RAPID EVOLUTION OF COMPLEX LIMIT CYCLES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2009

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Cornell University 2009

Limit cycles of planar polynomial vector fields have long been a focus of extensive research. Analogous to the real case, similar problems have been studied in the complex plane where a polynomial differential one-form gives rise to a foliation by Riemann surfaces. In this setting, a complex cycle is defined as a nontrivial element of the fundamental group of a leaf from the foliation. Whenever the polynomial foliation comes from a perturbation of an exact one-form, one can introduce the notion of a multi-fold cycle. This type of cycle has at least one representative that determines a free homotopy class of loops in an open fibred subdomain of the complex plane. The topology of this subdomain is closely related to the exact one-form mentioned earlier. The current dissertation is an introduction to the notion of multi-fold cycles of a close-to-integrable polynomial foliation. We explore the way these cycles correspond to periodic orbits of certain Poincaré maps associated with the foliation. We also discuss the tendency of a continuous family of multi-fold limit cycles to escape from certain large open domains in the complex plane as the foliation converges to its integrable part.

BIOGRAPHICAL SKETCH

Nikolay Dimitrov Dimitrov was born and raised in the city of Varna, Bulgaria. He completed his undergraduate studies at Sofia University ”Saint Kliment Ohridski” in 2001 with a major in pure mathematics. After graduation, he went to perform compulsory military service in the Bulgarian Naval Forces. Nikolay was dismissed in the summer of 2002 when he joined the Department of Mathematics at Cornell University to pursue a doctoral degree in theoretical mathematics.

To my grandparents,

Maika Pavlinka and Tatko Enko.

ACKNOWLEDGEMENTS

I am truly grateful to all the good people who have contributed to my personal growth and have helped me throughout the course of my life. I am very happy and fortunate to have met every one of them.

First, I would like to thank my adviser, Yulij Sergeevich Ilyashenko for all the help and support during my graduate studies. His dedication and lively engagement have been invaluable. I am also very thankful to my committee members John Hubbard, Reyer Sjamaar and Allen Hatcher. John Hubbard has always been a source of knowledge, encouragement and inspiration. His help during my job application process was of great importance. Reyer Sjamaar has always been there to provide advise and support whenever I needed them. Allen Hatcher kindly offered useful comments and suggestions in regards to my dissertation.

The completion of my graduate studies would not have been successful without the people from the Department of Mathematics at Cornell University. The members of the faculty have been inspirational, have provided a great amount of fascinating knowledge, and have helped broaden my horizon. The administration and staff were always friendly and welcoming every time I needed assistance.

I am immensely grateful to my parents Dimitar and Polya for their unconditional love, encouragement and support. My father's ingenuity and creativity, as well as my mother's strength and persistence are some of the qualities that I have drawn upon throughout my entire life.

I would like to thank my best friends who have always been there for me in happy and difficult times. Some of them are Ivan and Perikal, whom I know from Bulgaria, as well as Andrei, Radu, Jessica, Alimjon and Farhod whom I met in Ithaca. I am very thankful to my friend and partner Nicole, who has been my driving force these last few years and on several occasions showed me a highway when I was not able to see a path.

I would like to express my gratitude to my undergraduate advisors Ognian Christov and Emil Horozov from Sofia University, Bulgaria, whose encouragement and support opened the doors to the world of mathematics for me. I am also very grateful to Dana Schlomiuk from Université de Montréal and Dima Jakobson from McGill University who made it possible for me to continue my mathematical career as a post doctoral researcher at the Centre de Recherches Mathématiques and McGill University.

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CHAPTER 1

INTRODUCTION

Limit cycles of planar polynomial vector fields have long been a focus of extensive research. For instance, one of the major problems in this area of dynamical systems is the famous Hilbert's 16th problem [10]:

What may be said about the number and the location of the limit cycles of a polynomial vector field of degree n in the plane?

Since the original Hilbert's problem continues to be very persistent, some simplifications have been considered as well. Among them is the so called infinitesimal Hilbert's 16 problem [10], [11]:

What may be said about the number of limit cycles that can be born from periodic solutions of a polynomial Hamiltonian planar system by a small polynomial perturbation?

Recently, an answer to this question has been given in an article by Gal Binyamini, Dmitry Novikov and Sergei Yakovenko [2].

When studying a planar polynomial vector field, it is often helpful to extend it to the complex domain, an idea that can be attributed to Petrovskii and Landis [13], [14]. In this way a polynomial complex vector field is obtained and the holomorphic curves tangent to it form a partition of the complex plane by Riemann surfaces, called a *polynomial complex foliation with singularities*, or in short *polynomial complex foliation* [10], [11].

We are going to focus on polynomial perturbations of a polynomial Hamiltonian system in $\mathbb{C}^2$. More precisely we consider the complex line field

$$F^\varepsilon = \ker(dH + \varepsilon\omega) \tag{1.1}$$

with a one-form $\omega = Adx + Bdy$, where A, B and $H \in \mathbb{C}[x, y]$ are polynomials with complex coefficients and ε is a small complex parameter. As mentioned earlier, the holomorphic curves tangent to F^ε form a foliation of Riemann surfaces in $\mathbb{C}^2$ further denoted by $\mathcal{F}^\varepsilon(\mathbb{C}^2)$. Notice, that in the real case the phase curves of a planar vector field are topologically either lines or circles, i.e. curves with either a trivial or a non-trivial (isomorphic to $\mathbb{Z}$) fundamental group. This simple observation leads us to the definition of a marked complex cycle.

Definition 1. *A marked complex cycle of a complex foliation is a nontrivial element of the fundamental group of a leaf from the foliation with a marked base point.*

We denote a marked complex cycle by (Δ, q) where Δ is the homotopy class of loops on the leaf, all passing through the same base point q. In general, a real phase curve of a polynomial vector field in $\mathbb{R}^2$ extends to a Riemann surface tangent to the vector field's complexification in $\mathbb{C}^2$. Thus, a closed phase curve in $\mathbb{R}^2$ defines a loop on the corresponding complex leaf, giving rise to a nontrivial element from the fundamental group of that leaf [10]. In other words, a real closed phase curve is a marked complex cycle on its complexification.

When $\varepsilon = 0$ the foliation $\mathcal{F}^0(\mathbb{C}^2)$ consists of algebraic leaves of the form $S_u = \{p \in \mathbb{C}^2 \, : \, H(p) = u\}$ embedded in $\mathbb{C}^2$. From now on, we are going to refer to $\mathcal{F}^0(\mathbb{C}^2)$ as the *integrable foliation* and to $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ as the *perturbed foliation*. The idea is to study the complex cycles of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ using our knowledge of $\mathcal{F}^0(\mathbb{C}^2)$.

One of the most powerful tools for studying foliations and continuous dynamical systems in general, is the Poincaré map [10], [11]. To construct a *Poincaré map* for the foliation $\mathcal{F}^\varepsilon(\mathbb{C}^2)$, one can follow several steps. Start by choosing a point p_0 on a leaf S_{u_0} of $\mathcal{F}^0(\mathbb{C}^2)$ and a nontrivial loop $\delta_0 \subset S_{u_0}$ with a base point p_0. Take a small enough complex segment L passing through p_0 and transverse to the leaves of $\mathcal{F}^0(\mathbb{C}^2)$. Consider a tubular neighborhood A of δ_0 on the surface S_{u_0}. A tubular neighborhood $N(A)$ of A in $\mathbb{C}^2$ is diffeomorphic to a direct product $A \times \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc. Let ϱ be the projection of $N(A)$ onto A along $\mathbb{D}$. The direct product structure on $N(A)$ can be chosen so that $L = \varrho^{-1}(p_0)$. If ε is chosen small enough, then for any point $q \in L$ close to p_0 the loop δ_0 can be lifted to a curve $\delta(q)$ on the leaf of the perturbed foliation $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ passing through q, so that $\delta(q)$ covers δ_0 under the projection ϱ. By construction, $\delta(q)$ will have both of its endpoints on L, where $q \in L$ is one of them. Denote the second endpoint by $P_{\delta_0,\varepsilon}(q) \in L$. Thus, we obtain a correspondence $P_{\delta_0,\varepsilon} : L^{'} \to L$ where the open set $L^{'} \subset L$ is the domain of $P_{\delta_0,\varepsilon}$. The map $P_{\delta_0,\varepsilon}$ is holomorphic and close to identity. Notice that by construction, if δ_0 is homotopic on S_{u_0} to another loop $\delta_0^{'}$ passing through p_0, then for small enough ε the two maps $P_{\delta_0,\varepsilon}$ and $P_{\delta_0^{'},\varepsilon}$ will be equal.

The Poincaré map described above has the property that if two points from the cross-section L are in the same orbit of the map then they belong to the same leaf of the foliation. Moreover, a marked complex cycle of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$, with a base point on $L^{'}$ and a representative that covers m times the loop δ_0 under the projection ϱ, gives rise to an $m-$periodic orbit of $P_{\delta_0,\varepsilon}$. The inverse is also true [13], [14]. A periodic orbit of period m corresponds to a marked complex cycles of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ with a representative contained in $N(A)$, covering δ_0 under the projection ϱ a number of m times. Notice that since ϱ is a deformation retraction of $N(a)$ onto A, the representative will be free homotopic to δ_0^m inside $N(A)$.

Definition 2. *A marked complex cycle is called a $\delta_0, m-$fold cycle provided that it gives rise to an $m-$periodic orbit of some Poincaré map $P_{\delta_0,\varepsilon}$.*

When $m > 1$ and we do not want to specify the characteristics δ_0 and m we call such a cycle a *multi-fold* one. For any $m > 0,$ it is not difficult to see that $P^m_{\delta_0,\varepsilon} = P_{\delta_0^m,\varepsilon}$. Then a $\delta_0, m-$fold cycle is represented by a fixed point of the $m-$th iteration of $P_{\delta_0,\varepsilon}$ or equivalently by a fixed point of $P_{\delta_0^m,\varepsilon}$. Now, we can give a definition for a marked limit cycle.

Definition 3. *A marked limit cycle is a marked complex cycle represented by an isolated fixed point of the appropriate Poincaré map.*

The case when $m = 1$ has been extensively studied. In fact, the real cycles of a planar polynomial vector field of the form (1.1) extend to $1-$fold cycles of its complexificaion. The above mentioned infinitesimal Hilbert's 16th problem [2], [10] treats exactly the special case $m = 1$. The following classical result, known as Pontryagin's criterium [15] can be stated in the following form.

Theorem 1. *Let δ_u be an analytic family of simple closed loops on the corresponding leaves S_u from the integrable foliation $\mathcal{F}^0(\mathbb{C}^2)$, and consider the analytic function $I(u) = \int_{\delta_u} \omega$. If there exists u_0 such that $I(u_0) = 0$ and $I'(u_0) \neq 0$ then there exists a continuous family δ^ε of loops, each representing a 1-fold complex limit cycle of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$, such that $\delta^\varepsilon \to \delta_{u_0}$ as $\varepsilon \to 0,$ always staying close to δ_{u_0}.*

In contrast to 1-fold cycles, little is known about multi-fold ones. We are going to answer some questions related to the case $m > 1$. During an oral discussion, Ilyashenko proposed the following questions in the spirit of Petrovskii and Landis' works [13] and [14]:

1. *Are there examples of polynomial families (1) with Poncaré maps that have isolated periodic orbits of arbitrary period $m > 1$?*

2. *In the case $m > 1$, what may happen to a δ_0, m-fold limit cycle when ε approaches 0?*

3. *Does a multi-fold limit cycle settle on a leaf of $\mathcal{F}^0(\mathbb{C}^2)$ as $\varepsilon \to 0$?*

The current dissertation is an attempt to give some answers to the questions posed above. In order to do this, we construct a Poincaré map on a large cross-section of the foliation $\mathcal{F}^\varepsilon(\mathbb{C}^2)$. We show that a certain cycle of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ that generates a periodic orbit of the Poincaré map, have a representative that determines a specific free homotopy class of loops in an open fibred subdomain of $\mathbb{C}^2$. The topology and fiber structure of this subdomain is determined by $\mathcal{F}^0(\mathbb{C}^2)$. With the help of the construction of the global Poincaré map we see that the behavior of a multi-fold limit cycle is quite different from the behavior of a 1-fold limit cycle as ε tends to zero. By Theorem 1, the latter always stays close to some cycle from $\mathcal{F}^0(\mathbb{C}^2)$ and converges to it as ε converges to zero. In contrast to the behavior of a $1-$fold limit cycle, a multi-fold one tends either to escape from a very large domain in $\mathbb{C}^2$ when ε approaches 0 or to change the homotopy type of its representatives inside the fibred subdomain in $\mathbb{C}^2$. This last phenomenon is called *rapid evolution of the multi-fold limit cycle*. We also give an explicit example of a polynomial foliation of the form (1.1) with multi-fold limit cycles.

So far, the third question from the list above stays unanswered. The information we have on rapid evolution reveals an interesting insight. If the answer to that question is positive, then before a multi-fold limit cycle can reach an algebraic leaf as $\varepsilon \to 0$, its representatives should change their topological properties somewhere along the way. This means that there is a possibility that the cycle settles on a critical leaf of $\mathcal{F}^0(\mathbb{C}^2)$ or goes through one or several critical leaves of $\mathcal{F}^0(\mathbb{C}^2)$, settling on a regular leaf. Since the foliations are polynomial, they extend to foliations on $\mathbb{CP}^2$. Thus, another possibility is an interaction with the line at infinity.

CHAPTER 2

MAIN RESULTS

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2.1 Preliminaries

Fiber Bundle Structure on the Phase Space. In this section we define several fibred subdomains of the complex plain that will play an important role in our investigation.

Let the polynomial $H : \mathbb{C}^2 \to \mathbb{C}$ be of degree $n+1$ and have the following two properties:

• it has n^2 non-degenerate critical points in $\mathbb{C}^2$ with n^2 different critical values $\Sigma = \{a_1, ..., a_n^2\}$ in $\mathbb{C}$ and

• the projective closures of its leaves $S_u = H^{-1}(u)$ in $\mathbb{CP}^2$ are transverse to the line at infinity.

We are ready to define the first subdomain of $\mathbb{C}^2$. We are going to denote it by E. Consider the punctured domain $B = \mathbb{C} - \Sigma$ and its preimage $E = H^{-1}(B)$. Clearly, E is just $\mathbb{C}^2$ with all critical leaves of H removed. Choose $u_0 \in B$ and denote $S_{u_0} = H^{-1}(0)$. Then the map $H : E \to B$ defines a smooth locally trivial fibre bundle with fibres diffeomorphic to S_{u_0} [1], [11]. Denote by $\mathcal{F}^\varepsilon(E)$ the restriction of the foliation $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ on E. In other words, the leaves of $\mathcal{F}^\varepsilon(E)$ are the intersections of the leaves of $\mathcal{F}^\varepsilon(\mathbb{C}^2)$ with E. For simplicity, we are going to drop the notation for E in $\mathcal{F}^\varepsilon(E)$ and just write $\mathcal{F}^\varepsilon$ instead of $\mathcal{F}^\varepsilon(E)$. Thus

$\mathcal{F}^\varepsilon = \mathcal{F}^\varepsilon(E)$. When $\varepsilon = 0$, the restricted foliation $\mathcal{F}^0 = \mathcal{F}^0(E)$ consists of all leaves from $\mathcal{F}^0(\mathbb{C}^2)$ with the exception of the critical ones.

Before we go on with the construction of the other subdomains, we will need some facts concerning the topology of the fiber bundle $H : E \to B$. For each critical value $a_j \in \Sigma, j = 1...n^2$ consider a simple smooth path from u_0 to a small circle around a_j, so that the union of the path and the circle provides us with a counter clockwise oriented loop γ_j around a_j based at u_0. Also, suppose that for $i \neq j, \gamma_i \cap \gamma_j = \{u_0\}$. Then the homotopy classes of the loops $\{\gamma_1, ..., \gamma_{n^2}\}$ define generators of the fundamental group $\pi_1(B, u_0)$. For $u \in \gamma_j$ consider the fiber S_u. Then if the parameter u starts form u_0 and moves along the loop γ_j until it comes back to u_0 then the corresponding fibers S_u will also make one turn around the critical point a_j starting and ending up at S_{u_0}. According to Picard-Lefschetz's theory [1] this procedure gives rise to an isotopy class of maps (an element of the mapping class group of S_{u_0}) with a representative $\tilde{D}_{\gamma_j} : S_{u_0} \to S_{u_0}$ which is a Dehn twist around a simple closed geodesic we denote by δ_j for $j = 1, ..., n^2$. Moreover, the Dehn twist can be chosen so that the closed cylinder $supp(\tilde{D}_{\gamma_j}) \subset S_{u_0}$, on which $\tilde{D}_{\gamma_j}$ acts non trivially, is very thin with respect to the Poincaré metric on the fiber S_{u_0} and $supp(\tilde{D}_{\gamma_i}) \cap supp(\tilde{D}_{\gamma_j}) = \varnothing$ whenever $\delta_i \cap \delta_j = \varnothing$. Then on the closure of the complement $S_{u_0} - supp(\tilde{D}_{\gamma_j})$ the map $\tilde{D}_{\gamma_j}$ acts like the identity map. The cycles represented by the loops $\{\delta_1, ..., \delta_{n^2}\}$ give rise to a system of vanishing cycles on S_{u_0}, which can serve as a basis of the first homology group on S_{u_0} [1], [9]. Also, as a sphere with $n^2 + 1$ points removed, B has the structure of a Riemann surface with hyperbolic metric. For each cusp $a_j \in \Sigma$ let us choose a cut l_j connecting α_j to ∞ so that no two such cuts intersect. For simplicity, we may think that each cut l_j is geodesic and that u_0 is chosen so that it does not lie on any of the cuts. Later, in Section 4.1 we are going to find one possible way for those cuts to be chosen.

Construction of Domains. Now we are ready to define the subdomain $E_{\delta_0} \subset E$. Fix a point $p_0 \in S_{u_0}$ and some primitive element Δ_0 of the fundamental group $\pi_1(S_{u_0}, p_0)$. Choose a representative $\delta_0 \subset S_{u_0}$ of Δ_0 such that $\delta_0 \cap supp(\tilde{D}_{\gamma_j}) = \varnothing$ if the geometric intersection index $\delta_0 \cdot \delta_j = 0$. Define $J(\delta_0) = \{j = 1, ..., n^2 \,|\, \delta_0 \cdot \delta_j \neq 0\}$ to be the set of those indices for which the geometric intersection index of the corresponding vanishing cycle and δ_0 is non zero and consider the domain $B_{\delta_0} = B - (\sqcup_{j \in J(\delta_0)} l_j) \subset \mathbb{C}$ and $E_{\delta_0} = H^{-1}(B_{\delta_0}) \subset \mathbb{C}^2$.

Finally, we construct the rest of the domains. For a small number $\tilde{\rho} > 0$, let $B_1(\tilde{\rho}), ..., B_{n^2}(\tilde{\rho})$ be small disjoined closed discs of radius $\tilde{\rho}$ in $\mathbb{C}$ around the points $\alpha_1, ..., \alpha_{n^2}$ respectively and not containing the point u_0. Let $B_\infty(\tilde{\rho})$ be a very large disc centered at the origin and of radius $1/\tilde{\rho}$ so that it contains all of the small ones and the point u_0. Then one can define the domains $C_{\delta_0}(\tilde{\rho}) = B_{\delta_0} - \big(B_\infty(\tilde{\rho}) \sqcup \big(\sqcup_{j=1}^{n^2} B_j(\tilde{\rho})\big)\big)$ and $A(\tilde{\rho}) = B - \big(B_\infty(\tilde{\rho}) \sqcup \big(\sqcup_{j=1}^{n^2} B_j(\tilde{\rho})\big)\big)$. Fix four small positive numbers ρ_0, ρ_1, ρ_0' and ρ_1', satisfying the inequalities $\rho_0 > \rho_1 > \rho_0' > \rho_1' > 0$. Denote by C_{δ_0} and C_{δ_0}' the domains $C_{\delta_0}(\rho_0)$ and $C_{\delta_0}(\rho_0')$, respectively. Also, denote by A and A' the domains $A(\rho_1)$ and $A(\rho_1')$, respectively. Now, consider the preimages $E(C_{\delta_0}) = H^{-1}(C_{\delta_0})$ and $E(A') = H^{-1}(A')$.

2.2 Main Theorems and Statements

Multi-Fold Vertical Complex Cycles. Before stating the main results of this work, we are going to give another definition for a multi-fold cycle. It is of a more topological nature and, as point 4 from Theorem 2 shows, in certain situations Definition 2 and the new definition coincide.

Definition 4. *A loop contained in E_{δ_0} is called δ_0, m-fold vertical provided that it is free homotopic to δ_0^m inside E_{δ_0}. A marked complex cycle of $\mathcal{F}^\varepsilon$ is called δ_0, m-fold vertical provided that it has a δ_0, m-fold vertical representative contained in E_{δ_0}.*

The justification for this definition stems from the proposition that follows.

Proposition 1. *Let $\mathcal{F}^\varepsilon$ have a marked complex cycle (Δ, q) with a $\delta_0, m-$fold vertical representative δ contained in E_{δ_0}.*

1. *If δ is free homotopic inside E_{δ_0} to another loop $\delta_0^{'} \subset S_{u_0}$, then $\delta_0^{'}$ is free homotopic to δ_0^m on the fiber S_{u_0}.*

2. *If δ' is another representative of (Δ, q) contained in E_{δ_0}, then δ' is $\delta_0, m-$fold vertical.*

As we can see, a representative of a marked complex cycle can belong to only one free homotopy class in E_{δ_0}. Moreover, any other representative contained in E_{δ_0} belongs to the same class.

Existence of a Global Poincaré Map. The first main result of this dissertation is the construction of a large cross-section of the foliations from the family (1.1) and a Poincaré map defined on it. The result also shows that there is a connection between the periodic orbits of the Poincré map and some topological properties of the corresponding multi-fold cycles inside the fibered domain E_{δ_0}.

Theorem 2. *There exists a surface B_{p_0}, embedded in E, diffeomorphic to B and passing through p_0, such that B_{p_0} intersects transversely each noncritical leaf of $\mathcal{F}^0$ at exactly one point. Moreover, for a small enough $r > 0$, if ε is contained in a disc of radius r then the following statements are true:*

1. *The leaves of $\mathcal{F}^\varepsilon$ are transverse to $A'_{p_0} \subset B_{p_0}$, where $H(A'_{p_0}) = A'$.*

2. *Let $C'_{p_0} \subset B_{p_0}$ be such that $H(C'_{p_0}) = C'_{\delta_0}$. Then there exists a Poincaré map $P_{\delta_0,\varepsilon} : C'_{p_0} \to A'_{p_0}$ associated with the foliation $\mathcal{F}^\varepsilon$ and a complex structure on B_{p_0} so that $P_{\delta_0,\varepsilon}$ is holomorphic.*

3. *If $P_{\delta_0,\varepsilon}$ has a periodic orbit of period m in C'_{p_0} then the foliation $\mathcal{F}^\varepsilon$ has a marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$ with a base point q_ε belonging to C'_{p_0}. Moreover, the cycle has a representative δ_ε contained in $E(A')$ and passing through the points of the $m-$periodic orbit.*

4. *If δ'_ε is an arbitrary representative of the marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$, then δ'_ε is contained in E_{δ_0} and is $\delta_0, m-$fold vertical if and only if its image $H(\delta_\varepsilon)$ is contained in B_{δ_0} and is free homotopic to a point inside B_{δ_0}.*

What we gain with this theorem is that for a small enough ε we are able to construct a Poincaré transformation along δ_0 defined on a very large domain. In this way we can encode a lot of information about a big portion of the perturbed foliation $\mathcal{F}^\varepsilon$. In particular, it allows us to keep track of the behavior of *continuous families* of $\delta_0, m-$fold limit cycles with respect to the parameter ε. In addition, Theorem 2 reveals a link between the dynamical notion of a multi-fold cycle, as given by Definition 2 and the topological point of view introduced in Definition 4. Thus, there exists a strong connection between the dynamical properties of marked complex cycles, in terms of periodic orbits of the corresponding Poincaré map, and the topological properties of these cycles, in terms of free homotopy classes.

