Geometric Backlund Transformations in Homogeneous Spaces

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GEOMETRIC BÄCKLUND TRANSFORMATIONS IN HOMOGENEOUS SPACES

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A classical theorem of Bianchi states that two surfaces in space are the focal surfaces of a pseudospherical line congruence only if each surface has constant negative Gaussian curvature. Lie constructed a partial converse, explicitly calculating from one surface of constant negative curvature a pseudospherical line congruence and matching surface. We construct a generalization of these theorems to submanifolds of arbitrary homogeneous spaces. Applications are given to surfaces in the classical space forms and in a novel geometry related to the group of Lie sphere transformations.

BIOGRAPHICAL SKETCH

Matthew Noonan grew up in Kansas City, where he learned to play Go. He attended Hampshire College, where he learned to love mathematics. After some time, he completed graduate work at Cornell University, where he learned to build boats and play the banjo. This strikes him as very high quality of life, for which he is infinitely thankful.

To George, Maureen, and Deanne,

who support me in all my strange endeavors.

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I owe another great debt to the National Science Foundation and to the Cornell mathematics department, who supported me financially and in countless other ways throughout the last seven years.

The beautiful mathematics behind Bäcklund transformations and the related phenomena was first introduced to me by Franz Pedit and the GANG research group at the University of Massachusetts. The problems I met there have been a primary motivator in my research ever since. I gratefully acknowledge their support from the very start of my mathematical career and earnestly hope that their influence will be reflected in this work.

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CHAPTER 1

SYMMETRY METHODS FOR DIFFERENTIAL GEOMETRY

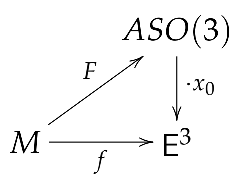
1.1 Classical Differential Geometry of Surfaces

Throughout this section we fix a 2-dimensional manifold M, immersed in E^3 by some map $f : M \to \mathsf{E}^3$.

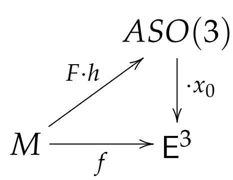
If we would like to understand the geometry of the surface $S = f(M)$ (that is, the properties of S invariant under Euclidean motions), it is reasonable to seek a description of S in terms of the Euclidean group $ASO(3) = \mathbb{R}^3 \rtimes SO(3)$. If we mark a preferred point $x_0 \in \mathsf{E}^3$ then we may replace f with any map $F : M \to ASO(3)$ such that

$$f(p) = F(p) \cdot x_0$$

Such an F is called a *framing* of f with respect to x_0.



Of course, the map F is not unique. If we identify $SO(3) \subset ASO(3)$ as the stabilizer of x_0 then for any map $h : M \to SO(3)$,



is another framing of f. Conversely, for any two framings $F, \hat{F}$ of f the difference $F^{-1} \cdot \hat{F}$ must take values in $SO(3)$.

Since F takes values in a Lie group, it is determined up to a global Euclidean motion by its Darboux derivative

$$\omega = F^*\vartheta = F^{-1}dF \in \Omega^1_M(\mathfrak{aso}(3))$$

where ϑ is the left-invariant Maurer-Cartan form on $ASO(3)$. If we choose a different framing $F' = F \cdot h$, the derivative undergoes a gauge transformation to

$$\omega' = \mathrm{Ad}(h^{-1})\omega + h^*\vartheta$$

We will make heavy use of the following lemma throughout this paper.

Lemma 1.1 (Fundamental Theorem of Nonabelian Calculus)**.** *Let M be a simply connected smooth manifold, and G a Lie group with Lie algebra $\mathfrak{g}$. Suppose ω is a 1-form on M taking values in $\mathfrak{g}$. Then there exists a function $F : M \to G$ such that*

$$F^{-1}dF = \omega$$

if and only if

$$d\omega + \omega \wedge \omega = 0$$

The map F is unique up to left multiplication by a constant $g \in G$.

1.1.1 Adapted Euclidean Frames

We will be using Lie groups extensively as a common language for doing geometric computations in homogeneous spaces. To gain some familiarity with the techniques, let us see how the classical differential geometry of surfaces in Euclidean 3-space may be approached via Lie groups.

Let us consider maps $f : \mathbb{R}^2 \to \mathsf{E}^3$, where E^3 is the homogeneous space $SO(3) \to ASO(3) \to \mathsf{E}^3$. Any framing of f is of the form

$$F = \begin{bmatrix} 1 & 0 \\ f & R \end{bmatrix}, \quad R \in SO(3)$$

The corresponding derivative is

$$F^*\theta = \begin{bmatrix} 0 & 0 \\ R^{-1}df & R^*\theta \end{bmatrix}$$

The rotation field R takes the standard basis of $\mathbb{R}^3$ to some orthonormal basis at $f(p)$. Classically, we say that the basis described by R is *adapted* if the tangent plane of f is spanned by $R\mathsf{e}_1$ and $R\mathsf{e}_2$. Right-multiplication by elements of the subgroup

$$H = \left\{ r \in ASO(3) : r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

preserves the condition "R is an adapted frame".

With these conventions, F is adapted if and only if the lower-left entry of $F^*\theta$ is zero. This entry is equal to $(R^{-1}df, \mathsf{e}_3) = (df, R\mathsf{e}_3)$, which is equivalent to the usual condition that F is adapted if and only if $R : \mathsf{e}_3 \mapsto N$, where N is the unit normal to f.

Let us now analyze the structure of an adapted frame in more detail. If F is adapted then its derivative $\omega = F^*\theta$ has the form

$$\omega = \begin{bmatrix} 0 & 0 & 0 \\ \tau & \rho & \nu \\ 0 & -\nu^T & 0 \end{bmatrix}$$

where $\tau, \nu \in \Omega^1_M(\mathbb{R}^2)$ and $\rho \in \Omega^1_M(\mathfrak{so}(2))$. The Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$ will generally split up into several differential equations plus a single algebraic *compatibility condition*. Computing directly gives

$$d\omega + \omega \wedge \omega = \begin{bmatrix} 0 & 0 & 0 \\ d\tau & d\rho & d\nu \\ 0 & -d\nu^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \tau & \rho & \nu \\ 0 & -\nu^T & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 \\ \tau & \rho & \nu \\ 0 & -\nu^T & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ d\tau + \rho \wedge \tau & d\rho + \rho \wedge \rho - \nu \wedge \nu^T & d\nu + \rho \wedge \nu \\ -\nu^T \wedge \tau & -d\nu^T - \nu^T \wedge \rho & 0 \end{pmatrix}$$

As expected, we have the three differential equations

$$\begin{aligned} \nabla \tau &= 0 \\ \nabla \nu &= 0 \qquad\qquad (\nabla \varphi = d\varphi + \rho \wedge \varphi) \\ \operatorname{curv} \nabla &= \nu \wedge \nu^T \end{aligned}$$

along with a single algebraic compatibility condition

$$-\nu^T \wedge \tau = 0$$

To train our intuition for these objects and equations, it is productive to translate them into the language of classical differential geometry.

Let us start by relating τ, ν and ρ to objects which we already understand.

1. The frame F with corresponding rotation field R is adapted exactly when R rotates e_3 to N. The vectors $R\mathsf{e}_1$ and $R\mathsf{e}_2$ therefore give an orthonormal framing Ψ of Tf at every point. Since ρ describes the derivative of R in the $\mathsf{e}_1 \wedge \mathsf{e}_2$ plane, it describes how the framing Ψ differs from its parallel

transport as we move around on $f(M)$. In other words: ρ is a connection form and $\nabla = d + \rho$ is, in fact, the Levi-Civita connection.

2. τ is given by $\tau = R^{-1}df$, and is therefore describes the differential df in terms of the orthonormal frame Ψ. It follows that the metric tensor (first fundamental form) can by recovered by defining

$$I = (\tau, \tau) = \tau^T \otimes \tau$$

Symmetry of I follows from the fact that $\alpha^T \wedge \alpha = 0$ for any vector-valued 1-form (see corollary A.2), so $\tau^T \otimes \tau$ actually belongs to the space of symmetric tensors.

3. ν is given by the equation

$$\nu = \begin{bmatrix} (R^{-1}dR \cdot \mathsf{e}_3, \mathsf{e}_1) \\ (R^{-1}dR \cdot \mathsf{e}_3, \mathsf{e}_2) \end{bmatrix}$$

Since $(R^{-1}dR \cdot \mathsf{e}_3, \mathsf{e}_\mathsf{i}) = (dR \cdot \mathsf{e}_3, R \cdot \mathsf{e}_\mathsf{i}) = (dN, R \cdot \mathsf{e}_\mathsf{i})$, this shows that ν is the derivative of N in terms of the frame Ψ. From this, we may derive the second fundamental form by

$$II = (dN, df) = (R^{-1}dN, R^{-1}df) = (\nu, \tau) = \nu^T \otimes \tau$$

Having established the relationship between the covariant calculus on surfaces and the objects which appear in the derivative of F, we can interpret the differential equations as follows:

1. $\nabla\tau = 0$: This implies the well-known fact that the covariant derivative of the metric is zero since

$$\nabla I = \nabla(\tau^T \otimes \tau) = \nabla\tau^T \otimes \tau + \tau^T \otimes \nabla\tau = 0$$

Stated another way, parallel transport preserves the metric tensor. But the claim $\nabla I = 0$ is slightly weaker than $\nabla \tau = 0$.

2. curv $\nabla = \nu \wedge \nu^T$: To understand this equation, let us try to understand the right-hand side better. Writing $\nu_i^j = (\mathrm{e_j}, \nu(\partial/\partial x^i))$,

$$\nu \wedge \nu^T = \begin{bmatrix} 0 & \nu_1^1 \nu_2^2 - \nu_1^2 \nu_2^1 \\ \nu_1^2 \nu_2^1 - \nu_1^1 \nu_2^2 & 0 \end{bmatrix}$$

Since $\det \nu = \det R \cdot \nu$ and $R \cdot \nu$ is the shape operator, $\nu \wedge \nu^T$ is an infinitesimal rotation of magnitude equal to KdA — the (extrinsically computed) Gaussian curvature. Thus, the equation curv $\nabla = \nu \wedge \nu^T$ is the *Gauss-Codazzi equation*, relating the Gaussian curvature to the curvature of ∇.

3. $\nabla \nu = 0$: Since ν is the derivative of N, $\nabla \nu$ measures the normal curvature — the way in which N twists when it is transported around an infinitesimal loop. Since the normal bundle is flat (parallelized by N, in fact), this curvature must be zero. This, along with $\nabla \tau = 0$, implies

$$\nabla II = \nabla(\nu^T \otimes \tau) = 0$$

Finally, we are left with the compatibility condition $-\nu^T \wedge \tau = 0$. We already determined that $-\nu^T \otimes \tau = II$, the second fundamental form. The equation $-\nu^T \wedge \tau = 0$ then forces II to be a symmetric tensor — this is the surface geometry version of "mixed partials must commute".

1.1.2 Application: Surfaces of Revolution

As an elementary exercise, let us compute the Euclidean geometric invariants for a surface of revolution described by the profile curve $r(x)$:

$$f(x,y) = \begin{bmatrix} r(x)\cos y \\ r(x)\sin y \\ x \end{bmatrix}$$

The $\partial/\partial x$ and $\partial/\partial y$ derivatives of f are orthogonal by construction, so the frame

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ f & \frac{f_x}{|f_x|} & \frac{f_y}{|f_y|} & \frac{f_x \times f_y}{|df|^2} \end{bmatrix}$$

is adapted. More explicitly,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} r'(x)\cos y \\ r'(x)\sin y \\ 1 \end{bmatrix}, \quad \left|\frac{\partial f}{\partial x}\right| = \sqrt{1+r'(x)^2}$$

$$\frac{\partial f}{\partial y} = \begin{bmatrix} -r(x)\sin y \\ r(x)\cos y \\ 0 \end{bmatrix}, \quad \left|\frac{\partial f}{\partial y}\right| = r(x)$$

and the rotation field for the adapted lift F is

$$R = \begin{bmatrix} \frac{r'(x)}{\sqrt{1+r'(x)^2}}\cos y & -\sin y & \frac{-\cos y}{\sqrt{1+r'(x)^2}} \\ \frac{r'(x)}{\sqrt{1+r'(x)^2}}\sin y & \cos y & \frac{-\sin y}{\sqrt{1+r'(x)^2}} \\ \frac{1}{\sqrt{1+r'(x)^2}} & 0 & \frac{r'(x)}{\sqrt{1+r'(x)^2}} \end{bmatrix}$$

From this we may compute

$$\tau = R^{-1}df = \begin{bmatrix} \sqrt{1+r'(x)^2}\,dx \\ r(x)\,dy \end{bmatrix}$$

$$I = \tau^T \otimes \tau = (1 + r'(x)^2)\, dx^2 + r(x)^2\, dy^2$$

and

$$R^*\theta = \begin{bmatrix} 0 & \frac{-r'(x)\ dy}{\sqrt{1+r'(x)^2}} & \frac{r''(x)\ dx}{1+r'(x)^2} \\ \star & 0 & \frac{-dy}{\sqrt{1+r'(x)^2}} \\ \star & \star & 0 \end{bmatrix}$$

giving

$$\nu = \begin{bmatrix} \frac{r''(x)\ dx}{1+r'(x)^2} \\ \frac{-dy}{\sqrt{1+r'(x)^2}} \end{bmatrix}$$

$$II = \nu^T \otimes \tau = \frac{r''(x)}{\sqrt{1+r'(x)^2}} dx^2 + \frac{-r(x)}{\sqrt{1+r'(x)^2}} dy^2$$

and, setting $\rho = i\lambda$,

$$\lambda = \frac{-r'(x)}{\sqrt{1+r'(x)^2}} dy$$

The mean and Gaussian curvatures may be computed from the map $II \cdot I^{-1}$:

$$H = \frac{1}{2}\left(\frac{-r''(x)}{(1+r'(x)^2)^{3/2}} + \frac{-1/r(x)}{\sqrt{1+r'(x)^2}} \right)$$

$$K = \frac{-r''(x)/r(x)}{(1+r'(x)^2)^2}$$

1.2 Homogeneous Spaces

Definition 1.1. A *homogeneous space* with *structure group* G is a smooth manifold M with a smooth, transitive action of a Lie group G denoted

$$m : G \times M \to M$$

When it does not cause confusion, $m(g, p)$ will simply be denoted $g \cdot p$. The stabilizer subgroup of $p \in M$ will be denoted H_p.

For the purposes of computation, it is often convenient to work on the Lie group G rather than the space M itself.

Definition 1.2. A *frame* relative to q is a smooth local section of $m(-,q)$. More explicitly, a frame over $U \subset M$ is a smooth map $\sigma : U \to G$ such that for all $p \in U$,

$$\sigma(p) \cdot q = p$$

Note that q does not need to be contained in U.

Often we will take advantage of a frame relative to q to induce isomorphisms between fibers of various bundles. As a simple example, suppose that $\operatorname{stab}_G q = H$. Then M carries a canonical H-bundle whose fiber over p is simply H_p. H_p is obviously isomorphic to H, but not canonically so. But if we have a frame $\sigma : U \to G$ then for any $p \in U$, the stabilizer $H_p = \mathrm{Ad}(\sigma(p))H_q$ since

$$\sigma(p) \cdot H_q \cdot \sigma(p)^{-1} \cdot p = \sigma(p) \cdot H_q \cdot q = \sigma(p) \cdot q = p$$

We will frequently make use of the infinitesimal version of this map, where $\mathrm{Ad}(\sigma(p))$ provides a Lie algebra isomorphism from $\mathfrak{h}_q$ to $\mathfrak{h}_p$.

1.2.1 Tangent Spaces

Let $m_q : G \to M$ be the map defined by

$$m_q(g) = m(g,q) = g \cdot q$$

Since $m_q(1) = q$, the pushforward $(m_q)_*$ maps $\mathfrak{g}$ to T_qM. Elements of H_q fix q, so the kernel of $(m_q)_*$ is just $\mathfrak{h}_q$, giving the short exact sequence

$$0 \to \mathfrak{h}_q \to \mathfrak{g} \overset{(m_q)_*}{\to} T_qM \to 0$$

It follows that T_qM may be canonically identified with the quotient $\mathfrak{g}/\mathfrak{h}_q$.

Example 1.1. Let $M = S^2$ be the Euclidean sphere with structure group $SO(3)$. We can pick a basis of $\mathfrak{so}(3)$ spanned by infinitesimal rotations e_1^2, e_2^3, e_3^1 which act on tangent vectors e_k to S^2 by

$$e_j^i e_k = \delta_k^i e_j - \delta_k^j e_i$$

Let r be the map

$$r(e_1^2) = e_3$$
$$r(e_2^3) = e_1$$
$$r(e_3^1) = e_2$$

Then $e_j^i e_k = r(e_j^i) \times e_k$, so we can immediately see that $\mathfrak{h}_p = r^{-1}\text{span}\,\{p\}$ and

$$T_pS^2 \cong r^{-1}T_pS^2 \mod \mathfrak{h}_p$$

Geometrically, $\mathfrak{h}_p$ is the set of infinitesimal rotations in the plane $p^\perp$.

Note that in this case there is actually a *best* element ρ_v of $\mathfrak{so}(3)$ representing each $v \in T_pS^2$ — the one such that $r(\rho_v) \cdot v = 0$. This is because the Euclidean sphere is a *reductive* homogeneous space, a condition which will be useful in the examples of chapter 3.

Any vector field on S^2 can therefore be described by a map $\psi : S^2 \to \mathfrak{so}(3)$, with $\psi = \psi'$ if and only if $(\psi - \psi')(p) \in \mathfrak{h}_p$ for all $p \in S^2$. Composing with r, we can think of any map $\psi : S^2 \to \mathbb{R}^3$ as a vector field, with $\psi = \psi'$ if and only if

$$\psi(p) \times p - \psi'(p) \times p$$

is the zero function.

The fact that on a homogeneous space we may think of sections of certain bundles as mere functions will dramatically simplify later calculations. It is considerations such as this which make the invariant approach to studying differential equations on homogeneous spaces particularly fruitful.

If we have chosen a frame $\sigma : U \to G$, things become even simpler. In this case,

$$Ad(\sigma(p)^{-1}) : \mathfrak{h}_p \to \mathfrak{h}_q$$

is an isomorphism, so we can describe any vector field on U by a map $\psi : U \to \mathfrak{g}$. The vector field $\tilde{\psi}$ is obtained by

$$\tilde{\psi}_p = (m_p)_* \mathrm{Ad}(\sigma(p)^{-1})\psi(p)$$

Two maps ψ, ψ' represent the same vector field if and only if $\psi - \psi'$ takes values in $\mathfrak{h}_q$.

Taking the equivalence relation into account, given a frame on U we may represent any vector field on U uniquely by a map $\psi : U \to \mathfrak{g}/\mathfrak{h}_q$. Furthermore, if ξ_a is any basis of $\mathfrak{g}/\mathfrak{h}_q$ then the constant maps $\xi_a(p) = \xi_a$ define a basis of $\Gamma(TU)$ as a $C^\infty(U, \mathbb{R})$-module. This basis will be extremely useful in computations.

This discussion shows that we can canonically think of $X_p \in T_pM$ as an element of $\mathfrak{g}/\mathfrak{h}_p$ via $(m_p)_*^{-1}$. We now look at how these individual isomorphisms may be collected into a bundle isomorphism.

We have already noted that M carries a canonical principal H-bundle with fiber H_p over p. There is an associated canonical vector bundle Iso with fiber

$$\mathrm{Iso}_p = T_1 H_p$$

Concretely, Iso_p is the set of infinitesimal motions of M fixing p — that is, the infinitesimal isotropy subgroup at p.

M also carries the trivial $\mathfrak{g}$-bundle $\underline{\mathfrak{g}}$. The maps $(m_p)_*$ above may be combined to give a bundle map

$$m_* : \underline{\mathfrak{g}} \to TM$$

defined as follows. If $\xi : U \to \mathfrak{g}$ is a local section of $\underline{\mathfrak{g}}$, then

$$m_*(\xi)_p = (m_p)_*\xi_p \ \in \ T_pM$$

The kernel is just Iso_p. If we consider the short exact sequence of bundle maps

$$0 \to \mathrm{Iso} \to \underline{\mathfrak{g}} \to \mathrm{Tan} \to 0$$

where Tan is the corresponding quotient bundle with fiber $\mathfrak{g}/\mathfrak{h}_p$ over p, then m_* descends to a bundle isomorphism

$$m_* : \mathrm{Tan} \xrightarrow{\sim} TM$$

1.2.2 Lie Calculus

For computations, it would be nice to have a way of doing all calculations with vector fields on Tan instead. To this end, let us see how the Lie bracket of vector fields appears to Tan.

First, suppose we have fixed a basis $\{\xi_i\}$ of $\mathfrak{g}$. Then each basis vector defines a constant section $\xi_i : M \to \mathfrak{g}$ of $\underline{\mathfrak{g}}$, and therefore a vector field $m_*\xi_i$ on M. These vector fields are not linearly independent, but they do span T_pM at each $p \in M$. As a result, any vector field can be written as a $C^\infty(M, \mathbb{R})$-linear combination of the $m_*\xi_i$.

How do the basis vector fields $m_*\xi_i$ act on functions? To find out, let ξ be the section of $\underline{\mathfrak{g}}$ corresponding to a fixed element $\xi \in \mathfrak{g}$, and let $f : U \to \mathbb{R}$ be a local function. To differentiate f at p with respect to ξ, we need a curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$, $\gamma'(0) = m_*\xi$. Such a curve is provided by

$$\gamma(t) = m(\exp(t\xi), p)$$

So we may now compute

$$\begin{aligned}(m_*\xi) \cdot f &= (f(m(\exp(t\xi), p)))'_{t=0} \\ &= df_p((m_p)_*\xi)\end{aligned}$$

This is not very interesting on its own; what is more important is the map $m(\exp(t\xi), p)$. We can think of this as a map

$$\Phi^\xi : M \times \mathbb{R} \to M$$

given by $\Phi^\xi(p,t) = m(\exp(t\xi), p)$. This is a flow on M such that $d\Phi^\xi/dt = m_*\xi$ at $t = 0$. Furthermore, since $m(g, m(g', p)) = m(g \cdot g', p)$ this flow is actually the flow on M induced by the vector field $m_*\xi$. Using this, we may compute the Lie derivative of one basis vector field with respect to another. Let X, Y be $m_*\xi, m_*\zeta$ for constants $\xi, \zeta \in \mathfrak{g}$ and write Φ_t for $\Phi^\xi(-, t)$. Then

$$\mathcal{L}_X Y = \lim_{t \to 0} \frac{1}{t} \left((\Phi_{-t})_* Y_{\Phi_t(p)} - Y_p \right)$$

Theorem 1.2. *Let $\xi, \zeta : M \to \mathfrak{g}$ be constant functions. Then*

$$\mathcal{L}_{m_*\xi}\, m_*\zeta = m_*[\zeta, \xi]$$

where $\mathcal{L}$ is the Lie derivative on M and $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$.

Proof. Let Φ_t denote the time-t flow of $m_*\xi$, so that

$$\Phi_t(p) = m(\exp(t\xi, p))$$

We will let $(m_*\zeta)_p$ act on a function f by

$$(m_*\zeta)_p \cdot f = \frac{d}{ds} f(m(\exp(s\zeta), p))\bigg|_{s=0} = df_p((m_p)_*\zeta)$$

The Lie derivative acts on a function f at the point p by

$$\begin{aligned}
\left(\mathcal{L}_{m_*\xi}\, m_*\zeta\right)_p \cdot f &= \lim_{t\to 0} \frac{(\Phi_{-t})_*(m_*\zeta)_{\Phi_t(p)} \cdot f - (m_*\zeta)_p \cdot f}{t} \\
&= \lim_{t\to 0} \frac{1}{t} \left(\frac{d}{ds} f(\Phi_{-t}(m(\exp(s\zeta), \Phi_t(p)))\bigg|_{s=0} \right. \\
&\qquad\qquad \left. - \frac{d}{ds} f(m(\exp(s\zeta), p))\bigg|_{s=0} \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(\frac{d}{ds} f\left(m(\exp(-t\xi), m(\exp(s\zeta), m(\exp(t\xi), p)))\right)\bigg|_{s=0} \right. \\
&\qquad\qquad \left. - df_p((m_p)_*\zeta) \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(\frac{d}{ds} f(m(\exp(-t\xi)\exp(s\zeta)\exp(t\xi), p))\bigg|_{s=0} \right. \\
&\qquad\qquad \left. - df_p((m_p)_*\zeta) \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(df_p((m_p)_*\mathrm{Ad}(\exp(-t\xi))\zeta) - df_p((m_p)_*\zeta) \right) \\
&= df_p\left((m_p)_* \left(\lim_{t\to 0} \frac{\mathrm{Ad}(\exp(-t\xi))\zeta - \zeta}{t} \right) \right) \\
&= df_p((m_p)_*[\zeta, \xi])
\end{aligned}$$

This demonstrates that for all $p \in M$,

$$\left(\mathcal{L}_{m_*\xi}\, m_*\zeta\right)_p = (m_p)_*[\zeta, \xi]$$

Stated more globally, the proceeding calculation shows that the diagram

$$\begin{array}{ccc}
\Gamma(TM) \otimes \Gamma(TM) & \xrightarrow{[\ ,\ ]} & \Gamma(TM) \\
\uparrow {\scriptstyle m_* \otimes m_*} & & \uparrow {\scriptstyle m_*} \\
\mathfrak{g} \otimes \mathfrak{g} & \xrightarrow[-[\ ,\ ]]{} & \mathfrak{g}
\end{array}$$

commutes, where the upper map is the commutator of vector fields and the lower map is the (negative of the) Lie bracket. □

Corollary 1.3. *If M is an effective geometry then m_* is a faithful representation of $\mathfrak{g}$ on $\Gamma(TM)$.*

1.2.3 Cotangent Spaces

In the previous section we showed that there is a canonical identification $T_pM \cong \mathfrak{g}/\mathfrak{h}_p$. Thus, it follows that there is a canonical isomorphism

$$T_p^*M \cong (\mathfrak{g}/\mathfrak{h}_p)^* = \mathfrak{h}_p^\perp$$

where $W^\perp$ is defined by

$$W^\perp = \{\varphi \in V^* \; : \; \varphi|_W = 0\}$$

for $W \subset V$. The spaces $\mathfrak{h}_p^\perp$ form a subbundle Cot of $\underline{\mathfrak{g}^*}$. A 1-form on M can therefore be described by a section of Cot. Equivalently, a 1-form on M is a map $\omega : M \to \mathfrak{g}^*$ such that

$$\omega_p(\mathfrak{h}_p) = 0$$

Working with 1-forms in this formalism is particularly nice since they are merely functions, not even equivalence classes of functions like the vector fields analyzed in the previous section. Likewise, a k-form is just a map $\Omega : M \to \bigwedge^k \mathfrak{g}^*$ such that

$$\mathfrak{h}_p \lrcorner \, \Omega_p = 0$$

The simplest tensorial operation is the contraction of a 1-form with a vector field to create a function. If $\xi : M \to \mathfrak{g}$ and $\omega : M \to \mathfrak{g}^*$ then the contraction

$m_*\xi \lrcorner m^*\omega$ is simply

$$m_*\xi \lrcorner m^*\omega = \omega(\xi)$$

The result is well-defined since ξ is defined up to sections of Iso and ω vanishes on these sections. This shows that we may evaluate a 1-form represented by $\omega : M \to \mathfrak{g}^*$ on a vector field represented by $\xi : M \to \mathfrak{g}$ simply by taking the composition $\omega(\xi)$. The following lemma and corollary are useful when we wish to evaluate ω on a vector field X but do not possess a map ξ with $m_*\xi = X$.

Lemma 1.4. *Let $\sigma : U \to G$ be a local frame. Then $\sigma^*\theta_R$ is a local section of m_*, where θ_R is the right-invariant Maurer-Cartan form $dg \cdot g^{-1}$ on G.*

Corollary 1.5. *Let $\omega : U \to \mathfrak{g}^*$, $\underline{\omega}$ the corresponding 1-form, and $X \in \Gamma(TM|_U)$. Then*

$$\underline{\omega}(X) = \omega\left((\sigma^*\theta_R)(X)\right)$$

where $\sigma : U \to G$ is any local frame.

Proof. From the lemma, $\sigma^*\theta_R$ is a local section of m_*, so we have

$$\underline{\omega}(m_*\xi) = \omega((\sigma^*\theta_R)(m_*\xi)) = \omega(\xi + h)$$

for some map $h : U \to \mathfrak{g}$ with $h_p \in \mathfrak{h}_p$. But ω_p is an element of $\mathfrak{h}_p^\perp$, so $\omega(\xi + h) = \omega(\xi)$. ☐

The Exterior Derivative

To complete this section, let us see how the exterior derivative behaves for the differential forms on G/H which are represented by constant maps to the Lie coalgebra $\mathfrak{g}^*$. Extending via C^∞-linearity will allow us to compute the exterior derivative of any 1-form on G/H.

