The Projective Geometry of Differential Operators

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THE PROJECTIVE GEOMETRY OF DIFFERENTIAL OPERATORS.

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THE PROJECTIVE GEOMETRY OF DIFFERENTIAL OPERATORS.

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This work studies the application of non-commutative projective geometry to the ring of differential operators on a smooth complex variety, or more generally a Lie algebroid on such a variety. Many classical results true about complex projective space have analogs which are proven, including Serre Finiteness, Serre Vanishing, Serre Duality, the Gorenstein property, the Koszulness property, and the Beilinson equivalence. Applications to the study of ideals, projective modules and the Grothendieck group are explored.

BIOGRAPHICAL SKETCH

Gregory Muller was born in New Jersey in 1982. He attended Bridgewater-Raritan High School from 1996-2000. He then attended Rutgers University from 2000-2004, graduating with a Bachelors of Arts in Mathematics with Highest Honors, with a minor in Physics. He was then a graduate student in Mathematics at Cornell University from 2004-2010, under the supervision of Yuri Berest.

To my loved ones and their unfailing support.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION.

1.1 Objects of Study.

This thesis studies the application of non-commutative projective geometry to the ring of differential operators on a smooth, irreducible affine variety of dimension n. We now briefly explain what this means, and why it is of interest.

1.1.1 Non-commutative Projective Geometry.

The ideas of modern algebraic geometry have been remarkably effective at studying commutative algebras, by assigning enriched spaces (schemes) to them, and then applying geometric intuition and techniques. One of the fundamental results along these lines is the Affine Serre Equivalence, which says that the category of modules of a commutative ring is equivalent to the category of quasi-coherent modules on the corresponding affine scheme. Thus, any module-theoretic question about a commutative ring can be translated into a sheaf-theoretic question on the scheme, and vice versa.

In the analogous world of graded algebras and projective schemes, it is *not* true that graded R-modules are equivalent to quasi-coherent sheaves on $Proj(R)$. However, there is a suitable quotient category $QGr(R)$ of the category $Gr(R)$ of graded R-modules which is equivalent to $QCoh(Proj(R))$; this is the Projective Serre Equivalence. An advantage of this equivalence is that the sheaf cohomology of a quasi-coherent sheaf on $Proj(R)$ can be understood in the language of graded modules as a 'section functor' $\omega : QGr(R) \to Gr(R)$ of the quotient functor $\pi : Gr(R) \to QGr(R)$ (see Section 3.2.2 for details).

The main idea is to notice that the construction of $QGr(R)$ at no point used the fact that R was commutative. Therefore, it is possible to take a non-commutative graded algebra A and associate to it a category $QGr(A)$ which plays the role of the category of quasi-coherent modules on the non-existent scheme $Proj(A)$; similarly, there is a category $qgr(A)$ which plays the role of the category of coherent modules. We have many constructions analogous to the commutative case; in particular, we have a section functor $\omega : QGr(A) \to Gr(A)$ and its higher derived functors which very naturally play the role of the sheaf cohomology functors. It is natural, then, to try to apply the techniques and intuition of commutative projective geometry to the study of non-commutative graded rings; this goes under the name *non-commutative projective geometry*. This idea has its origin in Gabriel's thesis [18], and was more explicitly explored by Artin and Zhang in [1]

1.1.2 Differential Operators and $QGr(\widetilde{\mathcal{D}})$.

Throughout, we assume the base field is $\mathbb{C}$. Let X be a smooth irreducible affine variety of dimension n, and let $\mathcal{D}(X)$ (or $\mathcal{D}$) denote the ring of algebraic differential operators on X. The ring $\mathcal{D}$ has a natural filtration by the order of an operator, with $\mathcal{D}_{<0} = 0$ and $\mathcal{D}_0 = \mathcal{O}_X$. We can then form the (graded) *Rees algebra* $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ by letting

$$\widetilde{\mathcal{D}}_i = \mathcal{D}_i \cdot t^i$$

where t is a central variable we introduce for bookkeeping. The algebra $\widetilde{\mathcal{D}}$ then a non-commutative graded algebra which contains all of the information of the ring of differential operators on X. So, especially in light of the previous section, it is natural to study the category $QGr(\widetilde{\mathcal{D}})$ and its geometric properties.

There is significant motivation for the study of the category $QGr(\widetilde{\mathcal{D}})$, aside from its intrinsic appeal. In [10], Berest and Wilson realized a classification of the right ideal classes in $\mathcal{D}(\mathbb{C})$ (the first Weyl algebra) in terms of homological information coming from 'information at infinity' in the projective geometry $QGr(\widetilde{\mathcal{D}})$.[1] This approach to classification of right ideal classes was extended to all smooth complex curves by Ben-Zvi and Nevins in [5]. They do this by proving a Beilinson equivalence for $QGr(\widetilde{\mathcal{D}})$, which is a derived equivalence from $QGr(\widetilde{\mathcal{D}})$ to a simpler algebra, and studying what happens to ideal classes.

A central philosophy in the approach of Ben-Zvi and Nevins is to notice that the algebra $\widetilde{\mathcal{D}}$ is a deformation of the symmetric algebra $Sym_X(\mathcal{T}_X \oplus \mathcal{O}_X)$, where $\mathcal{T}_X$ is the tangent bundle on X. We then notice that $Proj(Sym_X(\mathcal{T} \oplus \mathcal{O}_X)) = \overline{\mathcal{T}_X}$, the fiber-wise compactification of $\mathcal{T}_X$ into a $\mathbb{P}^{\dim(X)}$-bundle. Therefore, the category $QGr(\widetilde{\mathcal{D}})$ should be 'close' to the category $QCoh(\overline{\mathcal{T}_X})$, and so it should enjoy many of the properties of projective space, suitably redefined for the relative setting.

The thesis as a whole is in the slightly larger generality of Lie algebroids on a smooth variety. The justification for this is that it requires no extra work in the proofs, and in some cases the proofs require passing through this larger generality. It also has the advantage of including the commutative case as a special case, rather than as a 'nearby' case. See Section 2.2.5 for details. However, this introduction will stay in the narrower setting of differential operators.

[1]While right ideal classes in $\mathcal{D}(\mathbb{C})$ were first classified in [14], the classification of Berest and Wilson had several advantages, including naturally explaining the Calogero-Moser matrices that arose in the earlier classification [9] and setting the framework for [5].

1.2 Results.

This thesis proves for $QGr(\widetilde{\mathcal{D}})$ analogs of many of the known results about projective space $\mathbb{P}^n$ and $\mathbb{P}^n$-bundles. In what follows, we review in each case the commutative result, and then the non-commutative version proven here. The notation has been simplified from how the results appear in the body of the text, because the appropriate terminology hasn't been built up yet and so the statements look more directly analogous to their commutative counterparts.

1.2.1 The Gorenstein Property.

One important technical result about $\mathbb{P}^n$ is that it satisfies the *Gorenstein property.*

Lemma 1.2.1.1. *[12, 3.6.10.] Let $\mathbb{C}$ be an $R = \mathbb{C}[x_0, ...x_n]$-module by letting x_i act by zero. Then*

$$Ext^i_{Gr(R)}(\mathbb{C}, R(j)) = \left\{ \begin{array}{cc} \mathbb{C} & \text{if } i = -j = n+1 \\ 0 & \text{otherwise} \end{array} \right\}$$

where $Ext_{Gr(R)}$ denotes the higher derived functors of $Hom_{Gr(R)}$.

From this, one can deduce many important homological properties of $\mathbb{P}^n$ and $\mathbb{P}^n$-bundles.

The case of $\widetilde{\mathcal{D}}$ is no different, provided we make the appropriate relative statement. Let ω denote the canonical bundle to X.

Lemma (4.2.4.1). *(The relative Gorenstein property) Let $\mathcal{D}$ be the ring of differential operators, let $\widetilde{\mathcal{D}}$ denote its Rees algebra, and let $\mathcal{O}_X$ denote the structure* *sheaf of X (as a graded $\widetilde{\mathcal{D}}$ concentrated in degree zero).*

$$Ext^i_{Gr(\widetilde{\mathcal{D}})}(\mathcal{O}_X, \widetilde{\mathcal{D}}(j)) = \left\{ \begin{array}{ll} \omega & \text{if } i = -j = n+1 \\ 0 & \text{otherwise} \end{array} \right\}$$

1.2.2 Serre's Theorems.

For projective space $\mathbb{P}^n$, the cohomology of coherent sheaves satisfies three standard theorems of Serre.

Theorem 1.2.2.1. *[19, Theorem III.5.2 and Theorem III.7.1] Let $\mathcal{M}$ be any coherent sheaf of modules on $\mathbb{P}^n$. Then*

- *(Serre Finiteness) $H^i(\mathcal{M})$ is a finite dimensional $\mathbb{C}$-vector space for all i.*
- *(Serre Vanishing) $H^i(\mathcal{M}(j)) = 0$ if $i > n$ and any j, or if $i \neq 0$ and j is sufficiently large.*
- *(Serre Duality) If $\mathcal{M}$ is locally-free, then $H^i(\mathcal{M}) = H^{n-i}(\mathcal{M}^*(-n-1))^\vee$, where $\mathcal{M}^* = Hom(\mathcal{M}, \mathcal{O}_{\mathbb{P}^n})$ is the dual, and $\vee$ denotes the dual as a complex vector space.*

Again, we have relative versions of each of these results.[2]

Theorem (4.2.5.1). *Let $\mathcal{M} \in qgr(\widetilde{\mathcal{D}})$.*

- *(Serre Finiteness) $H^i(\mathcal{M})$ is a finitely-generated $\mathcal{O}_X$-module for all i.*
- *(Serre Vanishing) $H^i(\mathcal{M}(j)) = 0$ if $i > n$ and any j, or if $i \neq 0$ and j is sufficiently large.*

[2] Here, we are using H^i to denote the functor which will later be written $\mathbb{R}^i\omega_0$.

Serre Duality is a more delicate matter, because the relative version of the vector space dual $\vee$ is no longer an exact functor. Composing higher derived functors like cohomology functors often behaves badly; this is usually remedied by considering derived categories and derived functors there. The advantage of this change is that we no longer need to restrict to locally-free sheaves.

The 'bounded homotopy category' $K^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category whose objects are cochain complexes in $\mathcal{A}$ and whose morphisms are chain maps up to homotopy. The 'bounded derived category' $D^b(\mathcal{A})$ of $\mathcal{A}$ is the quotient category of $K^b(\mathcal{A})$ after formally inverting all quasi-isomorphisms (chain maps which induce isomorphisms on cohomology).

Let $\mathbb{R}H$ denote the derived cohomology functor, and let ${}^*\mathcal{M}$ denote the derived left dual of $\mathcal{M}$, as induced on the category $D^b(QGr(\widetilde{\mathcal{D}}))$ from the derived dual in $D^b(Gr(\widetilde{\mathcal{D}}))$. These constructions are derived versions of the more familiar functions; definitions can be found in 2.4.

Theorem (7.2.2.2). *(Serre Duality) Let $\mathcal{M} \in D^b(qgr(\widetilde{\mathcal{D}}))$. Then*

$$\mathbb{R}H(\mathcal{M}) = \mathbb{R}Hom_{-X}(\mathbb{R}H({}^*\mathcal{M}), \omega)[-n](n+1)$$

1.2.3 The Beilinson Equivalence.

A rather less-known fact about $\mathbb{P}^n$ is the Beilinson equivalence. Let $R_{\leq i} := \mathbb{C}[x_0, ...x_n]_{\leq i}$ denote the space of polynomials of degree i or less. Then let

$$Q_n := \begin{pmatrix} R_0 & R_{\leq 1} & R_{\leq 2} & \cdots & R_{\leq n} \\ 0 & R_0 & R_{\leq 1} & \cdots & R_{\leq n-1} \\ 0 & 0 & R_0 & \cdots & R_{\leq n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_0 \end{pmatrix}$$

The space Q_n is naturally an algebra, with multiplication following usual rules for matrix multiplication. The algebra Q_n is usually called the *nth Beilinson quiver algebra*, because it can be constructed as a quiver algebra in a natural way; see [13].

Theorem 1.2.3.1. *[4](The Beilinson equivalence) There is a natural equivalence of triangulated categories*

$$D^b(Coh(\mathbb{P}^n)) \simeq D^b(mod(Q_n))$$

This is the most basic example of a standard technique in the theory of derived categories called 'tilting', see Section 2.4.3.

The Beilinson equivalence has a particularly nice analog in the case of $qgr(\widetilde{\mathcal{D}})$. Let

$$E := \begin{pmatrix} \mathcal{D}_0 & \mathcal{D}_{\leq 1} & \mathcal{D}_{\leq 2} & \cdots & \mathcal{D}_{\leq n} \\ 0 & \mathcal{D}_0 & \mathcal{D}_{\leq 1} & \cdots & \mathcal{D}_{\leq n-1} \\ 0 & 0 & \mathcal{D}_0 & \cdots & \mathcal{D}_{\leq n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{D}_0 \end{pmatrix}$$

which is again an algebra in a natural way.

Theorem (6.1.3.1). *(The Beilinson equivalence for $qgr(\widetilde{\mathcal{D}})$) There is a natural equivalence of triangulated categories*

$$D^b(qgr(\widetilde{\mathcal{D}})) \simeq D^b(mod(E))$$

The algebra E is much nicer than $\widetilde{\mathcal{D}}$ in many ways; it is a finite module over $\mathcal{O}_X$ and has n idempotents which can be used to break modules down into pieces. As a consequence of this last fact, we can compute the Grothendieck group of $qgr(\widetilde{\mathcal{D}})$.

Theorem (8.2.1.1).

$$K_0(qgr(\widetilde{\mathcal{D}})) = K_0(X)^{\oplus(n+1)}$$

1.3 The Structure of the Thesis.

This work is arranged into several Chapters. Much of the content of Chapters 4, 5 and 6 appeared in [28].

2. Preliminaries. This chapter collects many disparate topics which are well-established in the literature, but are included for completeness and for the convenience of the reader. It also collects the notational conventions we establish for ease of reference.

3. Projective Geometry. This chapter reviews the basics of (commutative) projective geometry on the level of Hartshorne [19], and non-commutative projective geometry from Artin and Zhang [1]. It then discusses some specifics of the quotient category $QGr(\widetilde{\mathcal{D}})$.

4. Koszul Theory. This chapter proves analogs of the Koszul theory for $\widetilde{\mathcal{D}}$. The main results are the exactness of the Koszul complex (Theorem 4.2.2.1), the Gorenstein property (Lemma 4.2.4.1) and the Finiteness and Vanishing theorems of Serre (Theorem 4.2.5.1).

5. Tensor Products. The first section of this chapter establishes the necessary ground work for taking tensor products in non-commutative projective geometry. The second section then uses techniques from the Koszul theory to resolve the diagonal, produces canonical resolutions of objects in $qgr(\widetilde{\mathcal{D}})$ (Theorem 5.2.2.1).

6. The Beilinson Equivalence. This chapter proves the Beilinson equivalence for $qgr(\widetilde{\mathcal{D}})$ (Theorem 6.1.3.1) and writes down some explicit examples.

7. Duality. This chapter proves the Local duality theorem, and the Serre duality theorem, as well as computing the cohomology of the structure sheaf on $QGr(\widetilde{\mathcal{D}})$. This chapter only depends on Chapters 1-4.

8. A Cohomological Criterion for Projectivity. An application of Local Duality to the study of projective $\mathcal{D}$-modules is proven, as well as some useful lemmas and explicit computations.

9. Applications. The first section of this chapter discusses the application of projective geometry to ideals in $\mathcal{D}$, as well as some of the history and known results about these ideals. The second section then studies the Grothendieck group of $qgr(\widetilde{\mathcal{D}})$.

1.4 Notational Conventions and Assumptions.

- The base field of all schemes and vector spaces will be $\mathbb{C}$.

- X is a smooth, irreducible, affine variety over $\mathbb{C}$.
- $\mathcal{D}$ will denote the universal enveloping algebra of a Lie algebroid (X, L); we will be principally interested in the case when $\mathcal{D}$ is ring of algebraic differential operators on X. We will exclude the trivial case that $L = 0$.
- n will denote the the fiber dimension of the Lie algebroid; that is, the rank of the projective module L. Since we assume the Lie algebroid is non-trivial, it is always a positive integer. In the case that $\mathcal{D}$ is the ring of differential operators, n is equal to $\dim(X)$.
- For a ring R, $Mod(R)$ will denote the category of left R-modules, while $mod(R)$ will denote the category of finite left R-modules. All modules will be left modules unless otherwise specified.
- If M and N are R-bimodules, $Hom_{R-}(M, N)$ will denote the homomorphisms as left modules, while $Hom_{-R}(M, N)$ will denote homomorphisms as right modules. This notation will occasionally be used to highlight the distinction between left and right even when M or N is not an R-bimodule.
- For a graded ring R, $Gr(R)$ will denote the category of graded left R-modules, while $gr(R)$ will denote the category of finite graded left R-modules. For $M, N \in Gr(R)$, $\underline{Hom}_{Gr(R)}(M, N)$ will denote the graded Hom, which is $\oplus_{i\in\mathbb{Z}} Hom_{Gr(R)}(M, N(i))$. In general, an underline will be used to denote appropriate graded versions of certain constructions.
- Superscripts will denote cohomological indices, while subscripts will denote filtration orders and grading degrees.
- Soft brackets (n) will denote a shift in the grading degree, while hard brackets $[n]$ will denote a shift in the cohomological degree. Therefore, if M is a graded complex, then $M_p^i(q)[j] = M_{p+q}^{i+j}$.

CHAPTER 2

PRELIMINARIES.

2.1 Filtrations and Gradings.

For this section, let k be a field, and let A be a unital k-algebra. We will discuss various additional structures that can be put on A to introduce the notions of 'order' or 'degree'. Details may be found, for instance, in [16].

2.1.1 Graded Algebras and Modules.

A **grading** on A is, a family of k-subspaces $A_i \subseteq A$, $i \in \mathbb{Z}$, such that[1]

$$1 \in A_0, \quad A = \bigoplus_{i \in \mathbb{Z}} A_i \quad \text{and} \quad A_i \cdot A_j \subseteq A_{i+j}$$

A non-zero element a is called **homogeneous** if there is some i such that $a \in A_i$; this i is called the **degree** of a and is denote $deg(a)$. Zero is typically considered a homogeneous element which has either all degrees, or degree $-\infty$. By definition, degree satisfies $deg(ab) = deg(a) + deg(b)$ if $ab \neq 0$; in particular, the product of homogeneous elements is homogeneous. The space A_i is called the i**th graded component**, and

$$A_{\leq i} := \bigoplus_{j \in \mathbb{Z},\, j \leq i} A_j \quad \text{and} \quad A_{\geq i} := \bigoplus_{j \in \mathbb{Z},\, j \geq i} A_j$$

A graded algebra is called

- **Positively-graded** if $A_i = 0$ for all $i < 0$

[1] As a general rule of thumb, subscripts will denote the index of filtrations and gradings, so that superscripts can be used for cohomological degrees.

- **Connected** if it is positively graded, and A_0 is spanned by the unit of A.
- **Regular-graded** if it is generated as an algebra by A_0 and A_1. Note that it is then automatically positively graded.

The standard examples of graded algebras include the ring of polynomials in n variables, with the generators having degree 1.

If A is a graded k-algebra, and M is a left A-module, a **grading** on M is a family of k-subspaces M_i, $i \in \mathbb{Z}$ such that

$$M = \bigoplus_{i \in \mathbb{Z}} M_i \quad \text{and} \quad A_i \cdot M_j \subseteq M_{i+j}$$

Homogenous elements of M and their degree are defined the same as above. A **graded** or **homogeneous submodule** of M is an A-submodule N, such that the subspaces $N_i := N \cap M_i$ define a grading on N. The quotient of M by a graded submodule has a natural grading.

Given two graded A-modules M and N, a **graded morphism** (or a **morphism of degree zero**) is an A-module map $f : M \to N$ such that $f(M_i) \subseteq N_i$ for all i. Denote the k-space of all such maps by $Hom_{Gr(A)}(M, N)$. This defines the category of all graded (left) A-modules $Gr(A)$.

If M is a graded A-module, then let $M(i)$ denote the graded A-module which is isomorphic to M as an A-module, but the grading is given by $(M(i))_j := M_{i+j}$. This is functorial in the obvious way, and is called the i**th shift functor**, or the **shift functor** when $i = 1$. Define the **internal** or **graded** Hom to be the graded k-space

$$\underline{Hom}_{Gr(A)}(M, N) := \bigoplus_{i \in \mathbb{Z}} Hom_{Gr(A)}(M, N(i))$$

An element in $Hom_{Gr(A)}(M, N(i))$ is called a **morphism of degree** i.

If M is a graded left A-module and N is a graded right A-module, then the tensor product $N \otimes_A M$ may be given the structure of a graded k-space, where $(N \otimes_A M)_i$ is spanned by elements of the form $n \otimes_A m$ with n and m homogeneous and $deg(n) + deg(m) = i$; this is called the **graded tensor product over** A. We define the **degree zero tensor product** to be the k-space

$$N \circledcirc_A M = (N \otimes_A M)_0$$

which is the degree zero part of $N \otimes_A M$.

2.1.2 Filtered Algebras and Modules.

An **(ascending) filtration** on A is, for every integer i, a k-subspace $A_i \subseteq A$, such that

$$1 \in A_0, \quad A_i \subset A_{i+1} \quad \text{and} \quad A_i \cdot A_j \subseteq A_{i+j}$$

The filtration is **positive** if $A_{-1} = 0$, it is **exhaustive** if $\bigcup_{i \in \mathbb{Z}} A_i = A$, and it is **separated** if $\bigcap_{i \in \mathbb{Z}} A_i = 0$.

If a is a non-zero element in an exhaustive, separated filtered k-algebra A, then the smallest i such that $a \in A_i$ is called the **order** of a, and is denoted $ord(a)$. By the definition of ascending filtrations, order is sub-tropical (where $a, b \in A$):

$$ord(ab) \leq ord(a) + ord(b) \quad \text{and} \quad ord(a + b) \leq \max(ord(a), ord(b))$$

Examples of filtered k-algebras include rings of (continuous/ smooth/ analytic/ algebra) differential operators from an an appropriate space to k, and any quotient of a graded algebra (see next section).

If A is a filtered k-algebra, and M is a left A-module, an **(ascending) filtra-**

tion on M is, for every integer i, a k-subspace $M_i \subseteq M$, such that

$$M_i \subseteq M_{i+1} \text{ and } A_i \cdot M_j \subseteq M_{i+j}$$

Exhaustive and separated filtrations on M are defined the same as above, as is the order of an element of M. If $m, n \in M$ and $a \in A$, then

$$ord(am) \leq ord(a) + ord(m) \quad \text{and} \quad ord(m+n) \leq \max(ord(m), ord(n))$$

A **filtered submodule** of M is a submodule N, together with a filtration such that $N_i \subseteq M_i$ for all i. A submodule N of M may always be made into a filtered submodule by defining the **induced filtration** $N_i := N \cap M_i$, though not all filtered submodules arise this way. The quotient of M by a filtered submodule has a natural filtration. Notice that a submodule of a graded module had *at most* one compatible grading, while a submodule of a filtered module has *at least* one compatible filtration.

A graded algebra A may always be regarded as a filtered algebra, by using the **forgetful filtration** $\{A_{\leq i}\}$; similarly, graded modules may be made into filtered modules. Since all submodules can be filtered in a natural way, the quotient of a graded algebra or module still has a natural filtration, where the order of an element in the quotient is given by the smallest degree of any pre-image.

2.1.3 Rees Algebras and Modules.

There is a standard way to construct a graded algebra from a filtered algebra A called the **Rees construction**. Define the **Rees algebra** $\widetilde{A}$ of A to be the graded algebra such that

$$\widetilde{A} := \bigoplus_{i \in \mathbb{Z}} A_i \cdot t^i$$

where t is a formal variable which is central, and the graded multiplication maps $\widetilde{A}_i \otimes \widetilde{A}_j \to \widetilde{A}_{i+j}$ are given by the filtered multiplication maps $A_i \otimes A_j \to A_{i+j}$ and the ts are commuting and keeping track of degree.

If M is a filtered A-module, then the **Rees module** of M is defined as the graded $\widetilde{A}$-module

$$\widetilde{M} := \bigoplus_{i \in \mathbb{Z}} M_i \cdot t^i$$

with the action of $\widetilde{A}$ coming from the action maps $A_i \otimes M_j \to M_{i+j}$. Again, ts commute past other elements, and collect on the right, keeping track of degree.

An exhaustive filtered algebra A may be recovered from $\widetilde{A}$, by quotienting $\widetilde{A}$ by the two-sided ideal $\langle t-1 \rangle$ generated by $t-1$. By giving $\widetilde{A}$ the forgetful filtration and $\langle t-1 \rangle$ the induced filtration as a submodule, the quotient $A = \widetilde{A}/\langle t-1 \rangle$ is filtered; this recovers the original filtration on A. An exhaustive filtered A-module M may be recovered from $\widetilde{M}$ the same way, or equivalently by tensoring $A \otimes_{\widetilde{A}} \widetilde{M}$.

Therefore, the algebra $\widetilde{A}$ and the category of graded $\widetilde{A}$-modules contains all the information of A and its filtered modules. However, not every graded $\widetilde{A}$-module N is the Rees module of a filtered A-module. A necessary and sufficient condition is that $t \in \widetilde{A}$ is not a zero divisor on N; in this case, $N \simeq \widetilde{A \otimes_{\widetilde{A}} N}$.

2.1.4 Associated Algebras and Modules.

For an exhaustive filtered algebra A, the Rees algebra $\widetilde{A}$ and its graded modules contain all the information of A and its filtered modules; but it also contains some extraneous information, in the form of elements killed by t. We introduce an algebra which can be used to study this extraneous information separately. Define the **associated graded algebra** of A to be

$$\overline{A} := \widetilde{A}/\langle t \rangle$$

where $\langle t \rangle$ is the two-sided ideal generated by t. Since t is a homogeneous element, it generates a homogeneous ideal and so the quotient $\overline{A}$ is also graded. The graded components have a straight-forward presentation:

$$(\overline{A})_i = A_i/A_{i-1}$$

Any degree i element $a \in A$ can be assigned to $\sigma(a) \in \overline{A}_i$, where $\sigma(a)$ is the image of a in $A_i/A_{(i-1)}$. If the filtration is separated, then this defines a map σ on all of A called the **symbol map**; however, in general this map is not even additive!

Filtered A-modules M also have associated graded modules; they can be defined as the tensor $\overline{A} \otimes_{\widetilde{A}} \widetilde{M}$ or as quotients $\widetilde{M}/t\widetilde{M}$. If the filtration on M is exhaustive and separated, then there is also a symbol map $\sigma : M \dashrightarrow \overline{M}$; the dash is to imply this map is only a map of sets. A filtration on a finitely-generated A-module M is called a **good filtration** if $\overline{M}$ is a finitely generated $\widetilde{A}$-module.

One of the most useful properties of passing to associated graded modules is that isomorphisms can still be characterized in many cases.

Lemma 2.1.4.1. *Let M and N be exhaustive filtered A-modules such that $M_i = N_i = 0$ for $i \ll 0$, and let $f : M \to N$ be an A-module map such that $f(M_i) \subseteq N_i$ for all i (a filtered map). Then the induced map $\overline{f} : \overline{M} \to \overline{N}$ is an isomorphism if and only if f is.*

Proof. Assume that $\overline{f}$ is an isomorphism; this means

$$\overline{f}_i : M_i/M_{i-1} \to N_i/N_{i-1}$$

is an isomorphism for all i. Let m (resp. n) be the smallest integer such that M_m is non-zero (resp N_n). Since M_m is non-zero while M_{m-1}, $\overline{M}_m$ is necessarily non-zero; similarly, $N_n/N_{n-1} = \overline{N}_n$ is non-zero. We claim $n = m$. Otherwise, in degree $\min(m, n)$, the map $\overline{f}_{\min(m,n)}$ is an isomorphism between a non-zero group and zero. Therefore, $M_i = N_i = 0$ for $i < m$.

We now prove by induction that f restricts to an isomorphism of abelian groups between M_i and N_i for all $i \geq m$. For $i = m$, $M_m = \overline{M}_m$ and $N_m = \overline{N}_m$, and the map $\overline{f}_m$ is the restriction of f. Therefore, f is an isomorphism on M_m.

Now assume that f restricts to an isomorphism on M_i. Then there is a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M_i & \to & M_{i+1} & \to & \overline{M}_{i+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N_i & \to & N_{i+1} & \to & \overline{N}_{i+1} & \to & 0 \end{array}$$

By the inductive hypothesis, the left vertical arrow is an isomorphism. By the assumption that $\overline{f}_{i+1}$ is an isomorphism, the right vertical arrow is an isomorphism. Hence, by the Five Lemma (see [36]), the middle vertical arrow is an isomorphism. Thus, by induction, f is an isomorphism on every M_i, and so by the exhaustiveness of the filtration, f is an isomorphism. □

2.2 Differential Operators and Lie Algebroids.

In this section, we recall the basics of the ring of differential operators on a smooth, affine, irreducible variety X of dimension n over $\mathbb{C}$. We then introduce a generalized concept called a *Lie algebroid* which simultaneously generalizes both differential operators and Lie algebras.