Rapid Evolution of Marked Limit Cycles. Next, we explain the notion of a *continuous family* of $\delta_0, m-$fold limit cycles with respect to a parameter ε.

Definition 5. *A family $\{(\Delta_\varepsilon, q_\varepsilon)\}_\varepsilon$ of limit cycles of $\mathcal{F}^\varepsilon$ is called continuous with respect to ε, relative to an embedded in E surface L, if there exists a continuous family of representing loops from Δ_ε, so that the base point q_ε varies continuously on L.*

The next main result shows that for $m > 1$, a continuous family of $m-$fold limit cycles tends to escape from a very large domain in $\mathbb{C}^2$, namely $E(C_{\delta_0})$. We refer to this phenomenon as *rapid evolution* of the multi-fold family. This behavior is completely different from the behavior of a $1-$fold family. According to Theorem 1, the latter always stays in a neighborhood of an algebraic leaf of $\mathcal{F}^0$ as ε approaches 0.

Fix a positive integer $m > 1$ and let $D_r(0) = \{\varepsilon \in \mathbb{C} \ : \ |\varepsilon| \leq r\}$ for $r > 0$. We claim that as long as $r > 0$ is chosen small enough, rapid evolution of marked complex cycles occurs in the following form:

Theorem 3. *Assume that for some $\varepsilon_0 \in D_r(0)$ the foliation $\mathcal{F}^{\varepsilon_0}$ has a δ_0, m-fold vertical limit cycle which corresponds to an m-periodic orbit of $P_{\delta_0,\varepsilon_0}$ on the cross-section C'_{p_0}. Also, assume that the cycle has a $\delta_0, m-$fold vertical representative contained in $E(C_{\delta_0})$. Then, for any curve η connecting ε_0 to 0 and embedded in $D_r(0)$, there exists a relatively open subset σ of η, such that the cycle extends on σ to a continuous family $\{(\Delta_\varepsilon, q_\varepsilon)\}_{\varepsilon\in\sigma}$ of marked cycles of $\mathcal{F}^\varepsilon$. Moreover, as ε moves along σ in the direction of 0, it reaches a value $\varepsilon^* \in \sigma$ such that for any $\varepsilon \in \sigma$ past ε^* no $\delta_0, m-$fold vertical representative of $(\Delta_\varepsilon, q_\varepsilon)$ will be contained in $E(C_{\delta_0})$ anymore.*

To summarize the conclusions of Theorem 3, a limit δ_0, m-fold vertical cycle of the perturbed foliation, represented by a periodic orbit of the corresponding

Poincaré map, gives rise to a continuous family defined on σ. Eventually, as ε goes in the direction of 0 on σ, all representatives of the cycles from that family not only leave the domain $E(C_{\delta_0})$ but they do not come back to it as multi-fold vertical cycles of the same topological type. If they do come back, their topological characteristics δ_0 or m are changed.

Before we continue with the exposition, we are going to make a small comment. Denote by δ' the representative of the δ_0, m-fold vertical cycle from Theorem 3 contained in the domain $E(C_{\delta_0})$ when $\varepsilon = \varepsilon_0$. Notice that as soon as its image $H(\delta')$ is null-homotopic in B_{δ_0}, the loop δ' is forced by point 4 from Theorem 2 to be free homotopic inside E_{δ_0} to δ_0^m and cannot belong to any other free homotopy class in E_{δ_0}. Therefore, the fact that $P_{\delta_0,\varepsilon_0}$ is the Poincaré map with respect to δ_0, is directly related to the fact that δ' is $\delta_0, m-$vertical. Moreover, as Proposition 1 suggests, any other representatives of the same marked cycle, contained in E_{δ_0}, will also be $\delta_0, m-$fold vertical.

Outline of Proofs and Comments. We are going to give short outlines of the proofs of the above two results. To verify the claims of Theorem 2, one can use the pull back of the bundle E over the universal covering disc of the surface B. In this way, a covering bundle with an action of a deck group is obtained, and we can smoothly trivialize that bundle (notice the disc is contractible) so that the group will map both vertical and horizontal fibers to vertical and horizontal fibers, respectively. In fact, the group preserves the horizontal disc fibers passing through $S_{u_0} - \cup_{j=1}^{n^2} supp(\tilde{D}_{\gamma_j})$ because on the vertical fibers it is generated by the Dehn twists $\{\tilde{D}_{\gamma_j} \, : \, j = 1...n^2\}$, which act trivially outside $supp(\tilde{D}_{\gamma_j})$. In particular, if we take the horizontal disc passing through p_0 and project it to E, we will obtain the desired cross-section B_{p_0}. If we pull back the foliation in the trivial bundle then we obtain a foliation invariant with respect to the action of the deck group. The direct product structure on the trivial covering bundle allows us to lift δ_0 on the leaves of the pulled back foliation so that we get a Poincaré map $\hat{P}_{\delta_0,\varepsilon}$ on the disc. The invariance of the foliation implies the relation $\gamma \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\tilde{D}_\gamma^{-1}(\delta_0),\varepsilon} \circ \gamma$ for all $\gamma \in \pi_1(B, u_0)$. But for $\delta_j \cdot \delta_0 = 0$ we have $\gamma_j \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \gamma_j$ because $\tilde{D}_{\gamma_j}(\delta_0) = \delta_0$. Projecting everything back to $E,$ we get the desired cross-sections and Poincaré map. By construction the map branches over the cuts of B_{δ_0}. The complex structure on A'_{p_0} is defined as the transverse structure to the leaves of $\mathcal{F}^\varepsilon$ and extended by 0 on $B_{p_0} - A'_{p_0}$. The remaining claims follow from the constructions above.

When proving Theorem 3, one can use Theorem 2 in order to represent the family of limit cycles as an analytic family of $m-$periodic orbits of the corresponding Poincaré map inside the cross-sections C_{p_0}. Then one can apply a version of the known property that for $m > 1$, an analytic family of m-periodic orbits of a holomorphic map close to identity, tends to escape a domain inside the map's definition. In our case the domain happens to be C_{p_0}. Therefore, when the points from the periodic orbit leave C_{p_0} they also leave $E(C_{\delta_0})$. Because all representatives pass through the base point, and because the base point, which is a point from the periodic orbit, happens to be outside $E(C_{\delta_0}),$ no representative is entirely contained in $E(C_{\delta_0})$. In the case when the periodic orbit goes through a cut, it turns into a periodic orbit of another branch of the Poincaré map, obtained as a lift of a loop that can be sent to the original δ_0 by a Dehn twist. This implies that the new cycle will not have representatives free homotopic to δ_0^m inside E_{δ_0} anymore.

Perturbed Foliations with Multi-Fold Limit Cycles. An important problem in the study of multi-fold limit cycles is the existence of the latter in families of polynomial foliations of the form (1). Heuristically, we can follow these steps. Using Theorem 1, we can find a family of $\delta_0, 1$-fold cycles which gives a family of isolated fixed points for the corresponding Poincaré map $P_\varepsilon = P_{\delta_0,\varepsilon}$. For infinitely many values of ε in any neighborhood of 0, the derivative of P_ε evaluated at the fixed point will be an m-th root of unity. Thus, for such ε a local continuous family of m-periodic isolated orbits will bifurcate from the fixed point. This will happen as long as the resonant terms of the normal form of the map do not vanish, i.e. the map is not analytically equivalent to a rotation. Since having nonzero resonant terms is a very generic property of resonant maps, we can expect that the Poincaré transformations for most foliations of the form (1) will have a lot of isolated periodic orbits and thus, the foliations themselves will have many multi-fold limit cycles. The only obstacle in this strategy is the verification that some of the resonant term coefficients of the map's normal form are nonzero. This is hard to establish since the connection between the polynomial foliation and its Poincaré transformation is implicit and indirect.

Modifying the strategy above, we give an example of a polynomial foliation with limit multi-fold vertical cycles. Let H be the following polynomial with leaves transverse to infinity:

$$H = x^2 + y^2.$$

Choose polynomial forms ω_1 and ω_2 as follows:

$$\omega_1 = (H-1)(ydx - xdy) \quad \text{and} \quad \omega_2 = y\, dH.$$

Consider the two parameter family

$$\ker\Big(dH + \varepsilon(\omega_1 + a\omega_2)\Big), \tag{2.1}$$

where ε and a are the parameters. Consider the leaf

$$S_1 = \{(x, y) \in \mathbb{C}^2 \,|\, x^2 + y^2 = 1\}$$

tangent to the integrable line field $\ker(dH)$. Fix the loop $\delta_0 = S_1 \cap \mathbb{R}^2$. In this setting, the following result holds:

Theorem 4. *For any $m \in \mathbb{N}$ large enough there exists a complex parameter ε_m near $\frac{1}{m}$ and a parameter a_m such that for all ε in a neighborhood of ε_m, the polynomial foliation (2.1) has a limit $\delta_0, m-$fold vertical cycle. The cycle satisfies the properties of Theorem 2 and is subject to rapid evolution, as explained in Theorem 3.*

CHAPTER 3

MARKED CYCLES IN THE FIBRED DOMAIN

3.1 Unfolding the Fiber Bundle

First, we will try to understand the topology of the bundle $H : E \to B$ induced by the integrable foliation $\mathcal{F}^0$. The idea is to "unfold" E into something simple, a direct product in our case, keeping the "folding pattern" into a group of deck transformations.

Preliminary Notations and Constructions. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Consider the universal covering map $\pi : \mathbb{D} \to B$. Denote its group of deck transformations by Γ. Then, Γ is isomorphic to the fundamental group of B. Since the disc $\mathbb{D}$ is a conformal model of the hyperbolic plane, Γ is a discreet group of isometries acting properly discontinuously. Let $z_0 \in \mathbb{D}$ be a point such that $\pi(z_0) = u_0$. Each loop $\gamma_j \in \pi_1(B, u_0)$, chosen in Section 2.1, can be lifted to a path on $\mathbb{D}$ starting from point z_0. Denote by $z^{(j)}$ the second endpoint of this path. Abusing notation, for each $j = 1...n^2$ consider $\gamma_j \in \Gamma$ to be the parabolic isometry of $\mathbb{D}$ corresponding to the loop $\gamma_j \in \pi_1(B, u_0)$ that sends z_0 to $z^{(j)} = \gamma_j(z_0)$ [8], [12]. Then $\Gamma = \langle \gamma_1, ..., \gamma_{n^2} \rangle$ is a free group generated by n^2 transformations. Let $\hat{a}_j$ be the fixed point of the parabolic isometry γ_j on the boundary $\partial\mathbb{D}$ for all $j = 1, ..., n^2$. We can think of $\hat{a}_j$ as the lift of $a_j \in \Sigma$ on the ideal boundary $\partial\mathbb{D}$ of the hyperbolic plane $\mathbb{D}$. Assume that the subscripts in the notation of the critical values are chosen so that the loop $\gamma_{n^2}...\gamma_2\gamma_1$ on B is homotopic to a simple loop around the cusp ∞ of B. Thus, the corresponding isometry $\gamma_{n^2} \circ ... \circ \gamma_2 \circ \gamma_1 \in \Gamma$ is also parabolic with a fixed point which we shall denote by $\infty_1 \in \partial\mathbb{D}$. The latter can be thought of as a lift of the infinity point of $\mathbb{C} \cup \{\infty\}$ on the ideal boundary $\partial\mathbb{D}$. Similarly, for any $j = 2...n^2$ the isometry $\gamma_{j-1} \circ ... \circ \gamma_1 \circ \gamma_{n^2} \circ ... \circ \gamma_j \in \Gamma$ is parabolic with a fixed point $\infty_j \in \partial\mathbb{D}$. The ideal points $\hat{a}_1, \infty_1, \hat{a}_2, \infty_2, ..., \hat{a}_{n^2}$ and ∞_{n^2} are arranged in a cyclic order along the boundary $\partial\mathbb{D}$. The geodesic convex hull of those $2n^2$ points, with respect to the Poincaré metric on $\mathbb{D}$, is a closed (in the topology of $\mathbb{D}$) ideal $2n^2-$gon Q with geodesic edges, which is a fundamental domain for the deck group Γ [8],[12].

Statements and Proofs. From now on, we are going to use the shorter notation S for the fixed fiber S_{u_0}. Also, whenever we have a cartesian product $M_1 \times M_2$ of two sets, by pr_{M_i} we are going to denote the projection $pr_{M_i} : M_1 \times M_2 \to M_i$ where $pr_{M_i}(m_1, m_2) = m_i$ for $i = 1, 2$.

Theorem 5. *There is a smooth covering map $\Pi : \mathbb{D} \times S \to E$ with the following properties:*

1. *If $pr_{\mathbb{D}} : \mathbb{D} \times S \to \mathbb{D}$ is the projection $(z, p) \mapsto z$, then $H \circ \Pi = \pi \circ pr_{\mathbb{D}}$.*

2. *The deck group of $\Pi : \mathbb{D} \times S \to E$ is*

$$\hat{\Gamma} = \langle \ (z, p) \mapsto (\gamma_j(z), D_{\gamma_j}(p)) \mid j = 1...n^2 \ \rangle,$$

where $\gamma_j \in \Gamma$ are the earlier described generators of Γ and the maps $D_{\gamma_j} = \tilde{D}_{\gamma_j}^{-1}$ are Dehn twists along the vanishing cycles δ_j on the surface S. Thus, the factor bundle $(\mathbb{D} \times S)/\hat{\Gamma}$ is diffeomorphically isomorphic to the bundle E.

The essence of this theorem is that we can not only unfold the bundle $H : E \to B$ into a trivial covering bundle $pr_{\mathbb{D}} : \mathbb{D} \times S \to \mathbb{D}$, but we can do so by making sure the deck group $\hat{\Gamma}$ acts in a very special manner. It is natural to expect that any element of the group takes vertical fibers $\{z\} \times S$ to vertical fibers. What is important is that it also sends horizontal fibers $\mathbb{D} \times \{p\}$ to horizontal fibers.

Proof: Consider the pullback of the bundle $H : E \to B$ over the disc $\mathbb{D}$ under the covering map π. To carry out this construction, first define the total space $\pi^*E = \{(z, q) \in \mathbb{D} \times E \ : \ \pi(z) = H(q)\}$. Then, the restricted projection $\kappa = (pr_{\mathbb{D}})_{|_{\pi^*E}} : \pi^*E \to \mathbb{D}$ gives us the desired pullback bundle. Also, there is a map $\tilde{\Pi}' = (pr_E)_{|_{\pi^*E}} : \pi^*E \to E$ that satisfies the condition $H \circ \tilde{\Pi}' = \kappa \circ \pi$, and so it is a bundle map over the map π. Together with that, $\tilde{\Pi}' : \pi^*E \to E$ is a covering map.

Because $\mathbb{D}$ is contractible, the pullback bundle $\kappa : \pi^*E \to \mathbb{D}$ is trivializible, i.e. there is a smooth bundle isomorphism $\varsigma : \mathbb{D} \times S \to \pi^*E$, such that we have $\kappa \circ \varsigma = pr_{\mathbb{D}} \circ id_{\mathbb{D}}$ where $id_{\mathbb{D}}$ is the identity map on $\mathbb{D}$. Then, the composition $\tilde{\Pi} = \tilde{\Pi}' \circ \varsigma : \mathbb{D} \times S \to E$ satisfies the condition $H \circ \tilde{\Pi} = \pi \circ pr_{\mathbb{D}}$, and thus, it is a bundle map and a covering map at the same time. Without loss of generality, we can think that $\tilde{\Pi}(z_0, p) = p$. In other words, we identify the fiber $\{z_0\} \times S$ with the surface S.

We are going to look at the deck group $\tilde{\Gamma}$ of the covering map $\tilde{\Pi}$. Let $\tilde{\gamma} \in \tilde{\Gamma}$ be a deck transformation from that group. Then the diffeomorphism $\tilde{\gamma} : \mathbb{D} \times S \to \mathbb{D} \times S$ is of the form $\tilde{\gamma}(z, p) = (\gamma(z), \psi_\gamma(z, p))$, where $\gamma \in \Gamma$ is a deck transformation for the covering map π and $\psi : \mathbb{D} \times S \to S$ is a smooth map. If we factor $\mathbb{D} \times S$ by the action of the deck group $\tilde{\Gamma}$, we obtain the manifold $(\mathbb{D} \times S)/\tilde{\Gamma}$ which is isomorphic to E as a fiber bundle over B. For any $(z, p) \in \mathbb{D} \times S$ consider $\psi_{\gamma,z}(p) = \psi_\gamma(z, p)$. Then, $\psi_{\gamma,z} : S \to S$ is a diffeomorphism on the standard fiber S for any fixed $z \in \mathbb{D}$. If γ_j is one of the generators of Γ, as described before, then ψ_{γ_j, z_0} is isotopic to the Dehn twist $D_{\gamma_j} = \tilde{D}_{\gamma_j}^{-1}$. This follows from Picard-Lefchetz's theory as discussed previously in Section 2.1 and in [1].

By the properties of the ideal polygon Q, for each $j = 1, ..., n^2$ there are two adjacent geodesic edges that have $\hat{\alpha}_j$ as a common ideal vertex. One of those two edges, denoted by e_j, is mapped by γ_j to the other one, denoted by $\gamma_j(e_j)$. Then, both e_j and $\gamma_j(e_j)$ meet the ideal boundary $\partial\mathbb{D}$ at $\hat{\alpha}_j$. Now, for any $j = 1, ..., n^2$, consider an open tubular neighborhood I_j of e_j in $\mathbb{D}$, which is thin enough so that two properties hold. First, $\overline{I}_i \cap \overline{I}_j = \varnothing$ whenever $i \neq j$. Here, $\overline{I}_j$ is the closure of I_j in the hyperbolic plane $\mathbb{D}$. Second, $\overline{I_j} \cap \gamma_j(\overline{I}_j) = \varnothing$, where $j = 1, ..., n^2$. Notice, that $\gamma_j(I_j)$ is a tubular neighborhood of $\gamma_j(e_j)$. Let $I = \sqcup_{j=1}^{n^2} I_j$ and $J = \sqcup_{j=1}^{n^2} \gamma_j(I_j)$. Denote by $\tilde{Q}$ the union $Q \cup I \cup J$. We can see that $\tilde{Q}$ is an open neighborhood of the fundamental domain Q.

Define the smooth gluing map $\phi_0 : I \times S \to J \times S$ to be such that $\phi_0(z, p) = (\gamma_j(z), \psi_{\gamma_j}(z, p))$ for any $(z, p) \in I_j \times S$, where $j = 1, ..., n^2$. Since ϕ_0 respects the bundle structure of $\mathbb{D} \times S$, the quotients $(\tilde{Q} \times S)/\phi_0$ and $(\mathbb{D} \times S)/\tilde{\Gamma}$ are smoothly isomorphic as fiber bundles over B. Therefore, $(\tilde{Q} \times S)/\phi_0$ and E are smoothly isomorphic as bundles over B.

Notice, that I_j is diffeomorphic to a disc and so it deformation retracts onto a point $z_j \in I_j$ for $j = 1, ..., n^2$. For that reason, there exists a smooth deformation retraction $r^{(j)} : I_j \times [0, \frac{1}{3}] \to I_j$, so that $r_0^{(j)} = id_{I_j}$ and $r_{1/3}^{(j)} \equiv z_j$. Then, extend $r_t^{(j)}$ smoothly with respect to $t \in [0, \frac{2}{3}]$ by letting $r_t^{(j)}(z) = z_j(t)$, whenever $t \in [\frac{1}{3}, \frac{2}{3}]$ and $z \in I_j$. Here, $z_j(t)$ is a smoothly parameterized geodesic connecting z_j to z_0. Thus, the smooth map $r^{(j)} : I_j \times [0, \frac{2}{3}] \to I_j$ is a homotopy connecting the identity map on I_j to the constant map $r^{(j)}_{2/3}(z) = z_0$ for $z \in I_j$.

Define the isotopy $\phi : I \times S \times [0, \frac{2}{3}] \to J \times S$ as $\phi_t(z,p) = (\gamma_j(z), \psi_{\gamma_j}(r^{(j)}_t(z), p))$ for $(z,p) \in I_j \times S$ where $j = 1, ..., n^2$. When $t = 0$ we have the earlier defined map ϕ_0. When $t = 2/3$ we obtain the map $\phi_{2/3}(z,p) = (\gamma_j(z), \psi_{\gamma_j}(z_0, p))$ for $(z,p) \in I_j \times S$. Notice that the second component of $\phi_{2/3}$ does not depend on the variable z, but only on p. As we mentioned earlier, $\psi_{\gamma_j}(z_0,p) = \psi_{\gamma_j,z_0}(p)$ is isotopic to $D_{\gamma_j}(p)$. Let $\Psi^j_t(z,p) = \psi_{\gamma_j}(r^{(j)}_t(z), p)$ for $t \in [0, \frac{2}{3}]$ and $(z,p) \in I_j \times S$ where $j = 1, ..., n^2$. Let $\Psi^j_t(z,p)$ for $t \in [\frac{2}{3}, 1]$ be the isotopy on the surface S that connects the diffeomorphism $\psi_{\gamma_j,z_0}(p)$ to the Dehn twist $D_{\gamma_j} = \tilde{D}^{-1}_{\gamma_j}$. Notice, that in the case when $t \in [\frac{2}{3}, 1]$ the presence of the variable z in the expression $\Psi^j_t(z,p)$ is superficial as the isotopy, in fact, does not depend on z, but it takes place only on the surface S.

