 is a photograph of a turbulent wake—that is, a turbulent disturbance produced downstream by a body situated in a uniform flow. Turbulent disturbances arising in flows of different sorts have many features in common, and we can illustrate them with the wake.

The flow in the picture is from top to bottom. The bodies forming the wake are a pair of cylinders, one larger than the other, just visible at the top. Dye has been injected into the wakes of the cylinders, so as to fill the turbulent fluid. Also, dye streaks have been injected upstream in the fluid outside the turbulent part; this fluid is not undisturbed, but note the smooth sinuosity of the undulations; evidently there is little energy present at small scales in this part of the flow. By contrast, see how intense are the small-scale motions in the turbulent region.

The turbulence is gradually "eating" (more properly, entraining) the nonturbulent fluid: as a dye streak comes close to the boundary, it begins to be disturbed by the very-small-scale turbulence, is quickly gobbled up, and disappears in the turbulent fluid. The reason it disappears is that the effectiveness of transport is much greater in the turbulent fluid than it is in the nonturbulent region where there is only molecular transport (expressed as viscosity). This same transport effectiveness is also responsible for the fact that the injected dye spreads throughout the turbulent part of the flow, but not outside it.

Another interesting feature is the large-scale structure of the turbulent flow. At small scales there is not much apparent organization, but at large scales a recurring structure appears. While not reproduced in detail, this

Figure 1. Turbulent fluid flow: The wake of obstructions in a uniform flow. The disturbance is produced by a combination of two cylinders (at the top). Flow is from top to bottom. Visualization is by dye injection (see the text). An interesting feature, suggestive of strange attractors, is the similarity of large-scale structures in the turbulent wake. (Photograph courtesy of R. Dumas.)



structure seems to be globally similar in each oscillation.

Remarkably, the extreme complexity of the small-scale motion in the turbulent part of the flow can be reproduced in detail, at least for a short time, by an exact computer simulation. This is thus a *deterministic* flow, but one displaying extreme sensitivity to initial conditions, with exponentially diverging solutions for adjacent conditions. The global character of the flow, on the other hand, seems to be quite independent of the details of the initial conditions. These are all properties of *strange attractors* (see the other articles in this issue for a definition and simple examples).It must be emphasized, however, that until a great deal more is proven, strange attractors can serve as no more than a metaphor for what is going on here.

Many turbulent flows display coherent structures to some extent: some flows are almost completely organized and others almost completely disorganized. The structures often seem to arise from an instability, one associated either with the way in which the flow was set up initially (in this case, with the flow around the cylinders) or with the turbulent velocity profile and transport far from the point of initiation (far downstream of the cylinders, somewhat different coherent structures will be seen).

FROM ORDER TO CHAOS IN REAL-WORLD FLOWS

Whether a flow is laminar, turbulent, or somewhere in between determines the level of the forces and moments it transmits to solid structures (hence the resistance exerted on aircraft and land and sea vehicles, and therefore the power required to maintain their motion), the rapidity of mixing of liquids and gases (hence combustion rates, production rates of many manufacturing processes, effectiveness of pollutant dispersal), losses in fluid machinery (engines, turbines, compressors, pumps), rates of heat and mass transfer, and so forth. In short, most questions of fluid flow in technology—as well as in meteorology, oceanography, geophysics, astronomy, and the other sciences in which fluid motion plays an important part—have answers that depend critically upon whether the state of motion is laminar or turbulent. Needless to say, then, an understanding of when transitions between flow states occur, and the nature of turbulent fluid flows, is a central issue.

Our understanding of these matters is still fragmentary and inadequate. New experimental and theoretical ways of looking at irregularity in fluid motion and its interplay with ordered structures have evolved in recent years, made possible by the ready access to powerful computers. In this article we want to focus on a theoretical development that has attracted intense interest among fluid dynamicists who see in it a potential for tracking down the breakdown of order in fluid motion and its replacement by chaos. This development is the recognition of the existence of *deterministic chaos*: nonrepetitive fluctuating motion displaying extreme sensitivity to initial data and controlled by a strange attractor. The theoretical description of systems with complicated attractors falls in the purview of a mathematical discipline known as *dynamical systems theory*.

MODELS AND METAPHORS FOR DESCRIBING SYSTEMS

Before introducing a physical example of a fluid system exhibiting deterministic chaos, it is important to say that no problem in turbulence (at least as fluid mechanicians use the word) has yet fallen to dynamical systems theory. For the most part, what is known may be described as metaphorical.

We distinguish here between a metaphor and a model. In this context, a model is a description that comprises a mathematical statement of the physical laws governing the system, and embodies approximations of known accuracy that permit the prediction of system behavior. By contrast, a metaphor is a mathematical description that may get some important qualitative features of the system right, but either is not derived from the physical laws controlling the system, or employs approximations to them that are not appropriate. Of course, there is no reason to consider a metaphor unless the model is extremely difficult to solve (as is virtually always the case in fluid mechanics) and the metaphor is drastically simpler. Although metaphors in this sense do not suffice and are not the ultimate goal, they can provide conceptual frameworks that are suggestive and inspire specialists to explore real models from within those conceptual frameworks.

Here is an example of a metaphor.

"Many turbulent flows display coherent structures to some extent..."

Figure 2

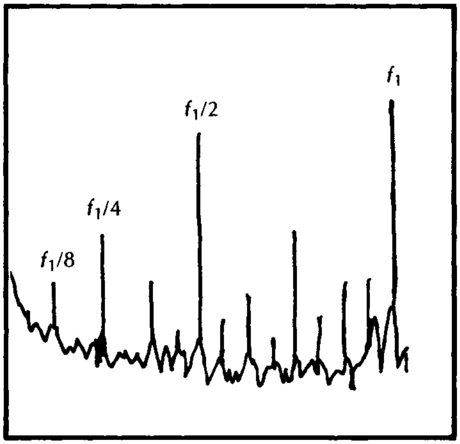


Figure 2. A schematic power spectrum of temperature perturbations, showing period-doubling (also known as subharmonic bifurcations). This schematic is based on Rayleigh-Bénard experiments of Libchaber and Maurer (1980).

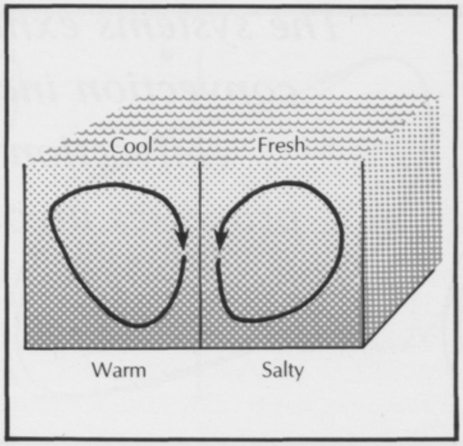
The frequency of the largest peak, f_1, is $2\pi/T$, where T is the period corresponding to f_1. The figure clearly shows three period-doublings following the development of period T. These have periods $2T$ (the peak at $f_1/2$), $4T$ ($f_1/4$), and $8T$ ($f_1/8$). A fourth period-doubling, too small to be seen clearly in this schematic, is at $f_1/16$; the flow is therefore periodic with period $16T$. The other strong peaks are harmonics of the fundamental frequency $f_1/16$.