Let us proceed by using the Lie derivative and Cartan's equation

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega)$$

Take X, Y to be a vector fields represented by $\xi, \zeta \in \mathfrak{g}$ and $\underline{\omega}$ a 1-form represented by $\omega \in \mathfrak{g}^*$. Then $\xi \lrcorner \omega$ is a constant function on G/H, so the second term in $\mathcal{L}_X \underline{\omega}$ is zero, leaving

$$\mathcal{L}_X \underline{\omega} = X \lrcorner d\underline{\omega}$$

In particular, this means that $(\mathcal{L}_X \underline{\omega})(Y) = d\underline{\omega}(X, Y)$. But on the other hand we have

$$\begin{aligned}(\mathcal{L}_X \underline{\omega})(Y) &= \mathcal{L}_X(\underline{\omega}(Y)) - \underline{\omega}(\mathcal{L}_X Y) \\ &= -\underline{\omega}(\mathcal{L}_X Y) \\ &= -\omega([\xi, \zeta])\end{aligned}$$

Altogether, this shows that for 1-forms represented by elements of $\mathfrak{g}^*$ the exterior differential may be computed using the *codifferential*

$$\delta : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

defined by

$$\delta\omega(\xi, \zeta) = -\omega([\xi, \zeta])$$

1.3 Exterior Differential Systems

The final piece of machinery which we will use is the *exterior differential system*. These are a way of encoding partial differential equations as geometric data on a manifold. An interesting argument for the utility of exterior differential systems may be found in Bryant, Griffith, and Hsu [3]; we will take the perspective that lemma 1.6 is sufficient justification for their introduction.

Definition 1.3. An *exterior differential system* on a manifold M is a differential ideal Θ of the exterior algebra $\Omega(M, \mathbb{R})$.

Example 1.2. Let $M = \mathbb{R}^3$ with coordinates x, y, p. Then the ideal Θ algebraically generated by the forms $dy - p\,dx$, $dx \wedge dp$, and $dx \wedge dy \wedge dp$ is closed under the exterior derivative, and therefore is a differential ideal. Θ is called the *contact ideal*. Integral curves of Θ are exactly those curves which are either locally of the form $y = f(x)$, $p = f'(x)$ or of the form $dy = dx = 0$.

Definition 1.4. An exterior differential system is called a *Pfaffian system* if it is differentially generated by 1-forms.

Example 1.3. The contact ideal on $\mathbb{R}^3$ is a Pfaffian system generated by the 1-form

$$\vartheta = dy - p\,dx$$

since $dx \wedge dp = d\vartheta$ and $dx \wedge dy \wedge dp = -\vartheta \wedge d\vartheta$.

Example 1.4. More generally, let M, N be smooth manifolds and define the *bundle of k-jets*

$$J^k(M, N) \xrightarrow{\pi^k} M \times N$$

by

$$J^k(M, N)_{p,q} = \bigoplus_{i=1}^{k} \mathrm{Hom}(S^i T_p M, T_q N)$$

where $S^i V$ is the i-th symmetric tensor power of V. To each smooth map $f : M \to N$ we may associate the graph $\Gamma_f : M \to M \times N$, so that $\Gamma_f(p) = (p, f(p))$. Using the graph, we can pull back the jet bundle to obtain a bundle

$$J^k_f = \Gamma^*_f J^k(M, N)$$

on M. The fiber over the point p is

$$\bigoplus_{i=1}^{k} \mathrm{Hom}(S^i T_p M, T_{f(p)} N)$$

We may think of an element of the fiber over p as a k-th order Taylor approximation of a function which maps p to $f(p)$. Conversely, we can always find a section of J^k_f by computing a Taylor series.

The importance of Pfaffian systems comes from the fact that they can encode every possible PDE — of any order, linear or nonlinear, on any manifold.

Lemma 1.6. *Let Δ be a partial differential equation of order k for maps from of the form $M \to N$ between smooth manifolds M, N. Then there exists a subset $\Sigma_\Delta \subset J^k(M,N)$ such that solutions to Δ are in one-to-one correspondence with sections of $J^k(M,N) \to M \times N$ which take values in Σ_Δ and annihilate the contact ideal on $J^k(M,N)$.*

1.3.1 Special Classes of Pfaffian Systems

Definition 1.5. A Pfaffian system I is *integrable* (also: *Frobenius*) if

$$dI = 0 \mod I$$

Lemma 1.7. *I is integrable if and only if the dual distribution $I^\perp$ is integrable.*

If Θ is integrable, it has the very useful property that through any point of M we may construct integrals of Θ by solving a series of ordinary differential equations. The ordinary differential equations are exactly the same as those which appear in the proof of the Frobenius theorem: each time we inductively extend the dimension of an integral manifold to an integrable distribution, we solve an ODE using the integral manifold as an initial condition. Thus, if a PDE corresponds to an integrable EDS it may be solved through a series of one-dimensional integrations.

Definition 1.6. Suppose that $\tilde{M}$, M are smooth manifolds, $\pi : \tilde{M} \to M$ a submersion, and Θ is an EDS on M. An EDS $\tilde{\Theta}$ on $\tilde{M}$ is an *extension* of Θ if

$$\pi^*\Theta = 0 \mod \tilde{\Theta}$$

Extensions are characterized by the following property: if $\tilde{f} : X \to \tilde{M}$ is an integral of $\tilde{\Theta}$, then $\pi \circ \tilde{f} : X \to M$ is an integral of Θ.

1.4 Geometric Exterior Differential Systems

Definition 1.7. A *geometric exterior differential system* (or gEDS) is a homogeneous space M equipped with a differential ideal $\Theta \subset \Omega(M, \mathbb{R})$ such that for all $g \in G$,

$$(L_g)^*\Theta = \Theta$$

For the remainder of this document, we will assume that any gEDS is given as a Pfaffian system.

Example 1.5. Let $UT\mathsf{E}^3$ be the homogeneous space of unit tangent vectors to points in E^3, with structure group $ASO(3)$. $UT\mathsf{E}^3$ is diffeomorphic to $\mathsf{E}^3 \times S^2$. Recall that $ASO(3) \cong \mathbb{R}^3 \rtimes SO(3)$. The action of an element $(T, R) \in ASO(3)$ on (p, n) is given by

$$(T, R) \cdot (p, n) = (T + R \cdot p, R \cdot n)$$

It follows that the stabilizer of a point (p, n) is the subgroup

$$H_{(p,n)} = \{(p - R \cdot p, R) \ : \ R \cdot n = n\}$$

Now, let ϑ be the 1-form $\langle n, dp \rangle$. Then

$$\begin{aligned}(L_{(T,R)})^* \vartheta &= \langle R \cdot n, d(T + R \cdot p) \rangle \\ &= \langle R \cdot n, R \cdot dp \rangle \\ &= \langle n, dp \rangle \\ &= \vartheta\end{aligned}$$

so ϑ is invariant under $ASO(3)$. The differential ideal Θ generated by ϑ encodes the differential equation for adapted lifts of maps to E^3: a map

$$(f, n) : \mathbb{R}^2 \to UT\mathsf{E}^3$$

with f nondegenerate is an integral of Θ if and only if n is the normal map of f.

By choosing a local frame, we get the following fundamental lemma:

Lemma 1.8. *There is a one-to-one correspondence between geometric exterior differential systems on a homogeneous space G/H and* $\mathrm{ad}^*(\mathfrak{h})$*-invariant subspaces $V \subset \mathfrak{h}^\perp \subset \mathfrak{g}^*$.*

The $\mathrm{ad}^*(\mathfrak{h})$-invariance ensures that we obtain the same subspace V no matter which local frame is used. For the rest of this document, we will essentially use "$\mathrm{ad}^*(\mathfrak{h})$-invariant subspace of $\mathfrak{h}^\perp$" as the *definition* of a gEDS.

1.4.1 Local Description of $\Theta^\perp$

To any EDS Θ there is an associated distribution $\Delta = \Theta^\perp$, where

$$\Theta^\perp_p = \bigcap_{\theta \in \Theta} \ker \theta_p$$

In the case when Θ is a geometric EDS, the distribution Δ is invariant under $(L_g)_*$ for any $g \in G$.

If we allow ourselves a local frame $\sigma : U \to G$ relative to some point $q \in M$, it becomes much easier to construct local sections of Δ. To that end, let $\psi : M \to \mathfrak{g}$ be a constant function such that $(m_*\psi)_q \in \Delta_q$. Generally it will not be true that $(m_*\psi)_p \in \Delta_p$ since

$$(L_g)_*(m_*\psi)_q = (m_*\mathrm{Ad}(g)\psi)_{g\cdot q}$$

so $\psi_{g\cdot q} \in \Delta_{g\cdot q}$ only if ψ is in the centralizer of g, so that $\psi = \mathrm{Ad}(g)\psi$. Thus, the vector fields corresponding to constant maps are not G-invariant. However, we can use the local section σ to twist these constant maps into invariant vector fields.

Lemma 1.9. *Let Θ be a gEDS, $\sigma : U \to G$ a local frame relative to $q \in M$, and take $\psi \in \mathfrak{g}$ such that $(m_q)_*\psi \in \Theta_q^\perp$. Then $p \mapsto (m_p)_*\mathrm{Ad}(\sigma(p))\psi$ is a nonvanishing section of $\Theta^\perp$ over U.*

Proof. To prove that $m_*\mathrm{Ad}(\sigma(p))\psi$ is an element of $\Theta_p^\perp$, we just use the intertwining relation

$$(m_*\mathrm{Ad}(\sigma(p))\psi)_p = (L_{\sigma(p)})_*(m_*\psi)_q$$

Let $p \in U$ and pick g such that $g \cdot q = p$. It follows that $g \cdot h = \sigma(p)$ for some $h \in H_q$. Since $(m_*\psi)_q \in \Theta_q^\perp$ and Θ is G-invariant,

$$\begin{aligned}
(L_{g^{-1}}^*\Theta)_p(m_*\mathrm{Ad}(\sigma(p))\psi) &= \Theta_q((L_{g^{-1}})_*m_*\mathrm{Ad}(\sigma(p))\psi) \\
&= \Theta_q((L_{g^{-1}})_*(L_{\sigma(p)})_*m_*\psi) \\
&= \Theta_q((L_h)_*m_*\psi) \\
&= 0
\end{aligned}$$

□

It follows that, by choosing a basis $\{\psi_i\}$ of $\Theta_q^\perp$, we may obtain vector fields

$$\tilde{\psi}_i = \mathrm{Ad}(\sigma(p))\psi$$

which span $\Theta^\perp$ over U.

CHAPTER 2

GEOMETRIC BÄCKLUND TRANSFORMATIONS

2.1 The Classical Bäcklund Transformation

Throughout this section, we will only be concerned with surfaces in Euclidean 3-space $\mathsf{E}^3 = ASO(3)/SO(3)$.

2.1.1 Pseudospherical Line Congruences and Bianchi's Theorem

A *line congruence* X is a 2-parameter family of lines in E^3. Generically, to each line congruence there are exactly two *focal surfaces* Y_1, Y_2, characterized by the property that each Y_i is tangent to the line congruence.

More precisely, let $\mathcal{L}_3$ denote the space of affine lines in E^3. Then a line congruence is a map from a 2-dimensional manifold U to $\mathcal{L}_3$, and Y is a focal surface of L if there is some parameterization $f : U \to \mathsf{E}^3$ of Y with normal field $n : U \to S^2$ such that for each $p \in U$ we have $f(p) \in L(p)$ and $n(p) \perp L(p)$.

Bianchi initiated the study of a special class of line congruences, where the focal surfaces are at a fixed distance.

Definition 2.1 (Pseudospherical line congruence)**.** A line congruence $X : U \to \mathcal{L}_3$ is *pseudospherical* if

1. Corresponding points on the focal surfaces are a unit distance apart.

2. The focal surfaces are perpendicular at corresponding points.

Since the two focal surfaces of a line congruence are each tangent to the congruence, we can characterize the pair of surfaces by the following four relations

Definition 2.2 (Bianchi relations)**.** We will call two surface elements (x, n) and $(\hat{x}, \hat{n})$ *Bianchi-related* if

- $|x - \hat{x}| = 1$
- $n \perp \hat{n}$
- $n \perp x - \hat{x}$
- $\hat{n} \perp x - \hat{x}$

The four Bianchi relations completely characterize the pair of focal surfaces in a pseudospherical line congruence.

Lemma 2.1. *Let $X, \hat{X} : U \to \mathsf{E}^3$ be a pair of surfaces, and write $(x, n), (\hat{x}, \hat{n}) : U \to \mathsf{E}^3 \times S^2$ for the surface elements of $X, \hat{X}$. Then there is a pseudospherical line congruence L such that $X, \hat{X}$ are the focal surfaces if and only if for each $p \in U$ the surface elements (x, n) and $(\hat{x}, \hat{n})$ are Bianchi-related.*

Proof. It is immediate from the definitions that the two focal surfaces of a pseudospherical line congruence are Bianchi-related. To prove the opposite implication, we only need to construct a pseudospherical line congruence given a pair of Bianchi-related surfaces. But this is also easy: to the point $p \in U$, associate the line

$$L(p) = x(p) + \lambda(\hat{x}(p) - x(p))$$

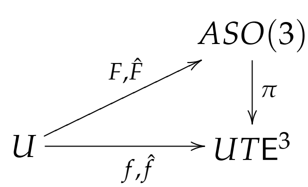
Since $x - \hat{x}$ is perpendicular to both n and $\hat{n}$, $L(p)$ is tangent to both X and $\hat{X}$ at p. Thus, $L : U \to \mathcal{L}_3$ is a line congruence with X and $\hat{X}$ as its focal surfaces. The condition that L is pseudospherical is now equivalent to the first two Bianchi relations. □

The odd appearance of "pseudospherical" in "pseudospherical line congruence" is explained by Bianchi's theorem:

Theorem 2.2 (Bianchi [2], 1879)**.** *Let $L : U \to \mathcal{L}_3$ be a pseudospherical line congruence. Then the two focal surfaces X, $\hat{X}$ have constant Gaussian curvature -1.*

Proof. We will prove this theorem in a somewhat roundabout way in order to emphasize the similarity with later proofs.

The surface elements for X and $\hat{X}$ give us a pair of maps $f, \hat{f} : U \to \mathsf{E}^3 \times S^2$. This space is the unit tangent bundle of E^3, and as a result may be thought of as the homogeneous space $UT\mathsf{E}^3 = ASO(3)/SO(2)$. Working locally, we can find framings $F, \hat{F} : U \to ASO(3)$ so that



commutes. We can also assume that F and $\hat{F}$ are adapted, so that $F(p) \cdot O = x(p)$ and $F(p) \cdot \mathsf{e}_3 = n(p)$, where O is the origin (and likewise for hatted equations).

By lemma 2.1, the maps f and $\hat{f}$ must satisfy the four Bianchi relations. It follows that there must be a map $\theta : U \to S^1$ such that $\hat{F} = F \cdot \beta_\theta$, where β_θ is a 90° unit-displacement screw motion in the direction $\cos(\theta)\mathsf{e}_1 + \sin(\theta)\mathsf{e}_2$.

Now, consider the derivatives $\omega = F^{-1}dF$ and $\hat{\omega} = \hat{F}^{-1}d\hat{F}$. In the standard matrix representation of $ASO(3)$, these have the form

$$\omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tau^1 & 0 & -\lambda & \nu^1 \\ \tau^2 & \lambda & 0 & \nu^2 \\ \alpha & -\nu^1 & -\nu^2 & 0 \end{bmatrix}, \qquad \hat{\omega} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hat{\tau}^1 & 0 & -\hat{\lambda} & \hat{\nu}^1 \\ \hat{\tau}^2 & \hat{\lambda} & 0 & \hat{\nu}^2 \\ \hat{\alpha} & -\hat{\nu}^1 & -\hat{\nu}^2 & 0 \end{bmatrix}$$

where the matrix elements are 1-forms. The adaptation conditions $F \cdot \mathsf{e}_3 = n$ and $\hat{F} \cdot \mathsf{e}_3 = \hat{n}$ are equivalent to $\alpha = \hat{\alpha} = 0$. As noted previously, the fact that f and $\hat{f}$ are Bianchi-related says that $\hat{F} = F \cdot \beta_\theta$, so we also have

$$\hat{\omega} = \beta_\theta^{-1} \cdot \omega \cdot \beta_\theta + \beta_\theta^{-1} d\beta_\theta \tag{2.1}$$

In this matrix representation, β_θ has the form

$$\beta_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos\theta & \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \sin\theta & \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ 0 & \sin\theta & -\cos\theta & 0 \end{bmatrix}$$

This means that, by using equation 2.1, we can also write $\hat{\omega} = \left[\begin{smallmatrix} 0 & 0 \\ x & \rho \end{smallmatrix}\right]$ where

$$x = \begin{bmatrix} \cos^2(\theta)\tau^1 + \cos(\theta)\sin(\theta)\tau^2 - \cos(\theta)\sin(\theta)\nu^1 + \sin^2(\theta)\nu^2 + \sin(\theta)\alpha \\ \cos(\theta)\sin(\theta)\tau^1 - \sin^2(\theta)\tau^2 + \cos^2(\theta)\nu^1 + \cos(\theta)\sin(\theta)\nu^2 - \cos(\theta)\alpha \\ \lambda - \sin(\theta)\tau^1 + \cos(\theta)\tau^2 + d\theta \end{bmatrix}$$

and ρ is a $\mathfrak{so}(3)$-valued 1-form whose contents are not important for these calculations.

Since $\hat{F}$ is assumed to be adapted, we know $\hat{\alpha} = 0$. The previous calculation gives a second form for $\hat{\alpha}$, so we have

$$0 = \hat{\alpha} = \lambda - \sin(\theta)\tau^1 + \cos(\theta)\tau^2 + d\theta$$

demonstrating that θ must be a solution of the differential equation

$$d\theta = \sin(\theta)\tau^1 - \cos(\theta)\tau^2 - \lambda \tag{2.2}$$

Take the exterior derivative of both sides to obtain:

$$0 = \cos(\theta)d\theta \wedge \tau^1 + \sin(\theta)d\tau^1 + \sin(\theta)d\theta \wedge \tau^2 - \cos(\theta)d\tau^2 - d\lambda \tag{2.3}$$

Since ω is the derivative of F, it must satisfy the Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$. This lets us replace the terms such as $d\tau^1$ with wedge products of entries in ω. In particular, modulo α we have $d\tau^1 = \lambda \wedge \tau^2$, $d\tau^2 = -\lambda \wedge \tau^1$, and $d\lambda = -\nu^1 \wedge \nu^2$. Applying these substitutions to equation 2.3 and replacing $d\theta$ terms using equation 2.2 yields the equation

$$\begin{aligned} 0 = &- \cos^2(\theta)\tau^2 \wedge \tau^1 - \cos(\theta)\lambda \wedge \tau^1 + \sin(\theta)\lambda \wedge \tau^2 \\ &+ \sin^2(\theta)\tau^1 \wedge \tau^2 - \sin(\theta)\lambda \wedge \tau^2 + \cos(\theta)\lambda \wedge \tau^1 \\ &+ \nu^1 \wedge \nu^2 \\ = &\ \tau^1 \wedge \tau^2 + \nu^1 \wedge \nu^2 \end{aligned}$$

In the more familiar terms described on page 4, if $x : U \to \mathbb{R}^3$ parameterizes X then the above equation reads $0 = dA + KdA$, where K is the Gaussian curvature of X. That is, X has constant Gaussian curvature $K = -1$.

Since the Bianchi relations and the above argument are symmetric under swapping hatted and un-hatted variables, it follows that $\hat{X}$ must also have $\hat{K} = -1$. □

2.1.2 Lie's Theorem

Bianchi's theorem tells us that there is a strong restriction on which kinds of surfaces may appear as the focal surfaces of a pseudospherical line congruence.

A natural follow-up question then presents itself: given a surface X of constant Gaussian curvature $K = -1$, does X appear as a focal surface for some pseudospherical line congruence?

This question was raised and investigated by Lie a year after the publication of Bianchi's result. Lie answered the question both *positively* and *constructively*: each surface X with $K = -1$ appears as a focal surface for some line congruence L^X, and L^X can be computed explicitly from X merely by integrating a sequence of *ordinary* differential equations.

More colloquially, if we are given a $K = -1$ surface X then it is easy to find a pseudospherical line congruence L^X which has X for a focal surface!

Theorem 2.3 (Lie [9], 1880)**.** *Let X be a surface in E^3 with $K = -1$. Then there exists a pseudospherical line congruence L^X such that X is a focal surface of L^X. Equivalently, there exists a second surface $\hat{X}$ with $\hat{K} = -1$ such that to each point of X there exists a Bianchi-related point of $\hat{X}$. Furthermore, L^X and $\hat{X}$ may be computed from X by integrating a sequence of ordinary differential equations.*

Proof. Let us reuse all notation from the proof of theorem 2.2, and assume that the $K = -1$ surface X and adapted frame $F : U \to ASO(3)$ are given. Our goal is to construct the surface $\hat{X}$ which is Bianchi-related to X by finding an adapted lift $\hat{F} : U \to ASO(3)$. As before, $\hat{F}$ will describe a surface Bianchi-related to X if and only if

$$\hat{F} = F \cdot \beta_\theta$$

for some map $\theta : U \to S^1$. And again as before, the derivative $\hat{\omega}$ of $\hat{F}$ would then be given by

$$\hat{\omega} = \beta_\theta^{-1} \cdot \omega \cdot \beta_\theta + \beta_\theta^{-1} d\beta_\theta$$

Since $\hat{\omega}$ is a gauge transformation of ω by β_θ,

$$d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0$$

and so no matter the choice of θ, by theorem 1.1 $\hat{\omega}$ may be integrated to a map $\hat{F}$. This map $\hat{F}$ might not describe a surface, however: we still need to ensure that $\hat{F}$ is adapted. We have already seen that $\hat{F}$ is adapted exactly when $\hat{\alpha} = 0$, so we must have

$$0 = \hat{\alpha} = d\theta - \sin(\theta)\tau^1 + \cos(\theta)\tau^2 + \lambda$$

This puts a restriction on θ, in the form of a differential equation. The choices of θ which result in adapted frames $\hat{F}$ are exactly the solutions to this differential equation.

We now turn our attention to the solution of this differential equation. We want to find a $\theta : U \to S^1$ which solves the first-order system of PDE

$$d\theta = \sin(\theta)\tau^1 - \cos(\theta)\tau^2 - \lambda \tag{2.4}$$

This is an overdetermined system of first-order PDE for θ, so by theorem B.3 there is a solution if and only if the equations are compatible. The compatibility condition is just

$$\begin{aligned} 0 &= d(\sin(\theta)\tau^1 - \cos(\theta)\tau^2 - \lambda) \\ &= \tau^1 \wedge \tau^2 + \nu^1 \wedge \nu^2 \\ &= 0 \end{aligned}$$

since $K = -1$ on X. So there are solutions θ to 2.4, and by theorem B.4 any solution θ can be constructed by picking an arbitrary value $\theta(p)$ at some $p \in U$ and integrating a sequence of ordinary differential equations.

Having found a solution θ to 2.4, we may form $\hat{F} = F \cdot \beta_\theta$. By construction the surface $\hat{X}$ corresponding to $\hat{F}$ is Bianchi-related to X, so by theorem 2.2 $\hat{X}$ satisfies $\hat{K} = -1$ and the line congruence connecting related points on X and $\hat{X}$ is pseudospherical. □

The construction of $\hat{X}$ from X is called the *Lie-Bäcklund transformation*.

2.1.3 Constructing Kuen's Surface

Lie's theorem gives us a way to easily create new $K = -1$ surfaces once a single one is known. Still, even finding a single $K = -1$ surface is nontrivial, as the defining PDEs for such surfaces are fundamentally nonlinear. We can construct special examples by enforcing a 1-parameter family of symmetries on our solutions; the defining PDEs are then reduced to an ODE for the profile curve.

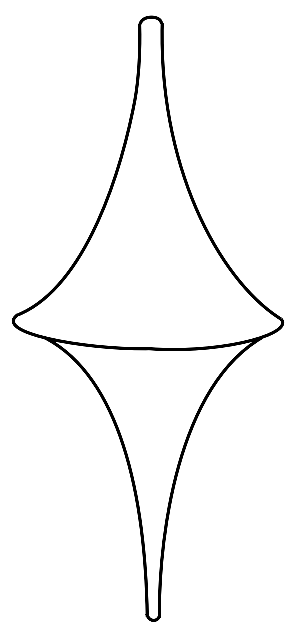


Figure 2.1: The pseudophere, a tractrix of revolution.

If we ask that our surface has a rotational symmetry, then we are led to the *pseudosphere* (figure 2.1), a surface of revolution with the tractrix as its profile curve.

The pseudosphere can be explicitly parameterized by

$$f(x,y) = \begin{bmatrix} \operatorname{sech} x \cos y \\ \operatorname{sech} x \sin y \\ x - \tanh x \end{bmatrix}$$

with unit normal

$$n(x,y) = \begin{bmatrix} -\tanh x \cos y \\ -\tanh x \sin y \\ -\operatorname{sech} x \end{bmatrix}$$

From this data we can easily compute an adapted frame $F : \mathbb{R}^2 \to ASO(3)$ over f, yielding

$$F(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \operatorname{sech} x \cos y & -\operatorname{sech} x \cos y & -\sin y & -\tanh x \cos y \\ \operatorname{sech} x \sin y & -\operatorname{sech} x \sin y & \cos y & -\tanh x \sin y \\ x - \tanh x & \tanh x & 0 & -\operatorname{sech} x \end{bmatrix}$$

The derivative of F is given by

$$\omega = F^{-1}dF = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tanh x\, dx & 0 & \operatorname{sech} x\, dy & \operatorname{sech} x\, dx \\ \operatorname{sech} x\, dy & -\operatorname{sech} x\, dy & 0 & -\tanh x\, dy \\ 0 & -\operatorname{sech} x\, dx & \tanh x\, dy & 0 \end{bmatrix} \tag{2.5}$$

We now go about the task of running F through the machinery of Lie's theorem 2.3. The goal is to construct $\hat{X}$ by finding an adapted frame $\hat{F}$ such that

$\hat{F} = F \cdot \beta_\theta$ for some map $\theta : \mathbb{R}^2 \to S^1$. As in the proofs of theorems 2.2 and 2.3, we may write the derivative of F as $\omega = F^{-1}dF$. The derivative of $\hat{F}$ must then be of the form

$$\hat{\omega} = \hat{F}^{-1}d\hat{F} = \beta_\theta^{-1}\omega\beta_\theta + \beta_\theta^{-1}d\beta_\theta$$

Since $\hat{\omega}$ is a gauge transformation of ω it also satisfies the Maurer-Cartan equation $d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0$, so we can always find an antiderivative $\hat{F} : \mathbb{R}^2 \to ASO(3)$ such that $\hat{\omega} = \hat{F}^{-1}d\hat{F}$ and, up to a Euclidean motion, such an $\hat{F}$ must also satisfy $\hat{F} = F \cdot \beta_\theta$. Thus, we only need to ensure that the lower-left matrix element of $\hat{\omega}$ is 0 so that the integral $\hat{F}$ corresponds to an adapted frame.