Using the notation above, define the isotopy $\phi : I \times S \times [0,1] \to J \times S$ so that $\phi_t(z,p) = (\gamma_j(z), \Psi^j_t(z,p))$ for $(z,p) \in I_j \times S$ where $j = 1, ..., n^2$. Thus, the maps $\phi_0(z,p) = (\gamma_j(z), \psi_{\gamma_j}(z,p))$ and $\phi_1(z,p) = (\gamma_j(z), D_{\gamma_j}(p))$ for $(z,p) \in I_j \times S$ and $j = 1, ..., n^3$ are isotopic. Notice that ϕ_t respects the vertical fibers $\{z\} \times S$. That is, the isotopy takes place only with respect to the second coordinate, along the fiber S while the first coordinate is kept the same. Therefore, $(\tilde{Q} \times S)/\phi_0$ and $(\tilde{Q} \times S)/\phi_1$ are smoothly isomorphic [7] as fiber bundles over B. As we already saw, $(\tilde{Q} \times S)/\phi_0$ and E are isomorphic as well. Hence, $(\tilde{Q} \times S)/\phi_1$ and E are isomorphic as bundles over B. Since by construction, $(\tilde{Q} \times S)/\phi_1$ and $(\mathbb{D} \times S)/\hat{\Gamma}$ are also isomorphic as bundles over B, we can conclude that there exists a smooth bundle isomorphism $\Phi : (\mathbb{D} \times S)/\hat{\Gamma} \to E$. If $\upsilon : \mathbb{D} \times S \to (\mathbb{D} \times S)/\hat{\Gamma}$ is the quotient map, then it is a bundle map over the covering map π. When we compose it with Φ, we obtain the desired bundle covering map $\Pi = \Phi \circ \upsilon$:

$\mathbb{D} \times S \to E$ which satisfies the condition $H \circ \Pi = \pi \circ pr_{\mathbb{D}}$ and has $\hat{\Gamma}$ as its group of deck transformations. This completes the proof of the theorem. $\square$

The results from Theorem 5 are a main tool in the proofs of Theorem 2 and 3. As it was mentioned already, a deck transformation $\hat{\gamma}(z,p) = (\gamma(z), D_\gamma(p))$ from $\hat{\Gamma}$, not only maps vertical fibers $\{z\} \times S$ to vertical fibers $\{\gamma(z)\} \times S$, but also maps horizontal fibers $\mathbb{D} \times \{p\}$ to horizontal fibers $\mathbb{D} \times \{D_\gamma(p)\}$. In particular, since D_γ acts on $S_{u_0} - (\cup_{j=1}^{n^2} supp(D_{\gamma_j}))$ as the identity map, whenever $p \in S_{u_0} - (\cup_{j=1}^{n^2} supp(D_{\gamma_j}))$, the horizontal disc $\mathbb{D} \times \{p\}$ is invariant under the action of $\hat{\Gamma}$. These facts lead us to the following conclusion.

Corollary 3.1. *The projection $\Pi(\mathbb{D} \times \{p\}) = B_p$ is a smoothly embedded surface in E, diffeomorphic to B. It intersects each leaf from the integrable foliation $\mathcal{F}^0$ transversely at a single point.*

In particular, this corollary applies to the point p_0. Thus, we have obtained the global cross-section B_{p_0}.

3.2 Properties of Multi-Fold Vertical Cycles

In this section, we give a proof of Proposition 1. We start with some notations which will be used at a later time.

Preliminary Notations and Constructions. Let denote by M an arbitrary path-connected topological space with a base point $x_0 \in M$. Let l be an arbitrary loop on M passing through x_0. Then, by $[l]_M$ we are going to denote the equivalence class of all loops homotopic to l in M, relative to the base point x_0.

Denote by $\hat{B}_{\delta_0} \subset \mathbb{D}$ the connected component of $\pi^{-1}(B_{\delta_0})$ that contains the point z_0. The domain $\hat{B}_{\delta_0}$ is open, the closure of $\cup_{\gamma\in\Gamma}\gamma(\hat{B}_{\delta_0})$ is equal to the whole disc $\mathbb{D}$, and for any two transformations γ_1 and γ_2 from Γ, either $\gamma_1(\hat{B}_{\delta_0}) \cap \gamma_2(\hat{B}_{\delta_0}) = \varnothing$ or $\gamma_1(\hat{B}_{\delta_0}) = \gamma_2(\hat{B}_{\delta_0})$.

Since $\hat{B}_{\delta_0}$ is homeomorphic to a disc, there exists a deformation retraction $\overline{R}_t : \hat{B}_{\delta_0} \to \hat{B}_{\delta_0}$ of $\hat{B}_{\delta_0}$ onto z_0, where $t \in [0,1]$. Then $\overline{R}_0 = id_{\hat{B}_{\delta_0}}, \overline{R}_1 \equiv z_0$ and $\overline{R}_t(z_0) = z_0$ for all $t \in [0,1]$. Using $\overline{R}_t$, we can define the continuous one-parameter family of maps $R_t : \hat{B}_{\delta_0} \times S \to \hat{B}_{\delta_0} \times S$ by denoting $R_t(z,p) = (\overline{R}_t(z), p)$, where $t \in [0,1]$ and $(z,p) \in \hat{B}_{\delta_0} \times S$. Notice, that $R_0 = id_{(\hat{B_{\delta_0}} \times S)}$ and $R_1(z,p) = (z_0, p)$. In addition, $R_t(z_0,p) = (\overline{R}_t(z_0), p) = (z_0, p)$ for any point $(z_0, p) \in \{z_0\} \times S$ and any $t \in [0,1]$. Then R_t is a deformation retraction of $\hat{B}_{\delta_0} \times S$ onto $\{z_0\} \times S$. For simplicity, let $R = R_1$. So $R(z,p) = (z_0,p)$ for any $(z,p) \in \hat{B}_{\delta_0} \times S$ and it can be rewritten as $R(z,p) = (z_0, pr_S(z,p))$.

Analogously, we can define a deformation retraction R'_t of $\mathbb{D}\times S$ onto $\{z_0\}\times S$. Again for simplicity, we denote $R'(z,p) = R'_1(z,p) = (z_0,p)$ for any point (z,p) from $\mathbb{D} \times S$. As in the case of R, we can write $R'(z,p) = (z_0, pr_S(z,p))$

Proof of Proposition 1. We start with point one from the proposition. By assumption, we know that the foliation $\mathcal{F}^{\varepsilon}$ has a marked cycle (Δ, q) with a representative δ contained in E_{δ_0} and free homotopic to δ_0^m inside E_{δ_0}. Assume that besides that, the representative δ is free homotopic inside E_{δ_0} to another loop δ'_0, also lying on the fibre S. This implies that there exists a free homotopy $\delta(t)$ inside E_{δ_0}, where $t \in [0,1]$, such that $\delta(0) = \delta'_0$ and $\delta(1) = \delta_0$. The loop δ'_0 lifts to the loop $\{z_0\} \times \delta'_0$ on the fiber $\{z_0\} \times S$ and so, $\Pi(\{z_0\} \times \delta'_0) = \delta'_0$. Then $\delta(t)$ lifts to a homotpy $\hat{\delta}(t)$ for which $\hat{\delta}(0) = \{z_0\} \times \delta'_0$. When $t = 1$ the loop $\hat{\delta}(1)$

belongs to the fiber $\{\gamma(z_0)\} \times S$ and maps to $\delta_0 = \Pi(\hat{\delta}(1))$, where $\gamma \in \Gamma$. Since the homotopy $\delta(t)$ takes place inside the domain E_{δ_0}, the lifted homotopy $\hat{\delta}(t)$ takes place in $\hat{B}_{\delta_0} \times S$, so in fact $\gamma \in \Gamma_0$. Because $\hat{\delta}(1)$ lies on the fiber $\{\gamma(z_0)\} \times S$, it has the form $\hat{\delta}(1) = \{\gamma(z_0)\} \times \delta_1$, where δ_1 is a loop on the surface S. Using this representation we compute

$$
\begin{aligned}
\Pi\Big(\{\gamma(z_0)\} \times \delta_1\Big) &= \Pi \circ \hat{\gamma}^{-1}\Big(\{\gamma(z_0)\} \times \delta_1\Big) \\
&= \Pi\Big(\{\gamma^{-1} \circ \gamma(z_0)\} \times D_{\gamma}^{-1}(\delta_1)\Big) \\
&= \Pi\Big(\{z_0\} \times D_{\gamma}^{-1}(\delta_1)\Big) \\
&= D_{\gamma}^{-1}(\delta_1) = \delta_0,
\end{aligned}
$$

that is $\delta_1 = D_{\gamma}(\delta_0)$. Now, consider the homotopy $pr_S(\hat{\delta}(t))$ which takes place only on the surface S. Notice that $pr_S(\hat{\delta}(t))$ is continuous with respect to $t \in [0,1]$. Moreover, for $t = 0$ we have $pr_S(\hat{\delta}(0)) = \delta_0'$ and for $t = 1$ we have $pr_S(\hat{\delta}(1)) = \delta_1 = D_{\gamma}(\delta_0)$. As we already noticed, $D_{\gamma}(\delta_0) = \delta_0$ whenever $\gamma \in \Gamma_0$, hence $pr_S(\hat{\delta}(t))$ is the desired homotopy on the surface S between the two loops δ_0' and δ_0.

Next, we prove the second part of the proposition. Since both δ and δ' are representatives from the same marked cycle (Δ, q), there exists a homotopy $\delta(t)$ on the leaf φ_q^{ε} that keeps the base point q fixed and connects δ to δ'. Ignoring the leaf φ_q^{ε}, we have a homotopy $\delta(t)$ inside E such that $\delta(0) = \delta$ and $\delta(1) = \delta'$.

Let $(\tilde{z}, \tilde{p}) \in \hat{B}_{\delta_0} \times S$ be such that $\Pi(\tilde{z}, \tilde{p}) = q$. Since δ is $\delta_0, m-$fold vertical, it lifts under the covering map Π to a loop $\hat{\delta}$ contained in $\hat{B}_{\delta_0} \times S$. By the homotopy lifting property of covering spaces [6], the homotopy $\delta(t)$ inside E lifts to a homotopy $\hat{\delta}(t)$ inside $\mathbb{D} \times S$, so that $\Pi(\hat{\delta}(t)) = \delta(t)$. Thus, $\hat{\delta}(t)$ connects $\hat{\delta}$ to $\hat{\delta}' = \hat{\delta}(1)$, where $\Pi(\hat{\delta}') = \delta'$.

Because of the assumption that δ' is contained in E_{δ_0}, it follows that $\hat{\delta}'$ is inside $\gamma(\hat{B}_{\delta_0}) \times S$ for some $\gamma \in \Gamma$. Then, the base point $(\tilde{z}, \tilde{p})$, which lies on the loop $\hat{\delta}'$, is simultaneously in $\gamma(\hat{B}_{\delta_0}) \times S$ and in $\hat{B}_{\delta_0} \times S$. Therefore $(\gamma(\hat{B}_{\delta_0}) \times S) \cap (\hat{B}_{\delta_0} \times S) \neq \varnothing$, which is possible only when $\gamma(\hat{B}_{\delta_0}) \cap \hat{B}_{\delta_0} \neq \varnothing$. But by construction, $\gamma(\hat{B}_{\delta_0}) \cap \hat{B}_{\delta_0} \neq \varnothing$ if and only if $\gamma(\hat{B}_{\delta_0}) = \hat{B}_{\delta_0}$. It follows from here that $\hat{\delta}'$ is contained in $\hat{B}_{\delta_0} \times S$.

As pointed out in the two paragraphs preceding the proof, the map $R : \hat{B}_{\delta_0} \times S \to \{z_0\} \times S$ defined by the expression $R(z,p) = (z_0, p)$ is a deformation retraction. Similarly, $R' : \mathbb{D} \times S \to \{z_0\} \times S$, defined by the same rule $\hat{R}'(z,p) = (z_0,p)$, is also a deformation retraction. The induced homomorphisms on the corresponding fundamental groups

$$R_* : \pi_1(\hat{B}_{\delta_0} \times S, (\tilde{z}, \tilde{p})) \to \pi_1(\{z_0\} \times S, (z_0, \tilde{p}))$$

$$R'_* : \pi_1(\mathbb{D} \times S, (\tilde{z}, \tilde{p})) \to \pi_1(\{z_0\} \times S, (z_0, \tilde{p})),$$

given by $R_*[\, l\,]_{(\hat{B}_{\delta_0} \times S)} = [R(l)]_{(\{z_0\}\times S)}$ and $R'_*[\, l'\,]_{(\mathbb{D}\times S)} = [R'(l')]_{(\{z_0\}\times S)}$ respectively, are isomorphisms since they come from deformation retractions [6]. Here, l and l' are arbitrary loops from $\hat{B}_{\delta_0} \times S$ and $\mathbb{D} \times S$ respectively, passing through $(\tilde{z}, \tilde{p})$. Because of the fact that R is simply the restriction of R' onto $\hat{B}_{\delta_0} \times S$ and that both loops $\hat{\delta}$ and $\hat{\delta}(1)$ are contained in $\hat{B}_{\delta_0} \times S$, it follows that

$$R'_*[\, \hat{\delta}\,]_{(\mathbb{D}\times S)} = [R'(\hat{\delta})]_{(\{z_0\}\times S)} = [R(\hat{\delta})]_{(\{z_0\}\times S)} = R_*[\, \hat{\delta}\,]_{(\hat{B}_{\delta_0}\times S)}$$

$$R'_*[\, \hat{\delta}'\,]_{(\mathbb{D}\times S)} = [R'(\hat{\delta}')]_{(\{z_0\}\times S)} = [R(\hat{\delta}')]_{(\{z_0\}\times S)} = R_*[\, \hat{\delta}'\,]_{(\hat{B}_{\delta_0}\times S)}.$$

Since $\hat{\delta}$ and $\hat{\delta}'$ are homotopic inside $\mathbb{D} \times S$ via $\hat{\delta}(t)$, we can see that $[\, \hat{\delta}\,]_{(\mathbb{D}\times S)} = [\, \hat{\delta}'\,]_{(\mathbb{D}\times S)}$. Therefore, $R'_*[\, \hat{\delta}\,]_{(\mathbb{D}\times S)} = R'_*[\, \hat{\delta}'\,]_{(\mathbb{D}\times S)}$. Combining all of those identities, we obtain

$$R_*[\, \hat{\delta}\,]_{(B_{\delta_0}\times S)} = R'_*[\, \hat{\delta}\,]_{(\mathbb{D}\times S)} = R'_*[\, \hat{\delta}'\,]_{(\mathbb{D}\times S)} = R_*[\, \hat{\delta}'\,]_{(B_{\delta_0}\times S)}.$$

Because R_* is a group isomorphism, $R_*[\,\hat{\delta}\,]_{(B_{\delta_0}\times S)} = R_*[\,\hat{\delta}'\,]_{(B_{\delta_0}\times S)}$ if and only if $[\,\hat{\delta}\,]_{(B_{\delta_0}\times S)} = [\,\hat{\delta}'\,]_{(B_{\delta_0}\times S)}$, which immediately implies that there exists a homotopy $\hat{\delta}_t$ inside $\hat{B}_{\delta_0} \times S$ such that $\hat{\delta}_0 = \hat{\delta}$ and $\hat{\delta}_1 = \hat{\delta}'$. The projection of $\hat{\delta}_t$ back to E gives rise to a homotopy $\delta_t = \Pi(\hat{\delta}_t)$ inside E_{δ_0} between the loops δ' and δ. By assumption, δ is free homotopic to δ^m inside E_{δ_0}. Therefore, δ' is also free homotopic to δ_0^m inside E_{δ_0}. $\square$

CHAPTER 4

THE POINCARÉ MAP, PERIODIC ORBITS, AND MARKED CYCLES

The goal of this chapter is to provide the proof of Theorem 2. It heavily relies on the results from the preceding chapter and establishes the link between the topological properties of the foliation and the dynamical properties of its Poincaré transformation, constructed on a very large cross-section.

4.1 Construction of a Globally Defined Poincaré Map

Cuts, Domains on the Disc, and the Lifted Foliation. As it was promised in Section 2.1, we begin with a description of each cut l_j that connects the cusp a_j to ∞ on B, for $j \in J(\delta_0)$. Let $l_j = \pi(e_j) = \pi(\gamma_j(e_j)) \subset B$ be the image of the two adjacent geodesic edges e_j and $\gamma_j(e_j)$ of the ideal polygon Q that meet the boundary of $\mathbb{D}$ at $\hat{a}_j$.

Now, having in mind all the constructions from Sections 2.1 and 3.1, we are ready to move on with the definition of the desired Poincaré map. Our first step will be to set up a few domains in $\mathbb{D}$ that will play an important role in the construction of the map. From this moment on, all interiors and closures of subsets of $\mathbb{D}$ will be relative to the topology of the open disc $\mathbb{D}$. Lift the domain A' onto $\mathbb{D}$ to obtain $\hat{A}' = \pi^{-1}(A')$. Take $\hat{C}'_{\delta_0}$ to be the connected component of $\pi^{-1}(C'_{\delta_0})$ that contain the point z_0. Define the compact domain $Q' = Q \cap \pi^{-1}(\overline{C'_{\delta_0}})$. We can think of Q' as the ideal geodesic polygon Q with its corners cut out along horocycle arcs. Attach to Q' the neighboring congruent pieces to form the compact domain

$$\hat{C}' = \cup\{\gamma(Q') \,:\, \gamma \in \{id_{\mathbb{D}}, \gamma_1, ..., \gamma_{n^2}, \gamma_1^{-1}, ..., \gamma_{n^2}^{-1}\}\}.$$

Similarly, let $Q_A = Q \cap \pi^{-1}(\overline{A})$ and let

$$\hat{C}_A = \cup\{\gamma(Q_A) \,:\, \gamma \in \{id_{\mathbb{D}}, \gamma_1, ..., \gamma_{n^2}, \gamma_1^{-1}, ..., \gamma_{n^2}^{-1}\}\}.$$

If we denote by $\hat{C}$ the intersection $Q \cap \pi^{-1}(\overline{C_{\delta_0}})$, then by construction $\hat{C} \subset \hat{C}_A \subset \hat{C}' \subset \hat{A}'$.

In the constructions that are going to follow we will need the group $\Gamma_0 = \langle \gamma_j \mid j \in J(\delta_0)\rangle$ and its lift $\hat{\Gamma}_0 = \langle \hat{\gamma}_j = \gamma_j \times D_{\gamma_j} \mid j \in J(\delta_0)\rangle$ which are subgroups of the deck groups Γ and $\hat{\Gamma}$ respectively. With the help of those groups we define the closed domains

$$\hat{X}_{\delta_0} = \cup_{\gamma\in\Gamma_0}\gamma(\hat{C}), \;\; \hat{X}'_{\delta_0} = \cup_{\gamma\in\Gamma_0}\gamma(\hat{C}') \;\text{ and }\; \hat{A} = \cup_{\gamma\in\Gamma_0}\gamma(\hat{C}_A).$$

Notice, that $\hat{X}_{\delta_0}$ is in fact the closure of $\hat{C}_{\delta_0}$.

Consider the pull-back $\hat{\mathcal{F}}^\varepsilon = \Pi^*\mathcal{F}^\varepsilon$. This is a foliation on $\mathbb{D} \times S$ invariant with respect to the action of $\hat{\Gamma}$. In other words, if $\hat{\gamma} \in \hat{\Gamma}$ and $\hat{\varphi}^\varepsilon_{(z,p)}$ is a leaf of the foliation $\hat{\mathcal{F}}^\varepsilon$ passing through the point $(z,p) \in \mathbb{D} \times S$, then $\hat{\gamma}(\hat{\varphi}^\varepsilon_{(z,p)}) = \hat{\varphi}^\varepsilon_{\hat{\gamma}(z,p)}$. Notice that the closure of the projection $\Pi(\hat{A}' \times \{p_0\}) = A'_{p_0}$ is compact in E and thus, the line field of the foliation $\mathcal{F}^\varepsilon$ is transverse to A'_{p_0} for all $|\varepsilon| \leq r$, where $r > 0$ is small enough.

Construction of the Poincaré Map.

Lemma 4.1. *For small enough $r > 0$ and for any $|\varepsilon| \leq r$ there exists a smooth Poincaré map $\hat{P}_{\delta_0,\varepsilon} : \hat{C}' \times \{p_0\} \to \hat{A}' \times \{p_0\}$ associated with the foliation $\hat{\mathcal{F}}^\varepsilon$ such that for any $\hat{\gamma} \in \hat{\Gamma}$ if both points (z, p_0) and $\hat{\gamma}(z, p_0)$ belong to $\hat{C}' \times \{p_0\}$ then $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{D_\gamma(\delta_0),\varepsilon} \circ \hat{\gamma}$. In particular, if $\hat{\gamma} \in \hat{\Gamma}_0$ then $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}$. Moreover, for an integer $m > 0$ the radius $r > 0$ can be chosen small enough so that $\hat{P}^k_{\delta_0,\varepsilon}(\hat{C} \times \{p_0\}) \subset \hat{C}_A \times \{p_0\}$, for $k = 1, ..., m$ and for all $\varepsilon \in D_r(0)$*

Proof: As usual, let $pr_S : \mathbb{D} \times S \to S$ be the projection $(z,p) \mapsto p$. By continuous dependance of $\hat{\mathcal{F}}^\varepsilon$ on parameters and initial conditions, we can choose the radius r of the parameter space so that the construction that follows holds for any $|\varepsilon| \leq r$. Choose an arbitrary point $(z, p_0) \in \hat{C}' \times \{p_0\}$. If $\hat{\varphi}^\varepsilon_{(z,p_0)}$ is the leaf of the perturbed foliation $\hat{\mathcal{F}}^\varepsilon$, passing through (z,p_0), lift the loop δ_0 to a curve $\hat{\delta}_\varepsilon(z,p_0)$ on $\hat{\varphi}^\varepsilon_{(z,p_0)}$ so that $\hat{\delta}_\varepsilon(z,p_0)$ covers δ_0 under the projection pr_S. Since r is chosen small enough, the lift $\hat{\delta}_\varepsilon(z,p_0)$ is contained in the domain $\hat{A}' \times S$ and both of its endpoints are on $\hat{A}' \times \{p_0\}$. The first endpoint is $(z,p_0) \in \hat{C}' \times \{p_0\}$ and the second we denote by $\hat{P}_{\delta_0,\varepsilon}(z,p_0) = (\tilde{P}_{\delta_0,\varepsilon}(z), p_0) \in \hat{A}' \times \{p_0\}$. Thus, we obtain the correspondence $\hat{P}_{\delta_0,\varepsilon} : \hat{C}' \times \{p_0\} \to \hat{A}' \times \{p_0\}$, which is a smooth map close to identity. Notice, that for some integer $m > 0$ if we decrease the radius of the parameter space enough, then by continuous dependance on parameters and initial conditions we can make sure that for any $\varepsilon \in D_r(0)$, all m iterations of $\hat{C} \times \{p_0\}$ under $\hat{P}_{\delta_0,\varepsilon}$ fall inside $\hat{C}_A \times \{p_0\}$.