2.2.1 Differential Operators.

The filtered ring of differential operators $\mathcal{D}(A)$ on a commutative k-algebra A can be defined by induction as follows. Let $\mathcal{D}(A)_{-1} := 0$, and for any $i \in \mathbb{N}$,

$$\mathcal{D}(A)_i := \{\delta \in End_k(\mathcal{O}_A) \mid \forall f \in A,\ [f, \delta] \in \mathcal{D}(A)_{i-1}\}$$

Equivalently, elements of $\mathcal{D}(A)_i$ are k-linear endomorphisms of A such that for any collection of $i + 1$ elements $\{f_j\} \in A$, the iterated commutator is zero:

$$[f_1, [f_2, [...[f_{i+1}, \delta]...]]] = 0$$

By the linearity of commutators, each of the $\mathcal{D}(A)_i$ is a subspace of $End_k(A)$. From the Leibniz rule for commutators ($[a, bc] = [a, b]c + b[a, c]$), it follows that $\mathcal{D}(A)_i \cdot \mathcal{D}(A)_j \subseteq \mathcal{D}(A)_{i+j}$, where multiplication is given by composition.

Define the **ring of differential operators** $\mathcal{D}(A)$ (or just $\mathcal{D}$, when A is clear) as the union over all $\mathcal{D}(A)_i$. This is a filtered ring which is exhaustive and positively-filtered (therefore, separated) by construction. Every element $f \in A$ commutes with every other element, and so $f \in \mathcal{D}_0$; in fact, the induced map

$$A \xrightarrow{\sim} \mathcal{D}_0$$

is an isomorphism of algebras.

Example. Let $A = \mathbb{C}[x_1, ...x_n]$ be the ring of complex polynomials in n variables. Then

$$\mathcal{D}(A) = \mathbb{C}[x_1, ...x_n, \partial_1, ...\partial_n]$$

is the nth Weyl algebra, where the above generators commute except for the relations $[\partial_i, x_i] = 1$.

It follows immediately from the definition of $\mathcal{D}$ that $[\mathcal{D}_0, \mathcal{D}_i] \subseteq \mathcal{D}_{i-1}$; from the Jacobi identity, the more general fact follows:

$$[\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j-1}$$

This means that for any two differential operators δ, δ', the products $\delta\delta'$ and $\delta'\delta$ are equal, up to terms of lower order. Since lower order terms are killed in the associated graded algebra, the algebra $\overline{\mathcal{D}}$ is commutative. This will be an eternally useful fact.

The above definition and observations were true for any commutative algebra A; we now turn to the case where $A = \mathcal{O}_X$, where X is a smooth, affine, irreducible variety of dimension n over $\mathbb{C}$. The ring of differential operators is typically denoted $\mathcal{D}(X)$ rather than $\mathcal{D}(\mathcal{O}_X)$. In this case, we have several facts (for proofs, see [27]).

- The ring $\mathcal{D}(X)$ is a simple Noetherian ring without zero divisors.
- The ring $\mathcal{D}(X)$ is generated as a $\mathbb{C}$-algebra by finitely many elements of degree zero and one.
- The associated graded algebra $\overline{D(X)}$ is canonically isomorphic to $\mathcal{O}(\mathcal{T}^*X)$, the ring of functions on the cotangent bundle to X.
- The ring $\mathcal{D}(X)$ has global dimension n.

This list of nice properties is also remarkably delicate, in terms of varying the hypotheses. If X is singular, then $\mathcal{D}(X)$ is in general no longer generated in degrees zero and one, and can be infinitely generated and non-Noetherian.[2] If $\mathbb{C}$ is replaced by an algebraically closed field of positive characteristic, then the ring $\mathcal{D}(X)$ will have a very large center, and hence it will be non-simple. Also, it can

[2] In fact, it is an outstanding conjecture of Nakai's [29] that $\mathcal{D}(X)$ is generated by elements of order zero and one if and only if X is smooth.

be infinitely-generated, non-Noetherian and possess zero divisors, even when X is smooth (see Smith, [34]).

The only assumption which is not critical is that X is affine; however, in the non-affine case, all the appropriate definitions must be sheafified. Provided this is done correctly, all the above properties are still true.

2.2.2 Lie Algebroids.

Lie algebroids are a simultaneous generalization of rings of differential operators and of Lie algebras. Studying them can be very useful for understanding those aspects of the representation theory of Lie algebras which have an analogous statement for the representation theory of differential operators. However, there are many interesting Lie algebroids which are neither Lie algebras nor differential operators. For a more detailed reference, consult [26].

The study of Lie algebroids is meant to be the study of families of infinitesmal symmetries, in the way that the study of Lie algebras is the study of infinitesmal symmetry.

Let X be as in the previous section; a smooth, irreducible, affine variety of dimension n over $\mathbb{C}$. An (algebraic) **Lie algebroid** on X is a finitely-generated projective[3] $\mathcal{O}_X$-module L with

- a **Lie bracket** on L which makes it into a Lie algebra over $\mathbb{C}$.

[3]Lie algebroids can be defined with the projectivity requirement. However, since such Lie algebroids are both qualitatively very different than projective ones, and not amenable to the techniques of this paper, such a possibility is ignored. Also, in the case when X is not affine, the condition must be relaxed to 'locally projective'.

- an **anchor map**, an $\mathcal{O}_X$-module map $\tau : L \to \mathcal{T}_X$.[4]

The bracket and the $\mathcal{O}_X$-module structure on L are *not* necessarily compatible in the simplest way; instead, the bracket and the $\mathcal{O}_X$-multiplication satisfy the relation:

$$[l, al'] = a[l, l'] + d_{\tau(l)}(a) \cdot l'$$

One consequence of the relation is that $\mathcal{O}_X \oplus L$ becomes a Lie algebra by the bracket $[(r, l), (r', l')] = (d_{\tau(l)}(r') - d_{\tau(l')}(r), [l, l'])$.

The idea is that sections of L describe families of 'infinitesimal symmetries' on X, which can be moving in directions both along X and in hidden 'internal' directions. The two basic examples reflect each of these possiblities:

1. *(Differential Operators, or the Tangent Lie Algebroid)* Let $L = \mathcal{T}_X$, endowed with the Lie bracket coming from the commutator of vector fields, and the anchor map being the identity map $\mathcal{T}_X \to \mathcal{T}_X$. Then L is a Lie algebroid; here, all the infinitesmal symmetries being described by sections of L are along X, since they are given by vector fields.
2. *(Lie algebras)* Let $X = \mathbb{C}$, and let $L = \mathfrak{g}$ be any finite-dimensional Lie algebra over $\mathbb{C}$. Since $T_{\mathbb{C}} = 0$, the anchor map is the zero map. This defines a Lie algebroid over $\mathbb{C}$; here, all the infinitesmal symmetries are internal, in that sections of L are describing directions which are not coming from directions along X.

A Lie algebroid is called **trivial** if L is the zero module, and it is called **abelian** if both the Lie bracket and the anchor map are zero.

[4]Here, and throughout this thesis, $\mathcal{T}_X$ will denote the tangent bundle to X

A Lie algebroid comes with instructions on how to commute two sections of L past each other (the bracket) and how to commute sections of L past sections of $\mathcal{O}_X$ (the anchor). This naturally leads to the consideration of the universal algebra generated by L and $\mathcal{O}_X$ which obey the given commutation relations. Let $\mathcal{D}(X, L)$ be the quotient of the universal enveloping algebra of the Lie algebra $\mathcal{O}_X \oplus L$ by the relations $(1, 0) = 1$ and $(a, 0) \otimes (a', l) = (aa', al)$ (1 the unit, $a \in \mathcal{O}_X$, and $l \in L$); this is called the **universal enveloping algebra** of L. The algebra $\mathcal{D}(X, L)$ will be denoted $\mathcal{D}$ when X and L are clear. In the case of the tangent Lie algebroid $(X, \mathcal{T})$, the enveloping algebra $\mathcal{D}$ is the ring of algebraic differential operators.

The ring $\mathcal{O}_X$ has a canonical structure of a left $\mathcal{D}$-module, by the action $a \cdot a' = aa'$ and $l \cdot a = d_{\tau(l)}(a)$ for $a, a' \in \mathcal{O}_X$ and $l \in L$. The 'action on 1' map $\mathcal{D} \to \mathcal{O}_X$ which sends σ to $\sigma \cdot 1$ is a left $\mathcal{D}$-module map which presents $\mathcal{O}_X$ as a quotient of $\mathcal{D}$ as a left module over itself. Note however, that there is no canonical right $\mathcal{D}$-module structure on $\mathcal{O}_X$.

The algebra $\mathcal{D}$ is naturally filtered by letting the image of $\mathcal{O}_X$ be degree 0 and the image of L be degree 1. The subspace $\mathcal{D}_1$ is a (not necessarily central) $\mathcal{O}_X$-bimodule which fits into a short exact sequence of $\mathcal{O}_X$-bimodules:

$$0 \to \mathcal{O}_X \to \mathcal{D}_1 \to L \to 0$$

The Rees algebra of $\mathcal{D}$ can be defined directly as a quotient of the tensor algebra $T_X\mathcal{D}_1$ by the relation $\partial \otimes \partial' - \partial' \otimes \partial = [\partial, \partial'] \otimes t$, where $\partial, \partial' \in \mathcal{D}_1$ and t denotes $1 \in \mathcal{O}_X \subset \mathcal{D}_1$ (as opposed to the unit of the algebra).

Other examples of Lie algebroids. We conclude with other interesting examples of Lie algebroids.

- *(Vector Bundles)* If L is any f.g. projective $\mathcal{O}_X$-module, then L can be given both a trivial Lie bracket and a trivial anchor map, making (X, L) into an **abelian** Lie algebroid. Geometrically, this corresponds to an algebraic vector bundle V with no meaningful extra structure. In this case, the universal enveloping algebra $\mathcal{D}$ is commutative, and is isomorphic to the ring of functions on the dual vector bundle V^*.

- *($\mathcal{O}_X$-Lie algebras)* A Lie algebroid (X, L) with trivial anchor map is the same thing as a Lie algebra object in the category of f.g. projective $\mathcal{O}_X$-modules. Geometrically, this amounts to an algebraic vector bundle with each fiber equipped with a Lie bracket, such that the brackets vary algebraically.

- *(Foliations)* If the variety X is equipped with a foliation by constant-dimensional submanifolds (called **leaves**), then there is a subbundle L of the tangent bundle consisting of the tangent bundles of the leaves. Sections of this bundle are vector fields which are tangent to the leaves. The commutator of two of these vector fields is still tangent to the leaves; hence, the space of sections of L is a Lie subalgebra of the space of vector fields (sections of $\mathcal{T}$). The inclusion of bundles $L \hookrightarrow \mathcal{T}$ defines the anchor map, which makes (X, L) into a Lie algebroid.

- *(Poisson Varieties)* If the variety X is equipped with a Poisson structure, then the cotangent bundle $\mathcal{T}^*$ has the structure of a Lie algebroid. The bracket of exact 1-forms is defined by

$$[df, dg] := d(\{f, g\})$$

This bracket is well-defined, and can be extended to non-exact 1-forms by first computing the bracket locally on exact forms, and checking that it patches together. The anchor map $\mathcal{T}^* \to \mathcal{T}$ is also defined locally on exact 1-forms, by sending df to the vector field corresponding to the derivation $\{f, -\}$.

2.2.3 The PBW Theorem for Lie Algebroids.

Because the commutator of a degree i element and a degree j element in $\mathcal{D}$ is of degree at most $i+j-1$, the associated graded algebra $\overline{\mathcal{D}}$ is commutative. In fact, the structure of the associated graded algebra is well-known.

Theorem 2.2.3.1. *(PBW theorem for Lie algebroids)[32, Theorem 3.1]The natural maps*

$$\overline{\mathcal{D}}_0 = \mathcal{D}_0 \xrightarrow{\sim} \mathcal{O}_X \quad \textit{and} \quad \overline{\mathcal{D}}_1 = \mathcal{D}_1/\mathcal{D}_0 \xrightarrow{\sim} L$$

extend to a canonical isomorphism of algebras

$$\overline{\mathcal{D}} \xrightarrow{\sim} Sym_X L$$

where $Sym_X L$ is the symmetric algebra of L over X.

The ring $Sym_X L$ is also isomorphic to $f^*(\mathcal{O}_{L^*})$, the total space of the dual bundle to L pushed forward along the bundle map $f : L^* \to X$.

The PBW property implies many of the most important properties of $\mathcal{D}$.

Corollary 2.2.3.1. *1. For all i, $\mathcal{D}_i$ is projective and finitely-generated as both a left and right $\mathcal{O}_X$-module (though not as a bimodule).*

2. $\mathcal{D}$ is a Noetherian ring without zero divisors.

Proof. (1) By the PBW theorem, $\mathcal{D}_i/\mathcal{D}_{i-1} = \overline{\mathcal{D}}_i = (Sym_X L)_i$ is f.g. projective, and so $\mathcal{D}_i$ has a finite composition sequence consisting entirely of f.g. projectives. Therefore, $\mathcal{D}_i$ is f.g. projective.

(2) Let

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$$

be an ascending chain of left ideals in $\mathcal{D}$. Each left ideal is naturally filtered as a submodule of $\mathcal{D}$, and the inclusions $I_i \hookrightarrow I_{i+1}$ are filtered maps. Therefore, there is an ascending chain of ideals in $\overline{\mathcal{D}}$

$$\overline{I}_0 \subseteq \overline{I}_1 \subseteq \overline{I}_2 \subseteq ...$$

Since $\overline{\mathcal{D}} = Sym_X L$, and $Sym_X L$ is Noetherian, all but a finite number of the above inclusions are isomorphisms. However, by Lemma 2.1.4.1, this implies that all but a finite number of the original inclusion maps were also isomorphisms. Thus, $\mathcal{D}$ is left Noetherian. A similar argument shows that $\mathcal{D}$ is right Noetherian.

The ring $\mathcal{D}$ has no zero divisors because $\overline{\mathcal{D}} = Sym_X L$ has no zero divisors. □

2.2.4 Localization.

As was mentioned before, Lie algebroids are compatible with localization; that is, the localization of a Lie algebroid naturally has a Lie algebroid structure. To wit, let X' be an affine open subscheme of affine X defined by a multiplicative subset S of $\mathcal{O}_X$, and let $L' := \mathcal{O}_{X'} \otimes_X L$.

Lemma 2.2.4.1. *If (X, L) is a Lie algebroid, then (X', L') has a unique Lie algebroid structure which is compatible with the inclusion $L \to L'$.*

Proof. For any $l \in L$ and $s \in S$, the anchor map defines the derivative of s along l to be $d_{\tau(l)}(s)$. Therefore, there is only one choice for the derivative of s^{-1} along l,

$$d_{\tau(l)}(s^{-1}) := -s^{-2} d_{\tau(l)}(s)$$

because $d_{\tau(l)}$ must be a derivation. In this way, the anchor map $L \to \mathcal{T}_X$ extends canonically to an anchor map $L \to \mathcal{T}_{X'}$. The $\mathcal{O}_{X'}$-module structure on $\mathcal{T}_{X'}$ means that this map extends uniquely to a map $L' \to \mathcal{T}_{X'}$.

Elements in L' are of the form $s^{-n} \otimes l$, for $s \in S$ and $l \in L$, and so the compatibility of the anchor map with the Lie bracket implies that

$$[s^{-n} \otimes l, s'^{-m} \otimes l'] = s^{-n} d_{\tau(l)}(s'^{-m}) \cdot l' + s'^{-m}[s^{-n} \otimes l, l']$$

$$= s^{-n} d_{\tau(l)}(s'^{-m}) \cdot l' - s'^{-m} d_{\tau(l')}(s^{-n}) \cdot l + s'^{-m} s^{-n}[l, l']$$

Since this final expression only depends on the Lie bracket in L, and the extended anchor map, the Lie bracket on L' is completely determined. □

The above technique for localizing Lie algebroids is clearly compatible with compositions of localizations, and defines a sheaf of Lie algebroids on X, for X affine. In the case of X not affine, this local data may be sheafified; we will call any sheaf of Lie algebroids obtained this way a **Lie algebroid on** X.

For X affine, and X' an affine open subscheme, $\mathcal{D}(X', L') = \mathcal{O}_{X'} \otimes_X \mathcal{D}(X, L) = \mathcal{D}(X, L) \otimes_X \mathcal{O}_{X'}$. This means that localizing enveloping algebras is the same on the left and on the right; so from now on we can refer to localizing them without refering to a side. An $\mathcal{O}_X$-bimodule which has the property that left localization is isomorphic to right localization will be called **nearly central**; since it means that as a sheaf on $X \times X$, it is supported scheme-theoretically on the diagonal.

The universal enveloping algebra of a non-affine Lie algebroid (X, L) will be defined as the sheaf of algebras $\mathcal{D}(X, L)$ which is affine-locally the enveloping algebra of (X, L). Since enveloping algebras are nearly central, this is a quasi-coherent sheaf as both a left and right $\mathcal{O}_X$-module.

It is worth noting that, while the global sections of a Lie algebroid (X, L) is again a Lie algebroid $(\Gamma(X), \Gamma(L))$, the global sections of $\mathcal{D}(X, L)$ is not necessarily the enveloping algebra of $(\Gamma(X), \Gamma(L))$. For example, take the tangent bundle on

$\mathbb{P}^1$. The global Lie algebroid is $(\mathbb{C}, \mathfrak{sl}_2)$ with trivial anchor map, but the global sections of $\mathcal{D}_{\mathbb{P}^1}$ is the algebra $\mathcal{U}\mathfrak{sl}_2/c$, where c is the Casimir element; see e.g. [15]

2.2.5 The Relevance of Lie Algebroids.

While the main objects of interest of this thesis are rings of differential operators $\mathcal{D}(X)$, there are two reasons to care about the larger generality of Lie algebroids. The first is that all the results presented here are true in this larger generality, and so there is an argument that can be made for stating things in the largest possible generality.

The second is that some of the proofs *require* the larger generality of Lie algebroids. For example, the exactness of the Koszul complex (Theorem 4.2.2.1) is proven first for abelian Lie algebroids and then deformed to the non-abelian case. This particular strategy of proof would not work if only the case of differential operators were considered.

However, because the majority of results and proofs are the same for both differential operators and other Lie algebroids, the distinction between the two cases will often be downplayed, with the letter $\mathcal{D}$ used to denote either the ring of differential operators, or the universal enveloping algebra of a Lie algebroid.

2.3 Quotients of Abelian Categories.

In this section, we review the techniques for 'quotienting' an abelian category by a subcategory which is to be sent to zero. For more details, see [31].

2.3.1 Quotients and Localizing Subcategories.

Given an abelian category $\mathcal{C}$, and a full subcategory $\mathcal{L}$, what is simplest category $\mathcal{C}/\mathcal{L}$ with a functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{L}$ such that every object in $\mathcal{L}$ becomes isomorphic to the zero object? Such a category $\mathcal{C}/\mathcal{L}$ is called the the **quotient category of $\mathcal{C}$ by $\mathcal{L}$**.[5] However, the set of subobjects $\mathcal{L}$ needs an additional property if the quotient is to be nice. The full subcategory $\mathcal{L}$ is called a **dense subcategory** of $\mathcal{C}$ if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$, B is in $\mathcal{L}$ if and only if A and C are.

The general idea behind the construction of $\mathcal{C}/\mathcal{L}$ is this. Consider the set of morphism $\Sigma_{\mathcal{L}}$ whose kernel and cokernel are in $\mathcal{L}$. Then, let $\mathcal{C}/\mathcal{L}$ be the category whose objects are the same as $\mathcal{C}$, but whose morphisms are generated by morphisms in $\mathcal{C}$ and by formal inverses to every morphism in $\Sigma_{\mathcal{L}}$ (this is called the **(additive) localization of $\mathcal{C}$ by $\Sigma_{\mathcal{L}}$**). Modulo some concerns about the resulting Homs being sets, this category can always be defined, and shown to have a universal property with respect to sending $\mathcal{L}$ to zero.

Theorem 2.3.1.1. *[31, Thm 4.3.8] Let $\mathcal{C}$ be a locally small[6] abelian category, and let $\mathcal{L}$ be a dense subcategory. Then there is an abelian category $\mathcal{C}/\mathcal{L}$ and an exact functor*

$$\pi : \mathcal{C} \to \mathcal{C}/\mathcal{L}$$

such that for any other additive functor $F : \mathcal{C} \to \mathcal{D}$ with $F(\mathcal{L}) \simeq 0$, there is a

[5]This category is also sometimes called the **localization category**. The reason for the seemingly-conflicting terms is that objects in $\mathcal{L}$ go to zero (like in quotients of rings), while morphisms in $\mathcal{L}$ and more generally $\Sigma_{\mathcal{L}}$ go to invertible morphisms (like in localizations of rings).

[6]Locally small here means that, for all $X \in \mathcal{C}$, the class of isomorphism classes of monomorphisms into X is a set.

unique additive functor $G : \mathcal{C}/\mathcal{L} \to \mathcal{D}$ such that $G \circ \pi = F$. Furthermore, a morphism $\pi(f)$ is an isomorphism if and only if $f \in \Sigma_{\mathcal{L}}$.

However, it can be difficult to work in the category $\mathcal{C}/\mathcal{L}$, because it is defined in a very abstract way. Morphisms in $\mathcal{C}/\mathcal{L}$ are defined as formal fractions of morphisms in $\mathcal{C}$ by those in $\Sigma_{\mathcal{L}}$. In order to get a more concrete category, it is often useful to try to embed $\mathcal{C}/\mathcal{L}$ in the category $\mathcal{C}$, which is typically easier to work in. The standard way to do this is to ask if the functor π has a right adjoint. If the functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{L}$ has a right adjoint ω, then $\mathcal{L}$ is called a **localizing subcategory** and ω is called the **section functor**. If ω exists, then $\pi\omega$ is the identity functor on $\mathcal{C}/\mathcal{L}$; hence the name 'section functor'.

Lemma 2.3.1.1. *[31, Prop. 4.5.2] Let $\mathcal{L}$ be a dense subcategory of a locally small abelian category $\mathcal{C}$ with enough injectives. Then $\mathcal{L}$ is localizing if and only if, for every $M \in \mathcal{C}$, the set of subobjects $N \subseteq M$ with $N \in \mathcal{L}$ has a greatest element.*

If $\mathcal{L}$ is localizing, then assigning to every $M \in \mathcal{C}$ its largest submodule $\tau(M) \in \mathcal{L}$ is functorial. In fact, the functor $\tau : \mathcal{C} \to \mathcal{L}$ is right adjoint to the inclusion functor $\iota : \mathcal{L} \to \mathcal{C}$ (which is exact by the density of $\mathcal{L}$). The functor $\iota\tau$ will often be denoted τ when no confusion will arise.

A straight-forward argument shows that an additive functor between two abelian categories which is a right adjoint is left exact, and left adjoints are right exact. This has the immediate consequence that the section functor ω and the maximal $\mathcal{L}$-subobject functor τ are left exact. Since ι is exact, $\iota\tau$ is also left exact.

2.3.2 Properties of Quotients.

When $\mathcal{L}$ is a localizing subcategory, the quotient cateogry $\mathcal{C}/\mathcal{L}$ often inherits the nice properties of $\mathcal{C}$.

Lemma 2.3.2.1. *Let $\mathcal{L}$ be a localizing subcategory of a locally small abelian cateogory $\mathcal{C}$. Then $\mathcal{C}/\mathcal{L}$ has each of the following properties if $\mathcal{C}$ has the corresponding property.*

1. *Enough injectives. [31, Prop 4.5.3]*
2. *The $Ab3$ condition. That is, the existence of arbitrary direct sums. [31, Prop. 4.6.1]*
3. *The $Ab4$ condition. That is, arbitrary direct sums of short exact sequences are still short exact. [31, Prop. 4.6.1]*
4. *The $Ab5$ condition. That is, the direct limit of a directed family of short exact sequences is still short exact. [31, Prop. 4.6.1]*
5. *The existence of a generator. That is, there is some object T such that for any distinct parallel morphisms $f, g : M \to N$, there is some $h : T \to M$ such that $fh \neq gh$. [31, Lemma 4.4.8.]*

It should be noted that the module category of any ring satisfies these conditions, and hence, so does the localization of any module category.

An important consequence of the $Ab5$ condition is that it means derived functors are compatible with direct limits.

Proposition 2.3.2.1. *Let $\mathcal{C}$ have enough injectives, let $\mathcal{D}$ be an $Ab5$-category, and let $F_j : \mathcal{C} \to \mathcal{D}$ be a direct system of left exact functors. Then for all i and all*

$M \in \mathcal{C}$,

$$\varinjlim \mathbb{R}^i F_j(M) = \mathbb{R}^i \left(\varinjlim F_j(M)\right)$$

Proof. The *Ab*5 condition says that direct limits of short exact sequences are short exact. As an immediate consequence, the ith cohomology of a direct limit of complexes is the direct limit of the ith cohomology of the complexes.

Let $I^\bullet$ be an injective resolution of M. Then

$$\varinjlim \mathbb{R}^i F_j(M) = \varinjlim H^i(F_j(I^\bullet)) = H^i(\varinjlim F_j(I^\bullet)) = \mathbb{R}^i(\varinjlim F_j(M))$$

Thus, it is proven. □

2.4 Derived Categories.

To any abelian category $\mathcal{A}$, we can define the corresponding derived category $D(\mathcal{A})$, which will be a quotient of the category of chain complexes in $\mathcal{A}$. The underlying idea behind the study of derived categories is that the cohomology of a chain complex is slightly too weak an invariant of the chain complex. We wish to instead identify two chain complexes when there is a chain map which induces an isomorphism on the cohomology; call such a map a **quasi-isomorphism**. Of course, by transitivity, this means we will ultimately identify chain complexes C and D where there are a chain of intermediate complexes I_i and a diagram of quasi-isomorphisms

$$C \to I_0 \leftarrow I_1 \to ... \leftarrow I_n \to D$$

We then say that C and D are **quasi-isomorphic**.

All complexes will be cohomological, and denoted by superscripts. Shifts in the grading of complexes will be denoted by hard brackets $[i]$. For more details on derived categories, see [13].

2.4.1 Derived Categories.

We review the standard construction of the derived category of $\mathcal{A}$, by passing through the homotopy category. Let $Com(\mathcal{A})$ denote the category of chain complexes in $\mathcal{A}$. Let $Com^+(\mathcal{A})$ be the full subcategory of complexes whose cohomology vanishes in sufficiently low degree, $Com^-(\mathcal{A})$ the full category of complexes whose cohomology vanishes in sufficiently high degree, and

$$Com^b(\mathcal{A}) := Com^+(\mathcal{A}) \cap Com^-(\mathcal{A})$$

Each of these categories is an abelian category in a natural way.

To each of these categories $Com^?(\mathcal{A})$, we associate a new category $K^?(\mathcal{A})$ with the same objects, but where a morphism is given by a homotopy class of chain maps between two complexes; these are called **homotopy categories**. These new categories are very similar to the old categories, with one crucial exception: they are no longer abelian. They are still additive categories, but because an injective map may be homotopic to a non-injective map, it no longer makes sense to speak of kernels, cokernels or images. A new structure has replaced these old ones that still contains information of exactness; that of *exact triangles*.

To any chain map $f : A \to B$, there is a chain complex M_f called the **mapping cone** of f given by

$$M_f^i = A^{i+1} \oplus B^i$$

and differential

$$d^i_{M_f} = \begin{pmatrix} d_A & 0 \\ f^{i+1} & d_B \end{pmatrix}$$

There are then natural chain maps $B \to M^f$ and $M^f \to A[1]$, which fit together into a diagram

$$A \to B \to M_f \to A[1]$$

called a 'distinguished triangle'. We then call any diagram

$$A \to B \to C \to A[1]$$

isomorphic to a distinguished triangle an **exact triangle**.

Now, to each of these categories $K^?(\mathcal{A})$, define the **derived category** $D^?(\mathcal{A})$ to be the localization of the category on the set of quasi-isomorphisms. That is, $D^?(\mathcal{A})$ is the universal additive category such that any map of additive categories $K^?(\mathcal{A}) \to \mathcal{C}$ factors through the map $K^?(\mathcal{A}) \to D^?(\mathcal{A})$. The objects of $D^?(\mathcal{A})$ are still complexes; however, a map between two complexes C and D in $D^b(\mathcal{A})$ is given by a map between two complexes C' and D' which are quasi-isomorphic to C and D, respectively. This means that we can freely replace a complex by a quasi-isomorphic one.

The categories $D^?(\mathcal{A})$ still have a notion of exact triangle, which is still defined as any triangle isomorphic to a distinguished triangle; however, because more complexes are isomorphic, more triangles are exact. For any object $C \in D^?(\mathcal{A})$, we can consider its ith cohomology $H^i(C) \in \mathcal{A}$; this is well-defined because equivalent complexes will all be quasi-isomorphic. For any exact triangle in $D^?(\mathcal{A})$,

$$A \to B \to C \to A[1]$$

there is an associated long exact sequence in $\mathcal{A}$ of cohomologies

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Often, one has a pair of derived categories $a \subset \mathcal{A}$, where the category a has nice finiteness properties but the category $\mathcal{A}$ has enough injectives. For example, when R is a ring, we have the abelian categories $mod(R) \subset Mod(R)$ of finite R-modules and all R-modules. In these cases, it is customary to denote by $D^?(a)$ the subcategory of $D^?(\mathcal{A})$ of complexes whose cohomology is in a. This allows us to replace objects in a with their injective resolutions, even though such a resolution might not have the right finiteness properties.