As in the proof of theorem 2.3, the lower-left matrix element of $\hat{\omega}$ is

$$d\theta - \sin(\theta)\tau^1 + \cos(\theta)\tau^2 + \lambda$$

Using the values for τ^i and λ in equation 2.5, we must solve the differential equation

$$0 = d\theta(x,y) - \sin(\theta(x,y))\tanh(x)\,dx + \cos(\theta(x,y))\operatorname{sech}(x)\,dy - \operatorname{sech}(x)\,dy$$

This is equivalent to the overdetermined system

$$\begin{aligned}\frac{\partial\theta}{\partial x}(x,y) &= \sin(\theta(x,y))\tanh(x)\\ \frac{\partial\theta}{\partial y}(x,y) &= (1-\cos(\theta(x,y)))\operatorname{sech}(x)\end{aligned}$$

which is guaranteed to be compatible by theorem 2.3. Following theorem B.4 we can integrate this system by picking a filtration on $\mathbb{R}^2$ and inductively integrating ODEs along lines. In this case, let us start by looking at our PDE system on the line $x = 0$. Here, the second equation becomes

$$\frac{d\theta}{dy}(0,y) = 1 - \cos(\theta(0,y))$$

This equation is separable, leading to the solution

$$\begin{aligned} y + C &= \int \frac{d\theta}{1 - \cos\theta} \\ &= \int \frac{1 + \cos\theta}{\sin^2\theta} d\theta \\ &= -\csc\theta - \cot\theta \\ &= -\cot\frac{\theta}{2} \end{aligned}$$

so we have

$$\theta(0, y) = -2\cot^{-1}(y + C) \tag{2.6}$$

Having found a solution along the y-axis, we now consider our PDE system along each line $y = \text{const}$. The first equation then becomes

$$\frac{d\theta}{dx}(x, y) = \sin(\theta(x, y))\tanh(x)$$

which is also separable. Integrating leads to the sequence of equations

$$\begin{aligned} \int \frac{d\theta}{\sin\theta} &= \int \tanh x \, dx \\ \log\tan\frac{\theta}{2} &= \log\cosh x + C(y) \\ \cot\frac{\theta}{2} &= A(y)\operatorname{sech} x \end{aligned}$$

so that in the end we have

$$\theta(x, y) = 2\cot^{-1}(A(y)\operatorname{sech} x)$$

We can determine the constant of integration $A(y)$ by comparing this equation with equation 2.6, finally leading to the general solution

$$\theta(x, y) = 2\cot^{-1}((y + C)\operatorname{sech} x)$$

Having found the function θ needed to ensure that $\hat{F}$ is adapted, we can form $\hat{F}$ by taking $F \cdot \beta_\theta$. The parameterization of $\hat{X}$ may be read off of $\hat{F}$ by looking at the first column.

When the constant $C = 0$, a simple but tedious calculation of $F \cdot \beta_\theta$ shows that the corresponding surface $\hat{X}$ is parameterized by

$$\hat{f}(x,y) = \begin{bmatrix} \frac{2}{1+u^2}(\operatorname{sech} x \cos y - u \sin y) \\ \frac{2}{1+u^2}(\operatorname{sech} x \sin y + u \cos y) \\ x - \frac{2}{1+u^2} \tanh x \end{bmatrix}, \qquad u = \operatorname{sech} x \cdot \left(\tan\left(\frac{\pi - \Theta}{2}\right) - y \right)$$

Lie's theorem guarantees that this surface satisfies $\hat{K} = -1$, which can be verified by yet another simple but tedious calculation. The resulting surface $\hat{X}$ is called *Kuen's pseudospherical surface*, and is highly nontrivial. The most interesting portion of the surface appears in figure 2.2.

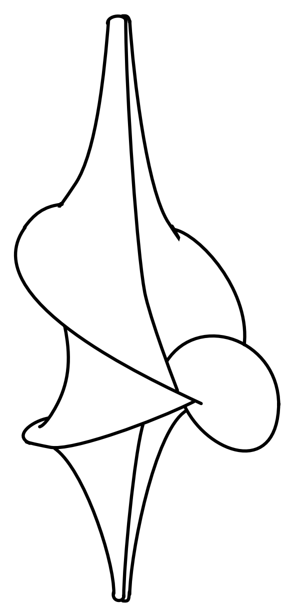


Figure 2.2: Kuen's surface, a transformation of the pseudosphere.

2.2 Interlude: the Big Picture

Before we try to generalize the theorems of Bianchi and Lie, let us step back and try to understand the bird's-eye view of the computations carried out in the previous section.

First, we began with a parameterized surface $x : X \to \mathsf{E}^3$ in Euclidean space. By considering both the parameterization x and the normal field $n : X \to S^2$, we could also think of X as a parameterized surface

$$f : X \to \mathsf{E}^3 \times S^2 \cong UT\mathsf{E}^3$$

This is a sort of Euclidean analog of "prolongation" — we have augmented our map with extra first-order data (the location of the tangent plane).

Of course, not every map $(x, n) : X \to UT\mathsf{E}^3$ will represent a prolongation of the surface $x(X)$. We are most interested in pairs where n really is the normal map of x. Symbolically, we require the *adaptation condition*

$$n(p) \perp x_* T_p X$$

The adaptation condition may be retooled using the machinery of differential forms. In particular, consider the 1-form θ on $UT\mathsf{E}^3$ given by

$$\theta_{(x,n)} = \langle n, dx \rangle$$

A pair $f = (x, n) : X \to UT\mathsf{E}^3$ will be adapted exactly when the pullback $f^*\theta$ vanishes. In other words $UT\mathsf{E}^3$ carries a hyperplane distribution (in this case, a contact structure) $\Delta = \ker \theta$, and a surface in $UT\mathsf{E}^3$ is the prolongation of a surface in E^3 exactly when it is tangent to Δ. These surfaces will be called *adapted*.

Note that for any element $(x,n) \in UT\mathsf{E}^3$ there exists a S^1-family of elements $(\hat{x},\hat{n})$ such that (x,n) is Bianchi-related to $(\hat{x},\hat{n})$. Altogether, this means that the Bianchi relations induce a S^1 bundle

$$\Sigma \xrightarrow{\pi} UT\mathsf{E}^3$$

Points of Σ are pairs $((x,n),(\hat{x},\hat{n}))$ such that (x,n) is Bianchi-related to $(\hat{x},\hat{n})$, and the bundle projection π just acts by

$$\pi\big((x,n),(\hat{x},\hat{n})\big) = (x,n)$$

From this perspective, it is clear that Σ carries a *second* bundle projection $\hat{\pi}$, acting by

$$\hat{\pi}\big((x,n),(\hat{x},\hat{n})\big) = (\hat{x},\hat{n})$$

Thus, Σ is a circle bundle over $UT\mathsf{E}^3$ in two distinct ways.

$$\begin{array}{ccc} & \Sigma & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\hat{\pi}} \\ UT\mathsf{E}^3 & & UT\mathsf{E}^3 \end{array}$$

Due to the definition of Σ, if we have a 2-dimensional surface $\tilde{X}$ in Σ which is transverse to both π and $\hat{\pi}$ then the two projections $\pi(\tilde{X})$ and $\hat{\pi}(\tilde{X})$ give a pair of Bianchi-related surfaces in $UT\mathsf{E}^3$. The converse also holds: to any pair of Bianchi-related surfaces $X, \hat{X}$ there exists a surface $\tilde{X}$ in Σ which simultaneously lifts them both.

Now suppose that $\tilde{X}$ is a 2-dimensional surface in Σ which is transverse to the fibers of π. Then the projection $\pi(\tilde{X})$ is a well-defined surface in $UT\mathsf{E}^3$, so it is natural to ask "when is the projection $\pi(\tilde{X})$ adapted?" As we saw above, a surface is adapted if and only if it is tangent to the hyperplane distribution Δ. This means that $\tilde{X}$ will project to an adapted surface exactly when $\tilde{X}$ is tangent to the distribution $\pi^{-1}\Delta$ on Σ. Likewise, if $\tilde{X}$ is transverse to $\hat{\pi}$ then it will project to an adapted surface by $\hat{\pi}$ exactly when it is tangent to $\hat{\pi}^{-1}\Delta$.

An integral manifold of $\pi^{-1}\Delta$ is an arbitrary section of π over an adapted surface in $UT\mathsf{E}^3$; that is, it is a surface in $UT\mathsf{E}^3$ along with a choice of Bianchi-related surface elements at each point. The analogous statement holds for $\hat{\pi}^{-1}\Delta$ as well.

We can combine these two distributions to form the intersection $\tilde{\Delta} = \pi^{-1}\Delta \cap \hat{\pi}^{-1}\Delta$ — a 4-dimensional distribution on the 6-dimensional space Σ. A surface $\tilde{X}$ transverse to both projections is tangent to $\tilde{\Delta}$ if and only if it projects to an adapted surface by *both* π and $\hat{\pi}$.

A priori, there is no reason to expect that $\tilde{\Delta}$ admits *any* integral surfaces. For example, instead of the Bianchi relations we could use the the four superficially similar relations

$$|x - \hat{x}| = 1, \quad \langle n, \hat{n} \rangle = 0, \quad \langle n, x - \hat{x} \rangle = -1, \quad \langle \hat{n}, x - \hat{x} \rangle = 0$$

Just as for the Bianchi relations, there is a S^1-family of surface elements related to any given point. As above, we can construct the double bundle Σ and the 4-dimensional distribution $\tilde{\Delta}$ on Σ. But thinking about these relations geometrically, it is clear that at most only one of (x, n) or $(\hat{x}, \hat{n})$ can be adapted. Thus, the distribution $\tilde{\Delta}$ must not admit any integral surfaces. This demonstrates that there is something special about the Bianchi relations which allows the Bianchi and Lie machines to operate. The special condition needed will be discussed in section 2.4.

So what, exactly, goes wrong with the above relations that goes so right with the Bianchi relations? In the Bianchi case, we have adapted surfaces X in $UT\mathsf{E}^3$

which admit a lift $\tilde{X}$ to Σ such that $\tilde{X}$ is tangent to $\tilde{\Delta}$ — the same cannot be said for the relations given in the last paragraph. The second focal surface $\hat{X}$ was then constructed by computing the lift $\tilde{X}$ of the adapted surface X and projecting down by $\hat{\pi}$:

$$\hat{X} = \hat{\pi}(\tilde{X})$$

Of course, it isn't true that just *any* adapted surface admits a lift to Σ. This is essentially the content of Bianchi's theorem: we can only lift an adapted surface to Σ if the surface has constant Gaussian curvature $K = -1$. To generalize Bianchi's theorem in section 2.4, we will need to develop a method for finding the additional geometric conditions (analogous to $K = -1$) on the adapted surfaces such that they will admit lifts to Σ.

2.3 Relations on Homogeneous Spaces

In order to generalize Bianchi and Lie's theorems and the Lie-Bäcklund transformation to other geometries, we will have to drop the notion of line congruence. In its place, we will use a generalization of Bianchi's four relations to arbitrary homogeneous spaces.

2.3.1 Invariant Relations

Recall that the relations $\sim$ on a set X may be put into one-to-one correspondence with the subsets $R_\sim \subset X \times X$ by declaring $(x, \hat{x}) \in R_\sim$ if and only if $x \sim \hat{x}$.

Let $M = G/H$ be a homogeneous space. A relation $\sim$ on M will be called *geometric* when the set $R_\sim \subset M \times M$ is invariant under the diagonal action of G. More explicitly, the relation $\sim$ is geometric when

$$\forall g \in G \qquad x \sim \hat{x} \implies g \cdot x \sim g \cdot \hat{x}$$

Example 2.1. Let M be any one of E^n, S^n, or H^n. In each of these cases, the group of geometric motions acts transitively on the set of geodesic line segments of a fixed length ℓ. Thus, the only invariant relations are of the form $x \sim \hat{x} \iff d(x, \hat{x}) = \ell$.

Example 2.2. Let $UT\mathsf{E}^3 = ASO(3)/SO(2)$ be the unit tangent bundle of E^3, so that a point of $UT\mathsf{E}^3$ is given by a pair $(x, n) \in \mathsf{E}^3 \times S^2$. Then each of the four Bianchi relations $|x - \hat{x}| = 1$, $n \perp \hat{n}$, $n \perp x - \hat{x}$, and $\hat{n} \perp x - \hat{x}$ is invariant under any Euclidean motion $g \in ASO(3)$. This demonstrates that the Bianchi relations are geometric for the Euclidean group.

Similar relations are available on $UT\mathsf{H}^3$ and $UT\mathsf{S}^3$ when the vector $x - \hat{x}$ is replaced with a geodesic connecting x to $\hat{x}$. These will be explored in more depth in chapter 3.

Example 2.3. Let $\mathbb{P}^1 = PSL(2, \mathbb{C})/H$ be the Möbius sphere. The group $PSL(2, \mathbb{C})$ acts 3-transitively on $\mathbb{P}^1$, so for any triple x, y, z there is a transformation $g \in PSL(2, \mathbb{C})$ such that

$$g \cdot x = x, \qquad g \cdot y = z$$

But if $\sim$ is a geometric relation, then we must have $x \sim y \iff g \cdot x \sim g \cdot y$. In particular, using the g above we would have $x \sim y \iff x \sim z$, so the only geometric relation on $\mathbb{P}^1$ is equality.

The 3-transitivity of the Möbius group makes $\mathbb{P}^1$ too floppy to support nontrivial geometric relations. However, there is a higher relation on $\mathbb{P}^1$ given by the cross-ratio of four points. This determines a generalized geometric relation corresponding to a $PSL(2,\mathbb{C})$-invariant subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A higher arity relation analogous to the cross-ratio will become important in chapter 4.

To classify the geometric relations on a given homogeneous space, it is helpful to know which relations are atomic, in the sense that all other relations are built from the atomic ones by unions.

Definition 2.3. A relation $\sim$ is *atomic* on $M = G/H$ if the corresponding subset $R_\sim \subset M \times M$ is the G-orbit of a single point $(x_0, \hat{x}_0)$.

Example 2.4. To illustrate the meaning of this definition and bring up some terminological complications, let us look at the example of $UT\mathsf{E}^3$ again. The atomic relations are all of the form $R(\ell, \alpha, \psi, \phi)$ where

$$\begin{array}{cc} |x - \hat{x}| = \ell & \langle n, \hat{n} \rangle = \cos\alpha \\ \langle n, x - \hat{x} \rangle = \ell \cos\psi & \langle \hat{n}, x - \hat{x} \rangle = \ell \cos\phi \end{array}$$

Other relations can be obtained by taking unions of subsets corresponding to various atomic relations. For example, the relation $n \perp \hat{n}$ corresponds to the union of the 3-dimensional family of relations $R(\star, 0, \star, \star)$ where each $\star$ is arbitrary.

This illustrates how the name *atomic* might be somewhat confusing; one relation probably seems more atomic than four. The idea is that atomic relations correspond to *minimal* G-invariant subsets of $M \times M$, which themselves correspond to *maximally determined* relations.

Lemma 2.4. *Let $\mathcal{R}_M$ denote the space of atomic invariant relations on a homogeneous space M with structure group G. Then $\mathcal{R}_M$ is isomorphic to the double coset space*

$$\mathcal{R}_M = H \backslash G / H$$

where H is the stabilizer of some point $x_0 \in M$.

Proof. Suppose $\sim$ is any atomic invariant relation, so that $R_\sim$ is the orbit of a single point (x, y) in $M \times M$. Fix a marked point $x_0 \in M$ such that H is the stabilizer of x_0. Since G acts transitively on M, the set $R_\sim$ contains points of the form (x_0, y). Any such point is of the form $(x_0, h \cdot y)$ for some $h \in H$ since H leaves x_0 fixed. This gives us a bijection between $\mathcal{R}_M$ and the left orbit space $H \backslash M$. Using the marked point $x_0 \in M$ we can write $M = G/H$ to obtain the desired result. □

Lemma 2.4 allows us to easily write down a complete set of atomic invariant relations for a given geometry. In the case of $UT\mathsf{E}^3$, the atomic relations $\mathcal{R}_{UT\mathsf{E}^3}$ are points of the double quotient $SO(2) \backslash ASO(3)/SO(2)$. By adjusting the $SO(2)$ multiples on the right and left, any atomic invariant relation may be represented by a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ X & 1 & 0 & 0 \\ Y & 0 & \cos\varphi & -\sin\varphi \\ Z & 0 & \sin\varphi & \cos\varphi \end{bmatrix} \tag{2.7}$$

which corresponds to the four relations

$$\begin{gathered} |x - \hat{x}|^2 = X^2 + Y^2 + Z^2 \qquad\qquad \langle n, \hat{n} \rangle = \cos\varphi \\ \langle n, x - \hat{x} \rangle = -Z \qquad \langle \hat{n}, x - \hat{x} \rangle = Y \sin\varphi - Z \cos\varphi \end{gathered} \tag{2.8}$$

Thus, we can immediately see that the four Bianchi relations have a natural provenance — they represent a single atomic invariant relation for $UT\mathsf{E}^3$.

2.3.2 The Double Bundle Induced by a Relation

Let us make a first attempt at mimicking Bianchi's theorem in the context of a general homogeneous space M. To replace the Bianchi relations, we must pick some atomic invariant relation $\sim$ in $\mathcal{R}_M$. Let us represent $\sim$ with the double-coset equivalence class of some transformation $\rho \in G$. From the definition of $\mathcal{R}_M$, we can immediately deduce the following lemma:

Lemma 2.5. *Suppose that x and $\hat{x}$ are two elements of M. We will say that a geometric motion $F \in G$ frames x relative to $\star$ if*

$$x = F \cdot \star$$

Then $x \sim \hat{x}$ if and only if there exist $F, \hat{F} \in G$ framing $x, \hat{x}$ (resp.) relative to $\star$ such that

$$\hat{F} = F \cdot \rho$$

A useful variant occurs when we take the point $\hat{x}$ to be $\star$. In this case, we obtain:

Lemma 2.6. *Suppose that $x \in M$ is such that $x \sim \star$. Then x is framed by a transformation of the form $h \cdot \rho$ with $h \in H$, unique up to right multiples of H. Furthermore, every point $\sim$-related to $\star$ is of the form $h \cdot \rho$ for some h.*

In particular, these lemmas may be interpreted as saying that the set of $\hat{x} \in M$ which are $\sim$-related to x carries a transitive action by the Lie group $\mathrm{stab}_G\, x \cong H$. Thus, to each atomic invariant relation $\sim$ there exists a bundle $\Sigma_\sim \to M$ of homogeneous spaces over M, where the principal group of each fiber is a conjugate of H. The fiber of $\Sigma_\sim$ over a point $x \in M$ is the set of all $\hat{x} \in M$ which are $\sim$-related to x.

Example 2.5. Suppose M is one of the space forms E^n, S^n, or H^n. We previously saw that the only atomic invariant relation is geodesic distance. If $\sim_\ell$ is the relation "$x \sim y$ iff $d(x,y) = \ell$" then the bundle $\Sigma_{\sim_\ell} \to M$ is a bundle of $(n-1)$-spheres over M. The fiber over a point $x \in M$ may be thought of as those points of M at a distance ℓ from x.

Example 2.6. Again, let us consider the homogeneous space $UT\mathsf{E}^3$. To a given surface element (x, n), there is a S^1 family of elements $(\hat{x}, \hat{n})$ which are Bianchi-related to (x, n). The group $SO(2)$ acts transitively and freely on these surface elements. As a result, the bundle $\Sigma_{\text{Bianchi}} \to UT\mathsf{E}^3$ is a circle bundle over $UT\mathsf{E}^3$.

2.4 Generalizing the Bianchi and Lie Theorems

In this section, we will answer the question: what is so special about the Bianchi relations which allows the Bianchi and Lie theorems to operate?

Throughout this section, we will suppose that M is a homogeneous space isomorphic to G/H, equipped with a geometric exterior differential system Θ differentially generated by a set $\{\theta_i\}_{i=1}^k$ of elements of $\mathfrak{g}^*$.

For any element $\rho \in G$, let us write $[\rho]$ for the equivalence class of ρ in the double-coset space $\mathcal{R}_M = H \backslash G / H$. We know that $[\rho]$ corresponds to some atomic invariant relation on M; we will write $\sim_\rho$ for this relation.

From now on we will focus on *symmetric* relations. The following lemma characterizes those elements of G giving rise to symmetric atomic invariant relations.

Lemma 2.7. *There is a natural involution on $\mathcal{R}_M$ given by sending the equivalence* *class* $[g]$ *to* $[g^{-1}]$*. The fixed points of this involution are exactly the symmetric atomic invariant relations, which we will denote by* $\mathcal{R}_M^{sym}$*.*

Proof. Let $[g] \in \mathcal{R}_M$ be given. Then $x \sim_g \tilde{g} \cdot x$ if and only if $[g] = [\tilde{g}]$. But

$$\tilde{g} \cdot x \sim_g x \quad \text{iff} \quad x \sim_g \tilde{g}^{-1} \cdot x$$

by G-invariance of $\sim_g$, so we also have $[g] = [\tilde{g}^{-1}]$. □

The following lemma will be occasionally useful for producing symmetric relations in homogeneous spaces where the geometric interpretation of the atomic relations is less clear or less useful for finding explicit transformations ρ to represent the relation.

Lemma 2.8 (Symmetric Relations and $\sqrt{H}$)**.** *Suppose that* $[\rho] \in \mathcal{R}_M^{sym}$ *is a symmetric invariant relation on the homogeneous space* $M = G/H$*. Then there is some* $\hat{\rho} \in G$ *such that* $[\hat{\rho}] = [\rho]$ *and* $\hat{\rho}^2 \in H$*. Conversely, if* $g \in G$ *is such that* $g^2 \in H$ *then* $[g]$ *is a symmetric invariant relation. Thus, the set* $\sqrt{H}$ *intersects each equivalence class of* $\mathcal{R}_M^{sym}$*.*

Proof. Suppose $\rho \in G$ represents some relation $[\rho] \in \mathcal{R}_M^{sym}$. Then there exist $h, h' \in H$ such that $h'\rho h = \rho^{-1}$. Left multiplying by h^{-1} gives $h'\rho = (h\rho)^{-1}$. Now set $\hat{\rho} = h\rho$; the previous equation then reads $h'h^{-1} \cdot \hat{\rho} = \hat{\rho}^{-1}$, so that $\hat{\rho}^2 = (h'h^{-1})^{-1} \in H$. This proves the first claim. The second claim is almost immediate: if $g^2 = h \in H$ then $g \cdot h^{-1} = g^{-1}$, so $[g] = [g^{-1}]$. □

Now let $[\rho]$ be a symmetric atomic invariant relation. From the previous section, we saw that $[\rho]$ induces a double bundle

$$\begin{array}{ccc} & \Sigma_\rho & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \hat{\pi}} \\ M & & M \end{array}$$

where the total space of Σ_ρ consists of pairs $(x, \hat{x}) \in M \times M$ such that

$$x \sim_\rho \hat{x}$$

and the bundle projections $\pi, \hat{\pi}$ are the projections onto the first and second factor, respectively.

Lemma 2.9. *Σ_ρ is a homogeneous space with structure group G. If $H_\rho \subset H$ is the stabilizer of the pair $(\star, \rho \cdot \star) \in M \times M$ then we can write*

$$\Sigma_\rho = G/H_\rho$$

The two projections π and $\hat{\pi}$ from Σ_ρ to M are G-equivariant maps, so Σ_ρ is actually a double bundle in the category of G-spaces.

Proof. That Σ_ρ is a homogeneous space with structure group G follows immediately from the fact that the set of $[\rho]$-related pairs $(x, \hat{x})$ is invariant under the diagonal action of G on $M \times M$. The G-equivariance of the projections is immediate from the definitions. □

Corollary 2.10. *Using the notation of the previous lemma, H_ρ must consist of those elements of H which commute with ρ.*

By assumption, each "downstairs" copy of M is carrying a gEDS Θ differentially generated by 1-forms. By analogy with the UTE^3 case, we will call integral manifolds of Θ *adapted*.

Since both M and Σ_ρ are homogeneous spaces with the same structure group G, a gEDS on either space is described by an invariant subspace

$$\Omega \subset \mathfrak{g}^*$$

To be a geometric EDS for M, the differential ideal must be invariant under the coadjoint action of H; for Σ, it must be invariant under the coadjoint action of H_ρ.

Definition 2.4. Let V be a vector space, X a subspace of V, and Y a subspace of V^*. Y is called a *transversal* with respect to X if the evaluation pairing

$$X \otimes Y \to \mathbb{R}$$

is non-degenerate.

This definition of transversal is closely related to the usual one: two subspaces $X, Y \subset W$ are transversal in the usual sense if and only if X and $Y^\perp$ are transversal in the sense given above.

Definition 2.5. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra, and $V \subset \mathfrak{g}^*$ a transversal to $\mathfrak{h}$. Let $\{\xi_i\}$ be a basis for $\mathfrak{h}$ and $\{\varphi^i\}$ the dual basis of V induced by the pairing. Then we will define the *relative curvature operator* $\Delta_V : V \to \bigwedge^2 \mathfrak{g}^*/V$ by

$$\Delta_V(\psi) = \delta\psi - \varphi^j \wedge (\xi_j \lrcorner \delta\psi) - \psi\left([\xi_j, \xi_k]\right)\ \varphi^j \wedge \varphi^k \mod V$$

Definition 2.6. Let $[\rho] \in \mathcal{R}_M^{sym}$ be a symmetric atomic invariant relation for the homogeneous space $M = G/H$, and assume M is equipped with a gEDS Θ. We will call $[\rho]$ *admissible* with respect to a subspace $V \subset \mathfrak{g}^*$ if the following conditions hold:

1. The conjugate of V by ρ^{-1} is a transversal to $\mathfrak{h}$, so the pairing

$$\mathfrak{h} \otimes \mathrm{Ad}^*(\rho^{-1})V \to \mathbb{R}$$

is non-degenerate.

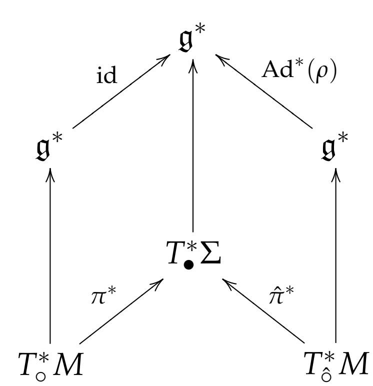
2. The Lie derivatives of the relative curvature operator $\Delta_{\mathrm{Ad}^*(\rho^{-1})V}$ vanish for each $\xi \in \mathfrak{h}$:

$$\mathcal{L}_{\mathfrak{h}} \Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0 \mod \Theta$$

Theorem 2.11 (Generalized Bianchi Theorem)**.** *Let M be a homogenous space with structure group G, $\star \in M$, H the stabilizer of $\star$, and Θ a Pfaffian gEDS on M generated by 1-forms $V \subset \mathfrak{g}^*$. Let $[\rho] \in \mathcal{R}_M^{sym}$ be an admissible relation with respect to V. Then $f, \hat{f} : X \to M$ are Θ-adapted and $\sim_\rho$-related if and only if they satisfy the differential equations*

$$f^* \Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0, \quad \hat{f}^* \Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0$$

Proof. Let Σ be the double bundle associated to the relation $[\rho]$, with bundle projections π and $\hat{\pi}$. Σ carries two geometric exterior differential systems, $\Omega = \pi^*\Theta$ and $\hat{\Omega} = \hat{\pi}^*\Theta$. Let $\tilde{\Omega}$ be the gEDS differentially generated by $\Omega \cup \hat{\Omega}$ on Σ. Since Ω is Pfaffian, it follows that $\tilde{\Omega}$ is as well. Let us look at the generating 1-forms of $\tilde{\Omega}$ in terms of the structure group G of Σ. By choosing a marked point $\circ \in \Sigma$ and defining $\circ = \pi(\bullet), \hat{\circ} = \hat{\pi}(\bullet)$, we get inclusions $T^*_\circ M \to \mathfrak{g}^*$, etc. Altogether, we get a whole commutative diagram of inclusions



Following the 1-forms $V \subset \mathfrak{g}^*$ which generate Θ around the left and right sides of the diagram, we see that $\tilde{\Omega}$ is differentially generated by $\tilde{V} = V \cup \mathrm{Ad}^*(\rho^{-1})(V)$. Stated more concretely, let $\tilde{F} : X \to G$ and define $\tilde{f} = \tilde{F} \cdot \bullet$, so that $f = \pi(\tilde{f}) = \tilde{F} \cdot \circ$ and $\hat{f} = \hat{\pi}(\tilde{f}) = \tilde{F} \cdot \hat{\circ}$. Then if f and $\hat{f}$ are to be a pair of Θ-adapted, $\sim_\rho$-related maps we must necessarily have

$$V(\tilde{F}^{-1}d\tilde{F}) = V(\rho^{-1}\tilde{F}^{-1}d\tilde{F}\rho) = 0$$

or, equivalently, $\tilde{V}(\tilde{F}^{-1}d\tilde{F}) = 0$.

Now, suppose that f is a Θ-adapted surface. We will try to understand under what conditions a $\sim_\rho$-related surface may be found. Let $F : X \to G$ be any framing of f and set $\omega = F^{-1}dF$. Any surface which is $\sim_\rho$-related to f must have a framing of the form $\hat{F} = F \cdot h$ for some $h : X \to H$, so the corresponding derivative $\hat{\omega} = \hat{F}^{-1}d\hat{F}$ is the conjugate of a gauge transformation of ω:

$$\hat{\omega} = \rho^{-1}\left(h^{-1}\omega h + h^{-1}dh\right)\rho$$

$\hat{\omega}$ is integrable and the resulting map $\hat{f}$ is $\sim_\rho$-related to f by construction, so the only issue is whether $\hat{f}$ is also Θ-adapted. Since Θ is differentially generated by V, $\hat{f}$ will be adapted so long as $V(\hat{\omega}) = 0$. But note that

$$V(\hat{\omega}) = \mathrm{Ad}^*(\rho^{-1})(V)(h^{-1}\omega h + h^{-1}dh)$$

In other words, there exists an adapted and related surface $\hat{f}$ to f if and only if there exists a gauge transformation

$$\tilde{\omega} = h^{-1}\omega h + h^{-1}dh$$

of ω such that

$$\tilde{V}(\tilde{\omega}) = 0$$

The integral of $\tilde{\omega}$ then yields the desired frame $\tilde{F}$ discussed in the previous paragraph.