One way in which a dynamical system can pass from an ordered state to a chaotic one as some control parameter is varied is through an infinite number of transitions that take the system through a series of periodic states. The first transition takes the system into a state that oscillates in time, so that any quantity, such as the velocity at some specific point in a fluid, repeats itself after time T, the period of the oscillation. The second transition leads to a doubling of the period, so that a time $2T$ is required for the fluid to return to any of its possible states. The third transition requires $4T$, the nth transition is to a state with period $2^{n-1}T$, and so forth. The period approaches infinity at a finite value of the control parameter, and a further increase puts the system into a chaotic state. By considering nonlinear recursion relations such as the logistic map (described in other articles in this issue), Cornell Professor of Physics Mitchell Feigenbaum has found the rate at which period infinity is approached, and has shown that the rate of approach is the same for a wide variety of nonlinear recursion relations. This feature, therefore, is *universal* for recursion relations in a certain class.

This sequential degeneration to chaos by period-doubling has been documented in a classical fluid-mechanics problem. Fluid heated from below can undergo an instability to convective motion because the hot fluid is less dense than the fluid above it; the light fluid is buoyant and has a tendency to rise that is resisted only by internal friction forces and by the tendency for heat to diffuse out of the rising fluid. (This is called the Bénard or sometimes the Rayleigh-Bénard problem, depending, presumably, on whether one prefers to give top billing to an experimentalist or a theoretician.) A. Libchaber and J. Maurer have experimentally verified a "period-doubling route to chaos" in a small container of liquid helium heated from below (see Figure 2). In this case, the logistic map or any other in the same class serves as a metaphor: the quality of period-doubling is shown and its rate is produced, but quantities of scientific or technological interest, such as the rate of heat transfer across the liquid layer, are not accessible.

There is another important metaphor associated with the Bénard problem. E. N. Lorenz, a theoretical meteorologist, was concerned with this question: Is it possible, with the most powerful computer imaginable, to predict the weather for an indefinitely long time, given the state of the atmosphere now? To address this question, Lorenz considered a set of three ordinary differential equations that have since been named for him. The Lorenz equations are derived from

Figure 3



the set of nonlinear partial differential equations that model (in our sense) thermal convection, which is one important process that occurs in the atmosphere. The Lorenz equations are given in the Figure 3 caption. The parameter σ is a material property and therefore fixed for any particular fluid, and R is a control parameter that measures the difference in temperature between the warmer lower surface of the layer and the cooler upper surface. The problem has been scaled so that no convection takes place for $R < 1$, and weak convection begins for R infinitesimally greater than one. The Lorenz equations are approximately right, and therefore constitute a valid model, for roll motions (independent of one of the horizontal directions) in fluid layers heated just slightly more than required to cause convective instability—in quantitative terms, for $|R-1| << 1$. They are not expected to be correct if $|R-1|$ is not small; if they do lose validity, their value is metaphorical.

Lorenz, in a remarkable study that was the first of its kind, showed that chaotic motion occurs for this set of

Figure 3. Convection in the ocean: A prototype double-diffusion problem. Warmer water tends to rise and saltier water tends to sink, but at markedly different rates, so that a growing oscillation may result.

A good approximation of the full mathematical model for diffusive convection in a horizontal layer of fluid (such as may be found in the ocean) is a metaphor proposed by G. Veronis in 1965: a set of five ordinary differential equations. In these equations R measures the level of heating from below; S measures the difference in saltiness of the fluid near the bottom and at the top of the layer; σ and τ are material properties; and γ is a geometrical parameter. The system is studied with σ, τ, and γ regarded as fixed.

$$dx_1/dt = \sigma\,[-x_1 + Rx_2 - Sx_3]$$
$$dx_2/dt = -x_2 + x_1(1-x_3)$$
$$dx_3/dt = \gamma\,[-x_3 + x_1x_2]$$
$$dx_4/dt = -\tau x_4 + x_1(1-x_5)$$
$$dx_5/dt = \gamma\,[-\tau x_5 + x_1x_4]$$

If the salt concentration is uniform, $S = 0$: in this case, the first three equations are independent of the last two, and form the metaphor for thermal convection that was proposed by E. N. Lorenz in 1963 (see the text).

equations (for $\sigma = 10$) when R exceeds 24.74. Chaos here means that if the system evolution is traced for a given initial condition and then again for a slightly different initial condition, after a sufficiently long period of time the outcomes of the two computations bear no resemblance to each other. In weather calculations there is inevitably a great deal of uncertainty in the specification of initial data: to completely specify the state of the atmosphere at a given time requires information about velocities, temperatures, pressures, moisture content, etc., at every point in the atmosphere, and clearly this detailed specification is and always will be beyond our capability. Lorenz showed by his simple example that even a small error will eventually lead to a prediction having no connection with reality, and this strongly suggests that long-term weather prediction is not possible.

Clearly, Lorenz's model is a metaphor for atmospheric motions, which involve much more complicated physics than thermal convection. It turns out to be a metaphor for two-dimensional Bénard convection as well, and not a very good

"The systems exhibiting doubly-diffusive convection include...the upper ocean, convection in stars, and industrial processes involving binary mixtures."

one at that. For values of R that are not close to one, chaotic thermal convection seems to occur only if the system is severely confined laterally, as in the Libchaber-Maurer experiments, and in that case the transition to chaos does not arise as indicated by the Lorenz equations. Numerical computations of the full model equations for two-dimensional thermal convection in the absence of the narrow confinements of the Libchaber-Maurer experiments show no chaos at all! Chaotic behavior arises only when three-dimensional motion is allowed.

A MODEL OF DOUBLE DIFFUSION CONVECTION

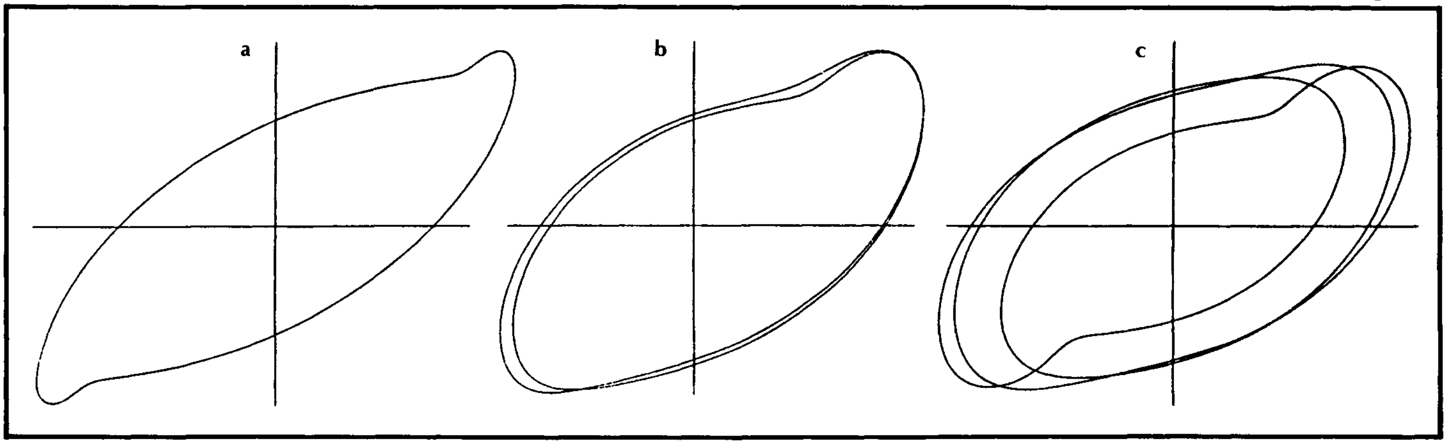
The first confirmation that period-doubling leading to chaos can arise as solutions of partial differential equations was reported in 1983 by D. R. Moore, J. Toomre, E. Knobloch, and N. O. Weiss. They considered a model of doubly-diffusive convection, a class of physical problems rich in importance and replete with many different kinds of complex motions. The systems exhibiting doubly-diffusive convection include (among others) the upper ocean, convection in stars, and industrial processes involving binary mixtures.