2.4.2 Derived Functors.

The abelian category $\mathcal{A}$ sits inside $D^?(\mathcal{A})$ by associating to $A \in \mathcal{A}$ the complex with A concentrated in degree 0. Short exact sequences in $\mathcal{A}$ give rise to exact triangles in $D^?(A)$. As an object in the derived category, an object A is isomorphic to any complex whose cohomology is A; in particular, any resolution of A is isomorphic to A. This means we can freely replace A by any resolution, which allows for many homological constructions to arise naturally.

Now let us assume that $\mathcal{A}$ has enough injective objects. Then, every complex in $Com^+(\mathcal{A})$ can be resolved by an injective complex; that is, there is a quasi-isomorphic complex with entirely injective objects.

Lemma 2.4.2.1. *[13, Coro. 2.7] Let $I \in Com^+(\mathcal{A})$ be an injective complex, and A any complex in $Com^+(\mathcal{A})$. Then $Hom_{K^+}(\mathcal{A})(I, A) \simeq Hom_{D^+(\mathcal{A})}(I, A)$; that is, every map $I \to A$ in the derived category comes from a homotopy class of chain maps $I \to A$.*

Therefore, $D^+(\mathcal{A}) \simeq K^+(Inj(\mathcal{A}))$, where $K^+(Inj(\mathcal{A}))$ is the homotopy category of bounded below injective complexes.

Thus, we can compute the maps $Hom_{D^+(\mathcal{A})}(A, B)$ by resolving A by an injective complex, and finding homotopy classes of maps.

Now let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two exact categories. This gives the **derived functor**

$$\mathbb{R}F : K^+(Inj(\mathcal{A})) \to K^+(\mathcal{B})$$

given by directly applying to the terms in the complex. This induces a map of derived categories, which we also call $\mathbb{R}F$,

$$D^+(\mathcal{A}) \tilde{\to} K^+(Inj(\mathcal{A})) \to K^+(\mathcal{B}) \to D^+(\mathcal{B})$$

In practice, $\mathbb{R}F(A)$ is computed by finding an injective resolution I of A, applying F to that resolution, and considering the derived object $F(I)$. If $A \in \mathcal{A}$, then define the i**th derived functor**

$$\mathbb{R}^i F(A) := H^i(\mathbb{R}F(A))$$

Under mild hypotheses on two left exact functors F and G, the composition derives well: $\mathbb{R}(F \circ G) = \mathbb{R}F \circ \mathbb{R}G$. Explicit computations of these compositions can often be technically complicated; this a philosophical origin of the study of 'spectral sequences', a subject we will pass by in respectful silence.

Derived functors take exact triangles to exact triangles; that is, if we have an exact triangle in $D^+(\mathcal{A})$

$$A \to B \to C \to A[1]$$

then we have an exact triangle

$$\mathbb{R}F(A) \to \mathbb{R}F(B) \to \mathbb{R}F(C) \to \mathbb{R}F(A)[1]$$

Then, for any short exact sequence in $\mathcal{A}$, there is a corresponding long exact sequence of the ith derived functors applied to that sequence.

An important example of a derived functor is $\mathbb{R}Hom$. Given an object $M \in D^b(\mathcal{A})$, the functor $Hom_{D^b(\mathcal{A})}(M, -)$ is left exact, and so $\mathbb{R}Hom_{D^b(\mathcal{A})}(M, -)$ may be defined, as above. For $N \in D^b(\mathcal{A})$, we define $\mathbb{R}Hom_{D^b(\mathcal{A})}(M, N)$ to be this functor, evaluated on N. A priori, this is a functor to the category of abelian groups, but if $\mathcal{A}$ or M has extra structure, this functor can be defined in a richer category.

One could also consider the functor $\mathbb{R}Hom_{D^b(\mathcal{A})}(-, N)$, which is a left exact functor on $D^b(\mathcal{A}^{op})$, and so it can be derived. We might worry that this would give a second, competiting definition of $\mathbb{R}Hom_{D^b(\mathcal{A})}(M, N)$. However, this is the same object; that is,

$$\left[\mathbb{R}Hom_{D^b(mathcalA)}(M, -)\right](N) = \left[\mathbb{R}Hom_{D^b(\mathcal{A})}(-, N)\right](M)$$

This is referred to as the 'balanced property of $\mathbb{R}Hom$' or, on cohomology, as the 'balanced property of Ext'. A proof of it can be found in [36].

In general, $\mathbb{R}F$ does *not* send objects in $D^b(\mathcal{A})$ to objects in $D^b(\mathcal{B})$. We say F has **finite homological dimension** when there is some i such that for all $j \geq i$ and $A \in \mathcal{A}$, $\mathbb{R}^j(A) = 0$. In this case, $\mathbb{R}F$ sends $D^b(\mathcal{A})$ to $D^b(\mathcal{B})$. In the case that $\mathcal{A}$ has finite global dimension, then *every* left exact functor has finite homological dimension.

When the category $\mathcal{A}$ has enough projectives, the dual statements to all the above theory hold. Bounded above complexes may be replaced by projective complexes, and right exact functors $G : \mathcal{A} \to \mathcal{B}$ may be applied to these complexes to get *left derived functors* $\mathbb{L}G$.

Identities which hold for a class of objects in $\mathcal{A}$ will often hold in the derived category for objects that can be resolved by complexes of those objects, where functors have been replaced by their derived analogs. If R is a Noetherian ring of finite global dimension, then every object in $mod(R)$ has a finite resolution by finite projectives, and so many of the best theorems that only hold for finite projectives hold here. Let $M \in D^b(mod(R))$, $N, N' \in D^b(Mod(R))$ and $B \in D^b(Bimod(R))$.

- (Reflexivity) $\mathbb{R}Hom_R(\mathbb{R}Hom_R(M,R),R) \simeq M$.
- (Dual Factoring) $\mathbb{R}Hom_R(M,N) \simeq \mathbb{R}Hom_R(M,R) \otimes^{\mathbb{L}}_R N$.
- ($Hom{-}\otimes$ Adjunction) $\mathbb{R}Hom_R(B \otimes^{\mathbb{L}}_R N, N') \simeq \mathbb{R}Hom_R(N, \mathbb{R}Hom_R(B,N'))$.

In an arbitrary abelian category, there might not be a notion of 'finitely-generated'. However, this can be replaced by the notion of a compact object; an object $A \in \mathcal{A}$ is **compact** if $Hom_{\mathcal{A}}(A,-)$ commutes with arbitrary direct sums. Note that in a module category, the compact objects are exactly the finitely generated ones. An object in $D^b(\mathcal{A})$ is called **perfect** if it is quasi-isomorphic to a finite complex of compact projective objects; the full subcategory of perfect objects is written $Perf(\mathcal{A})$. Then Reflexivity and Dual Factoring hold in $Perf(\mathcal{A})$.

2.4.3 Tilting and Derived Equivalence.

Another advantage of studying derived categories is that two very different-seeming abelian categories $\mathcal{A}$ and $\mathcal{B}$ can have equivalent derived categories, and that this equivalence is describing a hidden relation between the two categories. A particular striking example is given as follows.

Let $Coh(\mathbb{P}^1_{\mathbb{C}})$ be the category of coherent sheaves on $\mathbb{P}^1_{\mathbb{C}}$. Let Q denote the 4-dimensional $\mathbb{C}$-algebra spanned by elements e_0, e_1, x and y, with the multiplication

$$e_0^2 = e_0, \quad e_1^2 = e_1, \quad e_1 x = x e_0 = x, \quad e_1 y = y e_0 = y$$

and all other products zero. Then

Theorem 2.4.3.1. *[4] $D^b(Coh(\mathbb{P}^1_{\mathbb{C}}))$ and $D^b(mod(Q^{op}))$ are equivalent.*

The algebra Q is an example of a *quiver algebra*, which we won't define here. For larger n, $D^b(Coh(\mathbb{P}^n_{\mathbb{C}}))$ is equivalent to the category of finite modules of a finite-dimensional algebra Q_n called the *Beilinson algebra*. This means that any homological problem involving coherent sheaves on $\mathbb{P}^n_{\mathbb{C}}$ can be translated into a homological problem in the modules of a finite-dimensional algebra; or as Beilinson puts it, into a linear algebra problem.

Derived equivalences of this sort often arise in a uniform way. Start with a derived category $D^b(\mathcal{A})$, and produce a compact object T such that

1. Every object can be resolved by a finite complex consisting of finite sums of summands of T.
2. For $i > 0$, $Ext^i_{\mathcal{A}}(T, T) = 0$.
3. $End(T) := Hom_{\mathcal{A}}(T, T)$ is a Noetherian algebra.

In such a case, we say T is a **tilting object**, and we have

Theorem 2.4.3.2. *([2],[11]) Let T be a tilting object in the category $\mathcal{A}$. Then the derived functor*

$$\mathbb{R}Hom_{\mathcal{A}}(T, -) : D^b(\mathcal{A}) \to D^b(mod(End(T)^{op}))$$

is an equivalence of categories, with inverse functor $T \otimes^{\mathbb{L}}_{End(T)} -$.

CHAPTER 3

PROJECTIVE GEOMETRY.

In this chapter, we discuss the algebraic geometry of graded algebras. For a commutative algebra A, a grading can be interpreted as a $\mathbb{C}^*$-action on the spectrum $Spec(A)$. The space $Proj(A)$ is then the scheme of faithful $\mathbb{C}^*$-orbits, which has almost all the information of A, losing only the information of the fixed-point set in $Spec(A)$. The category of quasi-coherent modules on $Proj(A)$ can be constructed directly from the category of graded A-modules.

For non-commutative graded algebras, the absence of the scheme $Spec(A)$ prevents the above construction from working. However, it is still possible to construct a category which mimics the category of quasi-coherent modules on $Proj(A)$. It is this category which is the central object of study in the sequel. It was introduced by Artin and Zhang in [1], who also proved several basic and important results. One of the more compelling aspects of this approach is the way geometric constructions and intuition can still remain valid, even in the absence of a corresponding scheme.

We apply this general construction to the study of $\mathcal{D}$, to construct a category $QGr(\widetilde{\mathcal{D}})$ which emulates the category of quasi-coherent modules on $Proj(\mathcal{D})$. A reoccurring theme in the study of $QGr(\widetilde{\mathcal{D}})$ is that it behaves like a $\mathbb{P}^d$-bundle over X. It should be regarded as a non-commutative analog of a fiberwise compactification of $Spec(\mathcal{D})$, thought of as a bundle over X.

3.1 Commutative Projective Geometry.

For this section, let A be a regular-graded commutative k-algebra. We review the standard construction of the scheme $Proj(A)$, references can be found at, e.g. [19, pg. 160].

3.1.1 The Scheme.

Define an action of $\mathbb{C}^*$ on A as follows. For $\lambda \in \mathbb{C}^*$ and a a non-zero homogeneous element in A, define

$$\lambda \cdot a := \lambda^{deg(a)} a$$

It is immediate that this defines a group action of $\mathbb{C}^*$ on A, acting by algebra automorphisms. By the functoriality of $Spec$, this gives a group action of $\mathbb{C}^*$ on $Spec(A)$.[1]

Conversely, if Y is an affine scheme with an algebraic action of $\mathbb{C}^*$, it defines an action of $\mathbb{C}^*$ on $\mathcal{O}_Y$. Let $(\mathcal{O}_Y)_i$ denotes the subspace of $\mathcal{O}_Y$ consisting of functions f such that $\lambda \cdot f = \lambda^i f$ for all $\lambda \in \mathbb{C}^*$, and let

$$\breve{\mathcal{O}}_Y := \oplus_{i \in \mathbb{Z}} (\mathcal{O}_Y)_i$$

This algebra is naturally graded, and when X is of finite type over $\mathbb{C}$, we have that $\breve{\mathcal{O}}_Y = \mathcal{O}_Y$. Thus, the study of commutative graded algebras is closely related to the study of schemes with a $\mathbb{C}^*$-action.

We now define a scheme $Proj(A)$ which is meant to parametrize faithful $\mathbb{C}^*$ orbits in $Spec(A)$. Let $A_{>0}$ denote the ideal of A spanned by elements of strictly

[1] Technically, this is a group action of the opposite group $(\mathbb{C}^*)^{opp}$. However, because $\mathbb{C}^*$ is commutative, we implicitly identify it with its opposite group.

positive degree, called the **irrelevant ideal**.

- Let $Proj(A)$ denote the set of homogeneous prime ideals which do not contain $A_{>0}$.
- For any homogeneous ideal $I \subset A$, the set of homogeneous prime ideals $V(I)$ which contain I defines a closed subset of $Proj(A)$; extend this to define the **Zariski topology** on $Proj(A)$.
- For any homogeneous prime ideal I which doesn't contain $A_{>0}$, the complement $(V(I))^c$ is a **basic open set** of $Proj(A)$. Define a sheaf $\mathcal{O}_{Proj(A)}$ on $Proj(A)$ which, on $(V(I))^c$, is $(A_I)_0$, the degree zero subspace of the localization of A at the prime ideal I.

This defines a locally-ringed space, which can be shown to be a scheme [19, Prop 2.5.]. Furthermore, the natural map of algebras $A_0 \to A$ induces a natural map of schemes

$$Proj(A) \to Spec(A_0)$$

In the case that A is finitely-generated over A_0, this map is projective, in the sense that it can be expressed as a composition

$$Proj(A) \hookrightarrow Spec(A_0) \times_k \mathbb{P}^i_k \to Spec(A_0)$$

for some i large enough, and this composition is a proper morphism [19, Prop. 7.10].

3.1.2 The Module Category and the Serre Equivalence.

In the same way that a quasi-coherent sheaf πN on $Spec(A)$ maybe be assigned to an A-module N, there is a way to assign a quasi-coherent sheaf πM on $Proj(A)$

to any graded A-module M. However, in contrast with the affine case, this construction is not an equivalence of categories; some graded modules M are sent to zero by this construction.

Let $Qcoh(Proj(A))$ denote the abelian category of quasi-coherent sheaves of modules on $Proj(A)$. To any graded A-module M, there is a natural quasi-coherent sheaf πM on $Proj(A)$. For a homogeneous prime ideal I, let $\pi M|_{(V(I))^c}$ be $(A_I \otimes_A M)_0$, the degree zero part of the localization of M at I, and extend this to a sheaf πM on $Proj(A)$. By definition, $\pi A = \mathcal{O}_{Proj(A)}$.

This construction defines an exact functor

$$\mathcal{S} : Gr(A) \to Qcoh(Proj(A))$$

Some modules are killed by $\mathcal{S}$. Call M a $A_{>0}$**-torsion module** if, for every element $m \in M$, there is some $i \gg 0$ such that $(A_{>0})^i \cdot m = 0$. A module is $A_{>0}$-torsion if and only if $\mathcal{S}M = 0$.

Let $Tors(A)$ denote the full subcategory of $Gr(A)$ consisting of $A_{>0}$-torsion modules. Since every object in $Gr(A)$ has a maximal $A_{>0}$-torsion submodule, the subcategory $Tors(A)$ is a localizing subcategory in the sense of Section 2.3. Define the quotient category

$$QGr(A) := Gr(A)/Tors(A)$$

Then by the universality of quotient categories (Theorem 2.3.1.1), the functor $\mathcal{S}$ descends to a functor

$$\mathcal{S}' : QGr(A) \to QCoh(Proj(A))$$

Theorem 3.1.2.1 (The Projective Serre Equivalence)**.** *Let A be a regular-graded commutative k-algebra. Then the functor $\mathcal{S}'$ is an equivalence of abelian categories*

$$QGr(A) \xrightarrow{\sim} QCoh(Proj(A))$$

This equivalence allows many constructions in the graded category to be defined geometrically in $QCoh(Proj(A))$. Define the i**th Serre twist** $(\pi M)(i)$ of $\pi M \in QCoh(Proj(A))$ to be $\pi(M(i))$; this has a geometric construction not needed here. Continuing in this vein, for $\mathcal{M}, \mathcal{N} \in QCoh(Proj(A))$, define

$$\underline{Hom}_{QCoh(Proj(A))}(\mathcal{M}, \mathcal{N}) := \bigoplus_{i \in \mathbb{Z}} Hom_{QCoh(Proj(A))}(\mathcal{M}, \mathcal{N}(i))$$

Because $Tors(A)$ is a localizing subcategory, there is a right adjoint to $\mathcal{S}$

$$\omega : QCoh(Proj(A)) \xrightarrow{\sim} QGr(A) \to Gr(A)$$

which sends Serre twists to shifts. The meaning of this functor is easy to deduce. For $\mathcal{M} \in QCoh(Proj(A))$, the ith graded component of $\omega(\mathcal{M})$ is equal to

$$\begin{aligned} Hom_{Gr(A)}(A, (\omega(\mathcal{M})(i))) &= Hom_{Gr(A)}(A, (\omega(\mathcal{M}(i)))) \\ &= Hom_{QCoh(Proj(A))}(\pi A, \mathcal{M}(i)) \\ &= \Gamma(\mathcal{M}(i)) \end{aligned}$$

Here, Γ is the global sections functor $Hom_{QCoh(Proj(A))}(\mathcal{O}_{Proj(A)}, -)$. Therefore, $\omega(\mathcal{M})$ is the sum over all i of the global sections of $\mathcal{M}(i)$; hence, we call it the **graded global section functor**.

Since ω is a right adjoint functor, it is left exact, and so it can be right derived.[2] The sheaf cohomology functors H^i in algebraic geometry are the right derived functors of the global section functor; therefore, the groups $\mathbb{R}^i\omega(\mathcal{M})$ collect all the sheaf cohomology groups, summed over all twists. As such, the functors $\mathbb{R}^i\omega$ should be thought of as **graded cohomology functors**.

[2]Technically, to justify the ability to derive functors, we should prove that $QCoh(Proj(A))$ has enough projectives or enough injectives. The category (famously) does not have enough projectives, but it *does* have enough injectives.[19, Caution 6.5.2.]

3.1.3 Filtrations in Commutative Projective Geometry.

Projective geometry is useful for shedding light on the intrinsic geometric meaning of a grading on an algebra. Through the Rees construction, it can also be used to understand the geometric meaning of a filtration on an algebra.

Let A be an exhaustive, positively-filtered commutative algebra over k. The natural quotient map of graded algebras $\widetilde{A} \to \overline{A}$ induces a closed inclusion

$$Proj(\overline{A}) \hookrightarrow Proj(\widetilde{A})$$

The closed subscheme is defined by a single equation, so it is a hypersurface.

Now consider the localization $\widetilde{A}[t^{-1}]$ of $\widetilde{A}$ at t; that is, adjoining an inverse of t. The map $\widetilde{A}[t^{-1}] \to A$ which sends t to 1 induces an isomorphism

$$Spec(A) \xrightarrow{\sim} Proj(\widetilde{A}[t^{-1}])$$

which can be seen by showing that every homogeneous prime ideal in $\widetilde{A}[t^{-1}]$ is induced from some prime ideal in A.

So, the localization map $\widetilde{A} \to \widetilde{A}[t^{-1}]$ which sends t to 1 induces an open inclusion

$$Spec(A) \xrightarrow{\sim} Proj(\widetilde{A}[t^{-1}]) \hookrightarrow Proj(\widetilde{A})$$

Then the following proposition reveals the geometric nature of filtrations.

Proposition 3.1.3.1. *The closed subscheme $Proj(\overline{A})$ of $Proj(\widetilde{A})$ is the complement of the open subscheme $Spec(A)$.*

Therefore, a filtration defines a way of adding a closed hypersurface to $Spec(A)$.

This correspondence can go the other way, as well. If X is a scheme with a closed hypersurface X_∞ such that $X \backslash X_\infty$ is affine, then $\mathcal{O}_{X\backslash X_\infty}$ can be filtered by assigning an order to a function f given by the order of its pole along X_∞ in the function field of X. Provided X_∞ contains no irreducible components of X, then

$$X = Proj(\widetilde{\mathcal{O}_{X\backslash X_\infty}})$$

A relevant example of this comes from vector bundles over an affine scheme. Let $Spec(R)$ be some affine scheme, and let V be some vector bundle of rank r over $Spec(R)$. The total space of this vector bundle can be realized as the affine scheme

$$Spec(Sym_R(\Gamma(V))^*)$$

where Sym_R denotes the symmetric tensor algebra over R, and $(\Gamma(V))^*$ denotes the R-dual of the global sections of V. The ring $Sym_R(\Gamma(V))^*$ is naturally graded by word-length, and so it is filtered by the forgetful filtration (ie, the order of an element is its graded degree). Then taking the Rees algebra and then $Proj$ defines a scheme $\widehat{V}$. We have a diagram

$$\begin{array}{ccc} V & \hookrightarrow & \widehat{V} \\ \downarrow & & \downarrow \\ Spec(R) & \simeq & Spec(R) \end{array}$$

The k-fibers of the left map are k-vector spaces of dimension r. The k-fibers of the right map are copies of $\mathbb{P}^r_k$. Therefore, $\widehat{V}$ is the fiberwise projective compactification of V. This idea will be important later, in the non-commutative setting.

While adding extra pieces to an affine scheme might seem like it makes the situation more complicated (for instance, non-affine), there are two reasons for doing this. First, if an algebra A has a natural filtration, it would be morally reprehensible to ignore this extra information completely. Second (and less moralistic), certain aspects of the study of projective schemes are simpler than the study of affine schemes. In particular, affine schemes and their modules typically have infinite-dimensional spaces of global sections, and no higher cohomology. Projective schemes over R and their modules, however, often have finitely-generated global sections and non-trivial cohomologies, which will be used extensively (see, Serre Finiteness in 4.2.5).

3.2 Noncommutative Projective Geometry.

If A is a non-commutative algebra, then there is no general consensus as to what sort of object $Spec(A)$ should be, or even if can exist at all. However, instead of trying to build a locally ringed space to call $Spec(A)$, we can simply work with the category $Mod(A)$, thought of as the category of quasi-coherent sheaves on the non-existent $Spec(A)$. Since most questions one might ask about a scheme can be restated as a question about its category of modules, this allows many questions of a geometric flavor to be answered.

If A is positively-graded, then the similar complaints will prevent the construction of a scheme $Proj(A)$. As above, we can bypass the need for a space $Proj(A)$ and instead concern ourselves with its category of modules. The projective Serre equivalence provides a recipe for what this category should be.

3.2.1 The Categories $QGr(A)$ and $qgr(A)$.

Let A be a positively-graded algebra. Let $Gr(A)$ be the category of graded left A-modules, and let $Tors(A)$ be the full subcategory of modules such that, for every $m \in T \in Tors(A)$, $A_{\geq n} \cdot m = 0$ for some n. Let $gr(A)$ denote the category of finitely generated graded left modules, and $tors(A) := gr(A) \cap Tors(A)$. Then $Tors(A)$ (resp. $tor(A)$) is a localizing subcategory of $Gr(A)$ (resp. $gr(A)$), and so we define

$$QGr(A) := Gr(A)/Tors(A)$$

$$qgr(A) := gr(A)/tors(A)$$

For not-necessarily commutative A, we will think of $QGr(A) := Gr(A)/Tors(A)$ as the category of quasi-coherent modules on the undefined space $Proj(A)$. This perspective was first put forward by Artin and Zhang in [1], which also proved the majority of the results in this section.

The quotient functor $\pi : Gr(A) \to QGr(A)$ is exact, by Thm 2.3.1.1. As a rule of thumb, $\mathcal{M}, \mathcal{N}, \mathcal{O}, ...$ will denote objects in $QGr(A)$ without a specific choice of preimage under π in mind, while $\pi M, \pi N, \pi A, ...$ will denote objects in $QGr(A)$ where a specific pre-image has been chosen or emphasized.

The shifting functors descend to functors on $QGr(A)$ which are the non-commtutative analogs of the Serre twists; however for simplicity they will still be called 'shifts'. The graded Hom is well-defined, by

$$\underline{Hom}_{QGr(A)}(\mathcal{M}, \mathcal{N}) := \bigoplus_{i \in \mathbb{Z}} Hom_{QGr(A)}(\mathcal{M}, \mathcal{N}(i))$$

The category $QGr(A)$ has enough injectives (Lemma 2.3.2.1); however, $qgr(A)$ does not. This makes attempts to make homological constructions work in $qgr(A)$

almost impossible, and is the main justification for working with the larger category $QGr(A)$, even though the interesting objects of study typically lie in $qgr(A)$.

3.2.2 The Graded Global Section Functor.

Because $Tors(A)$ is localizing, the quotient functor π has a right adjoint $\omega : QGr(A) \to Gr(A)$ which is left exact. The same as the commutative case,

$$\omega(\mathcal{M}) = \underline{Hom}_{Gr(A)}(A, \omega(\mathcal{M})) = Hom_{QGr(A)}(\pi A, \mathcal{M})$$

and so $\omega(\mathcal{M})$ should be regarded as the **graded global section functor**.

Since it is a right adjoint, it is left exact, and so it can be right derived. The functors $\mathbb{R}^i\omega$ are the **graded cohomology functors**. More generally, we have a derived functor

$$\mathbb{R}\omega : D(QGr(A)) \to D(Gr(A))$$

which is right adjoint to the quotient functor $\pi : D(Gr(A)) \to D(QGr(A))$. For any $\mathcal{M} \in QGr(A)$, we have that $\pi\omega(\mathcal{M}) = \mathcal{M}$, and so it follows that $\pi(\mathbb{R}\omega(\mathcal{M})) = \mathcal{M}$ in the derived category.

If A is left Noetherian, then the composition $\omega\pi(M)$ can be computed as a limit [1, pg. 234]

$$\omega\pi(M) = \lim_{\to} \underline{Hom}_{Gr(A)}(A_{\geq n}, M)$$

Because graded module categories are $Ab5$ (see Section 2.3), the higher derived functors can also be computed as limits (by Prop 2.3.2.1)

$$\mathbb{R}^i\omega\pi(M) = \lim_{\to} \underline{Ext}^i_{Gr(A)}(A_{\geq n}, M)$$

Again in the case of A left Noetherian, there is a more useful definition of $\mathbb{R}\omega\pi(M)$.

Lemma 3.2.2.1. *Let A be left Noetherian. For $M \in Gr(A)$, there is an isomorphism in $D(Gr(A))$:*

$$\mathbb{R}\omega\pi(M) \simeq \mathbb{R}\omega\pi(A)\otimes_A^{\mathbb{L}} M$$

Proof. This follows from the isomorphisms

$$\mathbb{R}\underline{Hom}_{Gr(A)}(A_{\geq n}, M) \simeq \mathbb{R}\underline{Hom}_{Gr(A)}(A_{\geq n}, A)\otimes_A^{\mathbb{L}} M$$

by taking homologies and passing to the limit. □

Applying this for $M = \mathbb{R}\omega\pi(A)$,

Corollary 3.2.2.1. *There is an isomorphism in the derived category:*

$$\mathbb{R}\omega\pi(A)\otimes_A^{\mathbb{L}}\mathbb{R}\omega\pi(A) \simeq \mathbb{R}\omega\pi(\mathbb{R}\omega\pi(A)) = \mathbb{R}\omega\pi(A)$$

Proof. The first isomorphism is the preceding lemma, and the second follows from $\pi \circ \mathbb{R}\omega = Id$. □

3.2.3 The Torsion Functor.

Every module $M \in Gr(A)$ has a maximal submodule $\tau(M)$ in $Tors(A)$, called the **torsion of** M [1, pg.233]. Since $Tors(A)$ is a subcategory of $Gr(A)$, the object $\tau(M)$ can be thought of either in $Tors(A)$ or in $Gr(A)$, and the difference will rarely be noted. It can be explicitly defined by

$$\tau(M) := \lim_{\rightarrow} \underline{Hom}_{Gr(A)}(A/A_{\geq n}, M)$$

where the left A-module structure on $\tau(M)$ comes from the right A-module structure on $A/A_{\geq n}$.

The torsion τ is a left exact functor, and its derived functors $\mathbb{R}^i\tau(M)$ coincide with the ith local cohomology of M at the ideal $A_{\geq 1}$, when A is generated in degree 0 and 1. As for ω, the higher derived functors can be computed as limits

$$\mathbb{R}^i\tau(M) = \lim_{\rightarrow} \underline{Ext}^i_{Gr(A)}(A/A_{\geq n}, M)$$

If A is left Noetherian, the defining inclusion $\tau(M) \hookrightarrow M$ and the adjunction map $M \to \omega\pi(M)$ fit together to give an exact triangle in $D(Gr(A))$ [1, pg. 241]:

$$\mathbb{R}\tau(M) \to M \to \mathbb{R}\omega\pi(M) \to \mathbb{R}\tau(M)[1] \qquad \text{(Torsion)}$$

It is important enough to name; call this the **torsion exact triangle**.

Of course, the higher derived functors of the identity functor vanish, and so the higher cohomologies of the middle term in the torsion exact sequence are zero. Taking the long exact sequence of cohomology, we have an exact sequence

$$0 \to \tau(M) \to M \to \omega\pi(M) \to \mathbb{R}^1\tau(M) \to 0$$

and isomorphisms $\mathbb{R}^i\omega\pi(M) \simeq \mathbb{R}^{i+1}\tau(M)$ for $i \geq 1$. Since $\mathbb{R}^{i+1}\tau(M) \in Tors(A)$, the higher cohomology functors $\mathbb{R}^i\omega\pi(M)$ are torsion for $i > 0$.

In the case that the graded components A_k are f.g. projective A_0-modules, then the derived functors $\underline{Ext}^i_A(A/A_{\geq i}, M)$ can be built up out of copies of the simpler derived functors $\underline{Ext}^i_A(A_0, M)$. In particular, the vanishing of the latter implies the vanishing of the former.