This reduces the proof to finding necessary and sufficient conditions which ensure that such a gauge transformation can be found. Throughout, we maintain the notation $\omega = F^{-1}dF$ for $F : X \to G$. We will proceed by defining a (non-geometric) exterior differential system whose integral manifolds are the maps $h : X \to H$ which realize the required gauge transformations. The integrability of this EDS will have an obstruction; this obstruction will vanish when f annihilates the relative curvature operator $\Delta_{\mathrm{Ad}^*(\rho^{-1})V}$.

Since $\omega = F^{-1}dF$, it satisfies the Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$. Define $Y = X \times H$; we equip Y with the exterior differential system Λ differentially generated by the 1-forms

$$\varphi\left(h^{-1}\omega h + h^{-1}dh\right), \qquad \varphi \in \mathrm{Ad}^*(\rho^{-1})V$$

Note the following critical property: an integral manifold of Λ which is transverse to the projection $Y \to X$ is the graph of a map $h : X \to H$ such that the antiderivative of $h^{-1}\omega h + h^{-1}dh$ is an integral manifold of $\tilde{\Omega}$. The relationships between all these differential systems is depicted schematically in figure 2.3

We now find local conditions such that Λ admits integral manifolds through some point $(x, h) \in X \times H$. Fix a basis $\{\xi_i\}$ of $\mathfrak{h}$. By transversality of $\mathrm{Ad}^*(\rho^{-1})V$, there is a unique dual basis $\{\varphi^i\}$ of $\mathrm{Ad}^*(\rho^{-1})V$. This induces a basis

$$\psi^i = \varphi^i\left(h^{-1}\omega h + h^{-1}dh\right)$$

of the 1-forms in Λ. Since φ^i is dual to ξ_i, we have the relation

$$h^{-1}dh = -\varphi^i(h^{-1}\omega h)\xi_i \mod \Lambda$$

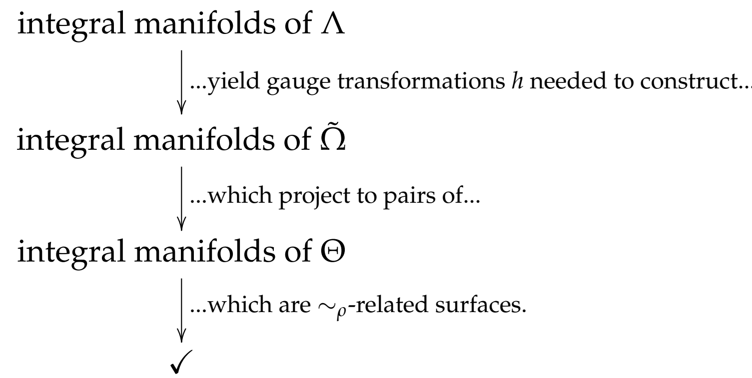


Figure 2.3: The relationship between the exterior differential systems appearing in theorems 2.11 and 2.12.

The exterior derivative of ψ^i at (x, h) is given by

$$\begin{aligned} d\psi^i &= d\left(\varphi^i\left(h^{-1}\omega h + h^{-1}dh\right)\right) \\ &= \varphi^i\left(-\left[h^{-1}dh \wedge h^{-1}\omega h\right] + h^{-1}d\omega h - h^{-1}dh \wedge h^{-1}dh\right) \\ &\overset{\omega}{=} \varphi^i\left(-\left[h^{-1}dh \wedge h^{-1}\omega h\right] - h^{-1}\omega h \wedge h^{-1}\omega h - h^{-1}dh \wedge h^{-1}dh\right) \\ &\overset{\Lambda}{=} \varphi^i\Big(\left[\varphi^j(h^{-1}\omega h)\xi_j \wedge h^{-1}\omega h\right] - h^{-1}\omega h \wedge h^{-1}\omega h \\ &\qquad\qquad - \left[\varphi^j(h^{-1}\omega h)\xi_j \wedge \varphi^k(h^{-1}\omega h)\xi_k\right]\Big) \\ &= \left(\delta\varphi^i - \varphi^j \wedge (\xi_j \lrcorner\, \delta\varphi^i) - \varphi^i\left([\xi_j, \xi_k]\right)\varphi^j \wedge \varphi^k\right)(h^{-1}\omega h \wedge h^{-1}\omega h) \end{aligned}$$

where $\overset{\omega}{=}$ and $\overset{\Lambda}{=}$ are equality modulo $d\omega + \omega \wedge \omega$ and Λ, respectively.

The form ω can, when evaluated at x, be any element of $\mathrm{Hom}(T_x X, \mathfrak{g})$. Combined with the above calculation, we find that that $d\psi^i = 0 \mod \Lambda, d\omega + \omega \wedge \omega$ at (x, h) if and only if the quantity

$$\mathrm{Ad}^*(h^{-1})\left(\delta\varphi^i - \varphi^j \wedge (\xi_j \lrcorner\, \delta\varphi^i) + \varphi^i\left([\xi_j, \xi_k]\right)\varphi^j \wedge \varphi^k\right) \tag{2.9}$$

vanishes.

We can interpret the expression 2.9 as an H-family of 2-forms which f must annihilate in order to admit a $\sim_\rho$-related $\hat{f}$. To ensure that we get the same 2-forms for each $h \in H$, the Lie derivative of expression 2.9 should vanish for each $\xi \in \mathfrak{h}$. But since H is connected, it suffices to check that the Lie derivatives vanish when $h = 1$. At $h = 1$, expression 2.9 is equal to $\Delta_{\mathrm{Ad}^*(\rho^{-1})V}$. Since $\sim_\rho$ is admissible, the Lie derivatives all vanish.

Altogether, this shows that if f and $\hat{f}$ are Θ-adapted and $\sim_\rho$-related, then

$$f^*\Delta_{\mathrm{Ad}^*(\rho^{-1})V} = \hat{f}^*\Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0$$

which completes the proof. □

Theorem 2.12 (Generalized Lie Theorem). *In the terminology of theorem 2.11, suppose that $f : U \to M$ is Θ-adapted and satisfies the differential equations Δ. Then there exists a map $\hat{f} : U \to M$ such that $f \sim_\rho \hat{f}$, $\hat{f}$ is Θ-adapted, and $\hat{f}$ also solves the differential equations Δ. Furthermore, $\hat{f}$ may be constructed from f by integrating a sequence of ordinary differential equations.*

Proof. The proof of the generalized Bianchi theorem also yields a proof of the generalized Lie theorem. Given a single Θ-adapted surface f with framing F which satisfies the differential equation $f^*\Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0$, we saw that the exterior differential system Λ satisfies the Frobenius condition $d\Lambda = 0 \mod \Lambda$. By lemma B.4 Λ admits $(\dim X)$-dimensional integral manifolds through any point, which may be constructed by integrating a sequence of ordinary differential equations. These integral manifolds give a gauge transformation of F which frames an integral manifold $\tilde{f}$ of $\tilde{\Omega}$; by construction the $\hat{\pi}$ projection of $\tilde{f}$ is a Θ-adapted map $\hat{f}$ which is $\sim_\rho$-related to f. Since $\sim_\rho$ is a symmetric relation, $\hat{f}$ must also satisfy the differential equation $\hat{f}^*\Delta_{\mathrm{Ad}^*(\rho^{-1})V} = 0$. □

Example 2.7. To relate the theorem to the classical Lie-Bäcklund transformation, take $M = UT\mathsf{E}^3 = ASO(3)/SO(2)$ and Θ the gEDS generated by the contact form $\vartheta = \langle n, dx \rangle$. Translating to the Lie coalgebra $\mathfrak{aso}(3)^*$, Θ is generated by the single 1-form e^3.

Comparing the Bianchi relations to equation 2.8 and their representing transformation 2.7, we see that the Bianchi relations may be represented by the transformation

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The transformation ρ acts on $\mathfrak{aso}(3)^*$ by pre-composition with the adjoint action, giving

$$\mathrm{Ad}^*(\rho^{-1}) \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathsf{e}^1 & 0 & -\mathsf{e}^2_1 & -\mathsf{e}^3_1 \\ \mathsf{e}^2 & \mathsf{e}^2_1 & 0 & -\mathsf{e}^3_2 \\ \mathsf{e}^3 & \mathsf{e}^3_1 & \mathsf{e}^3_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathsf{e}^1 & 0 & -\mathsf{e}^3_1 & \mathsf{e}^2_1 \\ \mathsf{e}^3 + \mathsf{e}^3_1 & \mathsf{e}^3_1 & 0 & -\mathsf{e}^3_2 \\ -\mathsf{e}^2_1 - \mathsf{e}^2 & -\mathsf{e}^2_1 & \mathsf{e}^3_2 & 0 \end{bmatrix}$$

Note in particular that $\mathrm{Ad}^*(\rho^{-1})\mathsf{e}^3 = -\mathsf{e}^2_1 - \mathsf{e}^2$. This is transverse to $\mathfrak{h}$, and the 1-form $\psi = \mathsf{e}^2_1 + \mathsf{e}^2$ is dual to the generator e^2_1 of $\mathfrak{h}$. In the notation of the generalized Bianchi and Lie theorems, the subspace $V = \mathrm{span}\,\{\mathsf{e}^3\}$ and $\mathrm{Ad}^*(\rho^{-1})V = \mathrm{span}\,\{\psi\}$.

Now, let us compute the relative curvature operator for the subspace $\mathrm{Ad}^*(\rho^{-1})V$. First, note that $\delta\mathsf{e}^2_1 = -\mathsf{e}^3_1 \wedge \mathsf{e}^3_2$ and $\delta\mathsf{e}^2 = -\mathsf{e}^2_1 \wedge \mathsf{e}^1 + \mathsf{e}^3_2 \wedge \mathsf{e}^3$. Since $\mathfrak{h}$ is 1-dimensional, the relative curvature operator has the particularly simple form

$$\begin{aligned}
\Delta_{-\mathsf{e}_1^2-\mathsf{e}^2}(\psi) &= -\delta(\mathsf{e}_1^2+\mathsf{e}^2) - (\mathsf{e}_1^2+\mathsf{e}^2)\wedge(\mathsf{e}_1^2 \lrcorner\, \delta(\mathsf{e}_1^2+\mathsf{e}^2)) \\
&= -\mathsf{e}_1^3\wedge\mathsf{e}_2^3 - \mathsf{e}_1^2\wedge\mathsf{e}^1 + \mathsf{e}_2^3\wedge\mathsf{e}^3 \\
&\qquad + (\mathsf{e}_1^2+\mathsf{e}^2)\wedge\Big(\mathsf{e}_1^2 \lrcorner\, (\mathsf{e}_1^3\wedge\mathsf{e}_2^3+\mathsf{e}_1^2\wedge\mathsf{e}^1-\mathsf{e}_2^3\wedge\mathsf{e}^3)\Big) \\
&= -\mathsf{e}_1^3\wedge\mathsf{e}_2^3 - \mathsf{e}_1^2\wedge\mathsf{e}^1 + \mathsf{e}_2^3\wedge\mathsf{e}^3 + (\mathsf{e}_1^2+\mathsf{e}^2)\wedge\mathsf{e}^1 \\
&= -\mathsf{e}_1^3\wedge\mathsf{e}_2^3 - \mathsf{e}_1^2\wedge\mathsf{e}^1 + \mathsf{e}_2^3\wedge\mathsf{e}^3 + \mathsf{e}_1^2\wedge\mathsf{e}^1 + \mathsf{e}^2\wedge\mathsf{e}^1 \\
&= -\mathsf{e}_1^3\wedge\mathsf{e}_2^3 - \mathsf{e}^1\wedge\mathsf{e}^2 + \mathsf{e}_2^3\wedge\mathsf{e}^3 \\
&= -\mathsf{e}_1^3\wedge\mathsf{e}_2^3 - \mathsf{e}^1\wedge\mathsf{e}^2 \mod \Theta
\end{aligned}$$

Is this relation admissible? To find out, we need to check that the Lie derivative $\mathcal{L}_{\mathsf{e}_1^2}\Delta_{-\mathsf{e}_1^2-\mathsf{e}^2}$ vanishes. But this is straightforward: the Lie derivatives of the component 1-forms act by 90° rotations, giving $\mathcal{L}_{\mathsf{e}_1^2}\mathsf{e}^1 = \mathsf{e}_1^2 \lrcorner\, \delta\mathsf{e}^1 = \mathsf{e}^2$, $\mathcal{L}_{\mathsf{e}_1^2}\mathsf{e}^2 = -\mathsf{e}^1$, $\mathcal{L}_{\mathsf{e}_1^2}\mathsf{e}_1^3 = \mathsf{e}_2^3$, and $\mathcal{L}_{\mathsf{e}_1^2}\mathsf{e}_2^3 = -\mathsf{e}_1^3$. Along with the fact that the Lie derivative acts as a derivation on forms so that $\mathcal{L}(\alpha\wedge\beta) = \mathcal{L}\alpha\wedge\beta + \alpha\wedge\mathcal{L}\beta$, this means that the Lie derivative of the relative curvature is zero. Altogether, this proves that $[\rho]$ is an admissible relation.

Since ρ is admissible the generalized Bianchi and Lie theorems apply when the maps $f, \hat{f}$ satisfy the differential equation $f^*\Delta_{-\mathsf{e}_1^2-\mathsf{e}^2} = 0$. Translating back to the language of $UT\mathsf{E}^3$, a map $f = (x, n)$ satisfies

$$f^*\Delta_{-\mathsf{e}_1^2-\mathsf{e}^2} = 0 \qquad \text{if and only if} \qquad |dx|^2 + |dn|^2 = 0$$

The Gaussian curvature may be characterized as the ratio $K = |dn|^2/|dx|^2$, so we have recovered the classical theorems of Bianchi and Lie.

2.4.1 Admissible Relations in $UT\mathsf{E}^3$

In this section, we will determine the space of admissible relations in $UT\mathsf{E}^3$ for the standard contact system Θ generated by e^3.

We already saw in equation 2.8 that every atomic invariant relation for $UT\mathsf{E}^3$ is of the form

$$|x-\hat{x}|^2 = X^2+Y^2+Z^2 \qquad \langle n,\hat{n}\rangle = \cos\varphi$$
$$\langle n, x-\hat{x}\rangle = -Z \qquad \langle \hat{n}, x-\hat{x}\rangle = Y\sin\varphi - Z\cos\varphi$$

Looking at the bottom two equations, it is immediate that such a relation will be symmetric exactly when

$$Y = Z\cdot(\csc\varphi+\cot\varphi)$$

so the space $\mathcal{R}^{sym}_{UT\mathsf{E}^3}$ is generically 3-dimensional. The relation corresponding to X, Z, φ is represented by the matrix

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ X & 1 & 0 & 0 \\ Z\cdot(\csc\varphi+\cot\varphi) & 0 & \cos\varphi & -\sin\varphi \\ Z & 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

Then the 1-form e^3 transforms under $\mathrm{Ad}^*(\rho^{-1})$ to

$$\hat{\mathrm{e}}^3 = Z\,(\csc\varphi+\cot\varphi)\ \mathrm{e}^3_2 - X\sin\varphi\ \mathrm{e}^2_1 + X\cos\varphi\ \mathrm{e}^3_1 - \sin\varphi\ \mathrm{e}^2 + \cos\varphi\ \mathrm{e}^3$$

Note that $\hat{\mathrm{e}}^3$ is transverse to $\mathfrak{h}$ when $X \neq 0$ and $\varphi \notin \pi\mathbb{Z}$; this also justifies *a posteriori* the divisions by $\sin\varphi$.

Note that $\hat{\mathrm{e}}^3$ is transverse to $\mathfrak{h}$, so $\psi = -X^{-1}\csc\varphi\ \hat{\mathrm{e}}^3$ gives a basis of $\mathrm{Ad}^*(\rho^{-1})V$ dual to the generator e^2_1 of $\mathfrak{h}$. Explicitly,

$$\psi = \mathrm{e}^2_1 - \cot\varphi\ \mathrm{e}^3_1 + X^{-1}\ \mathrm{e}^2 - X^{-1}Z\left(\frac{1+\cos\varphi}{\sin^2\varphi}\right)\mathrm{e}^3_2$$

The differential of ψ is given by

$$\delta\psi = -\mathsf{e}^3_1 \wedge \mathsf{e}^3_2 + \cot\varphi\, \mathsf{e}^3_2 \wedge \mathsf{e}^2_1 - X^{-1}\mathsf{e}^2_1 \wedge \mathsf{e}^1 - X^{-1}Z\left(\frac{1+\cos\varphi}{\sin^2\varphi}\right)\mathsf{e}^3_1 \wedge \mathsf{e}^2_1$$

so the relative curvature operator is

$$\begin{aligned}
\Delta &= \delta\psi - \psi \wedge (\mathsf{e}^2_1 \lrcorner\, \delta\psi) \\
&= -\mathsf{e}^3_1 \wedge \mathsf{e}^3_2 + \cot\varphi\, \mathsf{e}^3_2 \wedge \mathsf{e}^2_1 - X^{-1}\mathsf{e}^2_1 \wedge \mathsf{e}^1 - X^{-1}Z\left(\frac{1+\cos\varphi}{\sin^2\varphi}\right)\mathsf{e}^3_1 \wedge \mathsf{e}^2_1 \\
&\quad - \left(\mathsf{e}^2_1 - \cot\varphi\, \mathsf{e}^3_1 + X^{-1}\,\mathsf{e}^2 - X^{-1}Z\left(\frac{1+\cos\varphi}{\sin\varphi}\right)\mathsf{e}^3_2\right) \\
&\qquad\quad \wedge \left(X^{-1}Z\left(\frac{1+\cos\varphi}{\sin\varphi}\right)\mathsf{e}^3_1 - X^{-1}\,\mathsf{e}^1 - \cot\varphi\, \mathsf{e}^3_2\right) \\
&= \left(-1 - \cot^2\varphi - X^{-2}Z^2\left(\frac{1+\cos\varphi}{\sin\varphi}\right)^2\right)\mathsf{e}^3_1 \wedge \mathsf{e}^3_2 \\
&\quad + \left(X^{-2}Z\left(\frac{1+\cos\varphi}{\sin\varphi}\right)\right)\cdot(\mathsf{e}^3_1 \wedge \mathsf{e}^2 - \mathsf{e}^3_2 \wedge \mathsf{e}^1) \\
&\quad + \left(-X^{-2}\right)\mathsf{e}^1 \wedge \mathsf{e}^2 \\
&\quad + \left(-X^{-1}\cot\varphi\right)\cdot(\mathsf{e}^3_1 \wedge \mathsf{e}^1 + \mathsf{e}^3_2 \wedge \mathsf{e}^2)
\end{aligned}$$

The fourth term vanishes modulo Θ since $\delta\mathsf{e}^3 = -\mathsf{e}^3_1 \wedge \mathsf{e}^1 - \mathsf{e}^3_2 \wedge \mathsf{e}^2$. For ρ to be admissible, the Lie derivative of Δ must vanish. But, miraculously, each of the remaining three terms in Δ is $\mathfrak{h}$-invariant — despite what we would expect, there are no additional constraints on ρ:

Lemma 2.13 (No Admissibility Conditions on $UT\mathsf{E}^3$)**.** *All symmetric atomic invariant relations on $UT\mathsf{E}^3$ of the form*

$$\left.\begin{array}{cc} |x-\hat{x}|^2 = X^2 + 2Z^2\left(\frac{1+\cos\varphi}{\sin^2\varphi}\right) & \langle n, \hat{n}\rangle = \cos\varphi \\ \langle n, x-\hat{x}\rangle = -Z & \langle \hat{n}, x-\hat{x}\rangle = Z \end{array}\right\} \quad X \neq 0, \quad \varphi \notin \pi\mathbb{Z}$$

are admissible for the standard contact system.

Once the admissibility conditions have been characterized by lemma 2.13, an application of theorems 2.11 and 2.12 immediately gives a full characterization of the surfaces in $UT\mathsf{E}^3$ admitting geometric transformations.

Theorem 2.14 (Characterization of Geometric Transformations in $UT\mathsf{E}^3$)**.** *Let X, Z and φ be real numbers with $X \neq 0$ and $\varphi \notin \pi\mathbb{Z}$. Using (X, Z, φ) define the relation $\sim$ as in lemma 2.13. Then the following two statements are true:*

1. *If f and $\hat{f}$ are parameterized, adapted surfaces in $UT\mathsf{E}^3$ such that at corresponding points $f \sim \hat{f}$, then both f and $\hat{f}$ satisfy the affine Weingarten equation*
$$\left(\csc^2 \varphi + X^{-2}Y^2\right) K - 2\left(X^{-2}Y\right) H + X^{-2} = 0$$
where $Y = Z(1 + \cos\varphi)/\sin\varphi$.

2. *If f is an adapted surface in $UT\mathsf{E}^3$ which satisfies a Weingarten equation of the above form, then there is an adapted surface $\hat{f}$ satisfying the same equation which is $\sim$-related to f. $\hat{f}$ may be constructed from f by integrating a sequence of ordinary differential equations.*

Special cases of this theorem have appeared in the literature from time to time, dating at least back to an influential result of Bäcklund [1] demonstrated below and an 1894 result of Darboux [7] generalizing the Bianchi-Lie construction to surfaces of any constant negative curvature. More recently, Chen and Lie [5] [6] noted a link between Weingarten surfaces and the sine-Gordon equation, which was used to obtain a proof of theorem 2.14 and the same set of symmetric relations as we found in lemma 2.13.

The result of Bäcklund's referenced above is on the existence of a free parameter φ in the classical pseudospherical transformation, and may be immediately derived from theorem 2.14:

Corollary 2.15 (Bäcklund [1], 1883)**.** *Let $\mathcal{L}$ be a line congruence such that the corresponding normals of the two focal surfaces are at a constant angle φ, and the corresponding points are at a constant distance $1/\sin\varphi$. Then the two focal surfaces have constant Gaussian curvature -1.*

Proof. The vector connecting corresponding points on the focal surfaces of a line congruence is tangent to each surface, so $Z = 0$. The Weingarten equation in theorem 2.14 then becomes

$$\csc^2\varphi\, K + X^{-2} = 0$$

This reduces to $K = -1$ exactly when the distance X between the focal surfaces is $1/\sin\varphi$. □

The existence and special properties of 1-parameter families of relations which induce transformations of the same surfaces will be dealt with in a future work.

CHAPTER 3

EXAMPLES IN SPACE FORMS

To illustrate the utility of the methods described in chapter 2, let us set out to generalize theorem 2.14 to surfaces in spherical and hyperbolic 3-space. This is no more difficult than in the Euclidean case; for each geometry we must understand how to represent its unit tangent bundle as a homogeneous space equipped with a gEDS and we must have an ample supply of symmetric relations. Turning the crank to run this data through the generalized Bianchi and Lie theorems (theorems 2.11 and 2.12) will then give us a characterization of the geometric transformations and associated differential equations in each of these geometries.

The results of this chapter may be seen as an extension of Chen and Li's results [5] [6] to other space forms.

3.1 Spherical 3-Space

In this section, we will set out to apply the generalized Bianchi and Lie theorems to surfaces in spherical space $S^3 = SO(4)/SO(3)$.

3.1.1 The Structure of $SO(4)$

Let us begin by briefly reviewing the relationship between rotations on $\mathbb{R}^4$ and the quaternions.

Theorem 3.1. *Fix an identification of $\mathbb{R}^4$ with the quaternion algebra $\mathbb{H}$. Then any* *rotation $R \in SO(4)$ is of the form*

$$R(v) = q_L \cdot v \cdot \bar{q}_R$$

for some pair of quaternions $q_L, q_R \in \mathbb{H}$, called the left- and right-isoclinic parts of R. *The left- and right-isoclinic parts are unique up to multiplication of both parts simultaneously by -1.*

3.1.2 The Contact System on $UT\mathsf{S}^3$

Just as before we had to pass from E^3 to the unit tangent bundle $UT\mathsf{E}^3$ in order to study surface geometry, in order to study surfaces in S^3 we must move our attention to the unit tangent bundle $UT\mathsf{S}^3$. For the remainder of the section, we will identify $UT\mathsf{S}^3$ with the homogeneous space $SO(4)/H$, where

$$H = \{(e^{\mathsf{k}t}, e^{-\mathsf{k}t}) \ : \ t \in [0, 2\pi)\} \cong SO(2)$$

is the subgroup of rotations fixing both 1 and k.

The 1-forms on $SO(4)$ split naturally into two orthogonal families, vanishing on either the infinitesimal left- or right-isoclinic rotations. Each of these isoclinic parts is a 1-form taking values in the pure imaginary quaternions. From this, it follows that the 1-forms on $\mathfrak{so}(4)$ are spanned by those of the form

$$\varphi^L_\beta(\omega) = -\frac{1}{2}\left(\beta \cdot \omega_L + \omega_L \cdot \beta\right)$$
$$\varphi^R_\beta(\omega) = -\frac{1}{2}\left(\beta \cdot \omega_R + \omega_R \cdot \beta\right)$$

where β is a pure imaginary quaternion.

To proceed, we will need an analog of the Euclidean adapted frame for surfaces in S^3.

Definition 3.1. The *spherical contact lift* of a map $f : M \to \mathsf{S}^3$ with unit normal $n : M \to \mathsf{S}^3$ is the map $f^{(1)} : M \to UT\mathsf{S}^3$ defined by the rotation $F = (F_L, F_R)$ with $F_L \cdot \bar{F}_R = f$ and $F_L \cdot \mathsf{k} \cdot \bar{F}_R = n$. The rotation F is unique up to left multiplication by any map $h : M \to H$.

Note that, given a map $F : M \to SO(4)$, the Maurer-Cartan form $\omega = F^*\theta = F^{-1} \cdot dF$ splits into left- and right-isoclinic parts $\omega_L \oplus \omega_R$.

Lemma 3.2. *A rotation (q_L, q_R) with derivative $\omega = (\omega_L, \omega_R)$ is locally the spherical contact lift of the map $f = q_L \cdot q_R$ if and only if the form $\varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}}$ vanishes on ω.*

Proof. We already have that $f = q_L \cdot \bar{q}_R$ and the proposed normal is $n = q_L \cdot \mathsf{k} \cdot \bar{q}_R$. (q_L, q_R) will define a spherical contact lift of f precisely when n is an *actual* normal to f. In other words, we must check that $(n, df) = 0$. This can be done by translating both n and df by f^{-1} so that they each take values in the imaginary quaternions, then applying the fact that $(v, w) = -\frac{1}{2}(v \cdot w + w \cdot v)$ for $v, w \in \operatorname{Im} \mathbb{H}$. So we compute:

$$\begin{aligned}
\bar{f} \cdot df = d(q_L \cdot \bar{q}_R) &= q_R \cdot \bar{q}_L \cdot dq_L \cdot \bar{q}_R + q_r \cdot d\bar{q}_R \\
&= q_R \cdot (\omega_L + \overline{\omega_R}) \cdot \bar{q}_R \\
&= q_R \cdot (\omega_L - \omega_R) \cdot \bar{q}_R
\end{aligned}$$

since ω_R is pure imaginary. We also have

$$\bar{f} \cdot n = q_R \cdot \mathsf{k} \cdot \bar{q}_R$$

so that

$$\begin{aligned}(n, df) &= (\bar{f} \cdot n, \bar{f} \cdot df) \\ &= -\frac{1}{2} q_R \cdot \Big((\omega_L - \omega_R) \cdot \mathsf{k} + \mathsf{k} \cdot (\omega_L - \omega_R) \Big) \cdot \bar{q}_R \\ &= -\frac{1}{2} \Big((\omega_L - \omega_R) \cdot \mathsf{k} + \mathsf{k} \cdot (\omega_L - \omega_R) \Big) \\ &= \varphi^L_{\mathsf{k}}(\omega) - \varphi^R_{\mathsf{k}}(\omega)\end{aligned}$$

In other words, n is a normal for f exactly when the 1-form $\varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}}$ is annihilated by (q_L, q_R). This completes the proof. □

This lemma gives us a differential condition for testing if a map to $SO(4)$ could define the spherical contact lift of a surface, motivating the following definition:

Definition 3.2. The *spherical contact system* Θ on UTS^3 is the geometric exterior differential system differentially generated by the 1-form $\theta = \varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}}$. By analogy to the Euclidean case, integral manifolds of Θ will be called *adapted lifts*. Note that Θ is invariant under the coadjoint action of $\mathfrak{h}$.