The prototype double-diffusion problem is thermohaline convection in the ocean. The density of sea water increases with saltiness and decreases with temperature rise. Heavier water lying on top of lighter water is unstable; the heavier water will tend to fall and the lighter water to rise, setting up convection. If the water is saltier at the bottom, then it is possible to heat it from below without making it topheavy, since its density depends on both salt and temperature. If the heating rate is increased, though, the fluid below becomes buoyant enough to rise, and convection takes place. The process is different than in a fluid without salt because the rising salty fluid loses heat about eighty times faster than it loses salt. Accordingly, as it rises and cools, it finds itself embedded in fluid that is at about the same temperature, but fresher and therefore lighter, and so it sinks back down. On its downward trip the process is reversed and it overshoots its original position, whereupon it rapidly gains heat and rises back up. This continues on and on in a oscillatory fashion, and the result is *doubly-diffusive instability*. A similar behavior is exhibited in other fluids having two substances that diffuse at drastically different rates (like heat and salt here).

Moore and his collaborators found that the full partial differential equations of the double-diffusion model have the faithful metaphor given in the caption for Figure 3. This set of five ordinary differential equations, which represents the system by a five-dimensional space, emulates many of the qualitative features of solutions of the partial differential equations, an infinite-dimensional model. In particular, the investigators found that as the heating was increased (R was made larger), the period of the oscillation sometimes doubled and then redoubled (at an even larger value of R). This doubling process could continue indefinitely, following the Feigenbaum period-doubling scenario until chaotic conditions resulted.

Figure 4 shows a series of period-doubling transitions computed from a set of five equations very similar to

Figure 4



those in Figure 3, but constructed for Langmuir circulations, a different kind of doubly-diffusive instability that can take place in the ocean. The period-doubling transitions are almost identical in the two problems. All the graphs in Figure 4 are plots of x_4 against x_1. The first one (*a*) shows a periodic motion as time proceeds. The values of the two variables wind around the curve, returning to the starting point after one time period, *T*. Graph (*b*) shows what happens for a larger value of *R*. The original curve seems to have split into two connected curves of similar shape. To return to its original position, the system must pass through the same neighborhoods twice, taking twice as long: the period has doubled and is now $2T$. The third plot (*c*) shows the next doubling transition, with the period now $4T$. This process can be followed by computer up to about five or six period-doublings, or to a period of 32 or 64 times *T*. Beyond this, chaos (which in plots like these looks rather like a child's attempt to trace the curves in the figure round and round for hours on end) is encountered.

It is gratifying to see a simple system, like that illustrated in Figure 3, capture the important behavior of a complicated model. Unfortunately, it does so only when the model is solved under physically unenforceable lateral constraints. In work at Cornell (reported in 1985 by S. Leibovich, S. K. Lele, and I. M. Moroz) it has been shown that if these constraints are relaxed, the period-doublings and chaotic solutions obtained with both model and metaphor are unstable and disappear, and instead are replaced by very simple time-independent convection, the epitome of ordered patterns. This shows that chaos itself can sometimes be a delicate thing that may be lost by almost imperceptible modifications of the system or the assumptions made about it.

NOISE AND COHERENCE IN TURBULENCE

We have seen that chaos can arise in an entirely deterministic way. In the double-diffusion problem, for example, all the boundary conditions can be held fixed with time, and the system kept free from random forcing (exerted, say, by uncontrolled experimental conditions such as vibrations from passing trucks), yet chaotic motion is still possible. On the other hand, precise knowledge and control of experimental conditions is an ideal never realized in the real world.

Figure 4. Period-doubling arising in solutions of equations that have been derived for the problem of Langmuir circulations in the upper oceans. (The five-mode set of equations is similar to the set given in Figure 3.)

The five-dimensional phase space is projected onto two-dimensional figures, with the abscissa a measure of convective velocity and the ordinate a measure of the temperature variation. In a *the system winds around as time advances, returning to the same point after a time T, the period of the motion. In* b *the forcing, measured by a parameter R, has increased and the period has doubled; the orbit has split and it takes a time of 2T to return to the same state. In* c *the parameter R has been further increased, the period has doubled again, and it now takes a time of 4T to return to the same state. This process continues as R is increased until an infinite number of doublings has occurred at some value $R = R_c$. For R just above R_c, the motion is chaotic.*

Small uncontrolled, time-dependent variations about the nominal experimental parameters inevitably occur. Since we do not have adequate knowledge of these small experimental variations, the best we can do is to recognize their existence and regard them as random events, or *noise.*

By a simple change of variables, random fluctuation of boundary conditions can be treated as a random *force* in a system subject to the known, deterministic, nominal boundary conditions. Since a fluid experiencing turbulent flow amplifies all small disturbances, this continuous "jittering" force can be responsible for maintaining the turbulence. Before the discovery of strange attractors, this was a generally held perception of the origin of turbulence.

The discovery of strange attractors does not do away with the old view, but instead adds a new dimension to the problem. Noise must be accounted for, and the connections between noisy influences, the coherent structures in turbulence, and possible strange attractors are of great interest.

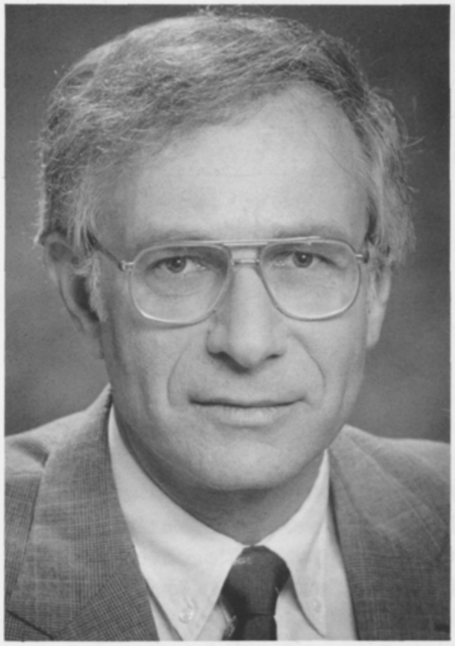
A possible way in which coherent structures in turbulence might arise with noise as a basic feature has been suggested recently. The suggestion comes by way of a metaphor from a group at Novosibirsk, in the Soviet Union. Consider a simple dynamical system with a two-lobed separatrix loop in the phase plane. One loop corresponds to a solution rising from zero in the past to a positive value in the present, and decaying to zero again in the future: a bump that appears, grows, matures, decays, and dies. The other loop corresponds to a similar solution that goes negative. Suppose the system is affected by a small random disturbance. While the system point is away from the origin, following approximately one loop or the other, it will continuously circulate around the phase plane. But when it arrives in the vicinity of the origin, the jitter from the random disturbance may start it off on either loop. The system thus will execute a series of loops, choosing the right or left at random, and each time producing approximately the characteristic solution, either positive or negative. This is very like the shot effect, which describes the noise signal produced by electrons arriving at the cathode of a vacuum tube, each arrival producing a characteristic signature, but the time of arrival being random. Such a signal can be described by a random distribution of deterministic signatures, each signature being characteristic of a trip around the loop that describes the solution.

The Soviet group has suggested that coherent structures in turbulence may be formed in somewhat this way. If a probe is placed in a turbulent flow similar to the one in Figure 1, on the passage of each coherent structure a characteristic signal will be recorded. Such a signal will look very like the output of the simple dynamical system described above. Perhaps coherent structures themselves are the signatures of trips around a complicated multidimensional fluid-dynamical strange attractor. This connection is at present no more than a gleam in the eye of the workers involved, but, in collaboration with Philip Holmes, we are examining the possibility.