Lemma 3.2.3.1. *Assume that A_k is a f.g. projective A_0-module for all k. Let $M \in Gr(A)$, and let i and j be integers such that*

$$\left(\underline{Ext}^i_{Gr(A)}(A_0, M)\right)_{\geq j} = 0$$

Then $(\mathbb{R}^i\tau(M))_{\geq j} = 0$*. In particular, if* $\underline{Ext}^i_{Gr(A)}(A_0, M) = 0$*, then* $\mathbb{R}^i\tau(M) = 0$*.*

Proof. For any k, there is a short exact sequence of A-modules:

$$0 \to A_k(-k) \to A_{\leq k} \to A_{\leq k-1} \to 0$$

where $A_k(-k)$ is the left A_0-module A_k concentrated in degree k, and given an A-module structure by allowing $A_{\geq 1}$ to act trivially. Applying $\underline{Hom}_{Gr(A)}(-, M)$ to this sequence gives an exact triangle of derived objects

$$\mathbb{R}\underline{Hom}_A(A_{\leq k-1}, M) \to \mathbb{R}\underline{Hom}_A(A_{\leq k}, M) \to \mathbb{R}\underline{Hom}_A(A_k, M(k)) \to$$

By adjunction,

$$\begin{aligned}
\mathbb{R}\underline{Hom}_A(A_k, M) &= \mathbb{R}\underline{Hom}_A(A_0 \otimes_{A_0} A_k, M) \\
&= \mathbb{R}Hom_{A_0}(A_k, \mathbb{R}\underline{Hom}_A(A_0, M)) \\
&= \mathbb{R}Hom_{A_0}(A_k, A_0) \otimes^{\mathbb{L}}_{A_0} \mathbb{R}\underline{Hom}_A(A_0, M) \\
&= Hom_{A_0}(A_k, A_0) \otimes_{A_0} \mathbb{R}\underline{Hom}_A(A_0, M)
\end{aligned}$$

In particular, if $\underline{Ext}^i_A(A_0, M)$ vanishes in degree j, then for all k, the space $\underline{Ext}^i_A(A_k, M)$ vanishes in degree j.

Considering now the long exact sequence of cohomology coming from the above exact triangle, we observe that the natural map

$$\underline{Ext}^i_A(A_{\leq k-1}, M) \to \underline{Ext}^i_A(A_{\leq k}, M)$$

is a surjection in degree j if $\underline{Ext}^i_A(A_0, M)$ vanishes in degree $j + k$.

Now assume that $\underline{Ext}^i_A(A_0, M)_{\geq j} = 0$. Then for any $j' > j$, we have a system of surjections

$$\underline{Ext}^i_A(A_0, M)_{j'} \to \underline{Ext}^i_A(A_{\leq 1}, M)_{j'} \to ... \to \underline{Ext}^i_A(A_{\leq k}, M)_{j'} \to ...$$

However, the first term $\underline{Ext}^i_A(A_0, M)_{j'}$ vanishes by assumption, and so the whole system vanishes. This implies the limit

$$\mathbb{R}^i\tau(M)_{j'} = \varinjlim \underline{Ext}^i_A(A_{\leq k}, M)_{j'} = 0$$

for all $j' > j$. □

The proof also implies a weaker vanishing result in negative degrees.

Corollary 3.2.3.1. *Let A be as above. Let $M \in Gr(A)$ and let i and j be such that*

$$\left(\underline{Ext}^i_{Gr(A)}(A_0, M)\right)_{\leq j} = 0$$

Then $\forall k$,

$$\left(\underline{Ext}^i_{Gr(A)}(A_{\leq k}, M)\right)_{\leq j-k} = 0$$

3.2.4 The χ-condition.

There is an important technical condition which controls the size of the modules $\underline{Ext}^i_{Gr(A)}(A/A_{\geq n}, M)$ which approximate the torsion of M.

Definition. *[1, pg. 243] A module $M \in Gr(A)$ is said to satisfy the χ-**condition** if, for all d and all i, there is an n_0 such that for all $n \geq n_0$, $\underline{Ext}^i_{Gr(A)}(A/A_{\geq n}, M)_{\geq d}$ is a finitely-generated A module.*

*If every finitely-generated A-module M satisfies the χ-condition, we say A satisfies the χ-**condition**.*

The χ-condition is relatively easy to satisfy; the Rees ring $\widetilde{\mathcal{D}}$ will satisfy it (Lemma 4.2.5.2). We then have the following theorem.

Theorem 3.2.4.1. *[1, pg. 273] Let A be left Noetherian and satisfy χ, and let $M \in gr(A)$. Then, for all $i \geq 1$, the dth graded component of the ith graded cohomology $\mathbb{R}^i\omega\pi(M)_d$ is a finitely generated A_0-module for all d, and is zero if d is sufficiently large.*

3.3 Projective Geometry of $\widetilde{\mathcal{D}}$.

We now focus on the projective geometry of $\widetilde{\mathcal{D}}$. As above, let $QGr(\widetilde{\mathcal{D}})$ and $qgr(\widetilde{\mathcal{D}})$ denote the categories $Gr(\widetilde{\mathcal{D}})/Tors(\widetilde{\mathcal{D}})$ and $gr(\widetilde{\mathcal{D}})/tors(\widetilde{\mathcal{D}})$, respectively. We know that $\mathcal{D}$ and $\overline{\mathcal{D}}$ are Noetherian rings with no zero divisors, which are generated in degrees zero and one, and so the same is true of $\widetilde{\mathcal{D}}$.Therefore, the results of the previous section are valid for $\widetilde{\mathcal{D}}$. Note, however, that $\widetilde{\mathcal{D}}$ is no longer simple, it has two-sided ideals of the form $\widetilde{\mathcal{D}}_{\geq n}$.

3.3.1 Behavior at Infinity.

The ring $\overline{\mathcal{D}}$ is commutative, and by the PBW theorem, it is isomorphic to $Sym_X L$. Therefore, $Spec(\overline{\mathcal{D}})$ is the dual vector bundle L^*, and so $Proj(\overline{\mathcal{D}})$ is the space $\mathbb{P}(L^*)$ of 1-dimensional subspaces of fibers of L^*.

Geometrically, the Rees algebra $\widetilde{\mathcal{D}}$ is defining a slightly larger space $QGr(\widetilde{\mathcal{D}})$ than the filtered algebra $\mathcal{D}$. There is the extra hyperplane defined by t; this will be referred to as the **hyperplane at infinity**, $QGr(\overline{\widetilde{D}})$. However, because $\overline{\mathcal{D}}$ is commutative, the category $QGr(\overline{\widetilde{D}})$ is the category of modules on an honest-to-God scheme $\mathbb{P}(L^*)$. This allows the conceptual geometry of $QGr(\widetilde{\mathcal{D}})$ to be linked to the actual geometry of $QGr(\overline{\widetilde{D}})$.

The quotient map of graded algebras $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}/\langle t\rangle \simeq \overline{\mathcal{D}}$ defines pullback, pushforward, and exceptional pullback functors.

$$i_\infty^* : Gr(\widetilde{\mathcal{D}}) \to Gr(\overline{\mathcal{D}}), \quad i_\infty^*(M) = {}_{\overline{\mathcal{D}}}\overline{\mathcal{D}}_{\widetilde{\mathcal{D}}} \otimes_{\widetilde{\mathcal{D}}} M$$

$$i_{\infty*} : Gr(\overline{\mathcal{D}}) \to Gr(\widetilde{\mathcal{D}}), \quad i_{\infty*}(N) = Hom_{\overline{\mathcal{D}}}({}_{\overline{\mathcal{D}}}\overline{\mathcal{D}}_{\widetilde{\mathcal{D}}}, N) = {}_{\widetilde{\mathcal{D}}}\overline{\mathcal{D}}_{\overline{\mathcal{D}}} \otimes_{\overline{\mathcal{D}}} N$$

$$i_\infty^! : Gr(\widetilde{\mathcal{D}}) \to Gr(\overline{\mathcal{D}}), \quad i_\infty^!(M) = Hom_{\widetilde{\mathcal{D}}}({}_{\widetilde{\mathcal{D}}}\overline{\mathcal{D}}_{\overline{\mathcal{D}}}, M)$$

Each of these functors sends $Tors(\widetilde{\mathcal{D}})$ to $Tors(\overline{\mathcal{D}})$ or vice versa, and so they each induce functors between the corresponding quotient categories $QGr(\widetilde{\mathcal{D}})$ and $QGr(\overline{\widetilde{D}})$; these functors will be denoted by the same symbol by abuse of notation.

In $Gr(\widetilde{\mathcal{D}})$, the multiplication-by-$t$ map $\widetilde{\mathcal{D}}(-1) \to \widetilde{\mathcal{D}}$ fits into a short exact sequence

$$0 \to \widetilde{\mathcal{D}}(-1) \to \widetilde{\mathcal{D}} \to \overline{\mathcal{D}} \to 0$$

Applying the exact functor π gives

$$0 \to \pi\widetilde{\mathcal{D}}(-1) \to \pi\widetilde{\mathcal{D}} \to \pi\overline{\mathcal{D}} \to 0$$

More generally, let M be a filtered $\mathcal{D}$-module. Then multiplication-by-t in $\widetilde{M}$ is an inclusion, so there is a short exact sequence

$$0 \to \widetilde{M}(-1) \to \widetilde{M} \to \overline{M} \to 0$$

Applying π gives

$$0 \to \pi\widetilde{M}(-1) \to \pi\widetilde{M} \to \pi\overline{M} \to 0$$

By applying $\mathbb{R}\omega$ to this short exact sequence, we get the **infinity exact triangle** which will come up frequently.

$$\mathbb{R}\omega\pi\widetilde{M}(-1) \to \mathbb{R}\omega\pi\widetilde{M} \to \mathbb{R}\omega\pi\overline{M} \to \mathbb{R}\omega\pi\widetilde{M}(-1)[1] \tag{Infinity}$$

The utility of this exact triangle is that it allows us to relate $\mathbb{R}\omega\pi\widetilde{M}$, the cohomology we wish to study, to $\mathbb{R}\omega\pi\overline{M}$, which is a cohomology computation on the scheme $\mathbb{P}(L^*)$.

3.3.2 Ideals.

So far, the driving force in the study of the projective geometry of $\widetilde{\mathcal{D}}$ has been the study of one-sided ideals in rings of differential operators on X. This idea was first introduced by Le Bruyn [24] in the case of $\mathcal{D}(\mathbb{A}^1)$ (the first Weyl algebra), though he uses a different filtration and hence a different theory of projective geometry. These ideas were expanded by Berest and Wilson [9], [10] and interpreted in terms of A_∞-algebras by Berest and Chalykh [6]. Ben-Zvi and Nevins [5] then reinterpreted the A_∞ classification of Berest and Chalykh in terms of the filtration and projective geometry featured here, also generalizing to the case of an arbitrary smooth curve X, paralleling results obtained by Berest and Chalykh [7] using more directly algebraic methods.

One of the main reasons ideals in rings of differential operators are interesting is because of the following classic result of Stafford.

Theorem 3.3.2.1. *[35] Let $\mathcal{D}$ be the ring of algebraic differential operators on $\mathbb{A}^n_{\mathbb{C}}$ (n-dimensional affine space), and let M be a projective $\mathcal{D}$-module. Then either M is a free $\mathcal{D}$-module, or M is isomorphic to a left ideal in $\mathcal{D}$.*

This means that every interesting projective $\mathcal{D}$-module is given by some ideal, when $\mathcal{D}$ is the ring of differential operators on $\mathbb{A}^n$. The analog of Stafford's theorem for an arbitrary smooth affine irreducible X is still an open question.

Since we are interested in ideals as modules, two ideals will be consider **equivalent** if there is an isomorphism between them as $\mathcal{D}$-modules. Two equivalent ideals will have filtrations that differ by a shift. The following lemma shows that an ideal I can be recovered up to isomorphism from $\pi\widetilde{I}$.

Lemma 3.3.2.1. *Let I be a left ideal in $\mathcal{D}$, with its natural filtration inherited from $\mathcal{D}$. Then $\omega\pi\widetilde{I} = \widetilde{I}$.*

Proof. Recall the Torsion exact sequence (Section 3.2.3)

$$0 \to \tau(\widetilde{I}) \to \widetilde{I} \to \omega\pi(\widetilde{I}) \to \mathbb{R}^1\tau(\widetilde{I}) \to 0$$

Thus, it will suffice to show that $\tau(\widetilde{I}) = \mathbb{R}^1\tau(\widetilde{I}) = 0$. Showing $\tau(\widetilde{I}) = 0$ is easy. After all, an element of $\tau(\widetilde{I})$ is an element in $\widetilde{I}$ which is killed by every element in $\widetilde{\mathcal{D}}$ of sufficiently high degree. However, $\mathcal{D}$ has no zero-divisors, so such an element must be zero.

Showing $\mathbb{R}^1\tau(\widetilde{I}) = 0$ is harder. Recall that

$$\mathbb{R}^1\tau(\widetilde{I}) = \lim_{n\to\infty} \underline{Ext}^1_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}_{\leq n}, \widetilde{I})$$

Since $\widetilde{\mathcal{D}}$ is a projective module over itself, the above ext groups can be computed using the resolution:[3]

$$0 \to \widetilde{\mathcal{D}}_{\geq n+1} \to \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}_{\leq n} \to 0$$

Therefore,

$$\underline{Ext}^1_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}_{\leq n}, \widetilde{I}) = \underline{Hom}_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}_{\geq n+1}, \widetilde{I}) / \underline{Hom}_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}, \widetilde{I})$$

So the theorem follows if it can be shown that every graded $\widetilde{\mathcal{D}}$-module map $f : \widetilde{\mathcal{D}}_{\geq n} \to \widetilde{I}$ extends to a graded $\widetilde{\mathcal{D}}$-module map $\hat{f} : \widetilde{\mathcal{D}} \to \widetilde{I}$.

Let f be such a map of degree i. Let $\sigma \in \widetilde{\mathcal{D}}_{n-1}$. Then $t\sigma \in \widetilde{\mathcal{D}}_n$ (it is the same differential operator, thought of in one higher degree). The goal is to show that $f(t\sigma) \in t \cdot \widetilde{I}_{(n+i-1)}$. Let σ' be any element in $\mathcal{D}^1$, the first order differential

[3] It isn't necessarily true that $\widetilde{\mathcal{D}}_{\geq n}$ is projective; to compute the n-th right derived functor of some object, one only needs a resolution which is projective in the first n steps. [36]

operators. Then

$$\sigma' f(t\sigma) = f(\sigma' t\sigma) = f(t\sigma'\sigma) = tf(\sigma'\sigma)$$

Notice that $f(\sigma'\sigma) \in \widetilde{I}_{n+i} \subset \widetilde{D}_{n+i}$, so it is a $(n+i)$-th degree differential operator. This means that $f(t\sigma) \in \widetilde{I}_{n+i} \subset \widetilde{\mathcal{D}}_{n+i}$ is a differential operator such that left multiplication by *any* first order differential operator is of degree at most $n+i$. Therefore, $f(t\sigma)$ must be of degree at most $n+i-1$. Since the filtration on I is inherited from the inclusion into $\mathcal{D}$, the differential operator $f(t\sigma)$ must be $(I)_{n+i-1}$.

From this construction, one concludes that any map $f : \widetilde{\mathcal{D}}_{\geq n} \to \widetilde{I}$ can be extended to a map $f' : \widetilde{\mathcal{D}}_{\geq n-1} \to \widetilde{I}$, and so by induction it can be extended to a map $\hat{f} : \widetilde{\mathcal{D}} \to \widetilde{I}$. Thus, the above Ext groups vanish, and so $\mathbb{R}^1\tau(\widetilde{I})$ is zero. □

As a consequence, the ideals in $\mathcal{D}$ can be classified by classifying their images in $QGr(\widetilde{\mathcal{D}})$.

3.3.3 Commutative Analogy.

The reoccurring theme of this thesis is the ways in which $\widetilde{\mathcal{D}}$ behaves like a regular graded-local commutative ring, and the ways in which the category $QGr(\widetilde{\mathcal{D}})$ behaves like the category $QCoh(\mathbb{P}_X L^*)$.[4] The main conceptual difference is that $\widetilde{\mathcal{D}}$ is not graded-local, and so the subring/module $\mathcal{O}_X$ plays the role of the ground field k. Hence, many of the results will be 'relative' versions of familiar commutative results, where k has been replaced by $\mathcal{O}_X$. This can be seen in the definition of the quadratic dual algebra, the relative Frobenius and Gorenstein theorems, the

[4]Here, $\mathbb{P}_X L^*$ is the projectivization of the dual bundle L^*. In the case of differential operators, this is $\mathbb{P}_X \mathcal{T}^*$, the projectivized cotangent bundle.

'relative quiver algebras' that appear in the Beilinson equivalence, and the relative versions of Matlis, Local and Serre duality. The proofs even adher closely to standard proofs in the graded-local commutative case.

In some ways, it is not surprising that there would be such similarities to the commutative case; after all, the theory of non-commutative projective geometry was designed to replicate the module-theoretic features of commutative projective geometry. Furthermore, every Lie algebroid is a deformation of an abelian one[5], and when (X, L) is abelian, $QGr(\widetilde{\mathcal{D}}) = QCoh(\mathbb{P}_X L^*)$. Therefore, the general case is a deformation of the commutative case.

However, in some ways it is also very surprising how much the category $QGr(\widetilde{\mathcal{D}})$ behaves like $QCoh(\mathbb{P}_X L)$. When $\mathcal{D}$ is the ring of differential operators, all modules of $\mathcal{D}$ are quite large. The most straight-forward theorem to this effect is that $\mathcal{D}$ has no finite-dimensional modules; this follows from a trace-based argument. Much stronger and deeper results are given by studying the *characteristic variety* of a $\mathcal{D}$-module M, which can be defined as the support of any 'good' deformation of M to a module on the commutative scheme $\mathcal{T}^*$. Then Bernstein's Inequality asserts that

$$dim(Char(M)) \geq dim(X)$$

while Gabber's Theorem [17] states that $Char(M)$ is always a coisotropic subvariety of the symplectic variety $\mathcal{T}^*$ (the latter theorem implies the former). This means that while the two categories $QGr(\widetilde{\mathcal{D}})$ and $QCoh(\mathbb{P}_X L^*)$ are deformations of each other, it is very far from being true that every module on $\mathbb{P}_X L^*$ deforms to an object in $QGr(\widetilde{\mathcal{D}})$.

This difference also manifests itself in the difference in homological dimension;

[5]Simply scale the bracket and anchor map to zero.

$dim(\mathbb{P}_X L^*) = 2n-1$ while $dim(QGr(\widetilde{\mathcal{D}})) = n$ (again, this is in the case of differential operators). From this perspective, it is very surprising that statements about the whole category $QGr(\widetilde{\mathcal{D}})$ are the appropriate deformations of the corresponding statements about $QCoh(\mathbb{P}_X L^*)$. Take, for instance, the Beilinson equivalence (Theorem 6.1.3.1), which is a derived equivalence to a kind of matrix algebra E; where the algebra is E is a deformation of the commutative case in the most obvious way.

CHAPTER 4

KOSZUL THEORY.

This section will develop the Koszul theory for the algebra $\mathcal{D}$ over X. The two main results of this will be:

- A canonical projective resolution of $\mathcal{O}_X$ as a left $\widetilde{\mathcal{D}}$-module, called the **left Koszul resolution**.
- For any $\pi M \in QGr(\widetilde{\mathcal{D}})$, a resolution of πM by objects of the form $\pi\widetilde{\mathcal{D}}(-i)$ for $i \in \{0, ...N\}$, called the **Beilinson resolution**.

4.1 The Quadratic Dual Algebra.

The key observation is that the definition of the universal enveloping algebra gives a surjective map $T_X\mathcal{D}_1 \to \widetilde{\mathcal{D}}$, whose kernel is generated by elements of degree 2 in $T_X\mathcal{D}$. This is similar to the case of 'quadratic algebras', which are quotients of T_kV by degree 2 elements (for k some field and V some k-space). [30]

In the theory of quadratic algebras, there is a notion of the quadratic dual, which, in the case of special algebras called **Koszul algebras**, is the same as the self-Ext algebra of the ground field k. Here, we develop the analogous techniques in this relative case, and reap the standard rewards.

4.1.1 The Construction of the Quadratic Dual Algebra.

A **relatively quadratic algebra over** X is an algebra with a surjective map from $T_X B$ for a $\mathcal{O}_X$-bimodule B, whose kernel is generated in degree 2. The following construction is valid for any relatively quadratic algebra, though the subsequent properties of the dual algebra will not always be true.

Let R be the $\mathcal{O}_X$-bimodule which is the kernel of the map $\mathcal{D}_1 \otimes_X \mathcal{D}_1 \to \mathcal{D}_2$. Note that R is the degree 2 part of the kernel of $T_X\mathcal{D}_1 \to \widetilde{\mathcal{D}}$, which generates the whole kernel as a two-sided ideal. By the definition of the universal enveloping algebra, this is the $\mathcal{O}_X$-bimodule generated by

$$\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1$$

for $\partial, \partial' \in \mathcal{D}_1$.

From now on, for M any right $\mathcal{O}_X$-module, let M^* denote the left $\mathcal{O}_X$-module $Hom_{-X}(M, \mathcal{O}_X)$ (as right $\mathcal{O}_X$-modules)[1]; analogously, for M any left $\mathcal{O}_X$-module, let ${}^*M$ denote the right $\mathcal{O}_X$-module $Hom_{X-}(M, \mathcal{O}_X)$. When M is a $\mathcal{O}_X$-bimodule, M^* and ${}^*M$ are also $\mathcal{O}_X$-bimodules, which are potentially non-isomorphic.

Let J^i be ${}^*(\mathcal{D}_i)$, which is called the **bimodule of** i**-jets**.[2] Since the $\mathcal{D}_i$ are finitely generated and projective as right $\mathcal{O}_X$-modules, there is an isomorphism

$${}^*(\mathcal{D}_1 \otimes_X \mathcal{D}_1) \simeq {}^*(\mathcal{D}_1) \otimes_X {}^*(\mathcal{D}_1) \simeq J^1 \otimes_X J^1$$

The map $\mathcal{D}_1 \otimes_X \mathcal{D}_1 \to \mathcal{D}_2$ then induces an inclusion $J^2 \hookrightarrow J^1 \otimes_X J^1$, which can be characterized as the subset of right $\mathcal{O}_X$-module maps $\mathcal{D}_1 \otimes_X \mathcal{D}_1 \to \mathcal{O}_X$ which kill

[1] Hom_{-X} will denote the Hom as right $\mathcal{O}_X$-modules, when there is also a left $\mathcal{O}_X$-structure. Similarly, Hom_{X-} will denote the Hom as left $\mathcal{O}_X$-modules.

[2] The reason for the superscript on J^i is that it will occur naturally as the degree -i part of a complex, and so this is in keeping with the convention that superscripts denote cohomological data.

$R \subset \mathcal{D}_1 \otimes_X \mathcal{D}_1$.

Now, let $\widetilde{\mathcal{D}}^!$ denote the quotient of the tensor algebra $T_X J^1$ by the two-sided ideal generated by J^2 as sitting inside the degree 2 part. The algebra $\widetilde{\mathcal{D}}^!$ is called the **Koszul dual** to $\widetilde{\mathcal{D}}$, or the **quadratic dual** algebra.[3] In contrast with the usual notation for graded algebras, the ith graded component of $\widetilde{\mathcal{D}}^!$ will be denoted $\widetilde{\mathcal{D}}^{!i}$. This is because in Section 4.2.3, it is shown that $\widetilde{\mathcal{D}}^! = \underline{Ext}^\bullet_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \mathcal{O}_X)$, where $\underline{Ext}$ is the graded Ext. Therefore, the grading on $\widetilde{\mathcal{D}}^!$ is naturally cohomological, and deserves a superscript.

4.1.2 The Structure of the Quadratic Dual Algebra.

We now explore the structure of $\widetilde{\mathcal{D}}^!$ as an algebra. Recall that L is the Lie algebroid, and is $\mathcal{T}$ in the case of differential operators. Note that J^1 fits into a short exact sequence of $\mathcal{O}_X$-bimodules,

$$0 \to L^* \to J^1 \to \mathcal{O}_X \to 0$$

The 'action on 1' map $\mathcal{D} \to \mathcal{O}_X$ is a map of left $\mathcal{D}$-modules. It restricts to a map of left $\mathcal{O}_X$-modules $e : \mathcal{D}_1 \to \mathcal{O}_X$, and so it determines an element $e \in J^1$ and its image in $\widetilde{\mathcal{D}}^!$. Since e acts as the identity on $\mathcal{O}_X \subset \mathcal{D}_1$, its image under the map $J^1 \to \mathcal{O}_X$ is the identity in $\mathcal{O}_X$.

Next, define the **L-exterior derivative** $\mu : L^* \to L^* \otimes_X L^* = (L \otimes_X L)^*$ by

$$\mu(\sigma)(l \otimes l') := \frac{1}{2}\left[d_{\tau(l)}(\sigma(l')) - d_{\tau(l')}(\sigma(l)) - \sigma([l, l'])\right]$$

[3]Note that we have made an asymmetric choice, in looking at the dual of $\mathcal{D}_1$ as a left $\mathcal{O}_X$-module, rather than as a right $\mathcal{O}_X$-module. Then, perhaps, this should be called the left Koszul dual. This choice was motivated by the fact that J^1 has much nicer properties than $(\mathcal{D}_1)^*$, which results in a nicer presentation of $\widetilde{\mathcal{D}}^!$. However, the right Koszul dual algebra would still have been sufficient for the purposes of this paper.

The name comes from the case when $L = \mathcal{T}$, where $\mu : \mathcal{T}^* \to \mathcal{T}^* \otimes_X \mathcal{T}^*$ is the usual exterior derivative.

Since explicit computations are looming, it is now worth explicitly describing some of the constructions already implicitly described.

- The way the $\mathcal{O}_X$-bimodule structure on $J^1 = {}^*(\mathcal{D}_1)$ was defined, $(ae)(\partial) = e(\partial a)$.
- From the isomorphism ${}^*(\mathcal{D}_1) \otimes_X {}^*(\mathcal{D}_1) = {}^*(\mathcal{D}_1 \otimes_X \mathcal{D}_1)$, for $\sigma, \sigma' \in {}^*(\mathcal{D}_1)$, $(\sigma \otimes \sigma')(\partial \otimes \partial') = \sigma'(\partial \cdot \sigma(\partial'))$.
- From the definition of $\mathcal{D}$, we see that for $\partial \in ker(e)$ and $a \in \mathcal{O}_X$, then $[\partial, a] = d_{\tau(\partial)}(a)$.

The following lemma explains how the element e commutates with other elements in $\widetilde{\mathcal{D}}^!$.

Lemma 4.1.2.1. *The element $e \in \widetilde{\mathcal{D}}^!$ satisfies*

1. $e^2 = 0$.
2. $ae - ea = \tau^\vee(da)$, *for* $a \in \mathcal{O}_X$, *and where* $\tau^\vee : \mathcal{T}^* \to L^*$ *is dual to the anchor map* $L \to \mathcal{T}$.
3. $\sigma e + e\sigma = \mu(\sigma)$, *for* $\sigma \in L^* \in J^1$.

Proof. The easy relation to show is (2), because it is a degree 1 relation. Consider the element $ae - ea \in J^1$, and apply it to any $\partial \in \mathcal{D}_1$.

$$(ae - ea)\partial = e(\partial a) - e(a\partial) = e([\partial, a]) = d_{\tau(\partial)}(a) = \iota_{da}(\tau(\partial)) = \tau^\vee(da)\partial$$

and so $(ae - ea) = \tau^\vee(da)$.

The other two relations are degree 2, so they are true if and only if they are in J^2; that is, if they kill $R \in \mathcal{D}_1 \otimes_X \mathcal{D}_1$. Remember that R is spanned by elements of the form $\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1$.

<u>(1) $e \otimes e$</u>.

$$(e \otimes e)(\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1)$$

$$\begin{aligned}
&= e(\partial e(\partial')) - e(\partial' e(\partial)) - e([\partial, \partial'] e(1)) \\
&= e(\partial')e(\partial) + e([\partial, e(\partial')]) - e(\partial)e(\partial') - e([\partial', e(\partial)]) - e([\partial, \partial']) \\
&= [\partial, e(\partial')] - [\partial', e(\partial)] - e([\partial, \partial'])
\end{aligned}$$

It suffices to check that this final expression vanishes in several cases.

- If both ∂ and ∂' are in $\mathcal{O}_X$, then all the commutators vanish.
- If one of ∂ and ∂' is in $\mathcal{O}_X$ and the other is in the kernel of e, then one of the terms vanish and the other two terms are identical.
- If both ∂ and ∂' are in the kernel of e, then this is also true of their commutator, and so all three terms vanish.

