A GEOMETRICALLY EXACT THIN-WALLED BEAM
THEORY CONSIDERING IN-PLANE CROSS-SECTION
DISTORTION

A Dissertation
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by
Fang Yiu
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A fully nonlinear theory of a three-dimensional thin-walled beam, in arbitrary rectangular coordinates with the pole of the sectorial area at an arbitrary point and the origin of the sectorial area at an arbitrary point of the beam section, is developed to incorporate transverse shear, torsion-induced warping, and local-buckling-induced cross-section distortion. Based on a geometrically-exact description of the kinematics of deformation, this theory allows large deformation and large overall motion with a general out-of-plane warping function and a general in-plane distortion function. The present theory can exactly reduce to the classical Vlasov theory for vanishing shearing and cross-section distortion in the case of small deformation. The nonlinear weak form of the governing equations of equilibrium is constructed and the linearization of the weak form is derived. A finite element code is developed to implement this generalized thin-walled beam element. The results given by the post-buckling analysis are compared with numerical and/or experimental results to investigate the local buckling effect on the member behavior.

*Keywords:* Geometrically-exact beam, thin-walled, large deformation, large overall motion, finite rotation, finite element, computational formulation, post-buckling, local buckling
Biographical Sketch

Fang Yiu graduated in May 1993 from Shanghai Railway Institute of Technology (now Tongji University) with a Bachelor of Science in Civil Engineering with highest honors. After graduation, she worked as a Structural Engineer for the Shanghai Design Institute of Telecommunications, where she designed various types of structures, such as steel towers, steel frames and concrete structures. In September 1995, she started graduate studies in the College of Structural Engineering at Tongji University, where she graduated in June 1997 with a Master of Science degree. Her research topic was ‘Stability Analysis of Steel Space Structures with System Parametric Uncertainties’. In August 1998, she entered the Ph.D. program at Cornell University. Her major concentration was Structural Engineering, and her minor was Theoretical and Applied Mechanics. She has developed her research interests in finite element analysis and computer simulations. In November 2002, she joined Mustang Engineering, L.P. as a structural engineer working on the first compliant-piled tower project in West Africa.
This thesis is dedicated to my parents who have given so much of themselves
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Chapter 1

Introduction

This work develops a three-dimensional thin-walled beam model based on a geometrically-exact description of the kinematics of deformation.

1.1 Background

There exists an increasing need for thin-walled structures. They have a wide range of applications in several areas of engineering, such as aircraft, bridges, ships, oil rigs, storage vessels, industrial buildings, and warehouses. Thin-walled structural members offer high performance with minimum weight. The accurate prediction of their ultimate strength is of fundamental importance in the design of thin-walled structures.

Structural elements that satisfy the relations $t/b < 0.1$ and $b/L < 0.1$, where $t =$ wall thickness, $b =$ typical cross section dimension, and $L =$ length of element, are referred to as thin-walled beams.

The classical theory of thin-walled beams with an arbitrary open cross section is based on the following assumptions [1]:

\begin{align*}
\text{t/b} &\leq 0.1 \\
\text{b/L} &\leq 0.1
\end{align*}
(a) The cross section is perfectly rigid in its own plane
(b) Shear strains in the middle surface can be neglected
(c) Normals to the middle surface remain un-deformed and normal during deformation

The analysis of open cross-section thin-walled flexural members is complicated by large elastic nonlinear displacements even at early stages of loading, which is due to low torsional stiffness and the consequent large twist.

The behavior of thin-walled open cross-sections is complex and unique. Subjected to in-plane compression, the plate elements of thin-walled members may undergo elastic local buckling and exhibit considerable post-buckling strength before failure. Such buckling in the plate elements of thin-walled members is referred to as local buckling. Unlike one-dimensional structural members, plate elements do not collapse after the elastic local buckling stress is reached. Additional load can be carried by the element after local buckling, accompanied by nonlinear redistribution of stress. This phenomenon is known as post-buckling strength.

The effects of local buckling on the strength and behavior of thin-walled structural members is referred to as interaction or coupling effect. The interaction could be between local buckling and either flexural, torsional or torsional-flexural buckling. The interaction of local and overall buckling, and its effect on member stability and strength should not be ignored.

In addition, it is well-known that the cross section of thin-walled beams exhibits significant out-of-plane warping in response to torsion and, in the case of axial constraints, is subjected to normal stresses. Consideration of warping stresses and warping deformations is important in the analysis of thin-walled members.
1.2 Motivation

As thin-walled members have complicated failure modes and buckling as well as post-buckling behavior, the majority of previous computational research efforts involve significant simplifications in order to result in tractable problems.

Current design procedures [57] ignore the interaction effects of adjacent plates and are conservative in predicting the load capacity, especially in the case of minor axis bending.

The thin-walled beam theory presented in this thesis incorporates shear, torsional warping, and cross-section distortional deformation. The proposed fully kinematically nonlinear thin-walled beam theory can handle arbitrary large deformations (displacements and strains) in a three-dimensional setting. The proposed generalized thin-walled beam theory permits a rigorous post-buckling behavior analysis within the context of a general nonlinear geometrically-exact model.

1.3 Objectives

The objectives of this research are to develop and implement a generalized thin-walled beam theory, currently considering a linear elastic material. Throughout this work, emphasis is placed on the consideration of the effects of cross-section distortion, warping of the cross section and shear deformation on the global finite bending deformation of thin-walled beams. Physical experiments are conducted to better understand the behavior of cold-formed steel plain C sections. Numerical studies are performed, based on computer implementations of the proposed formulation, to verify and assess the results. The objectives of this research are further categorized in two areas as follows:
Theoretical Aspects:

• To provide insights into the behavior of thin-walled structural members

• To develop the basic equations of a thin-walled beam theory that is more general than the classical theory

Experimental Aspects:

• To experimentally investigate the behavior of cold-formed steel channels

• To verify and assess the proposed thin-walled beam theory

1.4 Scope

In order to achieve the objectives stated in Section 1.3, this work is organized as described below:

Chapter 2 focuses on the theoretical aspects of the proposed thin-walled beam theory. Section 2.1 presents a review of previous theoretical research. Section 2.2 describes the kinematics of thin-walled beams. Sections 2.3 to 2.6 contain derivations of the deformation gradient, stress resultants and conjugate strains through the expression for mechanical power and finally equilibrium and constitutive equations. Section 2.7 presents the computational implementation of the proposed thin-walled beam theory, which includes the weak form (virtual work expression) of the governing equations, consistent linearization, and the details of the finite element implementation.

Chapter 3 describes the experimental part of this research. Section 3.1 reviews previous experimental studies. Section 3.2 presents geometrical imperfection mea-
urements and the details of the beam and beam-column test setup. Experimental results are presented and evaluated in Section 3.3.

Numerical examples for beams, columns, and beam-columns are discussed in Chapter 4. Examples 4.1 and 4.2 use the proposed thin-walled beam theory to analyze large-displacement three-dimensional beams excluding cross-section distortion. Examples 4.3 to 4.5 consider both warping and cross-section distortion and compare the results obtained by the proposed theory to numerical and/or experimental results available in the open literature or obtained experimentally in Chapter 3.

In Chapter 5, conclusions are drawn about the findings of this investigation, with some recommendations about possible directions for future work.

Appendix A contains the nomenclature used throughout the dissertation. Appendix B contains implementation details of the computer code generated for this research as well as instructions for its use.
Chapter 2

Theoretical Developments

In this chapter, we consider the basic kinematics and equilibrium equations of the geometrically-exact thin-walled beam. The component form of the stress power and constitutive equations relating stress resultants to kinematic degree of freedoms are derived. The weak form of the equations of equilibrium is constructed, including its linearization. Linearization of the weak form plays an important role in the finite element implementation.

2.1 Introduction

A great deal of research, both theoretical and experimental, has been devoted to the behavior of thin-walled members. We review here the theoretical developments in thin-walled beam theory. The literature review of the experimental work is deferred to Chapter 4.

Investigations into the stability behavior of straight thin-walled members with open cross sections have been carried out extensively since the early works of Vlasov [1] and Timoshenko and Gere [3]. A number of investigators [4, 5, 6, 7]
have presented finite element models for the analysis of thin-walled beams. For
brevity, only those works considering the interaction of local and overall buckling
of thin-walled members are reviewed.

Early work involving this interaction dealt with only specific types of cross sec-
tions. Rajasekaran and Murray [8] studied wide-flange cross sections. Wang and
Pao [9] developed a finite element model for channel sections. Apart from these early
works, a number of studies were carried out on arbitrary cross sections. Toneff [10]
extended the thin-walled beam element by Osterrieder [11]. Toneff developed a fi-
nite element model by choosing plate nodes as the degrees of freedom in individual
plate elements to represent local buckling. Shear deformations and Poisson’s effects
were ignored in this theory.

The finite strip method (FSM) is a specialization of the finite element method,
which uses a number of strips consisting of segments of the cross section extended in
the longitudinal direction and solved using small deflection plate theory. Originally
developed by Cheung [92], the FSM was extended by Hancock [94, 95, 96] to pre-
dict the behavior of hot-rolled steel members and cold-formed members using the
computer program BFINST. Schafer [93] implemented classical FSM in the software
package CUFSM (www.ce.jhu.edu/bschafer) and used it to explore elastic buckling
behavior.

The generalized beam theory (GBT) has received substantial attention in recent
years. First developed by Schardt [12], the principle of the GBT and its applica-
tions in cold-formed steel members are outlined in Davis et al. [13, 14, 15, 16, 17].
The GBT approach considers that the strips, into which a thin-walled member is
divided, can be analyzed using beam theory. Compared with FSM, this leads to
fewer degrees of freedom, although this is offset by the fact that more strips are gen-
eraly required for a given level of accuracy than would be the case with FSM. The GBT approach was further developed to consider composite (orthotropic) materials of thin-walled members in geometrically first-order (linear) and second-order (linear stability) analyses [18, 19].

Based on classical beam theory, Hikosaka et al. [20] analyzed open polygonal cross sections by subdividing a cross section into longitudinal beam strips with one degree of freedom at each node to specify the distortion of the cross section. This method made it possible to predict elastic buckling loads with a smaller number of degrees of freedom than FSM.

Rhodes [21] combined plate and beam theory in dealing with the out-of-plane and in-plane deformations of the walls of a cross section, and had features common to both the FSM approach and the GBT approach. However, this method inherited some shortcomings from FSM, in that it can only consider simply supported boundary conditions.

Some researchers resorted to FSM or to a finite element method in conjunction with empirical models to account for the effect of local buckling on overall buckling of thin-walled members. Lau and Hancock [22] used the spline finite strip method and allowed for boundary conditions other than simply supported ends to perform buckling analysis of thin flat-walled structures of finite length subjected to longitudinal compression and bending, transverse compression as well as shear. Basu and Akhtar [23] developed a p-version 3D finite-element buckling model to study the interaction of local and overall buckling modes including torsional-flexural buckling. This model also accounts for residual stresses.

The work reviewed so far mainly focuses on buckling analysis. Kwon [24] developed a nonlinear elastic analysis using a linear combination of $B_3$-spline functions
for studying the post-buckling behavior of thin-walled sections. However, the results were inconclusive due to numerical instabilities. Djugash [25] developed an elastic nonlinear and instability analysis of thin-walled members experiencing biaxial bending, warping and large deformation. The analysis includes the effects of geometric nonlinearity, out-of-plane bending and the consequent torsion, secondary effects of load applied away from the shear center, large displacements and rotations in three-dimensional space and initial imperfections. However, the analysis does not consider distortion of the cross section. Silvestre and Camotim [26] formulated a geometrically nonlinear GBT approach of isotropic thin-walled members to handle various types of loading and arbitrary initial geometrical imperfections. However, the interaction or coupling effect is not considered in the post-buckling analysis.

It is seen that no unified work is reported which permits a rigorous post-buckling behavior analysis within the context of a general nonlinear geometrically-exact model. A novel approach developed by Simo [27], systematically considers bifurcation and instability phenomena of rods and plates in nonlinear theories. This work has no restriction imposed on the magnitude of the displacement field. Recent research has extended this approach to a wide range of structures, such as dynamics of flexible satellites [28], beam model incorporating shear and torsion-warping deformation [29], multilayer beams [30, 31, 32], sandwich beams [33, 34, 35, 36], sandwich shells [37, 38, 39], multilayer shells [40], tapered I-beams [41], and plates [42]. However, no work considers in-plane distortion of thin-walled beams. The objective of this research is to extend this approach of geometrically nonlinear analysis to thin-walled structural members, with particular emphasis on the effects of cross-section distortion.
2.2 Kinematic Description of the Thin-walled Beam

Shown in Figure 2.1 is the profile of a thin-walled beam. The material configuration is defined by means of the material basis vectors $E_1, E_2, E_3$ with coordinates denoted by $X_1, X_2, X_3$. For simplicity we consider a straight beam of constant cross section with length $L$ along the $E_3$ axis. The spatial configuration is defined by the basis vectors $e_1, e_2, e_3$. For convenience, the basis vectors $E_I, e_I$ are chosen to be identical.

Let $O$ be the origin of the Lagrangian and Eulerian coordinates. $O'$ is the projection of the origin of the coordinate system $O$ onto the plane of the undeformed cross section. The kinematics of the thin-walled beam are described by a vector field consisting of the cross-section translation, a finite rotation about some point $P$, the warping displacement along $t_3$, and the distortion of the cross section relative to point $P$. The position vector of a material point in the deformed configuration initially located at $X = X_I E_I$, denoted by $\varphi$, can be expressed as follows

$$\varphi(X_1, X_2, X_3) = \varphi_p(X_3) + Q(X_3)[\bar{X} - P + U] + q(X_1, X_2, X_3) t_3(X_3). \quad (2.1)$$

Here, $\bar{X} = X_\alpha E_\alpha$, $U = U_\alpha(X_1, X_2, X_3) E_\alpha$, $P = X_{P\alpha} E_\alpha$ are 3-vectors whose third component is zero, e.g. $\bar{X}$ is the projection of a position vector on the cross-section plane; $t_I$ is a Cartesian triad which has $t_I(X_3) = Q(X_3) E_I$; $Q$ is an orthogonal two-point tensor; local buckling, characterized by distortion of the cross section [2], is described by the vector $U$. $q$ is the magnitude of the out-of-plane warping displacement.

\footnote{The convention that Greek indices take values of 1,2 and Latin indices take values of 1,2,3 is adopted throughout and summation is implied by repeated indices.}
Figure 2.1: Profile of a thin-walled beam.
Note that \( U \) is defined only on the points that are on the cross section. Point \( P \) is defined such that \( U(P, X_3) = 0 \), and \( q(P, X_3) = 0 \), i.e. a point on the cross section with zero distortional and zero warping displacements. Then for \( \bar{X} = P \) Equation 2.1 gives \( \varphi = \varphi_p \). These assumptions are made without loss of generality for the following reason. Suppose \( P \) is selected arbitrarily on the cross section, and suppose these assumptions are initially violated. Take \((X_1, X_2) = P\) in Equation 2.1 and define

\[
\tilde{\varphi}_p(X_3) = \varphi_p(X_3) + Q(X_3)U(P, X_3) + q(P, X_3)t_3(X_3).
\]

In addition define

\[
\tilde{U}(X_1, X_2, X_3) = U(X_1, X_2, X_3) - U(P, X_3)
\]

\[
\tilde{q}(X_1, X_2, X_3) = q(X_1, X_2, X_3) - q(P, X_3).
\]

Then it is clear that

\[
\varphi_p + Q(\bar{X} - P + U) + qt_3 = \tilde{\varphi}_p + Q(\bar{X} - P + \tilde{U}) + \tilde{q}t_3
\]

and the above assumptions are satisfied for \( \tilde{U} \) and \( \tilde{q} \), i.e. \( \tilde{U}(P, X_3) = 0 \) and \( \tilde{q}(P, X_3) = 0 \). Thus if the assumptions are initially violated, \( \varphi_p, q, \) and \( U \) may be redefined so that the assumptions hold and the possible motions are the same.