By construction, the cross-section $\hat{A}' \times \{p_0\}$ is $\hat{\Gamma}-$invariant. Now, assume $(z,p_0) \in \hat{C}' \times \{p_0\}$ is such that $\hat{\gamma}(z,p_0) = (\gamma(z), p_0) \in \hat{C}' \times \{p_0\}$ for some $\hat{\gamma} \in \hat{\Gamma}$. As pointed out earlier, the arc $\hat{\delta}_\varepsilon(z,p_0)$ is the lift of δ_0 on $\hat{\varphi}^\varepsilon_{(z,p_0)}$ under the projection pr_S. It connects the two points $(z,p_0) \in \hat{C}' \times \{p_0\}$ and $\hat{P}_{\delta_0,\varepsilon}(z,p_0) \in \hat{A}' \times \{p_0\}$. The image $\hat{\gamma}(\hat{\delta}_\varepsilon(z,p_0))$ lies on the leaf $\hat{\varphi}^\varepsilon_{\hat{\gamma}(z,p_0)}$ and its endpoints are $\hat{\gamma}(z,p_0) \in \hat{C}' \times \{p_0\}$ and $\hat{\gamma}(\hat{P}_{\delta_0,\varepsilon}(z,p_0)) \in \hat{A}' \times \{p_0\}$. We can see that $pr_S \circ \hat{\gamma}(z,p) = pr_S(\gamma(z), D_\gamma(p)) = D_\gamma(p) = D_\gamma \circ pr_S(z,p)$. The fact that $\hat{\delta}_\varepsilon(z,p_0)$ is the lift of δ_0 on the leaf $\hat{\varphi}^\varepsilon_{(z,p_0)}$ from $\mathcal{F}^\varepsilon$ means that $pr_S(\hat{\delta}_\varepsilon(z,p_0)) = \delta_0$. Similarly, to find out what the arc $\hat{\gamma}(\hat{\delta}_\varepsilon(z,p_0))$ is a lift of we just have to project it onto S. Using the property $pr_S \circ \hat{\gamma} = D_\gamma \circ pr_S$ we conclude that $pr_S \circ \hat{\gamma}(\hat{\delta}_\varepsilon(z,p_0)) = D_\gamma \circ pr_S(\hat{\delta}_\varepsilon(z,p_0)) = D_\gamma(\delta_0)$. That is, $\hat{\gamma}(\hat{\delta}_\varepsilon(z,p_0))$ is the lift of $D_\gamma(\delta_0)$ on the leaf $\hat{\varphi}^\varepsilon_{\hat{\gamma}(z,p_0)}$ under the projection pr_S. Therefore, the endpoint $\hat{\gamma}(\hat{P}_{\delta_0,\varepsilon}(z,p_0))$ can also be represented as $\hat{P}_{D_\gamma(\delta_0),\varepsilon}(\hat{\gamma}(z,p_0))$. Thus, we obtain the relation $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{D_\gamma(\delta_0),\varepsilon} \circ \hat{\gamma}$.

The base loop $\delta_0 \subset S$ is chosen so that whenever $\delta_0 \cdot \delta_j = 0$ then $\delta_0 \cap supp(D_{\gamma_j}) = \varnothing$. Because of this choice, if $\gamma \in \Gamma_0$ we have the identity $D_\gamma(\delta_0) = \delta_0$. That leads to the second equivariance relation $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}$. $\square$

Lemma 4.1 allows us to extend $\hat{P}_{\delta_0,\varepsilon}$ from a map on $\hat{C}' \times \{p_0\}$ to a $\hat{\Gamma}_0$ - equivarint map on the cross-section $\hat{X}'_{\delta_0} \times \{p_0\}$. In particular, since $\hat{C}''_{\delta_0} \times \{p_0\}$ is a $\hat{\Gamma}_0-$invariant open subdomain of $\hat{X}'_{\delta_0} \times \{p_0\}$, the map $\hat{P}_{\delta_0,\varepsilon}$ is well defined and $\hat{\Gamma}_0-$equivarint on it. This fact makes it possible for the $\hat{P}_{\delta_0,\varepsilon}$ to descend under the covering Π to a Poincaré map defined on C''_{p_0}.

Corollary 4.1. *The transformation $\hat{P}_{\delta_0,\varepsilon}$ constructed in lemma 4.1 gives rise to a Poincaré map $\hat{P}_{\delta_0,\varepsilon} : \hat{X}'_{\delta_0} \times \{p_0\} \to \hat{A}' \times \{p_0\}$ for the foliation $\hat{\mathcal{F}}^\varepsilon$ such that for any $\hat{\gamma} \in \hat{\Gamma}_0$ the equivariance relation $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}$ holds. In particular, the restriction of $\hat{P}_{\delta_0,\varepsilon}$ on $\hat{C}''_{\delta_0} \times \{p_0\}$ satisfies the same equivarance relation $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}$ for $\hat{\gamma} \in \hat{\Gamma}_0$.*

Proof: Notice that Γ_0 keeps both domains $\hat{X}'_{\delta_0}$ and $\hat{C}''_{\delta_0}$ invariant. In other words, $\gamma(\hat{X}'_{\delta_0}) = \hat{X}'_{\delta_0}$ and $\gamma(\hat{C}''_{\delta_0}) = \hat{C}''_{\delta_0}$ for any $\gamma \in \Gamma_0$. This immediately leads to the invariance of the cross-sections $\hat{X}'_{\delta_0} \times \{p_0\}$ and $\hat{C}''_{\delta_0} \times \{p_0\}$ under the action of $\hat{\Gamma}_0$.

Since $\hat{X}'_{\delta_0} = \cup_{\gamma \in \Gamma_0} \gamma(\hat{C}')$, we can define $\hat{P}_{\delta_0,\varepsilon}$ on $\gamma(\hat{C}') \times \{p_0\} = \hat{\gamma}(\hat{C}' \times \{p_0\})$ as the conjugated map $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}^{-1} : \gamma(\hat{C}') \times \{p_0\} \to \hat{A}' \times \{p_0\}$. By lemma 4.1, for $\hat{\gamma}_1$ and $\hat{\gamma}_2 \in \hat{\Gamma}_0$, the two maps $\hat{\gamma}_1 \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}_1^{-1}$ and $\hat{\gamma}_2 \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}_2^{-1}$ agree on the intersection $\hat{\gamma}_1(\hat{C}' \times \{p_0\}) \cap \hat{\gamma}_2(\hat{C}' \times \{p_0\})$ whenever it is nonempty. As C''_{δ_0} is a Γ_0-invariant subdomain of $\hat{X}'_{\delta_0}$, the second statement follows immediately. $\square$

Corollary 4.2. *The transformation* $\hat{P}_{\delta_0,\varepsilon} : \hat{C}'_{\delta_0} \times \{p_0\} \to \hat{A}' \times \{p_0\}$ *associated with the foliation* $\hat{\mathcal{F}}^{\varepsilon}$ *descends to a smooth Poincaré map* $P_{\delta_0,\varepsilon} : C'_{p_0} \to A'_{p_0}$ *for the foliation* $\mathcal{F}^{\varepsilon}$ *under the covering bundle map* $\Pi : \mathbb{D} \times S \to E$*. In other words, for any* $(z, p_0) \in \hat{C}'_{\delta_0} \times \{p_0\}$ *the relation* $\Pi \circ \hat{P}_{\delta_0,\varepsilon}(z, p_0) = P_{\delta_0,\varepsilon} \circ \Pi(z, p_0)$ *holds.*

Proof: The statement follows directly from corollary 4.1. □

At this point, it is not difficult to explain the role of the index set $J(\delta_0)$ and the choice of the cuts in the definition of B_{δ_0} and subsequently of C_{p_0} and C'_{p_0}. Whenever $j \in J(\delta_0)$, the loop δ_0 does not intersect the vanishing cycle δ_j and in fact is contained in $S - supp(D_{\gamma_j})$. Hence, it is true that $D_{\gamma_j}(\delta_0) = \delta_0$. As a result of this, the descended map $P_{\delta_0,\varepsilon}$ is univalent around the hole in C'_{p_0} associated to the singularity a_j. On the other hand, for i not in $J(\delta_0)$ the loop δ_0 intersects δ_i and so $D_{\gamma_i}(\delta_0)$ is not even free homotopic to δ_0. Therefore the map $P_{\delta_0,\varepsilon}$ is going to branch switching from $P_{\delta_0,\varepsilon}$ to $P_{D_\gamma(\delta_0),\varepsilon}$ when going through a cut.

On a side note, but still worth mentioning is a fact that follows from the constructions in the proof of lemma 4.1. It is not difficult to see that the Poincaré map does not change when the base loop δ_0 has been homotoped appropriately. In other words, if δ_0 is homotopic on S to another loop δ_0' passing through p_0, then the two maps $\hat{P}_{\delta_0,\varepsilon}$ and $\hat{P}_{\delta_0',\varepsilon}$ will be equal, as long as δ_0' is close enough to δ_0 on S or the radius r is kept small enough. Thus, if we slightly wiggle δ_0 on S so that the base point p_0 is kept fixed, the resulting Poincaré map will stay the same. This provides us with the opportunity to adjust the loop δ_0 if necessary. The same is true for $P_{\delta_0,\varepsilon}$.

4.2 Complex Structures on the Cross-Section

Apart from the smooth structure of a fiber bundle, the space E, being a subset of $\mathbb{C}^2$, has a complex structure with respect to which the foliation $\mathcal{F}^\varepsilon$ is holomorphic and depends analytically on the parameter ε. This fact provides the foliation with very specific properties. On the other hand, the Poincaré map $P_{\delta_0,\varepsilon} : C'_{p_0} \to A'_{p_0}$ for the perturbed foliation $\mathcal{F}^\varepsilon$ captures some topological properties of the foliation. Since some of those properties are strongly related to the holomorphic nature of the foliation, we would like our Poincaré map to reflect the complex analyticity of $\mathcal{F}^\varepsilon$. So far $P_{\delta_0,\varepsilon}$ is defined as a smooth map on the smooth surface C'_{p_0} and therefore our next step is to induce a complex structure on C'_{p_0} in which the Poincaré transformation is holomorphic.

Complex Atlas on the Cross-Section. Since the closure of A'_{p_0} is transverse to $\mathcal{F}^\varepsilon$, there is an open neighborhood $\tilde{A}_{p_0}$ of A'_{p_0} such that $\tilde{A}_{p_0}$ is transverse to $\mathcal{F}^\varepsilon$. Fix $\varepsilon \in D_r(0)$. Take a point $q_0 \in \tilde{A}_{p_0}$ and a complex cross-section L_{q_0} through q_0, transverse to $\mathcal{F}^\varepsilon$. More precisely, L_{q_0} is a complex segment, that is, it lies on a complex line through q_0 and is a real two dimensional disc.

The fact that the foliation $\mathcal{F}^\varepsilon$ is holomorphic and $\tilde{A}_{p_0}$ is smoothly embedded surface transverse to $\mathcal{F}^\varepsilon$ provides us with convenient flow-box charts. A chart of this kind consists of an open neighborhood $FB(q_0) \subset E$ of q_0 and a biholomorphic map

$$\beta_{q_0,\varepsilon} : \mathbb{D} \times \mathbb{D} \longrightarrow FB(q_0)$$

with the following properties:

1. $\beta_{q_0,\varepsilon}(0,0) = q_0$;

2. $\beta_{q_0,\varepsilon}(\{\zeta\} \times \mathbb{D})$ is a connected component of the intersection of $FB(q_0)$ with the leaf $\varphi^{\varepsilon}_{\beta_{q_0,\varepsilon}(\zeta,0)}$ through the point $\beta_{q_0,\varepsilon}(\zeta, 0)$ for any $\zeta \in \mathbb{D}$;

3. $\beta_{q_0,\varepsilon}(\mathbb{D} \times \{0\}) = L_{q_0}$;

4. The portion of $\tilde{A}_{p_0}$ passing through $FB(q_0)$ looks like the graph of a smooth map $\alpha_{q_0,\varepsilon} : \mathbb{D} \to \mathbb{D}$ in the chart $\mathbb{D} \times \mathbb{D}$. In other words

$$\beta^{-1}_{q_0,\varepsilon}(FB(q_0) \cap \tilde{A}_{p_0}) = \{(\zeta, \alpha_{q_0,\varepsilon}(\zeta)) \in \mathbb{D} \times \mathbb{D} \mid \alpha_{q_0,\varepsilon} : \mathbb{D} \to \mathbb{D} \text{ a smooth map}\}.$$

Denote by U_{q_0} the open subset $FB(q_0) \cap \tilde{A}_{p_0}$ of $\tilde{A}_{p_0}$. Let $pr_j : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ be $pr_j(\zeta_1, \zeta_2) = \zeta_j$, where $j = 1, 2$. Define the diffeomorphism

$$\begin{aligned}
\phi_{q_0,\varepsilon} \ &: \ U_{q_0} \longrightarrow \mathbb{D} \ \text{ by} \\
\phi_{q_0,\varepsilon} \ &: \ q \longmapsto pr_1 \circ (\beta^{-1}_{q_0,\varepsilon})\big|_{U_{q_0}}(q) \\
\phi^{-1}_{q_0,\varepsilon} \ &: \ \zeta \longmapsto \beta_{q_0,\varepsilon}(\zeta, \alpha_{q_0,\varepsilon}(\zeta)).
\end{aligned}$$

Consider the family of pairs $\mathcal{A}_\varepsilon(\tilde{A}_{p_0}) = \{(U_{q_0}, \phi_{q_0,\varepsilon}) \,|\, q_0 \in \tilde{A}_{p_0}\}$.

Lemma 4.2. *The collection of charts $\mathcal{A}_\varepsilon(\tilde{A}_{p_0})$ is a holomorphic atlas for the surface $\tilde{A}_{p_0}$.*

Proof: Let q_1, q_2 be two points from the surface $\tilde{A}_{p_0}$ with chart neighborhoods $U_{q_1} \cap U_{q_2} \neq \varnothing$. Let $V_j = \phi_{q_j,\varepsilon}(U_{q_1} \cap U_{q_2})$ for $j = 1, 2$. Consider the diffeomorphism $\phi_{q_2,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon} : V_1 \to V_2$. For a point $\zeta \in V_1$ compute

$$\begin{aligned}
\phi_{q_2,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon}(\zeta) &= pr_1 \circ (\beta^{-1}_{q_2,\varepsilon})\big|_{U_{q_2}} \circ \beta_{q_1,\varepsilon}(\zeta, \alpha_{q_1,\varepsilon}(\zeta)) \\
&= pr_1 \circ (\beta^{-1}_{q_2,\varepsilon} \circ \beta_{q_1,\varepsilon})(\zeta, \alpha_{q_1,\varepsilon}(\zeta)).
\end{aligned}$$

The map

$$\beta^{-1}_{q_2,\varepsilon} \circ \beta_{q_1,\varepsilon} \ : \ \beta^{-1}_{q_1,\varepsilon}(FB(q_1) \cap FB(q_2)) \ \longrightarrow \ \beta^{-1}_{q_2,\varepsilon}(FB(q_1) \cap FB(q_2))$$

is a holomorphic isomorphism. Let us take a local leaf $\{\zeta\} \times \mathbb{D}$ contained in the open set $\beta_{q_1,\varepsilon}^{-1}(FB(q_1) \cap FB(q_2))$. Then, $\beta_{q_1,\varepsilon}(\{\zeta\} \times \mathbb{D})$ lies on the leaf $\varphi^{\varepsilon}_{\beta_{q_1,\varepsilon}(\zeta,0)}$ from the foliation $\mathcal{F}^{\varepsilon}$. Since $\varphi^{\varepsilon}_{\beta_{q_1,\varepsilon}(\zeta,0)}$ passes through the intersection $FB(q_1) \cap FB(q_2)$, there exists $\zeta' \in \mathbb{D}$ such that $\varphi^{\varepsilon}_{\beta_{q_2,\varepsilon}(\zeta',0)} = \varphi^{\varepsilon}_{\beta_{q_1,\varepsilon}(\zeta,0)}$. It follows from here that $\beta_{q_2,\varepsilon}(\{\zeta'\} \times \mathbb{D})$ lies on the leaf $\varphi^{\varepsilon}_{\beta_{q_2,\varepsilon}(\zeta',0)} = \varphi^{\varepsilon}_{\beta_{q_1,\varepsilon}(\zeta,0)}$. Hence,

$$\beta_{q_2,\varepsilon}^{-1} \circ \beta_{q_1,\varepsilon}(\{\zeta\} \times \mathbb{D}) = \{\zeta'\} \times \mathbb{D}.$$

Therefore

$$pr_1 \circ \beta_{q_2,\varepsilon}^{-1} \circ \beta_{q_1,\varepsilon}(\zeta, \xi) = pr_1 \circ \beta_{q_2,\varepsilon}^{-1} \circ \beta_{q_1,\varepsilon}(\zeta, 0)$$

for all $\xi \in \mathbb{D}$. In particular,

$$\phi_{q_2,\varepsilon} \circ \phi_{q_1,\varepsilon}(\zeta) = pr_1 \circ \beta_{q_2,\varepsilon}^{-1} \circ \beta_{q_1,\varepsilon}(\zeta, \alpha_{q_1,\varepsilon}(\zeta)) = pr_1 \circ \beta_{q_2,\varepsilon}^{-1} \circ \beta_{q_1,\varepsilon}(\zeta, 0),$$

is a holomorphic transformation with respect to ζ. Notice, that in fact the transition map $\phi_{q_2,\varepsilon} \circ \phi_{q_1,\varepsilon}(\zeta)$ depends holomorphically on ε as well. $\Box$

Complex Analyticity of the Poincaré Map. The choice of complex structure on the surface $\tilde{A}_{p_0}$ is justified by the next lemma. As it turns out, the map $P_{\delta_0,\varepsilon}$ is holomorphic in the complex structure $\mathcal{A}_{\varepsilon}(\tilde{A}_{p_0})$.

Lemma 4.3. *The Poincaré map $P_{\delta_0,\varepsilon} : C'_{\delta_0} \to A'_{p_0}$ associated to the foliation $\mathcal{F}^{\varepsilon}$ is holomorphic in the complex structure defined by the atlas $\mathcal{A}_{\varepsilon}(\tilde{A}_{p_0})$ and depends analytically with respect to the parameter ε.*

Proof: Let $q_1 \in C'_{\delta_0}$ and $q_2 \in A'_{p_0}$ be two points such that $q_2 = P_{\delta_0,\varepsilon}(q_1)$. Find charts U_{q_1} and U_{q_2} such that $P_{\delta_0,\varepsilon}(U_{q_1}) \subset U_{q_2}$ and L_{q_1} and L_{q_2} are the corresponding cross-sections. According to the definition for a holomorphic transformation with respect to a complex atlas, $P_{\delta_0,\varepsilon}$ is considered holomorphic whenever

$$\phi_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon} : \mathbb{D} \longrightarrow \mathbb{D}$$

is holomorphic.

For an arbitrary $q_0 \in \tilde{A}_{p_0}$ define the map

$$\begin{aligned}
\bar{\phi}_{q_0,\varepsilon} &: U_{q_0} \longrightarrow L_{q_0}, \\
\bar{\phi}_{q_0,\varepsilon} &: q \longmapsto \beta_{q_0,\varepsilon}\big(pr_1 \circ \beta^{-1}_{q_0,\varepsilon}(q), 0\big), \\
\bar{\phi}^{-1}_{q_0,\varepsilon} &: q' \longmapsto \beta_{q_0,\varepsilon}\Big(pr_1 \circ \beta^{-1}_{q_0,\varepsilon}(q'), \alpha_{q_0,\varepsilon}\big(pr_1 \circ \beta^{-1}_{q_0,\varepsilon}(q')\big)\Big).
\end{aligned}$$

When $\bar{\phi}_{q_0,\varepsilon}$ is pre-composed with $\beta^{-1}_{q_0,\varepsilon}$, the following chain of equalities holds:

$$\begin{aligned}
\beta^{-1}_{q_0,\varepsilon} \circ \bar{\phi}_{q_0,\varepsilon}(q) &= \beta^{-1}_{q_0,\varepsilon} \circ \beta_{q_0,\varepsilon}\big(pr_1 \circ \beta_{q_0,\varepsilon}(q), 0\big) \\
&= pr_1 \circ \beta^{-1}_{q_0,\varepsilon}(q) \\
&= \phi_{q_0,\varepsilon}(q).
\end{aligned}$$

Let us look at the smooth map

$$\bar{\phi}_{q_0,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_0,\varepsilon} : L_{q_1} \rightarrow L_{q_2}.$$

As noted, $\bar{\phi}_{q_0,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_0,\varepsilon}(L_{q_1}) \subset L_{q_2}$. For $j = 1, 2$ and a point $q' \in L_{q_j}$, the image $\zeta' = pr_1(\beta^{-1}_{q_j,\varepsilon}(q'))$ belongs to $\mathbb{D}$. The straight segment

$$\Upsilon_{q_j,\varepsilon} = [0, \alpha_{q_j,\varepsilon}(\zeta')]$$

on $\mathbb{D}$ connects 0 to the point $\alpha_{q_j,\varepsilon}(\zeta')$ so $\{\zeta'\} \times \Upsilon_{q_j,\varepsilon}$ lies to the local leaf $\{\zeta'\} \times \mathbb{D}$. Therefore

$$\lambda^{\varepsilon}_j(q') = \beta_{q_j,\varepsilon}(\{\zeta'\} \times \Upsilon_{q_j,\varepsilon})$$

is an arc on $\varphi^{\varepsilon}_{q'} \cap FB(q_j)$ with one endpoint $q' \in L_{q_j}$ and the second one being

$$\beta_{q_j,\varepsilon}\Big(pr_1 \circ \beta^{-1}_{q_j,\varepsilon}(q'),\ \alpha_{q_j,\varepsilon}\big(pr_1 \circ \beta^{-1}_{q_j,\varepsilon}(q')\big)\Big) = \bar{\phi}^{-1}_{q_j,\varepsilon}(q') \in U_{q_j}.$$

Remember that the lifted Poincaré transformation $\hat{P}_{\delta_0,\varepsilon}$ was constructed in lemma 4.1 as a correspondence between the endpoints $(\tilde{z}, p_0)$ and $\hat{P}_{\delta_0,\varepsilon}(\tilde{z}, p_0)$ of the path $\hat{\delta}_\varepsilon(\tilde{z}, p_0)$. This path was obtained as the lift of $\delta_0 \subset S$ to the leaf $\hat{\varphi}^\varepsilon_{(\tilde{z},p_0)}$ of the foliation $\hat{\mathcal{F}}^\varepsilon$ under the projection pr_S. Let $\delta_\varepsilon(\tilde{q}) = \Pi(\hat{\delta}_\varepsilon(\tilde{z}, p_0))$, where $\tilde{q} = \Pi(\tilde{z}, p_0) \in C'_{p_0}$. Consider the path

$$\lambda^\varepsilon(q') = \lambda_1^\varepsilon(q') \cdot \delta_\varepsilon\big(\bar{\phi}^{-1}_{q_1,\varepsilon}(\tilde{q})\big) \cdot \Big(\lambda_2^\varepsilon\big(\bar{\phi}_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_1,\varepsilon}(q')\big)\Big)^{-1}.$$

The path connects the point $q' \in L_{q_1}$ to the point $P_{q_1,q_2,\varepsilon}(q') = \bar{\phi}_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_1,\varepsilon}(q')$. By construction, $\lambda^\varepsilon(q')$ lies on the leaf $\varphi^\varepsilon_{q'}$ and varies continuously with respect to both the endpoint $q' \in L_{q_1}$ and the parameter $\varepsilon \in D_r(0)$. The other endpoint $P_{q_1,q_2,\varepsilon}(q')$ belongs to the intersection $\phi^\varepsilon_{q'} \cap L_{q_2}$. As we already know, L_{q_1} and L_{q_2} are holomorphic cross-sections and $\phi^\varepsilon_{q'}$ is a leaf of the holomorphic foliation $\mathcal{F}^\varepsilon$ depending analytically on ε. Then, by analytic dependence of the foliation on parameters and initial conditions [11], it follows that $P_{q_1,q_2,\varepsilon}(q')$ depends analytically on (q', ε). In other words, the map

$$P_{q_1,q_2,\varepsilon}(q') = \bar{\phi}_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_1,\varepsilon} \;:\; L_{q_1} \longrightarrow L_{q_2}$$

is a holomorphic map depending holomorphically on ε. Conjugating with the holomorphic maps $\beta_{q_1,\varepsilon}$ and $\beta_{q_2,\varepsilon}$ we conclude that

$$(\beta^{-1}_{q_2,\varepsilon})|_{L_{q_2}} \circ \bar{\phi}_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \bar{\phi}^{-1}_{q_1,\varepsilon} \circ (\beta_{q_1,\varepsilon})|_{(\{0\}\times\mathbb{D})} = \phi_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon} : \mathbb{D} \to \mathbb{D}$$

is also holomorphic and depends analytically on ε. □

4.3 Periodic Orbits and Complex Cycles

We proceed with the study of the Poincaré maps $P_{\delta_0,\varepsilon}$ and $\hat{P}_{\delta_0,\varepsilon}$. More precisely, we are interested in the relationship between their periodic orbits and the complex cycles of the perturbed foliation $\mathcal{F}^{\varepsilon}$.