3.1.3 Symmetric Relations on UTS^3

To apply our main theorems to surfaces in the sphere, we must first construct some nontrivial symmetric relations on UTS^3.

Theorem 3.3. *The symmetric relations on UTS^3 contain a space homeomorphic to $(S^2 \times S^2)/\Delta_H$, where Δ_H is the group of rotations in the $\mathsf{e}_1 \wedge \mathsf{e}_2$ plane acting simultaneously on each factor.*

Proof. We will find the invariant relations on $UT\mathsf{S}^3 = SO(4)/H$ by computing $\sqrt{H}$ and applying lemma 2.8. The isotropy subgroup H is the set of rotations of the form $(e^{\mathsf{k}t}, e^{\mathsf{k}t})$, so an arbitrary rotation (q_L, q_R) squares to an element of H exactly when $q_L^2 = q_R^2 = e^{\mathsf{k}t}$ for some $t \in \mathbb{R}$. In particular, the i and j components of q_L^2 and q_R^2 must vanish.

To see when this may occur, let $q = a + b\mathsf{i} + c\mathsf{j} + d\mathsf{k}$ be an arbitrary quaternion. Then we have

$$(a + b\mathsf{i} + c\mathsf{j} + d\mathsf{k})^2 = (a^2 - b^2 - c^2 - d^2) + 2ab\mathsf{i} + 2ac\mathsf{j} + 2ad\mathsf{k}$$

For q^2 to be in the span of 1 and k, we must have that $ab = ac = 0$. If $a \neq 0$ then it must be that both b and c are zero; in this case, q is already an element of H. We may thus restrict our attention to the case when $a = 0$ so that q is a pure imaginary quaternion and $q^2 = -|q|^2$.

We may then assume that q_L and q_R are pure imaginary. If q_L^2 and q_R^2 are to lie in H then they must each have magnitude 1; it follows that $q_L^2 = q_R^2 = -1 \in H$.

To address the issue of uniqueness, note that the rotations (q_L, q_R) and $(e^{-\mathsf{k}t} q_L e^{\mathsf{k}t}, e^{-\mathsf{k}t} q_L e^{\mathsf{k}t})$ describe equivalent relations. Thus, the space of symmetric relations on $UT\mathsf{S}^3$ contains a copy of $(S^2 \times S^2)/\Delta_H$. □

3.1.4 Applying the Main Theorem

Lemma 3.4. *The space of 2-forms perpendicular to $\mathfrak{h}$ and invariant under the adjoint action of H is spanned by the four 2-forms*

$$\varphi_{\mathsf{i}}^L \wedge \varphi_{\mathsf{j}}^L, \quad \varphi_{\mathsf{i}}^R \wedge \varphi_{\mathsf{j}}^R, \quad \varphi_{\mathsf{i}}^L \wedge \varphi_{\mathsf{j}}^R + \varphi_{\mathsf{j}}^L \wedge \varphi_{\mathsf{i}}^R, \quad \varphi_{\mathsf{i}}^L \wedge \varphi_{\mathsf{i}}^R - \varphi_{\mathsf{j}}^L \wedge \varphi_{\mathsf{j}}^R$$

Each of these 2-forms has a geometric interpretation as follows:

- $\varphi^L_{\mathsf{i}} \wedge \varphi^L_{\mathsf{j}} + \varphi^R_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}}$: This form is the area element $|df|^2$, analogous to the Euclidean form $\tau^1 \wedge \tau^2$.
- $\varphi^L_{\mathsf{i}} \wedge \varphi^L_{\mathsf{j}} - \varphi^R_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}}$: This form is the differential of $-(\varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}})$; as such, it vanishes on any spherical contact lift. The vanishing of this form defines a quadratic equation in f, n, and their derivatives which must be satisfied by any surface. In the Euclidean case, this equation could be interpreted as the statement that the shape operator is symmetric.
- $\varphi^L_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}} + \varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{i}}$: This form is the Gaussian curvature element $K|df|^2$, analogous to the Euclidean form $\nu^1 \wedge \nu^2$.
- $\varphi^L_{\mathsf{i}} \wedge \varphi^R_{\mathsf{i}} - \varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{j}}$: This form is the mean curvature element $2H|df|^2$, analogous to the Euclidean form $\tau^1 \wedge \nu^2 - \tau^2 \wedge \nu^1$.

Now let q_L and q_R be two points in the 2-sphere $S^2 \subset \operatorname{Im} \mathbb{H}$, and define the relation $[\rho] \in \mathcal{R}^{sym}_{UTS^3}$ by $\rho = (q_L, q_R)$. If $\varphi = \varphi^L + \varphi^R$ is an element of $\mathfrak{so}(4)^*$, then $\operatorname{Ad}(\rho^{-1})$ acts on φ by

$$\operatorname{Ad}(\rho^{-1})(\varphi)(\xi) = \varphi^L(\bar{q}_L \cdot \xi_L \cdot q_L) + \varphi^R(\bar{q}_R \cdot \xi_R \cdot q_R)$$

where $\xi = \xi_L + \xi_R \in \mathfrak{so}(4)$ is arbitrary.

We are now ready to apply the theorems from chapter 2. Let $\psi = \hat{\varphi}^L_{\mathsf{k}} - \hat{\varphi}^R_{\mathsf{k}}$ be the $\operatorname{Ad}^*(\rho^{-1})$ transformation of $\varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}}$. Since rotations in $SO(4)$ act isometrically on the left- and right-isoclinic parts of a form, there exist $a^L, b^L, c^L, a^R, b^R, c^R$ with

$$(a^L)^2 + (b^L)^2 + (c^L)^2 = (a^R)^2 + (b^R)^2 + (c^R)^2 = 1$$

such that

$$\hat{\varphi}^L_{\mathsf{k}} = a^L \varphi^L_{\mathsf{i}} + b^L \varphi^L_{\mathsf{j}} + c^L \varphi^L_{\mathsf{k}}$$

$$\hat{\varphi}^R_{\mathsf{k}} = a^R \varphi^R_{\mathsf{i}} + b^R \varphi^R_{\mathsf{j}} + c^R \varphi^R_{\mathsf{k}}$$

The form ψ is transverse to $\mathfrak{h}$ exactly when $\psi(k) = c^L + c^R$ is nonzero. We will define the form $\hat{\theta} = \psi/(c^L + c^R)$ to be the dual to k.

The exterior derivative of ψ is given by

$$\begin{aligned}\delta\psi = &-2a^L \varphi^L_{\mathsf{j}} \wedge \varphi^L_{\mathsf{k}} - 2b^L \varphi^L_{\mathsf{k}} \wedge \varphi^L_{\mathsf{i}} - 2c^L \varphi^L_{\mathsf{i}} \wedge \varphi^L_{\mathsf{j}} \\ &- 2a^R \varphi^R_{\mathsf{j}} \wedge \varphi^R_{\mathsf{k}} - 2b^R \varphi^R_{\mathsf{k}} \wedge \varphi^R_{\mathsf{i}} - 2c^R \varphi^R_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}}\end{aligned}$$

so that the Lie derivative $\mathsf{k} \lrcorner \delta\psi$ is

$$\mathsf{k} \lrcorner \delta\psi = 2a^L \varphi^L_{\mathsf{j}} - 2b^L \varphi^L_{\mathsf{i}} - 2a^R \varphi^R_{\mathsf{j}} + 2b^R \varphi^R_{\mathsf{i}}$$

Lemma 3.5. *The relative curvature $\Delta_{\hat{\Theta}}\psi$ of ψ contains no 2-forms involving φ^L_{k} or φ^R_{k}.*

Proof. To see the general pattern, let us first compute the coefficients of $\varphi^L_{\mathsf{j}} \wedge \varphi^L_{\mathsf{k}}$ and $\varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{k}}$ in $\Delta_{\hat{\Theta}}\psi$. The coefficient of $\varphi^L_{\mathsf{j}} \wedge \varphi^L_{\mathsf{k}}$ is

$$\varphi^L_{\mathsf{j}} \wedge \varphi^L_{\mathsf{k}} \cdot \left(-2a^L + \frac{2a^L c^L}{c^L + c^R}\right)$$

with the first term coming from $\delta\psi$ and the second from $-\hat{\theta} \wedge (\mathsf{k} \lrcorner \delta\psi)$. Similarly, the relative curvature contains a term of the form

$$\varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{k}} \cdot \left(\frac{2a^L c^R}{c^L + c^R}\right)$$

coming only from the second term of $\Delta_{\hat{\Theta}}\psi$. Since we may work modulo $\theta = \varphi^L_{\mathsf{k}} - \varphi^R_{\mathsf{k}}$, the two terms may be combined into a single one with coefficient

$$-2a^L + \frac{2a^L c^L}{c^L + c^R} + \frac{2a^L c^R}{c^L + c^R} = -2a^L + 2a^L = 0$$

The same follows for other pairs: each term involving φ^L_{k} or φ^R_{k} has a matching term such that the sum is zero modulo θ. Thus, $\Delta_{\hat{\Theta}}$ consists solely of 2-forms involving i and j. □

Using this lemma, we find that the relative curvature of ψ is given by

$$\begin{aligned}\Delta_{\hat{\Theta}}\psi = &\left(\varphi^L_{\mathsf{i}} \wedge \varphi^L_{\mathsf{j}} + \varphi^R_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}}\right) \cdot \left(-2\frac{c^L c^R}{c^L + c^R}\right) \\ &+ \left(\varphi^L_{\mathsf{i}} \wedge \varphi^R_{\mathsf{i}} - \varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{j}}\right) \cdot \left(-2\frac{a^L b^R - a^R b^L}{c^L + c^R}\right) \\ &+ \left(\varphi^L_{\mathsf{i}} \wedge \varphi^R_{\mathsf{j}} - \varphi^L_{\mathsf{j}} \wedge \varphi^R_{\mathsf{i}}\right) \cdot \left(2\frac{a^L a^R + b^R b^L}{c^L + c^R}\right)\end{aligned}$$

Each of the 2-forms appearing in this sum is invariant; in summary, we have demonstrated

Theorem 3.6 (Geometric Transformations for Surfaces in the 3-Sphere)**.** *Let $\rho = (q_L, q_R) \in SO(4)$ represent a symmetric relation with $q_L, q_R \in S^2 \subset \operatorname{Im}\mathbb{H}$, and let the transformed forms $\hat{\varphi}^L_{\mathsf{k}}$, $\hat{\varphi}^R_{\mathsf{k}}$ be as above. Then ρ is an admissible relation as long as $c_L \neq -c_R$, and induces a Bäcklund transformation between surfaces satisfying the spherical Weingarten equation*

$$\left(\frac{c^L c^R}{c^L + c^R}\right) + 2H\left(\frac{a^L b^R - a^R b^L}{c^L + c^R}\right) - K\left(\frac{a^L a^R + b^R b^L}{c^L + c^R}\right) = 0$$

where H and K denote the mean and Gaussian curvature, respectively.

3.2 Hyperbolic 3-Space

To complete this chapter, we will apply the generalized Lie and Bianchi theorems to surfaces in hyperbolic 3-space.

3.2.1 The Structure of $PSL(2,\mathbb{C})$ and the Spin Representation

Throughout this section, we will make heavy use of the Hermitian model of hyperbolic 3-space. In particular, we will identify H^3 with the space of 2-by-2 Hermitian matrices with determinant 1 and let $g \in PSL(2,\mathbb{C})$ act on the Hermitian matrix X by

$$X \mapsto g \cdot X \cdot g^{\dagger}$$

where † is the conjugate transpose.

The determinant defines a quadratic form on the space of Hermitian matrices; it so happens that this quadratic form has signature $(1,3)$. Thus, the space of Hermitian matrices of determinant 1 is isomorphic to the hyperboloid of forward timelike vectors in Minkowski space of squared magnitude -1. This is a well-known model of hyperbolic 3-space.

The action of $PSL(2,\mathbb{C})$ on the set of Hermitian matrices given above clearly preserves the determinant, and therefore is an action by hyperbolic isometries. The homogeneous space H^3 is then obtained by taking the quotient $PSL(2,\mathbb{C})/SU(2)$, where $SU(2)$ is the stabilizer of the identity matrix.

3.2.2 The Contact System on $UT\mathsf{H}^3$

The space of Hermitian matrices is 4-dimensional, spanned by the Minkowski basis

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If $A \in PSL(2,\mathbb{C})$ is given such that $f = AA^\dagger \in \mathsf{H}^3$, then the three matrices $AXA^\dagger$, $AYA^\dagger$, $AZA^\dagger$ form an orthonormal basis of the tangent space to H^3 at f. By analogy with the Euclidean case, we will call the frame A a *hyperbolic contact lift* of f when Tf is perpendicular to $AZA^\dagger$.

Let H be the diagonal subgroup

$$H = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \; : \; t \in \mathbb{R} \right\} \cong SO(2) \subset SU(2)$$

By construction, if A is a hyperbolic contact lift of f then $A \cdot h$ is as well, for any map $h : M \to H$.

H acts on an infinitesimal element $\xi \in \mathfrak{sl}(2,\mathbb{C})$ by

$$\mathrm{Ad}(h)(\xi) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \cdot \begin{pmatrix} z & u \\ v & -z \end{pmatrix} \cdot \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} z & e^{2it}u \\ e^{-2it}v & -z \end{pmatrix}$$

From this calculation it is easy to see that the only two $\mathrm{Ad}(H)$-invariant elements of $\mathfrak{sl}(2,\mathbb{C})^*$ are the forms $\varphi_z^+ = \frac{z+\bar{z}}{2}$ and $\varphi_z^- = \frac{z-\bar{z}}{2}$ which extract the real and imaginary parts of z. Of these two forms, only φ_z^+ is orthogonal to $\mathfrak{h}$; we therefore define the contact form $\theta = \varphi_z^+$.

Lemma 3.7. *A map $A : M \to PSL(2,\mathbb{C})$ is the locally the hyperbolic contact lift of $f = AA^\dagger$ if and only if the form φ_z^+ vanishes on $A^{-1}dA$.*

Proof. Define the map $f = AA^\dagger$ and let $n = AZA^\dagger$ be the proposed normal. Then

$$df = dAA^\dagger + AdA^\dagger = A\left(A^{-1}dA + (A^{-1}dA)^\dagger\right)A^\dagger$$

Since the map $B \mapsto ABA^\dagger$ is an isometry, df takes values orthogonal to n if and only if $Z \perp (A^{-1}dA + (A^{-1}dA)^{\dagger})$. Writing

$$A^{-1}dA = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$$

we find that

$$A^{-1}dA + (A^{-1}dA)^{\dagger} = \begin{pmatrix} \alpha + \overline{\alpha} & \beta + \overline{\gamma} \\ \gamma + \overline{\beta} & -\alpha - \overline{\alpha} \end{pmatrix}$$

The Minkowski inner product of this matrix with Z is $-(\alpha + \overline{\alpha}) = -2\varphi_z^+(A^{-1}dA)$, demonstrating the claim. □

3.2.3 Symmetric Relations on $UT\mathsf{H}^3$

Once again we will compute $\sqrt{H}$ and apply lemma 2.8 to find representatives for the symmetric relations on $UT\mathsf{H}^3$.

Theorem 3.8. *Every trace-free element $A \in PSL(2, \mathbb{C})$ induces a symmetric relation $[A] \in \mathcal{R}^{sym}_{UT\mathsf{H}^3}$, and every nontrivial symmetric relation has a trace-free representative. Moreover, each nontrivial symmetric relation has a* unique *representative of the form*

$$\begin{pmatrix} a & \frac{-1-a^2}{c} \\ c & -a \end{pmatrix}$$

with $a \in \mathbb{C}$ and $c > 0 \in \mathbb{R}$.

Proof. Throughout this proof, we will set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$$

Let us now find conditions on A which ensure that $A^2 \in H$. We are trying to solve the equation

$$A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

This presents two distinct cases to consider: $\mathrm{tr} A = 0$ and $\mathrm{tr} A \neq 0$.

If $\mathrm{tr} A \neq 0$ then it must be that both b and c vanish. But then $a^2 = e^{it}$ and $d^2 = e^{-it}$; combined with $1 = \det A = ad$, we find that when the trace does not vanish A must already lie in H.

Now we consider the case when $\mathrm{tr} A = 0$. Since the trace vanishes we have $d = -a$, leading to the two equations

$$a^2 + bc = e^{it}, \qquad a^2 + bc = e^{-it}$$

But note that $\det A = -a^2 - bc = 1$, so these equations can be satisfied when $t = \pi$. Thus, all trace-free elements of $PSL(2, \mathbb{C})$ square to $-1 \in H$.

Conjugating A by a rotation $h = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \in H$, we find that

$$hAh^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \cdot \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a & \lambda^2 b \\ \overline{\lambda}^2 c & -a \end{pmatrix}$$

From this calculation it is evident that we may assume c is a strictly positive real number, for if c were zero then the determinant of A would be 0. It follows that each symmetric relation has a representative A of the form

$$A = \begin{pmatrix} a & \frac{-1-a^2}{c} \\ c & -a \end{pmatrix}$$

with $a \in \mathbb{C}$ and $c > 0 \in \mathbb{R}$ arbitrary. $\square$

3.2.4 Applying the Main Theorem

Every element of $\mathfrak{sl}(2,\mathbb{C})$ may be uniquely written as a $\mathbb{C}$-linear combination of the basis matrices X, Y, Z defined in the previous section. A convenient basis of $\mathfrak{sl}(2,\mathbb{C})^*$ is given by the forms

$$\begin{aligned}
\varphi_X^+(\alpha X + \beta Y + \gamma Z) &= \operatorname{Re} \alpha \\
\varphi_Y^+(\alpha X + \beta Y + \gamma Z) &= \operatorname{Re} \beta \\
\varphi_Z^+(\alpha X + \beta Y + \gamma Z) &= \operatorname{Re} \gamma \\
\varphi_X^-(\alpha X + \beta Y + \gamma Z) &= \operatorname{Im} \alpha \\
\varphi_Y^-(\alpha X + \beta Y + \gamma Z) &= \operatorname{Im} \beta \\
\varphi_Z^-(\alpha X + \beta Y + \gamma Z) &= \operatorname{Im} \gamma
\end{aligned}$$

The forms $\varphi_\star^+$ which compute the real parts vanish on the subalgebra $\mathfrak{su}(2)$, while the forms $\varphi_\star^-$ which compute the imaginary parts vanish on the Hermitian matrices.

To compute the codifferentials of these forms, note that

$$\left[\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & -\alpha' \end{pmatrix}\right] = \begin{pmatrix} \beta\gamma' - \gamma\beta' & 2(\alpha\beta' - \beta\alpha') \\ 2(\gamma\alpha' - \alpha\gamma') & \gamma\beta' - \beta\gamma' \end{pmatrix}$$

so computing modulo the contact form $\theta = \varphi_Z^+$ we have

$$\begin{aligned}
\delta\varphi_X^\pm &= 2\varphi_Y^\pm \wedge \varphi_Z^- \\
\delta\varphi_Y^\pm &= -2\varphi_X^\pm \wedge \varphi_Z^- \\
\delta\varphi_Z^\pm &= 2\left(\varphi_X^- \wedge \varphi_Y^\pm \pm \varphi_X^+ \wedge \varphi_Y^\mp\right)
\end{aligned}$$

In theorem 3.8, the symmetric relations on $UT\mathbb{H}^3$ were characterized as those which admit a representative $[A]$ with A a trace-free element of $PSL(2,\mathbb{C})$. We may therefore assume that A is given in the form

$$A = \begin{pmatrix} a & \frac{-1-a^2}{c} \\ c & -a \end{pmatrix}$$

with $a \in \mathbb{C}$ and $c > 0 \in \mathbb{R}$ arbitrary. The transformation A will yield an admissible relation so long as $a \notin \mathbb{R}$.

The coadjoint action of such an A on $\mathfrak{sl}(2, \mathbb{C})$ takes the form $\theta = \varphi_Z^+$ to

$$\begin{aligned} \mathrm{Ad}^*(A)(\varphi_Z^+) = & \left(-\mathrm{Re}\ a^3 + (1+c^2) \cdot \mathrm{Im}\ a\right) \varphi_X^+ \\ & + \left(-\mathrm{Im}\ a^3 + (1+c^2) \cdot \mathrm{Re}\ a\right) \varphi_X^- \\ & + \left(-\mathrm{Im}\ a^3 + (1-c^2) \cdot \mathrm{Re}\ a\right) \varphi_Y^+ \\ & + \left(\mathrm{Re}\ a^3 - (1-c^2) \cdot \mathrm{Im}\ a\right) \varphi_Y^- \\ & + \left(2c \cdot \mathrm{Im}\ a^2\right) \varphi_Z^- \qquad \pmod{\varphi_Z^+} \end{aligned}$$

The form $\hat{\theta}$ transverse to $\mathfrak{h}$ will then be given by

$$\begin{aligned} \hat{\theta} &= \frac{\mathrm{Ad}^*(A)(\varphi_Z^+)}{2c \cdot \mathrm{Im}\ a^2} \\ &= \varphi_Z^- + P\varphi_X^+ + Q\varphi_X^- + R\varphi_Y^+ + S\varphi_Y^- \end{aligned}$$

From the calculations above it follows that

$$\begin{aligned} \delta\hat{\theta}/2 = & \varphi_X^- \wedge \varphi_Y^- - \varphi_X^+ \wedge \varphi_Y^+ \\ & + A\varphi_Y^+ \wedge \varphi_Z^- + B\varphi_Y^- \wedge \varphi_Z^- - C\varphi_X^+ \wedge \varphi_Z^- - D\varphi_X^- \wedge \varphi_Z^- \end{aligned}$$

and

$$\xi \lrcorner\, \delta\hat{\theta}/2 = C\varphi_X^+ + D\varphi_X^- - A\varphi_Y^+ - B\varphi_Y^-$$

where ξ is the generator of $\mathfrak{h}$ dual to φ_Z^-. The relative curvature of $\hat{\theta}$ therefore has no terms involving φ_Z^- — each such term in $\delta\hat{\theta}$ is cancelled by an opposite one in $-\hat{\theta} \wedge (\xi \lrcorner \delta\hat{\theta})$. As in the Euclidean and spherical cases, the rest of the relative curvature consists of $\mathfrak{h}$-invariant 2-forms:

$$\begin{aligned}
\Delta_{\hat{\Theta}}\hat{\theta} &= \frac{1}{2}(\delta\hat{\theta} - \hat{\theta} \wedge (\xi \lrcorner \delta\hat{\theta})) \\
&= \left(-1 + P^2 + R^2\right) \varphi_X^+ \wedge \varphi_Y^+ \\
&\quad + \left(1 + Q^2 + S^2\right) \varphi_X^- \wedge \varphi_Y^- \\
&\quad + \left(QR - PS\right) (\varphi_X^+ \wedge \varphi_X^- - \varphi_Y^+ \wedge \varphi_Y^-) \\
&\quad + \left(PQ + RS\right) (\varphi_X^+ \wedge \varphi_Y^- + \varphi_X^- \wedge \varphi_Y^+)
\end{aligned}$$

By plugging these calculations into theorems 2.11 and 2.12, we have shown:

Theorem 3.9 (Geometric Transformations for Surfaces in Hyperbolic 3-space)**.** *Let $a \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ and $c > 0 \in \mathbb{R}$ be given, and let $[\mu(a,c)] \in \mathcal{R}^{sym}_{UT\mathsf{H}^3}$ be the corresponding symmetric relation on $UT\mathsf{H}^3$. Then two surfaces in H^3 are $\sim_{\mu(a,c)}$-related if and only if they each satisfy the Weingarten equation*

$$\left(-1 + P^2 + R^2\right) + 2H\left(QR - PS\right) + K\left(1 + Q^2 + S^2\right) = 0$$

where

$$\begin{aligned}
Q + iP &= \frac{a}{2c \cdot \operatorname{Im} a^2}\left((a^2+1) + c^2\right) \\
R - iS &= \frac{a}{2c \cdot \operatorname{Im} a^2}\left((a^2+1) - c^2\right)
\end{aligned}$$

Furthermore, if a surface in H^3 satisfies the above Weingarten equation then we may construct a second surface, $\sim_{\mu(a,c)}$-related to the first, by integrating a sequence of ordinary differential equations.

Proof. Apply theorems 2.11 and 2.12 to the calculations in this section. □

3.3 Spacetime Geometries

Although the calculations are not presented here, the same methods used to find transformations for surfaces in E^3, S^3, and H^3 may be mimicked to obtain analogous theorems for surfaces in both Minkowski space and its curved analogues: deSitter and anti-deSitter space. Once again it is found that there exist geometric transformations in these geometries for surfaces satisfying Weingarten equations. The Minkowski case was noted by TK Milnor in [10]; the deSitter and anti-deSitter cases appear to be new.

CHAPTER 4

EXTENDED APPLICATION: LIE SPHERE GEOMETRY

In the previous chapters, we have focused on the application of theorems 2.11 and 2.12 to the unit tangent bundles of Euclidean, spherical, and hyperbolic space. In each case the ambient space carried three G-invariant 2-forms outside of the contact ideal, corresponding to $|df|^2$, $H|df|^2$, and $K|df|^2$.

Since the main theorems of this thesis operate on a gEDS which is generated by 1-forms and the relative curvature operator increases the degree of forms by one, all differential equations resulting from these theorems can be represented by G-invariant 2-forms. In the case of the space forms analyzed previously, this means that the most general situation that we could hope for is a geometric transformation acting on surfaces which solve some affine Weingarten equation

$$(a + bH + cK)|df|^2 = 0$$

with some constraints on a, b, c to be expected, depending on the ambient geometry.

In particular, in the spaces we have analyzed so far it is impossible to find geometric transformations for certain otherwise interesting geometric PDE such as the constant distortion equation

$$d(k_1 - k_2)^2 = 0$$

To have access to higher-order equations like this, we must look at homogeneous spaces which are analogous to higher jet bundles. On these spaces, geometric quantities such as the mean and Gauss curvature will appear as coordinates rather than as derived quantities. In particular, this means that quantities involving derivatives of principal curvatures can be represented as 1-forms, thus becoming amenable to the techniques developed previously.

It would be nice to have a method for extending a homogeneous space to one which includes "higher geometric derivatives" in some canonical way, analogous to prolongation of jet bundles. In the category of manifolds equipped with an exterior differential system the jet bundle prolongation map $J^kM \xrightarrow{j} J^{k+1}M$ is functorial, so we may always build manifolds and EDSs which extend a given EDS to higher derivatives. But if we restrict to the category of homogeneous spaces with geometric exterior differential systems, it is not clear that a prolongation functor exists at all. Because of this unfortunate circumstance, we can only construct homogeneous spaces which can represent higher-order geometric differential equations in an *ad hoc* way.

The purpose of this chapter is twofold:

1. Apply the methods of the previous chapters to a homogeneous space where the principal curvatures of an immersed surface appear as coordinates.
2. Show how the transversality condition in theorems 2.11 and 2.12 may be weakened in the case of a dimensional mis-match where $\dim \Theta < \dim H$.

The geometry of choice here is *Lie sphere geometry*, which was coincidentally the subject of Sophus Lie's own thesis.

4.1 Background for Lie Sphere Geometry

Definition 4.1. A *Lie sphere* in $\mathbb{R}^3$ is any one of the following objects:

1. The point ∞ at infinity.
2. Any finite point p.
3. Any oriented plane Π.
4. Any oriented sphere S.

We will say that two Lie spheres X, Y are in *oriented contact* if $X \cap Y$ is nonempty and the orientations of X and Y match on the intersection. Points will be considered to have all orientations and every planes will be in oriented contact with ∞; furthermore, two planes will be in oriented contact at infinity precisely when they have the same normal. We will denote the set of Lie spheres by $\mathcal{L}$.

It will frequently be useful to consider a sphere of *signed radius* r to be a positively oriented sphere of radius $|r|$ when $r > 0$, a negatively oriented sphere of radius $|r|$ when $r < 0$, and a point when $r = 0$. From this perspective, the finite points are merely the oriented spheres of zero signed radius.

Note that by stereographic projection we could just as well think of Lie spheres as being oriented spheres of any radius (including zero) inside of S^3. From this perspective, it is clear that $\mathcal{L}$ has the structure of a smooth manifold.

Definition 4.2. A *Lie sphere transformation* is a diffeomorphism $\Phi : \mathcal{L} \to \mathcal{L}$ such that Φ preserves the relation of oriented contact. Throughout this chapter we will use G to denote the group of Lie sphere transformations.

It is clear that any Euclidean motion (and more generally, any Möbius transformation) acts on the set of Lie spheres and preserves contact — these will be called *point transformations* . What is perhaps less clear is that there exist Lie sphere transformations which are *not* point transformations.