Without loss of generality, and in order to make the problem well posed, we provide an additional constraint (besides setting \( U(P, X_3) = 0 \)) on the cross section as follows. We select another arbitrary point \( S \) on the cross section and a unit vector \( h \) in the \((E_1, E_2)\) plane and require that \( h \cdot U(S_1, S_2, X_3) = 0 \) for all \( X_3 \) (Figure 2.2). In practice, we usually choose \( S \) at an end point of the cross section. We usually choose \( h \) to be either \( E_1 \) or \( E_2 \). For reasons explained below, \( h \) should not be chosen parallel to \( S - P \).
To see why this assumption on \( S \) is without loss of generality, suppose it is violated, i.e., suppose for some \( X_3 \) that \( h \cdot U(S_1, S_2, X_3) \neq 0 \). We argue that it is possible to redefine \( U \) satisfying the assumption, along with a new \( Q \), and yielding the same deformation. Find a matrix \( \tilde{Q} \) such that

\[
 h \cdot [\tilde{Q}^T Q(X_3)(S - P + U(S_1, S_2, X_3)) - S + P] = 0 \quad (2.2)
\]

and such that \( Q(X_3) \) and \( \tilde{Q} \) have the same third column. The procedure for finding such a \( \tilde{Q} \) is as follows. The assumption about the third column implies that \( \tilde{Q}^T Q(X_3) \) has the form

\[
\begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

for some angle \( \theta \). If we write \( h = (\cos \alpha)E_1 + (\sin \alpha)E_2 \) for some \( \alpha \) and \( S - P + U(S_1, S_2, X_3) \) as \( \|S - P + U(S_1, S_2, X_3)\|(\cos \omega)E_1 + (\sin \omega)E_2 \) for some \( \omega \) and apply the angle-sum formulas to Equation 2.2, we see that Equation 2.2 is solved by choosing \( \theta \) so that

\[
\cos(\alpha + \theta - \omega) = \frac{h \cdot (S - P)}{\|S - P + U(S_1, S_2, X_3)\|} \quad (2.3)
\]

There is always a way to choose \( \theta \) to solve this equation provided that the numerator does not exceed the denominator in absolute value. This will be the case as long as \( h \) is not parallel nor nearly parallel to \( S - P \) and the displacement \( U \) is not too large.

Once \( \theta \) is found, \( \tilde{Q} \) is determined via

\[
\tilde{Q} = Q \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Finally, define

\[ \tilde{U}(X_1, X_2, X_3) = Q^T Q(\bar{X} - P + U(X_1, X_2, X_3)) - \bar{X} + P \]

It is easy to see that \( \tilde{U} \) lies in the \( E_1, E_2 \) plane, satisfies \( \tilde{U}(X_{P1}, X_{P2}, X_3) = 0 \), that \( h \cdot \tilde{U}(S_1, S_2, X_3) = 0 \) and that

\[ Q(\bar{X} - P + U) = \tilde{Q}(\bar{X} - P + \tilde{U}) \]

for all \( \bar{X} \) in the cross section. The reason that the assumption \( h \cdot U(S_1, S_2, X_3) = 0 \) is necessary for well-posedness is apparent from the above argument for why it is made without loss of generality. If the assumption were not made, then there would be an infinite family of \((\tilde{Q}, \tilde{U})\) obtained by trying many different choices of \( h \) all yielding the same deformation of the cross section.

The local-buckling-induced distortion displacements of the thin-walled beam, denoted in global coordinates by the vector \( U(X_1, X_2, X_3) \), are defined by the deformations of the cross section considered as a 2-D plane frame as shown in Figure 2.2. For each segment \( e = 1, 2, \ldots, n_{el} \) of the cross section, where \( n_{el} \) is the number of beam segments, the local displacements in segment-local coordinates \( 0 \leq \xi \leq l_e, -\frac{t_e}{2} \leq \eta \leq \frac{t_e}{2} \), where \( l_e, \ t_e \) are the length and thickness of the segment respectively, can be expressed as

\[ u_1 = u_1^0(\xi) - \eta u_2^0(\xi), \quad (2.4) \]
\[ u_2 = u_2^0(\xi). \] (2.5)

It can further be assumed that \( u_2^0 = g_\alpha(\xi) \cdot \phi(X_3) \), when suitable interpolation functions \( g_\alpha \) are chosen (see section 3.5). Here \( g_\alpha \) and \( \phi \) are r-vectors; \( r \) is the number of degrees of freedom (d.o.f.) in a beam segment, typically \( r = 6 \) (translations and rotations at both ends), and \( \phi(X_3) \) are element (segment) degrees of freedom in segment-local coordinates \((\xi, \eta, X_3)\). The distortion in segment-local coordinates \((u_1, u_2)^T\) can be transformed into beam-global coordinates \((X_1, X_2, X_3)\) by \((U_1, U_2)^e = \hat{G}^e \Phi^e(X_3)\), where \( \hat{G}^e \) is a \( 2 \times r \) matrix and \( \Phi^e \) is the r-vector of element degrees of freedom in global coordinates. For notational convenience, we write

\[(U_1, U_2, 0)^e = G^e(X_1, X_2)\Phi(X_3)\] (2.6)

where \( G^e \) is a \( 3 \times s \) matrix and \( \Phi \) is the s-vector of degrees of freedom of the cross section in global coordinates. See section 3.5 for details.

The torsion-induced warping displacement of the thin-walled beam is described by a general warping function

\[ q(X_1, X_2, X_3) = f(X_1, X_2)p(X_3) \] (2.7)

in the deformed configuration, where \( p(X_3) \) is an unknown scalar and \( f(X_1, X_2) \) is the warping function given by the law of sectorial areas [1], i.e. its value at some point \( Q \) of the cross section equals twice the area of the sector enclosed by \( MQ, MA \) and \( AQ \), as shown in Figure 2.3. The pole, or sectorial center \( M \) of the sectorial area is placed at an arbitrary point on the cross-sectional plane and the origin \( A \) of the sectorial area is chosen at an arbitrary point of the cross section, at which by definition \( f = 0 \).
Figure 2.3: Cross-section warping

Figure 2.4: Warping function of an I section

Figure 2.5: Warping function of a plain C section

\[
\begin{align*}
\text{W Section:} \\
\text{Plain C Section:} \\
\text{where, } a &= \frac{1}{2 + \frac{h}{3b}}
\end{align*}
\]
Consistent with our requirements for the center of rotation $P$ that $U(P, X_3) = 0$ and $q(P, X_3) = 0$, we take the origin $A$ to be the point $P$. It is shown in Vlasov [1, pg.53] that the principal sectorial center, i.e. the sectorial center for which $\int fX_1dA = 0$ and $\int fX_2dA = 0$, coincides with the shear center. In the proposed thin-walled beam theory, the sectorial pole is chosen to be the principal sectorial center (shear center). The sectorial origin, and hence the point $P$, is selected such that $\int fdA = 0$. The warping function $f(X_1, X_2)$ of typical cross-sections are shown in Figures 2.4, 2.5 and 2.6, where the dimensions refer to midline dimensions.

[REMARK 1] Shear deformation is considered in the proposed thin-walled beam model. The third vector of the moving basis $t_I$, $t_3$ is normal to the cross section. Define

$$\Xi_3 = \frac{\partial \varphi_P}{\partial X_3}$$

the tangent to the trajectory of point $P$. It is obvious (Figure 2.7) that $t_3$ and $\Xi_3$ are different. The difference disappears if shear deformation is not taken into account.
[REMARK 2] Let $\varphi_b = QU = U_\alpha t_\alpha (X_3)$ stand for the deformation contributed by distortion, and let $\varphi_B$ be the deformation associated with beam bending incorporating shear and torsional-warping. Hence, Equation 2.1 can be re-written

$$\varphi(X_1, X_2, X_3) = \varphi_B + \varphi_b, \quad (2.8)$$

where

$$\varphi_B = \varphi_P + Q(X_\alpha - X_{P\alpha})E_\alpha + f p t_3, \quad (2.9)$$

$$\varphi_b = QU_\alpha E_\alpha. \quad (2.10)$$

When $Q = I$, Equation 2.5 represents members with local buckling only. When $U_\alpha = 0$, Equation 2.5 represents undistorted beam members with global instability. Without local buckling deformations, the proposed theory is then reduced to the Simo and Vu-Quoc beam model [29], which in the case of small deformation is reduced to the Vlasov theory [1] when transverse shear deformation vanishes.
[REMARK 3] Consistent with Vlasov’s theory [1] and engineering practice [44], segments of the cross section with constant thickness are assumed line elements when computing geometric properties. The accuracy of this assumption for a given section depends on the thickness and the cross-section configuration [44]. In what follows we develop the theory for cross sections consisting of straight segments. A curved cross section can be approximated by a piece-wise linearized polygon. The number of polygonal sides depends on the desired accuracy.

It follows from Equation 2.1 and Equation 2.6 that the thin-walled beam configuration space can be uniquely defined as

\[ C = \{(\varphi_p, Q, p, \Phi) : [0, L] \rightarrow R^3 \times SO(3) \times R \times R^s\} \]

in which, \( SO(3) \) is the special orthogonal group, i.e. \( 3 \times 3 \) orthogonal matrices with determinant equal to 1; \( s \) is the number of total d.o.f. of the cross-section that describe distortional deformation. In other words, knowing \( \varphi_p, Q, p, \Phi \), we can compute the beam kinematics through the equations derived hereto.

### 2.3 Deformation Gradient

Differentiating the deformation map \( \varphi \) with respect to the spatial coordinates \( X_1, X_2, X_3 \), we obtain the following equations:

\[
\frac{\partial \varphi}{\partial X_1} = \left[t_1(X_3) + \frac{\partial f}{\partial X_1} p t_3\right] + \frac{\partial U_\alpha}{\partial X_1} t_\alpha = \frac{\partial \varphi_B}{\partial X_1} + \frac{\partial \varphi_b}{\partial X_1}, \tag{2.11}
\]

\[
\frac{\partial \varphi}{\partial X_2} = \left[t_2(X_3) + \frac{\partial f}{\partial X_2} p t_3\right] + \frac{\partial U_\alpha}{\partial X_2} t_\alpha = \frac{\partial \varphi_B}{\partial X_2} + \frac{\partial \varphi_b}{\partial X_2}, \tag{2.12}
\]

where \( \varphi_B \) and \( \varphi_b \) are defined in Equations 2.9 and 2.10.
Define the infinitesimal rotation matrix $\mathbf{\omega}$ with axial vector $\mathbf{\omega}$:

$$
\mathbf{\omega} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
$$

Using $\frac{\partial \mathbf{t}_a}{\partial X_3} = \mathbf{\omega} \times \mathbf{t}_a$, $\frac{\partial \mathbf{t}_a}{\partial X_3} = \mathbf{\omega} \times \mathbf{t}_3$,

$$
\frac{\partial \varphi}{\partial X_3} = \frac{\partial \varphi_P}{\partial X_3} + \mathbf{\omega} \times [(X_a + U_a - X_{Pa}) \mathbf{t}_a] + \frac{\partial U_a}{\partial X_3} \mathbf{t}_a + f \frac{\partial p}{\partial X_3} \mathbf{t}_3 + f p \mathbf{\omega} \times \mathbf{t}_3
$$

$$
= \frac{\partial \varphi_B}{\partial X_3} + \frac{\partial \varphi_b}{\partial X_3},
$$

(2.13)

where

$$
\frac{\partial \varphi_B}{\partial X_3} = \frac{\partial \varphi_P}{\partial X_3} + \mathbf{\omega} \times (\varphi_B - \varphi_P) + f \frac{\partial p}{\partial X_3} \mathbf{t}_3,
$$

$$
\frac{\partial \varphi_b}{\partial X_3} = \mathbf{\omega} \times \varphi_b + \frac{\partial U_a}{\partial X_3} \mathbf{t}_a.
$$

Using $\mathbf{t}_a \otimes \mathbf{E}_a = \mathbf{Q} - \mathbf{t}_3 \otimes \mathbf{E}_3$, and noting that $\mathbf{Q}(\mathbf{a} \times \mathbf{b}) = \mathbf{Qa} \times \mathbf{Qb}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $\mathbf{Q}$ orthogonal matrices, the deformation gradient can be expressed as

$$
\mathbf{F} = \varphi_I \otimes \mathbf{E}_I
$$

$$
= \varphi_B, I \otimes \mathbf{E}_I + \varphi_b, I \otimes \mathbf{E}_I
$$

$$
= \mathbf{F}_B + \mathbf{F}_b,
$$

(2.14)

where

$$
\mathbf{F}_B = \mathbf{Q} \left\{ \mathbf{I} + f_{a,p} \mathbf{E}_3 \otimes \mathbf{E}_a + \left[ \mathbf{Q}^T (\varphi_P' - \mathbf{t}_3) + \mathbf{Q}^T (\varphi_B - \varphi_P) + f p' \mathbf{E}_3 \right] \otimes \mathbf{E}_3 \right\},
$$

$$
\mathbf{F}_b = \mathbf{Q} \left\{ U_{a,\beta} \mathbf{E}_a \otimes \mathbf{E}_\beta + \left[ \mathbf{Q}^T \varphi_b + U_{a,\beta} \mathbf{E}_a \right] \otimes \mathbf{E}_3 \right\}.
$$

In the above $\mathbf{\Omega} = \mathbf{Q}^T \mathbf{\omega}$ and $\otimes$ denotes the tensor product. Following the notation in [29], we define $\mathbf{\Gamma} = \mathbf{Q}^T (\varphi_P' - \mathbf{t}_3)$, the physical meaning of which will become
apparent in the next section. Since \( \frac{\partial Q}{\partial t} = \bar{\omega}Q \), and
\[
\frac{\partial [Q^T(\varphi_B - \varphi_P)]}{\partial t} = \frac{\partial}{\partial t}[(X_\alpha - X_{P\alpha})E_\alpha + fp E_3] = f\dot{p} E_3,
\]
\[
\frac{\partial [Q^T \varphi_b]}{\partial t} = \dot{U}_\alpha E_\alpha,
\]
it follows
\[
\dot{F} = \dot{F}_B + \dot{F}_b,
\]
where
\[
\dot{F}_B = \bar{\omega}F_B + Q\left\{ f_{,\alpha}\dot{p} E_3 \otimes E_\alpha + \left[ \bar{\Gamma} + \Omega \times Q^T(\varphi_B - \varphi_P) + \Omega \times \dot{\varphi} E_3 + f\dot{p}' E_3 \right] \otimes E_3 \right\},
\]
\[
\dot{F}_b = \bar{\omega}F_b + Q\left\{ \dot{U}_{\alpha,\beta} E_\alpha \otimes E_\beta + \left[ \bar{\Omega} \times Q^T \varphi_b + \Omega \times \dot{U}_\alpha E_\alpha + \dot{U}_{\alpha,3} E_\alpha \right] \otimes E_3 \right\}.
\]
Here we have used \( S(u \otimes v) = (Su \otimes v) \).

[REMARK 4] If we consider geometric imperfections, then the line of \( P \) is an arbitrary curve, and the basis \( E_I \) becomes a function of \( X_3 \).