<u>(3) $\sigma \otimes e + e \otimes \sigma - \mu(\sigma)$</u>.

$$(\sigma \otimes e + e \otimes \sigma)(\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1)$$

$$\begin{aligned}
&= [e(\partial \sigma(\partial')) - e(\partial' \sigma(\partial))] + [\sigma(\partial e(\partial')) - \sigma(\partial' e(\partial)) - \sigma([\partial, \partial'] e(1))] \\
&= e(\partial \sigma(\partial')) - e(\partial' \sigma(\partial)) + e(\partial')\sigma(\partial) - e(\partial)\sigma(\partial') - \sigma([\partial, \partial']) \\
&= e([\partial', \sigma(\partial)]) - e([\partial, \sigma(\partial')]) - \sigma([\partial, \partial']) \\
&= [\partial', \sigma(\partial)] - [\partial, \sigma(\partial')] - \sigma([\partial, \partial']) \\
&= d_{\tau(\partial)}(\sigma(\partial')) - d_{\tau(\partial')}(\sigma(\partial)) - \sigma([\partial, \partial'])
\end{aligned}$$

Compare to

$$\mu(\sigma)(\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1)$$

$$\begin{aligned}
&= \frac{1}{2}\left[d_{\tau(\partial)}(\sigma(\partial')) - d_{\tau(\partial')}(\sigma(\partial)) - \sigma([\partial, \partial'])\right] \\
&- \frac{1}{2}\left[d_{\tau(\partial')}(\sigma(\partial)) + d_{\tau(\partial)}(\sigma(\partial')) + \sigma([\partial', \partial])\right] \\
&- \frac{1}{2}\left[d_{\tau([\partial,\partial'])}(\sigma(1)) - d_{\tau(1)}([\partial, \partial']) - \sigma([[\partial, \partial'], 1])\right] \\
&= d_{\tau(\partial)}(\sigma(\partial')) - d_{\tau(\partial')}(\sigma(\partial)) - \sigma([\partial, \partial'])
\end{aligned}$$

Therefore, $\sigma \otimes e + e \otimes \sigma - \mu(\sigma)$ kills $R \in \mathcal{D}_1 \otimes_X \mathcal{D}_1$, and so it is a relation in $\widetilde{\mathcal{D}}^!$. □

For any element $\widetilde{\mathcal{D}}^!$, the above (graded) commutators allow e to collected on one side (for instance, to the right). Since $e^2 = 0$, an element in $\widetilde{\mathcal{D}}^!$ can have at most one e in it. The following theorem then establishes that $\widetilde{\mathcal{D}}^!$ is a rank 2 module over the subalgebra of elements without an e.

Theorem 4.1.2.1. *The map $L^* \to J^1$ extends to an inclusion $\Lambda^\bullet_X L^* \to \widetilde{\mathcal{D}}^!$. This map fits into a short exact sequence of graded $\Lambda^\bullet_X L^*$-bimodules*

$$0 \to \Lambda^\bullet_X L^* \to \widetilde{\mathcal{D}}^! \to \Lambda^\bullet_X L^*(-1) \to 0$$

where $e \in \widetilde{\mathcal{D}}^!$ goes to $1 \in \Lambda^\bullet_X L^*(-1)$.

Proof. First, it is easy to see that, for $\sigma, \sigma' \in L^*$, $\sigma \otimes \sigma' + \sigma' \otimes \sigma$ is a relation in $\widetilde{\mathcal{D}}^!$.

$$(\sigma \otimes \sigma' + \sigma' \otimes \sigma)(\partial \otimes \partial' - \partial' \otimes \partial - [\partial, \partial'] \otimes 1)$$

$$\begin{aligned}
&= \sigma'(\partial\sigma(\partial')) + \sigma(\partial\sigma'(\partial')) - \sigma'(\partial'\sigma(\partial)) - \sigma(\partial\sigma'(\partial')) \\
&= \sigma(\partial')\sigma'(\partial) + \sigma'(\partial')\sigma(\partial) - \sigma(\partial)\sigma'(\partial') - \sigma'(\partial')\sigma(\partial) = 0
\end{aligned}$$

It is not much harder to see that any relation in $L^* \otimes_X L^*$ is fixed by the map which sends $\sigma \otimes \sigma'$ to $\sigma' \otimes \sigma$. Therefore, elements of the form $\sigma \otimes \sigma' + \sigma' \otimes \sigma$

generate the relations in $L^* \otimes_X L^*$. It follows that the submodule $L^* \subset J^1 \subset \widetilde{\mathcal{D}}^!$ generates a copy of the algebra $\Lambda^\bullet_X L^*$.

Now, let C denote the cokernel of $\Lambda^\bullet_X L^* \to \widetilde{\mathcal{D}}^!$, as a $\Lambda^\bullet_X L^*$-bimodule. Note that the previous lemma showed that the (graded) commutator of e with any element of J^1 lies in $L^* \subset J^1$. Therefore, the image of e in $\widetilde{\mathcal{D}}^! \to C$ is (graded) central. Furthermore, since $e^2 = 0$, e generates C, and so there is a surjective map $\Lambda^\bullet_X L^*(-1) \to C$ which sends 1 to e.

For this not to be an isomorphism, there would have to be a relation of the form $\sigma e - \Upsilon$, for $\sigma \in L^*$ and $\Upsilon \in L^* \otimes_X L^*$. Let ∂ be an element in $\mathcal{D}_1$ which is not killed by σ. Then

$$(\sigma \otimes e)(\partial \otimes 1 - 1 \otimes \partial) = e(\partial\sigma(1)) - e(\sigma(\partial)) = \sigma(\partial)$$

By construction, this is not zero. However, Υ must kill $\partial \otimes 1 - 1 \otimes \partial$ since L^* kills $1 \in \mathcal{D}_1$. Therefore, there cannot be such a relation, and the map $\Lambda^\bullet_X L^*(-1) \to C$ is an isomorphism. □

Since $\Lambda^\bullet_X L^*$ is an algebra which is finitely generated projective as a $\mathcal{O}_X$-module on either side and zero in large enough degree, we can deduce identical facts about $\widetilde{\mathcal{D}}^!$.

Corollary 4.1.2.1. *For all i, $\widetilde{\mathcal{D}}^{!i}$ is a finitely generated, projective $\mathcal{O}_X$-module on the left and right.*

Corollary 4.1.2.2. *If $i > n+1$, then $\widetilde{\mathcal{D}}^{!i} = 0$.*

4.1.3 The Relative Frobenius Property.

Let ω_L denote $\Lambda^n_X L^*$, the top exterior power of the dual to L. From the Lemma, it is clear that $\widetilde{\mathcal{D}}^{!n+1} = \omega_L$. This now gives a pairing between elements of $\widetilde{\mathcal{D}}^!$ whose degree adds to $n+1$. We then have

Lemma 4.1.3.1. *(The relative Frobenius property) For any i, the multiplication map*

$$\widetilde{\mathcal{D}}^{!i} \otimes_X \widetilde{\mathcal{D}}^{!(n+1-i)} \to \omega_L$$

is a 'perfect pairing'. That is, the adjoint maps

$$\widetilde{\mathcal{D}}^{!(n+1-i)} \to Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \omega_L), \quad \textit{and} \quad \widetilde{\mathcal{D}}^{!i} \to Hom_{-X}(\widetilde{\mathcal{D}}^{!(n+1-i)}, \omega_L)$$

are isomorphisms of $\mathcal{O}_X$-bimodules.

Proof. Explicitly, the adjoint map $\widetilde{\mathcal{D}}^{!(n+1-i)} \to Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \omega_L)$ takes an element $\mu \in \widetilde{\mathcal{D}}^{!(n+1-i)}$ and sends it to the map $\gamma \in \widetilde{\mathcal{D}}^{!i} \to \mu \cdot \gamma \in \omega_L$. Consider the short exact sequence of $\mathcal{O}_X$-bimodules

$$0 \to \Lambda^{n+1-i}_X L^* \to \widetilde{\mathcal{D}}^{!n+1-i} \to \Lambda^{n+i}_X L^* \to 0$$

If $\mu \in \Lambda^{(n+1-i)}_X L^* \in \widetilde{\mathcal{D}}^{!(n+1-i)}$, then $\mu \cdot \gamma$ only depends on the image of γ under the map $\widetilde{\mathcal{D}}^{!i} \to \Lambda^{i-1}_X L^*$. Similarly, if we know that $\gamma \in \Lambda^i_X L^* \subset \widetilde{\mathcal{D}}^{!i}$, then $\mu \cdot \gamma$ only depends on the image of μ under the map $\widetilde{\mathcal{D}}^{!(n+1-i)} \to \Lambda^{n+i}_X L^*$. This means that the adjoint map above splits into a map of short exact sequences

$$\begin{array}{ccccc} \Lambda^{n+1-i}_X L^* & \to & \widetilde{\mathcal{D}}^{!(n+1-i)} & \to & \Lambda^{n-i}_X L^* \\ \downarrow & & \downarrow & & \downarrow \\ Hom_{X-}(\Lambda^{i-1}_X L^*, \omega_L) & \to & Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \omega_L) & \to & Hom_{X-}(\Lambda^i_X L^*, \omega_L) \end{array}$$

The left and right maps are isomorphisms, because they are both adjoint to multiplication maps of the form $\Lambda^j_X L^* \otimes_X \Lambda^{n-j}_X L^* \to \omega_L$. Therefore, the middle map is an isomorphism. The proof for the other adjoint map is identical. $\square$

This can be restated in a more compact form.

Corollary 4.1.3.1. *There are isomorphisms of $\mathcal{O}_X$-bimodules*

$$(\widetilde{\mathcal{D}}^{!i})^* = \omega_L^* \otimes_X \widetilde{\mathcal{D}}^{!(n+1-i)}, \quad {}^*(\widetilde{\mathcal{D}}^{!i}) = \widetilde{\mathcal{D}}^{!(n+1-i)} \otimes_X \omega_L^*$$

Proof.

$$\widetilde{\mathcal{D}}^{!(n+1-i)} \simeq Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \omega_L) = Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \mathcal{O}_X) \otimes_X \omega_L = {}^*(\widetilde{\mathcal{D}}^{!i}) \otimes_X \omega_L$$

Similarly, $\widetilde{\mathcal{D}}^{!(n+1-i)} = \omega_L \otimes_X (\widetilde{\mathcal{D}}^{!})^*$. Since ω_L is a line bundle, tensoring these with ω_L^* on the left or right gives the theorem. □

4.2 Koszul Complexes.

The quadratic dual algebra and its properties allows for the construction of several important complexes, called **Koszul complexes**.

4.2.1 The Left Koszul Complex.

The multiplication map $m_{\widetilde{\mathcal{D}}^!} : \widetilde{\mathcal{D}}^{!i-1} \otimes_X J^1 \to \widetilde{\mathcal{D}}^{!i}$ induces a right dual map

$$m_{\mathcal{D}^!}^{\vee} : (\widetilde{\mathcal{D}}^{!i})^* \to (\widetilde{\mathcal{D}}^{!i-1} \otimes_X J^1)^* \simeq (J^1)^* \otimes_X (\widetilde{\mathcal{D}}^{!i-1})^* \simeq \mathcal{D}_1 \otimes_X (\widetilde{\mathcal{D}}^{!i-1})^*$$

Define a composition map,

$$k^{-i} : \widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^* \to \widetilde{\mathcal{D}}(-i) \otimes_X \mathcal{D}_1 \otimes_X (\widetilde{\mathcal{D}}^{!i-1})^* \to \widetilde{\mathcal{D}}(-i+1) \otimes_X (\widetilde{\mathcal{D}}^{!i-1})^*$$

where the first map is the above map $m_{\mathcal{D}^!}^{\vee}$, and the second map is the multiplication map $m_{\mathcal{D}} : \widetilde{\mathcal{D}}(-i) \otimes_X \mathcal{D}_1 \to \widetilde{\mathcal{D}}(-i+1)$. Let $K_{(X,L)}^{-i}$ (or K^{-i} when X and L are clear) denote the left $\widetilde{\mathcal{D}}$-module $\widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^*$. Note that $K^{-i} = 0$ if $i < 0$ or $i > n+1$.

Theorem 4.2.1.1. *The map $k^{-i} : K^{-i} \to K^{1-i}$ makes $K^\bullet$ into a complex of left $\widetilde{\mathcal{D}}$-modules called the **left Koszul complex**.*

Proof. The square of the Koszul boundary, $(k)^2$, is $m_{\mathcal{D}} m_{\mathcal{D}^!}^\vee m_{\mathcal{D}} m_{\mathcal{D}^!}^\vee$. However, the middle two maps can be commuted, since they involve disjoint terms in the tensor product. Therefore, $k^2 = (m_{\mathcal{D}})^2 (m_{\mathcal{D}^!}^\vee)^2$, which is the composition

$$\widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^* \to \widetilde{\mathcal{D}}(-i) \otimes_X \mathcal{D}_1 \otimes_X \mathcal{D}_1 \otimes_X (\widetilde{\mathcal{D}}^{!i-2})^* \to \widetilde{\mathcal{D}}(-i+2) \otimes_X (\widetilde{\mathcal{D}}^{!i-2})^*$$

The map $(m_{\mathcal{D}^!}^\vee)^2$ is the map

$$Hom_{X-}(\widetilde{\mathcal{D}}^{!i}, \mathcal{O}_X) \to Hom_{X-}(\widetilde{\mathcal{D}}^{!i-2} \otimes_X J^1 \otimes_X J^1, \mathcal{O}_X)$$

right dual to multiplication. Everything in the image of this map necessarily kills $\widetilde{\mathcal{D}}^{!i-2} \otimes_X J^2 \subset \widetilde{\mathcal{D}}^{!i-2} \otimes_X J^1 \otimes_X J^1$, which translates to the image of $(m_{\mathcal{D}_!}^\vee)^2$ being contained in $R \otimes_X (\widetilde{\mathcal{D}}^{!i-2})^*$. Then, it is clear that the multiplication map $(m_{\mathcal{D}})^2$ kills anything in $\widetilde{\mathcal{D}}(-i) \otimes_X R \otimes_X (\widetilde{\mathcal{D}}^{!i-2})^*$. Therefore, $k^2 = 0$. □

The construction of the left Koszul complex commutes with localization in the natural way, as per the following lemma.

Lemma 4.2.1.1. *Let $X' \subset X$ be an open subscheme of X, and L' the localization of L. Then the left Koszul complex $K^\bullet_{(X',L')}$ of the Lie algebroid (X', L') is equal to the localization of the left Koszul complex $K^\bullet_{(X,L)}$.*

Proof. On the level of terms of the complex,

$$\begin{aligned}
&\mathcal{O}_{X'} \otimes_X \widetilde{\mathcal{D}(X,L)}(-i) \otimes_X (\widetilde{\mathcal{D}(X,L)}^{!i})^* \\
&= \widetilde{\mathcal{D}(X',L')}(-i) \otimes_X (\widetilde{\mathcal{D}(X,L)}^{!i})^* \\
&= \widetilde{\mathcal{D}(X',L')}(-i) \otimes_{X'} \mathcal{O}_{X'} \otimes_X (\widetilde{\mathcal{D}(X,L)}^{!i})^* \\
&= \widetilde{\mathcal{D}(X',L')}(-i) \otimes_{X'} (\widetilde{\mathcal{D}(X',L')}^{!i})^*
\end{aligned}$$

Note that the key is that the enveloping algebra is nearly central (see Section 2.2.4), and so localizing on the left localizes on the right. Finally, it is immediate to show that the Koszul boundary is the correct one, because the Koszul boundary was defined in terms of multiplication in $\mathcal{D}(X, L)$, and localization is an algebra homomorphism. □

4.2.2 The Exactness of the Koszul Complex.

We are finally ready for the most meaningful fact about the left Koszul complex, that it resolves $\mathcal{O}_X$ as a left $\widetilde{\mathcal{D}}$-module.

Theorem 4.2.2.1. *The natural quotient map $K^0 = \widetilde{\mathcal{D}} \to \mathcal{O}_X$ makes $K^\bullet$ into a resolution of $\mathcal{O}_X$; that is, the complex $K^\bullet$ is exact in negative degrees, and its cohomology in degree zero is exactly the image of the augmentation map.*

Proof. The strategy of the proof will be a succession of cases of increasing generality.

- $X = Spec(\mathbf{k})$ **($\mathbf{k}$ a field), L abelian.** This is the classical case of Koszul duality for $Sym_{\mathbf{k}}L$ and $\Lambda_{\mathbf{k}}L$. A proof can be found in [36], page 114.

- X **a regular local ring,** L **abelian.** Because X is local, L being projective implies that it is free, specifically that $L = \mathcal{O}_X \otimes_{\mathbf{k}} L/m$ where $\mathbf{k}$ is the residue field. The Rees algebra $\widetilde{\mathcal{D}}$ is isomorphic to the symmetric algebra $Sym_X L = \mathcal{O}_X \otimes_{\mathbf{k}} Sym_{\mathbf{k}} L/m$. The quadratic dual algebra $\widetilde{\mathcal{D}}^!$ is then the corresponding exterior algebra $Alt_X L^* = \mathcal{O}_X \otimes_{\mathbf{k}} Alt_{\mathbf{k}} L^*/m$. The left Koszul complex $K^\bullet_{(X,L)}$ is then $\mathcal{O}_X \otimes_{\mathbf{k}} K^\bullet_{(Spec(\mathbf{k}),L/m)}$, where L/m is the Lie algebroid restricted to the residue field $\mathbf{k}$. Since the theorem is true for $K^\bullet_{(Spec(\mathbf{k}),L/m)}$ by the previous case, it is then true here.

- X **arbitrary,** L **abelian.** Let $\pi : X_p \to X$ be the open embedding corresponding to localization at some prime p, and let $L_p = \pi^* L$. By the lemma before the theorem, $\pi^* K^\bullet_{(X,L)} = K^\bullet_{(X_p,L_p)}$. Since localization is exact, we have that

$$\pi^* H^i\left(K^\bullet_{(X,L)}\right) = H^i\left(\pi^* K^\bullet_{(X,L)}\right)$$

  The two facts together imply that $\pi^* H^i\left(K^\bullet_{(X,L)}\right) = H^i\left(K^\bullet_{(X_p,L_p)}\right)$. The previous case of the theorem implies that this second group vanishes for $i > 0$, and is isomorphic to $\mathcal{O}_{X_p}$ for $i = 0$. Since this fact is true at any prime p, it is true everywhere, and so the theorem is true.

- X **arbitrary,** L **arbitrary.** Consider a family of Lie algebroids $(X, L_\hbar)$, $\hbar \in \mathbb{C}$, where the bracket $[-,-]_\hbar := \hbar[-,-]$ and $\tau_\hbar := \hbar\tau$. In this notation, L_1 is the original Lie algebroid, and L_0 is the Lie algebroid with zero bracket and anchor. This gives a graded algebra $\widetilde{\mathcal{D}}_\hbar$, which is isomorphic as a left $\widetilde{\mathcal{D}}$-module to $\widetilde{\mathcal{D}} \otimes \mathbb{C}[\hbar]$. There is a corresponding quadratic dual algebra $\widetilde{\mathcal{D}}_\hbar^\perp$ and a left Koszul complex $K^\bullet_\hbar$.

  The left Koszul complex $K^\bullet_\hbar$ is filtered by $\hbar$-degree; this filtration is bounded below and exhaustive, so the associated spectral sequence converges. The spectral sequence coming from this filtration has

$$E^0_{pq} = F_p(K^q_\hbar)/F_{p-1}(K^q_\hbar)$$

Each column is isomorphic to $K^\bullet_0$, and so by the previous step, is a resolution of $\mathcal{O}_X$. The E^1 is then concentrated the ray $p \geq 0$ and $q = 0$, and so the boundary is zero. Thus, the natural map $gr(K^\bullet_\hbar) \to gr(\mathcal{O}_X \otimes \mathbb{C}[\hbar])$ becomes an augmentation map. By Lemma 2.1.4.1, the original map $K^\bullet_\hbar \to \mathcal{O}_X \otimes \mathbb{C}[\hbar]$ is an augmentation map.

Let $\mathbb{C}_1 := \mathbb{C}[\hbar]/(\hbar - 1)$. Since $(X, L_\hbar)$ is flat over $\mathbb{C}[\hbar]$, we have that

$$H^i(K^\bullet_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1) = H^i(K^\bullet_\hbar) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1$$

Therefore, $K^\bullet_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1$ is a resolution of $\mathcal{O}_X \otimes \mathbb{C}[\hbar] \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1 = \mathcal{O}_X$. However, $K^\bullet_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1$ is the left Koszul resolution corresponding to (X, L) with the undeformed bracket and anchor map.

□

Since $\mathcal{D}_{!i}$ is a f.g. projective right $\mathcal{O}_X$-module, then $(\mathcal{D}_{!i})^*$ is a f.g. projective left $\mathcal{O}_X$-module. Therefore, K^i is a projective left $\widetilde{\mathcal{D}}$-module, and the left Koszul resolution is a projective resolution of $\mathcal{O}_X$ as a $\widetilde{\mathcal{D}}$-module.

Recall the derived torsion functor $\mathbb{R}\tau$ and the derived global section functor $\mathbb{R}\omega\pi$ from Section 3.2.3 and 3.2.2, respectively.

Corollary 4.2.2.1. *The functor $\mathbb{R}\tau$ has dimension $n+1$, and the functor $\mathbb{R}\omega\pi$ has dimension n; that is, $\mathbb{R}^i\tau(M) = 0$ if $i > n+1$ and $\mathbb{R}^i\omega\pi(M) = 0$ if $i > n$.*

Proof. The Koszul resolution is a length $n+1$ projective resolution of $\mathcal{O}_X$, and so, for all M, $\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M) = 0$ when $i > n+1$. Therefore, by Lemma 3.2.3.1,

$\mathbb{R}^i\tau(M) = 0$ when $i > n+1$. Since $\mathbb{R}^i\omega\pi(M) \simeq \mathbb{R}^{i+1}\tau(M)$ when $i > 0$, the statement follows. □

There is also a right Koszul complex $K^\bullet_{right}$ whose terms are $(\widetilde{\mathcal{D}}^{!i})^* \otimes_X \widetilde{\mathcal{D}}(-i)$, with boundary right dual to the multiplication map $\mathcal{D}_1 \otimes_X \widetilde{\mathcal{D}}_{i-1} \to \widetilde{\mathcal{D}}^i$. This is again a projective resolution of $\mathcal{O}_X$, this time as a right $\widetilde{\mathcal{D}}$-module. The proofs are analogous.

4.2.3 The Quadratic Dual as an *Ext* Algebra.

The following theorem about $\widetilde{\mathcal{D}}^!$ follows from the exactness of the Koszul complex, which partially explains the significance of $\widetilde{\mathcal{D}}^!$ a postieri.

Theorem 4.2.3.1. *$\widetilde{\mathcal{D}}^!$ is isomorphic to $\underline{Ext}^\bullet_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, \mathcal{O}_X)$ as a graded algebra, where $J^1 = {}^*(\mathcal{D}_1) \subset \widetilde{\mathcal{D}}^!$ has graded degree -1.*

Proof. It is easy to see this isomorphism, on the level of graded $\mathcal{O}_X$-modules.

Lemma 4.2.3.1. *$\widetilde{\mathcal{D}}^!$ is isomorphic to $\underline{Ext}^\bullet_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, \mathcal{O}_X)$ as a graded $\mathcal{O}_X$-module.*

Proof. The left Koszul resolution $K^\bullet$ is a left projective resolution of $\mathcal{O}_X$. Therefore,

$$\begin{aligned}
\mathbb{R}\underline{Hom}^\bullet_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, \mathcal{O}_X) &= \underline{Hom}_{\widetilde{\mathcal{D}}-}(K^\bullet, \mathcal{O}_X) \\
&= \bigoplus_{i=0}^{n} \underline{Hom}_{\widetilde{\mathcal{D}}-}(\widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^*, \mathcal{O}_X) \\
&= \bigoplus_{i=0}^{n} Hom_{X-}((\widetilde{\mathcal{D}}^{!i})^*, \mathcal{O}_X)(i) \\
&= \bigoplus_{i=0}^{n} \widetilde{\mathcal{D}}^{!i}(i)
\end{aligned}$$

Since each term in the complex is concentrated in a different graded degree, the boundary vanishes, and so the cohomology is isomorphic to $\widetilde{\mathcal{D}}^!$. $\square$

Showing that this is an isomorphism of algebras will require more work. Let $\mathbf{B}^\bullet$ denote the **normalized left bar resolution** of $\mathcal{O}_X$ (see [36], page 284 for details). This is the complex of graded left $\widetilde{\mathcal{D}}$-modules with $\mathbf{B}^{-i} = \widetilde{\mathcal{D}} \otimes_X (\widetilde{\mathcal{D}}_{\geq 1})^{\otimes_X i}$ where the boundary sends $a_1 \otimes_X a_2 \otimes_X \ldots \otimes_X a_n$ to

$$\sum_{i=1}^{n-1} (-1)^i a_1 \otimes_X a_2 \otimes_X \ldots \otimes_X a_i a_{i+1} \otimes_X \ldots \otimes_X a_n$$

The complex $\mathbf{B}^\bullet$ is a left projective resolution of $\mathcal{O}_X$, with the augmentation map $\mathbf{B}^0 = \widetilde{\mathcal{D}} \to \mathcal{O}_X$ the natural projection onto graded degree zero.

Therefore, $\underline{Ext}^\bullet_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, \mathcal{O}_X)$ is the cohomology algebra of the differential graded algebra (dga) $\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathbf{B}^\bullet)$, where the multiplication is the composition of maps. The augmentation map $\mathbf{B}^\bullet \to \mathcal{O}_X$ gives a quasi-isomorphism of complexes $\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathbf{B}^\bullet) \to \underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathcal{O}_X)$. Since

$$\begin{aligned}
\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^{-i}, \mathcal{O}_X) &= \underline{Hom}_{\widetilde{\mathcal{D}}-}(\widetilde{\mathcal{D}} \otimes_X (\widetilde{\mathcal{D}}_{\geq 1})^{\otimes_X i}, \mathcal{O}_X) \\
&= \underline{Hom}_{X-}((\widetilde{\mathcal{D}}_{\geq 1})^{\otimes_X i}, \mathcal{O}_X) \\
&= [\,^*(\widetilde{\mathcal{D}}_{\geq 1})]^{\otimes_X i}
\end{aligned}$$

Thus, $\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathcal{O}_X)$ is isomorphic to $T_X\,^*(\widetilde{\mathcal{D}}_{\geq 1})$ as a graded $\mathcal{O}_X$-module, and the natural multiplication on the tensor algebra makes it into a dga.

In fact, the quasi-isomorphism

$$\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathbf{B}^\bullet) \to \underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbf{B}^\bullet, \mathcal{O}_X) = T_X\,^*(\widetilde{\mathcal{D}}_{\geq 1})$$

is a map of dgas. To see this, let us construct a section of this map. Let $\phi \in [\,^*(\widetilde{\mathcal{D}}_{\geq 1})]^{\otimes_X i}$, then for any $j > i$, there is a natural map

$$\widetilde{\mathcal{D}} \otimes_X (\widetilde{\mathcal{D}}_{\geq 1})^{\otimes_X j} \to \widetilde{\mathcal{D}} \otimes_X (\widetilde{\mathcal{D}}_{\geq 1})^{\otimes_X (j-i)}$$

given by applying ϕ to the first i terms on the left. It is easy but tedious to verify that this gives a map of dgas $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1}) \to \underline{Hom}_{\widetilde{\mathcal{D}}^-}(\mathbf{B}^\bullet, \mathbf{B}^\bullet)$ which is a section of the above map. Therefore, $\underline{Ext}^\bullet_{\widetilde{\mathcal{D}}^-}(\mathcal{O}_X, \mathcal{O}_X)$ is the cohomology algebra of the dga $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1})$.

The dga $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1})$ has both a cohomological degree (coming from the usual grading on a tensor algebra) and a graded degree (coming from the grading on $\widetilde{\mathcal{D}}_{\geq 1}$). Because ${}^*(\widetilde{\mathcal{D}}_{\geq 1})$ is concentrated in graded degree ≤ -1, $[{}^*(\widetilde{\mathcal{D}}_{\geq 1})]^{\otimes_X i}$ is concentrated in graded degree $\leq -i$. Therefore, if one restricts the complex $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1})$ to graded degree $-i$, the resulting complex is non-zero in cohomological degrees j, $0 \leq j \leq i$.

However, we do actually know the cohomology of this complex, due to Lemma 4.2.3.1. Specifically, we know that in graded degree $-i$, the cohomology is concentrated in cohomological degree i. Since the corresponding complex is concentrated in cohomological degrees $\leq i$, the cohomology must be the cokernel of the boundary map. We therefore have a map of dgas $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1}) \to \widetilde{\mathcal{D}}^!$, which is a quasi-isomorphism.

Note that, for an element in $T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1})$ to have graded degree $-i$ and cohomological degree i, it must be the tensor product of i elements of graded degree -1 elements; therefore, $(T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1}))^{(-i,i)} = [{}^*(\widetilde{\mathcal{D}}_1)]^{\otimes_X i} = (J^1)^{\otimes_X i}$. If we let $T_X J^1$ be a dga with zero boundary, this extends to a map of dgas $T_X J^1 \to T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1})$, which is the identity in degree $(-i, i)$ and zero elsewhere.

The composition

$$T_X J^1 \to T_X{}^*(\widetilde{\mathcal{D}}_{\geq 1}) \to \widetilde{\mathcal{D}}^!$$

is then a surjection of dgas; since their boundaries are zero, we can think of them as algebras again. Since it is an isomorphism in graded degree -1 on the J^1's, its kernel must be exactly generated by $J^2 \subset J^1 \otimes_X J^1$. The theorem follows. $\square$

4.2.4 The Relative Gorenstein Property.

The following lemma should be regarded as a relative version of the Gorenstein property for graded algebras. Recall that $\omega_L := \Lambda^n_X L^*$.