First, we start with a more general result.

Lemma 4.4. *Let $r > 0$ be the radius obtained in lemma 4.1. Let*

$$\hat{P}_{\delta_0,\varepsilon} : \hat{X}'_{\delta_0} \times \{p_0\} \to \hat{A}' \times \{p_0\}$$

be the map defined in corollary 4.1, where $\varepsilon \in D_r(0)$. Then, the following statements are true:

1. *Let $\hat{P}_{\delta_0,\varepsilon}$ has a periodic orbit $((z_1, p_0), ..., (z_m, p_0))$ in $\hat{X}'_{\delta_0} \times \{p_0\}$. Then the foliation $\mathcal{F}^{\varepsilon}$ has a marked complex cycle $(\Delta_{\varepsilon}, q_{\varepsilon})$ with a base point $q_{\varepsilon} = \Pi(z_1, p_0)$ and a representative δ_{ε} contained in $E(A')$.*

2. *For an arbitrary representative δ'_{ε} of the marked complex cycle $(\Delta_{\varepsilon}, q_{\varepsilon})$, if δ'_{ε} is contained in E_{δ_0} then it is $D_{\gamma}(\delta_0), m-$fold vertical for some $\gamma \in \Gamma$. Moreover, if z_1 belongs to $\hat{C}'_{\delta_0} \subset \hat{X}'_{\delta_0}$, then $\gamma \in \Gamma_0$ and thus, δ'_{ε} is $\delta_0, m-$fold vertical. Otherwise, if z_1 is in $\hat{X}'_{\delta_0} - \hat{C}'_{\delta_0}$, then $\gamma \in \Gamma - \Gamma_0$ and therefore δ'_{ε} is not $\delta_0, m-$fold vertical.*

Proof: Consider the map $\hat{P}_{\delta_0,\varepsilon} : X'_{\delta_0} \times \{p_0\} \to \hat{A}' \times \{p_0\}$ and its periodic orbit $(z_1, p_0), ..., (z_m, p_0)$ on $\hat{X}'_{\delta_0} \times \{p_0\}$. For convenience, let $(z_{m+1}, p_0) = (z_1, p_0)$. Notice that since all m points belong to the same orbit, they lie on the same leaf $\hat{\varphi}^{\varepsilon}_{(z_1,p_0)}$ from the foliation $\hat{\mathcal{F}}^{\varepsilon}$. Let $\delta(z_i, z_{i+1})$, for $i = 1, ..., m$, be the lift of δ_0 on the leaf $\hat{\varphi}^{\varepsilon}_{(z_1,p_0)}$ so that $\delta(z_i, z_{i+1})$ covers δ_0 under the projection pr_S and connects the points (z_i, p_0) and (z_{i+1}, p_0). By the construction of the map $\hat{P}_{\delta_0,\varepsilon}$ in the proof of lemma 4.1, all arcs $\delta(z_i, z_{i+1})$ are contained in $\hat{A}' \times S$. Therefore, the path $\hat{\delta}_{\varepsilon} = \cup_{i=1}^{m-1} \delta(\hat{q}_i, \hat{q}_{i+1})$ is contained in $\hat{A}' \times S$ and goes through all the points $(z_1, p_0), ..., (z_m, p_0)$. Also, its two endpoints are (z_1, p_0) and $(z_{m+1}, p_0) = (z_1, p_0)$ so in fact $\hat{\delta}_{\varepsilon}$ is a loop.

When mapping $\hat{\delta}_\varepsilon$ with Π back onto E we obtain a loop $\delta_\varepsilon = \Pi(\hat{\delta}^\varepsilon)$ lying on the leaf $\varphi^\varepsilon_{q_\varepsilon} = \Pi(\hat{\varphi}^\varepsilon_{(z_1,p_0)})$ from the perturbed foliation $\mathcal{F}^\varepsilon$. Moreover, δ_ε is contained in $E(A') = \Pi(\hat{A}' \times S)$. As discussed in [13] and [14], the loop δ_ε is non trivial on $\varphi^\varepsilon_{q_\varepsilon}$ and defines a marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$.

Let us now look at an arbitrary representative δ'_ε of the marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$ and let us assume δ'_ε is contained in E_{δ_0}. By assumption, δ'_ε and δ_ε are representatives of the same marked cycle $(\Delta_\varepsilon, q_\varepsilon)$ for the foliation $\mathcal{F}^\varepsilon$. This implies that there exists a homotopy $\delta(t)$ on the leaf $\varphi^\varepsilon_{q_\varepsilon}$ between the two loops, keeping the base point q_ε fixed. Since the leaf is contained in E, the homotopy $\delta(t)$ takes place inside E. As pointed out earlier, δ_ε lifts to a loop $\hat{\delta}_\varepsilon$ contained in $\hat{A}' \times S$ and passing through (z_1, p_0). By the homotopy lifting property for covering spaces [6], $\delta(t)$ lifts to a homotopy $\hat{\delta}(t)$ inside $\mathbb{D} \times S$ so that $\Pi(\hat{\delta}(t)) = \delta(t)$. Since $\hat{\delta}(0) = \hat{\delta}_\varepsilon$ is a loop, then $\hat{\delta}(1)$ is also a loop that passes through (z_1, p_0) and $\Pi(\hat{\delta}(1)) = \delta(1) = \delta'_\varepsilon$. Let $\hat{\delta}'_\varepsilon = \hat{\delta}(1)$. Thus, $\hat{\delta}'_\varepsilon$ is homotopic inside $\mathbb{D} \times S$ to $\hat{\delta}_\varepsilon$ via $\hat{\delta}(t)$ relative to the base point (z_1, p_0).

It follows from the notations in Section 3.2 that $\Pi(\gamma(B_{\delta_0}) \times S) = E_{\delta_0}$ for any $\gamma \in \Gamma$. Since δ'_ε is contained in E_{δ_0}, the loop $\hat{\delta}'_\varepsilon$ is contained in $\gamma(\hat{B}_{\delta_0}) \times S$, where γ is chosen so that $z_1 \in \gamma(\hat{B}_{\delta_0})$. Notice that $\gamma(\hat{B}_{\delta_0}) = \hat{B}_{\delta_0}$ if and only if $\gamma \in \Gamma_0$. Consider the following deformation retractions

$$\begin{aligned} R'_\gamma &= \hat{\gamma} \circ R' \circ \hat{\gamma}^{-1} : \mathbb{D} \times S \longrightarrow \{\gamma(z_0)\} \times S \quad \text{and} \\ R_\gamma &= \hat{\gamma} \circ R \circ \hat{\gamma}^{-1} : \gamma(\hat{B}_{\delta_0}) \times S \longrightarrow \{\gamma(z_0)\} \times S, \end{aligned}$$

where R' and R are defined in Section 3.2. Then, $R'_\gamma(\hat{\delta}(t)) = \{\gamma(z_0)\} \times pr_S(\hat{\delta}(t))$ is a homotopy on $\{\gamma(z_0)\} \times S$ between the loops $\{\gamma(z_0)\} \times pr_S(\hat{\delta}_\varepsilon)$ and $\{\gamma(z_0)\} \times$

$pr_S(\hat{\delta}'_\varepsilon)$. By construction, $pr_S(\hat{\delta}_\varepsilon) = \delta_0^m$. Therefore,

$$\begin{aligned}
\Pi(\{\gamma(z_0)\} \times pr_S(\hat{\delta}_\varepsilon)) &= \Pi \circ \hat{\gamma}(\{z_0\} \times D_\gamma^{-1}(\delta_0^m)) \\
&= \Pi(\{z_0\} \times D_\gamma^{-1}(\delta_0^m)) \\
&= D_\gamma^{-1}(\delta_0^m) \text{ and} \\
\Pi(\{\gamma(z_0)\} \times pr_S(\hat{\delta}'_\varepsilon)) &= \Pi \circ \hat{\gamma}(\{z_0\} \times D_\gamma^{-1} \circ pr_S(\hat{\delta}'_\varepsilon)) \\
&= D_\gamma^{-1}(pr_S(\hat{\delta}'_\varepsilon)).
\end{aligned}$$

are homotopic on the fiber S. With the help of the fact that the loop $\hat{\delta}'_\varepsilon$ is contained in $\gamma(\hat{B}_{\delta_0}) \times S$, we deduce that $\{\gamma(z_0)\} \times pr_S(\hat{\delta}'_\varepsilon) = R'_\gamma(\hat{\delta}'_\varepsilon) = R_\gamma(\hat{\delta}'_\varepsilon)$. But R_γ is a deformation retraction of $\gamma(\hat{B}_{\delta_0}) \times S$ onto $\{\gamma(z_0)\} \times S$, so $\hat{\delta}'_\varepsilon$ is free homotopic to $\{\gamma(z_0)\} \times pr_S(\hat{\delta}'_\varepsilon)$ inside $\gamma(\hat{B}_{\delta_0}) \times S$. This fact immediately implies that $\delta'_\varepsilon = \Pi(\hat{\delta}'_\varepsilon)$ is free homotopic to $D_\gamma^{-1}(pr_S(\hat{\delta}'_\varepsilon)) = \Pi(\{\gamma(z_0)\} \times pr_S(\hat{\delta}'_\varepsilon))$ inside $E_{\delta_0} = \Pi(\gamma(\hat{B}_{\delta_0}) \times S)$. Therefore, δ'_ε is free homotopic to $D_\gamma^{-1}(\delta_0^m)$ inside E_{δ_0}. Since z_1 is from $\hat{X}'_{\delta_0}$, there are two options. Either $z_1 \in \hat{C}'_{\delta_0} \cap \hat{X}'_{\delta_0}$ or $z_1 \in \hat{X}'_{\delta_0} - \hat{C}'_{\delta_0}$. In the first case, $\hat{C}'_{\delta_0} \subset \hat{B}_{\delta_0}$ so $\gamma \in \Gamma_0$ and therefore $D_\gamma^{-1}(\delta_0^m) = \delta_0^m$ which means that δ'_ε is $\delta_0, m-$fold vertical. In the second case, due to the identity $\hat{C}'_{\delta_0} = \hat{B}_{\delta_0} \cap \hat{X}'_{\delta_0}$, the point z_1 does not belong to the domain $\hat{B}_{\delta_0}$, so $\gamma \in \Gamma - \Gamma_0$ and therefore $D_\gamma^{-1}(\delta_0^m)$ is not even free homotopic to δ_0^m on the fiber S which implies that δ'_ε is not $\delta_0, m-$fold vertical. $\square$

The lemma above leads to a corollary that settles part of Theorem 2.

Corollary 4.3. *Let $r > 0$ be the radius obtained in lemma 4.1. Let $P_{\delta_0,\varepsilon} : C'_{p_0} \to A'_{p_0}$ be the Poincaré map for $\mathcal{F}^\varepsilon$ as described in corollary 4.2, where $\varepsilon \in D_r(0)$. Then, the following statements are true:*

1. *If $P_{\delta_0,\varepsilon}$ has a periodic orbit of period m in C'_{p_0} then the foliation $\mathcal{F}^\varepsilon$ has a marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$ with a base point q_ε belonging to C'_{p_0}.*

2. *The marked complex cycle* $(\Delta_\varepsilon, q_\varepsilon)$ *has a representative* δ_ε *contained in* $E(A')$ *and passing through the points of the* $m-$*periodic orbit.*

3. *If* δ'_ε *is an arbitrary representative of the marked complex cycle* $(\Delta_\varepsilon, q_\varepsilon)$*, then* δ'_ε *is contained in* E_{δ_0} *and is* $\delta_0, m-$*fold vertical if and only if its image* $H(\delta_\varepsilon)$ *is contained in* B_{δ_0} *and is free homotopic to a point inside* B_{δ_0}*.*

Proof: Let us assume that the map $P_{\delta_0,\varepsilon} : C'_{p_0} \to A'_{p_0}$ has a periodic orbit of period $m > 0$ on C'_{p_0}. Denote this orbit by $q_1, ..., q_m$. Consider its lift $\hat{q}_1, ..., \hat{q}_{m+1}$ on $\hat{C}'_{\delta_0} \times \{p_0\}$ so that $\hat{P}_{\delta_0,\varepsilon}(\hat{q}_i) = \hat{q}_{i+1}$ for $i = 1, ..., m$. Then, there exists $\hat{\gamma} \in \hat{\Gamma}_0$ such that $\hat{P}_{\delta_0,\varepsilon}(\hat{q}_m) = \hat{q}_{m+1} = \hat{\gamma}(\hat{q}_1)$. The fact that all $m + 1$ points belong to the same orbit implies that they lie on the same leaf $\hat{\varphi}^{\varepsilon}_{\hat{q}_1}$ from the foliation $\hat{\mathcal{F}}^{\varepsilon}$. Analogously to the proof of lemma 4.4, let $\delta(\hat{q}_i, \hat{q}_{i+1})$ be the lift of δ_0 on the leaf $\hat{\varphi}^{\varepsilon}_{\hat{q}_1}$ so that $\delta(\hat{q}_i, \hat{q}_{i+1})$ covers δ_0 under the projection pr_S and connects the points $\hat{q}_i$ and $\hat{q}_{i+1}$ for $i = 1, ..., m$. Because of the way the map $\hat{P}_{\delta_0,\varepsilon}$ is defined, all arcs $\delta(\hat{q}_i, \hat{q}_{i+1})$ are contained in $\hat{A}' \times S$. Therefore, the curve $\hat{\delta}_\varepsilon = \cup_{i=1}^{m-1} \delta(\hat{q}_i, \hat{q}_{i+1})$ is contained in $\hat{A}' \times S$ and goes through all the points $\hat{q}_1, ..., \hat{q}_m$.

The image $\delta_\varepsilon = \Pi(\hat{\delta}^\varepsilon)$ inside E is a loop lying on the leaf $\varphi^{\varepsilon}_{q_1} = \Pi(\hat{\varphi}^{\varepsilon}_{\hat{q}_1})$ from the perturbed foliation $\mathcal{F}^{\varepsilon}$. Moreover, δ_ε is contained in $E(A') = \Pi(\hat{A}' \times S)$ and passes through the points of the periodic orbit $q_1, ..., q_m$. As pointed out in the proof of the previous lemma, the loop δ_ε is non trivial on $\varphi^{\varepsilon}_{q_1}$ and defines a marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$, where q_ε can be chosen to be any point from the $m-$periodic orbit of $P_{\delta_0,\varepsilon}$. Without loss of generality, we can think that $q_\varepsilon = q_1$. Thus, we have proved points 1 and 2 from the current statement.

Let us now look at an arbitrary representative δ'_ε of the marked complex cycle $(\Delta_\varepsilon, q_\varepsilon)$ and its projection $H(\delta'_\varepsilon)$ on B. Clearly, δ'_ε is contained in E_{δ_0} exactly when its image $H(\delta'_\varepsilon)$ is contained in B_{δ_0}. As we know $\Pi(\hat{q}_1) = q_1 = q_\varepsilon$, so the loop $\delta'_\varepsilon \ni q_1$ lifts as a path $\hat{\delta}'_\varepsilon$ starting from $\hat{q}_1$ on $\mathbb{D} \times S$ under the covering map Π. The projection $\tilde{\delta}'_\varepsilon = pr_{\mathbb{D}}(\hat{\delta}'_\varepsilon)$ on the disc $\mathbb{D}$ is the lift of $H(\delta'_\varepsilon)$ under the universal covering map π. This is true because of the identity $H \circ \Pi = \pi \circ pr_{\mathbb{D}}$.

Assume first that the loop $H(\delta'_\varepsilon)$ is contained in B_{δ_0} and is homotopic to a point inside B_{δ_0}. For that reason, the lift $\tilde{\delta}'_\varepsilon$ is a loop in $\hat{B}_{\delta_0}$ and therefore $\hat{\delta}'_\varepsilon$ is also a loop contained in $\hat{B}_{\delta_0} \times S$.

By assumption, δ'_ε and δ_ε are representatives of the same marked cycle $(\Delta_\varepsilon, q_\varepsilon)$. This implies that there exists a homotopy $\delta(t)$ on the leaf $\varphi^\varepsilon_{q_\varepsilon}$ between the two loops, keeping the base point q_ε fixed. Since the leaf is contained in E, the homotopy $\delta(t)$ takes place inside E. As pointed out earlier, δ'_ε lifts to a loop $\hat{\delta}'_\varepsilon$ contained in $\hat{B}_{\delta_0} \times S$ and passing through $\hat{q}_1$. The homotopy lifting property for covering spaces applies again [6], leading to a lifted homotopy $\hat{\delta}(t)$ inside $\mathbb{D} \times S$ such that $\Pi(\hat{\delta}(t)) = \delta(t)$. Since $\hat{\delta}(0) = \hat{\delta}'_\varepsilon$ is a loop, then $\hat{\delta}(1)$ is also a loop that passes through $\hat{q}_1$ and such that $\Pi(\hat{\delta}(1)) = \delta_\varepsilon$. Therefore, $\hat{\delta}(1) = \hat{\delta}_\varepsilon$. It follows from here that $\hat{q}_1 = \hat{q}_{m+1} = \hat{\gamma}(\hat{q}_1)$. But $\hat{\gamma}$ can have a fixed point inside $\mathbb{D} \times S$ only if $\hat{\gamma} = id_{(\mathbb{D}\times S)}$. Therefore, the lifted map $\hat{P}_{\delta_0,\varepsilon}$ has a periodic orbit of period m and $pr_{\mathbb{D}}(\hat{q}_1) \in \hat{C}'_{\delta_0}$. By point 2 from lemma 4.4, it follows that the representative δ'_ε is $\delta_0, m-$fold vertical.

It is easier to see that the converse is also true. If δ'_ε is free homotopic to δ_0^m inside E_{δ_0} then its projection $H(\delta_\varepsilon)$ is necessarily free homotopic to a point inside B_{δ_0}. If the homotopy between δ'_ε and δ_0^m is denoted by $\delta_\varepsilon(t)$, then it is enough to project with H and obtain the homotopy $H(\delta_\varepsilon(t))$ connecting the loop $H(\delta_\varepsilon)$ to the point $H(\delta_0) = u_0$. $\square$

Proof of Theorem 2. All pieces of the theorem are already proved. We only need to put them together. The existence of a global cross-section B_{p_0} transverse to the unperturbed foliation $\mathcal{F}^0$ follows from Corollary 3.1. Then we can see in the beginning of Section 4.1 that A'_{p_0} is transverse to the perturbed foliation $\mathcal{F}^\varepsilon$. By Corollary 4.2, we are able to construct the desired Poincaré map. Lemma 4.2 and Lemma 4.3 provide us with a complex structure on the cross-section with respect to which the map is holomorphic. Corollary 4.3 establishes the correspondence between periodic orbits and multi-fold cycles and explains the link between the dynamical features of the Poincaré transformation and the topological properties of the multi-fold cycles with respect to the fibred domain E_{δ_0}. □

CHAPTER 5

RAPID EVOLUTION OF MARKED COMPLEX CYCLES

Our next goal is to explore the behavior of multi-fold limit cycles of $\mathcal{F}^\varepsilon$ as the parameter ε approaches zero. We would like to show their escape from large sub-domains of the complex plane $\mathbb{C}^2$ as explained in Theorem 3. This phenomenon is what we call a rapid evolution of marked limit cycles and this will be the topic of the current discussion. Before we can give a proof of Theorem 3 we will need some auxiliary statements.

5.1 Continuous Families of Orbits and Cycles

The Quotient Surface. We begin with some useful constructions. Fix a positive integer $m > 0$ and for convenience, consider an embedded arc η in the parameter disc $D_r(0)$, where $r > 0$ is the radius chosen in Lemma 4.1. Define the surface

$$Y' = (\hat{A}' \times \{p_0\})/\hat{\Gamma}_0.$$

By construction, $\hat{X}'_{\delta_0} \times \{p_0\}$, $\hat{A} \times \{p_0\}$ and $\hat{X}_{\delta_0} \times \{p_0\}$ are $\hat{\Gamma}_0-$invariant sub-surfaces of $\hat{A}' \times \{p_0\}$, so the quotients

$$X'_{\delta_0} = (\hat{X}'_{\delta_0} \times \{p_0\})/\hat{\Gamma}_0\,, \quad Y = (\hat{A} \times \{p_0\})/\hat{\Gamma}_0 \text{ and } X_{\delta_0} = (\hat{X}_{\delta_0} \times \{p_0\})/\hat{\Gamma}_0$$

are sub-surfaces of Y' such that $X_{\delta_0} \subset Y \subset \hat{X}'_{\delta_0}$. Denote by

$$\pi^{(0)} \,:\, \hat{A}' \times \{p_0\} \longrightarrow Y'$$

the corresponding quotient map. Since $\hat{P}_{\delta_0,\varepsilon} \,:\, \hat{X}'_{\delta_0} \times \{p_0\} \to \hat{A}' \times \{p_0\}$ is $\hat{\Gamma}_0$ - equivariant, that is $\hat{\gamma} \circ \hat{P}_{\delta_0,\varepsilon} = \hat{P}_{\delta_0,\varepsilon} \circ \hat{\gamma}$ for any $\hat{\gamma} \in \hat{\Gamma}_0$, it descends to a diffeomorphism

$$\tilde{P}_{\delta_0,\varepsilon} \,:\, X'_{\delta_0} \longrightarrow Y'$$

so that $\pi^{(0)} \circ \hat{P}_{\delta_0,\varepsilon} = \tilde{P}_{\delta_0,\varepsilon} \circ \pi^{(0)}$. Because by construction

$$\hat{P}^k_{\delta_0,\varepsilon}(\hat{X}_{\delta_0} \times \{p_0\}) \subset \hat{A} \times \{p_0\}\,, \text{ for } \varepsilon \in D_r(0) \text{ and } k = 1, ..., m,$$

the descended map has the corresponding property

$$\tilde{P}^k_{\delta_0,\varepsilon}(X_{\delta_0}) \subset Y\,, \text{ for } \varepsilon \in D_r(0) \text{ and } k = 1, ..., m.$$

Complex Structure on the Lifted and the Quotient Surfaces. Denote the restriction of Π on the surface $\hat{A}' \times \{p_0\}$ by

$$\Pi_{p_0} = \Pi|_{(\hat{A}' \times \{p_0\})} \,:\, \hat{A}' \times \{p_0\} \longrightarrow A'_{p_0}.$$

Then, the map Π_{p_0} is a covering map.