**Proof of** \( \bar{\omega}t_\alpha = \omega \times t_\alpha, \bar{\omega}t_3 = \omega \times t_3 \)

Infinitesimal rotation matrix \( \bar{\omega} \) with its axial vector \( \omega \):
\[
\bar{\omega} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\]
\[
Q = t_I \otimes E_I
\]
\[
\frac{\partial t_\alpha}{\partial X_3} = \frac{\partial}{\partial X_3} (QE_\alpha) = \bar{\omega}QE_\alpha = \bar{\omega}t_\alpha = \omega \times t_\alpha
\]
\[
\Omega = Q^T \omega.
\]
Similarly,
\[
\bar{\omega}t_3 = \omega \times t_3.
\]
2.4 Mechanical Power. Stress resultants and stress couples. Conjugate strains

Let \( P \) denote the first Piola-Kirchhoff stress tensor,

\[
P = T_\alpha \otimes E_\alpha + T_3 \otimes E_3,
\]

where, \( T_\alpha = PE_\alpha \) and \( T_3 = PE_3 \) are traction vectors per unit reference area acting on the deformed faces that have \( E_\alpha \) and \( E_3 \) as their normal in the undeformed configuration, respectively. The stress power in terms of the first Piola-Kirchhoff stress tensor is given by

\[
P = \int P : \dot{F} dX_1 dX_2 dX_3.
\]

Note that \( P : \dot{\omega}F = tr[P(\dot{\omega}F)^T] = tr[P F^T \dot{\omega}^T] = tr[J \sigma \dot{\omega}^T] = 0 \), where \( \sigma = \frac{PF^T}{J} \) is the (symmetric) Cauchy stress tensor, \( J = detF \) and \( A : B = tr(AB^T) = 0 \) whenever \( A \) is a symmetric and \( B \) is a skew-symmetric matrix. It then follows that

\[
P = \int (T_\gamma \otimes E_\gamma + T_3 \otimes E_3) : (\dot{\Phi}_B + \dot{\Phi}_b) dA dX_3
\]

\[
= \int \left\{ T_3 \cdot Q \left\{ \dot{\Gamma} + [\dot{\Omega} \times Q^T (\varphi_B - \varphi_P)] + \Omega \times \dot{f} \dot{p} E_3 + \dot{f} \dot{p}' E_3 \right\}
+ T_\alpha \cdot Q f_{\alpha,\hat{p}} E_3 + T_3 \cdot Q (\dot{\Omega} \times Q^T \varphi_b + \dot{\Omega} \times \dot{U}_{\alpha,3} E_\alpha + \Omega \times \dot{U}_{\alpha} E_\alpha)
+ T_\beta \cdot Q \dot{\Omega} \dot{U}_{\alpha,\beta} E_\alpha \right\} dA dX_3
\]

where we have used the formulas

\[
(u \otimes v) : (x \otimes y) = (u \cdot x)(v \cdot y)
\]

\[
u \cdot (v \times w) = v \cdot (w \times u),
\]

by which,

\[
T_\gamma \otimes E_\gamma : Q f_{\alpha} E_3 \otimes E_\alpha = (T_\gamma \cdot Q f_{\alpha} E_3)(E_\gamma \cdot E_\alpha) = T_\gamma \cdot QE_3 \delta_{\alpha \gamma} f_{\alpha} = T_\gamma \cdot QE_3 f_{\gamma}
\]
\[ T_\gamma \otimes E_\gamma : (\cdot) \otimes E_3 = 0 \]

\[ T_3 \otimes E_3 : (\cdot) \otimes E_3 = T_3 \cdot (\cdot) \]

\[ T_\gamma \otimes E_\gamma : Q \mathbf{U},_\beta \otimes E_\beta = (T_\gamma \cdot Q \mathbf{U},_\beta)(E_\gamma \cdot E_\beta) = T_\gamma \cdot Q \mathbf{U},_\gamma = T_\gamma \cdot Q \mathbf{U},_\gamma, \]

Using \( \varphi = \varphi_B + \varphi_b \), we rewrite the mechanical power as

\[
P = \int \left\{ (Q^T T_3) \cdot \dot{\mathbf{T}} + \left\{ Q^T (\varphi - \varphi_P) \times T_3 \right\} \cdot \dot{\mathbf{\Omega}} + \left\{ t_3 \cdot (f_\alpha T_\alpha + f T_3 \times \omega) \right\} \dot{p} + (t_3 \cdot f T_3) \dot{p}' \right\} dX_3 dA + P_{\text{distort}}. \]

From Equation 2.6,

\[ U_\alpha = G^e_\alpha \cdot \Phi, \quad (2.17) \]

where \( G^e_\alpha \) are s-vectors such that

\[
G^e_{3x3} = \begin{bmatrix} G^e_1^T \\ G^e_2^T \\ 0 \end{bmatrix}, \quad (2.18)
\]

hence the distortion related terms can be written as follows:

\[
P_{\text{distort}} = \left\{ \int Q^T T_\beta \cdot \dot{\mathbf{U}},_\beta + \int Q^T T_3 \cdot \dot{\mathbf{U}},_3 + \int Q^T (T_3 \times \omega) \cdot \dot{\mathbf{U}} \right\} dAdX_3
\]

\[
= \sum_e \int \left\{ (t_\alpha \cdot T_\beta) G^e_{\alpha,\beta} + t_\alpha \cdot (T_3 \times \omega) G^e_\alpha \right\} \cdot \dot{\Phi} + \left\{ (t_\alpha \cdot T_3) G^e_\alpha \right\} \cdot \dot{\Phi}' \right\} dX_3 dA.
\]

We now define stress resultants \( N, M, N_f, M_f, N_u, M_u \) such that the mechanical power can be written as

\[
P = \int \left\{ N \cdot \dot{\mathbf{T}} + M \cdot \dot{\mathbf{\Omega}} + N_f \dot{p} + M_f \dot{p}' + N_u \cdot \dot{\Phi} + M_u \cdot \dot{\Phi}' \right\} dX_3. \quad (2.19)
\]
i.e. \(\mathbf{N}, \mathbf{M}, N_f, M_f, \mathbf{N}_u, \mathbf{M}_u\) are conjugate to the ‘strain’ measures \(\mathbf{\Gamma}, \mathbf{\Omega}, p, p', \mathbf{\Phi}, \mathbf{\Phi}'\), which are functions of \(X_3\) and time \(t\).

Therefore, the stress resultants for the cross section have the following expressions:

\[\mathbf{N} = Q^T \int T_3 dA\]
\[\mathbf{M} = Q^T \int (\varphi - \varphi_P) \times T_3 dA\]
\[N_f = t_3 \cdot \int (f_{,x}T_3 + fT_3 \times \omega) dA\]
\[M_f = t_3 \cdot \int fT_3 dA\]
\[\mathbf{N}_u = \sum_e \int \left\{ (t_3 \cdot T_{\beta}) G^{e}_{\alpha,\beta} + [t_3 \cdot (T_3 \times \omega)] G^{e}_{\alpha} \right\} dA\]
\[\mathbf{M}_u = \sum_e \int (t_3 \cdot T_3) G^{e}_{\alpha} dA\]

(REMARK 6) The above definitions have clear physical interpretations in linear theory. \(\mathbf{N}\) refers to shear and axial stresses; \(\mathbf{M}\) is bending about \(X_1, X_2\) and twisting about \(X_3\); \(N_f\) is warping (non-uniform) moment; \(M_f\) is bi-moment. \(\mathbf{N}_u\) is the beam-generalized bending and twisting induced by distortion, and \(\mathbf{M}_u\) is the variation of these generalized forces along the beam axial direction.

### 2.5 Equilibrium Equations

Since \(DIV \mathbf{P} = \mathbf{T}_{l,l}\), the local equilibrium equation \(DIV \mathbf{P} + \rho_o \mathbf{B} = \rho_o \ddot{\varphi}\) is expressed as

\[
\mathbf{T}_{l,l} + \rho_o \mathbf{B} = \rho_o \ddot{\varphi}
\]

(2.20)

where \(\mathbf{B}\) denotes the body force per unit reference volume acting on the beam and \(\rho_o\) is the density in the reference configuration.

Let us define the applied force, torque, and bi-moment per unit of reference length
as follows.

$$\mathbf{n} = \int \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 \mathbf{B} \right) dA = \int T_\Gamma \gamma_\Gamma d\Gamma + \int \rho_0 \mathbf{B} dA,$$

$$\mathbf{m} = \int (\varphi - \varphi_P) \times \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 \mathbf{B} \right) dA = \int (\varphi - \varphi_P) \times T_\Gamma \gamma_\Gamma d\Gamma + \int (\varphi - \varphi_P) \times \rho_0 \mathbf{B} dA,$$

$$\bar{M}_f = t_3 \cdot \left\lvert \int f T_\Gamma \gamma_\Gamma d\Gamma + \int f \rho_0 \mathbf{B} dA \right\rvert,$$

$$\mathbf{M}_u = \sum_e \int \rho_0(t_\alpha \cdot \mathbf{B}) G^e_\alpha dA + \int (t_\alpha \cdot T_\Gamma) G^e_\alpha \gamma_\Gamma d\Gamma.$$

The local governing equations for the thin-walled beam model are obtained from Equation 2.20 and are found to be

$$\frac{\partial \mathbf{n}}{\partial X_3} + \mathbf{n} = \int \rho_0 \ddot{\varphi} dA \quad (2.21)$$

$$\frac{\partial \mathbf{m}}{\partial X_3} + \frac{\partial \varphi_P}{\partial X_3} \times \mathbf{n} + \mathbf{m} = \int \rho_0 (\varphi - \varphi_P) \times \ddot{\varphi} dA \quad (2.22)$$

$$\frac{\partial M_f}{\partial X_3} - N_f + \bar{M}_f = t_3 \cdot \int \rho_0 \ddot{\varphi} dA \quad (2.23)$$

$$\frac{\partial \mathbf{M}_u}{\partial X_3} - \mathbf{N}_u + \mathbf{M}_u = \sum_e \int \rho_0(t_\alpha \cdot \ddot{\varphi}) G^e_\alpha dA \quad (2.24)$$

where $\mathbf{n} = QN = \int T_3 dA$ and $\mathbf{m} = QM = \int (\varphi - \varphi_P) \times T_3 dA$ are the spatial descriptions of vectors $\mathbf{N}$ and $\mathbf{M}$.

In the static case,

$$\frac{\partial \mathbf{n}}{\partial X_3} + \mathbf{n} = 0$$

$$\frac{\partial \mathbf{m}}{\partial X_3} + \frac{\partial \varphi_P}{\partial X_3} \times \mathbf{n} + \mathbf{m} = 0$$

$$\frac{\partial M_f}{\partial X_3} - N_f + \bar{M}_f = 0$$

$$\frac{\partial \mathbf{M}_u}{\partial X_3} - \mathbf{N}_u + \mathbf{M}_u = 0.$$
Proof of Equilibrium Equations

From the local equilibrium equations

\[ T_{1,1} + \rho_0 B = \rho_0 \ddot{\varphi} \]

and the definition \( n = QN = \int T_3 \, dA \), it follows

\[ \frac{\partial n}{\partial X_3} = \int_A \frac{\partial T_3}{\partial X_3} \, dA = -\int_A \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 B \right) \, dA + \int_A \rho_0 \dot{\varphi} \, dA. \]

With the definition distributed applied force

\[ \bar{n} = \int_A \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 B \right) dA = \int T_\Gamma \gamma_\Gamma d\Gamma + \int \rho_0 B dA \]

this may be written as

\[ \frac{\partial n}{\partial X_3} + \bar{n} = \int_A \rho_0 \ddot{\varphi} \, dA. \]

Here, \( \gamma_\Gamma \) denotes the components of unit vector normal to the boundary \( \Gamma \). In the static case, \( \frac{\partial n}{\partial X_3} + \bar{n} = 0 \). Similarly, from the definition

\[ m = \int_\Omega (\varphi - \varphi_P) \times T_3 dA, \]

\[ \frac{\partial m}{\partial X_3} = \int_\Omega \frac{\partial (\varphi - \varphi_P)}{\partial X_3} \times T_3 dA + \int_\Omega (\varphi - \varphi_P) \times \frac{\partial T_3}{\partial X_3} dA \]

\[ = \int_\Omega \frac{\partial \varphi}{\partial X_3} \times T_3 dA - \int_\Omega \frac{\partial \varphi_P}{\partial X_3} \times T_3 dA + \int_\Omega (\varphi - \varphi_P) \times [-\left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 B \right) + \rho_0 \ddot{\varphi}] dA \]

\[ = \int_\Omega \frac{\partial \varphi}{\partial X_3} \times T_3 dA - \int_\Omega \frac{\partial \varphi_P}{\partial X_3} \times n - \int_\Omega (\varphi - \varphi_P) \times \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 B \right) dA \]

\[ + \int_\Omega (\varphi - \varphi_P) \times \rho_0 \ddot{\varphi} dA. \]

Define the applied distributed moment

\[ m = \int_A (\varphi - \varphi_P) \times \left( \frac{\partial T_\alpha}{\partial X_\alpha} + \rho_0 B \right) dA = \int_A (\varphi - \varphi_P) \times T_\Gamma \gamma_\Gamma d\Gamma + \int_A (\varphi - \varphi_P) \times \rho_0 B dA. \]
Thus,
\[
\frac{\partial m}{\partial X_3} + \frac{\partial \varphi_P}{\partial X_3} \times n + \dot{m} = \int \rho_0 (\varphi - \varphi_P) \times \dot{\varphi} dA,
\]
where we note that \( \int \frac{\partial \varphi}{\partial X_3} \times T_3 dA = 0 \), since \( \frac{\partial \varphi}{\partial X_3} = F_E_3 \), \( T_3 = P E_3 \), and \( F P^T = P F^T \), we therefore have \( \frac{\partial \varphi}{\partial X_3} \times T_3 = F_E_3 \times P E_3 = 0 \).

In the static case, \( \frac{\partial m}{\partial X_3} + \frac{\partial \varphi_P}{\partial X_3} \times n + \dot{m} = 0 \).

In the case of bi-moment, we have
\[
M_f = t_3 \cdot \int f T_3 dA.
\]
Recalling that \( \frac{\partial t_3}{\partial X_3} = \omega \times t_3 \) in Appendix B.1,
\[
\frac{\partial M_f}{\partial X_3} = \omega \times t_3 \cdot \int f T_3 dA + t_3 \cdot \int f \frac{\partial T_3}{\partial X_3} dA
\]
\[
= t_3 \cdot \int (f \alpha T_\alpha + f T_3 \times \omega) dA + t_3 \cdot \int f [- \frac{\partial t_\alpha}{\partial X_\alpha} + \rho_0 B + \rho_0 \dot{\varphi}] dA.
\]
Define the applied bi-moment:
\[
\bar{M}_f = t_3 \cdot \int f [- \frac{\partial t_\alpha}{\partial X_\alpha} + \rho_0 B] dA = t_3 \cdot \left[ \int f T_\Gamma \gamma \gamma d\Gamma + \int f \rho_0 B dA \right]
\]
\[
\frac{\partial M_f}{\partial X_3} - N_f + \bar{M}_f = t_3 \cdot \int \rho_0 \dot{\varphi} dA.
\]
In the static case,
\[
\frac{\partial M_f}{\partial X_3} - N_f + \bar{M}_f = 0.
\]
For the beam-generalized bending and twisting,
\[
M_a = \int (t_\alpha \cdot T_3) G_\alpha dA,
\]
\[
\frac{\partial M_u}{\partial X_3} = \int (\omega \times t_\alpha \cdot T_3) G_\alpha dA + \int (t_\alpha \cdot T_{3,3}) G_\alpha dA \\
= \int [t_\alpha \cdot (T_3 \times \omega)] G_\alpha dA + \int t_\alpha \cdot [- (T_{\beta,\beta} + \rho_0 B) + \rho_0 \ddot{\varphi}] G_\alpha dA \\
= \int [t_\alpha \cdot (T_3 \times \omega)] G_\alpha dA - \int (t_\alpha \cdot T_{\beta,\beta}) G_\alpha dL - \int (t_\alpha \cdot T_\beta) G_{\alpha,\beta} dA \\
- \int \rho_0 (t_\alpha \cdot B) G_\alpha dA + \int \rho_0 (t_\alpha \cdot \ddot{\varphi}) G_\alpha dA \\
= N_u - \int (t_\alpha \cdot T_{\beta,\beta}) G_\alpha dL - \int \rho_0 (t_\alpha \cdot B) G_\alpha dA + \int \rho_0 (t_\alpha \cdot \ddot{\varphi}) G_\alpha dA.
\]

Let
\[
\bar{M}_u = \int \rho_0 (t_\alpha \cdot B) G_\alpha dA + \int (t_\alpha \cdot T_\Gamma) G_\alpha \gamma_\Gamma d\Gamma.
\]

Therefore,
\[
\frac{\partial M_u}{\partial X_3} - N_u + \bar{M}_u = \int \rho_0 (t_\alpha \cdot \ddot{\varphi}) G_\alpha dA.
\]

In the static case,
\[
\frac{\partial M_u}{\partial X_3} - N_u + M_u = 0.
\]

### 2.6 Constitutive Equations

We presume that the rotation \(Q\) is the dominant part of the deformation gradient. Let
\[
F = Q + \varepsilon \Lambda
\]
i.e. assume that
\[
F - Q = \varepsilon \Lambda
\]
\[
= Q \left\{ f_{\alpha P} E_3 \otimes E_\alpha + [\Gamma + \Omega \times Q^T (\varphi - \varphi_P) + f P_3 E_3] \otimes E_3 \\
+ U_{\alpha,3} E_\alpha \otimes E_3 + U_{\alpha,3} E_\alpha \otimes E_3 \right\}
\]
is \( O(\varepsilon) \), which means \( \| F - Q \| = O(\varepsilon) \), where \( O(\varepsilon) \) is defined by \( \frac{O(\varepsilon)}{\varepsilon} \) bounded above by a constant when \( \varepsilon \to 0 \). This is equivalent to requiring that \( H = Q^T F - I = O(\varepsilon) \) as in Simo [29], which is an infinitesimal strain. In particular, we require that \( \| U_{\alpha, \beta} \| = O(\varepsilon) \). We assume that the leading term of the second Piola-Kirchhoff stress \( S \) is selected linearly to the leading term of the Lagrange strain \( E \). Note that

\[
E = \frac{1}{2} \left[ F^T F - I \right] = \varepsilon (Q^T \Lambda)^s + O(\varepsilon^2)
\]

\[
= (Q^T F)^s - I + O(\varepsilon^2) = H^s + O(\varepsilon^2),
\]

and

\[
S = JF^{-1} \sigma F^{-T} = (1 + O(\varepsilon))(Q^T + O(\varepsilon))\sigma (Q + O(\varepsilon))
\]

\[
= Q^T \sigma Q + O(\varepsilon)O(\| \sigma \|) = \Sigma + O(\varepsilon^2),
\]

where we have used

\[
J = \text{det} F = \text{det} Q + O(\varepsilon) = 1 + O(\varepsilon),
\]

\[
F = Q + O(\varepsilon),
\]

\[
F^{-1} = Q^T + O(\varepsilon).
\]

In the above equations, \( \Sigma = Q^T \sigma Q \), \( \| \sigma \| = O(\varepsilon) \), and \( H^s \) is the symmetric part of \( H \). We postulate a linear isotropic relationship between the second Piola-Kirchhoff stress tensor \( S \) and the Lagrange strain tensor \( E \) leading to

\[
\Sigma_{ij} = \lambda H^s_{\rho\theta} \delta_{ij} + 2G H^s_{ij} = [\lambda \delta_{ij} \delta_{\rho\theta} + 2G \delta_{i\rho} \delta_{j\theta}] H^s_{\rho\theta}
\]

where \( \Sigma = \Sigma_{ij} E_i \otimes E_j \); \( H^s = H^s_{\rho\theta} E_\rho \otimes E_\theta \); \( G \) and \( \lambda \) denote Lamé’s constants.