Lemma 4.2.4.1. *(The relative Gorenstein property)*

$$\underline{Ext}^i_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \widetilde{\mathcal{D}}) = \begin{cases} \omega_L(n+1) & i = n+1 \\ 0 & otherwise \end{cases}$$

Proof. Resolve $\mathcal{O}_X$ by the left Koszul resolution $K^\bullet$. Using Corollary 4.1.3.1, which says that $(\widetilde{\mathcal{D}}^{!i})^* = \omega_L^* \otimes_X \widetilde{\mathcal{D}}^{!(n+1-i)}$ and ${}^*(\widetilde{\mathcal{D}}^{!i}) = \widetilde{\mathcal{D}}^{!(n+1-i)} \otimes_X \omega_L^*$,

$$\begin{aligned} \underline{Hom}_{\widetilde{\mathcal{D}}_-}(K^i, \widetilde{\mathcal{D}}) &= \underline{Hom}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^*, \widetilde{\mathcal{D}}) \\ &= \underline{Hom}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}(-i) \otimes_X \omega_L^* \otimes_X \widetilde{\mathcal{D}}^{!(n+1-i)}, \widetilde{\mathcal{D}}) \\ &= Hom_{X-}(\omega_L^* \otimes_X \widetilde{\mathcal{D}}^{!(n+1-i)}, \mathcal{O}_X) \otimes_X \widetilde{\mathcal{D}}(i) \\ &= {}^*(\widetilde{\mathcal{D}}^{!(n+1-i)}) \otimes_X \omega_L \otimes_X \widetilde{\mathcal{D}}(i) \\ &= \omega_L \otimes_X (\widetilde{\mathcal{D}}^{!(n+1-i)})^* \otimes_X \widetilde{\mathcal{D}}(i) \end{aligned}$$

Since the duality map is adjoint to the multiplication map, the boundary map on this complex is the right Koszul differential. Therefore,

$$\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \widetilde{\mathcal{D}}) = \omega_L(n+1)[-n-1] \otimes_X K^\bullet_{right}$$

Since $K^\bullet_{right}$ is a resolution of $\mathcal{O}_X$, the theorem follows. $\square$

Corollary 4.2.4.1. *The derived torsion functor $\mathbb{R}^i\tau(\widetilde{\mathcal{D}})_j$ vanishes if $i \neq n+1$ or if $j > -n-1$.*

Proof. This follows from the Gorenstein property and Lemma 3.2.3.1. □

4.2.5 Serre Finiteness and Vanishing for $\widetilde{\mathcal{D}}$.

The Gorenstein property can also be used to show that the ring $\widetilde{\mathcal{D}}$ satisfies the χ-condition (Definition 3.2.4), which will in turn imply the Serre Finiteness and Vanishing Theorems.

Lemma 4.2.5.1. *Let M be a f.g. $\widetilde{\mathcal{D}}$-module. For n large enough, the induced map*

$$\underline{Ext}^i_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \widetilde{\mathcal{D}}_{\geq n} \cdot M) \to \underline{Ext}^i_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, M)$$

is zero for all i.

Proof. We use the left Koszul resolution, where we get a map on complexes

$$\underline{Hom}_{\widetilde{\mathcal{D}}_-}(K^\bullet, \widetilde{\mathcal{D}}_{\geq n} \cdot M) \to \underline{Hom}_{\widetilde{\mathcal{D}}_-}(K^\bullet, M)$$

For any $m \in \widetilde{\mathcal{D}}_{\geq n}M$, there are $\delta \in \mathcal{D}_1$ and $m' \in M$ such that $m = \delta m$. Therefore, any composition

$$(\widetilde{\mathcal{D}}^{!i})^* \to \widetilde{\mathcal{D}}_{\geq n}M(-i) \hookrightarrow M(-i)$$

(where the second map is the natural inclusion) can be factored as

$$(\widetilde{\mathcal{D}}^{!i})^* \to \mathcal{D}_1 \otimes_X M(-i-1) \to M(-i)$$

A cohomology class in $H^i(\underline{Hom}_{\widetilde{\mathcal{D}}_-}(K^\bullet, \widetilde{\mathcal{D}}_{\geq n} \cdot M))$ is represented by a map $(\widetilde{\mathcal{D}}^{!i})^* \to \widetilde{\mathcal{D}}_{\geq n}M(-i)$, and its image is the composition with the inclusion to $M(-i)$. Thus, the image of any representative of the cohomology class has a preimage in $(\widetilde{\mathcal{D}}^{!i+1})^* \otimes_X M(-i-1)$, and so it is exact. □

Lemma 4.2.5.2. *Let M be a f.g. $\widetilde{\mathcal{D}}$-module. Then there is some n such that for all i and j,*

(a) $\left(\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M)\right)_{\geq n} = 0.$

(b) $\left(\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\widetilde{\mathcal{D}}_{\leq j}, M)\right)_{\geq n} = 0.$

(c) $(\mathbb{R}^i\tau(M))_{\geq n} = 0.$

Therefore, $\widetilde{\mathcal{D}}$ satisfies the χ-condition.

Proof. We consider the long exact sequence coming from applying $\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, -)$ to

$$0 \to \widetilde{\mathcal{D}}_{\geq n}M \to M \to M/\widetilde{\mathcal{D}}_{\geq n}M \to 0$$

By the preceeding lemma, the map

$$\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, \widetilde{\mathcal{D}}_{\geq n} \cdot M) \to \underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M)$$

is zero, and so the map

$$\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M) \to \underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M/\widetilde{\mathcal{D}}_{\geq n})$$

is an inclusion. Now, because M is finitely generated, there is some n' such that $\widetilde{\mathcal{D}}_{\geq n}M \subseteq M_{\geq n'}$, and so the above inclusion factors through

$$\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M) \to \underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M/M_{\geq n'})$$

Thus, this map is an inclusion.

Now, $M/M_{\geq n'}$ is concentrated in finitely many graded degrees. This means that the Koszul complex which computes $\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M/M_{\geq n'})$ is not acylic in finitely many graded degrees, and so $\underline{Ext}^i_{\widetilde{\mathcal{D}}-}(\mathcal{O}_X, M/M_{\geq n'})$ is as well. This proves the first part. The other two parts follow from Lemma 3.2.3.1.

Part (a) of the theorem is the χ° condition of Artin and Zhang, and by Proposition 3.8 in [1, pg.243], this together with the fact that $\mathcal{D}_i$ is a finitely-generated $\mathcal{O}_X$-module for all i imply the χ-condition. □

Theorem 4.2.5.1. *Let $\mathcal{M} \in qgr(\widetilde{\mathcal{D}})$.*

- *(Serre Finiteness)* $\mathbb{R}^i\omega(\mathcal{M})_j$ *is a finitely-generated* $\mathcal{O}_X$*-module for all* i, j.
- *(Serre Vanishing)* $\mathbb{R}^i\omega(\mathcal{M})_j = 0$ *if* $i > n$ *and any* j*, or if* $i \neq 0$ *and* j *is sufficiently large.*

Proof. The χ-condition and Theorem 3.2.4.1 immediately imply finiteness and vanishing in sufficiently high graded degree. Vanishing in cohomological degree greater than n was shown in Corollary 4.2.2.1. □

CHAPTER 5

TENSOR PRODUCTS.

5.1 Tensoring and Fourier-Mukai Transforms.

We need to generalize an important technique from commutative projective geometry to the non-commutative setting; that of the Fourier-Mukai transform. Let X be a scheme, and let K be any module on $X \times X$, or more generally any derived object in $D^b(Mod(X \times X))$ (equivalently, K is a derived $\mathcal{O}_X$-bimodule). Let p_1 and p_2 denote the projections of $X \times X$ onto the first and second coefficient, respectively.

Given any $M \in D^b(Mod(X))$, $K \otimes^{\mathbb{L}}_X p_2^*(M) \in D^b(Mod(X \times X))$, and so it can be pushed forward along the projection $p_1 : X \times X \to X$ onto the first factor to give $\mathbb{R}p_{1*}(K \otimes^{\mathbb{L}}_X p_2^*(M)) \in D^b(Mod(X))$. The functor $M \to \mathbb{R}p_{1*}(K \otimes^{\mathbb{L}}_X p_2^*(M))$ is called the **Fourier-Mukai transform of** K. These have been studied extensively, for references check [20].

5.1.1 Tensor Products.

For A a positively-graded algebra, the categories $Gr(A)$ and $gr(A)$ don't have a tensor product in the sense of a bifunctorial map $Gr(A) \times Gr(A) \to Gr(A)$. The tensor product here is a bifunctorial map

$$\otimes_A : Gr(A^{op}) \times Gr(A) \to Gr(\mathbb{C})$$

Subsequently taking the degree zero part gives a map

$$\circledcirc_A : Gr(A^{op}) \times Gr(A) \to Vect$$

Naively, one would hope that this descends to some kind of map $\circledcirc_A : QGr(A^{op}) \times QGr(A) \to Vect$. However, for this to descend to a map on quotient categories, we would need that $T \circledcirc_A M = M' \circledcirc_A T' = 0$ for $T \in Tors(A^{op})$ and $T' \in Tors(A)$. This is just not true; take, for example, $A_0 \circledcirc_A A$ or $A \circledcirc_A A_0$, which are both isomorphic to A_0 as a vector space.

So, instead of trying to push the multiplication forward along π, we can pull the multiplication back along ω. Given $\pi M \in QGr(A^{op})$ and $\pi N \in QGr(A)$, define

$$\pi M \circledcirc_A \pi N := \omega \pi M \circledcirc_A \omega \pi N = (\omega \pi M \otimes_A \omega \pi N)_0$$

The derived analog of this bifunctor is $(\mathbb{R}\omega\pi M \otimes^{\mathbb{L}}_A \mathbb{R}\omega\pi N)_0$ (for $\pi M \in D^b(QGr(A^{op}))$ and $\pi N \in D^b(QGr(A))$). Note that this is neither the left nor right derived functor of the previous functor, and so in particular they might not agree in cohomological degree zero.

5.1.2 The Category of Quotient Bimodules.

The point of these tensoring constructions is to be able to define the Fourier-Mukai transforms on this category; however, we still need to know where the kernels of the transforms live. Let $A^e := A \otimes A^{op}$, which has the property that left A^e-modules are the same as A-bimodules; note that it is a bigraded algebra. Let $\mathbb{G}r(A^e)$ be the category of bigraded A^e-modules, which is the same as the category of bigraded A-bimodules. Let $\mathbb{T}ors(A^e)$ be the subcategory of $\mathbb{G}r(A^e)$ such that, for every $m \in T \in \mathbb{T}ors(A^e)$, there is some n such that $A_{\geq n} m A_{\geq n} = 0$. Let $qgr(A^e)$ denote the quotient category $\mathbb{G}r(A^e)/\mathbb{T}ors(A^e)$. For an account of non-commutative projective geometry of polygraded algebras, at least in the case of connected algebras, see [3].

The category $qgr(A^e)$ satisfies all the same properties that were listed for $QGr(A)$, or at least analogous properties. The only difference is the structure of the functors ω and τ, which may be given by (where $\underline{Hom}$ now denotes a bi-graded Hom)

$$\omega\pi(M) := \lim_{n\to} \underline{Hom}_{\mathbb{G}r(A^e)}(A_{\geq n} \otimes A_{\geq n}, M)$$

$$\tau(M) := \lim_{n\to} \underline{Hom}_{\mathbb{G}r(A^e)}((A \otimes A)/(A_{\geq n} \otimes A_{\geq n}), M)$$

In certain nice cases, the derived functor $\mathbb{R}\omega\pi$ has a more useful definition.

Lemma 5.1.2.1. *Let A be left and right noetherian. For $M \in \mathbb{G}r(A^e)$, there is an isomorphism in $D(\mathbb{G}r(A^e))$:*

$$\mathbb{R}\omega\pi(M) \simeq \mathbb{R}\omega\pi(A) \otimes^{\mathbb{L}}_A M \otimes^{\mathbb{L}}_A \mathbb{R}\omega\pi(A)$$

Proof. Consider the directed system $A_{\geq m} \otimes A_{\geq m'}$, as m and m' run over the integers, with the maps being the natural inclusions. This directed system has a sub-directed system $A_{\geq n} \otimes A_{\geq n}$ which is coinitial, in the sense that any object $A_{\geq m} \otimes A_{\geq m'}$ has a inclusion from some $A_{\geq n} \otimes A_{\geq n}$ (for instance, $n = \max(m, m')$). Therefore, there is an isomorphism of direct limits:

$$\lim_{n\to} \mathbb{R}\underline{Hom}_{\mathbb{G}r(A^e)}(A_{\geq n} \otimes A_{\geq n}, M) \simeq \lim_{m\to} \lim_{m'\to} \mathbb{R}\underline{Hom}_{\mathbb{G}r(A^e)}(A_{\geq m} \otimes A_{\geq m'}, M)$$

By adjunction, this second $\mathbb{R}\underline{Hom}$ becomes

$$\lim_{m\to} \lim_{m'\to} \mathbb{R}\underline{Hom}_{Gr(A)}(A_{\geq m}, \mathbb{R}\underline{Hom}_{Gr(A^{op})}(A_{\geq m'}, M))$$

$$= \lim_{m\to} \lim_{m'\to} \mathbb{R}\underline{Hom}_{Gr(A)}(A_{\geq m}, A) \otimes^{\mathbb{L}}_A \mathbb{R}\underline{Hom}_{Gr(A^{op})}(A_{\geq m'}, M)$$

$$= \lim_{m\to} \lim_{m'\to} \mathbb{R}\underline{Hom}_{Gr(A)}(A_{\geq m}, A) \otimes^{\mathbb{L}}_A M \otimes^{\mathbb{L}}_A \mathbb{R}\underline{Hom}_{Gr(A^{op})}(A_{\geq m'}, A)$$

The last two equalities use that $A_{\geq m}$ is noetherian as a left and right A-module. This final expression is then equal to $\mathbb{R}\omega\pi(A) \otimes^{\mathbb{L}}_A M \otimes^{\mathbb{L}}_A \mathbb{R}\omega\pi(A)$. □

5.1.3 Fourier-Mukai Transforms.

Now, given any object $K \in D^b(\mathbb{G}r(A^e))$, define the derived functor F_K on $D^b(QGr(A))$ by:

$$F_K(\pi M) := \pi(\mathbb{R}\omega\pi(K) \otimes_A^{\mathbb{L}} \mathbb{R}\omega(\pi M))_{\bullet,0}$$

This has a simpler form for nice A.

Lemma 5.1.3.1. *If A is left and right noetherian, then*

$$F_K(\pi M) = \pi(K \circledcirc_A^{\mathbb{L}} \mathbb{R}\omega\pi(M)) = \pi(\mathbb{R}\omega\pi(K) \circledcirc_A^{\mathbb{L}} M)$$

Proof. By Lemma 5.1.2.1 and Lemma 3.2.2.1, this is equal to

$$\pi(\mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} K \otimes_A^{\mathbb{L}} \mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} \mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} M)_{\bullet,0}$$

By Corollary 3.2.2.1, this is

$$\pi(\mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} K \otimes_A^{\mathbb{L}} \mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} M)_{\bullet,0} \tag{5.1}$$

Applying Lemma 3.2.2.1 twice and using that $\pi\mathbb{R}\omega\pi = \pi$, this is equal to

$$\pi(\mathbb{R}\omega\pi(A) \otimes_A^{\mathbb{L}} K \otimes_A^{\mathbb{L}} \mathbb{R}\omega\pi(M))_{\bullet,0} = \pi(\mathbb{R}\omega\pi(K \otimes_A^{\mathbb{L}} \mathbb{R}\omega\pi(M))) = \pi(K \circledcirc_A^{\mathbb{L}} \mathbb{R}\omega\pi(M))$$

Instead, we could apply Lemma 5.1.2.1 to Equation (5.1) to get

$$\pi(\mathbb{R}\omega\pi(K) \otimes_A^{\mathbb{L}} M)_{\bullet,0} = \pi(\mathbb{R}\omega\pi(K) \circledcirc_A^{\mathbb{L}} M)$$

This concludes the proof. □

Given any exact triangle $A \to B \to C \to A[1]$ in $D^b(\mathbb{G}r(A^e))$, there is an associated exact triangle of functors $F_A \to F_B \to F_C \to F_A[1]$, in the sense that for any $\pi M \in D^b(QGr(A^e))$, there is an exact triangle:

$$F_A(\pi M) \to F_B(\pi M) \to F_C(\pi M) \to F_A(\pi M)[1]$$

Therefore, a functor F_K may be resolved by other, simpler functors by resolving πK into simpler objects in $D^b(\mathbb{G}r(A^e))$.

5.1.4 The Diagonal Object.

Even the identity functor on $D^b(QGr(A))$ arises as a Fourier-Mukai transform. Let $\widetilde{\Delta}$ be the bigraded A-bimodule such that $\widetilde{\Delta}_{i,j} = A^{i+j}$, where $A^k = 0$ in negative degrees. $\widetilde{\Delta}$ has the property that $\widetilde{\Delta} \circledcirc_A M = (\widetilde{\Delta} \otimes_A M)_{\bullet,0} = M$ for all $M \in Gr(A)$. As an immediate corollary, $\widetilde{\Delta}$ is flat as a right A-module. If A is noetherian, the Fourier-Mukai transform $F_{\widetilde{\Delta}}(\mathcal{M})$ is $\pi(\widetilde{\Delta} \circledcirc_A \mathbb{R}\omega(\mathcal{M}))$, which is $\pi(\mathbb{R}\omega(\mathcal{M})) = \mathcal{M}$. Therefore, $F_{\widetilde{\Delta}}$ is the identity functor.

However, $\widetilde{\Delta}$ is not the only object in $\mathbb{G}r(A^e)$ whose corresponding Fourier-Mukai transform is the identity. After all, all that matters is the image under π in $qgr(A^e)$. Let Δ be the **diagonal object**, the bigraded A-bimodule such that $\Delta_{i,j} = A^{i+j}$ when $i \geq 0$ and $j \geq 0$, and zero otherwise. There is a natural inclusion $\Delta \hookrightarrow \widetilde{\Delta}$, and $(\widetilde{\Delta}/\Delta)_{i,j} = 0$ if $i \geq 0$ and $j \geq 0$. $\omega\pi(\widetilde{\Delta}/\Delta) = 0$, and so $\pi(\Delta) = \pi(\widetilde{\Delta})$. Then, the Fourier-Mukai transform F_Δ is also the identity.

The point of this is now that producing a resolution of Δ in $\mathbb{G}r(A^e)$ will give a resolution of the identity, which in turn will give a resolution of any object in $Gr(A)$.

5.2 The Resolution of the Diagonal.

The homological computations of the past several sections finally start to yield results relevant to the projective geometry of $\widetilde{\mathcal{D}}$. The next step is to combine the left and right Koszul complexes into a Koszul bicomplex, which can then be used to extract a resolution of the diagonal bimodule Δ. The reader should prepare emotionally for bigraded bicomplexes, and the quadruple indices that entails.

5.2.1 The Koszul Bicomplex.

Let $\mathbb{K}^{i,j}$ be the $\widetilde{\mathcal{D}}$-bimodule $\widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!(i+j)})^* \otimes_X \widetilde{\mathcal{D}}(-j)$. The left Koszul boundary map acts on the first two terms, and sends $\mathbb{K}^{i,j}$ to $\mathbb{K}^{i-1,j}$; the right Koszul boundary map acts on the last two terms, and sends $\mathbb{K}^{i,j}$ to $\mathbb{K}^{i,j-1}$.

Lemma 5.2.1.1. *These two boundary maps, k_{left} and k_{right}, make $\mathbb{K}^{i,j}$ into a bicomplex of $\widetilde{\mathcal{D}}$-bimodules called the* ***Koszul bicomplex*** *(making sure to obey the Koszul sign rule for commuting odd-degree maps).*

Proof. It is immediate that the two boundaries square to zero themselves. Thus, all that remains to check is that $(k_{left} + k_{right})$ squares to zero, which by the Koszul sign rule is equivalent to k_{left} and k_{right} commuting.

Since multiplication in $\widetilde{\mathcal{D}}^!$ is associative, the multiplication map $J^1 \otimes_X \widetilde{\mathcal{D}}^{!i-2} \otimes_X J^1 \to \widetilde{\mathcal{D}}^{!i}$ doesn't depend on the order of multiplication. Dualizing gives the desired fact that k_{left} and k_{right} commute. □

The terms of the Koszul bicomplex are bigraded $\widetilde{\mathcal{D}}$-bimodules, and so an element in this complex can have a graded bidegree (it's bigrading as a $\widetilde{\mathcal{D}}$-bimodule) as well as a homological bidegree (which term of the bicomplex it is in). The space of elements with graded bidegree (p, q) and homological bidegree (i, j) will be denoted $\mathbb{K}^{i,j}_{p,q}$, and it is equal to $\mathcal{D}_{p-i} \otimes_X (\widetilde{\mathcal{D}}^{!(i+j)})^* \otimes_X \mathcal{D}_{q-j}$.

5.2.2 The Resolution of the Diagonal Object.

Define the complex $\mathcal{K}_\Delta$ to be such that $\mathcal{K}^i_\Delta = \ker(d_r : \mathbb{K}^{i,0} \to \mathbb{K}^{i,-1})$, together with the boundary d_l inherited from $\mathbb{K}$. Because $\mathbb{K}^{0,-1} = 0$, we have that $\mathcal{K}^0_\Delta = \mathbb{K}^{0,0} = \widetilde{\mathcal{D}} \otimes_X \widetilde{\mathcal{D}}$.

As in Section 5.1.4, let $\Delta \in \mathbb{G}r(\widetilde{\mathcal{D}}_e)$ be the **diagonal object**, the bigraded A-bimodule such that $\Delta_{i,j} = A^{i+j}$ when $i \geq 0$ and $j \geq 0$, and zero otherwise. There is a canonical surjection $\widetilde{\mathcal{D}} \otimes_X \widetilde{\mathcal{D}} \to \Delta$, which in bidegree (p, q) is the multiplication map $\mathcal{D}_p \otimes_X \mathcal{D}_q \to \mathcal{D}_{p+q}$.

Theorem 5.2.2.1. *The canonical surjection $\mathcal{K}_\Delta \to \Delta$ makes $\mathcal{K}_\Delta$ into a resolution of Δ. Accordingly, the complex $\mathcal{K}_\Delta$ is called a* ***resolution of the diagonal****.*

Proof. First, we show that the map $\mathcal{K}^0_\Delta \to \Delta$ gives an augmentation of the complex; that is, it kills the image of $\mathcal{K}^1_\Delta$ in $\mathcal{K}^0_\Delta$. By definition, $\mathcal{K}^1_\Delta$ is the kernel of

$$\widetilde{\mathcal{D}}(-1) \otimes_X \mathcal{D}_1 \otimes_X \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}(-1) \otimes_X \widetilde{\mathcal{D}}(1)$$

This map is given by multiplying the last two terms. However, since the composition map $\mathcal{K}^1_\Delta \to \mathcal{K}^0_\Delta \to \Delta$ is given by multiplying all the terms of $\mathcal{K}^1_\Delta$ together, and because multiplication in $\mathcal{D}$ is associative, this composition must be zero.

Now, define the **truncated Koszul bicomplex** $\widehat{\mathbb{K}}^{i,j}$ to be equal to $\mathbb{K}^{i,j}$ when $j \geq 0$, and 0 otherwise. For a fixed graded bidegree (p, q), the term $\widehat{\mathbb{K}}^{i,j}_{p,q}$ vanishes for $i > p$, $j > q$ or $i+j < 0$. Therefore, in any fixed graded bidegree, the bicomplex $\widehat{\mathbb{K}}$ is bounded. This means that both the horizontal-then-vertical spectral sequence and the vertical-then-horizontal spectral sequence converge to total cohomology of $\widehat{\mathbb{K}}$.

Taking horizontal cohomology first, the rows are all right Koszul complexes tensored with $\widetilde{\mathcal{D}}$, and so we get

$$E_1^{i,j} = \left\{ \begin{array}{cc} \widetilde{\mathcal{D}}(j) \otimes_X \mathcal{O}_X(-j) & if\ j = -i \geq 0 \\ 0 & otherwise \end{array} \right\}$$

Therefore, the spectral sequence collapses on the first page, and we have

$$H^0(Tot(\widehat{\mathbb{K}})) = \bigoplus_{j=0}^{\infty} \widetilde{\mathcal{D}}(j) \otimes_X \mathcal{O}_X(-j),\ H^{\neq 0}(Tot(\widehat{\mathbb{K}})) = 0$$

Taking vertical cohomology first, the rows are either left Koszul complexes tensored with $\widetilde{\mathcal{D}}$, or they are left Koszul complexes which have been brutally truncated. Therefore,

$$E_1^{i,j} = \left\{ \begin{array}{cc} \mathcal{O}_X(j) \otimes_X \widetilde{\mathcal{D}}(-j) & if\ j = -i \geq 1 \\ \mathcal{K}_\Delta^i & if\ i \geq 0, j = 0 \\ 0 & otherwise \end{array} \right\}$$

Therefore, the spectral sequence collapses on the second page, and we have

$$H^0(Tot(\widehat{\mathbb{K}})) = H^0(\mathcal{K}_\Delta) \oplus \left(\bigoplus_{j=1}^{\infty} \mathcal{O}_X(j) \otimes_X \widetilde{\mathcal{D}}(-j) \right),\ H^{i \neq 0}(Tot(\widehat{\mathbb{K}})) = H^i(\mathcal{K}_\Delta)$$

Comparing the two results, $\mathcal{K}_\Delta$ is exact outside degree zero, and we have that

$$H^0(\mathcal{K}_\Delta) \oplus \left(\bigoplus_{j=1}^{\infty} \mathcal{O}_X(j) \otimes_X \widetilde{\mathcal{D}}(-j) \right) = \bigoplus_{j=0}^{\infty} \widetilde{\mathcal{D}}(j) \otimes_X \mathcal{O}_X(-j)$$

Looking in graded bidegree (p, q), we have that $H^0(\mathcal{K}_\Delta) = \mathcal{D}_{p+q}$ if and only if $p, q \geq 0$. Therefore, the map $H^0(\mathcal{K}_\Delta) \to \Delta$ induced by the augmentation is an isomorphism. □

The power of this theorem comes from the structure of $\mathcal{K}_\Delta$. To see this structure, define Ω^i_R to be the kernel of the i-th boundary in the right Koszul complex:

$$d_r : (\widetilde{\mathcal{D}}^{!i})^* \otimes_X \widetilde{\mathcal{D}}(-i) \to (\widetilde{\mathcal{D}}^{!i-1})^* \otimes_X \widetilde{\mathcal{D}}(-i+1)$$

Since $\widetilde{\mathcal{D}}^{!j} = 0$ for $j > n+1$, $\Omega^j_R = 0$ for $j > n$. It is clear from the definition of $\mathcal{K}_\Delta$ that $\mathcal{K}^i_\Delta = \widetilde{\mathcal{D}}(-i) \otimes_X \Omega^i_R(i)$.

Corollary 5.2.2.1. *The resolution of the diagonal then has the form:*

$$\Delta \leftarrow \widetilde{\mathcal{D}} \otimes_X \widetilde{\mathcal{D}} \leftarrow \widetilde{\mathcal{D}}(-1) \otimes_X \Omega^1_R(1) \leftarrow ... \leftarrow \widetilde{\mathcal{D}}(-i) \otimes_X \Omega^i_R(i) \leftarrow ... \leftarrow \widetilde{\mathcal{D}}(-n) \otimes_X \Omega^n_R(n)$$

There is a mirror image version of this, where $\mathcal{K}_\Delta$ is replaced by $ker(d_l : \mathbb{K}^{0,i} \to \mathbb{K}^{-1,i})$. Defining

$$\Omega^i_L := ker\left(d_l : \widetilde{\mathcal{D}}(-i) \otimes_X (\widetilde{\mathcal{D}}^{!i})^* \to \widetilde{\mathcal{D}}(-i+1) \otimes_X (\widetilde{\mathcal{D}}^{!i-1})^*\right),$$

all the same arguments work to show that the following is also a resolution of the diagonal:

$$\Delta \leftarrow \widetilde{\mathcal{D}} \otimes_X \widetilde{\mathcal{D}} \leftarrow \Omega^1_L(1) \otimes_X \widetilde{\mathcal{D}}(-1) \leftarrow ... \leftarrow \Omega^i_L(i) \otimes_X \widetilde{\mathcal{D}}(-i) \leftarrow ... \leftarrow \Omega^n_L(n) \otimes_X \widetilde{\mathcal{D}}(-n)$$

5.2.3 The Beilinson Resolution.

The resolution of the diagonal then gives a resolution for every object πM in $QGr(\widetilde{\mathcal{D}})$.