Specifically, we are looking at the sub and buffer layer of a turbulent boundary layer. A turbulent boundary layer is formed whenever a fluid flows over a solid surface sufficiently fast; the sub and buffer layer is the region very close to the wall, where the influence of molecular transport (viscosity) is important. Farther from the wall, the transport is due almost entirely to turbulence; viscosity can be neglected.

The near-wall region of a turbulent boundary layer is known to be disturbed by coherent eddies, more or less in the form of rolls with the axes lying in the streamwise direction. We hope that examination of this region will be particularly profitable, since (because of the proximity of the wall and the consequent importance of viscosity) the range of scales present is relatively small. With this limited complexity, it is possible to represent the flow by a small number of elementary motions. We are examining the equations describing the interactions of the elementary motions, to see what connection we can find with ideas of dynamical systems theory. The equations we obtain are similar to those investigated by Lorenz in connection with weather

Leibovich



Lumley

prediction, and bear some more-than-metaphorical similarity to the equations describing Langmuir circulations. This gives us reason to hope that some related phenomena will be found.

THE USEFULNESS OF GOOD METAPHORS

A good metaphor will imitate a number of features of the physical problem; a really bad metaphor will imitate none. Unfortunately, there is no way to know in advance which variety one has. The natural way to search for a metaphor is to start from a valid model and (drastically) simplify it. With luck, the simplification will still provide a model for the problem, but this is exceptional and a metaphor is likely to result. Even a metaphor inspired by the physics of a problem is dangerous, however, and a system description that it provides should not be accepted even on qualitative grounds unless it is already known to imitate behavior predicted by a valid model. A high degree of caution is warranted because a very good metaphor can become a very bad one by virtue of an almost imperceptible change in the problem, as the example of doubly-diffusive convection shows.

The crucial unanswered question is this: Will the conceptual framework gained by the study of *metaphors* permit us to treat *models* by similar methods, and will it help us to organize physical phenomena in an understandable way?

Sidney Leibovich and John Lumley are professors in Cornell's Sibley School of Mechanical and Aerospace Engineering. Both are specialists in fluid dynamics; Leibovich is working primarily on problems of wave propagation and air-sea interaction, and Lumley on problems of turbulence, including modeling. They are co-investigators, with Philip Holmes, in a study—funded by the Office of Naval Research—of structures and chaotic dynamics in turbulent wall layers.

Leibovich did his graduate work at Cornell and joined the faculty in 1966 after a year as a NATO postdoctoral fellow in mathematics at the University of London; his undergraduate degree is from the California Institute of Technology. He has been a visiting scientist at the Weizmann Institute of Technology in Israel, and a British Science Research Council senior visiting fellow at the University of St. Andrews in Scotland. He is a fellow of the American Physical Society and the American Society of Mechanical Engineers, and is an associate editor of the Journal of Fluid Mechanics *and of* Acta Mechanica.

Lumley came to Cornell in 1977 as the Willis H. Carrier Professor of Engineering. He studied at Harvard University as an undergraduate, earned his doctorate at The Johns Hopkins University, spent two years there in postdoctoral research, and taught at The Pennsylvania State University for eighteen years. At Pennsylvania he was in charge of research on turbulence and transition at the Applied Research Laboratory, which is operated for the Naval Sea Systems Command. Lumley has been a Fulbright senior lecturer at the University of Liege in Belgium and a Guggenheim fellow at the University of Aix-Marseille and the Ecole Centrale de Lyon in France. He is a fellow of the American Physical Society and of the American Academy of Arts and Sciences. In 1982 he was awarded the Fluid and Plasma Dynamics Award of the American Institute of Aeronautics and Astronautics.

MATHEMATICS OF CHAOS

by John Guckenheimer

Many natural phenomena are like the weather: they are unpredictable to some degree. To deal with such phenomena, we generally try to distinguish between features that obey *deterministic* laws and those that obey *random* or stochastic laws. If the system is deterministic, it should be possible to make accurate predictions, but actually a deterministic system can have inherent dynamics that severely restrict long-term prediction.

A focus of experimental attempts to understand erratic and irregular dynamics has been the study of chaotic motions, and problems in this area have been a major motivation for mathematical study. We ask whether it is possible to model such a system mathematically, and if so, whether the mathematics can reveal properties that cause unpredictability in physical phenomena. This article explores the mathematics of chaos, particularly with reference to physical systems being studied at Cornell.

EXPERIMENTS IN CHAOTIC FLUID DYNAMICS

Suppose a fluid is subjected to a force that tends to set it in motion in opposition to its internal frictional forces (expressed, for example, as viscosity). If the force is strong enough, the energy it supplies cannot be dissipated by regular fluid motion, and the result is instability of the regular flow to perturbations. If the force is very large, the fluid motion can become *turbulent*—that is, highly irregular as a function of position and changing in a chaotic manner with time.

The earliest experiments with systems of this kind were studies of the motion of a thin layer of fluid—say, water—heated from below. (An analogous natural system is the earth's atmosphere, which to some extent can be considered as a thin layer of fluid heated by contact with the sun-warmed earth.) If the fluid is motionless, then the bottom of the layer becomes hotter and therefore lighter. Because of the buoyant effect of gravity, these lighter parcels of fluid want to go upward. If the heating is gentle, the buoyant forces will not be strong enough to overcome the internal friction (viscosity) of the fluid and the heat will conduct through the layer with no motion of the fluid. But beyond a certain threshold in the rate of heating, an *instability* will occur and the fluid will begin to move. (An important factor in weather is atmospheric convection driven by the sun's heating of the earth.)

The entire bottom of a fluid layer cannot rise uniformly because no portion can move unless it is displaced by another portion. In careful experiments, one sees regular patterns of fluid motion when heating is continued just beyond the threshold of instability. The particular pattern formed depends very much upon such details as the shape of the boundary of the fluid layer and whether the top of the fluid is also confined by a rigid plate or is free to move vertically. The patterns most commonly observed are hexagonal cells (discovered by Bénard) and parallel cylindrical rolls. These flows are *steady*; the fluid velocity at each point does not change with time. Some work along this line has been done at Cornell: Eric Siggia of the Department of Physics has investigated the instability of the roll solutions to three-dimensional effects.

With still stronger heating, these patterns become more complicated and begin to fluctuate in time. Eventually

"...a deterministic system can have inherent dynamics that severely restrict long-term prediction."

the time dependence of the flow becomes irregular or *chaotic*. The fluid motion can no longer be described by the superposition of a finite number of flow patterns, each varying periodically with its own frequency. Experimentally, this is measured through the *power spectrum* of the flow: the spectrum undergoes a transition in form from *discrete* to *continuous*.

FLOW BETWEEN CONCENTRIC ROTATING CYLINDERS

In another kind of fluid-dynamics experiment the fluid is contained between concentric rotating cylinders. This system has some features that make mathematical analysis harder, but the experiment is easier to control precisely and to visualize.

In the "standard" apparatus for such an experiment the outer cyclinder is rotated while the inner cylinder remains fixed. What happens as the rotational speed is increased? Before there is an instability, the motion is described as *Couette* flow: the fluid moves along circles centered on and perpendicular to the axis of the apparatus. At the first instability, a pattern of bands, the *Taylor cells*, is established along the axis of the cylinders. Each cell is a solid ring of fluid undergoing steady motion that is no longer azimuthal. Next, waves develop on the boundaries of the Taylor cells. At sufficiently high rotational speed (Reynold's number in nondimensional terms), modulations appear in the wavy vortex flow. The modulated wavy vortex flow is *quasiperiodic* and can be represented by the superposition of two periodic velocity fields whose periods depend upon the Reynold's number. Further increases in rotational speed result in chaotic flow without destruction of the overall Taylor-cell structure.