By using \( Q^T P = \Sigma \), \( P = T_I \otimes E_I \), and \( (u \otimes v)w = (w \cdot v)u \), we get \( Q^T T_\alpha = \Sigma_{I\alpha} E_I \) and \( Q^T T_3 = \Sigma_{I3} E_I \). We have \( \Gamma_\alpha = \Gamma \cdot E_\alpha \) for the conjugate strain measure.
Omitting higher order $O(\varepsilon^2)$ terms, the following is found from $H^s_{ij} = E_i \cdot (H^s E_j)$:

\[
2H^s_{\alpha\beta} = U_{\alpha,\beta} + U_{\beta,\alpha} \tag{2.25}
\]

\[
2H^s_{\alpha 3} = \Gamma_{\alpha} + \varepsilon_{\alpha 3\beta} \Omega_{3}(X_\beta - X_{P\beta}) + U_{\alpha,3} + f_{\alpha P} \tag{2.26}
\]

\[
H^s_{33} = \Gamma_{3} + \varepsilon_{3\alpha\beta} \Omega_{4}(X_\alpha - X_{P\alpha}) + f_{P3} \tag{2.27}
\]

where we have used the formulas

\[
A_{\rho\theta} = E_{\rho} \cdot (AE_{\theta})
\]

\[
2(u \otimes v)^s w = [u \otimes v + v \otimes u]w.
\]

\[
(u \otimes v) S = (u \otimes S^T v)
\]

\[
(u \otimes v) w = (w \cdot v) u,
\]

\[
u \cdot S v = v \cdot S^T u.
\]

We introduce the plane stress assumptions $\Sigma^{l}_{22} = \Sigma^{l}_{21} = \Sigma^{l}_{23} = 0$ throughout the thickness of each plate, where $\Sigma^{l}$ is $\Sigma$ expressed in local coordinates. It is a reasonable assumption for a thin-walled beam member with each plate subjected to in-plane forces. Let $T$ be the transformation matrix defined as

\[
\begin{bmatrix}
\cos(\xi, X_1) & \cos(\eta, X_1) & 0 \\
\cos(\xi, X_2) & \cos(\eta, X_2) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\Sigma^{l}_{11} = \frac{E}{1 - \nu^2}(TH^s T^T)_{11} + \frac{E\nu}{1 - \nu^2}(TH^s T^T)_{33}
\]

\[
\Sigma^{l}_{33} = \frac{E\nu}{1 - \nu^2}(TH^s T^T)_{11} + \frac{E}{1 - \nu^2}(TH^s T^T)_{33}
\]

\[
\Sigma^{l}_{\alpha 3} = 2G(TH^s T^T)_{\alpha 3}
\]
The plane stress assumption is depicted in Figure 2.8. The stresses in beam-global coordinates are therefore obtained by \( \Sigma = T^T \Sigma' T \).

![Figure 2.8: Plane stress assumption](image)

**Stress Resultants:**

The following expressions are for the beam cross section. Note that non-linear terms are omitted in these expressions. We take the centroidal coordinates for cross sections, i.e. \( \int X_\beta \, dA = 0 \), \( \int X_1 X_2 \, dA = 0 \), \( \int dA = A \), and the principal sectorial coordinates for warping, i.e. \( \int f \, dA = 0 \), \( \int X_\alpha f \, dA = 0 \). We have used \( \Sigma_{33} = \Sigma'_{33} \) in derivations.

\[
N = Q^T n = Q^T \int T_3 dA = \int \Sigma_{33} E_3 dA 
\]

\[
M = Q^T m = Q^T \int (\varphi - \varphi_P) \times T_3 dA
\]

\[
= \int [(X_\alpha - X_{P\alpha} + U_\alpha) E_\alpha + f p E_3] \\
\times \left\{ (T^T \Sigma' T)_{33} E_3 + \frac{E \nu}{1 - \nu^2} (T \Sigma^* T)_{11} + \frac{E}{1 - \nu^2} (T \Sigma^* T)_{33} \right\} dA
\]
\[ N_f = t_3 \cdot \int (f_\alpha T_\alpha + f T_3 \times \omega) dA \] (2.30)
\[ = \int f_\alpha \Sigma_{3\alpha} dA = \int f_\alpha (T^T \Sigma^l T)_{\alpha 3} dA \]

\[ M_f = t_3 \cdot \int f T_3 dA = \int f \Sigma_{33} dA \] (2.31)
\[ = \int f \left[ \frac{E}{1 - \nu^2} (T^h T)^{11} + \frac{E}{1 - \nu^2} (T^h T)^{33} \right] dA \]

\[ N_u = \sum_e \int (t_\alpha \cdot T_3) G^e_{\alpha,3} dA + \sum_e \int [t_\alpha \cdot (T_3 \times \omega)] G^e_\alpha dA \] (2.32)
\[ = \sum_e \int (t_\alpha \cdot Q \Sigma_{I3} E_I) G^e_{\alpha,3} dA = \sum_e \int \Sigma_{\alpha 3} G^e_\alpha dA \]
\[ = \sum_e \int (T^T \Sigma^l T)_{\alpha 3} G^e_\alpha dA \]

\[ M_u = \sum_e \int (t_\alpha \cdot T_3) G^e_\alpha dA \] (2.33)
\[ = \sum_e \int (t_\alpha \cdot Q \Sigma_{I3} E_I) G^e_\alpha dA = \sum_e \int \Sigma_{\alpha 3} G^e_\alpha dA \]
\[ = \sum_e \int (T^T \Sigma^l T)_{\alpha 3} G^e_\alpha dA \]

Let \( \bar{T} = \begin{bmatrix} \cos(\xi, X_1) & \cos(\eta, X_1) \\ \cos(\xi, X_2) & \cos(\eta, X_2) \end{bmatrix} \) where \( (\xi, \eta), (X_1, X_2) \) are segment-local and beam-global axes respectively, \( \bar{I} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Define \( K^e = (\bar{T}^T \bar{I} \bar{T})_{\alpha 3} G^e_{\alpha,3} \), \( S^e_\alpha = (\bar{T}^T \bar{I} \bar{T})_{\alpha 3} G^e_{\alpha} \), \( C^e_\alpha = (\bar{T}^T \bar{I} \bar{T})_{\alpha 3} f, \beta \), and \( D^e_{\alpha 3} = (\bar{T}^T \bar{I} \bar{T})_{\alpha 3} \), where \( G^e_\alpha \) are interpolation functions for the distortional displacements, given in Equations 2.17 and 2.18, and \( f \) is the warping function introduced in Equation 2.7. The components of stress resultants \( N, M, N_f, M_f, N_u, M_u \) in global coordinates are found to be:

\[ N_1 = \sum_e G \int D_{1\alpha}^e dA \Gamma_\alpha + \sum_e G \int S_{11}^e T dA \Phi' + \sum_e G \int C_{11}^e dA \Omega \]
\[ + \sum_e G \int \varepsilon_{\alpha 3}(X_\alpha - X_P) D_{1\beta}^e dA \Omega_3 \]

\[ N_2 = \sum_e G \int D_{2\alpha}^e dA \Gamma_\alpha + \sum_e G \int S_{22}^e T dA \Phi' + \sum_e G \int C_{22}^e dA \Omega \]
\[ + \sum_e G \int \varepsilon_{\alpha 3}(X_\alpha - X_P) D_{2\beta}^e dA \Omega_3 \]
\[ N_3 = \frac{E}{1-\nu^2}A_3 + \sum_e \frac{E\nu}{1-\nu^2} \int \mathbf{K}^e dA_3 + \frac{E}{1-\nu^2} \varepsilon_{\alpha3} \mathbf{X}_{P_3}^\alpha A_3 \Omega_3 \]

\[ M_1 = \frac{E}{1-\nu^2} \int \varepsilon_{\beta\alpha3}(X_2 - X_{P_2})(X_\alpha - X_{P_3}) dA_\beta \]
\[ + \sum_e \int \frac{E\nu}{1-\nu^2}(X_2 - X_{P_2}) \mathbf{K}^e dA_\Phi - \frac{E}{1-\nu^2} X_{P_2} A_3 \Gamma_3 \]

\[ M_2 = \frac{E}{1-\nu^2} \int \varepsilon_{\alpha3\beta}(X_1 - X_{P_1})(X_\alpha - X_{P_3}) dA_\beta \]
\[ - \sum_e \int \frac{E\nu}{1-\nu^2}(X_1 - X_{P_1}) \mathbf{K}^e dA_\Phi + \frac{E}{1-\nu^2} X_{P_1} A_3 \Gamma_3 \]

\[ M_3 = \sum_e G \left[ \int \varepsilon_{\alpha3\gamma\beta}(X_\alpha - X_{P_\alpha})(X_\gamma - X_{P_\beta}) D_{\beta\gamma}^e dA \right] \Omega_3 \]
\[ + \sum_e G \int \varepsilon_{\alpha3\beta}(X_\alpha - X_{P_\alpha}) C_{\beta}^e dA p \]
\[ + \sum_e G \int \int \varepsilon_{\alpha3\beta}(X_\alpha - X_{P_\alpha}) S_{\beta}^e dA \Phi' \]
\[ + \sum_e G \int \varepsilon_{\alpha3}(X_\alpha - X_{P_\alpha}) D_{\beta\gamma}^e dA \Gamma_\beta \]

\[ N_f = \sum_e G \left[ \int C_{\alpha}^e dA_\Gamma_\alpha + \int G(f_\alpha S_{\alpha}^T) dA \Phi' \right] \]
\[ + \sum_e \int G(C_{\alpha}^e f_\alpha) dA p + \sum_e G \left[ \int \varepsilon_{\alpha3\beta}(X_\alpha - X_{P_\alpha}) C_{\beta}^e dA_\Omega_3 \right] \]

\[ M_f = \frac{E}{1-\nu^2} \int f^2 dA p' + \sum_e \frac{E\nu}{1-\nu^2} \int f \mathbf{K}^e dA_\Phi \]

\[ N_u = \sum_e \frac{E\nu}{1-\nu^2} \int \mathbf{K}^e dA_3 + \sum_e \frac{E\nu}{1-\nu^2} \int \varepsilon_{\beta\alpha3}(X_\alpha - X_{P_\alpha}) \mathbf{K}^e dA_\Omega_3 \]
\[ + \sum_e \frac{E\nu}{1-\nu^2} \int f \mathbf{K}^e dA p' + \sum_e \frac{E}{1-\nu^2} \int \mathbf{K}^e \mathbf{K}^e dA_\Phi \]

\[ M_u = \sum_e G \left[ \int f_\alpha S_{\alpha}^e dA p + \int G(\varepsilon_{\alpha3}(X_\alpha - X_{P_\alpha}) S_{\beta}^e) dA_\Omega_3 \right] \]
\[ + G \sum_e \int S_{\alpha}^e dA_\Gamma_\alpha + \sum_e G \left[ \int S_{\alpha}^e G_{\alpha}^T dA \Phi' \right] \]
The constitutive equations between stress resultants \( \mathbf{N}, \mathbf{M}, \mathbf{N}_f, \mathbf{M}_f, \mathbf{N}_u, \mathbf{M}_u \) and the conjugate ‘strain’ measures \( \Gamma, \Omega, p, p', \Phi, \Phi' \) may be written in the form:

\[
\begin{bmatrix}
\mathbf{N} \\
\mathbf{M} \\
\mathbf{N}_f \\
\mathbf{M}_f \\
\mathbf{N}_u \\
\mathbf{M}_u
\end{bmatrix} = \begin{bmatrix}
\mathbf{C}_n & \mathbf{C}_{nm} & \mathbf{C}_{nw1} & \mathbf{O} & \mathbf{C}_{nl1} & \mathbf{C}_{nl2} \\
\mathbf{C}_{nm}^T & \mathbf{C}_m & \mathbf{C}_{mw1} & \mathbf{O} & \mathbf{C}_{ml1} & \mathbf{C}_{ml2} \\
\mathbf{C}_{nw1}^T & \mathbf{C}_{mw1}^T & \mathbf{C}_w & \mathbf{O} & \mathbf{O} & \mathbf{C}_{w12} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C}_{w2} & \mathbf{C}_{w21} & \mathbf{O} \\
\mathbf{C}_{nl1}^T & \mathbf{C}_{ml1}^T & \mathbf{C}_{w11} & \mathbf{O} & \mathbf{O} & \mathbf{C}_{l2} \\
\mathbf{C}_{nl2}^T & \mathbf{C}_{ml2}^T & \mathbf{C}_{w12}^T & \mathbf{O} & \mathbf{O} & \mathbf{C}_l \\
\end{bmatrix} \begin{bmatrix}
\Gamma \\
\Omega \\
p \\
p' \\
\Phi \\
\Phi'
\end{bmatrix}
\]

(2.34)

The sub-matrices \( \mathbf{C}_{nl1}, \mathbf{C}_{nl2}, \mathbf{C}_{ml1}, \mathbf{C}_{ml2}, \mathbf{C}_{w1l1}, \mathbf{C}_{w2l1}, \mathbf{C}_l, \) and \( \mathbf{C}_{l2} \) represent distortion effects which can not be neglected when modeling local instability.

The sub-matrices in Equation 2.34 are as follows. \( \mathbf{C}_n, \mathbf{C}_{nm}, \mathbf{C}_m \) represent global bending contributions, and are given by

\[
\mathbf{C}_n = \begin{bmatrix}
\sum_e G \int D_{11}^e dA & \sum_e G \int D_{12}^e dA & 0 \\
\sum_e G \int D_{21}^e dA & \sum_e G \int D_{22}^e dA & 0 \\
0 & 0 & \frac{EA}{1-\nu^2}
\end{bmatrix}
\]

in which \( \frac{EA}{1-\nu^2} \) becomes the usual axial stiffness when \( \nu = 0; \)

\[
\mathbf{C}_{nm} = \begin{bmatrix}
0 & 0 & \sum_e G \int -(X_2 - X_{P2})D_{11}^e + (X_1 - X_{P1})D_{12}^e dA \\
0 & 0 & \sum_e G \int -(X_2 - X_{P2})D_{21}^e + (X_1 - X_{P1})D_{22}^e dA \\
-\frac{E}{1-\nu^2}X_{P2}A & \frac{E}{1-\nu^2}X_{P1}A & 0
\end{bmatrix}
\]

in which \( X_{P1}, X_{P2} \) are the coordinates of point \( P \)which has zero warping and zero distortion displacements;

\[
\mathbf{C}_m = \begin{bmatrix}
\frac{E}{1-\nu^2} \int (X_2 - X_{\rho2})^2 dA & -\frac{E}{1-\nu^2}X_{P1}X_{P2}A & 0 \\
-\frac{E}{1-\nu^2}X_{P1}X_{P2}A & \frac{E}{1-\nu^2} \int (X_1 - X_{\rho1})^2 dA & 0 \\
0 & 0 & \sum_e G \int \frac{(X_1 - X_{\rho1})^2 + (X_2 - X_{\rho2})^2}{2} dA \\
\end{bmatrix}
\]

\[
-\sum_e G \int (X_1 - X_{\rho1})(X_2 - X_{\rho2})D_{21}^e dA \\
-\sum_e G \int (X_1 - X_{\rho1})(X_2 - X_{\rho2})D_{12}^e dA
\]
in which \( \frac{E}{(1-\nu^2)} \int X_2^2 dA \) and \( \frac{E}{(1-\nu^2)} \int X_1^2 dA \) become the usual principal bending stiﬀnesses with respect to \( t_1 \) and \( t_2 \) when \( \nu = 0 \).