Lemma 4.1. *There are inclusions $ASO(3) \hookrightarrow Möb(3) \hookrightarrow G$, but none of these inclusions are isomorphisms.*

Proof. The inclusion of $ASO(3) \hookrightarrow Möb(3)$ is clearly not an isomorphism since $Möb(3)$ contains inversions interchanging ∞ and the origin, for example. More generally, the image of $ASO(3)$ in $Möb(3)$ is the stabilizer of ∞.

To see that G contains new transformations outside of $Möb(3)$, consider the *normal shift* $\Phi^{\perp}(t)$ which operates as follows:

- Fix ∞.
- Move the planes with normal n by tn.
- Add t to the signed radius of each sphere (including the finite points). That is, take a sphere of signed radius r to the sphere with the same center and radius $r + t$; take points to spheres of signed radius t.

To check that the normal shift preserves oriented contact is mildly tedious; in lieu of a proof, the reader may like to meditate on the configuration in figure 4.1.

Note that in the left figure, all Lie spheres shown in contact are in fact in *oriented* contact, except for the right sphere and the horizontal plane. The image of these Lie spheres on the right maintains the same pattern of oriented contact. Also note that *unoriented* contact is not preserved — the right sphere and horizontal plane begin in unoriented contact but end up not in contact at all.

This demonstrates that G contains elements beyond those accounted for by the Möbius transformations. As it happens, G is generated by the Möbius transformations and the normal shift; we omit the proof as we will not need to use this fact. □

The normal shift provides some insight into the nature of Lie sphere transformations; in particular, note that the property "X is a point" is not G-invariant, since points maps to sphere under a normal shift. Thus, Lie sphere geometry is fundamentally a geometry of *contact transformations* , not merely a prolonged geometry of point transformations. The proper philosophy to take is that *the fundamental objects of Lie sphere geometry are contact elements, not points*.

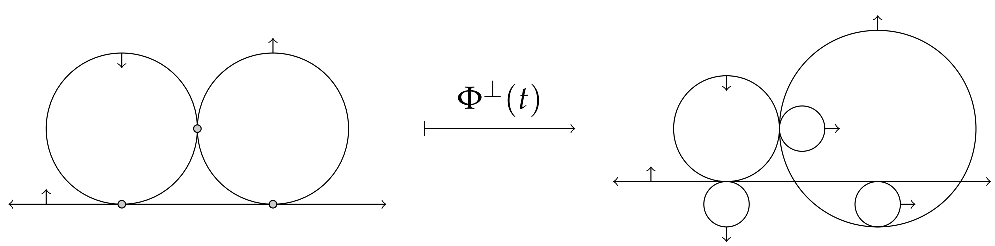


Figure 4.1: The action of normal shifts as Lie sphere transformations

4.2 Projective Model of $\mathcal{L}$

There is a remarkable projective model for the manifold $\mathcal{L}$ of Lie spheres, extending the more well-known projective model of conformal geometry. In this model $\mathcal{L}$ will appear as the projectivized null quadric of a certain bilinear form. Likewise, the group of Lie sphere transformations will be identified with the group of linear transformations preserving this bilinear form.

Throughout this section, we will endow $\mathbb{R}^6$ with a metric $\langle , \rangle$ of signature $(4,2)$. When taking this metric into account, we use the notation $\mathbb{R}^{4+2}$ instead of $\mathbb{R}^6$.

Definition 4.3. The *null quadric* of $\langle , \rangle$ is the set

$$Q = \{v \in \mathbb{R}^6 \setminus \{0\} \ : \ \langle v, v \rangle = 0\} \subset \mathbb{R}^{4+2}$$

We will also frequently make use of the *projectivized null quadric*

$$[Q] = Q/\mathbb{R}^\times \subset \mathbb{RP}^5$$

Let us now fix once and for all a basis $\mathsf{e}_1, \ldots, \mathsf{e}_6$ of $\mathbb{R}^{4+2}$ such that

$$\langle \mathsf{e}_i, \mathsf{e}_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ +1 & \text{if } i = j \text{ and } i \in \{2,3,4,5\} \\ -1 & \text{if } i = j \text{ and } i \in \{1,6\} \end{cases}$$

From here out, any 6-tuple should be understood to be written in this basis. We will also have occasion to make use of the matrix G_{lie} of $\langle , \rangle$ in this basis, which is simply

$$G_{\text{lie}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Theorem 4.2. *There is a diffeomorphism $\varphi : \mathcal{L} \to [Q]$ between the manifold of Lie spheres and the projectivized null quadric.*

Proof. We define a map φ_0 on a case-by-case basis, depending on the type of Lie sphere.

- If $S(x,r)$ is a sphere with center $x = (x_1, x_2, x_3)$ and signed radius r, then

$$\varphi_0(S(x,r)) = \left(\frac{1+|x|^2-r^2}{2}, \frac{1-|x|^2+r^2}{2}, x_1, x_2, x_3, r\right)$$

- As a special case, the finite point $P(x)$ at x gets mapped to

$$\varphi_0(P(x)) = \varphi_0(S(x,0)) = \left(\frac{1+|x|^2}{2}, \frac{1-|x|^2}{2}, x_1, x_2, x_3, 0\right)$$

- The (hyper)plane $H(x,n)$ with normal n passing through the point x gets mapped to

$$\varphi_0(H(x,n)) = ((n,x), -(n,x), n_1, n_2, n_3, 1))$$

- The point at infinity ∞ gets mapped to

$$\varphi_0(\infty) = (1, -1, 0, 0, 0, 0)$$

Checking that φ_0 as defined takes values in the null quadric is straightforward. The spheres and points map to null vectors since

$$\begin{aligned}\langle \varphi_0(S(x,r)), \varphi_0(S(x,r))\rangle &= -\left(\frac{1+|x|^2-r^2}{2}\right)^2 + \left(\frac{1-|x|^2+r^2}{2}\right)^2 + |x|^2 - r^2 \\ &= -\frac{1}{4}\cdot 2\left(|x|^2-r^2\right) + \frac{1}{4}\cdot 2\left(-|x|^2+r^2\right) + |x|^2 - r^2 \\ &= 0\end{aligned}$$

while the planes map to null vectors since

$$\langle \varphi_0(H(x,n)), \varphi_0(H(x,n))\rangle = -(n,x)^2 + (n,x)^2 + |n|^2 - 1 = 0$$

Finally, $\langle \varphi_0(\infty), \varphi_0(\infty)\rangle = -(1^2) + (-1)^2 = 0$. Altogether, this shows that φ_0 takes values in Q. We will use $\varphi = [\varphi_0]$ to denote its projectivization.

φ is clearly injective; to see that it is surjective, let us take an arbitrary point $[y] = [y_1 : y_2 : y_3 : y_4 : y_5 : y_6] \in [Q]$ and attempt to decode it into one of the above representations of a Lie sphere. There are several cases to consider:

1. If $y_6 \neq 0$ and...

   (a) ...$y_1 \neq -y_2$ then we may multiply y by $1/(y_1 + y_2)$ to get a null vector $y'' \in Q$ such that $y_1'' + y_2'' = 1$. But then

   $$\begin{aligned}0 = \langle y', y'\rangle &= -(y_1')^2 + (1-y_1')^2 + (y_2')^2 + (y_3')^2 + (y_4')^2 - (y_5')^2 \\ &= 1 - 2y_1' + (y_2')^2 + (y_3')^2 + (y_4')^2 - (y_5')^2\end{aligned}$$

   Solving for y_1' and writing $x = (y_2', y_3', y_4')$, $r = y_5'$, we have demonstrated that

   $$y_1' = \frac{1+|x|^2-r^2}{2}, \qquad y_2' = 1 - y_1' = \frac{1-|x|^2+r^2}{2}$$

   so $y' = S(x,r)$ represents a sphere of nonzero radius.

(b) ...$y_2 = -y_1$ then we may multiply y by $1/y_6$ to get a null vector $y'' \in Q$ with 1 in its last coordinate. But then

$$0 = \langle y'', y'' \rangle = -(y_1'')^2 + (-y_1'')^2 + (y_3'')^2 + (y_4'')^2 + (y_5'')^2 - 1$$

so the vector $n = (y_3'', y_4'', y_5'')$ is of unit length; y'' is then of the form $H(x, n)$ for some point x.

2. If $y_6 = 0$ then $y_1 \neq 0$, and there are also two subcases:

(a) If $y_2 \neq -y_1$ then we can divide y by $1/(y_1 + y_2)$ to get a null vector $y' \in Q$ such that $y_1' + y_2' = 1$. In that case we have

$$\begin{aligned} 0 = \langle y', y' \rangle &= -(y_1')^2 + (1 - y_1')^2 + (y_2')^2 + (y_3')^2 + (y_4')^2 \\ &= 1 - 2y_1' + (y_2')^2 + (y_3')^2 + (y_4')^2 \end{aligned}$$

Solving for y_1' and writing $x = (y_2', y_3', y_4')$, we have demonstrated that

$$y_1' = \frac{1 + |x|^2}{2}, \qquad y_2' = 1 - y_1' = \frac{1 - |x|^2}{2}$$

so $y' = P(x)$ represents a point.

(b) If $y_2 = -y_1$ then we can divide through by y_1 to get $y'' = (1, -1, y_3'', y_4'', y_5'', 0)$. But then

$$0 = \langle y'', y'' \rangle = -(1^2) + (-1)^2 + (y_3'')^2 + (y_4'')^2 + (y_5'')^2$$

so it must be that $y'' = (1, -1, 0, 0, 0, 0) = \infty$.

Since this decoding procedure never fails, the map $\varphi : \mathcal{L} \to [Q]$ is a bijection; we will let $\mathcal{L}$ inherit the smooth structure from $[Q]$ so that φ is a diffeomorphism.

□

It will be useful later to drop the map φ_0 and think of $\mathcal{L}$ as $[Q]$ directly. To this end, we will define the maps

$$\text{Point}(x) = P(x)$$
$$\text{Sphere}(x, r) = S(x, r)$$
$$\text{Plane}(x, n) = H(x, n)$$

The symbol ∞ will be overloaded and used to denote $\varphi_0(\infty)$.

The fact that $\mathcal{L}$ is diffeomorphic to $[Q]$ is interesting enough, but the connection between the two runs deeper — the contact relation in $\mathcal{L}$ is reflected in the geometry of $[Q]$!

Theorem 4.3. *Two Lie spheres X and Y are in oriented contact if and only if $\langle \varphi(X), \varphi(Y) \rangle = 0$.*

Proof. We may proceed by direct computation. Let $S = \text{Sphere}(x, r_x)$ and $S' = \text{Sphere}(y, r_y)$ be oriented spheres, and $H = \text{Plane}(y, n)$, $H' = \text{Plane}(y', n')$ planes. Then

$$\begin{aligned}
\langle S, S' \rangle &= -\left(\frac{1 + |x|^2 - r_x^2}{2} \cdot \frac{1 + |y|^2 - r_y^2}{2} \right) \\
&\quad + \left(\frac{1 - |x|^2 + r_x^2}{2} \cdot \frac{1 - |y|^2 + r_y^2}{2} \right) + (x, y) - r_x r_y \\
&= -\frac{1}{2}(|x|^2 - r_x^2 + |y|^2 - r_y^2) + (x, y) - r_x r_y \\
&= \frac{(r_x - r_y)^2 - |x - y|^2}{2}
\end{aligned}$$

If r_x and r_y have the same sign then the two spheres S, S' must be nested one inside the other. Two spheres in such a configuration are in contact precisely when the difference of their radii is equal to the distance between their centers. Similarly, if r_x and r_y have opposite sign then the interiors of their spheres do not intersect. In this configuration, two spheres are in contact precisely when the sum of their unsigned radii is equal to the distance between their centers. But in this case the sum of unsigned radii is the difference of the signed radii. This shows that two spheres are in oriented contact if and only if $\langle S, S' \rangle = 0$.

Next consider the case of two planes. Then we find

$$\langle H, H' \rangle = -(n, y)(n', y') + (n, y)(n', y') + (n, n') - 1 = (n, n') - 1$$

which vanishes only when $n = n'$. This is exactly the condition that H and H' are in oriented contact at infinity.

Now consider a plane and a sphere:

$$\begin{aligned}
\langle S, H \rangle &= -\left(\frac{1 + |x|^2 - r_x^2}{2}\right) \cdot (n, y) - \left(\frac{1 - |x|^2 + r_x^2}{2}\right) \cdot (n, y) + (n, x) - r_x \\
&= -(n, y) + (n, x) - r_x \\
&= (n, (x - r_x n) - y)
\end{aligned}$$

which vanishes only when $x - r_x n$ is in H. But this too is just the oriented contact condition for a plane and signed sphere.

Finally, we consider the infinite point ∞: $\langle \infty, S \rangle = 1$ for all spheres, and

$$\langle \infty, H \rangle = -(n, y) + (n, y) = 0$$

so ∞ is in oriented contact with all planes. □

This theorem has some remarkable immediate corollaries, obtained by jumping between the abstract Lie sphere model and the projective model.

Corollary 4.4. *Through each point $[q] \in [Q]$ there is an S^2-family of distinct projective lines which lie in $[Q]$.*

Proof. Suppose that q represents the Lie sphere S; by performing a Möbius inversion and a normal shift, we may assume that S is an actual sphere. To any point $P \in S$, take a vector $p \in Q$ which represents P. Since P is in oriented contact with S, we have $\langle p, q \rangle = 0$. But then $\alpha p + \beta q \in Q$ for all α, β, so there is a projective line on $[Q]$ containing both $[p]$ and $[q]$. □

Corollary 4.5. *The space Λ of projective lines on $[Q]$ is diffeomorphic to the unit tangent bundle of the 3-sphere.*

Proof. By the proof of the previous lemma, each projective line on $[Q]$ corresponds to a complete family of Lie spheres in oriented contact at some point. The point of contact and the normal to the plane of contact give the desired diffeomorphism. □

Finally, we come to the most important corollary:

Theorem 4.6 (The Group of Lie Sphere Transformations)**.** *The group G of Lie sphere transformations is isomorphic to $SO(4,2)/\pm 1$.*

Proof. Any Lie sphere transformation must act on Q so as to preserve the metric $\langle , \rangle$, and conversely any transformation preserving $\langle , \rangle$ acts on Q and preserves contact. The only elements of $SO(4,2)$ which act trivially on $[Q]$ are the scalars ± 1. □

Corollary 4.7. *The space $\mathcal{L}$ of Lie spheres is a homogeneous space for $SO(4,2)$.*

Proof. This is simply a special case the observation that $SO(p,q)$ acts transitively on its projective null quadric. □

4.3 The Space of Contact Elements

We may now introduce one of the first interesting homogeneous spaces built from the group of Lie sphere transformations.

Lemma 4.8. *The space* Λ *of projective lines in* $[Q]$ *is isomorphic to the homogeneous space* G/H_{10}*, where* H_{10} *is the 10-dimensional stabilizer of some particular projective line* $[A_0 : B_0]$.

Proof. The only interesting fact to check is that G acts transitively on $[Q]$. We will explicitly construct a Lie sphere transformation which takes an arbitrary line ℓ to another arbitrary line ℓ'. Any projective line contains a point (either finite or infinite) and a plane; if one of our lines contains the infinite point ∞, we may apply a Möbius inversion M sending ∞ to some finite point. This mean that we may assume $\ell = [\text{Point}(x) : \text{Plane}(x, n)]$ and $\ell' = [\text{Point}(x') : \text{Plane}(x', y')]$ with x, x' both finite. But there is a Euclidean motion E taking x to x' and n to n'. Conjugating E by M results in the desired Lie sphere transformation. The dimension of $H_1 0$ will be determined in section 4.5. □

Just as Euclidean (or spherical, or hyperbolic) frames were essential to studying surfaces in the classical geometries, we need a good notion of a frame to aid our study of surfaces in Λ. We will choose a slightly unusual definition, however.

Definition 4.4. A *Lie frame* is a map $\Psi : M \to GL(\mathbb{R}^{4+2})$ such that

$$\Psi^T G_{\text{lie}} \Psi = G_{\text{frame}}$$

where

$$G_{\text{frame}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Lie frames are naturally adapted to problems in Lie sphere geometry since the first and last two columns are null-vectors, and therefore represent a quadruple of Lie spheres. Geometrically, these four spheres S_1, A, B, S_2 represented by the first, second, fifth, and sixth columns of Ψ have the pattern of contact represented in figure 4.2. Note in particular that S_1 and S_2 are in oriented contact —

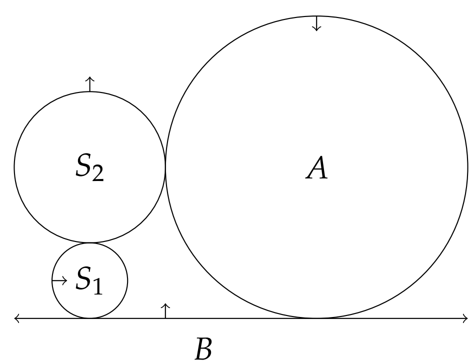


Figure 4.2: Contact configuration for a Lie frame

when we later find Lie frames well-adapted to surface geometry, this geometry will be encoded by S_1 and S_2.

It will be convenient to write ψ_i for the ith column of Ψ. Given a Lie frame Ψ, we may compute its Darboux derivative $\theta = \Psi^{-1} d\Psi$. Since Ψ is a Lie frame, we have

$$\Psi^{-1} = G_{\text{frame}} \Psi^T G_{\text{lie}}$$

Taking A to be the matrix with elements $a_i^j = \langle \psi_j, d\psi_i \rangle$, we may write $\theta = G_{\text{frame}} A$. Then the matrix of Θ is explicitly given by

$$\theta = \begin{bmatrix} \theta_1^1 & \theta_2^1 & \theta_3^1 & \theta_4^1 & \theta_5^1 & \theta_6^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 & \theta_4^2 & \theta_5^2 & \theta_6^2 \\ \theta_1^3 & \theta_2^3 & \theta_3^3 & \theta_4^3 & \theta_5^3 & \theta_6^3 \\ \theta_1^4 & \theta_2^4 & \theta_3^4 & \theta_4^4 & \theta_5^4 & \theta_6^4 \\ \theta_1^5 & \theta_2^5 & \theta_3^5 & \theta_4^5 & \theta_5^5 & \theta_6^5 \\ \theta_1^6 & \theta_2^6 & \theta_3^6 & \theta_4^6 & \theta_5^6 & \theta_6^6 \end{bmatrix} = \begin{bmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 \\ a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & a_6^1 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & a_6^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & a_6^4 \\ a_1^6 & a_2^6 & a_3^6 & a_4^6 & a_5^6 & a_6^6 \\ a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & a_6^5 \end{bmatrix}$$

These matrix elements are subject to several algebraic and differential equations. First, the matrix elements of θ must satisfy the differential equations

$$d\theta_i^j + \theta_k^j \wedge \theta_i^k = 0$$

implied by the Maurer-Cartan equation. On the algebraic side, differentiation of the equation $\Psi^T G_{\text{lie}} \Psi = G_{\text{frame}}$ leads to the skew-symmetry condition

$$A^T + A = d\Psi^T G_{\text{lie}} \Psi + \Psi^T G_{\text{lie}} d\Psi = 0$$

on the forms in A.

Throughout the remainder of this chapter, when a Lie frame is present we will use the names S_1, A, X_1, X_2, B, S_2 to refer to the column vectors of the frame.

Definition 4.5. Let $f : M \to \mathbb{R}^3$ be an immersed surface. The point $\text{Point}(f)$ and the tangent plane $\text{Plane}(f, n)$ at a given location on the surface are each Lie spheres in oriented contact, so they define a unique contact element $[\text{Point}(f) : \text{Plane}(f, n)] \in \Lambda$. A Lie frame Ψ is called a *Legendrian frame* for f if $[S_1 : S_2] = [\text{Point}(f) : \text{Plane}(f, n)]$.

Lemma 4.9. *Let Ψ be a Lie frame, and associate to Ψ the map $f : M \to \mathbb{R}^3$ given by taking the unique finite point in each projective line $[S_1 : S_2]$ of Lie spheres. Then Ψ is locally a Legendrian frame for f if and only if $\Psi^*\theta_1^5 = 0$.*

Proof. Since the property "Ψ is a Legendrian frame" is invariant under H_{10}, we may freely transform Ψ so that $S_1 = \text{Point}(f)$ and $S_2 = \text{Plane}(f, n)$ with n the unit normal of f. Then we have

$$\langle dS_1, S_2 \rangle = -(f, df)(f, n) + (f, df)(f, n) + (df, n) = (df, n) = 0$$

But $\langle dS_1, S_2 \rangle = a_1^6 = \theta_1^5$, so when Ψ is Legendrian we have $\theta_1^5 = 0$. Conversely, the above calculation also shows that in general $\theta_1^5 = (df, n)$, which vanishes only if

$$[S_1 : S_2] = [\text{Point(f)} : \text{Plane}(f, n)]$$

□

The H_{10}-invariant gEDS differentially generated by θ_1^5 will be called the *standard contact system*.

Since the space Λ of contact elements is 5-dimensional and the stabilizer H_{10} of a contact element is 10-dimensional, we have a very large amount of flexibility in choosing a Legendrian frame along any given surface. In the next section, we will see how to build a bundle over Λ which carries an extension of the standard contact system and can trap more detailed information about the geometry of a given surface.

4.4 The Space of Kissing Elements

If $f : M \to \mathbb{R}^3$ is an immersed surface, then we have a high degree of flexibility in choosing a Legendrian frame Ψ for f. We may encode more of the geometry of f into the frame by asking Ψ to describe the principal curvatures of our surface.

Definition 4.6. Let $f : M \to \mathbb{R}^3$ be an immersed, umbilic-free surface with principal curvature functions k_1, k_2. A *kissing frame* for f is a Legendrian frame Ψ such that X_1 and X_2 point in the first and second principal directions and S_1 and S_2 are spheres with signed radius equal to $-1/k_1$ and $-1/k_2$, respectively. These spheres are called the *curvature spheres* of f.

The property of being a kissing frame is invariant under a 7-dimensional subgroup $H_7 \subset H_{10}$, isomorphic to the group of transformations which fixes a given projective line on $[Q]$ pointwise.

Definition 4.7. The space $\mathcal{K}$ of pairs of distinct Lie spheres in oriented contact is a bundle over Λ. A surface $\tilde{f} : M \to \mathcal{K}$ is said to be a *kissing lift* if it projects to a Legendrian surface $\pi\tilde{f}$ in Λ and the two spheres represented by each point of $\tilde{f}$ are the curvature spheres of $\pi\tilde{f}$ with principal directions corresponding to X_1 and X_2, respectively. Any simply connected, umbilic-free Legendrian surface in Λ has two unique kissing lifts to $\mathcal{K}$, corresponding to the ordering of the principal curvatures.

Theorem 4.10. *Let Ψ be a Legendrian frame for f. Then Ψ is a kissing frame if and only if $\Psi^*\theta_1^3 = 0$ and $\Psi^*\theta_6^4 = 0$.*

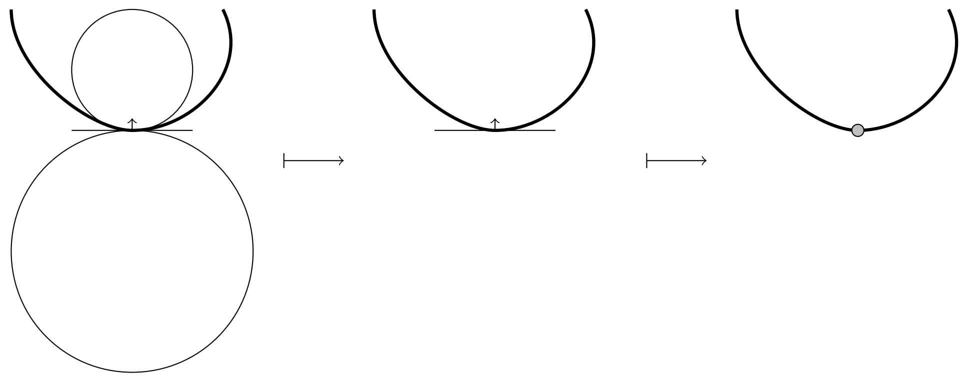


Figure 4.3: An element of the bundle $\mathcal{K} \to \Lambda \to \mathbb{R}^3 \cup \infty$ over a slice of a surface.

Proof. Let $S(r) = \text{Sphere}(f + rn, r)$ be the unique sphere of signed radius r in oriented contact with both $\text{Point}(f)$ and $\text{Plane}(f, n)$. By an appropriate gauge transformation in H_{10} we may assume that

$$X_i = d\text{Point}(f)(\xi_i)$$

for some orthonormal basis ξ_1, ξ_2 of TM. Since X_1, X_2 lie in the principal directions, ξ_1 and ξ_2 do as well. Now consider the inner product

$$\langle dS(r), X_i \rangle = (df, \xi_i) + dr(n, \xi_i) + r(dn, \xi_i) = (df + rdn, \xi_i)$$

Recall that he shape operator σ is the unique symmetric linear operator on f_*TM such that $dn = \sigma df$, and the principal curvatures k_i are the eigenvalues of σ. Since we have chosen X_i to lie along principal directions we must have $\sigma\xi_i = k_i\xi_i$. So altogether we find

$$\begin{aligned}
\langle dS(r), X_i \rangle &= (df + rdn, \xi_i) \\
&= (df, \xi_i) + r(\sigma df, \xi_i) \\
&= (df, \xi_i) + r(df, \sigma\xi_i) \\
&= (1 + rk_i)(df, \xi_i)
\end{aligned}$$

This means that $\langle dS(r), X_i \rangle$ will vanish only when $r = -1/k_i$. In particular, if S_1 and S_2 are to be spheres of radius $-1/k_1$ and $-1/k_2$ then we must have

$$0 = \langle dS_1, X_1 \rangle = a_1^3 = \theta_1^3$$

and

$$0 = \langle dS_2, X_2 \rangle = a_6^4 = \theta_6^4$$

This completes the proof. □

Definition 4.8. The H_7-invariant gEDS $\Theta_{\mathcal{K}}$ generated by $\{\theta_1^5, \theta_1^3, \theta_6^4\}$ is called the *kissing system* . By theorem 4.10, the surfaces in $\mathcal{K}$ which are kissing lifts of surfaces in $\mathbb{R}^3$ are precisely those surfaces which annihilate the kissing system $\Theta_{\mathcal{K}}$.

Cecil [4] and others have used Lie frames adapted to one curvature sphere to analyze Dupin hypersurfaces in $\mathbb{R}^n$. Kissing frames, by contrast, represent *both* curvature spheres of a surface in $\mathbb{R}^3$. The use of kissing frames to study surface geometry appears to be new — perhaps this approach has simply been ignored, as it only may be applied to the study of surfaces in three dimensions.

4.5 The Structure of $\mathfrak{so}(4,2)$

To make future computations in Lie sphere geometry easier, let us first spend some time understanding the structure of the Lie algebra $\mathfrak{so}(4,2)$. With respect to our chosen basis of $\mathbb{R}^{4+2}$, we get the standard matrix representation

$$\mathfrak{so}(4,2) = \left\{ \begin{bmatrix} 0 & x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0 & 0 & x_5 & x_6 & x_7 & x_8 \\ x_1 & -x_5 & 0 & x_9 & x_{10} & x_{11} \\ x_2 & -x_6 & -x_9 & 0 & x_{12} & x_{13} \\ x_3 & -x_7 & -x_{10} & -x_{12} & 0 & x_{14} \\ -x_4 & x_8 & x_{11} & x_{13} & x_{14} & 0 \end{bmatrix} \quad : \quad x_0, x_1, \ldots, x_{14} \in \mathbb{R} \right\}$$

Throughout this section, we will write X_i for the generator of $\mathfrak{so}(4,2)$ given by setting $x_i = 1$ and $x_j = 0$ when $j \neq i$ in the above matrix. When there is a notational clash with the Lie frame columns X_1, X_2, we will call frame columns $\vec{X}_1, \vec{X}_2$.