Describing the chaotic flow in these experiments is difficult at best. If the irregularity is random and unpredictable, we can only hope to develop statistical measures of the flow properties; the detailed structure of the flow itself will not be expressed in terms of a few definite patterns (called modes), as is the case with quasiperiodic flow. There is, however, another possibility. It may be that the flow is determined by only a few modes, but that it exhibits chaotic behavior because of a *sensitive dependence on initial conditions*. It is a fact that physical systems with only a few degrees of freedom can have chaotic dynamics (see the article in this issue by Francis C. Moon), and David Ruelle and Floris Takens put forward the suggestion in 1971 that fluid turbulence can be explained similarly. Experimental evidence obtained so far is consistent with the hypothesis that some fluid flows can be described in these terms. At Cornell, in a large research project headed by John Lumley, Sidney Leibovich, and Philip Holmes, turbulent shear flows that occur near a wall are being investigated from this point of view. (See the article in this issue by Leibovich and Lumley.)

DYNAMICAL SYSTEMS WITH ATTRACTORS

Let us put the Ruelle-Takens hypothesis into better perspective by considering how fluid problems are analyzed theoretically. The *state* of a fluid is described by specifying its *velocity field* $\underset{\sim}{V}$ as a function of spatial coordinates and

Figure 1

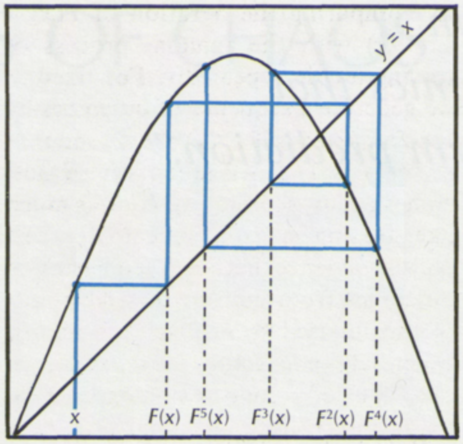


Figure 1. An illustration of mathematical modeling of chaotic behavior by a simple deterministic system. In the model F(x) = ax(1–x), where F gives the rule for determining a new state F(x) from the state x. The value of a lies between 0 and 4, and x is between 0 and 1. The blue line, tracing the sequence x, F(x), F(F(x))... represents the trajectory of x as a sequence of points along the graph.

time. (In the convection problem we also need temperature, but apart from small temperature-dependent effects, we assume that the fluid density is constant.) Since $\underset{\sim}{V}$ depends on the continuous spatial variables, the system has an infinite number of degrees of freedom. Its evolution is described by a complicated set of differential equations, the Navier-Stokes equations, along with boundary conditions and the equation for conservation of mass; with perhaps some technical difficulty we can find the velocity $\underset{\sim}{V}(t)$ at any time t in the future from the initial velocity field, $\underset{\sim}{V}(0)$. The Ruelle-Takens hypothesis is that there is a special set L of velocity fields with the following three properties:

- any two elements of L can be distinguished from each other with a finite number of parameters;
- if a velocity field $\underset{\sim}{V}$ is in L at time 0, the velocity fields $\underset{\sim}{V}(t)$ are in L for all $t>0$;
- velocity fields that start near L have evolutions that become indistinguishable from elements of L after some period of time.

The existence of the set L, which is called an *attractor*, has powerful implications. By restricting attention to an attractor, we reduce our analysis to one described by a finite number of parameters. In terms of these parameters, the evolution of the system is given by a set of ordinary differential equations.

What kinds of chaotic attractors are possible for a dynamical system? The answer is far from complete, but our understanding has increased significantly during the past twenty years. There are simple mathematical models that illustrate the features of these attractors, showing how a simple deterministic system can act in an inherently unpredictable manner. I shall discuss such a system—one whose study has been a primary object of my research during the past ten years.

MATHEMATICAL MODELS WITH ATTRACTORS

The most easily understood examples of chaotic attractors require a further reduction—the assumption that time is discrete. This has the effect of replacing the continuous evolution of the system by a discrete one. If $x = (x_1,...,x_n)$ denotes the parameter values that specify a state of the system L at any given time, then there is a rule $F(x)$ (transformation, map, or function) which determines the state of the system one unit of time later. (There are other ways to make this reduction.) Then the evolution of the system from the initial state determined by x is described by the sequence of iterates x, $F(x)$, $F(F(x))$,... $F^n(x)$. (Here $F^n(x)$ denotes the process of applying F to x a certain number (n) of successive times; it is not a product.)

In the simple example described here, we take x to be a single variable and $F(x)$ to be a quadratic function of x: $F(x) = ax(1\text{-}x)$. Provided that a is between 0 and 4, the function F will take values between 0 and 1 if x is between 0 and 1. These transformations—with a treated as an experimental parameter—constitute our simple mathematical model for chaotic behavior. This model displays a bewildering array of complicated behavior whose analysis is quite sophisticated. The physical significance is that some features of the model illuminate the way in which chaotic dynamics can occur; as I will discuss, these features have been seen in laboratory experiments. An added bonus is that there is a universal character to many of the results within the scope of systems that depend upon a single variable.

Iterating the transformation F graphically helps one understand the model (see Figure 1). We start with a point x on the x axis. Directly above it lies the point with y coordinate $F(x)$. If we draw the horizontal segment from this point to the line $y=x$, we have located a point with x coordinate $F(x)$. Directly above (or below) this point lies the point with

y coordinate $F(F(x))$. By alternately tracing horizontal and vertical segments in this manner, without lifting the pen, we obtain the evolution of x as a sequence of points along the graph. Each point represents a pair—$F^n(x)$, $F(F^n(x))$—with $F(F^n(x)) = F^{n+1}(x)$.

The sequence x, $F(x)$, $F(F(x))$... is called the *trajectory* of x. There is a special kind of trajectory in which the point x returns to itself after some iterate n; we say that x is periodic with period n. If $F(x) = x$, then x is a fixed point. (A fixed point might correspond to either an equilibrium solution with no motion or a periodic motion for a continuous time system). A periodic trajectory or a fixed point is *stable* if nearby trajectories tend to join it; it represents an *attractor*. For the transformation $F(x) = ax(1-x)$ there is a stable fixed point when $a < 3$. Most periodic trajectories are not stable; indeed, for each value of a there is *at most* one stable periodic trajectory.

The transformation $F(x) = 4x(1-x)$ illustrates the kind of chaotic dynamics these simple models can have. Here $F(1/2) = 1$, so that the graph rises from 0 to 1 and then falls back to 0 again. Think of this process as one of stretching the interval from 0 to 1 and laying it back upon itself so that two points fall on top of each one. When the process is repeated, four points fall on top of each one. When the process is repeated n times, 2^n points fall on top of each one. In other words, the graph of F^n will rise to 1 and fall to 0 a total of 2^{n-1} times. There will be at least 2^n crossings of the graph of F^n with the line $y=x$. Each will be a solution of the equation $F^n(x) = x$ and hence be part of a periodic trajectory.

Let us perform a thought experiment by comparing the iteration of $F(x) = 4x(1-x)$ with the random process of tossing a coin repeatedly. For fixed x, we generate a sequence of outcomes by assigning heads if $F^n(x) < 1/2$, and tails if $F^n(x) > 1/2$. All sequences of outcomes occur, so iterating F looks much like tossing a coin repeatedly. The correspondence between sequences of outcomes from coin tosses and points of the interval from 0 to 1 is almost exact. By converting a sequence of outcomes of a coin-tossing experiment to a binary decimal expansion of a number between 0 and 1, we can recover an x whose iterates have $F^n(x) < 1/2$ when the n^{th} coin toss was heads, and $F^n(x) > 1/2$ when the n^{th} coin toss was tails. The only difficulty arises when $F^n(x) = 1/2$ or when there is a sequence of coin tosses that ends in an infinite string of heads or an infinite string of tails. There is, however, a difference between coin tosses and iterating the transformation F. In coin tosses we have no control over the point x, whereas in a fluid experiment we have much control over initial conditions.