Lemma 5.1. *Let $\mathcal{A}_\varepsilon(A'_{p_0}) = \{(U_{q_0}, \phi_{q_0,\varepsilon}) \;:\; q_0 \in A'_{p_0}\}$ be the complex atlas for A'_{p_0} as defined in Lemma 4.2. Then $\hat{A}' \times \{p_0\}$ has a complex atlas*

$$\mathcal{A}_\varepsilon(\hat{A}' \times \{p_0\}) = \{(\hat{U}_{\hat{q}_0}, \hat{\phi}_{\hat{q}_0,\varepsilon}) \;:\; \hat{q}_0 \in \hat{A}' \times \{p_0\}\},$$

such that the covering map Π_{p_0} is holomorphic. The new atlas makes the lifted Poincaré map $\hat{P}_{\delta_0,\varepsilon}$ holomorphic, depending analytically on ε. Analogously, the surface Y' has a complex structure given by the atlas

$$\mathcal{A}_\varepsilon(Y') = \{(\tilde{U}_{x_0}, \tilde{\phi}_{x_0,\varepsilon}) \;:\; x_0 \in Y'\},$$

such that the quotient map $\pi^{(0)}$ is holomorphic. This new atlas makes the map $\tilde{P}_{\delta_0,\varepsilon}$ holomorphic, depending analytically on ε.

Proof: The proof of this fact is straightforward. All we have to do is to pull back the complex structure given by $\mathcal{A}_\varepsilon(A'_{p_0})$ to the surface $\hat{A}' \times \{p_0\}$ in the first case, and to push forward the same structure on the surface Y' in the second case.

Pick an arbitrary point $\hat{q}_0 \in \hat{A}' \times \{p_0\}$ such that $q_0 = \Pi_{p_0}(\hat{q}_0) \in A'_{p_0}$. Let $(U_{q_0}, \phi_{q_0,\varepsilon})$ be a chart form $\mathcal{A}_\varepsilon(A_{p_0})$ around the point q_0. The neighborhood U_{q_0} can be chosen small enough so that Π_{p_0} is invertible, that is there exists and open neighborhood $\hat{U}_{\hat{q}_0}$ on $\hat{A}' \times \{p_0\}$ around the point $\hat{q}_0$ such that $\Pi_{p_0}|_{\hat{U}_{\hat{q}_0}} : \hat{U}_{\hat{q}_0} \to U_{q_0}$ is a diffeomorphism. Then, define the map

$$\hat{\phi}_{\hat{q}_0,\varepsilon} = \phi_{q_0,\varepsilon} \circ \Pi_{p_0}|_{\hat{U}_{\hat{q}_0}} \ : \ \hat{U}_{\hat{q}_0} \longrightarrow \mathbb{D} \ \text{ with an inverse}$$
$$\hat{\phi}^{-1}_{\hat{q}_0,\varepsilon} = (\Pi_{p_0}|_{\hat{U}_{\hat{q}_0}})^{-1} \circ \phi_{q_0,\varepsilon} \ : \ \mathbb{D} \longrightarrow \hat{U}_{\hat{q}_0}.$$

This definition allows us to obtain the collection of charts $\mathcal{A}_\varepsilon(\hat{A}' \times \{p_0\})$. Then for two charts $(\hat{U}_{\hat{q}_1}, \hat{\phi}_{\hat{q}_1,\varepsilon})$ and $(\hat{U}_{\hat{q}_2}, \hat{\phi}_{\hat{q}_2,\varepsilon})$ from that collection, if $\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2} \neq \varnothing$ then the transition transformation

$$\begin{aligned}
\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} \ &: \ \hat{\phi}_{\hat{q}_1,\varepsilon}(\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}) \longrightarrow \hat{\phi}_{\hat{q}_2,\varepsilon}(\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}) \ \text{ equals} \\
\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} &= (\phi_{q_2,\varepsilon} \circ \Pi_{p_0}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}}) \circ (\phi_{q_1,\varepsilon} \circ \Pi_{p_0}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}})^{-1} \\
&= \phi_{q_2,\varepsilon} \circ (\Pi_{p_0}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}}) \circ (\Pi_{p_0}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}})^{-1} \circ \phi^{-1}_{q_1,\varepsilon} \\
&= \phi_{q_2,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon},
\end{aligned}$$

which, as we know form Lemma 4.2, is holomorphic.

Now, assume $\hat{q}_1$ and $\hat{q}_2$ are two points from $\hat{A}' \times \{p_0\}$ such that $\hat{P}_{\delta_0,\varepsilon}(\hat{q}_1) = \hat{q}_2$. Choose two charts $(\hat{U}_{\hat{q}_1}, \hat{\phi}_{\hat{q}_1,\varepsilon})$ and $(\hat{U}_{\hat{q}_2}, \hat{\phi}_{\hat{q}_2,\varepsilon})$ from $\mathcal{A}_\varepsilon(\hat{A}' \times \{p_0\})$ so that the image

$\hat{P}_{\delta_0,\varepsilon}(\hat{U}_{\hat{q}_1})$ is a subset of $\hat{U}_{\hat{q}_2}$. Then for the composition

$$\begin{aligned}
&\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} \ : \mathbb{D} \longrightarrow \mathbb{D} \quad \text{we compute} \\
&\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} = (\phi_{q_2,\varepsilon} \circ \Pi_{p_0}|_{\hat{U}_{\hat{q}_2}}) \circ \hat{P}_{\delta_0,\varepsilon} \circ (\phi_{q_1,\varepsilon} \circ \Pi_{p_0}|_{\hat{U}_{\hat{q}_1}})^{-1} \\
&\qquad\qquad\qquad\qquad = \phi_{q_2,\varepsilon} \circ \big((\Pi_{p_0}|_{\hat{U}_{\hat{q}_2}}) \circ \hat{P}_{\delta_0,\varepsilon} \circ (\Pi_{p_0}|_{\hat{U}_{\hat{q}_1}})^{-1}\big) \circ \phi^{-1}_{q_1,\varepsilon} \\
&\qquad\qquad\qquad\qquad = \phi_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon}.
\end{aligned}$$

Since it was established in Lemma 4.3 that the map $\phi_{q_2,\varepsilon} \circ P_{\delta_0,\varepsilon} \circ \phi^{-1}_{q_1,\varepsilon}$ is holomorphic and depends analytically on ε, the composition $\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon}$ has the same complex analytic properties.

For the surface Y', we define the complex charts in an analogous manner. Let $\hat{q}_0 \in \hat{A}' \times \{p_0\}$ and choose a chart $(\hat{U}_{\hat{q}_0}, \hat{\phi}_{\hat{q}_0,\varepsilon})$ around $\hat{q}_0$ from the atlas $\mathcal{A}_\varepsilon(\hat{A}' \times \{p_0\})$ such that $\pi^{(0)}|_{\hat{U}_{\hat{q}_0}} : \hat{U}_{\hat{q}_0} \to \tilde{U}_{x_0}$ is a diffeomorphism and $\pi^{(0)}(\hat{q}_0) = x_0$. Define the map

$$\begin{aligned}
&\tilde{\phi}_{x_0,\varepsilon} = \hat{\phi}_{\hat{q}_0,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_0}})^{-1} \ : \ \tilde{U}_{x_0} \longrightarrow \mathbb{D} \quad \text{with an inverse} \\
&\tilde{\phi}^{-1}_{x_0,\varepsilon} = (\pi^{(0)}|_{\hat{U}_{\hat{q}_0}}) \circ \hat{\phi}^{-1}_{\hat{q}_0,\varepsilon} \ : \ \mathbb{D} \longrightarrow \tilde{U}_{x_0}.
\end{aligned}$$

We obtain the collection of charts $\mathcal{A}_\varepsilon(Y') = \{(\tilde{U}_{x_0}, \tilde{\phi}_{x_0,\varepsilon}) \ : \ x_0 \in Y'\}$. Then for two charts $(\tilde{U}_{x_1}, \tilde{\phi}_{x_1,\varepsilon})$ and $(\tilde{U}_{x_2}, \tilde{\phi}_{x_2,\varepsilon})$ from $\mathcal{A}_\varepsilon(Y')$ such that $\tilde{U}_{x_1} \cap \tilde{U}_{x_2} \neq \varnothing$, the transition map

$$\begin{aligned}
&\tilde{\phi}_{x_2,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1,\varepsilon} \ : \ \tilde{\phi}_{x_1,\varepsilon}(\tilde{U}_{x_1} \cap \tilde{U}_{x_2}) \longrightarrow \hat{\phi}_{\hat{q}_2,\varepsilon}(\tilde{U}_{x_1} \cap \tilde{U}_{x_2}) \quad \text{can be written as} \\
&\tilde{\phi}_{x_2,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1,\varepsilon} = \big(\hat{\phi}_{\hat{q}_2,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}})^{-1}\big) \circ \big(\hat{\phi}_{\hat{q}_1,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}})^{-1}\big)^{-1} \\
&\qquad\qquad = \hat{\phi}_{\hat{q}_2,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}})^{-1} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1} \cap \hat{U}_{\hat{q}_2}}) \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} \\
&\qquad\qquad = \hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon},
\end{aligned}$$

which is holomorphic, as it was shown above.

Take two points x_1 and x_2 from Y' such that $\tilde{P}_{\delta_0,\varepsilon}(x_1) = x_2$. Choose two charts $(\tilde{U}_{x_1}, \tilde{\phi}_{x_1,\varepsilon})$ and $(\tilde{U}_{x_2}, \tilde{\phi}_{x_2,\varepsilon})$ from $\mathcal{A}_\varepsilon(Y')$ so that $\tilde{P}_{\delta_0,\varepsilon}(\tilde{U}_{x_1}) \subset \tilde{U}_{x_2}$. Then for the composition

$$\tilde{\phi}_{x_2,\varepsilon} \circ \tilde{P}_{\delta_0,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1,\varepsilon} : \mathbb{D} \longrightarrow \mathbb{D} \quad \text{we have the equalities}$$

$$\begin{aligned}
\tilde{\phi}_{x_2,\varepsilon} \circ \tilde{P}_{\delta_0,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1,\varepsilon} &= \big(\hat{\phi}_{\hat{q}_2,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_2}})^{-1}\big) \circ \tilde{P}_{\delta_0,\varepsilon} \circ \big(\hat{\phi}_{\hat{q}_1,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1}})^{-1}\big)^{-1} \\
&= \hat{\phi}_{\hat{q}_2,\varepsilon} \circ \big((\pi^{(0)}|_{\hat{U}_{\hat{q}_2}})^{-1} \circ \tilde{P}_{\delta_0,\varepsilon} \circ (\pi^{(0)}|_{\hat{U}_{\hat{q}_1}})\big) \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon} \\
&= \hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon}.
\end{aligned}$$

We already know that the map $\hat{\phi}_{\hat{q}_2,\varepsilon} \circ \hat{P}_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{\hat{q}_1,\varepsilon}$ is holomorphic and depends analytically on ε. Therefore the composition $\tilde{\phi}_{x_2,\varepsilon} \circ \tilde{P}_{\delta_0,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1,\varepsilon}$ is also complex analytic. □

Continuous Families of Periodic Orbits and Marked Cycles. The holomorphic nature of the Poincaré map guarantees that every time the map has an isolated periodic orbit for some particular value of ε, there will be a continuous family of periodic orbits defined near that particular value of ε. In other words, an isolated periodic orbit gives rise to a local continuous family of periodic orbits due to the complex analytic properties of the Poincaré map. In addition, there will be a continuous family of marked complex cycles as well.

Lemma 5.2. *Let ε' belong to the parameter disc $D_r(0)$, where the radius $r > 0$ is chosen as in Lemma 4.1.*

1. Assume $\hat{P}_{\delta_0,\varepsilon'}$ has an isolated $m-$periodic orbit $(z_1, p_0), ..., (z_m, p_0)$ on the cross-section $\hat{X}'_{\delta_0} \times \{p_0\}$. Then $\pi^{(0)}$ maps that orbit to an isolated $m-$periodic orbit $x_1, ..., x_m$ for the map $\tilde{P}_{\delta_0,\varepsilon'}$ on the surface X'_{δ_0}.

2. There exists $r' > 0$ with $D_{r'}(\varepsilon') \subset D_r(0)$, such that for any embedded in

$D_{r'}(\varepsilon')$ curve η', passing through ε', there exists a continuous family of periodic orbits $\big((z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)\big)_{\varepsilon\in\eta'}$ for the map $\hat{P}_{\delta_0,\varepsilon}$ on $\hat{X}'_{\delta_0} \times \{p_0\}$, which for $\varepsilon = \varepsilon'$ becomes $(z_1, p_0), ..., (z_m, p_0)$. Moreover, the continuous family of $\hat{P}_{\delta_0,\varepsilon}$ is mapped by $\pi^{(0)}$ to a continuous family of periodic orbits $(x_1(\varepsilon), ..., x_m(\varepsilon))_{\varepsilon\in\eta'}$ for the transformation $\tilde{P}_{\delta_0,\varepsilon}$ on the surface X'_{δ_0}, which for $\varepsilon = \varepsilon'$ becomes the orbit $x_1, ..., x_m$.

3. If $\hat{P}_{\delta_0,\varepsilon}$ has a continuous family of periodic orbits on $\hat{X}'_{\delta_0} \times \{p_0\}$ for ε varying on some curve $\tilde{\eta}$ embedded in $D_r(0)$, then the perturbed foliation $\mathcal{F}^\varepsilon$ has a continuous family of marked cycles $\{(\Delta_\varepsilon, q_\varepsilon)\}_{\varepsilon\in\tilde{\eta}}$.

Proof: By assumption, $(z_1, p_0), ..., (z_m, p_0)$ is an isolated $m-$periodic orbit of $\hat{P}_{\delta_0,\varepsilon'}$ on $\hat{X}'_{\delta_0} \times \{p_0\}$. The image of this orbit under the covering map $\pi^{(0)}$ is denoted by $x_1, ..., x_m$. Because of the property $\pi^{(0)} \circ \hat{P}_{\delta_0,\varepsilon'} = \tilde{P}_{\delta_0,\varepsilon'} \circ \pi^{(0)}$, the orbit $x_1, ..., x_m$ is also isolated and periodic with possibly a smaller or equal period. Clearly, $\tilde{P}^m_{\delta_0,\varepsilon'}(x_1) = \tilde{P}^m_{\delta_0,\varepsilon'}(\pi^{(0)}(z_1, p_0)) = \pi^{(0)} \circ \hat{P}^m_{\delta_0,\varepsilon'}(z_1, p_0) = \pi^{(0)}(z_1, p_0) = x_1$.

Assume there exists a smaller $k = 1, ..., m-1$ such that $x_1 = x_{k+1}$. Then, there exists $\hat{\gamma} \in \hat{\Gamma}_0$ such that $(z_{k+1}, p_0) = \hat{\gamma}(z_1, p_0) = (\gamma(z_1), p_0)$ for the corresponding $\gamma \in \Gamma_0$. On the other hand, $(z_{k+1}, p_0) = \hat{P}^k_{\delta_0,\varepsilon'}(z_1, p_0)$. Thus, $\hat{P}^k_{\delta_0,\varepsilon'}(z_1, p_0) = \hat{\gamma}(z_1, p_0)$. Applying $\hat{P}^k_{\delta_0,\varepsilon'}$ to the last equality we obtain

$$\begin{aligned}\hat{P}^{2k}_{\delta_0,\varepsilon'}(z_1, p_0) &= \hat{P}^k_{\delta_0,\varepsilon'} \circ \hat{\gamma}(z_1, p_0)\\ &= \hat{\gamma} \circ \hat{P}^k_{\delta_0,\varepsilon'}(z_1, p_0)\\ &= \hat{\gamma}^2(z_1, p_0).\end{aligned}$$

In general, $\hat{P}^{jk}_{\delta_0,\varepsilon'}(z_1, p_0) = \hat{\gamma}^j(z_1, p_0)$ for any $j \in \mathbb{N}$. In particular, when $j = m$ we have $(z_1, p_0) = \hat{P}^{mk}_{\delta_0,\varepsilon'}(z_1, p_0) = \hat{\gamma}^m(z_1, p_0) = (\gamma^m(z_1), p_0)$. As it turns out, $z_1 = \gamma^m(z_1)$ which means that γ^m has a fixed point in the interior of the hyperbolic disc $\mathbb{D}$. As a subgroup of a Fuchsian group associated to a Riemann surface, Γ_0 can have no elliptic elements but only parabolic and hyperbolic [8],[12]. Therefore, $\gamma^m = id_{\mathbb{D}}$ and more precisely, $\gamma = id_{\mathbb{D}}$. Thus, as it turns out, $(z_{k+1}, p_0) = (z_1, p_0)$ which is not the case.

As $\hat{P}^m_{\delta_0,\varepsilon'}(z_1, p_0) = (z_1, p_0)$, we choose a chart $(\hat{U}_{z_1}, \hat{\phi}_{z_1,\varepsilon})$ form the atlas $\mathcal{A}_\varepsilon(\hat{A}' \times \{p_0\})$ around the point (z_1, p_0) and a smaller neighborhood $\hat{U}'_{z_1}$ of the same point such that $\hat{U}'_{z_1} \subset \hat{U}_{z_1}$ and $\hat{P}^m_{\delta_0,\varepsilon'}(\hat{U}'_{z_1}) \subset \hat{U}_{z_1}$. Let $D' = \hat{\phi}_{z_1,\varepsilon'}(\hat{U}'_{z_1}) \subset \mathbb{D}$ where $\hat{\phi}_{z_1,\varepsilon'}(z_1, p_0) = 0 \in D'$. If $r' > 0$ is chosen small enough, then

$$P^{(m)}_\varepsilon = \hat{\phi}_{z_1,\varepsilon} \circ \hat{P}^m_{\delta_0,\varepsilon} \circ \hat{\phi}^{-1}_{z_1,\varepsilon} \, : \, D' \longrightarrow \mathbb{D}$$

for $\varepsilon \in D_{r'}(\varepsilon') \subset D_r(0)$. Notice that $P^{(m)}_{\varepsilon^{**}}(0) = 0$. The complex valued function

$$\tilde{F} : D' \to \mathbb{C} \;\text{ defined as }\; \tilde{F}(\zeta, \varepsilon) = P^{(m)}_\varepsilon(\zeta) - \zeta$$

is holomorphic with respect to $\zeta \in D'$ and with respect to $\varepsilon \in D_{r'}(\varepsilon')$. By Hartogs' Theorem [5], it is holomorphic with respect to $(\zeta, \varepsilon) \in D' \times D^{r'}(\varepsilon')$. Since $P^{(m)}_{\varepsilon'}(0) = 0$, the point $(0, \varepsilon')$ is a zero of $\tilde{F}$, that is $\tilde{F}(0, \varepsilon') = 0$.

Let us look at the zero locus of $\tilde{F}$ in $D' \times D^{r'}(\varepsilon')$. The fact that the periodic orbit is isolated means that (z_1, p_0) is an isolated fixed point for the map $\hat{P}^m_{\delta_0,\varepsilon'}$. Therefore 0 is an isolated fixed point for $P^{(m)}_{\varepsilon'}$ and thus, it is an isolated zero for the holomorphic function $\tilde{F}(\zeta, \varepsilon')$. By Weierstrass Preparation Theorem [5],[3], we can write

$$\tilde{F}(\zeta, \varepsilon) = \prod_{j=1}^{s} (\zeta - \alpha_j(\varepsilon))\theta(\zeta, \varepsilon),$$

where $\theta(0, \varepsilon') \neq 0$ and $\{\alpha_j(\varepsilon) \, : \, j = 1, ..., s\}$ depend analytically on ε, satisfying the equalities $\alpha_1(\varepsilon') = ... = \alpha_s(\varepsilon') = 0$ and possibly branching into each other.

Now, let η' be some curve embedded in the disc $D_{r'}(\varepsilon')$ and passing through ε'. For ε varying on η', we can choose a branch, denoted for simplicity by

$\alpha_1(\varepsilon)$. Then the desired continuous family for $\hat{P}_{\delta_0,\varepsilon}$ can be constructed by setting $(z_1(\varepsilon), p_0) = \hat{\phi}^{-1}_{z_1,\varepsilon}(\alpha_j(\varepsilon))$ and $(z_{j+1}, p_0) = \hat{P}^j_{\delta_0,\varepsilon}(z_1(\varepsilon), p_0)$ for $j = 1, ..., m-1$. Its image under the covering $\pi^{(0)}$ will provide the continuous family of periodic orbits for $\tilde{P}_{\delta_0,\varepsilon}$.

The third point of the statement follows directly form Lemma 4.4 with the remark that the representative δ_ε is constructed to depend continuously on the parameter ε. □

5.2 Proof of Theorem 3.

By assumption, the Poincaré map $P_{\delta_0,\varepsilon_0}$ has an isolated periodic orbit $(q_1, ..., q_m)$ on the cross-section C'_{p_0} and the perturbed foliation $\mathcal{F}^{\varepsilon_0}$ has a marked limit cycle (Δ, q_1) with a $\delta_0, m-$fold vertical representative δ' contained inside the domain $E(C_{\delta_0})$. Since the loop δ' passes through the point q_1, the latter in fact belongs to the surface $C_{p_0} \subset E(C_{\delta_0})$. Because $\Pi(\hat{X}_{\delta_0} \times S) = \overline{E(C_{\delta_0})}$, there exists a point $(z_1, p_0) \in \hat{X}_{\delta_0} \times S$ such that $\Pi(z_1, p_0) = q_1$.

As already discussed in the proof of Corollary 4.3, the fact that $H(\delta') \subset C_{\delta_0}$ is null-homotopic implies that δ' lifts to a loop $\hat{\delta}'$ on $\hat{X}_{\delta_0} \times S$ that passes through the point (z_1, p_0) and its image $\Pi(\hat{\delta}') = \delta'$. Let $(z_{j+1}, p_0) = \hat{P}^j_{\delta_0,\varepsilon_0}(z_1, p_0)$ for $j = 1, ..., m-1$. The orbit $(z_1, p_0), ..., (z_m, p_0)$ belongs to $\hat{A} \times \{p_0\}$. The loop δ' can be regarded as a path from the point q_1 to itself so its lift $\hat{\delta}$, being also a loop, is a path from (z_1, p_0) to itself. For that reason, we can conclude $\hat{P}^m_{\delta_0,\varepsilon_0}(z_1, p_0) = (z_1, p_0)$ which means that $(z_1, p_0), ..., (z_m, p_0)$ is an $m-$periodic orbit on $\hat{A} \times \{p_0\}$. Together with that, the orbit is isolated because the original orbit $q_1, ..., q_m$ is isolated.