\( C_{nw1}, C_{nw1} \) represent global bending and warping contributions and are given by

\[
C_{nw1} = \begin{bmatrix}
\sum_e G \int C_1^e dA \\
\sum_e G \int C_2^e dA \\
0 
\end{bmatrix},
\]

\[
C_{nw1} = \begin{bmatrix}
0 \\
0 \\
\sum_e G \int (X_1 - X_{p1}) C_2^e dA - \sum_e G \int (X_2 - X_{p2}) C_1^e dA 
\end{bmatrix}.
\]

\( C_{w1}, C_{w2} \) represent warping contributions and are given by

\[
C_{w1} = \sum_e G \int C_\alpha^e f_\alpha dA;
\]

\[
C_{w2} = \frac{E}{(1-\nu^2)} \int f^2 dA, \text{ where } C_{w2} \text{ becomes the usual warping constant when } \nu = 0.
\]

\( C_{nl1}, C_{nl2}, C_{ml1}, C_{ml2} \) represent global bending and distortion contributions and are given by

\[
C_{nl1} = \begin{bmatrix}
0 \\
0 \\
\sum_e \frac{E\nu}{(1-\nu^2)} \int K^{eT} dA
\end{bmatrix}, \quad C_{nl2} = \begin{bmatrix}
\sum_e G \int S_1^{T} dA \\
\sum_e G \int S_2^{T} dA \\
0
\end{bmatrix},
\]

\[
C_{ml1} = \begin{bmatrix}
\sum_e \frac{E\nu}{(1-\nu^2)} \int (X_2 - X_{p2}) K^{eT} dA \\
\sum_e -\frac{E\nu}{(1-\nu^2)} \int (X_1 - X_{p1}) K^{eT} dA \\
0
\end{bmatrix},
\]

\[
C_{ml2} = \begin{bmatrix}
0 \\
0 \\
\sum_e \int [(X_1 - X_{p1}) S_2^{T} - (X_2 - X_{p2}) S_1^{T}] dA
\end{bmatrix}.
\]

\( C_{wl2}, C_{w2l1} \) represent warping and distortion contributions and are given by

\[
C_{wl2} = \sum_e G \int (f_1 S_1^{T} + f_2 S_2^{T}) dA;
\]
\[ C_{w2l1} = \sum e \frac{E\nu}{1-\nu^2} \int fK^eT dA. \]

Finally \( C_{l1}, C_{l2} \) represent distortion contributions and are given by

\[ C_{l1} = \sum e \frac{E}{1-\nu^2} \int K^eK^eT dA; \]

\[ C_{l2} = \sum e G \int (S_1^eG_1^eT + S_2^eG_2^eT) dA. \]

### 2.7 Computer Implementation

The implementation of the proposed beam theory in the context of a finite element method is presented in this section.

#### 2.7.1 Weak Form of the Equilibrium Equations

**Admissible Variations. Tangent Space**

Consider a given configuration of the thin-walled beam \( \varphi = \{ \varphi_p, Q, p, \Phi \} \). The space of the deformation map is defined by \( C := \{ \mathbb{R}^3 \times \text{SO}(3) \times R \times R^s \} \), where \( s \) refers to the number of degrees of freedom describing cross-sectional distortion.

The orthogonal matrix \( Q \) is uniquely represented through the Rodrigues formula 2.35 [27] by a set of three parameters \( \theta \), referred to as the ‘rotation vector’,

\[ Q = \cos(\theta)I + \frac{\sin(\theta)}{\theta} \hat{\theta} + \frac{1 - \cos(\theta)}{\theta^2} \hat{\theta}^2, \quad (2.35) \]

where \( \hat{\theta} \) denotes a skew-symmetric tensor of which \( \theta \) is the axial vector, and \( \hat{\theta} \) and \( \theta \) are given by

\[ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}, \]

\[ \hat{\theta}h = \theta \times h \quad \forall h \in \mathbb{R}^3 \quad (2.36) \]
and \( \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \) is the magnitude of the rotation vector.

The admissible variations to the deformation map \( \varphi \) are denoted by \( \delta \varphi := \{\delta \varphi_P, \delta \vartheta, \delta p, \delta \Phi \} \). The space of admissible variations, denoted by \( T_\Lambda C \), i.e. the tangent space to the current deformation \( \varphi \), is defined as \( T_\Lambda C := \{R^3 \times R^3 \times R \times R^8\} \).

**Weak Form of the Equilibrium Equations**

Let \( \delta \varphi \) be an arbitrary variation in the tangent space at the configuration \( \varphi \). By multiplying the balance laws of Equations 2.21 to 2.24 by \( \delta \varphi \) and integrating by parts with all variations vanishing at the Dirichlet boundary, the weak form for a thin-walled beam is obtained. The computational problem then is to find \( \varphi \), such that \( G(\varphi, \delta \varphi) - G_{ext}(\delta \varphi) = 0 \) for all admissible variations \( \delta \varphi \), where \( G(\varphi, \delta \varphi) \) is the weak form of the stiffness operator contributed by forces/couples,

\[
G(\varphi, \delta \varphi) = -\int_0^L [\mathbf{n} \cdot \left( \frac{d \delta \varphi_P}{dX_3} - \delta \vartheta \times \frac{d \varphi_P}{dX_3} \right) + \mathbf{m} \cdot \frac{d \delta \vartheta}{dX_3} + N_f \delta p] + M_f \delta p + N_u \cdot \delta \Phi + M_u \cdot \delta \Phi] dX_3 \tag{2.37}
\]

and \( G_{ext}(\delta \varphi) \) is the weak form of the applied forces/couples,

\[
G_{ext}(\delta \varphi) = \int_0^L [\bar{\mathbf{n}} \cdot \delta \varphi_P + \bar{\mathbf{m}} \cdot \delta \vartheta + \bar{M}_f \delta p + \bar{N}_u \cdot \delta \Phi] dX_3. \tag{2.38}
\]

**2.7.2 Linearization of the Weak Form**

To construct the linearization of the weak form at a given configuration \( \varphi \) in the direction of an incremental field \( \Delta \varphi := \{\Delta \varphi_P, \Delta \vartheta, \Delta p, \Delta \Phi \} \in T_\Lambda \varphi \), we consider a perturbed configuration \( \varphi_\epsilon \) such that \( \varphi_\epsilon |_{\epsilon = 0} = \varphi \), \( \frac{d \varphi_\epsilon}{d \epsilon} |_{\epsilon = 0} = \Delta \varphi \).

**Linearized Strain Measures**

Let \( \mathbf{E}_\varphi = (\mathbf{\Gamma}, \mathbf{\omega}, \mathbf{p}, \mathbf{p}', \mathbf{\Phi}, \mathbf{\Phi}') \) be the generalized strain measures defined in Section 2.4. The linearization of \( \mathbf{E}_\varphi \) at a given configuration \( \varphi \) in the direction of an
incremental field $\Delta \varphi$, is

$$DE_\varphi(\varphi) \cdot \Delta \varphi = \frac{d}{d\epsilon} E_\varphi(\varphi,\epsilon)|_{\epsilon=0}$$

$$\equiv \Pi B \Delta \varphi,$$

where $B \Delta \varphi = \{(\frac{d\varphi_p}{dx_3} - \Delta \theta \times \frac{d\varphi_P}{dx_3}), \frac{d\theta}{dx_3}, \Delta p, \frac{d\Delta p}{dx_3}, \Delta \Phi, \frac{d\Delta \Phi}{dx_3}\}^T$ and

$$\Pi = \begin{bmatrix}
Q_{3\times 3} \\
I_{2\times 2} \\
I_{s\times s}
\end{bmatrix}.$$

### Linearized Weak Form

The linearization of the weak form $G(\varphi, \delta \varphi)$ at a configuration $\varphi$ in the direction of an incremental field $\Delta \varphi$ is $DG(\varphi, \delta \varphi) \cdot \Delta \varphi = \frac{d}{d\epsilon}|_{\epsilon=0} G(\varphi, \delta \varphi) = DG_M(\varphi, \delta \varphi) \cdot \Delta \varphi + DG_G(\varphi, \delta \varphi) \cdot \Delta \varphi$, where $DG_M(\varphi, \delta \varphi) \cdot \Delta \varphi$ is the material part, and $DG_G(\varphi, \delta \varphi) \cdot \Delta \varphi$ is the geometric part. We follow procedures similar to the derivation in [29] to obtain

$$DG_M(\varphi, \delta \varphi) \cdot \Delta \varphi = \int B \delta \varphi \cdot c B \Delta \varphi dx_3,$$

$$DG_G(\varphi, \delta \varphi) \cdot \Delta \varphi = \int L \delta \varphi \cdot b L \Delta \varphi dx_3,$$

in which $b = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & [-n \times] \\
0_{3\times 3} & 0_{3\times 3} & [-m \times] \\
[n \times] & 0_{3\times 3} & [n \otimes \varphi_p' - (\varphi_p' \cdot n)I] \end{bmatrix}$, $L \Delta \varphi = \{\frac{d\Delta \varphi_p}{dx_3}, \frac{d\Delta \theta}{dx_3}, \Delta \theta\}^T$ and

$$c = \Pi C \Pi^T$$,

where $C$ is defined in Equation 2.34.

### 2.7.3 Distortional Displacements

Distortion of the cross-section is defined by a combination of translation, bending and stretching of each segment. Details on the selection of the prescribed distortion functions in $G^e$ (Equation 2.6) are as follows.
In segment-local coordinates, the distortional components of a beam segment \( pq \) with length \( l_e \) and thickness \( t_e \) are defined by six degrees of freedom \( \phi = \{ u_p, v_p, \theta_p, u_q, v_q, \theta_q \}^T \) shown in Figure 2.9 and the rate of change of these degrees of freedom in the \( X_3 \) beam longitudinal direction. The distortional displacements of each segment can be expressed in terms of \( u_1 \) and \( u_2 \), the displacements in the \( \xi \) and \( \eta \) directions such that \( u_1(0) = u_p, u_1(l_e) = u_q, u_2(0) = v_p, u_2(l_e) = v_q \), when suitable shape functions are chosen. We write \( u_1 = g_1(\xi, \eta) \cdot \phi, u_2 = g_2(\xi, \eta) \cdot \phi. \)

\[ u_1 = u_1^0(\xi) - \eta u_2^0(\xi), \]

\[ u_2 = u_2^0(\xi). \]

The kinematic assumptions of the classical Euler-Bernoulli beam theory are

We employ standard polynomial interpolations (cubic Legendre polynomials)

\[ u_1^0 = \begin{bmatrix} 1 - \lambda & 0 & 0 & \lambda & 0 & 0 \end{bmatrix}^T \cdot \phi, \]

\[ u_2^0 = \begin{bmatrix} 0 & 1 - 3\lambda^2 + 2\lambda^3 & \xi(1 - 2\lambda + \lambda^2) & 0 & 3\lambda^2 - 2\lambda^3 & \xi(\lambda^2 - \lambda) \end{bmatrix}^T \cdot \phi, \]

where \( \lambda = \frac{\xi}{L} \), therefore,

\[ g_1 = \begin{bmatrix} 1 - \lambda & \frac{\eta}{L}(-6\lambda + 6\lambda^2) & \eta(1 - 4\lambda + 3\lambda^2) & \lambda & \frac{\eta}{L}(6\lambda - 6\lambda^2) & \eta(3\lambda^2 - 2\lambda) \end{bmatrix}^T. \]
\[ g_2 = \begin{bmatrix} 0 & 1 - 3\lambda^2 + 2\lambda^3 & \xi(1 - 2\lambda + \lambda^2) & 0 & 3\lambda^2 - 2\lambda^3 & \xi(\lambda^2 - \lambda) \end{bmatrix}^T. \]

The distortional displacements can also be described in beam-global coordinates as

\[
U = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} = T^T \begin{bmatrix} g_1^T \\ g_2^T \\ 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \Phi(X_3) = G^e(X_1, X_2)\Phi(X_3),
\]

where \( \Phi(X_3) \) are distortional degrees of freedom in global coordinates and \( T \) is the transformation matrix

\[
T = \begin{bmatrix} \cos(\xi, X_1) & \cos(\eta, X_1) & 0 \\ \cos(\xi, X_2) & \cos(\eta, X_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

It follows \( G^e = \sum_e \int s(X_1, X_2)G^e dA \) in the material elasticity tensor \( C \) (Equations 2.28 - 2.33) are implemented as

\[
\sum_e \int s(X_1(\xi, \eta), X_2(\xi, \eta))G^e(\xi, \eta)d\xi d\eta,
\]

in which \( e = 1, 2, \ldots, n_{el} \), where \( n_{el} \) is the number of beam segments of the cross section.

### 2.8 Summary

Thin-walled members exhibit significant cross-sectional distortion due to local buckling. A fully nonlinear thin-walled beam theory is developed to permit a rigorous post-buckling behavior analysis within the context of a general geometrically-exact model.
Chapter 3

Experimental Studies

Experimental work is important to verify the proposed thin-walled beam theory. One application of the proposed theory is to analyze cold-formed members, in which distortion of the cross section due to local buckling is significant. Available experimental work on plain channels is reviewed. Additionally, beam and beam-column tests were carried out as part of this thesis. This chapter also contains the details of the experimental setup and validated test results through finite element analysis.

3.1 Introduction

Cold-formed steel members have applications in the marine, aerospace and civil engineering industries, in which they are capable of achieving substantial economies. Despite the complexity of the behavior of cold-formed steel members, their use has been increasing due to continued research efforts and incorporation of the findings into design specifications.

The behavior of a plate element in the post-buckling range is complex, and its analysis is quite involved, which has led investigators to resort to experiment-
Figure 3.1: (1) Stress distribution of a buckled plate (2) Idealized stress distribution based empirical methods. Von Karman [52] introduced and proposed an empirical equation for effective width, which is defined as follows. Figure 3.1 (1) shows the membrane (axial) stress distribution of a buckled plate under uniform compression with two edges supported. The maximum stress occurs at the plate edges while the stresses near the center of the buckled plate are relatively small. Figure 3.1 (2) shows the effective plate, in which two strips of combined width $b_e$ at the edge of the plate carry the maximum membrane stress.

Based on extensive test results, Winter [53] modified Von Karman’s equation to include the effect of imperfections and residual stresses. Design recommendations by DeVries [54], and later by Peköz [55], who developed a unified approach which treats both stiffened and unstiffened compression elements under various stress gradients, were adopted in AISC [56] and AISI specifications [57].

Extensive research on coupled local and flexural buckling has been carried out by Dewolf et al. [58, 59], Kalyanaraman et al. [60], Kalyanaraman and Peköz [61], Mulligan and Peköz [62, 63], Davids and Hancock [65], Loughlan [66], LaBoube and Yu [67] have conducted research on lateral buckling of beams. Coupled local and
torsional-flexural buckling has also been studied extensively. Chajes [68] studied axially loaded compression members and Peköz [69] extended the work to eccentrically loaded members. Loh and Peköz [70] and Talja [71, 72] investigated concentric and eccentric compression; Jayabalan [73] and Rao [74] studied the eccentric compression case.

A plain channel is one type of singly symmetric cold-formed steel member, used as bracing member in racks and tracks in steel framed housing. Though plain channels are seemingly simple, their accurate design presents special challenges. Available experimental work on beams, columns, and beam-columns is reviewed in the following.

**Beams** El Mahi and Rhodes [75], Enjily [76], Jayabalan [73], Cohen [77] performed experiments involving bending about the minor axis of plain C sections. El Mahi and Rhodes [75] and Cohen [77] tested beams with the stiffened element in tension, while Enjily [76] and Jayabalan [73] tested beams with both stiffened and unstiffened elements in compression. In addition, Cohen proposed an iterative effective width approach and post-yield strain reserve capacity model. Reck [60] and Talja [72] performed experiments involving bending about the major axis of plain C sections. Yiu and Peköz [78] carried out beam tests with minor axis bending with the stiffened element in compression. Details of these experiments are presented in Section 4.2.

**Columns** Mulligan and Peköz [62], Talja [71], Young and Rasmussen [79] performed experiments involving columns. Mulligan and Peköz [62] studied stub columns with flat-ended (clamped) boundaries under uniform compression. Talja [71] tested flat-ended long columns with uniform compression, while Young and Rasmussen [79] performed experiments involving flat-ended and pin-ended (sim-
ply supported) columns.

**Beam-Columns** Jayabalan [73] tested flat-ended beam-columns with the maximum compression occurring either at the free edge of the cross-section or at the supported edge. Rao [74] tested pin-ended beam-columns. Yiu and Peköz [80] tested beam-columns with a) bending about the symmetry axis and b) axial loading with biaxial bending. Details of these experiments are presented in Section 4.2.

Yiu and Peköz [78] proposed effective width design procedures for plain channel sections based on the previous test results along with extensive finite element studies. The design procedures developed are applicable to cross sections in the range of practical sections used in the industry.