Theorem 5.2.3.1. *Every object $\pi(M) \in QGr(\widetilde{\mathcal{D}})$ has a resolution of the form:*

$$\pi\left(\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_X \mathbb{R}\omega\pi(M)_0\right) \leftarrow ... \leftarrow \pi\left(\widetilde{\mathcal{D}}(-i) \otimes^{\mathbb{L}}_X \left(\Omega^i_R(i) \circledcirc^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M)\right)\right) \leftarrow ...$$

Proof. The resolution of the diagonal gives a complex of Fourier-Mukai transforms. Applying each of these to some $\pi M \in QGr(\widetilde{\mathcal{D}})$, we get

$$F_\Delta(\pi M) \leftarrow F_{\widetilde{\mathcal{D}} \otimes_X \widetilde{\mathcal{D}}}(\pi M) \leftarrow ... \leftarrow F_{\widetilde{\mathcal{D}}(-i) \otimes_X \Omega^i_R(i)}(\pi M) \leftarrow ... F_{\widetilde{\mathcal{D}}(-n) \otimes_X \Omega^n_R(n)}(\pi M)$$

The first object is πM, by the design of Δ. The Fourier-Mukai transform is

$$F_{\widetilde{\mathcal{D}}(-i) \otimes_X \Omega^i_R(i)}(\pi M) = \pi(\mathbb{R}\omega\pi(\widetilde{\mathcal{D}}(-i) \otimes_X \Omega^i_R(i)) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M))_{\bullet,0}$$

By Lemma 5.1.2.1,

$$= \pi\left(\mathbb{R}\omega\pi(\widetilde{\mathcal{D}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \left(\widetilde{\mathcal{D}}(-i) \otimes^{L}_{X} \Omega^i_R(i)\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(\widetilde{\mathcal{D}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M)\right)_{\bullet,0}$$

which simplifies to

$$\pi\left(\mathbb{R}\omega\pi(\widetilde{\mathcal{D}}(-i)) \otimes^{\mathbb{L}}_{X} \left(\Omega^i_R(i) \circledcirc^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M)\right)\right) = \pi\left(\widetilde{\mathcal{D}}(-i) \otimes^{\mathbb{L}}_{X} \left(\Omega^i_R(i) \circledcirc^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M)\right)\right)$$

□

Note that when $\pi M \in qgr(\widetilde{\mathcal{D}})$, the object $\left(\Omega^i_R(i) \circledcirc^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi(M)\right)$ is a derived object in $D^b(Coh(X))$ (it is bounded by the Serre Finiteness Theorem 3.2.4.1). Since X is affine and smooth, every object in $Coh(X)$ has a surjection from a finitely-generated free module $\mathcal{O}^m_X$.

Corollary 5.2.3.1. *Every object $\pi M \in qgr(\widetilde{\mathcal{D}})$ has a surjection from a finite sum of the objects $\pi\widetilde{\mathcal{D}}$, $\pi\widetilde{\mathcal{D}}(-1)$, ... $\pi\widetilde{\mathcal{D}}(-n)$.*

CHAPTER 6

THE BEILINSON EQUIVALENCE.

This section contains the first major result about the category $qgr(\widetilde{\mathcal{D}})$. We show that the derived category $D^b(qgr(\widetilde{\mathcal{D}}))$ is equivalent to the derived category $D^b(E)$ of an algebra E, which is smaller and more tractable than the ring $\mathcal{D}$. This can be used to turn questions about the abstract abelian category $qgr(\widetilde{\mathcal{D}})$ into questions about complexes of E-modules.

6.1 Tilting and the Beilinson Equivalence.

The previous section proved that any $\pi M \in qgr(\widetilde{\mathcal{D}})$ has a finite resolution by finite sums of the objects $\pi\widetilde{\mathcal{D}}$, $\pi\widetilde{\mathcal{D}}(-1)$, ... and $\pi\widetilde{\mathcal{D}}(-n)$. This means that the derived category $D^b(qgr(\widetilde{\mathcal{D}}))$ can be completely understood by studying these $n+1$ objects and the relations between them; specifically, by studying the derived endomorphism algebra of their sum. This typically goes by the name of 'tilting theory'. The end result will be an equivalence of derived categories between $D^b(qgr(\widetilde{\mathcal{D}}))$ and $D^b(E)$, for E a rather simple algebra.

Typically in tilting theory, the derived equivalent algebra E is a quiver algebra; which can be thought of as a finitely-generated algebra over a semi-simple ring $\oplus_{i=0}^{n}\mathbb{C}$ for some n. However, as has been a reoccuring theme in the study of $QGr(\widetilde{\mathcal{D}})$, the role of the ground field is being played by the $\widetilde{\mathcal{D}}$-module $\mathcal{O}_X$. Therefore, as one would expect, the algebra E is finitely-generated over the algebra $\oplus_{i=0}^{N}\mathcal{O}_X$; in fact, it is finitely generated as a module over this subalgebra. Thus, the algebra E is behaving like a 'loop-free quiver algebra over X'.

6.1.1 The Tilting Object T and the Algebra E.

Instead of studying the $n+1$ different objects $\pi\widetilde{\mathcal{D}}(-i)$, it is simpler to study one object which contains them all, in the most straight-forward way. Define the **tilting object** T by

$$T := \bigoplus_{i=0}^{n} \pi\widetilde{\mathcal{D}}(-i)$$

By Corollary 5.2.3.1, if $\pi M \in qgr(\widetilde{\mathcal{D}})$, there is always some surjection $T^{\oplus i} \to \pi M$ for large enough i. This property says that the object T is called a **generator** for the category $qgr(\widetilde{\mathcal{D}})$.

We then turn to study the derived endomorphism algebra of T. The relative Gorenstein property is the key lemma in computing the structure of $\mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(T,T)$.

Theorem 6.1.1.1. *For $i > 0$, $Ext^i_{qgr(\widetilde{\mathcal{D}})}(T,T) = 0$, and*

$$E := Hom_{qgr(\widetilde{\mathcal{D}})}(T,T) = \begin{pmatrix} \mathcal{O}_X & \mathcal{D}_1 & \mathcal{D}_2 & \cdots & \mathcal{D}_n \\ 0 & \mathcal{O}_X & \mathcal{D}_1 & \cdots & \mathcal{D}_{n-1} \\ 0 & 0 & \mathcal{O}_X & \cdots & \mathcal{D}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{O}_X \end{pmatrix}$$

Proof. Replacing $T = \oplus_{i=0}^{n} \pi \widetilde{\mathcal{D}}(-i)$ gives that

$$\begin{aligned}
\mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(T,T) &= \mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(\oplus_{i=0}^{n} \pi\widetilde{\mathcal{D}}(-i), \oplus_{i=0}^{n} \pi\widetilde{\mathcal{D}}(-i)) \\
&= \bigoplus_{0\leq i,j\leq n} \mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(\pi\widetilde{\mathcal{D}}(-i), \pi\widetilde{\mathcal{D}}(-j)) \\
&= \bigoplus_{0\leq i,j\leq n} \mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(\pi\widetilde{\mathcal{D}}, \pi\widetilde{\mathcal{D}}(i-j)) \\
&= \bigoplus_{0\leq i,j\leq n} \mathbb{R}Hom_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}, \mathbb{R}\omega\pi\widetilde{\mathcal{D}}(i-j)) \\
&= \bigoplus_{0\leq i,j\leq n} [\mathbb{R}\omega\pi(\widetilde{\mathcal{D}})]_{j-i}
\end{aligned}$$

The derived object $\mathbb{R}\omega\pi(\widetilde{\mathcal{D}})$ fits into the torsion exact triangle in $D^b(gr(\widetilde{\mathcal{D}}))$

$$\mathbb{R}\tau(\widetilde{\mathcal{D}}) \to \widetilde{\mathcal{D}} \to \mathbb{R}\omega\pi(\widetilde{\mathcal{D}}) \to$$

However, by Lemma 4.2.4.1, the derived torsion $\mathbb{R}\tau(\widetilde{\mathcal{D}})$ vanishes above graded degree $-n-1$. Therefore, $\mathbb{R}\omega\pi(\widetilde{\mathcal{D}})_k \simeq \widetilde{\mathcal{D}}_k = \mathcal{D}_k$ for $k \geq -n$, and so

$$\mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(T,T) = \bigoplus_{0\leq i,j\leq n} \mathcal{D}_{j-i}$$

Therefore, the higher Exts vanish completely, and the endomorphism algebra of T is given by the above algebra. □

6.1.2 The Tilting Functor.

Given any $\pi M \in qgr(\widetilde{\mathcal{D}})$, $\mathbb{R}Hom_{qgr}(T, \pi M)$ has a right action by $Hom_{qgr}(T,T)$ by composition, and so it is a left E module. In this way, the functor $\mathbb{R}Hom_{qgr}(T,-)$ defines a functor from $D^b(qgr(\widetilde{\mathcal{D}}))$ to $D^b(mod(E))$.

This functor can be expressed in terms of the functor $\mathbb{R}\omega\pi$. After all, as derived right $\mathcal{O}_X$-modules,

$$\begin{aligned}
\mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(T, \pi M) &= \mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(\oplus_{i=0}^{n} \pi\widetilde{\mathcal{D}}(-i), \pi M) \\
&= \bigoplus_{0=i}^{n} \mathbb{R}Hom_{qgr(\widetilde{\mathcal{D}})}(\pi\widetilde{\mathcal{D}}(-i), \pi M) \\
&= \bigoplus_{0=i}^{n} \mathbb{R}Hom_{gr(\widetilde{\mathcal{D}})}(\widetilde{\mathcal{D}}, \mathbb{R}\omega\pi M(i)) \\
&= \bigoplus_{0=i}^{n} [\mathbb{R}\omega\pi(M)]_{-i}
\end{aligned}$$

The extra structure needed to make $\bigoplus_{i=0}^{n}[\mathbb{R}\omega\pi(M)]_{-i}$ into a derived left E-module is the collection of action maps

$$\widetilde{\mathcal{D}}_{j-i} \otimes_X [\mathbb{R}\omega\pi(M)]_{-j} \to [\mathbb{R}\omega\pi(M)]_{-i}$$

which come from $\mathbb{R}\omega\pi(M)$'s left $\widetilde{\mathcal{D}}$-module structure.

6.1.3 The Equivalence Theorem.

Either way one writes it, it defines an equivalence of derived categories.

Theorem 6.1.3.1. *(The Beilinson Equivalence) The functor* $\mathbb{R}Hom_{qgr}(T, -) = \bigoplus_{0=i}^{n}[\mathbb{R}\omega\pi(-)]_{-i}$ *defines an equivalence of triangulated categories (in fact, of dg categories)*

$$D^b(qgr(\widetilde{\mathcal{D}})) \simeq D^b(mod(E))$$

with inverse given by $T \otimes_E^{\mathbf{L}} -$.

Proof. The theorem will follow from the following lemma.

Lemma 6.1.3.1. *Let $\mathcal{A}$ be an abelian category, and let T be an object in $\mathcal{A}$ which is:*

- ***Compact***: *The functor $Hom_{\mathcal{A}}(T,-)$ commutes with direct sums.*

- ***Generator***: *For any object $M \in \mathcal{A}$, there is a surjection $T^{\oplus I} \to \mathcal{A}$ for some index set I.*

- ***Finite Dimension***: *There is some i such that $Ext^j_{\mathcal{A}}(T,M) = 0$ for all $j > i$ and $M \in \mathcal{A}$.*

- *$Ext^i_{\mathcal{A}}(T,T) = 0$ for $i > 0$.*

Then $\mathbb{R}Hom_{\mathcal{A}}(T,-)$ defines a quasi-equivalence of triangulated categories (and in fact, an equivalence of dg categories)

$$D^b(\mathcal{A}) \simeq D^b(mod(End(T)^{op}))$$

with inverse $T \otimes^{\mathbf{L}}_{End(T)^{op}} -$.

Proof. Theorem 4.3 in [22] (see also Theorem 8.5 in [23]) provides a a quasi-equivalence of dg categories $D^b(\mathcal{A}) \simeq Perf(Mod(End(T)^{op}))$, where $Perf(Mod(E))$ is the category of perfect complexes. However, by the finite dimensionality, the image of the functor takes bounded complexes to bounded complexes. Therefore, $Perf(Mod(End(T)^{op})) \simeq D^b(mod(End(T)^{op}))$. □

The compactness of T is immediate, because π is a compact functor and T is π of a f.g. object. The fact that T is a generator was Corollary 5.2.3.1. The Serre Finiteness Theorem (Theorem 3.2.4.1) proves that $\mathbb{R}\omega\pi$ has finite homological dimension, and so then $\mathbb{R}Hom_{qgr}(T,-)$ does as well. Finally, the vanishing of higher Exts was Theorem 6.1.1.1. □

One interpretation of this theorem is that an object $\pi M \in qgr(\widetilde{\mathcal{D}})$ can be completely determined by knowing $\mathbb{R}\omega\pi(M)$ in degrees $-n$ to 0, together with knowing the action maps

$$\widetilde{\mathcal{D}}_{j-i} \otimes_X [\mathbb{R}\omega\pi(M)]_{-j} \to [\mathbb{R}\omega\pi(M)]_{-i}$$

In fact, any object in $D^b(qgr(\widetilde{\mathcal{D}}))$ can be constructed by giving $n+1$ objects $N_{-i} \in D^b(\mathcal{O}_X)$, together with action maps $\widetilde{\mathcal{D}}_{j-i} \otimes_X N_{-j} \to N_{-i}$ which are required to be associative in the natural way.

6.2 Examples.

The generality of Lie algebroids means that this theorem encompasses a wide array of different examples. We review some of these examples now.

6.2.1 Polynomial Algebras.

This is the case $X = Spec(\mathbb{C})$, and L abelian. The bundle L is then a vector space with trivial Lie bracket. If $\{x_1, x_2, ...x_n\}$ is a basis for L, $\mathcal{D}$ is $\mathbb{C}[x_1, x_2, ..., x_n]$ and $\widetilde{\mathcal{D}} = \mathbb{C}[t, x_1, x_2, ...x_n]$. Therefore, $qgr(\widetilde{\mathcal{D}}) = Coh(\mathbb{P}^n)$, by the projective Serre equivalence. Then the main theorem becomes the derived equivalence of $\mathbb{P}^n$ and the algebra

$$\begin{pmatrix} \mathbb{C} & \mathbb{C}\oplus L & \mathbb{C}\oplus L\oplus Sym^2 L & \cdots & \mathbb{C}\oplus L\oplus ...\oplus Sym^n L \\ 0 & \mathbb{C} & \mathbb{C}\oplus L & \cdots & \mathbb{C}\oplus L\oplus ...\oplus Sym^{n-1} L \\ 0 & 0 & \mathbb{C} & \cdots & \mathbb{C}\oplus L\oplus ...\oplus Sym^{n-2} L \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbb{C} \end{pmatrix}$$

This algebra is usually written as the path algebra of a quiver Q_n, called the n**th Beilinson quiver**. The equivalence $D^b(mod(\mathbb{P}^n)) \simeq D^b(mod(Q_n))$ is the original

Beilinson equivalence [4].

6.2.2 Lie Algebras.

This is the case $X = Spec(\mathbb{C})$, and $L = \mathfrak{g}$, some Lie algebra. The enveloping algebra is then the usual enveloping algebra $U\mathfrak{g}$ of the Lie algebra, and $\widetilde{\mathcal{U}\mathfrak{g}}$ is the homogenization. The categories $qgr(\widetilde{\mathcal{U}\mathfrak{g}})$ were first introduced by [25] under the name **quantum space of a Lie algebra**. The main theorem becomes the derived equivalence of this category and the algebra

$$\begin{pmatrix} \mathbb{C} & (\mathcal{U}\mathfrak{g})_1 & (\mathcal{U}\mathfrak{g})_2 & \cdots & (\mathcal{U}\mathfrak{g})_n \\ 0 & \mathbb{C} & (\mathcal{U}\mathfrak{g})_1 & \cdots & (\mathcal{U}\mathfrak{g})_{n-1} \\ 0 & 0 & \mathbb{C} & \cdots & (\mathcal{U}\mathfrak{g})_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbb{C} \end{pmatrix}$$

This algebra again can be written as the path algebra of a quiver, which will look like the nth Beilinson quiver with its relations deformed by the Lie bracket.

6.2.3 Example: Differential Operators.

In this case, X is any irreducible smooth affine variety, and L is the tangent bundle $\mathcal{T}$. Then, $\mathcal{D}$ is the ring of differential operators. The category $qgr(\widetilde{\mathcal{D}})$ is then derived equivalent to the algebra

$$\begin{pmatrix} \mathcal{O}_X & \mathcal{D}_1 & \mathcal{D}_2 & \cdots & \mathcal{D}_d \\ 0 & \mathcal{O}_X & \mathcal{D}_1 & \cdots & \mathcal{D}_{d-1} \\ 0 & 0 & \mathcal{O}_X & \cdots & \mathcal{D}_{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{O}_X \end{pmatrix}$$

Not much else can be said in this level of generality. However, for a powerful application of this in the form of curves, see Section 8.1.

6.2.4 Non-Examples.

It is worth noting that $\widetilde{\mathcal{D}}$ is not the most general class of graded algebra for which the techniques here work, and for which a similar version of the main theorem applies. For example, let $PP_\hbar$ denote the algebra over $\mathbb{C}$ generated by w_1, w_2, and w_3, subject to the relations

$$[w_1, w_3] = [w_2, w_3] = 0, \quad [w_1, w_2] = 2\hbar w_3^2$$

One can check that this is not the homogenization of any universal enveloping algebra of a Lie algebra.

However, in [21], a similar Koszul theory is developed, as well as a similar Beilinson equivalence, which is then used for a monad-theoretic construction of the moduli space of certain kinds of modules.

Another non-example of a relatively quadratic algebra which has an identical Koszul theory and Beilinson transform is the $\widetilde{\mathcal{D}}^{op}$, the opposite algebra of the enveloping algebra of a Lie algebroid. This is equivalent to showing that the category of graded right $\widetilde{\mathcal{D}}$-modules has a quotient $qgr(\widetilde{\mathcal{D}})^{op}$ which satisfies all the theorems of this paper. Every proof in this paper works in this case, occasionally with slight modification (actually, the proof of the relative Gorenstein property is a little bit shorter).

So then, **what is the most general setting where the above proof of the Beilinson equivalence holds?** The answer is that the proofs in this paper will work for any relatively quadratic algebra A, such that

- A is **Koszul**, in that the left and right Koszul complexes are resolutions of $\mathcal{O}_X$.
- $A^!$ is a finitely generated projective left and right $\mathcal{O}_X$-module and relatively Frobenius over $\mathcal{O}_X$. That is, Corollaries 4.1.2.1, 4.1.2.2 and 4.1.3.1 hold.

CHAPTER 7

DUALITY

This chapter uses the homological consequences of the Koszul theory to prove a number of dualizing results about the category $QGr(\widetilde{\mathcal{D}})$. This section in particular should feel strongly analogous to the case of a commutative algebra which is graded local; ie, connected. For the commutative analog of these results, see Bruns and Herzog [12].

7.1 Local Duality.

In this section, we regard the torsion functor τ as analogous to the local cohomology of a connected, commutative graded algebra. The main result is a non-commutative, relative analog of Local Duality theorem of Grothendieck [12].

7.1.1 The Graded Dualizing Object $\mathbb{J}$.

We start by introducing the object which will be doing the dualizing.

Lemma 7.1.1.1. *As graded $\mathcal{O}_X$-bimodules, there is a canonical isomorphism*

$$\underline{Hom}_{X-}(\widetilde{\mathcal{D}}, \omega) = \underline{Hom}_{-X}(\widetilde{\mathcal{D}}, \omega)$$

Proof. For any i, define $\mathcal{O}_X$-bimodule maps $\omega \otimes \mathcal{O}_X \to Hom_{X-}(\mathcal{D}_i, \omega)$ and $\omega \otimes \mathcal{O}_X \to Hom_{-X}(\mathcal{D}_i, \omega)$ by $\mu \otimes f \to (\delta \to \delta(f)\mu)$ and $\mu \otimes f \to (\delta \to f\delta(\mu))$. These maps are surjective, and have isomorphic kernel. The theorem follows. $\square$

Therefore, let $\mathbb{J} := \underline{Hom}_{X-}(\widetilde{\mathcal{D}}, \omega) = \underline{Hom}_{-X}(\widetilde{\mathcal{D}}, \omega)$. Each of these has an obvious structure of a left or right $\widetilde{\mathcal{D}}$-module, which together make $\mathbb{J}$ into a graded $\widetilde{\mathcal{D}}$-bimodule.

The importance of $\mathbb{J}$ is the functor $\underline{Hom}_{\widetilde{\mathcal{D}}-}(-, \mathbb{J})$ from left $\widetilde{\mathcal{D}}$-modules to right $\widetilde{\mathcal{D}}$-modules, and the analogous $\underline{Hom}$ as a map of right modules. As the following lemma shows, this functor is equivalent to $\underline{Hom}_{X-}(-, \omega)$, which is then equipped with a right $\widetilde{\mathcal{D}}$-module structure.

Lemma 7.1.1.2. *Let M be a left $\widetilde{\mathcal{D}}$-module, and N a right $\widetilde{\mathcal{D}}$-module. Then*

$$\underline{Hom}_{\widetilde{\mathcal{D}}-}(M, \mathbb{J}) \simeq \underline{Hom}_{X-}(M, \omega)$$

as right $\mathcal{O}_X$-modules, and

$$\underline{Hom}_{-\widetilde{\mathcal{D}}}(N, \mathbb{J}) \simeq \underline{Hom}_{-X}(N, \omega)$$

as left $\mathcal{O}_X$-modules.

Proof. The lemma follows from the $(Hom, \otimes)$ adjunction.

$$Hom_{\widetilde{\mathcal{D}}-}(M, \underline{Hom}_{\widetilde{\mathcal{D}}-}(\widetilde{\mathcal{D}}, \omega)) \simeq Hom_{X-}(\widetilde{\mathcal{D}} \otimes_{\widetilde{\mathcal{D}}} M, \omega) = Hom_{X-}(M, \omega)$$

Summing over all twists gives the graded isomorphism. The proof for N is similar. □

The next lemma shows that the derived endomorphism algebra of $\mathbb{J}$ is just $\widetilde{\mathcal{D}}$.

Lemma 7.1.1.3. *As algebras and as $\widetilde{\mathcal{D}}$-bimodules,*

$$\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{J}, \mathbb{J}) = \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{J}, \mathbb{J}) = \widetilde{\mathcal{D}}$$

Proof. As $\widetilde{\mathcal{D}}$-bimodules, we have

$$\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{J},\mathbb{J}) = \mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\underline{Hom}_{-X}(\widetilde{\mathcal{D}},\omega), \underline{Hom}_{X-}(\widetilde{\mathcal{D}},\omega))$$

$$= \mathbb{R}\underline{Hom}_{X-}(\underline{Hom}_{-X}(\widetilde{\mathcal{D}},\omega),\omega) = \widetilde{\mathcal{D}}$$

The last equality follows from the projectivity of $\widetilde{\mathcal{D}}$ as a right $\mathcal{O}_X$-module which is f.g. in each graded degree. □

7.1.2 Matlis Duality.

We now establish the Matlis duality theorem, which says that the functors $\mathbb{R}Hom_{\widetilde{\mathcal{D}}-}(-,\mathbb{J})$ and $\mathbb{R}_{-\widetilde{\mathcal{D}}}(-,\mathbb{J})$ are mutual inverses on sufficiently nice modules.

Theorem 7.1.2.1. *(Matlis duality) Let N be a f.g. left $\widetilde{\mathcal{D}}$-module. Then*

$$\mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N,\mathbb{J}),\mathbb{J}) \simeq N$$

Proof.

$$\mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N,\mathbb{J}),\mathbb{J}) = \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N,\widetilde{\mathcal{D}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{J},\mathbb{J})$$

$$= \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N,\widetilde{\mathcal{D}}), \mathbb{R}\underline{Hom}(\mathbb{J},\mathbb{J}))$$

$$= \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N,\widetilde{\mathcal{D}}),\widetilde{\mathcal{D}}) = N$$

□

7.1.3 $\mathbb{R}\tau(\widetilde{\mathcal{D}})$ and $\mathbb{R}\omega\pi\widetilde{\mathcal{D}}$.

Next, to relate Matlis duality to cohomology computations, we need to relate $\mathbb{J}$ to cohomology.

Lemma 7.1.3.1. *There is an isomorphism of $\widetilde{\mathcal{D}}$-bimodules*

$$\mathbb{R}\tau(\widetilde{\mathcal{D}}) = \mathbb{J}[-d-1](d+1)$$

Proof. First, we show they are isomorphic as graded $\mathcal{O}_X$-bimodules. The Gorenstein Lemma says that $\underline{Ext}^i_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \widetilde{\mathcal{D}})$ vanishes outside graded degree $-d-1$ and cohomological degree $d+1$. Therefore, Lemma 3.2.3.1 and Corollary 3.2.3.1 imply that $\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j}, \widetilde{\mathcal{D}})$ is concentrated in graded degrees between $-d-j-1$ and $-d-1$. Consider the short exact sequence of left $\widetilde{\mathcal{D}}$-modules [1]

$$0 \to \mathcal{D}_j(-j) \to \widetilde{\mathcal{D}}_{\leq j} \to \widetilde{\mathcal{D}}_{\leq j-1} \to 0$$

Applying $\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}_-}(-, \widetilde{\mathcal{D}})$ and taking the long exact sequence, the above vanishing conditions imply that there is a short exact sequence

$$0 \to \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j-1}, \widetilde{\mathcal{D}}) \to \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j}, \widetilde{\mathcal{D}}) \to \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\mathcal{D}_j, \widetilde{\mathcal{D}}(j)) \to 0$$

However, because $\mathcal{D}_j$ is a f.g. projective left $\mathcal{O}_X$-module, we have that

$$\begin{aligned}
\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\mathcal{D}_j, \widetilde{\mathcal{D}}(j)) &= \underline{Hom}_{X-}(\mathcal{D}_j, \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\mathcal{O}_X, \widetilde{\mathcal{D}}(j))) \\
&= \underline{Hom}_{X-}(\mathcal{D}_j, \omega_L(j+d+1)) \\
&= \mathbb{J}_{-j}(j+d+1)
\end{aligned}$$

As a consequence, $\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\mathcal{D}_j, \widetilde{\mathcal{D}}(j))$ is concentrated in graded degree $-j-d-1$. Therefore, the map

$$\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j-1}, \widetilde{\mathcal{D}})_k \to \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j}, \widetilde{\mathcal{D}})_k$$

is an isomorphism of $\mathcal{O}_X$-bimodules when $k \neq -d-j-1$. Since $\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j-1}, \widetilde{\mathcal{D}})$ is concentrated between graded degree $-d-j$ and $-d-1$, the map

$$\underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\widetilde{\mathcal{D}}_{\leq j}, \widetilde{\mathcal{D}})_{-j-d-1} \to \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}_-}(\mathcal{D}_j, \widetilde{\mathcal{D}}(j))_{-j-d-1} = \mathbb{J}_{-j}$$

[1] Here $\mathcal{D}_j$ is a graded $\widetilde{\mathcal{D}}$-module concentrated in degree zero on which $\widetilde{\mathcal{D}}_{\geq 1}$ acts trivially; that is, it is the module induced from the left $\mathcal{O}_X$-module $\mathcal{D}_j$.

is an isomorphism of $\mathcal{O}_X$-bimodules.

Therefore, the limit

$$\varinjlim \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}-}(\widetilde{\mathcal{D}}_{\leq j}, \widetilde{\mathcal{D}})_k$$

is an isomorphism in all degrees except when $j = k + d + 1$. Therefore,

$$\mathbb{R}^{d+1}\tau(\widetilde{\mathcal{D}})_k = \underline{Ext}^{d+1}_{\widetilde{\mathcal{D}}-}(\mathcal{D}_{k+d+1}, \widetilde{\mathcal{D}}(j))_k = \mathbb{J}_{k+d+1}$$

Putting this together in each graded degree, we have an isomorphism $\mathbb{R}^{d+1}\tau(\widetilde{\mathcal{D}}) = \mathbb{J}(d+1)$. Since $\mathbb{R}^i\tau(\widetilde{\mathcal{D}})$ vanishes in all other degrees, this implies that $\mathbb{R}\tau(\widetilde{\mathcal{D}}) = \mathbb{J}[-d-1](d+1)$. □

As a corollary, we deduce the structure of the derived global sections of $\pi\widetilde{\mathcal{D}}$, which is playing the role of the structure sheaf.

Corollary 7.1.3.1. *There is an exact triangle in $D^b(Gr(\widetilde{\mathcal{D}}))$*

$$\widetilde{\mathcal{D}} \to \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{J}[-d](d+1) \to \widetilde{\mathcal{D}}[1]$$

7.1.4 Local Duality.

We recall Watt's Theorem, from homological algebra.

Lemma 7.1.4.1. *(Watt's Theorem [33]) Let R and S be rings, and let*

$$F : Mod(R) \to Mod(S)$$

be a contravariant functor which is additive, left exact and preserves direct sums. Then $F(R)$ is naturally a $R - S^{op}$-bimodule, and we have a natural equivalence of functors

$$F(-) \simeq Hom_R(-, F(R))$$

We can then prove local duality. This proof was heavily influenced by a similar proof of Yekutieli and Zhang [37].

Theorem 7.1.4.1. *(Local Duality) Let N be a f.g. left $\widetilde{\mathcal{D}}$-module. Then*

$$\mathbb{R}\tau(N) \simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(N, \widetilde{\mathcal{D}}), \mathbb{J})[-d-1](d+1)$$

Proof. First, note that $\mathbb{R}\tau$ has homological dimension $d+1$. Since the left Koszul complex is a projective resolution of $\mathcal{O}_X$ by projective $\widetilde{\mathcal{D}}$-modules of length $d+1$, the *Ext* groups $\underline{Ext}^i_{\widetilde{\mathcal{D}}}(\mathcal{O}_X, N) = 0$ for $i > d+1$. Thus, by Lemma 3.2.3.1, $\mathbb{R}^i\tau(N) = 0$ for $i > d+1$.