The generators $\{X_i\}$ are orthogonal with respect to the Killing form B. Furthermore, for any i, j we have $B(X_i, X_i)^2 = B(X_j, X_j)^2 \neq 0$. The generators of negative Killing norm form an 8-dimensional subspace

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & x_0 & x_1 & x_2 & x_3 & 0 \\ x_0 & 0 & 0 & 0 & 0 & x_8 \\ x_1 & 0 & 0 & 0 & 0 & x_{11} \\ x_2 & 0 & 0 & 0 & 0 & x_{13} \\ x_3 & 0 & 0 & 0 & 0 & x_{14} \\ 0 & x_8 & x_{11} & x_{13} & x_{14} & 0 \end{bmatrix} \quad : \quad x_i \in \mathbb{R} \right\}$$

while those of positive Killing norm span the 7-dimensional subalgebra

$$\mathfrak{k} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & x_5 & x_6 & x_7 & 0 \\ 0 & -x_5 & 0 & x_9 & x_{10} & 0 \\ 0 & -x_6 & -x_9 & 0 & x_{12} & 0 \\ 0 & -x_7 & -x_{10} & -x_{12} & 0 & 0 \\ -x_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad : \quad x_i \in \mathbb{R} \right\} \cong \mathfrak{so}(4) \oplus \mathfrak{so}(2)$$

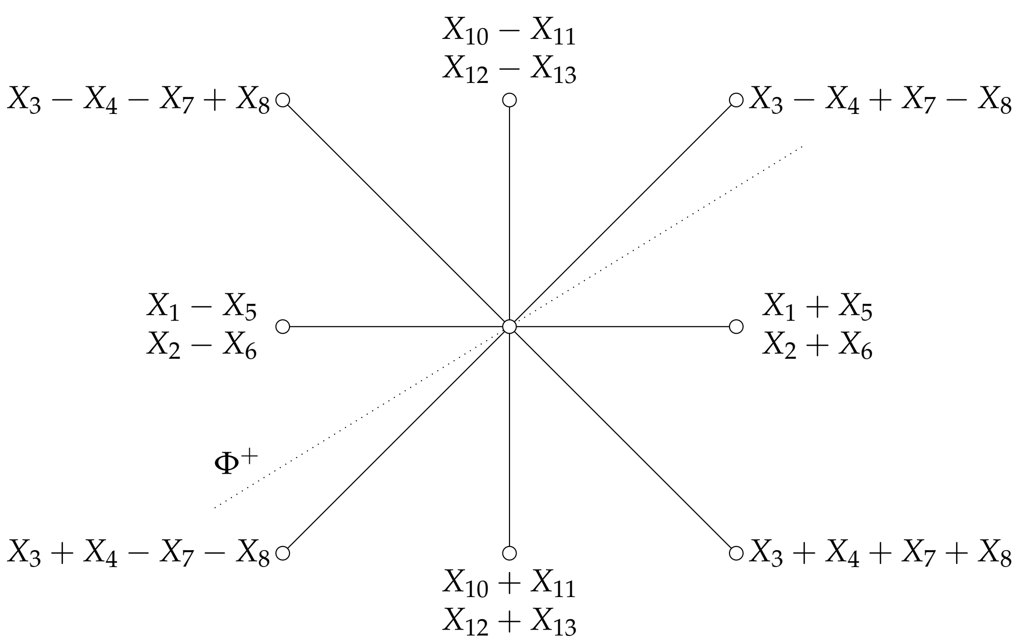


Figure 4.4: The relative roots of $\mathfrak{a}$ acting on $\mathfrak{so}(4,2)$, labelled by generators. The center vertex corresponds to the generators X_0, X_9, X_{14}.

Let us now identify the maximal abelian subalgebra $\mathfrak{a}$ of the negative eigenspace $\mathfrak{p}$. Let

$$X = \begin{bmatrix} 0 & v^T & 0 \\ v & 0 & w \\ 0 & w^T & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & \hat{v}^T & 0 \\ \hat{v} & 0 & \hat{w} \\ 0 & \hat{w}^T & 0 \end{bmatrix}$$

be two arbitrary elements of $\mathfrak{p}$, with $v, w, \hat{v}, \hat{w}$ arbitrary vectors in $\mathbb{R}^4$. Then we have

$$[X, Y] = \begin{bmatrix} 0 & 0 & v \cdot \hat{w} - \hat{v} \cdot w \\ 0 & (v\hat{v}^T - \hat{v}v^T) + (w\hat{w}^T - \hat{w}w^T) & 0 \\ w \cdot \hat{v} - \hat{w} \cdot v & 0 & 0 \end{bmatrix}$$

Lemma 4.11. *The maximal abelian subalgebras of $\mathfrak{p}$ are two-dimensional and have generators of the form*

$$X = \begin{bmatrix} 0 & v^T & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & w^T & 0 \end{bmatrix}$$

with $v \perp w$.

We will make the convention that the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is

$$\mathfrak{a} = \text{span}\{X_0, X_{14}\}$$

corresponding to $v = (1, 0)$ and $w = (0, 1)$ above.

The eigenspaces of $\mathfrak{a}$ acting on $\mathfrak{so}(4, 2)$ and the corresponding eigenvalues are represented by the root diagram in figure 4.4. The particular choice of positive roots Φ^+ is explained in the next section, where the Iwasawa decomposition induced by this choice of roots is related to the structure of the contact space $\Lambda = G/H_{10}$ and the kissing space $\mathcal{K} = G/H_7$.

The purpose of all of this machinery is to give a method for decomposing a general Lie sphere transformation uniquely into a composition of simpler, more understandable transformations.

Lemma 4.12 (Iwasawa Decomposition of $SO(4,2)$)**.** *There is a decomposition*

$$SO(4,2) = KAN = KNA$$

where $K \cong SO(4) \times SO(2)$, $A \cong \mathbb{R}^2$, and N is a 6-dimensional nilpotent group. The Lie algebras of these groups are:

$$\mathfrak{k} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & x_5 & x_6 & x_7 & 0 \\ 0 & -x_5 & 0 & x_9 & x_{10} & 0 \\ 0 & -x_6 & -x_9 & 0 & x_{12} & 0 \\ 0 & -x_7 & -x_{10} & -x_{12} & 0 & 0 \\ -x_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{a} = \begin{bmatrix} 0 & x_0 & 0 & 0 & 0 & 0 \\ x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & 0 & x_{14} & 0 \end{bmatrix}$$

$$\mathfrak{n} = \begin{bmatrix} 0 & 0 & x_1 & x_2 & x_3 - x_8 & -x_3 + x_8 \\ 0 & 0 & -x_1 & -x_2 & -x_3 - x_8 & x_3 + x_8 \\ x_1 & x_1 & 0 & 0 & -x_{11} & x_{11} \\ x_2 & x_2 & 0 & 0 & -x_{13} & x_{13} \\ 0 & 0 & x_{11} & x_{13} & 0 & 0 \\ 0 & 0 & x_{11} & x_{13} & 0 & 0 \end{bmatrix}$$

We will show in the next section that this Iwasawa decomposition interacts nicely with the isotropy subgroups H_{10} and H_7.

4.6 1-Parameter Subgroups of Lie Sphere Transformations

In this section we detail the action of the a complete set of 1-parameter subgroups of G on a standard Lie frame.

Definition 4.9. The *standard Lie frame* Ψ_0 is the Lie frame with columns $\infty, -O, \mathsf{e}_1, \mathsf{e}_2, -\frac{1}{2}\Pi_-, \Pi_+$ where ∞ is the point at infinity, $O = \text{Point}(0)$, and $\Pi_\pm = \text{Plane}(0, \pm\mathsf{e}_3)$.

In the next set of equations, the element of $\mathfrak{g}$ appearing in the subscript denotes the generator of the corresponding 1-parameter subgroup.

The transformations in A act on the standard frame Ψ_0 by

$$\lambda_1(t)\Psi_0 : \quad \left\{ \begin{array}{l} \infty' = e^{-t}\infty \\ O' = e^t O \end{array} \right\}_{X_0}$$

$$\lambda_2(t)\Psi_0 : \quad \left\{ \begin{array}{l} \Pi_+' = e^t\Pi_+ \\ \Pi_-' = e^{-t}\Pi_- \end{array} \right\}_{X_{14}}$$

The transformations in N act on the standard frame Ψ_0 by

$$\alpha_1(t)\Psi_0: \quad \left\{ \begin{array}{l} O' = O + \frac{t^2}{2}\infty + t\mathsf{e}_1 \\ \mathsf{e}_1' = \mathsf{e}_1 + t\infty \end{array} \right\}_{X_1 - X_5}$$

$$\alpha_2(t)\Psi_0: \quad \left\{ \begin{array}{l} O' = O + \frac{t^2}{2}\infty + t\mathsf{e}_2 \\ \mathsf{e}_2' = \mathsf{e}_2 + t\infty \end{array} \right\}_{X_2 - X_6}$$

$$\beta_1(t)\Psi_0: \quad \left\{ \begin{array}{l} \Pi_-' = \Pi_- + t^2\Pi_+ - 2t\mathsf{e}_1 \\ \mathsf{e}_1' = \mathsf{e}_1 - t\Pi_+ \end{array} \right\}_{X_{10} - X_{11}}$$

$$\beta_2(t)\Psi_0: \quad \left\{ \begin{array}{l} \Pi_-' = \Pi_- + t^2\Pi_+ - 2t\mathsf{e}_2 \\ \mathsf{e}_2' = \mathsf{e}_2 - t\Pi_+ \end{array} \right\}_{X_{12} - X_{13}}$$

$$\gamma(t)\Psi_0: \quad \left\{ \begin{array}{l} O' = O + t\Pi_+ \\ \Pi_-' = \Pi_- - 2t\infty \end{array} \right\}_{X_3 - X_4 - X_7 + X_8}$$

$$\delta(t)\Psi_0: \quad \left\{ \begin{array}{l} \infty' = \infty + 2t\Pi_+ \\ \Pi_-' = \Pi_- - 4tO \end{array} \right\}_{X_3 - X_4 + X_7 - X_8}$$

As we will have less use for the rotations in K, we will not enumerate their one-parameter subgroups here except for two important cases:

$$\varepsilon(t)\Psi_0 : \quad \left\{ \begin{aligned} \mathsf{e}_1' &= \mathsf{e}_1 \cos t - \mathsf{e}_2 \sin t \\ \mathsf{e}_2' &= \mathsf{e}_1 \sin t + \mathsf{e}_2 \cos t \end{aligned} \right\}_{X_9}$$

$$\zeta(t)\Psi_0 : \quad \left\{ \begin{aligned} \infty' &= \infty \cos t + \Pi_+ \sin t \\ \Pi^{+'} &= -\infty \sin t + \Pi_+ \cos t \\ O' &= O \cos t + \tfrac{1}{2}\Pi_- \sin t \\ \Pi^{-'} &= -2O \sin t + \Pi_- \cos t \end{aligned} \right\}_{X_7 - X_4}$$

The only other significant fact we need about K is that all the one-parameter subgroups of K except for $\varepsilon(t)$ and $\zeta(t)$ disturb the projective line $[\infty : \Pi_+]$, and therefore belong to neither H_{10} nor H_7.

From the actions of these one parameter subgroups, it is easy to see the following two facts:

Lemma 4.13. *The subalgebra $\mathfrak{h}_{10}$ is spanned by $\mathfrak{a} \oplus \mathfrak{n} \oplus \{X_9, X_7 - X_4\}$, where $\mathfrak{n}$ is the nilpotent subalgebra spanned by the eigenvectors corresponding to the positive roots in figure 4.4.*

Proof. By inspection, we can see that none of the 1-parameter subgroups in A and N disturb the line $[\infty : \Pi_+]$. The transformations in A map the equivalence classes $[\infty]$ and $[\Pi_+]$ to themselves. As for the transformations in N, only $\delta(t)$ has any effect on ∞ or Π_+, but it only acts to replace Π_+ with a linear combination of Π_+ and ∞. In particular, the point $[\infty : \Pi_+] \in \Lambda$ remains fixed.

Now we consider the other two alleged generators X_9 and $X_7 - X_4$. X_9 fixes ∞, O, Π_-, and Π_+ so commutes with A. For the elements of N, it it straightforward to see that conjugation by X_9 mixes α_1 with α_2, β_1 with β_2, and leaves

γ and δ fixed. Therefore the semidirect product $X_9 \rtimes A \cdot N$ is also a subgroup of H_{10}. Finally, we consider the generator $X_7 - X_4$ of ζ. The adjoint action of $X_7 - X_4$ on $\mathfrak{n}$ acts as a unit-speed rotation mixing α_1 with β_1 and α_2 with β_2. With regards to the subalgebra $\mathfrak{a}$, we have $[X_7 - X_4, \mathfrak{a}] \subset \mathfrak{a}$. Finally, note that X_9 and $X_7 - X_4$ commute; along with the previous observation that no other 1-parameter subgroups of K fix $[\infty : \Pi_+]$, this shows that there is a 10-dimensional subgroup H_{10} of G whose Lie algebra is spanned by $\mathfrak{a}$, $\mathfrak{n}$, X_9 and $X_7 - X_4$. $\square$

Lemma 4.14. *The isotropy subalgebra $\mathfrak{h}_7$ is a codimension-1 subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent subalgebra spanned by the eigenvectors corresponding to the positive roots in figure 4.4.*

Proof. H_7 is the subgroup of H_{10} which fixes the equivalences classes $[\infty]$ and $[\Pi_+]$ and the kissing system $\Theta_{\mathcal{K}} \subset \mathfrak{so}(4,2)^*$, so to identify H_7 we will first discard any transformations in H_{10} which do not have ∞ and Π_+ for eigenvectors.

A quick survey of the generators of H_{10} shows that the 1-parameter subgroup $\delta(t)$ corresponding to the generator $X_3 - X_4 + X_7 - X_8$ is problematic, since it sends ∞ to $\infty + 2t\Pi_+$. This could only be cancelled by composing with a cleverly chosen $\zeta(s)$, but this would still not save us: $\zeta(s)$ would then perturb Π_+, and this perturbation could not be counteracted since Π_+ is an eigenvector for all the other generators of H_{10}. This means that $\mathfrak{h}_7$ can contain neither $X_7 - X_4$ nor $X_3 - X_4 + X_7 - X_8$.

Next we may consider the subgroup $\varepsilon(t)$. This subgroup simply acts as rotations in the $\mathsf{e}_1 \wedge \mathsf{e}_2$ plane, so it acts on a Lie frame Ψ by rotating the frame vectors $\vec{X}_1$ and $\vec{X}_2$. It is clear that this subgroup fixes both ∞ and Π_+; however, it fails to fix the contact forms $\langle dS_1, \vec{X}_1 \rangle$ and $\langle dS_2, \vec{X}_2 \rangle$. This excludes X_9 from being in $\mathfrak{h}_7$.

The only remaining subgroups of concern are the α and β subgroups, since these affect $\mathsf{e}_1, \mathsf{e}_2$ and therefore change the frame vectors $\vec{X}_1, \vec{X}_2$. But these subgroups all act in such a way that $\vec{X}_i \mapsto \vec{X}_i \pm f(t) S_i$. Along with the fact that S_i is a Lie sphere ($\langle S_i, S_i \rangle = 0$), we have

$$\langle dS_i, \vec{X}_i \rangle \mapsto \langle dS_i, \vec{X}_i \pm f(t) S_i \rangle = \langle dS_i, \vec{X}_i \rangle$$

Thus, the Lie algebra $\mathfrak{h}_7$ is spanned by $\mathfrak{a}$ and the elements of $\mathfrak{n}$ which are perpendicular to $X_3 - X_4 + X_7 - X_8$ with respect to the Killing form. This set is indeed closed under the Lie bracket since the given generators are all roots for the action of $\mathfrak{a}$ and $X_3 - X_4 + X_7 - X_8$ has top level in $\mathfrak{n}$, forcing $[X_3 - X_4 + X_7 - X_8, \mathfrak{n}] \subset (X_3 - X_4 + X_7 - X_8)^{\perp}$. □

We will also need to make more detailed use of the transformations in H_7, so let us spend some time looking in more detail at how they act. Note that any Lie sphere transformation is determined by its action on the finite point spheres, since the image of some set of points Z by a transformation g may be recovered as an envelope of $g \cdot Z$.

1. The pair α_1, α_2. These simply act as Euclidean translations. The point sphere $\mathrm{Point}(x)$ gets sent by $\alpha_i(t)$ to the point sphere $\mathrm{Point}(x + t\mathsf{e}_i)$.
2. The pair β_1, β_2. These transformations have a more complex action. The point sphere $\mathrm{Point}(x)$ is inflated by $\beta_i(t)$ to a sphere of signed radius $-tx_i - \frac{1}{2}t^2 x_3$. The center of this sphere is at $x + tx_3\mathsf{e}_i - (tx_1 + \frac{1}{2}t^2 x_3)\mathsf{e}_3$.
3. The transformation γ. This transformation is more basic. The point spheres $\mathrm{Point}(x)$ simply get inflated to spheres of radius t and center $x + t\mathsf{e}_3$.

4. The transformation λ_1: This transformation is particularly simple: $\lambda_1(t)$ acts as a uniform dilation, taking $\text{Point}(x)$ to $\text{Point}(e^{-t}x)$.

5. The transformation λ_2: This transformation has an action similar to γ. The point sphere $\text{Point}(x)$ is inflated to a sphere of radius $\sinh(t)$ and center $x + \cosh(t)\mathrm{e}_3$.

To finish this section, we will catalog the roots of $\mathfrak{a}$ acting on $\mathfrak{g}^* \wedge \mathfrak{g}^* \mod \Theta_{\mathcal{K}}$ which additionally lie in

$$\bigcap_{n \in \mathfrak{n} \cap \mathfrak{h}_7} \ker(n \lrcorner \delta \mod \Theta_{\mathcal{K}}) \tag{4.1}$$

These are precisely the G-invariant 2-forms on $\mathcal{K}$.

Theorem 4.15. *The coadjoint action of $\mathfrak{a}$ on certain matrix elements $\theta^i_j \wedge \theta^k_l$ is represented by the root diagram in figure 4.5. The given forms also span the intersection of the kernels shown in 4.1, and therefore descend to a complete set of invariant 2-forms on $\mathcal{K}$.*

4.7 An Invariant Cross-Ratio

Since G is a group of contact transformations rather than point transformations, it takes some practice to understand the action of G on the space of Lie spheres intuitively. In this section, we will derive a useful invariant of G acting on $\mathcal{L}$.

Lemma 4.16 (Sphere Separation)**.** *To any countable collection of Lie spheres $S_0, S_1, S_2, \dots$ there is a Lie sphere transformation $g \in G$ such that none of $g \cdot S_1, g \cdot S_2, \dots$ are in contact with S_0.*

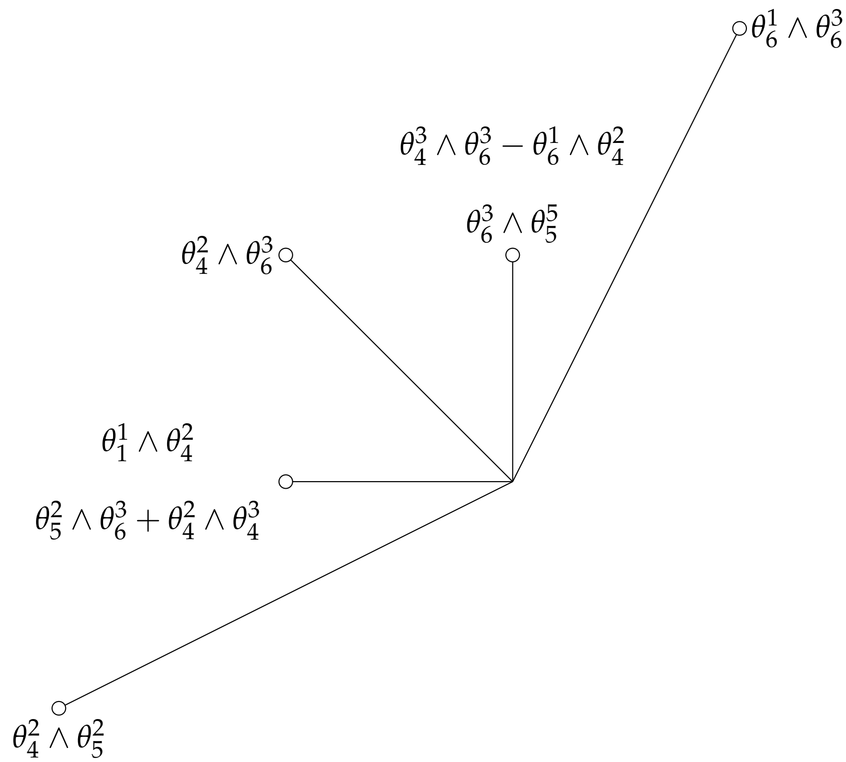


Figure 4.5: The seven $\mathfrak{h}_7$-invariant 2-forms, displayed as a root diagram for the action of $\mathfrak{a}$.

Proof. Since G acts transitively on the set of Lie spheres, we may assume without loss of generality that we have chosen a reference frame in which S_0 is a point sphere. Let $\hat{S}_0$ be any point sphere not contained in any of $S_0, S_1, S_2 \dots$. Then g may be taken to be a Möbius involution interchanging S_0 and $\hat{S}_0$. Since no S_i contains $\hat{S}_0$ and $\hat{S}_0 = g \cdot S_0$ we have

$$0 \neq \langle S_i, \hat{S}_0 \rangle = \langle S_i, g \cdot S_0 \rangle = \langle g \cdot S_i, S_0 \rangle$$

demonstrating that none of $g \cdot S_i$ are in contact with S_0. □

Definition 4.10. Let S_1 and S_2 be a pair of distinct oriented spheres. There are precisely two double cones C, C' which are in unoriented contact with both S_1 and S_2. Of these two cones, exactly one (C, say) will also be in *oriented* contact with both S_1 and S_2, for the right choice of orientation on the cone. The distance T from $S_1 \cap C$ to $S_2 \cap C$ is called the *outer tangential distance* between S_1 and S_2.

The outer tangential distance is depicted graphically in figure 4.6. The faint grey line in the figure lies on the cone C' with mismatched orientation.

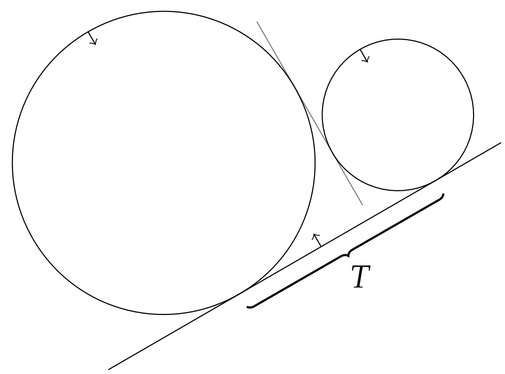


Figure 4.6: Measuring the outer tangential distance T.

Lemma 4.17. *Let $S, \hat{S}$ be the standard vectors representing a pair of Lie spheres. Then the inner product $\langle S, \hat{S} \rangle$ is equal to:*

1. *0 if S and $\hat{S}$ are in oriented contact.*
2. *The squared outer tangential distance between S and $\hat{S}$, if both are spheres.*
3. -1 *if one of S and $\hat{S}$ is ∞ and the other is a sphere.*
4. $\cos(\theta) - 1$ *if both S and $\hat{S}$ are planes meeting in an angle θ.*
5. *The distance of closest oriented separation if one of S and $\hat{S}$ is a plane and the other is a sphere.*

Proof. The proof is by direct calculation using the maps Point, Sphere, Plane and the vector ∞. □

Lemma 4.18. *Let S_1, S_2 and $\hat{S}_1, \hat{S}_2$ be four Lie spheres. Then the cross-ratio of the outer tangential distances*

$$[S_1, S_2; \hat{S}_1, \hat{S}_2] = \frac{\langle S_1, \hat{S}_1 \rangle \langle S_2, \hat{S}_2 \rangle}{\langle S_2, \hat{S}_1 \rangle \langle S_1, \hat{S}_2 \rangle} \in \mathbb{RP}^1$$

is G-invariant.

Proof. Since G preserves the Lie metric $\langle , \rangle$ it is sufficient to note that the cross-ratio is well-defined — since each term appears once in the numerator and once in the denominator, the choice of null vector representing each Lie sphere does not affect the quotient. In particular, we can assume that each sphere is represented by a vector in standard form. By the sphere separation lemma 4.16 we can also assume that none of $S_1, S_2, \hat{S}_1, \hat{S}_2$ are planes or ∞, so that the given quantity is in fact the cross-ratio of the outer tangential distances. □

Cross-ratios of various types have been studied in relation to Lie sphere geometry. For one particular example, see [4] for a discussion on the invariant *Lie curvatures* of immersed submanifolds of dimension ≥ 4, formed by taking cross-ratios of principal curvatures. Of course, the quantity appearing above is not *literally* a cross-ratio; still, it has the same qualitative feel. This cross-ratio of outer tangential distances appears to be a new invariant in Lie sphere geometry — we will apply it fruitfully later on to obtain an interesting second-order relation between surfaces in Euclidean space.

Theorem 4.19. *Let X_4 be the space of quadruples $S_1, S_2, \hat{S}_1, \hat{S}_2$ of Lie spheres such that S_1, S_2 are in contact, $\hat{S}_1$, $\hat{S}_2$ are in contact, and $(S_1, S_2) \neq (\hat{S}_1, \hat{S}_2)$. Then cross-ratio gives a map $R : X_4 \to \mathbb{RP}^1$, and G acts transitively on the level sets of R.*

Proof. We may proceed by first moving $\hat{S}_1$ and $\hat{S}_2$ to a standard position and then attempting to move S_1 and S_2 into some standard form. So without loss of generality, assume that we have applied a Lie sphere transformation so that $\hat{S}_1 = \infty$, $\hat{S}_2 = \Pi_+$. The object is to now find a transformation in H_{10} which moves S_1, S_2 to a standard position, as shown in figure 4.7. We will actually be able to realize the transformation to standard position by an element of H_7.

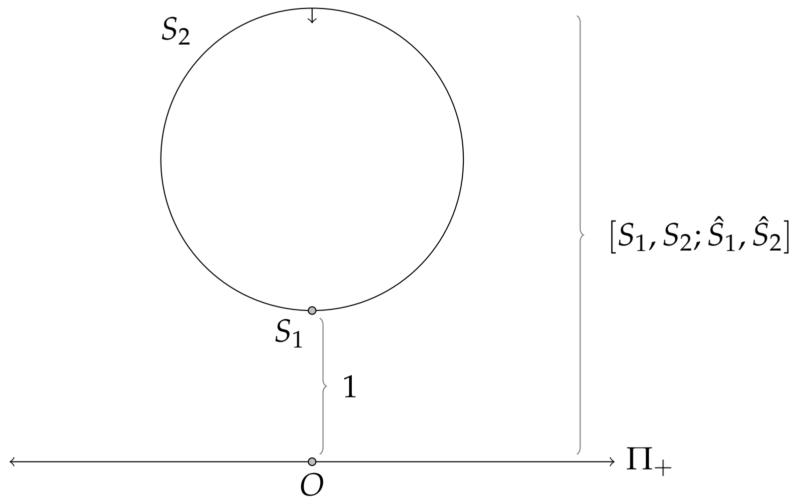


Figure 4.7: Standard position for theorem 4.19.

Since neither of S_1, S_2 is in contact with either of $\hat{S}_1, \hat{S}_2$, they each represent standard spheres or points after $\hat{S}_1, \hat{S}_2$ are moved to ∞, Π_+. In particular, neither S_1 nor S_2 is a plane and we can associate with each a finite center and a finite radius. As a result, the standard position may be achieved through the following steps:

1. Apply β_i transformations to vertically stack the centers of S_1 and S_2. This is possible since the centers move at a rate proportional to the oriented height above Π_+, and so by continuity there is a transformation which aligns the centers when the oriented heights are distinct. On the other hand, when the oriented heights above Π_+ are equal the centers are already vertically stacked.

2. Apply α_i transformations to center S_1 and S_2 above O.
3. Apply the γ transformation to shrink S_1 to a point.
4. Apply $\pm\exp(tX_0)$ to dilate the configuration until S_1 is a distance of 1 above Π_+. This is possible since S_1 is not in contact with $\hat{S}_2$, so the height of S_1 above Π_+ is nonzero.

The cross-ratio of this configuration is given by

$$\frac{\langle S_1, \infty\rangle\langle S_2, \Pi_+\rangle}{\langle S_2, \infty\rangle\langle S_1, \Pi_+\rangle} = \frac{-1 \cdot L}{-1 \cdot 1} = L$$

where L is the lower height of S_2. The demonsrates that the constructed configuration is in standard form, which completes the proof. □

4.8 A Geometric Interpretation of Forms in $\mathfrak{so}(4,2)^*$

In this section, we aim to interpret the basic invariant 1-forms in the kissing space $\mathcal{K}$.