### 3.2 Tests and Results

The beam and beam-column experiments of Yiu and Peköz [78] are described in this section. Imperfections were measured before the tests were performed.

#### 3.2.1 Imperfection Measurements

A lathe was used to measure geometric imperfections of different length specimens. Two end plates of specimens were clamped to the chuck end supports of a lathe. They were centered with respect to the centroid of the specimen cross section. Because there was no ruler in the lathe, grids needed to be marked to locate measuring positions. A marker was attached to the tool support, which was moved horizontally to mark lines. In order to mark vertical lines to these horizontal lines, plastic rulers were stuck to the top of the specimen with tape. A center head of a combination square set was used to mark the vertical lines on two sides of the cross section with
the alignment of the vertical lines on the top. Thus, a grid with the interval of 1 inch was marked before measuring the imperfections.

A DCDT (with measuring range) was mounted on the tool support of the lathe. The tool support was moved horizontally to position the DCDT at 1 inch intervals along each horizontal line. The DCDT can be adjusted up and down, back and forth to measure the imperfection of different sides of the cross section. The measuring system, comprised of an IBM-PC clone computer, an NI-LPC 16 channel data acquisition card, and a power supply was monitored while the DCDT initial position was set. The horizontal positioner for the tool support was used to move the DCDT from position to position for measurements, as shown in Figure 3.2. The data acquisition program stored the data in a file. Measurements were made at 1 inch increments horizontally along 3 lines on each outside surface of the specimen. With the lathe, imperfections of longer specimens can be measured.

Figure 3.2: Measuring Imperfections by DCDT
3.2.2 Beam Tests and Results

Two plain channel beams were tested, which have the dimensions common in industry. The purpose of the test is, firstly, to study the behavior of plain-channel cross sections of minor axis beam bending with stiffened elements in tension, and secondly, to evaluate the proposed thin-walled beam theory.

Two plain-channel beams with end plates were tested. The typical cross section is presented in Figure 3.3. The measured cross-section dimensions, as well as material properties, were listed in Table 3.1. Here, $t$ is thickness, $L$ is the beam span of the pure bending part. Thickness was measured with a metric micrometer, and web and flange width were measured with a Vernier calliper. All these measurements were taken as the average value of three readings at different locations. The round corners of adjacent plates are small and not measured.

Beams were tested on the flat table of the test machine. The test machine was a Baldwin 400kip open loop load frame. The magnitude and speed of loading were controlled by adjusting the loading and unloading valves of the control panel. Load was measured with a pressure sensor working in parallel with the test machine force measuring system. Displacement was measured with DC-DC linear variable differential transformers (LVDT) mounted between the table of the test machine and appropriate points on the bottom of the deflecting beam. Measurements were made with an HP3497 data acquisition system controlled by an IBM PC clone computer. During the test, a load vs. displacement curve was plotted on the computer screen and individual measurement values were printed on the computer screen. All data was stored on the computer disk drive for later analysis. Two very stiff arms of the C cross section were firmly attached to the two end plates of each beam in order to ensure the application of pure moment in the plain channel beam. Load
Table 3.1: Beam Cross-section Dimensions and Material Properties

<table>
<thead>
<tr>
<th></th>
<th>$b_1$(in)</th>
<th>$b_2$(in)</th>
<th>$t$(in)</th>
<th>$L$(in)</th>
<th>$F_y$(kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specimen 1</td>
<td>2.237</td>
<td>1.0265</td>
<td>0.06488</td>
<td>35.25</td>
<td>58.4</td>
</tr>
<tr>
<td>Specimen 2</td>
<td>2.3055</td>
<td>1.567</td>
<td>0.0758</td>
<td>59.25</td>
<td>35.25</td>
</tr>
</tbody>
</table>

Figure 3.3: A plain C cross section

Figure 3.4: Beam Test Setup

was applied through a load spreading beam onto plates outside of the pure bending range. The test setup is sketched in Figure 3.4. The following are the observations from the experiments. The unstiffened components in compression buckled almost simultaneously at the two ends of the beam in Specimen 1, shown in Figure 3.7. The buckling of one of the unstiffened components followed by the buckling of the other unstiffened components and twisting were observed in Specimen 2, shown in Figure 3.8.
### 3.2.3 Beam-Column Tests and Results

Four beam-column tests were performed, two on axial loading with bending about the symmetry axis and two on axial loading with bi-axial bending. Among these four experiments, two are on short columns and two on long columns. The measured dimensions and material properties are listed in Table 3.2. The round corners of adjacent plates are small and not measured.

The beam-columns were pin-ended, which allowed rotations about the x-x and y-y axis with restraining twist rotations and warping. The effective length coefficients were $K_x = K_y = 1.0$ and $K_t = 0.5$. End plates were welded to the column. Thus, cross-section warping at the ends was restrained by the end plates. Hence, the eccentric load did not produce a bimoment at the ends. As loading was eccentric in these experiments, twisting at the ends was prevented as a result of cross-sectional warping being restrained. The test setup is sketched in Figure 3.5.

The steel plate was used to transfer the load into the column. The hinge at the supports was accomplished by a steel ball. The load was applied to the column through the steel ball. Circular dimples were machined on the end plates so that the steel balls rested in a particular position with respect to the specimen throughout the test. Washers were welded onto the end plates to prevent the steel balls from accidentally dislodging from their positions. The column was loosely chained to the testing machine at the top and the bottom. A spirit level was used to ensure that the column was vertical. Displacement transducers were mounted to measure the midheight deflections as well as the deflections at the supports. In this way, it was possible to compensate for the midheight deflection from the possible movements of the steel ball at the supports when the applied load was increased. Beam-column test results are presented in Table 3.3 and Figures 3.9, 3.10.
Figure 3.5: Beam-Column Test Setup

Table 3.2: Details of Beam-Column Test Specimens

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$b_2$ (in)</th>
<th>$b_1$ (in)</th>
<th>$t$ (in)</th>
<th>$L$ (in)</th>
<th>$F_y$ (kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC1-30</td>
<td>1.979</td>
<td>3.271</td>
<td>0.0666</td>
<td>30.00</td>
<td>36</td>
</tr>
<tr>
<td>BC2-30</td>
<td>2.036</td>
<td>3.277</td>
<td>0.0750</td>
<td>30.75</td>
<td>36</td>
</tr>
<tr>
<td>BC1-65</td>
<td>2.028</td>
<td>3.278</td>
<td>0.0650</td>
<td>65.00</td>
<td>36</td>
</tr>
<tr>
<td>BC2-65</td>
<td>2.075</td>
<td>3.278</td>
<td>0.0630</td>
<td>64.75</td>
<td>36</td>
</tr>
</tbody>
</table>
Table 3.3: Beam-Column Test Results

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Eccentricity $e_x$(in)</th>
<th>Eccentricity $e_y$(in)</th>
<th>Ultimate load $P$(kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC1-30</td>
<td>1.437</td>
<td>1.636</td>
<td>2.138</td>
</tr>
<tr>
<td>BC2-30</td>
<td>0.000</td>
<td>1.639</td>
<td>6.435</td>
</tr>
<tr>
<td>BC1-65</td>
<td>1.467</td>
<td>1.639</td>
<td>1.437</td>
</tr>
<tr>
<td>BC2-65</td>
<td>0.000</td>
<td>1.639</td>
<td>4.187</td>
</tr>
</tbody>
</table>

3.3 Test Evaluations Using ABAQUS

The experimental results presented in previous sections are first assessed by ABAQUS before they are used to evaluate the proposed beam theory. ABAQUS shell element $S9R5$ is selected for modeling the behavior of the members. $S9R5$ [81] is a nine-node flexible shell element, with five degrees of freedom per node. The element is derived using Mindlin plate theory, i.e. a theory considering transverse shear deformation. Solution is obtained using a modified Riks method. The ABAQUS model uses an elasto-plastic material. The aspect ratio of elements is between 1.02 and 1.21. Geometric imperfections are considered in the model. The lowest eigenmode of the member is selected for geometric imperfection distribution. Its magnitude is chosen to be at 50 percent of the cumulative distribution [82] of experimentally measured values of $d/t$, where $d$ is the maximum magnitude of the imperfections and $t$ is the plate thickness. Details of the geometric imperfection studies can be found in [80].

An ABAQUS model of a pin-ended beam-column with two thick end plates, shown in Figure 3.6, was used to simulate the restraints of warping and cross-section distortion at two ends in the beam-column specimens. The thickness of the end plates was the actual thickness of the end plates used in the experiments.
Similarly, an ABAQUS model of a simply-supported beam used the stiff beams at the two ends to serve the purpose of restraining the warping and distortion of cross-section at the supports to simulate the experiment, shown in Figure 3.4, as well as for applying the bending moment at supports. The comparison of ABAQUS and test results are listed in Tables 3.4, 3.5, which shows good agreement.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$M_{Test}$ (kips $\cdot$ in)</th>
<th>$M_{ABAQUS}$ (kips $\cdot$ in)</th>
<th>$M_{Test}/M_{ABAQUS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam 1</td>
<td>3.032</td>
<td>3.225</td>
<td>0.940</td>
</tr>
<tr>
<td>Beam 2</td>
<td>6.512</td>
<td>6.232</td>
<td>1.045</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$P_{Test}$ (kips)</th>
<th>$P_{ABAQUS}$ (kips)</th>
<th>$P_{Test}/P_{ABAQUS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC1-30</td>
<td>2.138</td>
<td>2.520</td>
<td>0.848</td>
</tr>
<tr>
<td>BC1-65</td>
<td>1.437</td>
<td>1.600</td>
<td>0.898</td>
</tr>
<tr>
<td>BC2-30</td>
<td>6.435</td>
<td>6.165</td>
<td>1.044</td>
</tr>
<tr>
<td>BC2-65</td>
<td>4.187</td>
<td>4.860</td>
<td>0.861</td>
</tr>
</tbody>
</table>
Figure 3.7: Beam Test 1

Figure 3.8: Beam Test 2
Figure 3.9: Beam-Column Test 1

Figure 3.10: Beam-Column Test 2
3.4 Conclusions

Six experiments, two on beams and four on beam-columns, were performed to study the behavior of plain-channel cross sections. ABAQUS results show that these beam and beam-column tests were able to provide reliable experimental data to evaluate the proposed beam theory.
Chapter 4

Numerical Examples

Results of the proposed beam theory are compared with analytical results from the open literature, as well as with results from ABAQUS, along with experimental results on thin-walled members presented in Chapter 3. A full Newton-Raphson iterative solution procedure was employed in all the analyses reported herein. A general form of the classical arc-length method [87] was adopted to trace the post-buckling branch. Analyses were carried out with varying mesh sizes, based on which the mesh size for which convergence was observed was chosen.

A perturbation-type force [51], 5% of the applied load is used throughout to simulate global and local imperfections. Global perturbation loads are applied perpendicular to the weak axis of the beam to simulate member global imperfections, while local perturbation loads with a pattern that resembles the local buckling mode shape at mid-span are used to simulate local imperfections. Timoshenko [3] used an equivalent load method to perturb a perfect column into a member with initial imperfection. Ibrahimbegović [51] performed a study of the Euler formula for pure bending of beams using perturbation-type forces and found that the computed values of buckling loads are affected very little by the perturbation-type forces.
In the proposed beam element, external loads are applied at the centroid. For the finite element analysis they are equivalently applied at the reference point $P$, since the degrees of freedom describing global beam deformation are located at that point and no external loads are applied to the distortional degrees of freedom at other nodal points of the cross-section.

All warping and distortional degrees of freedom are restrained at the two ends. This is consistent with experiments as described in Chapter 3. Two thick and rigid plates are typically welded at the two ends of the specimen in thin-walled beam tests providing restraints on both warping and distortional deformations.

Examples 4.1 and 4.2 use the proposed thin-walled beam theory to analyze large displacement three-dimensional beams excluding distortions. Example 4.1 shows that the proposed theory can be reduced to classical 3D beam theory without considering warping or cross-section distortion. Example 4.2 takes into account warping but excludes cross-section distortion. Examples 4.3 and 4.4 are studies of a plain C and a lipped C stub column respectively. Stub columns exclude member buckling and are subject to local-type buckling only. Example 4.5 is a beam column under axial loading and bi-axial bending and exhibits coupling of local and global behaviors.

4.1 Large Displacement 3D Beam Analysis of a 45-degree Bend

A concentrated vertical tip load is applied to a curved beam with 45-degree arc in the z direction as shown in Figure 4.1. This example has been considered by Bathe and Bolourchi [88], whose results have been verified by other researchers [49, 25].
The bend has a radius of 100 (no units were given in the original reference [88]). The area and the bending stiffness with respect to the principal axes are $A = 1$, $I_x = 0.0833$, $I_y = 0.0833$, respectively. The Young’s modulus is $E = 10^4$; Poisson’s ratio is $\nu = 0.0$. The warping and distortional degrees of freedom are constrained at all nodes. The beam bend is modeled with eight beam elements. The cantilever bend is fixed at node 1 and free at node 9 (see Figure 4.1), i.e. the boundary conditions $u_x = u_y = u_z = \theta_x = \theta_y = \theta_z = 0$ are applied at node 1. The static load path is obtained in five steps of 100 units of load using the arc-length method.

Results are shown in Figure 4.2, in which $u$, $v$ and $w$ are the deformed coordinates of node 9 in the $x$, $y$ and $z$ directions respectively. It is seen that the proposed thin-walled beam theory agrees well with the results of Bathe and Bolourchi [88], i.e. it reduces to three-dimensional beam theory when warping and distortion of the cross section are not taken into account.

![Diagram of a 45-degree beam bend](image)

**Figure 4.1**: A 45-degree beam bend
4.2 Nonlinear Analysis of a Beam-Column without Distortion

An I section W14 × 43 [56] beam column with length $L = 264.6$ in is subjected to biaxial bending, as shown in Figure 4.3. This numerical example was also considered by Soltis and Christiano [90].

The width and the thickness of the flange are 7.995 in and 0.530 in respectively, while those of the web are 13.66 in and 0.305 in. The prescribed warping function $f$ is taken as in Figure 2.4. Distortion is not considered in this case, i.e. $\Phi$ was set to 0 at all nodes. Due to the double symmetry of the I section, $x_p = y_p = 0$.

Cross sectional properties are $A = 12.6 \text{in}^2$, $I_x = 428 \text{in}^4$, and $I_y = 45.2 \text{in}^4$. The boundary conditions $U_x = U_y = U_z = \theta_z = 0$ are applied at end A and $U_x = U_y = \theta_z = 0$ at end B. The eccentricity of the applied load of 28 kips is $e_x = 5$ in and $e_y = 0.5$ in. Material properties are $E = 30000 \text{ksi}$ and $\nu = 0.0$. The
beam column is modeled with eight elements. The load is applied at the centroid with $N = 28 \text{kips}$, $M_x = 140 \text{kips\cdot in}$, and $M_y = 14 \text{kips\cdot in}$ in seven steps.

The comparison in Figure 4.4 shows results of the proposed fully nonlinear theory and those obtained by Soltis and Christiano [90] using a third order analysis. The load versus mid-span displacement curves exhibit nonlinearity due to beam-column behavior. It is observed that this nonlinearity is more pronounced with the proposed theory compared to the third order analysis of Soltis and Christiano [90].

![Figure 4.3: Beam column under biaxial loading](image)

![Figure 4.4: Comparison with the results of Soltis and Christiano [90]](image)
4.3 Post-buckling Analysis of a Thin-walled Plain C Column

Stub columns are defined so that they are short enough to preclude the effects of overall buckling, but sufficiently long to exhibit cross-section distortion due to local-type buckling behavior. By analyzing stub columns, the proposed generalized thin-walled beam element can be verified and assessed.

Based on the recommendations of Technical Memorandum No. 3 of the Structural Stability Research Council (SSRC), ‘Stub-Column Test Procedures’, reprinted in [91], columns with $L < 20r_{\text{min}}$ and $L > 3W_{\text{max}}$, where $r_{\text{min}}$ is the least radius of gyration, $W_{\text{max}}$ is the maximum cross-section width and $L$ is the length of the column, are defined as stub columns.

A column test specimen $SC/1\ 60 \times 60$ [62] that satisfies the above stub column criteria is selected. The column has cross-sectional geometry $W1 = 3.058\text{ in}$, $W2 = 3.080\text{ in}$, $t = 0.0479\text{ in}$, $L = 9.141\text{ in}$, where $t$ is the thickness, and $L$ is the length of the column. A physical test of this column was performed by Mulligan and Peköz [62]. The finite deformation of the column obtained in the test is used to verify the proposed thin-walled beam theory. The test setup and the cross section of the specimen are sketched in Figure 4.5. The geometric properties of the stub column are $A = 0.4416\text{ in}^2$, $I_y = 0.4655\text{ in}^4$ and $I_z = 0.8051\text{ in}^4$. The material properties are $E = 29500\text{ ksi}$ and $\nu = 0.3$. The prescribed warping function $f$ is taken as in Figure 2.5.