For some number of iterations, our control over initial conditions will be reflected in our ability to predict the initial set of outcomes. After this time (which depends logarithmically on the experimental precision), the uncertainty in our choice of x will have been amplified into total uncertainty about future outcomes. This is the property called *sensitive dependence on initial conditions*. Initial conditions which start close to one another have trajectories which diverge and evolve in ways that are largely independent of one another. Even though our system is deterministic, experimental imprecision means that only short-term prediction is possible.

Our transformations F have stable periodic orbits for $a < 3$ and a chaotic attractor when $a = 4$. What happens in between is surprisingly complicated. Figure 2 illustrates this behavior. The parameter a varies from 3 to 4 along the horizontal direction. On each of one thousand vertical segments, a trajectory is plotted in the attractor of the transformation F corresponding to that value of a. Note that there are intervals of a values above which only a few points are plotted. These correspond to values of a for which the attractor is a stable periodic orbit. For many other values of a, the plotted points appear to fill vertical segments. These values of a yield transformations with chaotic attractors. The set of values of a which do yield chaotic attractors has a complicated structure, apparently all holes. Though it has not yet been proved, it seems almost certain that between any two values of a with chaotic attractors there are values of a with stable periodic orbits. Still, a value of a chosen at random has a positive probability of corresponding to a chaotic attractor. This result, proved by the Russian mathematican Jakobson, provides at least a partial explanation for the

Figure 2a

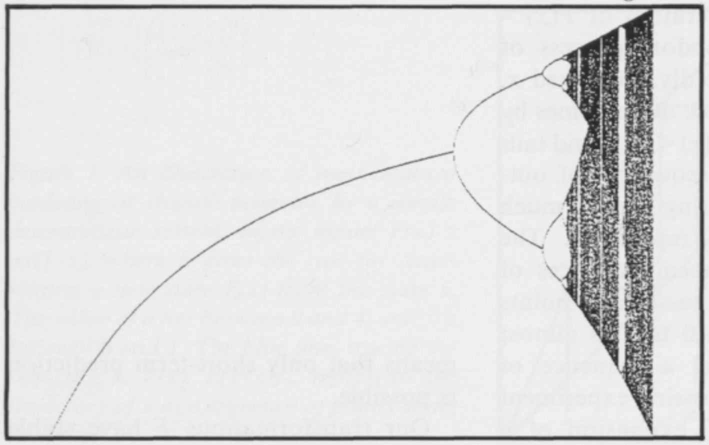


Figure 2b

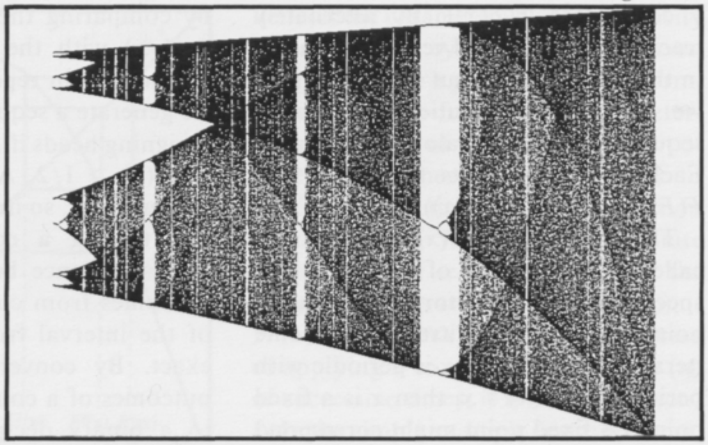


Figure 3a

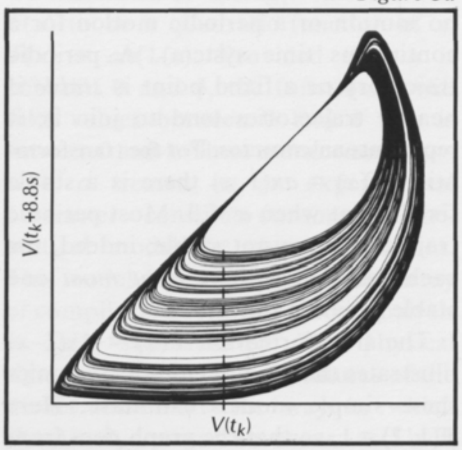


Figure 3b

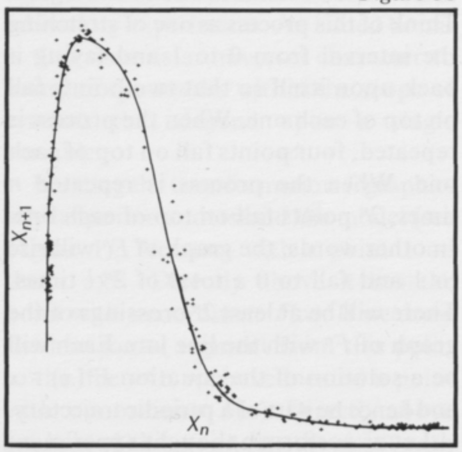


Figure 2. Attractors for the mappings of the transformations $F(x) = ax(1-x)$ as a function of a.

*In Figure 2*a *the parameter a varies from 0 to 4 along the horizontal axis, a region in which the mappings show both stable periodic orbits and chaotic attractors. Plotted along each vertical segment are many iterates of the initial point, 1/2. The first few thousand iterates are not plotted; this is to allow the point to settle onto the attractor of the transformation. Over some horizontal intervals, only a few points are plotted; here the transformation has a stable periodic orbit consisting of the points shown. Chaotic attractors occur where whole vertical segments are dark.*

*Figure 2*b *shows the right-hand portion of 2*a *with the a axis stretched.*

Figure 3. Modeling chaotic dynamical behavior of a "real" system. Data obtained in experiments on the Belousov-Zhabotinskii chemical reaction are given in a *as a two-dimensional projection of a three-dimensional phase portrait of a chaotic state observed in the reactor. The dotted line indicates a chosen coordinate normal to the trajectory.*

A one-dimensional map constructed from these data is shown in b. *The points plotted are obtained in the following way. The x coordinate is the value along the dotted line in* a *for the n^{th} intersection, and the y coordinate is the value along the dotted line for the $n+1^{st}$ intersection. Because the points fall along a curve, these data can be modeled by the iteration of a one-dimensional transformation.*

occurrence of chaotic attractors in physical systems.

PREDICTIONS OF CHAOTIC PHYSICAL BEHAVIOR

There is another aspect of our model that is of special interest in connection with physical phenomena. As we have seen, a transition from regular to chaotic behavior occurs in the model as *a* is increased from 3 to 4. Mitchell Feigenbaum of Cornell's physics department discovered in 1976 that the transition to chaos for our model possesses universal exponents that are reminiscent of those in the theory of critical phenomena. (He won the Wolf prize in physics for his theory of this phenomenon.)