First, let us suppose that a surface described by a map $f : M \to \mathbb{R}^3$ is given, with unit normal field $n : M \to S^2$. We will further assume that f is umbilic-free. Then f may be assigned a Lie frame Ψ_f in the following way:

Define

$$p = \text{Point}(f) = (\frac{1+|f|^2}{2}, \frac{1-|f|^2}{2}, f, 0)$$
$$\tau^{\pm} = \text{Plane}(f, \pm n) = ((f, \pm n), (f, \mp n), \pm n, 1)$$

and two unit tangent vector fields $\xi_i : M \to S^2$ such that (ξ_1, ξ_2, n) form a positive orthonormal basis of $\mathbb{R}^3$ and ξ_i points in the ith principal direction, corresponding to the sectional curvature k_i. Since f is umbilic-free, we have $k_1 \neq k_2$. As a result, we may define the *standard kissing frame* to be the Lie frame Ψ_f with columns S_1, A, X_1, X_2, B, S_2 as follows:

$$S_i = \tau^+ - k_i p$$
$$X_i = ((f, \xi_i), -(f, \xi_i), \xi_i, 0)$$
$$A = \frac{1}{k_1 - k_2}\left(\infty + \frac{1}{2}k_2\tau^-\right)$$
$$B = \frac{-1}{k_1 - k_2}\left(\infty + \frac{1}{2}k_1\tau^-\right)$$

Using the relations $\langle \tau^+, \tau^- \rangle = -2$, $\langle \infty, p \rangle = -1$ it is straightforward to verify that Ψ_f is a Lie frame.

Lemma 4.20. *The Lie frame Ψ_f is a kissing frame for f.*

Proof. Throughout this proof, we let $\theta = \Psi_f^{-1} d\Psi_f$ be the Maurer-Cartan form associated to Ψ_f.

To see that Ψ_f is Legendre, we compute

$$\begin{aligned}
\theta^2_6 &= \langle S_1, dS_2 \rangle \\
&= \langle \tau^+ - k_1 p, d\tau^+ - dk_2 p - k_2 dp \rangle \\
&= (k_1 - k_2)\langle \tau^+, dp \rangle \\
&= (k_1 - k_2)\left(-(f, n)(f, df) + (f, n)(f, df) + (n, df)\right) = 0
\end{aligned}$$

To show that Ψ_f is a kissing frame, we will use the fact that the principal directions ξ_i are eigenvectors of the shape operator σ with eigenvalues k_i. We have previously noted that the shape operator σ is the unique symmetric linear transformation on f_*TM such that $dn = \sigma df$. Using this fact, we may compute the kissing forms θ^2_3 and θ^4_6 (corresponding to $i = 1$ and $i = 2$, respectively):

$$\begin{aligned}
\langle S_i, dX_i \rangle &= \langle \tau^+ - k_i p, dX_i \rangle \\
&= \langle \tau^+ - k_i p, ((df, \xi_i) + (f, d\xi_i), -(df, \xi_i) - (f, d\xi_i), d\xi_i, 0) \rangle \\
&= (n, d\xi_i) + k_i(df, \xi_i) = -(dn, \xi_i) + k_i(df, \xi_i) \\
&= -(\sigma df, \xi_i) + k_i(df, \xi_i) \\
&= -(df, \sigma\xi_i) + k_i(df, \xi_i) = 0
\end{aligned}$$

This proves that Ψ_f annihilates the kissing system $\Theta_{\mathcal{K}} = \{\theta^2_6, \theta^2_3, \theta^4_6\}$, so Ψ_f is a kissing frame for f. □

The next result catalogues the geometric meaning of each matrix element in the derivative $\Psi_f^{-1} d\Psi_f$ of the standard kissing frame.

Theorem 4.21. *Let $f : M \to \mathsf{E}^3$ define a regular surface of class C^3, and let Ψ_f be its standard kissing frame, with Maurer-Cartan form $\theta = \Psi_f^{-1} d\Psi_f$. Let ξ_1, ξ_2 be unit vectors pointing along the principal directions, corresponding to the principal curvatures k_1, k_2. Then the matrix elements of θ are:*

- $\theta^1_2 = \theta^2_1 = \theta^5_5 = \theta^6_6 = 0$
- $\theta^2_6 = -\theta^5_1 = 0$, $\theta^2_3 = -\theta^3_1 = 0$, $\theta^4_6 = -\theta^5_4 = 0$
- $\theta^3_4 = -\theta^4_3 = (\xi_1, d\xi_2)$
- $\theta^3_6 = -\theta^5_3 = (k_1 - k_2)(\xi_1, df)$
- $\theta^2_4 = -\theta^4_1 = (k_1 - k_2)(\xi_2, df)$
- $\theta^2_5 = -\theta^6_1 = \theta^1_1 = -\theta^2_2 = \frac{dk_1}{k_1 - k_2}$
- $\theta^1_6 = -\theta^5_2 = \theta^5_5 = -\theta^6_6 = \frac{dk_2}{k_1 - k_2}$

- $\theta^1_5 = -\theta^6_2 = 0$
- $\theta^3_5 = -\theta^6_3 = \frac{1}{2}\frac{k_1^2}{k_1-k_2}(\xi_1, df)$
- $\theta^4_5 = -\theta^6_4 = \frac{1}{2}\frac{k_1 k_2}{k_1-k_2}(\xi_2, df)$
- $\theta^1_3 = -\theta^3_2 = \frac{1}{2}\frac{k_1 k_2}{k_1-k_2}(\xi_1, df)$
- $\theta^1_4 = -\theta^4_2 = \frac{1}{2}\frac{k_2^2}{k_1-k_2}(\xi_2, df)$

Several of these forms have appeared in the literature; particularly notable is the Lie-invariant quadratic differential

$$q = \theta^2_5 \theta^1_6 = \frac{dk_1 dk_2}{(k_1 - k_2)^2}$$

The form q has taken on a role in Lie sphere geometry analogous to the metric in Euclidean geometry. In [8], Ferapontov defines *Lie minimal* surfaces as the stationary surfaces for the functional $\int q$; a study of surfaces which are Lie minimal, diagonally cyclidic, or both then leads to connections with several important integrable systems, including the Tzitzeica and modified Veselov–Novikov equations. These connections to known integrable systems were one of the original motivations for looking into applications of theorems 2.11 and 2.12 to geometries based upon Lie sphere transformations.

Note that these 1-forms are not individually $\mathfrak{h}_7$-invariant — they depend on the particular choice of Ψ_f as the standard kissing frame. However, we may still use the preceding equations to give geometric interpretations to the seven $\mathfrak{h}_7$-invariant 2-forms as follows:

Theorem 4.22. *The seven Lie-invariant 2-forms appearing in theorem 4.15 are proportional to the following geometric quantities:*

- $\theta^2_5 \wedge \theta^3_6 + \theta^2_4 \wedge \theta^3_4 \quad \propto \quad dk_1 \wedge (\xi_1, df) - (k_1 - k_2)(\xi_1, d\xi_2) \wedge (\xi_2, df)$

- $\theta^3_6 \wedge \theta^3_4 - \theta^2_4 \wedge \theta^1_6 \quad \propto \quad dk_2 \wedge (\xi_2, df) - (k_1 - k_2)(\xi_1, d\xi_2) \wedge (\xi_1, df)$
- $\theta^1_1 \wedge \theta^2_4 \quad \propto \quad dk_1 \wedge (\xi_2, df)$
- $\theta^2_5 \wedge \theta^2_4 \quad \propto \quad dk_1 \wedge (\xi_2, df)$
- $\theta^5_5 \wedge \theta^3_6 \quad \propto \quad dk_2 \wedge (\xi_1, df)$
- $\theta^1_6 \wedge \theta^3_6 \quad \propto \quad dk_2 \wedge (\xi_1, df)$
- $\theta^3_5 \wedge \theta^2_4 \quad \propto \quad (k_1 - k_2)^2 (\xi_1, df) \wedge (\xi_2, df)$

Once again, these forms are not strictly invariant — they are invariant under $\mathfrak{h}_7 \cap \mathfrak{n}$ and are roots of the coadjoint action of $\mathfrak{a}$ on $\mathfrak{g}^* \wedge \mathfrak{g}^* \mod \Theta_{\mathcal{K}}$ with nonzero weight. In other words, the *forms* are not invariant but the *lines through each form* are. In particular, the vanishing of any of the forms defines a Lie-invariant differential equation on $\mathcal{K}$.

4.9 Modifying the Main Theorems

Now that the structure of the homogeneous spaces $\mathcal{K}$ and Λ has been explained and their fundamental geometric exterior differential systems described, we turn to the problem of applying the main theorems 2.11 and 2.12 to these new geometries.

The space of contact elements Λ does not have any interesting Lie-invariant relations, for the same reason that the Möbius group does not have interesting invariants between points when acting on the sphere: the group action is *too transitive* (and, in fact, *2-transitive*). If a group G is acting 2-transitively on the space X then for any x, y, x', y' with $x \neq y$, $x' \neq y'$ we can find a transformation

$g \in G$ such that $x' = g \cdot x, y' = g \cdot y$. But then if $\sim$ is some invariant relation on G we have $x \sim y$ implies $x' = g \cdot x \sim g \cdot y = y'$. In other words, if G acts 2-transitively on X then the only G-invariant relations are the trivial cases "is equal" and "is not equal". Since G acts 2-transitively on Λ, there are therefore no interesting Lie-geometric relations on Λ which could be exploited to find a generalized Bianchi or Lie theorem.

The situation is more interesting in the case of $\mathcal{K}$, however. Points of $\mathcal{K}$ represent a pair of Lie spheres in oriented contact, and lemma 4.18 demonstrates that the cross-ratio of outer tangential distances in a quadruple of Lie spheres is G-invariant. This cross-ratio therefore descends to an invariant relation on $\mathcal{K}$: if (S_1, S_2) and $(\hat{S}_1, \hat{S}_2)$ are a pair of elements of $\mathcal{K}$ then the quantity

$$\frac{\langle S_1, \hat{S}_1 \rangle \langle S_2, \hat{S}_2 \rangle}{\langle S_2, \hat{S}_1 \rangle \langle S_1, \hat{S}_2 \rangle}$$

is invariant. We then have an invariant relation, akin to constant geodesic distance in the classical geometries.

Since G is 15-dimensional and H_7 is 7-dimensional, we know that $\mathcal{R}_{\mathcal{K}} = H_7 \backslash G / H_7$ must be at least 1-dimensional and, by lemma 4.18, the invariant cross-ratio provides a coordinate. Unlike the previous sections, we will not attempt to describe the entire space of symmetric relations on $\mathcal{K}$; instead, we will look at a single symmetric relation and use it as an archetype for the extension of theorems 2.11 and 2.12 to Lie sphere geometry. A complete list of symmetric relations could be obtained by applying lemma 2.8 as before, along with an analysis of the Iwasawa decomposition for elements of the form $NANA$.

From this point onward, let us fix a symmetric relation $[\rho] \in \mathcal{R}_{\mathcal{K}}^{sym}$, where

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

ρ^2 preserves ∞ and Π_+, and therefore is an element of H_7. By lemma 2.8, $[\rho]$ must then be a symmetric relation. With this particular relation, the standard kissing configuration $[\Psi_0] = \star \in \mathcal{K}$ and the ρ-related configuration $\rho \cdot \star$ have invariant cross-ratio -1.

Theorem 4.23. *Let $f : M \to \mathbb{R}^3$ an immersion of a simply-connected surface in $\mathbb{R}^3$. Then there exists a 7-parameter family of surfaces $\hat{f}$ such that the cross-ratio of the outer tangential distances between the curvature spheres of f and those of $\hat{f}$ is equal to -1 at every point. The surfaces $\hat{f}$ may be constructed explicitly by integrating a sequence of five ordinary differential equations.*

Proof. Recall that the kissing system $\Theta = \Theta_{\mathcal{K}}$ is the gEDS generated by $\{\theta^2_6, \theta^2_3, \theta^4_6\}$. The ρ-transformed system $\hat{\Theta} = \mathrm{Ad}^*(\rho)(\Theta)$ is then given by

$$\begin{aligned} \hat{\theta}^2_6 &= \theta^1_1 + \theta^5_5 + \theta^1_6 + \theta^2_5 \\ \hat{\theta}^2_3 &= \theta^1_3 + \theta^3_5 + \frac{1}{2}\theta^3_6 \\ \hat{\theta}^4_6 &= -\theta^1_4 - \theta^4_5 - \frac{1}{2}\theta^2_4 \end{aligned}$$

Since the dimensions of Θ and H_7 do not match, $\mathfrak{h}_7$ cannot be transverse to $\hat{\Theta}$. As a result, the requirements of theorems 2.11 and 2.12 are not satisfied and the theorems, as stated, become inapplicable. However, all hope is not lost; if we could find a larger gEDS $\tilde{\Theta}$ which extends $\hat{\Theta}$ and is transverse to $\mathfrak{h}_7$, then the proofs of the main theorems would still be able to operate.

Deferring any and all motivation, let us define $\tilde{\Theta}$ to be the subspace of $\mathfrak{g}^*$ spanned by

$$\begin{aligned}
\tilde{\theta}^1_1 &= \theta^1_1 + \theta^2_5 \\
\tilde{\theta}^5_5 &= \theta^1_6 + \theta^5_5 \\
\tilde{\theta}_u &= \theta^1_3 + \theta^3_5 + \frac{1}{2}\theta^3_6 \\
\tilde{\theta}_v &= \theta^1_4 + \theta^4_5 + \frac{1}{2}\theta^2_4 \\
\tilde{\theta}^1_5 &= \theta^1_5 + \frac{1}{2}\theta^2_5 + \frac{1}{2}\theta^1_6
\end{aligned}$$

These forms are both transverse to and invariant under the 5-dimensional subalgebra $\mathfrak{h}_5$ generated by $\dot{\gamma}$, $\dot{\lambda}_i$, and $\dot{\alpha}_i + \dot{\beta}_i$. Also note that $\tilde{\Theta}$ is an extension of $\hat{\Theta}$, so integral manifolds to $\tilde{\Theta}$ are also integral to $\hat{\Theta}$. Then as in the proof of theorem 2.11, we may construct an EDS Λ on $G \times H_5$ such that integral manifolds of Λ yield the gauge transformations needed to construct integral manifolds to $\tilde{\Theta}$. We omit the calculations here, as they have the dual misfortunes of being both tedious and unenlightening. If the reader would like to verify the calculations by hand, the tables in appendix C should come in handy. The important point is that the relative curvatures of $\tilde{\Theta}$ vanish modulo Θ, so there is in fact *no obstruction whatsoever* to finding integral manifolds to Λ. This completes the proof.

Since the actual calculations were omitted, they will be replaced by this attempt at explaining how the system $\tilde{\Theta}$ was conjured up. The 1-forms listed above were chosen with extensive help from the computer algebra system `SAGE` by searching for spanning sets which are transverse to subalgebras of $\mathfrak{h}_7$, extend $\hat{\Theta}$, and on which the relative curvatures take values in the space of $\mathfrak{h}_7$-invariant 2-forms. Since modifying one basis vector will have quadratic effect on each of the relative curvatures, this was not an easy search to carry out either abstractly or symbolically, even with `SAGE`. Instead, a more dynamic approach was taken. `SAGE` was then used to rapidly recompute the relative curvature as vectors were added or adjusted in the spanning set; this had the distinct feel of fitting together puzzle pieces or working a Rubik's cube, where later moves appear to disrupt earlier work. Although no clever algorithm manifested itself, a few hours of play with this setup was sufficient to solve the puzzle and discover these suitable forms for $\tilde{\Theta}$. $\square$

APPENDIX A

VECTOR-VALUED DIFFERENTIAL FORMS

A.1 Forms with Values in an Inner Product Space

Suppose that V is a vector space over $\mathbb{F}$ equipped with a non-degenerate inner product $\langle , \rangle$ and let ${}^T : V \to V^*$ and ${}^T : V^* \to V$ be the induced isomorphisms. We will use $O(V)$ to denote the group of linear isomorphisms of V preserving $\langle , \rangle$.

Lemma A.1. *If $A, B \in \Omega^1_M(V)$ then $A^T \wedge B \in \Omega^2_M(\mathbb{F})$ and*

$$A^T \wedge B = -B^T \wedge A$$

Furthermore, if $g \in O(V)$ then $(g \cdot A)^T \wedge (g \cdot B) = A^T \wedge B$.

Proof. Choose coordinates on M and write the components of ω as $\omega_i dx^i$. By definition,

$$\begin{aligned} A^T \wedge B &= \sum_{i<j} \left(\langle A_i, B_j \rangle - \langle A_j, B_i \rangle \right) dx^i \wedge dx^j \\ &= -\sum_{i<j} \left(\langle B_i, A_j \rangle - \langle B_j, A_i \rangle \right) dx^i \wedge dx^j \\ &= -B^T \wedge A \end{aligned}$$

Finally, since $g \in O(V)$ we have $g^T = g^{-1}$, so

$$(g \cdot A)^T \wedge (g \cdot B) = A^T \cdot g^{-1} \wedge g \cdot B = A^T \wedge B$$

□

Corollary A.2. *For any vector-valued 1-form A,*

$$A^T \wedge A = 0$$

Corollary A.3. *For any vector-valued 1-form A, $A^T \otimes A$ is a quadratic differential.*

Proof. By the previous corollary, the antisymmetric part of $A^T \otimes A$ vanishes; in other words, $A^T \otimes A$ is symmetric. □

Lemma A.4. *If $A, B \in \Omega^1_M(V)$ then $A \wedge B^T + B \wedge A^T \in \Omega^2_M(\mathfrak{o}(V))$.*

Proof. At the very least $A \wedge B^T + B \wedge A^T \in \Omega^2_M(V \otimes V^*)$, so we only need to prove that this 2-form is skew-symmetric with respect to ^{T}. But just as in the proof of the previous lemma,

$$\left(A \wedge B^T + B \wedge A^T\right)^T = -B \wedge A^T - A \wedge B^T$$

so in fact $A \wedge B^T + B \wedge A^T$ takes values in $\mathfrak{o}(V)$. □

Corollary A.5. *For any vector-valued 1-form A,*

$$A \wedge A^T \in \Omega^2_M(\mathfrak{o}(V))$$

A.2 $\mathbb{R}^2$-valued Forms

Let us now specialize to the case of $\mathbb{R}^2$ with the standard inner product and M a 2-dimensional manifold with local coordinates x, y. We will use i to denote the linear transformation given by the matrix

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in an orthonormal basis.

If $A \in \Omega^1_M(\mathbb{R}^2)$ then after choosing coordinates x, y on M and a orthonormal basis $\mathsf{e}_1, \mathsf{e}_2$ on $\mathbb{R}^2$, we may write

$$A = \begin{pmatrix} A^1_x & A^1_y \\ A^2_x & A^2_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

We will use $[A]$ to denote the square matrix on the right-hand side.

Lemma A.6. $A^T \wedge iA = -2\det[A]\, dx \wedge dy$

Proof. We proceed by direct computation.

$$\begin{aligned} A^T \wedge iA &= \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} A^1_x & A^2_x \\ A^1_y & A^2_y \end{pmatrix} \wedge \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^1_x & A^1_y \\ A^2_x & A^2_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \\ &= \begin{pmatrix} dx & dy \end{pmatrix} \wedge \begin{pmatrix} A^1_x & A^2_x \\ A^1_y & A^2_y \end{pmatrix} \begin{pmatrix} -A^2_x & -A^2_y \\ A^1_x & A^1_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \\ &= \begin{pmatrix} dx & dy \end{pmatrix} \wedge \begin{pmatrix} 0 & -A^1_x A^2_y + A^2_x A^1_y \\ A^1_x A^2_y - A^2_x A^1_y & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \\ &= \det[A] \cdot \begin{pmatrix} dx & dy \end{pmatrix} \wedge \begin{pmatrix} -dy \\ dx \end{pmatrix} \\ &= -2\det[A]\, dx \wedge dy \end{aligned}$$

□

APPENDIX B

LEMMAS FOR PDE SYSTEMS

Definition B.1. A k-dimensional *distribution* Δ on a manifold M is a choice of k-dimensional subspace Δ_p in each tangent space T_pM, smooth over M.

Theorem B.1 (Frobenius). *We call a distribution* Δ integrable *or* Frobenius *if, for any two vector fields* X, Y *on* M*,*

$$X \in \Delta \quad and \quad Y \in \Delta \implies [X, Y] \in \Delta$$

If Δ *is an integrable* k*-dimensional distribution on* M *then through each point of* M *there exists a* k*-dimensional submanifold* S *such that for all* $q \in S$*,*

$$T_qS = \Delta_q$$

Theorem B.2 (Frobenius for forms). *Let* $\Theta \subset \Omega^1(M)$ *be a Pfaffian system on* M*. Call* Θ integrable *if*

$$d\theta = 0 \mod \Theta$$

for all $\theta \in \Theta$*. Suppose that* M *has dimension* m *and* Θ *has dimension* k*. If* Θ *is integrable then through each point of* M *there exists a* $(m-k)$*-dimensional submanifold* S *such that for all* $q \in S$

$$i^*\Theta = 0$$

where i *is the inclusion of the abstract manifold* S *into* M*.*

Theorem B.3. *Consider the system of first-order PDE*

$$\frac{\partial y}{\partial x^i} = A^i(y, x_1, \ldots, x_n)$$

for the function $y : U \to \mathbb{R}$*. Then there are local solutions to the system through any point if and only if*

$$\frac{\partial A^i}{\partial x^j} + A^j \frac{\partial A^i}{\partial y} = \frac{\partial A^j}{\partial x^i} + A^i \frac{\partial A^j}{\partial y}$$

If this equation holds, we call the system compatible.

Proof. Let $\mathbb{R}^{n+1}$ carry the variables $y, x_1, \ldots, x_n$. Define the 1-form θ by

$$\theta = dy - A^i dx_i$$

Let y be any smooth function of the variables $x_1, \ldots, x_n$, and let i_y be the inclusion of the graph of y into $\mathbb{R}^{n+1}$, so

$$i_y(x_1, \ldots, x_n) = (y(x_1, \ldots, x_n), x_1, \ldots, x_n)$$

Then

$$\begin{aligned} i_y^* \theta &= dy(x_1, \ldots, x_n) - A^i dx_i \\ &= \frac{\partial y}{\partial x_i} dx_i - A^i dx_i \end{aligned}$$

This shows that y solves the system of PDE if and only if the form θ vanishes on the graph of y.

The exterior derivative of θ is given by

$$\begin{aligned} d\theta &= -dA^i \wedge dx_i \\ &= \left(\frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} \right) dx_i \wedge dx_j - \frac{\partial A^i}{\partial y} dy \wedge dx_i \\ &= \left(\frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} + \frac{\partial A^i}{\partial y} A^j - \frac{\partial A^j}{\partial y} A^i \right) \quad \text{mod } \theta \end{aligned}$$

so the Pfaffian system generated by θ is integrable if and only if the compatibility equations

$$\frac{\partial A^i}{\partial x^j} + A^j \frac{\partial A^i}{\partial y} = \frac{\partial A^j}{\partial x^i} + A^i \frac{\partial A^j}{\partial y}$$

hold. When these conditions do hold, the Frobenius theorem guarantees n-dimensional integral manifolds of θ through any point of $\mathbb{R}^{n+1}$. By the above discussion, any integral manifold of θ must be the graph of some solution to our PDE system. This proves the theorem. □

Theorem B.4. *Let $U \subset \mathbb{R}^n$ be contractible, and suppose that the system*

$$\frac{\partial y}{\partial x^i} = A(u, x_1, \dots, x_n)$$

is compatible on U. Then to any point $p \in U$ and $y^* \in \mathbb{R}$ there exists a unique solution $y : U \to \mathbb{R}$ with $y(p) = y^*$. This solution may be constructed by integrating a sequence of n ordinary differential equations.

Lemma B.5 (Cartan's Lemma)**.** *Suppose φ_i, ω_i are elements of a vector space such that*

$$\varphi_1 \wedge \omega_1 + \cdots + \varphi_n \wedge \omega_n = 0$$

and $\varphi_1 \wedge \cdots \wedge \varphi_n \neq 0$. Then there exists a symmetric matrix $A = [A_i^j]$ such that

$$\omega_i = A_i^j \varphi_j$$

Proof. The condition $\varphi_1 \wedge \cdots \wedge \varphi_n \neq 0$ is equivalent to the statement that $\{\varphi_1, \dots, \varphi_n\}$ is a linearly independent set. Extend this set to a basis $\{\varphi_i\} \cup \{\overline{\varphi}_a\}$ of V^*. In this basis we have $\omega_i = A_i^j \varphi^j + B_i^a \overline{\varphi}_a$ for some matrices A and B. But for a given i and a, the only term of the form $\varphi_i \wedge \overline{\varphi}_a$ in the sum $\varphi_1 \wedge \omega_1 + \cdots + \varphi_n \wedge \omega_n$ comes from the contribution of $\varphi_i \wedge \omega_i$. So the only way this sum can vanish is if the matrix $B = 0$.

Now consider the terms of the form $\varphi_i \wedge \varphi_j$. Each term of this form appears twice: once in $\varphi_i \wedge \omega_i$ with coefficient A_j^i and once in $\varphi_j \wedge \omega_j$ with coefficient $-A_i^j$. So for the sum to vanish, we must have $A_i^j - A_j^i = 0$. □

APPENDIX C

USEFUL TABLES FOR LIE SPHERE GEOMETRY

Table C.1: Codifferential acting on the matrix elements of the Darboux derivative of a Lie frame modulo the kissing system.

Form (φ)	Codifferential ($\delta\varphi \mod \Theta_{\mathcal{K}}$)
θ^1_1	$\theta^1_4 \wedge \theta^2_4 + \theta^1_6 \wedge \theta^2_5$
θ^1_3	$-\theta^1_1 \wedge \theta^1_3 + \theta^3_5 \wedge \theta^1_6 + \theta^3_6 \wedge \theta^1_5 + \theta^1_4 \wedge \theta^3_4$
θ^1_4	$-\theta^1_1 \wedge \theta^1_4 - \theta^1_3 \wedge \theta^3_4 + \theta^4_5 \wedge \theta^1_6$
θ^1_5	$-\theta^1_1 \wedge \theta^1_5 - \theta^1_3 \wedge \theta^3_5 - \theta^4_5 \wedge \theta^1_4 - \theta^5_5 \wedge \theta^1_5$
θ^1_6	$-\theta^1_1 \wedge \theta^1_6 - \theta^1_3 \wedge \theta^3_6 + \theta^5_5 \wedge \theta^1_6$
θ^2_3	$\theta^3_6 \wedge \theta^2_5 + \theta^2_4 \wedge \theta^3_4$
θ^2_4	$\theta^1_1 \wedge \theta^2_4$
θ^2_5	$\theta^1_1 \wedge \theta^2_5 - \theta^4_5 \wedge \theta^2_4 - \theta^5_5 \wedge \theta^2_5$
θ^2_6	0
θ^3_4	$\theta^1_3 \wedge \theta^2_4 + \theta^3_6 \wedge \theta^4_5$
θ^3_5	$\theta^1_3 \wedge \theta^2_5 - \theta^3_5 \wedge \theta^5_5 - \theta^4_5 \wedge \theta^3_4$
θ^3_6	$\theta^3_6 \wedge \theta^5_5$
θ^4_5	$\theta^3_5 \wedge \theta^3_4 - \theta^4_5 \wedge \theta^5_5 + \theta^1_4 \wedge \theta^2_5 - \theta^1_5 \wedge \theta^2_4$
θ^4_6	$\theta^3_6 \wedge \theta^3_4 - \theta^1_6 \wedge \theta^2_4$
θ^5_5	$-\theta^3_5 \wedge \theta^3_6 + \theta^1_6 \wedge \theta^2_5$

Table C.2: Coadjoint action of $\mathfrak{h}_7$ on the matrix elements of the Darboux derivative of a Lie frame.

Form	$\dot{\gamma}$	$\dot{\beta}_1$	$\dot{\beta}_2$	$\dot{\alpha}_1$	$\dot{\alpha}_2$	$\dot{\lambda}_1$	$\dot{\lambda}_2$
θ^3_4	0	0	$-\theta^3_6$	θ^2_4	0	0	0
θ^2_4	0	0	0	0	0	θ^2_4	0
θ^3_6	0	0	0	0	0	0	$-\theta^3_6$
θ^1_6	0	0	0	$-\theta^3_6$	0	$-\theta^1_6$	$-\theta^1_6$
θ^2_5	0	0	θ^2_4	0	0	θ^2_5	θ^2_5
θ^1_5	$\theta^1_1 - \theta^5_5$	θ^1_3	θ^1_4	$-\theta^3_5$	$-\theta^4_5$	$-\theta^1_5$	$-\theta^1_5$
θ^3_5	0	$-\theta^5_5$	θ^3_4	θ^2_5	0	0	θ^3_5
θ^4_5	$-\theta^2_4$	$-\theta^3_4$	$-\theta^5_5$	0	θ^2_5	0	θ^4_5
θ^1_3	θ^3_6	$-\theta^1_6$	0	θ^1_1	θ^3_4	$-\theta^1_3$	0
θ^1_4	0	0	$-\theta^1_6$	$-\theta^3_4$	θ^1_1	$-\theta^1_4$	0
θ^1_1	0	0	0	0	θ^2_4	0	0
θ^5_5	0	$-\theta^3_6$	0	0	0	0	0

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