The concentric load $N = 1.9\text{ kips}$ is equivalently applied at the reference point $p$ of the cross-section as the combined loading of $N = 1.9\text{ kips}$ and $M = 1.945\text{ kips-in}$. A perturbation load of $0.095\text{ kips}$ is applied to simulate overall imper-
fections as in Figure 4.5(a). The load pattern shown in Figure 4.5(b) is applied at mid-span to simulate local geometric imperfections. Initial geometric imperfections of the column were measured before the experiments, and it was reported that the maximum global imperfections varied between \( \frac{L}{1200} \) and \( \frac{L}{1205} \), while the maximum local imperfections were 0.016 – 0.095 in [62]. In comparison, the perturbation load of 0.095 kips produces a maximum global displacement of 0.000640 in and a maximum local displacement of 0.001296 in. Five load steps are applied in load control.

The stub column is discretized into six elements along the column longitudinal axis. The cross section is discretized into 12 segments. The applied boundary conditions of the column are \( u_x = u_y = u_z = \theta_x = \theta_y = \theta_z = p = \Phi = 0 \) at end A and \( u_y = u_z = \theta_x = p = \Phi = 0 \) at end B.

The axial displacement of the reference point at end B is monitored and compared to the experimentally measured displacement at end B [62]. In Figure 4.6 applied load F versus beam axial deformation results obtained by the proposed theory are compared to those obtained experimentally and to those obtained with ABAQUS using the model described in Section 3.3. Eigen-value analysis of the ABAQUS model yields a local buckling load of 5.7365 kips. This result is also marked in Figure 4.6. Both ABAQUS and the proposed theory show reduction of the column axial stiffness at a similar load level, which is a little beyond 5.7365 kips. In the post-buckling range there is reasonable agreement between the proposed and the ABAQUS results. It is seen that the proposed theory produces slightly higher stiffness reduction than ABAQUS.

The concept of effective width, introduced by von Karman [52], provides a physical understanding of the post-buckling behavior. With this concept, the cross section is not fully effective after local buckling occurs, therefore a shift of the neu-
tral axis is expected. 2D deformation shapes of the cross section at mid-span are plotted in Figure 4.7. Cross-section translation perpendicular to the weak axis of the cross-section is caused by global imperfection. After the cross section undergoes local buckling, the cross-section deformation is prominent resulting in a shift of the neutral axis.

Local deformation shapes of the cross-section at mid-span obtained with the proposed theory and with ABAQUS are plotted in Figure 4.8. The dashed line is the result of the proposed theory at 8.85 kips and the solid line is the result of ABAQUS at 8.086 kips. The applied load versus the distortional degree of freedom $U_{ZD}$ at point D of the cross-section at mid-span, predicted by the ABAQUS and by the proposed theory, is plotted in Figure 4.9. In order to exclude beam contributions from the ABAQUS model, we have plotted $u_{ZD} - u_{ZP}$ which is not the same as $U_{ZD}$ of the proposed theory, hence the discrepancy in the result.

Figure 4.5: Column SC/1 60 × 60 with concentric loading
Figure 4.6: Applied load F vs. beam axial deformation at end B

Figure 4.7: Local deformation shapes at mid-span as the load is increased
We recall that in the proposed theory, in addition to the reference point $P$, a support point $S$ is also selected on the cross-section (Figure 4.10). However, the results should be independent of the choice of support point. To check this fact, two locations of the support point $S$, in which the rotational degree of freedom was constrained, were selected as shown in Figure 4.10. The applied load vs axial displacement curves are identical for the two locations as shown in Figures 4.11.
Local cross-section deformation shapes at step 3 shown in Figures 4.12 are within a rigid body translation as expected.

Cross-section membrane (axial) stresses after local buckling occurred at 8.85 kips are plotted in Figure 4.13. These stresses are at the Gauss point of the 3rd element from the top (3.809 in) down from the application point of the load. The stress distribution is almost symmetric but for small variations due to the overall imperfections applied. As expected, the stress distribution is nonlinear in both stiffened and unstiffened elements. However, more nonlinearity in the stress distribution was expected.

Figure 4.10: Choices of different support points S
Figure 4.11: Applied load F vs. beam axial deformation for different support points S

Figure 4.12: Local deformation shapes at mid-span at load step 3 for different support points S
Figure 4.13: Membrane (axial) stress distribution at a cross-section located 3.809 in from the top of the column

4.4 Post-buckling Analysis of a Thin-walled Lipped C Column

A lipped C column, shown in Figure 4.14(a), which meets the stub column criteria is chosen for this example. For lipped C section, distortional buckling is often differentiated from local buckling, in which the corner nodes of the cross-section do not translate. In this example, distortional, or stiffener buckling is observed, i.e. the member fails due to partial instability of the cross-section, in which the lip stiffeners are inadequate to prevent the lateral movement of the flanges that they support [97].

The lipped C cross section has dimension $b_{\text{web}} = b_{\text{flange}} = 5.9055$ in, $b_{\text{lip}} =$
0.2953 in and thickness \( t = 0.0591 \) in. The length of the column is 25.0787 in. The geometric properties of the lipped C section are \( A = 1.083 \) in\(^2\), \( I_{y} = 4.599 \) in\(^4\), and \( I_{z} = 7.396 \) in\(^4\). The material properties are \( E = 29500 \) ksi and \( \nu = 0.3 \). The prescribed warping function \( f \) is taken as in Figure 2.6. The applied boundary conditions are \( u_{x} = u_{y} = u_{z} = \theta_{x} = \theta_{y} = p = \Phi = 0 \) at A and \( u_{y} = u_{z} = \theta_{x} = \theta_{y} = p = \Phi = 0 \) at B. The stub column is divided into eight generalized thin-walled beam elements. Taking into account the boundary conditions of the segment frame, 42 degrees of freedom are used to describe the cross-section distortion.

The load \( F = 1.4 \) kips (Figure 4.14) applied at the centroid of the lipped C section, is equivalent to the combined loading of \( N_{x} = 1.4 \) kips, \( M_{y} = 2.9386 \) kips-in applied at the reference point \( P \). Similar to example 4.3, a perturbation load of 0.07 kips is applied to simulate overall imperfections and local geometric imperfections as shown in Figure 4.14(b) to trigger the distortional buckling mode. The loading is applied in five steps using the arc-length method.

The results from the proposed theory are compared to those from ABAQUS in Figure 4.15, which shows the applied load \( F \) versus column axial deformation at end B. The distortional buckling load of the ABAQUS model obtained from an eigen-value analysis is 5.15 kips and is also shown in Figure 4.15. Good comparison is obtained.

Deformed shapes of the cross section at mid-span obtained with the proposed theory are plotted in Figure 4.16 at load levels of 0 kips, 1.4 kips, 3.441 kips, 5.975 kips and 8.668 kips. Prominent deformation in the post-buckling range resulting in a shift of the neutral axis is also observed.

Figure 4.17 shows some discrepancy between the ABAQUS result at 8.5731 kips (solid line) and the result of the proposed theory at 8.667 kips (dashed line). Cor-
respondingly at mid-span the applied load versus distortional deformation $U_z$ at mid-span does not show good agreement (Figure 4.18). The distortional deformation for the ABAQUS model was calculated as in the previous example.

Figure 4.14: A lipped C column

Figure 4.15: Applied load F versus column axial deformation at end B
Figure 4.16: Local deformation shapes at mid-span as the load is increased

Figure 4.17: Local deformation shapes at load step 4 at mid-span
4.5 Nonlinear Analysis of a Thin-walled Plain C Beam-Column

A plain C beam-column under axial loading and bi-axial bending is shown in Figure 4.19. This physical test was reported in Chapter 3.2. We consider this un-symmetric loading case in order to study the interaction of local buckling with beam-column behavior by the proposed thin-walled beam theory.

Cross sectional dimensions are \( W_1 = 3.271 \text{ in}, \ W_2 = 1.979 \text{ in} \) and \( t = 0.0666 \text{ in} \). The geometric properties are \( A = 0.4815 \text{ in}^2, \ I_y = 0.8992 \text{ in}^4 \) and \( I_z = 0.2032 \text{ in}^4 \). Linear elastic material properties are \( E = 29500 \text{ksi} \) and \( \nu = 0.3 \). The warping function \( f \) is prescribed as in Figure 2.5. The length of the column is 30in.

To simulate the simply supported end condition with respect to flexural rotation and the fixed end condition with respect to torsion of the member, the boundary
conditions are $u_x = u_y = u_z = \theta_x = 0$ at A and $u_y = u_z = \theta_x = 0$ at B. Warping and distortional deformations are restrained at the two ends in the experiment; therefore, $p = \Phi = 0$ at ends A and B. The beam-column is discretized into 10 thin-walled beam elements. The cross section is further discretized with four segments in the web and three segments in each flange. Therefore, 30 degrees of freedom are used to describe the cross-section distortion.

The load is applied at the tip of the free edge of the cross-section, as shown in Figure 4.19(b). Equivalently, loads $N_x = 0.5$ kips, $M_y = 0.8178$ kips-in, and $M_z = 0.9895$ kips-in are applied at the reference point $P$ of the cross section. A perturbation load of 0.025 kips (5% of $N_x$) was applied to simulate overall imperfections and local geometric imperfections, as in Example 4.3. The loading is applied in six steps using the arc-length method.

Eigen-value analysis of the ABAQUS model gives a local buckling load of 1.34 kips. Mid-span displacements in the experiments were reported in the literature [78]. A model consisting of S9R5 shell elements in ABAQUS and the experimental results are used to validate the proposed theory before and after local buckling occurs. Figure 4.20 shows good agreement between the experimental results and those obtained with the proposed method and with ABAQUS. There is a slightly better comparison of $u_y$ between the proposed theory and the experiment, and a slightly better comparison of $u_z$ between the ABAQUS model and the experiment.

Local deformed shapes at different load steps are plotted in Figures 4.21, 4.22, and 4.23 at the quarter point C, mid-span D and three-quarter point E from the beam end A, respectively. The history of local deformed shapes indicates local buckling. The lower web experiences more local deformation than the upper web,
as expected, since bending about the major axis applies more in-plane compression to the lower web.

Local deformation shapes of the cross-section at mid-span obtained with the proposed theory at 2.13kips (dashed line) and with ABAQUS at 2.20kips (solid line) are plotted in Figure 4.24. It is apparent that the deformed shapes are smooth with similar local shapes. The applied load versus the cross-section distortional deformation $U_z$ at point G of the cross-section at mid-span is also plotted in Figure 4.25 as obtained from both ABAQUS and the proposed generalized beam theory. Close agreement is observed in this case.

Figure 4.19: Beam-column with bi-axial loading
Figure 4.20: Applied load F vs. mid-span beam deformation

Figure 4.21: Deformed shapes of cross-section at C (quarter point from end A)
Figure 4.22: Deformed shapes of cross-section at D (mid-span)

Figure 4.23: Deformed shapes of cross-section at D (three-quarter point from end A)
Figure 4.24: Local deformation shapes at step 4 at mid-span

Figure 4.25: Applied load F vs. mid-span cross-section distortional deformation $U_{ZG}$
4.6 Conclusions

Nonlinear analysis using a finite element program based on the fully nonlinear thin-walled beam theory proposed in Chapter 2 gives results for plain C and lipped C sections under various load conditions.

Comparison studies with numerical results in the literature, results obtained from ABAQUS and/or experimental work are performed and good agreement is generally obtained. The applied load versus member deformation curves show stiffness degradation and nonlinear behavior after local buckling occurs. Deformed shapes during the loading history are plotted and compared with ABAQUS. It is found that the 2D plane frame approach of the proposed theory is able to capture local buckling accurately. As material nonlinearity is not considered, the ultimate load cannot be obtained and loads in the range of material nonlinear behavior cannot be accurately predicted.
Chapter 5

Conclusions and Future Work

In this chapter, the contributions of this work are summarized, conclusions are drawn, and directions for further research are suggested.

5.1 Summary and Conclusions

The objective of this research is the post-buckling analysis of thin-walled members considering local or distortional buckling. This is achieved by a geometrically exact description of the deformation map. The formulation of the equations of motion in the general dynamic case was obtained. The resulting nonlinear equations of equilibrium were implemented in the static case.

The principal conclusions from this research can be summarized as follows:

- The geometrically-exact thin-walled beam theory has been used to study the effects of local buckling on stub columns of plain C and lipped C sections, and beam columns of plain C section, in which distortion due to local buckling is significant.
Based on this theory, a finite element program has been developed for non-linear analysis. The numerical solutions are compared with ABAQUS results and/or experimental work, and reasonably close agreement is found. The same is true for 2D deformed shapes of the cross-section obtained at different load steps during the loading history.

5.2 Future Work

Future research includes the investigation of geometric imperfections. It is well known that geometric imperfections influence the behavior and ultimate strength of thin-walled members. The proposed theory can readily be extended to take into account the global geometric imperfections by treating the basis $E_i$ as a function of $X_3$. A common approach is to assume that the most detrimental type of imperfection is the one that has the same shape as the first eigen-mode. Therefore, the scheme below can be used.

- Perform eigen-value analysis to obtain the first eigen mode and take it as the initial geometric imperfection shape.
- Superimpose the first eigen mode onto the geometrically perfect cross-section member.
- Perform nonlinear analysis on the initially imperfect member.

Another area of future research includes the study of more complex material behavior. It is necessary to further extend the theory to consider both geometric and material non-linearity for a realistic analysis.
In addition, the structural members that are tested and analyzed in this dissertation are limited. A further test program should be performed on a variety of thin-walled sections that have low torsional rigidity, such as channels with simple and compound lips (lips bent both inward and outward), hat sections, Z sections, and angle sections.

Future work can extend this research into aerospace structures, where it has potential to find applications in dynamic analysis. The present formulation has considered the effect of inertia. Implementation in the dynamic case is needed in the future.
Appendix A

Nomenclature

\((.,)_I\) differentiation \(\frac{\partial}{\partial X_I}\)

\(\dot{.}\) differentiation \(\frac{\partial}{\partial t}\)

\((.)'\) differentiation \(\frac{\partial}{\partial X_3}\)

\(E_1, E_2, E_3\) material basis vectors

\(e_1, e_2, e_3\) spatial basis vectors

\(\varphi\) position vector of a material point in the deformed configuration

\(P\) sectorial pole (warping center) of the cross-section

\(\varphi_P\) deformation map of line P

\(\varphi_B\) deformation map of classical beam incorporating shear and torsional warping

\(\varphi_b\) deformation map contributed by distortion

\(F\) deformation gradient

\(Q\) orthogonal two-point tensor representing rotation of the cross-section

\(\omega\) infinitesimal rotation matrix with its axial vector \(\omega\)

\[
\omega = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\]
$q$ general out-of-plane warping function, $q = fp$

$f$ prescribed warping function

$p$ unknown warping magnitude

$u_1, u_2, u_3$ distortion of the cross-section in segment-local coordinates

$U_1, U_2$ distortion of the cross-section in beam-global coordinates

$G^e$ a matrix containing prescribed in-plane distortion functions for segment $e$,

$U_a^e = G^e \Phi$

$E$ Lagrange strain tensor

$P$ first Piola-Kirchhoff stress tensor

$\sigma$ Cauchy stress tensor

$S$ second Piola-Kirchhoff stress tensor

$B$ body force

$P$ stress power

$\rho_0$ density in the reference configuration

$\Gamma, \Omega, p, p', \Phi, \Phi'$ generalized strain measures defined by Equation 2.19

$N, M$ total assigned force and moment

$N_f, M_f$ warping moment and bi-moment

$N_u, M_u$ generalized forces associated with distortion

$n, m, \tilde{M}_f, \tilde{M}_u$, applied force, torque, bi-moment and generalized plane frame force

$P : \dot{F} = tr(P \dot{F}^T)$

$\otimes$ the tensor product of the vectors

$\times$ the vector product
Appendix B

Implementation Details and User Manual

This appendix consists of four parts: details of the computer implementation in FEAP in B.1, fortran FEAP user subroutine Elmt12.f in B.2, data structures in B.3, description of FEAP inputs for elmt12.f in B.4, and pre- and post- processors in B.5.