Let η be an embedded in $D_r(0)$ curve, connecting ε_0 to 0. For convenience, define a natural linear order $\preceq$ on it so that $0 \prec \varepsilon_0$. By point 2 from Lemma 5.2, there exists $D_{r_0}(\varepsilon_0) \subset D_r(0)$ for some $r_0 > 0$, such that if $\eta_0 = \eta \cap D_{r_0}(\varepsilon_0)$, then there is a continuous family of periodic orbits $\big((z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)\big)_{\varepsilon\in\eta_0}$ of the map $\hat{P}_{\delta_0,\varepsilon}$ on the cross-section $\hat{X}'_{\delta_0} \times \{p_0\}$.

Define $\eta_{max} \subseteq \eta$ as the maximal relatively open subset of η on which the continuous family $\big((z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)\big)_{\varepsilon\in\eta_{max}}$ of periodic orbits for $\hat{P}_{\delta_0,\varepsilon}$ exists on $\hat{X}'_{\delta_0} \times \{p_0\}$. Since $\eta_0 \neq \varnothing$ is a relatively open in η, the inclusion $\eta_0 \subseteq \eta_{max}$ holds and therefore $\eta_{max} \neq \varnothing$.

By point 3 from Lemma 5.2 there is a continuous family of marked complex cycles $\{(\Delta_\varepsilon, q_\varepsilon)\}_{\varepsilon\in\eta_{max}}$ with $q_\varepsilon = \Pi(z_1(\varepsilon), p_0)$. Near $\varepsilon_0 \in \eta_{max}$ the cycles $(\Delta_\varepsilon, q_\varepsilon)$ have $\delta_0, m-$fold vertical representatives δ'_ε contained in $E(C_{\delta_0})$ because for $\varepsilon = \varepsilon_0$ the cycle $(\Delta_{\varepsilon_0}, q_{\varepsilon_0})$ has a $\delta_0, m-$fold vertical representative, namely $\delta' = \delta'_{\varepsilon_0}$, contained inside the domain $E(C_{\delta_0})$. We are interested to find out what happens to the cycles as ε varies on η_{max}.

Let η' be the set of all ε from η_{max} for which the periodic orbits from the family $\big((z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)\big)_{\varepsilon\in\eta_{max}}$ are contained in $\hat{A} \times \{p_0\}$. As we already saw, at ε_0 the orbit $(z_1(\varepsilon_0), p_0), ..., (z_m(\varepsilon_0), p_0)$ is inside $\hat{A}\times\{p_0\}$ and by continuity, for ε near ε_0 the orbits $(z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)$ are also contained in $\hat{A} \times \{p_0\}$. This fact shows that $\eta' \neq \varnothing$ and in fact it has a nonempty interior.

Let $\varepsilon^{**} = \inf_\eta(\eta_{max})$ be the infimum of η_{max} with respect to the linear ordering on η. Then, $D_{\frac{1}{N}}(\varepsilon^{**}) \cap \eta_{max} \neq \varnothing$ for all $N \in \mathbb{N}$. Similarly, define $\varepsilon^* = \inf_\eta(\eta')$ as the infimum of η'. The inclusion $\eta' \subseteq \eta_{max}$ implies that $\varepsilon^{**} \preceq \varepsilon^*$. We are going to show that $\varepsilon^{**} \neq \varepsilon^*$.

Assume $\varepsilon^{**} = \varepsilon^*$, that is for all $N \in \mathbb{N}$ there exists $\varepsilon_N \in D_{\frac{1}{N}}(\varepsilon^{**}) \cap \eta_{max}$ such that $(z_1(\varepsilon_N), p_0), ..., (z_m(\varepsilon_N), p_0)$ is contained in $\hat{A} \times \{p_0\}$. As explained in point 2 of Lemma 5.2 the periodic family $\big((z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)\big)_{\varepsilon \in \eta_{max}}$ is mapped by $\pi^{(0)}$ to a periodic family $(x_1(\varepsilon), ..., x_m(\varepsilon))_{\varepsilon \in \eta_{max}}$ of the map $\tilde{P}_{\delta_0,\varepsilon}$ on the surface X'_{δ_0}. Also, the corresponding orbits $x_1(\varepsilon_N), ..., x_m(\varepsilon_N)$ are inside $Y \subset X'_{\delta_0}$ for $N \in \mathbb{N}$. In particular, the sequence $\{x_1(\varepsilon_N)\}_{N \in \mathbb{N}}$ is contained in the compact set Y. Then, there exists $x^* \in Y$ and a subsequence $\{x_1(\varepsilon_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} x_1(\varepsilon_n) = x_1^*$ and $\lim_{n \to \infty} \varepsilon_n = \varepsilon^{**}$. By continuity, the identity $\tilde{P}^m_{\delta_0,\varepsilon_n}(x_1(\varepsilon_n)) = x_1(\varepsilon_n)$ converges to $\tilde{P}^m_{\delta_0,\varepsilon^{**}}(x_1^*) = x_1^*$ as $n \to \infty$. Generate a periodic orbit $x_1^*, ..., x_m^*$ by setting $x_{j+1}^* = \tilde{P}^j_{\delta_0,\varepsilon^{**}}(x_1^*)$ for $j = 1, ..., m-1$. Since $x_{j+1}(\varepsilon_n) = \tilde{P}^j_{\delta_0,\varepsilon^{**}}(x_1(\varepsilon_n))$, the limit for each $x_j(\varepsilon_n)$ is x_j^* as $n \to \infty$. Thus, the periodic orbit $x_1^*, ..., x_m^*$ is the limit of periodic orbits $x_1(\varepsilon_n), ..., x_m(\varepsilon_n)$.

We will show that under the current assumptions $\varepsilon^{**} = 0$. Assume that $\varepsilon^{**} \neq 0$. Then $\{\varepsilon \in \eta : \varepsilon \prec \varepsilon^{**}\} \neq \varnothing$. We proceed in a very similar fashion to that in the proof of Lemma 5.2. The point $x_1^* \in Y$ is fixed by the map $\tilde{P}^m_{\delta_0,\varepsilon^{**}}$. Take a complex chart $(\tilde{U}_{x_1^*}, \tilde{\phi}_{x_1^*,\varepsilon^{**}})$ form the atlas $\mathcal{A}_\varepsilon(Y')$ around the point x_1^* and a smaller neighborhood $\tilde{U}'_{x_1^*} \subset \tilde{U}_{x_1^*}$ of the same point such that $\tilde{P}^m_{\delta_0,\varepsilon^{**}}(\tilde{U}'_{x_1^*}) \subset \tilde{U}_{x_1^*}$. Let $D' = \tilde{\phi}_{x_1^*,\varepsilon^{**}}(\tilde{U}'_{x_1^*}) \subset \mathbb{D}$ where $\tilde{\phi}_{x_1^*,\varepsilon^{**}}(x_1^*) = 0 \in D'$. Choose $r^* > 0$ small enough such that

$$P_\varepsilon^{(m)} = \tilde{\phi}_{x_1^*,\varepsilon} \circ \tilde{P}^m_{\delta_0,\varepsilon} \circ \tilde{\phi}^{-1}_{x_1^*,\varepsilon} \,:\, D' \longrightarrow \mathbb{D}$$

for $\varepsilon \in D_{r^*}(\varepsilon^{**}) \subset D_r(0)$. Notice that $P^{(m)}_{\varepsilon^{**}}(0) = 0$. The complex valued function

$$\tilde{F} : D' \to \mathbb{C} \;\text{ defined as }\; \tilde{F}(\zeta, \varepsilon) = P_\varepsilon^{(m)}(\zeta) - \zeta$$

is holomorphic with respect to $(\zeta, \varepsilon) \in D' \times D^{r^*}(\varepsilon^{**})$. Since $P^{(m)}_{\varepsilon^{**}}(0) = 0$, the point $(0, \varepsilon^{**})$ is a zero of $\tilde{F}$, that is $\tilde{F}(0, \varepsilon^{**}) = 0$.

We are interested in the zero locus of $\tilde{F}$ in $D' \times D^{r^*}(\varepsilon^{**})$. If we assume for a moment that $\tilde{F}(\zeta, \varepsilon) \equiv 0$ on D' then we would have the identity $P_\varepsilon^{(m)}(\zeta) \equiv \zeta$ on D' and therefore $\tilde{P}^m_{\delta_0,\varepsilon}(x) \equiv x$ on the open subset $\tilde{U}'_{x_1^*} \subset X'_{\delta_0}$. Because of the analyticity of $\tilde{P}^m_{\delta_0,\varepsilon}(x)$ with respect to both x and ε, the identity $\tilde{P}^m_{\delta_0,\varepsilon}(x) \equiv x$ will hold on all of X'_{δ_0} and for all $\varepsilon \in D_r(0)$. In particular, it will be true for $\varepsilon = \varepsilon_0$. But for that value the map $\tilde{P}^m_{\delta_0,\varepsilon}$ has an isolated fixed point $x_1(\varepsilon_0) \in Y \subset X_{\delta_0}$ which leads to a contradiction. Therefore $\tilde{F}$ is not identically zero.

There are two cases for $\tilde{F}$. Either $\tilde{F}(\zeta, \varepsilon^{**}) \equiv 0$ or $\tilde{F}(\zeta, \varepsilon^{**}) \not\equiv 0$ for $\zeta \in D'$. For both of those options $\tilde{F}$ can be written as

$$\tilde{F}(\zeta, \varepsilon) = (\varepsilon - \varepsilon^{**})^b F(\zeta, \varepsilon)$$

where $F(\zeta, \varepsilon^{**}) \not\equiv 0$ and $b \geq 0$. When $b > 0$ we have the first case and when $b = 0$ we have the second case.

Let us look at the zero locus of F. By Weierstrass Preparation Theorem [3], [5], F can be written as

$$F(\zeta, \varepsilon) = \prod_{j=1}^{s} (\zeta - \alpha_j(\varepsilon))\theta(\zeta, \varepsilon),$$

where $\theta(0, \varepsilon^{**}) \neq 0$ and $\{\alpha_j(\varepsilon) \, : \, j = 1, ..., s\}$ depend analytically on ε, satisfying the equalities $\alpha_1(\varepsilon') = ... = \alpha_s(\varepsilon') = 0$ and possibly branching into each other. Without loss of generality, we can think that D' is chosen small enough so that $\nu(\zeta, \varepsilon) \neq 0$ for all $(\zeta, \varepsilon) \in D' \times D_{r^*}(\varepsilon^{**})$. Let $\tilde{\alpha}_j(\varepsilon) = \tilde{\phi}^{-1}_{x_1^*,\varepsilon}(\alpha_j(\varepsilon))$. Since $x_1(\varepsilon_n) \to x_1^*$, there exists $N_0 \in \mathbb{N}$ such that $x_1(\varepsilon_n) \in \tilde{U}'_{x_1^*}$ for $n > N_0$. By the continuity of $x_1(\varepsilon)$, for each $\varepsilon \in D_{r^*}(\varepsilon^{**}) \cap \eta_{max}$ we have that $x_1(\varepsilon) = \tilde{\alpha}_j(\varepsilon)$ for some $j = 1, .., m$. Thus, $x_1(\varepsilon)$ converges to x_1^* as $\varepsilon \to \varepsilon^{**}$ always staying on the zero locus of F. Thus we can extend $x_1(\varepsilon)$ continuously on η past ε^{**} by setting $x_1(\varepsilon) = \tilde{\alpha}_j(\varepsilon)$ for $\varepsilon \in D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta \, : \, \eta \preceq \varepsilon^{**}\}$. By construction, the identity $\tilde{P}^m_{\delta_0,\varepsilon}(\tilde{\alpha}_1(\varepsilon)) = \tilde{\alpha}_1(\varepsilon)$

holds and if we set $x_{j+1}(\varepsilon) = \tilde{P}^{j}_{\delta_0,\varepsilon}(\tilde{\alpha}_1(\varepsilon))$ we obtain a continuation of the family $x_1(\varepsilon), ..., x_m(\varepsilon)$ on the relatively open arc $D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta \,:\, \eta \preceq \varepsilon^{**}\}$. As a result we have a continuous family $\big(x_1(\varepsilon), ..., x_m(\varepsilon)\big)_{\varepsilon \in \tilde{\eta}}$ of periodic orbits for $\tilde{P}_{\delta_0,\varepsilon}$ where $\tilde{\eta} = (D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta \,:\, \eta \preceq \varepsilon^{**}\}) \cup \eta_{max}$ is relatively open in η.

Since the family $(z_1(\varepsilon), p_0), ..., (z_m(\varepsilon), p_0)$ is the lift of $x_1(\varepsilon), ..., x_m(\varepsilon)$ for $\varepsilon \in \eta_{max}$ and the latter extends on $\tilde{\eta} \supset \eta_{max}$, the former also extends on $\tilde{\eta}$ as a family of periodic orbits for $\hat{P}_{\delta_0,\varepsilon}$ on the cross-section $\hat{X}'_{\delta_0} \times \{p_0\}$. This conclusion contradicts the maximality of η_{max}, stemming from the assumption that $\varepsilon^{**} \neq 0$. Therefore $\varepsilon^{**} = 0$ and $x_1(0), ..., x_m(0)$ is a periodic orbit of $\tilde{P}_{\delta_0,0} = id_{X'_{\delta_0}}$. For that reason, $x_1(0) = ... = x_m(0) = x^*$ inside X'_{δ_0}.

Take a complex chart $(\tilde{U}_{x^*}, \tilde{\phi}_{x^*,0})$ around the point x^* and choose a smaller neighborhood $\tilde{U}'_{x^*} \subset \tilde{U}_{x^*}$ of x^* such that $\tilde{P}^{k}_{\delta_0,\varepsilon}(\tilde{U}'_{x^*}) \subset \tilde{U}_{x^*}$ for all $k = 1, ..., m$ and $\varepsilon \in D_{r_0}(0)$, where $r_0 >$ is small enough. Let $D' = \tilde{\phi}_{x^*,0}(\tilde{U}'_{x^*}) \subset \mathbb{D}$ and

$$P_\varepsilon = \tilde{\phi}_{x^*,\varepsilon} \circ \tilde{P}_{\delta_0,\varepsilon} \circ \tilde{\phi}^{-1}_{x^*,\varepsilon} \,:\, D' \longrightarrow \mathbb{D}.$$

Denote by $\zeta_j(\varepsilon) = \tilde{\phi}_{x^*,\varepsilon}(x_j(\varepsilon))$ for $\varepsilon \in D_{r_0}(0) \cap \eta_{max} = \eta_0$ and $j = 1, ..., m$. Then $\zeta_1(\varepsilon), ..., \zeta_m(\varepsilon)$ is a periodic orbit for P_ε in D'. Notice,that due to the holomorphic nature of the map $\tilde{P}_{\delta_0,\varepsilon}$, those $\varepsilon \in \eta_{max}$ for which $x_i(\varepsilon) = x_j(\varepsilon)$, where $1 \leq i < j \leq m$, are isolated because the family at ε_0 consists of an $m-$periodic point. As before $P_\varepsilon(\zeta)$ is holomorphic with respect to (ζ, ε). Then we can write the map as

$$P_\varepsilon(\zeta) = \zeta + \varepsilon^l I(\zeta) + \varepsilon^{l+1} R(\zeta, \varepsilon)$$

where $I(\zeta) \not\equiv 0$ and $l \geq 1$. If we iterate the map m times we obtain the representation

$$P^m_\varepsilon(\zeta) = \zeta + \varepsilon^l m I(\zeta) + \varepsilon^{l+1} R_{(m)}(\zeta, \varepsilon).$$

For $\varepsilon \in \eta_0 - \{0\}$ the equations

$$P_\varepsilon(\zeta) - \zeta = \varepsilon^l(I(\zeta) + \varepsilon R(\zeta, \varepsilon)) = 0 \quad \text{and}$$

$$P_\varepsilon^m(\zeta) - \zeta = \varepsilon^l(mI(\zeta) + \varepsilon R_{(m)}(\zeta, \varepsilon)) = 0$$

are divisible by ε^l and thus, become

$$I(\zeta) + \varepsilon R(\zeta, \varepsilon) = 0 \quad \text{and} \quad mI(\zeta) + \varepsilon R_{(m)}(\zeta, \varepsilon) = 0 \tag{5.1}$$

The function $I(\zeta)$ is not identically zero, so it has isolated zeroes. Choose $D'' \subset D'$ to be a small closed disc centered at zero, so that no zeroes of $I(\zeta)$ are contained in $D'' - \{0\}$. In particular, $I(\zeta) \neq 0$ for $\zeta \in \partial D''$. We can decrease the parameter radius $r_0 > 0$ enough so that by Rouche's Theorem [4] the equations (5.1) will have the same number of zeroes, counting multiplicities, as the equation $I(\zeta) = 0$. Clearly, all zeroes of $P_\varepsilon(\zeta) - \zeta$ are zeroes of $P_\varepsilon^m(\zeta) - \zeta$ because the fixed points of P_ε are fixed points of P_ε^m but not the other way around. On the other hand, as already noted, for almost every $\varepsilon \in D_{r_0}(0)$ there is an $m-$periodic orbit $\zeta_1(\varepsilon), ..., \zeta_m(\varepsilon)$ for the map P_ε inside D''. Thus, we can see that $P_\varepsilon^m(\zeta) - \zeta$ has at least m zeroes more than $P_\varepsilon^m(\zeta) - \zeta$, which contradicts the fact that both of these should have the same number of zeroes. The contradiction comes from the assumption that $\varepsilon^{**} = \varepsilon^*$. Therefore we can conclude that $\varepsilon^{**} \neq \varepsilon^*$ and in fact $\varepsilon^{**} \prec \varepsilon^*$.

Let $\eta_1 = \{\varepsilon \in \eta_{max} : \varepsilon^{**} \prec \varepsilon \prec \varepsilon^*\}$. Then for any $\varepsilon_1 \in \eta_1$ at least one $(z_{j_0}(\varepsilon_1), p_0)$ is contained in $\hat{X}'_{\delta_0} \times \{p_0\}$ but not in $\hat{A} \times \{p_0\}$. It follows form here that $(z_1(\varepsilon_1), p_0)$ is not contained in $\hat{X}_{\delta_0} \times \{p_0\}$, otherwise if $(z_1(\varepsilon_1), p_0)$ were in $\hat{X}_{\delta_0} \times \{p_0\}$, then $(z_{j_0}(\varepsilon_1), p_0) = P_{\delta_0,\varepsilon_1}^{j_0-1}(z_1(\varepsilon_1), p_0)$ would be inside $\hat{A} \times \{p_0\}$, which is not the case.

By point 3 of Lemma 5.2 there exists a continuous family $\{(\Delta_\varepsilon, q_\varepsilon)\}_{\varepsilon \in \eta_{max}}$ of marked cycles, where $q_\varepsilon = \Pi(z_1(\varepsilon), p_0)$. For any $\varepsilon_1 \in \eta_1 \subset \eta_{max}$ there are two options. The first one is that $q_{\varepsilon_1} \in C'_{p_0} - C_{p_0}$. Then, no representative of $(\Delta_{\varepsilon_1}, q_{\varepsilon_1})$ is contained in $E(C_{\delta_0})$ because all of them pass through q_{ε_1} and q_{ε_1} is not in $E(C_{\delta_0})$. The second option is that $(z_1(\varepsilon_1), p_0)$ belongs to $\Pi^{-1}(C_{p_0}) = \cup_{\gamma \in \Gamma} \left(\gamma(\hat{C}_{\delta_0}) \times \{p_0\}\right)$ but does not belong to $\hat{C}_{\delta_0} \times \{p_0\}$. In this case, there exists $\gamma \in \Gamma - \Gamma_0$ such that $(z_1(\varepsilon_1), p_0) \in \gamma(\hat{C}_{\delta_0}) \times \{p_0\}$. By point 2 of Lemma 4.4, any representative δ'_{ε_1} of the marked complex cycle $(\Delta_{\varepsilon_1}, q_{\varepsilon_1})$, that is contained in E_{δ_0}, is not $\delta_0, m-$fold vertical. Thus, Theorem 3 is true with $\sigma = \eta_{max}$. $\square$

CHAPTER 6

FOLIATIONS WITH MULTI-FOLD LIMIT CYCLES

In this chapter we discuss an example, such that for any $m \in \mathbb{N}$, a family of polynomial foliations of the form 1.1 has a limit m-fold vertical cycle. More specifically we are going to look at the two-parameter family 2.1 already introduced in Section 2.

6.1 The Foliation and Its Poincaré Map

Introduction of the Foliation. As defined earlier, the foliation $\mathcal{F}^{a,\varepsilon}$ is given by the complex line field

$$F^{a,\varepsilon} = \ker\Big(dH + \varepsilon\big((H-1)(ydx - xdy) + ay\,dH\big)\Big), \tag{6.1}$$

with a transverse to infinity integrable part $H = x^2 + y^2$ and parameters ε and a. The leaf

$$S_1 = \{(x, y) \in \mathbb{C}^2 \,|\, x^2 + y^2 = 1\}$$

tangent to $\ker(dH)$ is diffeomoprhic to a cylinder with a nontrivial loop on it $\delta_0 = S_1 \cap \mathbb{R}^2$. It is very important to point out that, in fact, S_1 is tangent to the line field $F^{a,\varepsilon}$ and therefore is a leaf of the foliation $\mathcal{F}^{a,\varepsilon}$ for all $(a, \varepsilon) \in \mathbb{C}^* \times \mathbb{C}^*$.

Define $A(\delta_0)$ as a tubular neighborhood of δ_0 on the surface S_1 and $N(\delta_0)$ as a tubular neighborhood of $A(\delta_0)$ in $\mathbb{C}^2$. Let

$$\mathbb{B}_{r_0} = \{\zeta \in \mathbb{C} \,:\, |\text{Im}(\zeta)| < r_0\}$$

be a an infinite horizontal band in $\mathbb{C}$ of width r_0 and let

$$D_{r_0}(1) = \{\xi \in \mathbb{C} \,:\, |\xi - 1| \leq r_0\}$$

be the disc of radius r_0 centered at 1. Consider the map

$$f_1 : \mathbb{B}_{r_0} \times D_{r_0}(1) \to N(\delta_0) \text{ defined by } f_1 : (\zeta, \xi) \mapsto (\xi \cos\zeta, \xi \sin\zeta).$$

Without loss of generality, we can think that $f_1(\mathbb{B}_{r_0} \times D_{r_0}(1)) = N(\delta_0)$. In other words, f_1 can be thought of as the universal covering map of $N(\delta_0)$. Notice, that implies $f_1(\mathbb{B}_{r_0} \times \{1\}) = A(\delta_0) \subset S_1$.