The transition to chaotic behavior in the model is accomplished by a sequence of *subharmonic* or *period-doubling* bifurcations that accumulate at a parameter value $a_c = 3.57$. (The first three of these are readily visible in Figure 2*a*.) In each of these bifurcations there arises a new stable periodic orbit whose period is twice the period of the old stable periodic orbit. If one denotes by a_n the parameter values at which the *n*th of these period-doubling bifurcations appears, then the ratios

$$\frac{a_{n+1} - a_n}{a_n - a_{n-1}}$$

approach a number $\delta = 4.69920$.... This number is *universal*. It does not depend on the exact quadratic expression of our model transformation and is the same for a very large class of transformations. Indeed, every stable periodic orbit in the family of transformations $F_a(x) = ax(1-x)$ loses its stability in a period-doubling bifurcation. Every cascade of period-doubling bifurcations (of which there are infinitely many) yields the same limiting ratio δ for the distance between successive bifurcations in the cascade.

This remarkable behavior near a critical parameter value a_c (which is the accumulation value of a sequence of period-doubling bifurcations) is reflected in data from fluid experiments. Albert Libchaber, who shared the Wolf prize with Feigenbaum, provided convincing evidence locating the beginning of such a cascade of period-doubling bifurcations in convection experiments of the sort that I described. These cascades are by no means the only "route" to chaos, but when this route is followed, our simple one-dimensional model gives predictions that appear to be not only qualitatively but quantitatively correct.

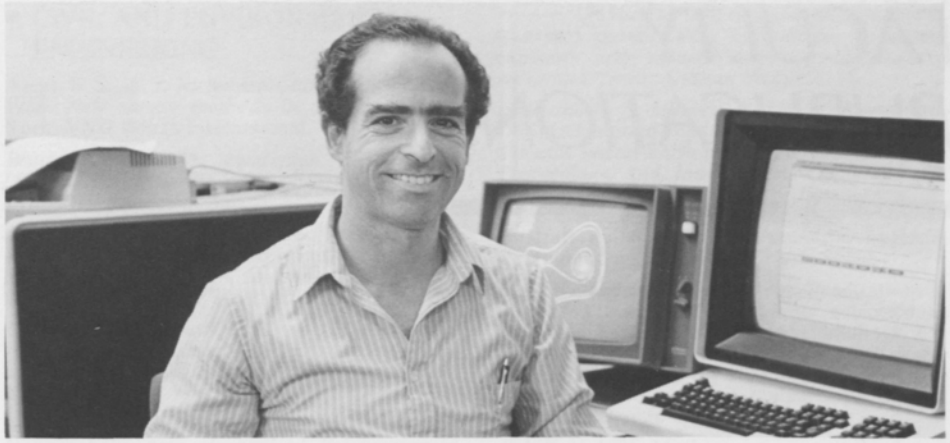


Figure 3, drawn from work of Swinney and Roux on chemical reactors, is an illustration. These researchers studied chaotic behavior in a stirred tank into which steady streams of chemicals flow and then exit. Though the reactor is fed at a steady rate, in some circumstances the reactions that take place are chaotic. Figure 3*a* shows a trajectory constructed from experimental data taken while the system was in a chaotic state. Figure 3*b* is a scatter plot of successive intersections of the experimental trajectory with the dotted line of Figure 3*a*. Denoting X_n as the coordinate of the n^{th} intersection, each point is a pair (X_n, X_{n+1}). The fact that the points lie along a curve indicates that the iteration of the one-dimensional transformation described by this curve is a good model for the dynamics of this chemical reaction.

At first glance, one-dimensional transformations seem to be highly oversimplified models of chaotic dynamical behavior. Experimental evidence like that found in the work on chemical reactors shows, however, that there are physical systems for which such a model is a good approximation. As mathematicians, scientists, and engineers are discovering together, the amazing mathematics of chaos has relevance to the real world.

John Guckenheimer, professor of mathematics and of theoretical and applied mechanics, has been a member of the Cornell faculty since 1985. As his article indicates, his research has centered on bifurcations of dynamical systems, both theoretical and applied.

He received the B.A. degree from Harvard University in 1966 and the Ph.D. from the University of California at Berkeley in 1970. He taught at the University of California at Santa Cruz from 1973 until he came to Cornell. He has had fellowships or visiting appointments at the Instituto de Matemática Pura e Aplicada in Rio de Janeiro, University of Warwick, Institute for Advanced Study, Courant Institute for Mathematical Sciences, Institut des Hautes Etudes Scientifiques, Mathematical Sciences Research Institute, and Institute Mittag-Leffler. He was a Guggenheim fellow in 1984.

FACULTY PUBLICATIONS

Current research activities at the Cornell University College of Engineering are represented by the following publications and conference papers that appeared or were presented during the three-month period June through August, 1985. (Earlier entries omitted from previous Quarterly *listings are included here with the year of publication in parentheses.) The names of Cornell personnel are in italics.*

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COMMUNICATIONS

Computer Science at Cornell

Editor: I particularly enjoyed the Autumn issue of the *Quarterly* describing the evolution of the Department of Computer Science. I thought I might add a few points concerning the origin of the department that would be of interest.

David Gries' account of the creation of the department is understandably brief, since none of the present faculty members were there during the gestation period in 1964–65. Many people were involved, and several key decisions were made that I believe contributed to the department's remarkable success.

I think the key participants were Bill Maxwell and Chris Pottle in the engineering faculty, Paul Olum and Anil Nerode (in addition to Bob Walker, of course) from the Department of Mathematics, Frank Long, then vice-president of research, and Andy Schultz, then dean of engineering. Contrasting Cornell's choices to those made at other universities, I think this group wisely avoided two major problems.

The first key decision involved the immediate full commitment to both a department and a graduate field. While many other institutions temporized with "programs", "committees", and "centers", Cornell moved boldly to establish the only organizational means that would give the new area the necessary independence of action. I particularly recall the meeting of the graduate faculty that led to the establishment of the Field of Computer Science. There was considerable discussion regarding the substance and significance of the proposed field, and concern for how long the "fad" would last. The pivotal point in the discussion was Paul Olum's assertion that if one were to rank the existing fields of the the Cornell Graduate School by "intellectual content", and insert Computer Science in the list, it would be "much closer to the top than the bottom."

The second decision concerned the location of the new department—should it be in the College of Engineering or the College of Arts and Sciences? Basically, the issue was whether the new activity should have stronger ties with Electrical Engineering or with Mathematics. The choice would, of course, be strongly self-fulfilling, and the emphasis of the new department would be influenced by its organizational position. While other institutions made this difficult choice, and narrowed the scope of their computer science activity accordingly, Cornell simply invented a two-college department. There were predictions of doom—a department that had to have agreement of two deans would be impossibly handicapped—but it turned out that most things could be done with the concurrence of *either* dean and the arrangement worked beautifully.

At some other institutions, computer science remained a captive of another department—often Electrical Engineering—and at those universities the progress in computer science has been slower than it might otherwise have been. Cornell was fortunate in that all the "parent" departments—Mathematics, Industrial Engineering and Operations Research, and Electrical Engineering—had the wisdom to understand that computer science would be better treated as an independent discipline.

Bob Walker and I have always agreed that the most important thing we did for the new department was help persuade Juris Hartmanis to become its chairman. Without deprecating the contribution of others, I would say that the history of academic computer science, and Cornell's role in particular, would be much different if Hartmanis had decided to stay in the ivory tower of GE Research.

The key person throughout the crucial early years was Andy Schultz. Much of the boldness of the original concept was his, and certainly the success of the organizational groundwork reflected his enthusiastic support. Andy was instrumental in securing the initial grant from the Sloan Foundation that accelerated the department's growth. Once the department was established, the team of Hartmanis and Schultz was extraordinarily effective. From the outset, Andy understood better than any of us just how important computer science would become to every university, and he made sure that Cornell would be in the vanguard of that development.

Overall, the development of Computer Science at Cornell is a remarkable success story, and everyone who has been associated with it can be justly proud of the result.