B.1 Details of Computer Implementation in FEAP

The finite element formulation of the generalized thin-walled beam theory presented in Chapter 2 was implemented within the Finite Element Analysis Program (FEAP) [83, 84]. A user-defined subroutine elmt12.f was developed and added to the user library. Implementation details of the generalized beam element are discussed in this Section.
FEAP Program

The FEAP program has posed several challenges for the computer implementation of the proposed theory. The first challenge is the number of degrees of freedom per node. Subroutines PINPUT and TINPUT are standard subprograms used by FEAP to input each data record. PINPUT permits up to 16 individual expressions on one input record (with up to 255 characters on each record), which sets a limit of 16 on the number of degree of freedom per node. The idea of a ‘super-node’ at each end of the beam element could not therefore be implemented. Instead, \( n_{sec} \) nodes per cross-section at each end are used in the implementation.

Another challenge is the requirement of FEAP that each node within an element have the same number of degrees of freedom. The implementation technique used is to have 7 degrees of freedom for every node. At the reference point \( P \) all 7 degrees of freedom are active. At the remaining \( n_{sec}-1 \) nodes on the cross section, only 3 degrees of freedom are actually used that describe the distortion. The remaining 4 degrees of freedom are restrained (i.e. set to zero) in the boundary conditions section of the FEAP input.

Generalized Thin-walled Beam Element elmt12.f

Consider a thin-walled beam with \( n_{sec} \) nodes per cross section at each end. For example, a thin-walled beam element of plain C cross section with two ends at \( i \), \( j \) and \( n_{sec} = 5 \) is shown in Figure B.1. Node 1 is the reference point \( P \) of the proposed theory at end \( i \) and node 6 is the reference point \( P \) at end \( j \). The element connectivity is 1 2 3 4 5 6 7 8 9 10. The reference point \( P \), i.e. node 1 or node 6, has 7 degrees of freedom describing three translations \( u_x, u_y, u_z \), three rotations \( \theta_x, \theta_y, \theta_z \), and warping \( p \). The number of distortional degrees of freedom depends on
the cross-section configuration. For example, a plain C section has 15 distortional degrees of freedom as shown in Figure B.2. Before considering boundary conditions, the total number of degrees of freedom at each end is \( 7 + 3 \times nsec \).

![Figure B.1: A thin-walled beam element](image1)

![Figure B.2: Distortional degrees of freedom for a plain C section](image2)

**B.2 Fortran FEAP User Subroutine Elmt12.f**

The main structure of the Fortran FEAP program elmt12.f is shown in Figure B.3. In our computer runs in Chapter 4 we used Matlab to produce the matrices described in B.2.1 for the plain C and lipped C sections shown in Figure B.4.
B.2.1 Data Files

A user is to provide the sub-matrices of the material elasticity tensor in Equation 2.34 in the format described below. Cn.dat, Cw.dat, and Cd.dat should be put in the same directory as the executable code feapV.exe along with the FEAP input, e.g. iBeam.txt and NOF.txt. These inputs should be named exactly as follows.

**Cn.dat**

- $C_n$ a $3 \times 3$ matrix, $3 \times 10.6$ per row;
- $C_m$ a $3 \times 3$ matrix, $3 \times 10.6$ per row;
- $C_{nm}$ a $3 \times 3$ matrix, $3 \times 10.6$ per row;

**Cw.dat**

- $C_{nw}$ a $3 \times 1$ matrix, $1 \times 10.6$ per row;
- $C_{mw}$ a $3 \times 1$ matrix, $1 \times 10.6$ per row;
- $C_{w1}$ a scalar, $1 \times 10.6$;
- $C_{w2}$ a scalar, $1 \times 10.6$;

**Cd.dat**

The notation $C_{xx}(i,:)$ refers to the $i$th row of matrix $C_{xx}$.

- $C_{nl1}(3,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{nl2}(1,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{nl2}(2,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{ml1}(1,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{ml1}(2,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{ml2}(3,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{w1l2}(1,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{w2l1}(1,:)$, a $1 \times (3 \ast nsec)$ matrix, $(3 \ast nsec) \times 10.6$;
- $C_{l1}(:, :)$, a $(3 \ast nsec) \times (3 \ast nsec)$ matrix, $(3 \ast nsec)^2 \times 10.6$ row-wise;
$C_{t2}(,;$), a $(3 \times nsec) \times (3 \times nsec)$ matrix, $(3^{nsec})^2 e^{10.6}$ row-wise;

**NOF.dat**

It provides $nsec$, which is the size of a dynamically allocated array. The format of the data is (1I10). Fortran dynamic allocation techniques are used so that the generalized beam element has the capacity of analyzing a thin-walled member of arbitrary open polygonal cross-section. Without using dynamic memory allocation, the program may have to be recompiled each time with different cross-section discretizations.

![Flowchart diagram](image)

**Figure B.3: Elmt12.f Flowchart**
Figure B.4: Different Cross-sections

B.3 Description of FEAP Inputs for Elmt12.f

In this section information is provided on the input file generated by the pre-processor described in Section B.4.1. Although possible, it is not intended that the user write the input file without the use of the pre-processor, as this is often cumbersome due to the complicated data structure employed.

The FEAP input data file can be divided into 3 sections as follows.

General Information

The input data file must contain the following control data:

A start-title record which must have "FEAP" as the first four non-blank characters. Problem information consisting of:

- NUMNP - number of nodal points;
- NUMEL - number of elements;
- NUMMAT - number of material property sets;
• NDM - space dimension of mesh;

• NDF - maximum number of unknowns per node (number of d.o.f.);

• NEN - maximum number of nodes per element;

• MATErial - material property set;

• User12 - User-defined element 12 in the user element library;

• FINIt - finite deformation rather than small deformation;

• CROSs - cross-section properties;

• ELASStic - finite deformation elastic material

• REFEnence - orientation of the cross-section defined by reference vector or node

For example, the input file for the stub column of Example 4.3 is as follows:

FEAP ** 3D Beam: Mulligan SC/60x60

3 5 6 1 3 7 10

MATErial, 1

User 12

Finite

CROss section 0.4370 0.4534 0.7720 0 1.2254

ELASStic NEOHook 29500 0.3

REFEnence NODE 0 0 1

The mesh of the column in this example has 35 nodes, 6 elements, and 1 material set. It is a three-dimensional problem with 7 degrees of freedom at each node and 10 nodes for each thin-walled beam element (Figure B.1). One type of material
is indicated. The cross sectional properties are $A = 0.4370 \text{in}^2$, $I_{xx} = 0.4534 \text{in}^4$, $I_{yy} = 0.7720 \text{in}^4$, $I_{xy} = 0.0 \text{in}^4$, $J_{zz} = 1.2254 \text{in}^4$. A finite deformation neoHookean elastic material is specified with elastic modulus $E = 29500 \text{ksi}$, and Poisson ratio $\nu = 0.3$. The cross-section orientation is defined by the reference node 0 0 1.

**Mesh input data**

These consist of the specification of nodes, element connectivity, material set for each element, boundary restraints and applied loads.

- COORdi- nodal coordinates;
- ELEMent- element connectivity, element type and its associated material set;
- BOUNdary- boundary restraints to degrees of freedom;
- FORC- applied forces

For example for the stub column of Example 4.3:

```
COORdinates
10000
20000
30000
40000
50000
60 1.5235 0 0
70 1.5235 0 0
80 1.5235 0 0
90 1.5235 0 0
100 1.5235 0 0
```
This fragment of input data defines a node number, a generation increment to next node number (0 in this case), and the x coordinate value of each node, while the y and z coordinates are set to zero and are generated internally by the program. There are 35 lines defining 35 nodes in sets of $n_{sec} = 5$.

ELEMents

1 0 1 1 2 3 4 5 6 7 8 9 10
2 0 1 6 7 8 9 10 11 12 13 14 15
3 0 1 11 12 13 14 15 16 17 18 19 20
4 0 1 16 17 18 19 20 21 22 23 24 25
5 0 1 21 22 23 24 25 26 27 28 29 30
6 0 1 26 27 28 29 30 31 32 33 34 35

This segment of inputs gives the element number, the generation parameter, the material data set associated with the element, and the list of nodes connected to the element. There are 6 elements, with material set 1 and 10 nodes connected to each element. (see Figure B.2).

BOUNdary restraints

1 0 1 1 1 1 1 1 1
2 0 1 1 1 1 1 1 1
3 0 1 1 1 1 1 1 1
This segment of inputs assigns restraints to degrees of freedom. Each record defines a node number, a generation number, and the restraint code for each degree of freedom associated with the node. For example, node 1 (a P node) as well as nodes 2, 3, 4, 5 are assigned the condition 1 1 1 1 1 1 1 for $u_x = u_y = u_z = \theta_x = \theta_y = \theta_z = p = \Phi = 0$ at end A. Node 31 (a P node) has 0 1 1 1 0 0 1 to apply $u_y = u_z = \theta_x = p = 0$ at end B. Nodes 32, 33, 34, 35 have 1 1 1 1 1 1 1 which restrains the distortional degrees of freedom $\Phi$, as explained below. Figure B.5 shows a typical non-boundary cross-section with nodes N1-N5. Node N3, being the P node, is assigned the code 0 0 0 0 0 0 0 for beam global deformation. Regarding distortions the following conditions are assigned: N1 0 0 0 1 1 1 1, N2 0 0 0 1 1 1 1, N3 0 0 0 1 1 1 1, N4 0 0 0 1 1 1 1 and N5 0 0 0 1 1
1. Applying the boundary conditions of the 2D frame implies crossing out (i.e., restraining) the degrees of freedom corresponding to two distortional degrees of freedom at point $P$ and one degree of freedom at the support point $S$ (here identified with $N5$). Thus we have $N1 0 0 0 1 1 1 1$, $N2 0 0 0 1 1 1 1$, $N3 - 0 1 1 1 1$, $N4 0 0 0 1 1 1 1$ and $N5 0 0 - 1 1 1 1$, where - means cross-out (restrained) terms. In this input data specification, the crossed out numbers are simply omitted, and the restraining 0's and 1's are compressed together so that they can all fit in $nsec - 1$ lines of this file. For example, the above sequence has 35 restraint codes, three of which have crossed out. The remaining 32 codes are redistributed to $N1$, $N2$, $N4$, $N5$. These nodes have space for 28 code, so in addition the final four 1's are suppressed. Thus, these boundary conditions are specified as $N1 0 0 0 1 1 1 1$, $N2 0 0 0 1 1 1$, $N4 0 1 1 1 0 0$ and $N5 0 1 1 1 1 0 0$ as shown in the above input segment for nodes 7,8,9,10.

![Figure B.5: A plain frame describing distortional displacements](image-url)

FORC

\[
\begin{align*}
31 & 0 -2 0 0 0 0 2.0476 0 \\
16 & 0 0 -0.1 0 0 0 0 \\
17 & 0 0 0.1 0 0 0 0
\end{align*}
\]
This segment of inputs imposes non-zero forces. Since the coordinates of the reference point \( P \) are \( x_p = 0 \text{in}, \ y_p = 1.0238 \text{in} \), the uniform compression \( N = 2 \text{kips} \) is equivalent to the combined loading of \( N = 2 \text{kips} \) and \( M = 2.0476 \text{kips}\cdot\text{in} \) applied at point \( P \). A perturbation load of 0.1 \( \text{kips} \), 5\% of the external load, was applied to simulate overall imperfections at the mid-span node 16 and at nodes 17 and 20 to simulate local geometric imperfections (See Figure 4.5).

**Instructions for problem solving and output solution**

These contain macro commands that form the algorithm defining the particular solution method employed and commands for printing the required output variables. These commands may be used in batch mode (included in the input file) or given interactively during runtime. All the commands are interpreted by FEAP by reading the first four characters (e.g. ARCL for ARCLength command).

The arclength method is specified by ARC,\( n \), where \( n \) is the desired arclength scheme (see FEAP user manual). The basic solution step in FEAP is the command sequence

\[
\text{TANG} \\
\text{FORM} \\
\text{SOLV}
\]

which for simplicity may be replaced by the single command

\[
\text{TANG,}1.
\]

This command instructs the program to form the tangent stiffness matrix and force residual vector, and to solve the linearized system of equilibrium equations. For each time step, the program repeats this process indicated in the sequence.
DT,,v1
LOOP,,ni
TIME
LOOP,,100
TANG,,1
NEXT
NEXT,TIME

where v1 is the specified time step and ni is the Newton iterations.

The conventions of input and output data

The input data file must have the i-letter as first character (e.g. ibeam). FEAP automatically creates the output file by posing the o-letters in front of the basic name (e.g. in the previous example obeam). All input records for FEAP are in free format. Each data item is separated by a comma, an equal sign or a blank character.

B.3.1 Example

The following is a complete listing of FEAP input data for the stub column example 4.3.

FEAP ** 3D Beam: Mulligan SC/60x60

3 5 6 1 3 7 10

MATErial, 1

User 12

Finite

CROss section 0.4370 0.4534 0.7720 0 1.2254
ELASTic NEOHook 29500 0.3

REFERENCE NODE 0 0 1

! blank termination record

COORDinates

1 0 0 0
2 0 0 0
3 0 0 0
4 0 0 0
5 0 0 0
6 0 1.5235 0 0
7 0 1.5235 0 0
8 0 1.5235 0 0
9 0 1.5235 0 0
10 0 1.5235 0 0
11 0 3.047 0 0
12 0 3.047 0 0
13 0 3.047 0 0
14 0 3.047 0 0
15 0 3.047 0 0
16 0 4.5705 0 0
17 0 4.5705 0 0
18 0 4.5705 0 0
19 0 4.5705 0 0
20 0 4.5705 0 0
21 0 6.094 0 0
22 0 6.094 0 0
23 0 6.094 0 0
24 0 6.094 0 0
25 0 6.094 0 0
26 0 7.6175 0 0
27 0 7.6175 0 0
28 0 7.6175 0 0
29 0 7.6175 0 0
30 0 7.6175 0 0
31 0 9.141 0 0
32 0 9.141 0 0
33 0 9.141 0 0
34 0 9.141 0 0
35 0 9.141 0 0

! blank termination record

ELEMents

1 0 1 1 2 3 4 5 6 7 8 9 10
2 0 1 6 7 8 9 10 11 12 13 14 15
3 0 1 11 12 13 14 15 16 17 18 19 20
4 0 1 16 17 18 19 20 21 22 23 24 25
5 0 1 21 22 23 24 25 26 27 28 29 30
6 0 1 26 27 28 29 30 31 32 33 34 35

! blank termination record

BOUNDary restraints

1 0 1 1 1 1 1 1
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2011111</td>
<td>3011111</td>
<td>4011111</td>
<td>5011111</td>
<td>6000000</td>
<td>7000011</td>
<td>8000011</td>
<td>9001110</td>
<td>1000110</td>
<td>1100110</td>
</tr>
</tbody>
</table>
FORC

END

BATCH

ARCL,,3

DT,,1

LOOP,,7

TIME

LOOP,,100

TANG,,1

NEXT,iteration

Disp ALL

NEXT,TIME
B.4 Pre- and Post- Processors

The details of FEAP commands and solution algorithms are available in the FEAP manual [83, 84]. In order to assist the user with the preparation of input data, including the matrices of the material elasticity, a FEAP Preprocessor and a FEAP Postprocessor interfacing with Excel packages were developed (see Figures B.6 and B.7 respectively). Note that the Preprocessor is currently specific to plain C sections. Only minor changes, however, are needed for implementing cross-sections other than plain C.

Excel Visual Basic for Applications (VBA) programs are used to handle the user interface of the Pre- and Post-Processor with the FEAP program elmt12.f. Excel VBA helps generate the FEAP input data and the NOF.dat file that is needed for FEAP elmt12.f, by interfacing with the Excel spreadsheet and collecting information that is provided by a user on the geometric properties, loading and boundary conditions. Furthermore, Excel VBA can efficiently handle a large amount of output data and automate the postprocessing.

B.4.1 Preprocessor

A user simply fills out the excel worksheet filenames before clicking the button in the same worksheet. The following information is required.

- The directory where FeapV.exe is located
• The name of the FEAP input file, iBeam.txt for instance

• Geometric information:
  the width of the web
  the thickness of the web
  the width of the flange
  the thickness of the flange

• The number of generalized beam elements

• The number of load steps

• Boundary Condition: (1: restrained; 0: free)
  b.c. at A
  b.c. at B

• Loads at Each Step:
  applied loads at A
  applied loads at B
  imperfection equivalent loads at mid-span

Besides FEAP input data, NOF.dat is also generated by Preprocessor.

**B.4.2 Postprocessor**

It is developed to extract the FEAP output data file. The user-interface is very easy to follow. The following information is required.

• The directory FEAP output file is to be located

• The name of the FEAP output file
• The number of load steps

• The number of the generalized beam element

• The number of segments of the web

• The number of segments of the flange
Figure B.6: FEAP Element12 Preprocessor
Figure B.7: FEAP Element 12 Postprocessor
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