The pull-back $f_1^* F^{a,\varepsilon}$ on $\mathbb{B}_{r_0} \times D_{r_0}(1)$ of the line field $F^{a,\varepsilon}$ is

$$f_1^* F^{a,\varepsilon} = \ker \big(d(\xi^2) - \varepsilon(\xi^2 - 1)\xi^2 \, d\zeta + a\varepsilon \, \xi \sin\zeta \, d(\xi^2) \big).$$

For $0 < r_1 < 1$, define the map

$$f_2 : \mathbb{B}_{r_0} \times D_{r_1}(0) \to \mathbb{B}_{r_0} \times D_{r_0}(1) \text{ where } f_2 : (z, w) \mapsto \Big(z, \frac{1}{\sqrt{1-w}} \Big).$$

Composing the maps f_1 and f_2 we obtain

$$f = f_1 \circ f_2 \, : \, \mathbb{B}_{r_0} \times D_{r_1}(0) \longrightarrow N(\delta_0).$$

Then the pull-back $f^* F^{a,\varepsilon}$ is

$$f^* F^{a,\varepsilon} = \ker \left(\frac{1}{(1-w)^2} \Big(dw - \varepsilon \, w dz + \varepsilon a \, \frac{\sin z}{\sqrt{1-w}} \, dw \Big) \right)$$

and since $\frac{1}{(1-w)^2}$ is well defined and nonzero for $w \in D_{r_1}(0)$, the line field becomes

$$f^* F^{a,\varepsilon} = \ker \Big(dw - \varepsilon \, w dz + \varepsilon a \, \frac{\sin z}{\sqrt{1-w}} \, dw \Big).$$

The holomorphic function $\mu_\varepsilon(z) = e^{-\varepsilon z}$ is nonzero everywhere, so

$$\begin{aligned} f^* F^{a,\varepsilon} &= \ker \Big(e^{-\varepsilon z} dw - \varepsilon \, w e^{-\varepsilon z} \, dz + \varepsilon a \, \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} \, dw \Big) \\ &= \ker \Big(d(w e^{-\varepsilon z}) + \varepsilon a \, \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} \, dw \Big) \\ &= \ker \big(dJ^{(\varepsilon)} + a\omega^{(\varepsilon)} \big) \end{aligned}$$

$$\text{where } \; J^{(\varepsilon)} = w e^{-\varepsilon z} \; \text{ and } \; \omega^{(\varepsilon)} = \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} \, dw.$$

Construction of the Poincaré Map. Our next step is to define the Poincaré transformation for the foliation $\mathcal{F}^{a,\varepsilon}$, using the local chart f on the tubular neighborhood $N(\delta_0)$ of the loop δ_0. Denote the desired map by

$$P_{a,\varepsilon} \,:\, D_{r_1}(0) \longrightarrow \mathbb{C}.$$

We are going to explain how it is constructed.

Define the path $\hat{\delta}_0 = \{(t,0) \in \mathbb{B}_{r_0} \times \{0\} \,:\, t \in [0,2\pi]\}$. Then $f(\hat{\delta}_0) = \delta_0$. The segment $\hat{\delta}_0$ can be lifted to a path $\delta_{a,\varepsilon}(u)$ on the leaf of $\mathcal{F}^{a,\varepsilon}$ passing through the point $(0,u) \in \{0\} \times D_{r_1}(0)$, so that if $pr_1 : (z,w) \mapsto z$ then $pr_1(\delta_{a,\varepsilon}(u)) = \hat{\delta}_0$. The lift $\delta_{a,\varepsilon}(u)$ has two endpoints. The first one is $(0,u)$ and the second one we denote by $(2\pi, P_{a,\varepsilon}(u))$. When a=0, the map $P_{0,\varepsilon}(u)$ comes from the foliation $\mathcal{F}^{0,\varepsilon}$ which in our tubular neighborhood is given by $\ker(d(we^{-\varepsilon z}))$. Then, $\delta_{0,\varepsilon} = \{(t, ue^{\varepsilon t}) \,:\, t \in [0,2\pi]\}$ and so $P_{0,\varepsilon} = e^{2\pi\varepsilon}u$. Since $\hat{\delta}_{a,\varepsilon}(0) = \hat{\delta}_0$, the equality $P_{a,\varepsilon}(0) = 0$ holds for all (a,ε). As a result, the Poincaré transformation can be written down as

$$P_{a,\varepsilon}(u) = e^{2\pi\varepsilon}u + aI(u,\varepsilon)u + a^2G(u,a,\varepsilon)u$$

and its k-th iteration can be expressed as

$$P^k_{a,\varepsilon}(u) = e^{2k\pi\varepsilon}u + aI_{(k)}(u,\varepsilon)u + a^2G_{(k)}(u,a,\varepsilon)u.$$

If $m = \frac{i}{m}$ then after m iterations the map becomes

$$P^m_{a,\frac{i}{m}}(u) = u + aI_{(m)}\Big(u, \frac{i}{m}\Big)u + a^2G_{(m)}\Big(u, a, \frac{i}{m}\Big)u.$$

Notice that in this case, by lifting $\hat{\delta}_0^m$ we obtain the path

$$\delta^{(m)}_{a,\frac{i}{m}}(u) = \{(t, e^{\frac{i}{m}t}u) \,|\, t \in [0, 2\pi m]\} \tag{6.2}$$

with endpoints $(0,u)$ and $(2\pi m, u)$.

In order to study the periodic orbits of $P_{a,\varepsilon}(u)$, we are going to look at the difference $P^m_{a,\frac{i}{m}}(u) - u$. Since $\left(dJ^{(i/m)} + \omega^{(i/m)}\right)|_{\delta_{a,i/m}(u)} = 0$, it can be concluded that

$$\int_{\delta_{a,i/m}(u)} \left(dJ^{(i/m)} + a\omega^{(i/m)}\right) = 0 \quad \text{and hence}$$
$$\int_{\delta_{a,i/m}(u)} dJ^{(i/m)} = -a \int_{\delta_{a,i/m}(u)} \omega^{(i/m)}.$$

The one-form $dJ^{(i/m)}$ is exact and yields

$$\begin{aligned} P^m_{a,i/m}(u) - u &= P^m_{a,i/m}(u)e^{-2\pi} - ue^0 \\ &= J^{(i/m)}(2\pi m, u) - J^{(i/m)}(0,u) \\ &= \int_{\delta_{a,i/m}(u)} dJ^{(i/m)} \\ &= -a \int_{\delta_{a,i/m}(u)} \omega^{(i/m)}. \end{aligned} \tag{6.3}$$

Dividing equation (6.3) by a and taking into account that the limit of the left hand side is $I_{(m)}(u, i/m)u$, as well as $\delta_{a,i/m}(u) \to \delta_{0,i/m}(u)$, when $a \to \infty$, we can conclude that

$$I_{(m)}(u, i/m)u = -\int_{\delta_{0,i/m}(u)} \omega^{(i/m)}.$$

Now, remembering that $\delta_{0,i/m}(u)$ is of the form (6.2) compute

$$\begin{aligned} I_{(m)}(u, i/m)u &= -\int_{\delta_{0,i/m}(u)} \frac{e^{-\frac{i}{m}z}\sin z}{\sqrt{1-w}}\, dw \\ &= -\int_0^{2\pi m} \frac{e^{-\frac{i}{m}t}\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}} \left(\frac{i}{m}ue^{\frac{i}{m}t}\right) dt \\ &= -\frac{iu}{m}\int_0^{2\pi m} \frac{\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}}\, dt. \end{aligned}$$

Since both sides of the equation are divisible by u,

$$I_{(m)}(u, i/m) = -\frac{i}{m}\int_0^{2\pi m} \frac{\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}}\, dt \tag{6.4}$$

To solve the integral, notice that $1/\sqrt{1-w}$ is well defined and holomorphic in the disc $D_{r_1}(0) \not\ni 1$ so it expands as convergent series

$$(1-w)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} b_k w^k,$$

where $b_k = (-1)^k \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\dots\left(-\frac{1}{2}-(k-1)\right)}{k!} \neq 0$. Thus, the integral can be written as

$$\begin{aligned} \int_0^{2\pi m} \frac{\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}}\, dt &= \int_0^{2\pi m} \left(\sum_{k=0}^{\infty} b_k e^{i\frac{k}{m}t} u^k \right) \sin t\, dt \\ &= \sum_{k=0}^{\infty} b_k \left(\int_0^{2\pi m} e^{i\frac{k}{m}t} \sin t\, dt \right) u^k. \end{aligned} \tag{6.5}$$

The value of the integral depends on the coefficients of (6.5) that depend on the integral

$$\begin{aligned} \int_0^{2\pi m} e^{i\frac{k}{m}t} \sin t\, dt &= \frac{1}{2i} \int_0^{2\pi m} e^{i\frac{k}{m}t} (e^{it} - e^{-it})\, dt \\ &= \frac{1}{2i} \int_0^{2\pi m} \left(e^{i\frac{k+m}{m}t} - e^{i\frac{k-m}{m}t} \right) dt \end{aligned}$$

When $k \neq m$ the primitive of the function $\left(e^{i\frac{k+m}{m}t} - e^{i\frac{k-m}{m}t}\right)$ under the integral is again $2\pi m$-periodic, leading to the conclusion that the integral is zero. When $k = m$ the integral becomes

$$\begin{aligned} \int_0^{2\pi m} e^{it} \sin t\, dt &= \frac{1}{2i} \int_0^{2\pi m} e^{it} (e^{it} - e^{-it})\, dt \\ &= \frac{1}{2i} \int_0^{2\pi m} \left(e^{i2t} - 1 \right) dt \\ &= \frac{1}{(2i)^2} \left(e^{2it} \right)_0^{2\pi m} - \frac{\pi m}{i} \\ &= i\pi m \end{aligned}$$

The computations above lead to $I_{(m)}\left(u, \frac{i}{m}\right) = -\frac{i}{m}\, b_m\, i\pi m\; u^m = \pi b_m\, u^m$. Finally, for $\varepsilon = \frac{i}{m}$, setting $c = \pi b_m \neq 0$, the Poincaré map takes the form

$$P^m_{a,\frac{i}{m}}(u) = u + a\, c u^{m+1} + a^2 G_{(m)}\big(u, a, i/m\big) u. \tag{6.6}$$

6.2 Existence of Periodic Orbits and Multi-Fold Cycles

Proof of Theorem 4. This section establishes the result of Theorem 4. From the discussion in the Introduction, the existence of a multi-fold limit cycle of $\mathcal{F}^{a,\varepsilon}$ follows from the existence of an isolated m-periodic orbit of the Poincar e transformation $P_{a,\varepsilon}$. representative of the cycle in this case will be contained in the the tubular neighborhood $N(\delta_0)$ and therefore free homotopic to δ_0^m in it. This means the limit cycle will be δ_0, m-fold. Thus, the main objective will be to show that $P_{a,\varepsilon}$ has an isolated m-periodic orbit.

Assume we can show that the periodic orbit exists. After fixing the appropriate $a,$ so that the presence of the periodic orbit is secured, Theorem 2 will apply to the family $\mathcal{F}^{a,\varepsilon}$ and by picking $p_0 = (1, 0),$ we can construct a global smooth cross-section B_{p_0} diffeomorphic to the punctured plain $B = \mathbb{C}^*$. In fact, the topology of the integrable leaves is so simple (they are cylinders) that $B_{\delta_0} = B$ and so $E_{\delta_0} = E$. The regions $C'_{\delta_0}, C_{\delta_0}$ and A' will be nested annuli of very large width and we will have, as Theorem 2 implies, a global Poincaré transformation on a cross-section $C'_{p_0} \subset B_{p_0}$. It is easy to notice that, as Lemma 4.3 reveals, the map $P_{a,\varepsilon}$ can be regarded simply as a representation of the $P_{\delta_0,\varepsilon}$ in one of the complex charts introduced in Lemma 4.2. Theorem 2 shows, that the complex cycle corresponding to the m-periodic orbit of $P_{a,\varepsilon}$ will be in fact limit δ_0, m-fold vertical and will satisfy the premises of Theorem 3. Thus, the limit multi-fold vertical cycle of $\mathcal{F}^{a,\varepsilon}$ will be subject to rapid evolution as described in Theorem 3.

In the context of the preceding two paragraphs, a small remark is in order. The theory, developed in the chapters before the current one, has to undergo a small correction. Originally, our assumption was that B is a hyperbolic Riemann surface covered by the disc $\mathbb{D}$. In our example, B is in fact non-hyperbolic and is covered by $\mathbb{C}$. Since $\mathbb{C}$ is still contractible, all the proofs and construction will be essentially the same and the correction will be merely a matter of change in some notations.

Existence of Periodic Orbits. We have the radii $r_1 > 0, r_2 > 0$ and $\bar{r}_3 > 0$ so that for any $(a, \varepsilon) \in D_{r_2}(0) \times D_{\bar{r}_3}$ the map

$$P_{a,\varepsilon} \,:\, D_{r_1}(0) \longrightarrow \mathbb{C}$$

is well defined. Let $m > 0$ be such that $i/m \in D_{\bar{r}_3}(0)$.

Lemma 6.1. *There exist ε_m near $\frac{i}{m}$ and a parameter a_m such that for all ε in a neighborhood of ε_m, the map $P_{a,\varepsilon}$ has an isolated periodic orbit of period m.*

Proof: The verification of the claim depends on four facts. Putting them together will help us determine the values of the parameters a and ε. As before, in order to find a periodic orbit for the map $P_{a,\varepsilon}(u)$, we are going to look at the equation

$$P^m_{a,\varepsilon}(u) - u = 0. \tag{6.7}$$

Whenever $a \neq 0$ we can rewrite (6.7) in the form

$$\frac{e^{2\pi m\varepsilon} - 1}{a}\, u + I_{(m)}(u, \varepsilon)u + a\, G_{(m)}(u, a, \varepsilon)u = 0.$$

Furthermore, having in mind that $u = 0$ is always a solution of (6.7), we can divide by u and obtain

$$g(u, a, \varepsilon) = \frac{e^{2\pi m\varepsilon} - 1}{a} + I_{(m)}(u, \varepsilon) + a\, G_{(m)}(u, a, \varepsilon) = 0 \tag{6.8}$$

for $u \in D_{r_1}(0), a \in D_{r_2}(0) - \{0\}$ and $\varepsilon \in D_{\bar{r}_3}(0)$.

Fact 1. Let us focus on the equation

$$g\Big(u,a,\frac{i}{m}\Big) = I_{(m)}\Big(u,\frac{i}{m}\Big) + aG_{(m)}\Big(u,a,\frac{i}{m}\Big) = 0 \tag{6.9}$$

If necessary, decrease the radius $r_2 > 0$ enough so that

$$\mathcal{M}(r_1,r_2) = \max\Big\{|a|\,\Big|G_{(m)}\Big(u,a,\frac{i}{m}\Big)\Big| \;:\; |u| = r_1 \text{ and } a \in D_{r_2}(0)\Big\} < |c|\,r_1^m.$$

Since $I_{(m)}\Big(u,\frac{i}{m}\Big) = cu^m$, it follows that for $|u| = r_1$ and for any $a \in D_{r_2}(0)$

$$|c|\,|u|^m = |c|\,r_1^m > \mathcal{M}(r_1,r_2) \geq |a|\,\Big|G_{(m)}\Big(u,a,\frac{i}{m}\Big)\Big|,$$

so by Rouche's Theorem [], equation (6.9) has exactly k zeroes $u_1(a),..,u_m(a)$ in $D_{r_1}(0)$, counted with multiplicities.

Fact 2. Let $\mu(\varepsilon) = \min\{|e^{2\pi k\varepsilon} - 1| \;:\; 1 \leq k \leq m-1\}$. Regarded as a function, $\mu(\varepsilon)$ is continuous and $\mu(i/m) > 0$. Hence, there exists $r_3 > 0$, such that $\overline{D_{r_3}(i/m)} \subset D_{\bar{r}_3}(0)$. Moreover, there exists a constant $\mu > 0$, such that $\mu(\varepsilon) > \mu$ for any $\varepsilon \in D_{r_3}(i/m)$. If needed, decrease $r_2 > 0$ so that

$$\max\big\{\,|a|\,\big|I_{(k)}(u,\varepsilon) + aG_{(k)}(u,a,\varepsilon)\big| \;:\; 1 \leq k \leq m-1\big\} < \mu$$

for all $u \in D_{r_1}(0), a \in D_{r_2}(0)$ and $\varepsilon \in D_{r_3}(i/m)$.

Fact 3. Equation (6.8) can take the form

$$g(u,a,\varepsilon) = g\Big(u,a,\frac{i}{m}\Big) + \Big(g(u,a,\varepsilon) - g\Big(u,a,\frac{i}{m}\Big)\Big) = 0 \tag{6.10}$$

For some specific $a \in D_{r_2}(0) - \{0\}$, Fact 1 reveals that whenever $|u| = r_1$, the following inequalities hold:

$$\Big|I_{(m)}\Big(u,\frac{i}{m}\Big) + a\,G_{(m)}\Big(u,a,\frac{i}{m}\Big)\Big| \geq \Big|I_{(m)}\Big(u,\frac{i}{m}\Big)\Big| - |a|\,\Big|G_{(m)}\Big(u,a,\frac{i}{m}\Big)\Big| > 0.$$

Hence, $\mu_1(a) = \min\Big\{\Big|I_{(m)}\Big(u, \frac{i}{m}\Big) + a\, G_{(m)}\Big(u, a, \frac{i}{m}\Big)\Big| \ :\ |u| = r_1\Big\} > 0$ Notice, that for any nonzero $a \in D_{r_2}(0)$ one can find a radius $r_3(a) > 0$, continuously depending on a, such that

$$\max\left\{\left|g(u,a,\varepsilon) - g\Big(u, a, \frac{i}{m}\Big)\right| \ :\ |u| = r_1,\ \ \varepsilon \in D_{r_3(a)}\big(i/m\big)\right\} < \mu_1(a),$$

Because of the last inequality, it follows by Rouche's Theorem that equation (6.8) has as many solutions as equation (6.9). Thus, due to Fact 1, (6.8) has exactly m solutions $u_1(a,\varepsilon), ..., u_m(a,\varepsilon)$, counted with multiplicities. If we set

$$W = \bigsqcup_{0 \neq a \in D_{r_2}(0)} \Big(\{a\} \times D_{r_3(a)}(i/m)\Big),$$

then W is open and $\overline{W} \ni (0, \frac{i}{m})$.

Fact 4. Let $g_0(a,\varepsilon) = (e^{2\pi m \varepsilon} - 1) + a\, I_{(m)}(0,\varepsilon) + a^2\, G_{(m)}(0,a,\varepsilon)$. Notice, that $g_0\big(0, \frac{i}{m}\big) = 0$ and $\frac{\partial g_0}{\partial \varepsilon}\big(0, \frac{i}{m}\big) = 2\pi m \neq 0$. Hence, by the inverse function theorem, it follows that for possibly decreased $r_2 > 0$ there exists a holomotphic function $\chi : D_{r_2}(0) \to D_{r_3}\big(\frac{i}{m}\big)$ such that $\chi(0) = \frac{i}{m}$ and $g_0(a, \chi(a)) = 0$ for all $a \in D_{r_2}(0)$. From here, we can see that the zero locus of g_0 inside the product domain $D_{r_2}(0) \times D_{r_3}\big(\frac{i}{m}\big)$ is

$$Z = \{(a,\varepsilon) \ :\ g_0(a,\varepsilon) = 0\} = \{(a, \chi(a)) \ :\ a \in D_{r_2}(0)\}.$$

The set Z is relatively closed in $D_{r_2}(0) \times D_{r_3}\big(\frac{i}{m}\big)$ so its complement $\big(D_{r_2}(0) \times D_{r_3}\big(\frac{i}{m}\big)\big) - Z \neq \varnothing$ is open. Therefore, $W \cap \Big[\big(D_{r_2}(0) \times D_{r_3}\big(\frac{i}{m}\big)\big) - Z\Big] \neq \varnothing$ is open as well.

Completing the Proof. Let $(a_m, \varepsilon_m) \in W \cap \Big[\big(D_{r_2}(0) \times D_{r_3}\big(\frac{i}{m}\big)\big) - Z\Big]$. Apply the results from Fact 4

$$g_0(a_m, \varepsilon_m) = (e^{2\pi m \varepsilon_m} - 1) + a_m\, I_{(m)}(0, \varepsilon_m) + a_m^2\, G_{(m)}(0, a_m, \varepsilon_m) \neq 0.$$

Hence, the equation

$$P^m_{\varepsilon_m,a_m}(u) - u = (e^{2\pi m\varepsilon_m} - 1)\, u + a_m\, I_{(m)}(u,\varepsilon_m)u + a_m^2\, G_{(m)}(u,a_m,\varepsilon_m)u = 0$$

has $u_0 = 0$ as a simple root.

Since $(a_m,\varepsilon_m) \in W$, it follows from Fact 3 that whenever $|u| = r_1$ the following inequality holds

$$\left| I_{(m)}\Big(u,\frac{i}{m}\Big) + a_m\, G_{(m)}\Big(u,a_m,\frac{i}{m}\Big)\right| \geq \mu_1(a_m) > \left| g(u,a_m,\varepsilon_m) - g\Big(u,a_m,\frac{i}{m}\Big)\right|$$

Therefore, by Rouche's Theorem, the equation

$$a_m\, g(u,a_m,\varepsilon_m) = (e^{2\pi m\varepsilon_m} - 1) + a_m\, I_{(m)}(u,\varepsilon_m) + a_m^2\, G_{(m)}(u,a_m,\varepsilon_m) = 0 \quad (6.11)$$

has as many solutions as

$$a_m\, g\Big(u,a_m,\frac{i}{m}\Big) = a_m\, I_{(m)}\Big(u,\frac{i}{m}\Big) + a_m^2\, G_{(m)}\Big(u,a_m,\frac{i}{m}\Big) = 0. \quad (6.12)$$

By Fact 1, equation (6.12) has m roots $u_1(a_m),...,u_m(a_m)$ contained in $D_{r_1}(0)$. For that reason, equation (6.11) has m solutions $u_1(a_m,\varepsilon_m),...,u_m(a_m,\varepsilon_m)$ contained in $D_{r_1}(0)$, and as it was established, none of them is zero. For simplicity, let $u_j = u_j(a_0,\varepsilon_0)$, where $j = 1,..,m$.

By Fact 2, for $1 \leq k \leq m-1$ and for $u \in D_{r_1}(0)$,

$$|e^{2\pi k\varepsilon_m} - 1| \geq \mu(a_m) > \mu > |a_m|\left| I_{(k)}(u,\varepsilon_m) + a_m G_{(k)}(u,a_m,\varepsilon_m)\right|.$$

Having in mind that $u_j \in D_{r_0}(0)$ and each of them is nonzero for $j = 1,..,m$, we estimate

$$\left|P^k_{a_m,\varepsilon_m}(u_j) - u_j\right| = |u_j|\left|(e^{2\pi k\varepsilon_m} - 1) + a_m\, I_{(k)}(u_j,\varepsilon_m) + a_m^2\, G_{(k)}(u_j,a_m,\varepsilon_m)\right|$$
$$\geq |u_j|\left(|e^{2\pi k\varepsilon_m} - 1| - |a_m|\left| I_{(k)}(u_j,\varepsilon_m) + a_m^2\, G_{(k)}(u_j,a_m,\varepsilon_m)\right|\right) > 0.$$

For that reason, $P^k_{a_m,\varepsilon_m}(u_j) \neq u_j$ for $1 \leq k \leq m-1$. Hence, the orbit u_1,...,u_m consists of different points and therefore is periodic of period m in $D_{r_1}(0)$. □

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