Richard Conway, Professor
Johnson Graduate School of Management
Cornell University

Corrections

A reader points out that the distance light travels in one nanosecond (a billionth of a second) is 11.8 inches, rather than the approximately 4 inches mentioned in the article on computer architecture in the Autumn 1985 issue (Vol 20, No. 2, page 36). Accordingly, our correspondent notes, the time required for a signal to travel the 56-inch length of a CRAY-XMP supercomputer would be about 4.7 nanoseconds, assuming that the route followed were direct (it is not known how direct the route is). We note that the essential point of the article—that the speed of light is an important physical limitation on the size of fast computers—remains valid.

In the same issue the photographs of Kenneth Birman and Sam Toueg on page 22 were transposed.

The Power of Chaos

Chaos as an integrating phenomenon may seem to be a contradiction, but in fact it is one of the most pervasive scientific subjects to emerge in recent times. Ideas and discoveries that began in theoretical mathematics soon interested not only mathematicians but physicists and other scientists, and the possibility of broad applicability was recognized. Chaotic dynamics has quickly become an interdisciplinary study.

The activity at Cornell is representative. People in mathematics, physics, and several engineering fields are engaged in research related to chaotic theory, and much of it involves collaborations that cross departmental and college lines. The authors of articles in this issue, for example, are faculty members in mathematics, theoretical and applied mechanics, and mechanical and aerospace engineering; and it is not coincidental that they discuss, from different viewpoints, some of the same theoretical and practical problems.

Others at the University who are pursuing or initiating studies in this area include Mitchell Feigenbaum, Eric Siggia, and James Sethna in physics, James Thorp and David Delchamps in electrical engineering, and Richard Rand and Andrew Ruina in mechanics. (Feigenbaum, whose work is cited in several of the *Quarterly* articles, is a MacArthur Foundation fellow and recently was co-recipient of the Wolf Foundation Prize in Physics. Sethna was one of ninety young scientists who received Sloan Research Fellowships last year. Delchamps and Ruina were among the first recipients of the Presidential Young Investigator Awards, which provide National Science Foundation research grants.)

The practical potential of study in nonlinear dynamics is suggested by current and planned research in the College of Engineering. Work in mechanical vibration and fluid mechanics is discussed in these pages. Thorp, whose specialty is large-scale electric power systems, is beginning a program, using Cornell's new supercomputer facility, that will investigate the nature of the stability domain in power systems. A question of practical concern, he says, is how long a disturbance can be allowed to exist in a power system; it is apparent that there is no clear stability boundary, and that evaluation may require techniques based on knowledge of fractal behavior. Delchamps is studying similar problems in his specialty area of control systems. Rand's group is developing ways of using computer algebra (MACSYMA) to accomplish the lengthy computation involved in predicting bifurcations in dynamical systems. A graduate student working with Ruina is relating a theory of friction to earthquake prediction, and has found that his mathematical model of sliding plates exhibits chaotic characteristics, in keeping with the unpredictability of earthquakes.

In preparing this issue of the *Quarterly* we have felt, even at the considerable remove of the editor's desk, the allure of strange attractors, the fascination of fractals, the beauty of chaos with hidden order. Surely the scientists and engineers among our readers will recognize in addition the powerful implications of chaotic theory.—GMcC

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The poem by Philip Holmes on page 12 first appeared in *Poetry Review* (London), Vol. 68, No. 4, 1979, and is included in *The Green Road*, his third collection of poems, to be published in 1986 by Anvil Press, London. An earlier and shorter version of Holmes' article appeared in the July 1985 issue of *Forefronts*, the newsletter of the Center for Theory and Simulation in Science and Engineering, Cornell University.

The photographs of Newton on page 2 and of Poincaré on page 7 are courtesy of the History of Science Collections, Cornell University Libraries.

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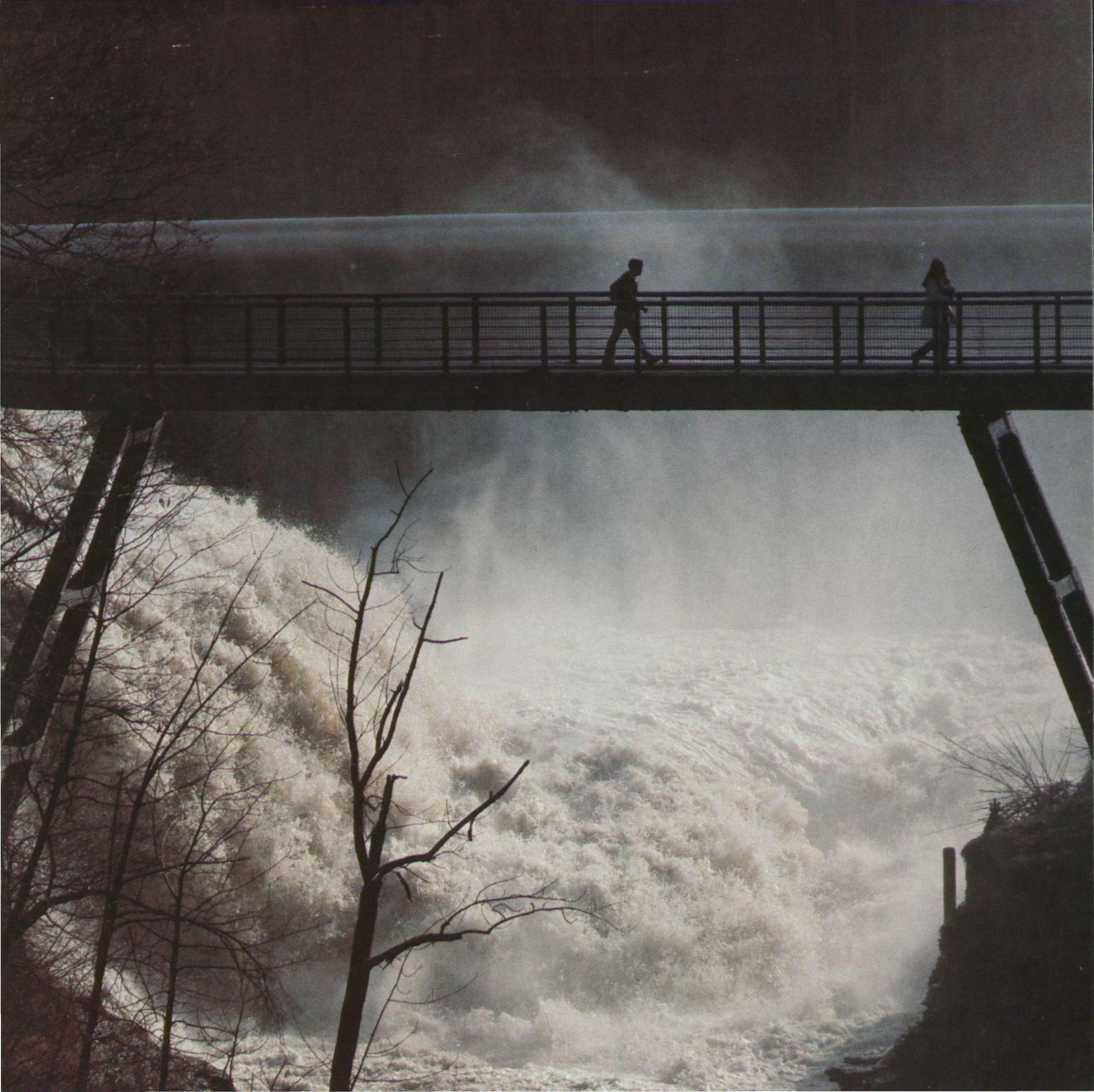

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