Therefore, $\mathbb{R}\tau(N)$ is concentrated between degrees 0 and $d+1$. Therefore,

$$\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{R}\tau(N), \mathbb{J})[-d-1](d+1)$$

is zero in negative cohomological degrees. Therefore,

$$\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{R}^{d+1}\tau(-), \mathbb{J})(d+1)$$

is a left exact functor. Since it also commutes with direct sums, by Watts' Theorem, its representable by the functor

$$\underline{Hom}_{\widetilde{\mathcal{D}}-}(-, \underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}^{d+1}\tau(\widetilde{\mathcal{D}}), \mathbb{J})(d+1)) = \underline{Hom}_{\widetilde{\mathcal{D}}-}(-, \underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{J}, \mathbb{J}))$$

Since $\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{J}, \mathbb{J}) = \widetilde{\mathcal{D}}$, this is just the usual dual.

The higher derived functors of $\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{R}^{d+1}\tau(-), \mathbb{J})(d+1)$ vanish on $\widetilde{\mathcal{D}}(i)$ for all i (by the previous lemma and the fact that $\widetilde{\mathcal{D}}$ is a graded projective $\mathcal{O}_X$-module). Therefore, by the universality of derived functors, we have an equivalence of functors

$$\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(-, \widetilde{\mathcal{D}}) \simeq \mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{R}\tau(-), \mathbb{J})[-d-1](d+1)$$

Applying Matlis duality to both sides, we get the theorem. □

7.2 Serre Duality.

Using the tools of the previous section, we prove the appropriate form of Serre duality for the category $qgr(\widetilde{\mathcal{D}})$.

7.2.1 Serre Duality for $\pi\widetilde{\mathcal{D}}$.

The first step is to prove Serre duality for the structure sheaf $\pi\widetilde{\mathcal{D}}$. Recall that we have a quasi-isomorphism (Corollary 3.2.2.1)

$$\mathbb{R}\omega\pi\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{R}\omega\pi\widetilde{\mathcal{D}}$$

Composing this with the map $\mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{J}[-d](d+1)$, we get a map

$$\mathbb{R}\omega\pi\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{J}[-d](d+1)$$

Lemma 7.2.1.1. *The map $\mathbb{R}\omega\pi\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{J}[-d](d+1)$ is a derived perfect pairing. That is, the natural adjoint map*

$$\mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\omega\pi\widetilde{\mathcal{D}}, \mathbb{J}[-d](d+1))$$

is a quasi-isomorphism in $D^b(Gr(\widetilde{\mathcal{D}}))$.

Proof. For this proof, let F denote $\mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(-, \mathbb{J})[-d](d+1)$. By Corollary 7.1.3.1, we have an exact triangle

$$\widetilde{\mathcal{D}} \to \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \to \mathbb{J}[-d](d+1) \to$$

Applying F to this, we get another exact triangle

$$F(\mathbb{J}[-d](d+1)) \to F(\mathbb{R}\omega\pi\widetilde{\mathcal{D}}) \to F(\widetilde{\mathcal{D}}) \to$$

By Lemma 7.1.1.3, we know that the action map $\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{J} \to \mathbb{J}$ is a perfect pairing; that is, that the map

$$\widetilde{\mathcal{D}} \to \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{J}, \mathbb{J}) =: F(\mathbb{J}[-d](d+1))$$

is a quasi-isomorphism. This isomorphism fits into a commutative square

$$\begin{array}{ccc} \widetilde{\mathcal{D}} & \to & \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \\ \downarrow & & \downarrow \\ F(\mathbb{J}[-d](d+1)) & \to & F(\mathbb{R}\omega\pi\widetilde{\mathcal{D}}) \end{array}$$

Similarly, the multiplication map $\mathbb{J} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{D}} \to \mathbb{J}$ gives an quasi-isomorphism

$$\mathbb{J}[-d](d+1) \to \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}, \mathbb{J}[-d](d+1)) =: F(\widetilde{\mathcal{D}})$$

This fits into a commutative diagram (in fact, a map of exact triangles)

$$\begin{array}{ccccccc} \widetilde{\mathcal{D}} & \to & \mathbb{R}\omega\pi\widetilde{\mathcal{D}} & \to & \mathbb{J}[-d](d+1) & \to \\ \downarrow & & \downarrow & & \downarrow & \\ F(\mathbb{J}[-d](d+1)) & \to & F(\mathbb{R}\omega\pi\widetilde{\mathcal{D}}) & \to & F(\widetilde{\mathcal{D}}) & \to \end{array}$$

The first and the third maps are quasi-isomorphisms, so by the Five-Lemma for triangulated categories (see, for instance, [13]), the middle map is also a quasi-isomorphism. □

7.2.2 Serre Duality.

From this, we deduce the first form of Serre Duality.

Theorem 7.2.2.1. *(Serre Duality, Version 1) Let $M \in gr(\widetilde{\mathcal{D}})$, and let ${}^*M = \mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(M, \widetilde{\mathcal{D}})$. Then*

$$\mathbb{R}\omega\pi M \simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\omega\pi({}^*M), \mathbb{J})[-d](d+1)$$

Proof. The theorem follows from a string of known identities.

$$\begin{aligned}
\mathbb{R}\omega\pi M &\simeq \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} M \\
&\simeq \mathbb{R}\omega\pi\widetilde{\mathcal{D}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}({}^*M, \widetilde{\mathcal{D}}) \\
&\simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}({}^*M, \mathbb{R}\omega\pi\widetilde{\mathcal{D}}) \\
&\simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}({}^*M, \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\omega\pi\widetilde{\mathcal{D}}, \mathbb{J}[-d](d+1))) \\
&\simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}({}^*M \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}} \mathbb{R}\omega\pi\widetilde{\mathcal{D}}, \mathbb{J}[-d](d+1)) \\
&\simeq \mathbb{R}\underline{Hom}_{-\widetilde{\mathcal{D}}}(\mathbb{R}\omega\pi({}^*M), \mathbb{J}[-d](d+1))
\end{aligned}$$

□

This can be rewritten in a form more familiar to the commutative case. For this, let

$${}^*\mathcal{M} := \pi(\mathbb{R}\underline{Hom}_{\widetilde{\mathcal{D}}-}(\mathbb{R}\omega\mathcal{M}, \widetilde{\mathcal{D}}))$$

Note that this is an object in $D^b(QGr(\widetilde{\mathcal{D}}^{op}))$, the derived category of the quotient category of right $\widetilde{\mathcal{D}}$-modules.

Theorem 7.2.2.2. *(Serre Duality, Version 2) Let $\mathcal{M} \in qgr(\widetilde{\mathcal{D}})$. Then* [2]

$$\mathbb{R}\omega\mathcal{M} \simeq \mathbb{R}\underline{Hom}_{-X}(\mathbb{R}\omega({}^*\mathcal{M}), \omega_X)[-d](d+1)$$

Proof. This is a straightforward rewriting of the previous theorem, using Lemma 7.1.1.2. □

While in general, the left-hand side of this identity contains three derived functors, in many cases several of these vanish. If M is projective (this is the analog of $\mathcal{M}$ being locally free), then the dual ${}^*M$ has no higher derived functors. If X is

[2]Here, ω_X denotes the canonical bundle of X, while ω denotes the section functor of the quotient categories. Apologies for the confusing notation.

a point, then $\mathbb{J}$ is injective, and the outer $\mathbb{R}\underline{Hom}$ has no higher derived functors. Therefore, in either of these cases, the above quasi-isomorphism gives a spectral sequence; while in the intersection of these cases, it gives an outright isomorphism on cohomology.

CHAPTER 8

APPLICATIONS.

8.1 Ideals.

We review some of the applications of this theory to the study of right ideals in rings of differential operators.

8.1.1 The Affine Line.

Let $X = \mathbb{A}^1$, the affine line, so that $\mathcal{O}_X = \mathbb{C}[x]$. In this case, $\mathcal{D}$ is the first Weyl algebra, generated by x and ∂. For a right ideal I in $\mathcal{D}$, the inherited filtration from $\mathcal{D}$ is *almost* an invariant of an ideal class. Two equivalent ideals will have filtrations which differ by a shift. This shift can be fixed with the following observation.

Lemma 8.1.1.1. *[14] Every right ideal I in $\mathcal{D}$ is equivalent to an ideal J such that $J_0 \neq 0$ and $J_{-1} = 0$.*

An ideal I such that $I_0 \neq 0$ is called **fat**. The fat ideals will be the representatives in an ideal class of 'minimum shift', so we can fix the shift in the filtration by requiring that a representative be fat.

Let I be a fat ideal. The Beilinson equivalence says that to understand I, it suffices to understand $\mathbb{R}\omega\pi\widetilde{I}_0$ and $\mathbb{R}\omega\pi\widetilde{I}_{-1}$, together with an action of $\mathcal{D}_1$ between them. However, we know that $\omega\pi\widetilde{I} = \widetilde{I}$ and

$$0 = \mathbb{R}^{\geq 2}\omega\pi\widetilde{I}_0 = \mathbb{R}^{\geq 2}\omega\pi\widetilde{I}_{-1}$$

so the only cohomology groups in question are $\mathbb{R}^1\omega\pi\widetilde{I}_0$ and $\mathbb{R}^1\omega\pi\widetilde{I}_{-1}$. Let

$$V := \mathbb{R}^1\omega\pi\widetilde{I}_{-1}$$

Then we have

Lemma 8.1.1.2. *[5, Theorem 4.6.] The $\mathbb{C}[x]$-module V is a finite-dimensional vector space. Furthermore, the Infinity long exact sequence (Section 3.3.1) becomes*

$$0 \to I_0 \to \mathbb{C}[x] \to V \to \mathbb{R}^1\omega\pi\widetilde{I}_0 \to 0$$

Denote by i the map $\mathbb{C}[x] \to V$ occurring in the lemma. As a consequence, we have that $\mathbb{R}\omega\pi\widetilde{I}_0$ is equivalent to the complex given by $\mathbb{C}[x] \to V$, with $\mathbb{C}[x]$ in degree zero; and that $\mathbb{R}\omega\pi\widetilde{I}_{-1}$ is equivalent to $V[-1]$.

The final data to describe I is the action map

$$\mathcal{D}_1 \otimes^{\mathbb{L}}_X \left\{ \begin{array}{c} V \\ \uparrow \\ 0 \end{array} \right\} \to \left\{ \begin{array}{c} V \\ \uparrow \\ \mathbb{C}[x] \end{array} \right\}$$

We may compute this as follows. Let n denote the dimension of V as a vector space, and choose a basis e_i for V. The action of $\mathbb{C}[x]$ on V is determined by the action of x, which may be expressed as a matrix X in the chosen basis. This gives a free resolution of V as

$$V \leftarrow \mathbb{C}[x]^n \leftarrow_{x-X} \mathbb{C}[x]^n \leftarrow 0$$

Choosing a representative of the action map, we get a commutative square in the homotopy category:

$$\begin{array}{ccc} \mathcal{D}_1 \otimes_X \mathbb{C}[x]^n & \to_{a_1} & V = \mathbb{C}^n \\ \uparrow_{x-X} & & \uparrow_i \\ \mathcal{D}_1 \otimes_X \mathbb{C}[x]^n & \to_{a_0} & \mathbb{C}[x] \end{array}$$

Let $h_\partial : \mathbb{C}^n \to \mathbb{C}[x]$ be the unique linear map such that, for all $v \in \mathbb{C}^n$,

$$a_0(\partial \otimes v) + xh_\partial(v) - h_\partial(Xv) \in \mathbb{C} \subset \mathbb{C}[x]$$

This is possible by starting with the highest degree term in $a_0(\partial \otimes \mathbb{C}^n)$ and proceeding by downward induction. Let $h : \mathcal{D}_1 \otimes_X \mathbb{C}[x]^n \to \mathbb{C}[x]$ be the $\mathcal{O}_X$-module map defined by $h(f \otimes g) = 0$ and $h(\partial \otimes v) = h_\partial(v)$. Then the chain homotopy of the above diagram defined by h sends a_0 to a_0' such that $a_0'(\partial \otimes v) \in \mathbb{C}$. We apply this homotopy, and by abuse of notation denote the resulting maps by a_0 and a_1.

These maps restricted to $\mathcal{D}_0 \subseteq \mathcal{D}_1$ must be the natural maps coming from the previous resolution of V. Therefore, we need only determine the maps a_0 and a_1 on elements of the form $\partial \otimes e_i$. We let $j : \mathbb{C}^n \to \mathbb{C}$ be defined by

$$j(v) := a_0(\partial \otimes v)$$

and $Y \in Mat_{n,n}(\mathbb{C})$ be defined by

$$Y(v) := a_1(\partial \otimes v)$$

The commutativity of the above diagram implies that we have the matrix identity

$$Id + XY - YX = ij$$

This is the **Calogero-Moser equation**, and a pair of matrices (X, Y) satisfying it are called **Calogero-Moser matrices** (i and j are usually surpressed). A different choice of a basis for the space V will conjugate the matrices X and Y. Define the n**th Calogero-Moser space** CM_n to be the algebraic quotient of space of $n \times n$ Calogero-Moser matrices by the conjugation action of PGL_n.

Then by the Beilinson equivalence, we have a natural injection from the set of ideal classes in $\mathcal{D}$ to the union over all the Calogero-Moser spaces.

Theorem 8.1.1.1. *[9, Theorem 1.1] The map constructed above, from right ideal classes in $\mathcal{D}$ to $\prod_{n \in \mathbb{N}} CM_n$, is a bijection.*

Furthermore, if $G = Aut(\mathcal{D})$, then there are natural G actions on $\mathcal{D}$ and on each CM_n; it can be shown that the bijection is G-equivariant, and that it is transitive on each of the CM_n [9, Theorem 1.3.].

It is worth mentioning that this parametrization of ideals in the Weyl algebra was first discovered by very different means. Cannings and Holland [15] first classified ideal classes by considering their images when acting on $\mathcal{O}_X$ in terms of an 'adelic Grassmannian' (though they did not call it such), and Berest and Wilson [9] first characterized this parameterization in terms of Calogero-Moser matrices.

The connection with projective geometry was introduced by Lebruyn in [24], and developed by Berest and Wilson in [10]; though in that case it was with the filtration on $\mathcal{D}$ with x and ∂ both having order 1 (the Bernstein filtration). The advantage of this filtration over the present case is that the cohomology groups considered are automatically finite-dimensional vector spaces, making the appearance of matrices more natural.

8.1.2 Smooth Affine Curves.

To generalize the above story to general smooth affine curves, it is necessary to develop techniques that generalize appropriately. The Bernstein filtration has no analog in $\mathcal{D}$ for an arbitrary curve, and so the differential filtration we have been considering throughout is more naturally suited to this case.

The discussion of the previous section still works; every ideal is equivalent to a fat one and $\mathbb{R}^1\omega\pi\widetilde{I}_{-1}$ is always finite dimensional. This leads to the following classification of right ideals in $\mathcal{D}$.

Theorem 8.1.2.1. *[5, Theorem 4.3.] Let I be an ideal in $\mathcal{D}$ for X a smooth affine curve. Then*

1. $(\mathbb{R}\omega\pi(\widetilde{I}))_{-1} = V[-1]$, *where V is a finite-length sheaf on X.*
2. $(\mathbb{R}\omega\pi(\widetilde{I}))_0 = Cone(i : J \to V)$, *where J is some ideal on X and i is some $\mathcal{O}_X$-module map.*
3. *The action map* $a : \mathcal{D}^1 \otimes_X (\mathbb{R}\omega\pi(\widetilde{I}))_{-1} \to (\mathbb{R}\omega\pi(\widetilde{I}))_0$ *restricts on $\mathcal{O}_X$ to the natural map*

$$\left\{\begin{matrix} V \\ \uparrow \\ 0 \end{matrix}\right\} \begin{matrix} \to_{Id_V} \\ \\ \to_0 \end{matrix} \left\{\begin{matrix} V \\ \uparrow \\ J \end{matrix}\right\}$$

Furthermore, any choice of such V, J, i and a will determine a derived E-module which corresponds to an ideal under the inverse Beilinson equivalence.

It is worth mentioning that a simultaneous characterization of these ideal classes was obtained by Berest and Chalykh [7] using a different technique of *deformed preprojective algebras*. Deformed preprojective algebras have frequently come up in this theory, and offer interesting generalizations in the direction of replacing $\mathcal{O}_X$ with a quiver algebra. However, as such directions are perpendicular to our discussion, we instead direct the interested reader to [7].

8.1.3 Projective Ideals.

One possible direction in which to take the previous story is to investigate right ideals in $\mathcal{D}$ when X has dimension greater than 1. However, differences from the 1-dimensional case appear immediately. Not every ideal is equivalent to a fat one, and not every ideal is projective. This presents problems for most known classification techniques; but also for the applications of ideal classes. In the one dimensional case, ideal classes can be used to produce everything from algebras Morita equivalent to $\mathcal{D}$, to new examples of wave operators which satisfy Huygen's principle.

Therefore, we address the potentially simpler question, of how to classify the projective ideal class in $\mathcal{D}$; this has the advantage of being a more well-behaved class of ideals, while being closer to the applications known in the 1-dimensional case. Also, in light of Stafford's theorem (Theorem 3.3.2.1), this are intrinsically interesting for the Weyl algebra because they are the *only* non-free projectives.

However, in general, very little is known about projective $\mathcal{D}$-modules for higher dimensional X so far. The only general classes of examples are those induced from the 1-dimensional cases. When X can be written as $X = X' \times X''$, for X' a curve, then any ideal I in $\mathcal{D}(X')$ induces a projective ideal $I \otimes \mathcal{D}(D'')$ in $\mathcal{D}(X)$. Furthermore, if X is 2-dimensional, then $\mathcal{D}$ has global dimension 2, and intersections of projective ideals are still projective.

8.1.4 Quasi-Invariants.

There is also a very specific but interesting class of projective ideals in the higher Weyl algebras coming from the theory of quasi-invariants. Let $\mathfrak{h}$ be a d-dimensional vector space with a non-degenerate inner product, and let W be a Coxeter group acting on $\mathfrak{h}$ by reflections. Each simple reflection $s_i \in W$ defines an invariant hyperplane H_i; let v_i denote the normal vector to H_i. Assign to every invariant hyperplane H_i a positive integer c_i so that this is invariant under the action of W.

Then the **ring of quasi-invariants** Q_c is the subring of the ring $\mathcal{O}_{\mathfrak{h}} = \mathbb{C}[\mathfrak{h}]$ consisting of functions f such that

$$\forall H_i,\ \forall j, 1 \leq j \leq c_i;\ (\partial_{v_i}^{2j-1} f)(H_i) = 0$$

That is, for every hyperplane H_i, the first c_i odd derivatives of f normal to H_i vanish along H_i. Note that if we required *every* odd derivative normal to H_i vanishes along H_i, then the function would be globally invariant by reflection across H_i. Hence the name 'quasi-invariants'; they are those functions which appear invariant across H_i to $2c_i$th order.

The significance of the ring Q_c is that the ring of differential operators on $Spec(Q_c)$ is isomorphic to the $eH_{1,c}e$, the spherical subalgebra of the rational Cherednik algebra at c. This is another story about which we will say very little, except that it is a very well-developed theory studying non-commutative deformations of the ring $(\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*]) \rtimes W$. In particular, there are many powerful tools which do not generalize to other settings well.

Let I_c denote the right ideal of differential operators δ in $\mathcal{D}(\mathfrak{h})$ such that $\delta(\mathcal{O}_{\mathfrak{h}}) \subseteq Q_c$. Using representation theory of the Cherednik algebra, Berest, Etingof and Ginzburg [8] showed that I_c is a projective ideal, with $End_{\mathcal{D}}(I_c) = \mathcal{D}(Q_c)$. In dimension greater than 2, these constitute essentially the only known examples of projective ideals which are not constructed from 1-dimensional examples.

8.1.5 Projective Geometry and Projective Ideals.

A main justification for the theory of projective geometry developed in this thesis has been to create the tools for producing an analogous classification to the 1-dimensional case. We review what such a classification would look like for a d-dimensional variety.

First, the shift on ideal classes must be fixed; since not every projective ideal is fat, this is a more delicate question than the 1-dimensional case. The Beilinson equivalence then reduces to studying the $(d+1)^2$ cohomology groups $\mathbb{R}^i\omega\pi\widetilde{I}_{-j}$, and the various connecting morphisms between them. The hope is that projectivity, possibly in conjunction with other conditions, will imply that many of these cohomology groups vanish, and the rest are given by 'small' modules (not necessary finite-dimensional over $\mathbb{C}$, but with small support). Such hopes are born out by explicit computation with examples, but so far no general theory is forthcoming.

8.2 Grothendieck Groups and Chern Classes.

An application of the Beilinson equivalence is computing the Grothendieck group $K_0(qgr(\widetilde{\mathcal{D}}))$ of the category $qgr(\widetilde{\mathcal{D}})$, because the Grothendieck group depends only on the bounded derived category. Furthermore, $K_0(mod(E))$ is easy to compute because, like a quiver, it can be shown that the Grothendieck group depends only on the diagonal part of E (the vertices) and not on the above diagonal part (the arrows).

8.2.1 Grothendieck Group.

Lemma 8.2.1.1. $K_0(mod(E)) = K_0(coh(X))^{\oplus(n+1)}$.

Proof. Let $M \in mod(E)$, and let e_{-i} denote the idempotent in E which is $1 \in \mathcal{O}_X$ in the $(n+1-i, n+1-i)$ entry in the matrix. Recall that M can be described by the $\mathcal{O}_X$-modules $M_{-i} := e_i M \in coh(X)$, together with a collection of action maps $\mathcal{U}^1 \otimes_X M_{-i} \to M_{-i+1}$. Note that M has a filtration by submodules $M_{\geq -i} := (\sum_{j=0}^{i} e_{-i})M$, with the action maps the same as M where they aren't necessarily zero. The successive quotients $M_{\geq -i}/M_{\geq -i+1} = M_{-i}$, and so $[M] = \sum_{i=0}^{n}[M_{-i}]$. Therefore, $K_0(mod(E))$ is generated by the class of modules of the form M_{-i} for some M.

Let N and N' be two $\mathcal{O}_X$ modules, and let $e_{-i}N$ and $e_{-i}N'$ be the corresponding E-modules. Then $[e_{-i}N] = [e_{-i}N']$ only if $[N] = [N']$ in $K_0(coh(X))$. Furthermore, $[e_{-i}N] = [e_{-j}N']$ for $i \neq j$ only if both are the zero class. Therefore, the group $K_0(mod(E))$ decomposes into $K_0(coh(X))^{\oplus(n+1)}$, where $[M]$ goes to $([M_0], [M_{-1}], ...[M_{-n}])$. □

Theorem 8.2.1.1. $K_0(qgr(\widetilde{\mathcal{D}})) \simeq K_0(coh(X))^{\oplus(n+1)}$.

Explicitly, under this isomorphism, $[\pi M]$ goes to

$$([\mathbb{R}\omega\pi(M)_0], [\mathbb{R}\omega\pi(M)_{-1}], ...[\mathbb{R}\omega\pi(M)_{-n}])$$

8.2.2 Chern Classes.

This decomposition can be used to define the notion of a $K_0(coh(X))$-valued ith Chern class for an object in $qgr(\widetilde{\mathcal{D}})$. Let the i-th Chern class of πM be defined as

$$c_i(\pi M) := \sum_{j=0}^{n} \binom{i}{j} [\mathbb{R}\omega\pi(M)_{-j}] \in K_0(coh(X))$$

where $\binom{i}{j} = 0$ if $j > i$. In the case of $\mathbb{P}^n$, this will coincide with the usual Chern class of a module, see [12].

This amounts to a change of basis of $K_0(qgr(\widetilde{\mathcal{D}}))$ from the natural basis coming from the idempotents e_i, to a basis corresponding in form to powers of a hyperplane divisor (if hyperplane divisors existed). A hyperplane divisor should have a resolution of the form $\pi\widetilde{D}(-1) \to \pi\widetilde{D}$, and intersections of hyperplane divisors will have resolutions corresponding to tensor products of this resolution, we can deduce what its class in the Grothendieck group should be.

Note that the Chern class introduced here is distinct from the 'local second Chern class' of a $\mathcal{D}$-bundle introduced in [5].

BIBLIOGRAPHY

[1] M. Artin and J. J. Zhang. Noncommutative projective schemes. *Adv. Math.*, 109(2):228–287, 1994. (In §1.1.1, 3, 3, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.4.1, and 4.2.5)

[2] Dagmar Baer. Tilting sheaves in representation theory of algebras. *Manuscripta Math.*, 60(3):323–347, 1988. (In §2.4.3.2)

[3] Vladimir Baranovsky, Victor Ginzburg, and Alexander Kuznetsov. Wilson's Grassmannian and a noncommutative quadric. *Int. Math. Res. Not.*, (21):1155–1197, 2003. (In §5.1.2)

[4] A. A. Beĭlinson. Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978. (In §1.2.3.1, 2.4.3.1, and 6.2.1)

[5] David Ben-Zvi and Thomas Nevins. Perverse bundles and Calogero-Moser spaces. *Compos. Math.*, 144(6):1403–1428, 2008. (In §1.1.2, 1, 3.3.2, 8.1.1.2, 8.1.2.1, and 8.2.2)

[6] Yuri Berest and Oleg Chalykh. A_∞-modules and Calogero-Moser spaces. *J. Reine Angew. Math.*, 607:69–112, 2007. (In §3.3.2)

[7] Yuri Berest and Oleg Chalykh. Ideals of rings of differential operators on algebraic curves (with an appendix by George Wilson). *To appear*, 2008. (In §3.3.2 and 8.1.2)

[8] Yuri Berest, Pavel Etingof, and Victor Ginzburg. Cherednik algebras and differential operators on quasi-invariants. *Duke Math. J.*, 118(2):279–337, 2003. (In §8.1.4)

[9] Yuri Berest and George Wilson. Automorphisms and ideals of the Weyl algebra. *Math. Ann.*, 318(1):127–147, 2000. (In §1, 3.3.2, 8.1.1.1, and 8.1.1)

[10] Yuri Berest and George Wilson. Ideal classes of the Weyl algebra and noncommutative projective geometry. *Int. Math. Res. Not.*, (26):1347–1396, 2002. With an appendix by Michel Van den Bergh. (In §1.1.2, 3.3.2, and 8.1.1)

[11] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989. (In §2.4.3.2)

[12] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993. (In §1.2.1.1, 7, 7.1, and 8.2.2)

[13] Andrei Căldăraru. Derived categories of sheaves: a skimming. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 43–75. Amer. Math. Soc., Providence, RI, 2005. (In §1.2.3, 2.4, 2.4.2.1, and 7.2.1)

[14] R. C. Cannings and M. P. Holland. Right ideals of rings of differential operators. *J. Algebra*, 167(1):116–141, 1994. (In §1 and 8.1.1.1)

[15] S. C. Coutinho and M. P. Holland. Locally free $D(\mathbf{P^n})$-modules. *Math. Proc. Cambridge Philos. Soc.*, 112(2):233–245, 1992. (In §2.2.4 and 8.1.1)

[16] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry. (In §2.1)

[17] Ofer Gabber. The integrability of the characteristic variety. *Amer. J. Math.*, 103(3):445–468, 1981. (In §3.3.3)

[18] Pierre Gabriel. Des catégories abéliennes. *Bull. Soc. Math. France*, 90:323–448, 1962. (In §1.1.1)

[19] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. (In §1.2.2.1, 3, 3.1, 3.1.1, and 2)

[20] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006. (In §5.1)

[21] Anton Kapustin, Alexander Kuznetsov, and Dmitri Orlov. Noncommutative instantons and twistor transform. *Comm. Math. Phys.*, 221(2):385–432, 2001. (In §6.2.4)

[22] Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994. (In §6.1.3)

[23] Bernhard Keller. Derived categories and tilting. In *Handbook of tilting theory*, volume 332 of *London Math. Soc. Lecture Note Ser.*, pages 49–104. Cambridge Univ. Press, Cambridge, 2007. (In §6.1.3)

[24] Lieven Le Bruyn. Moduli spaces for right ideals of the Weyl algebra. *J. Algebra*, 172(1):32–48, 1995. (In §3.3.2 and 8.1.1)

[25] Lieven Le Bruyn and Michel Van den Bergh. On quantum spaces of Lie algebras. *Proc. Amer. Math. Soc.*, 119(2):407–414, 1993. (In §6.2.2)

[26] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005. (In §2.2.2)

[27] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small. (In §2.2.1)

[28] Greg Muller. The Beilinson equivalence for differential operators. *Journal of Pure and Applied Algebra*, 214(12):2124 – 2143, 2010. (In §1.3)

[29] Yoshikazu Nakai. High order derivations. I. *Osaka J. Math.*, 7:1–27, 1970. (In §2)

[30] Alexander Polishchuk and Leonid Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005. (In §4.1)

[31] N. Popescu. *Abelian categories with applications to rings and modules*. Academic Press, London, 1973. London Mathematical Society Monographs, No. 3. (In §2.3, 2.3.1.1, 2.3.1.1, 1, 2, 3, 4, and 5)

[32] George S. Rinehart. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.*, 108:195–222, 1963. (In §2.2.3.1)

[33] Joseph J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979. (In §7.1.4.1)

[34] S. P. Smith. Differential operators on commutative algebras. In *Ring theory (Antwerp, 1985)*, volume 1197 of *Lecture Notes in Math.*, pages 165–177. Springer, Berlin, 1986. (In §2.2.1)

[35] J. T. Stafford. Module structure of Weyl algebras. *J. London Math. Soc. (2)*, 18(3):429–442, 1978. (In §3.3.2.1)

[36] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. (In §2.1.4, 2.4.2, 3, 4.2.2, and 4.2.3)

[37] Amnon Yekutieli and James J. Zhang. Serre duality for noncommutative projective schemes. *Proc. Amer. Math. Soc.*, 125(3):697–707, 1997. (In §7.1.4)