

GEOMETRIC FLOWS ON MANIFOLDS WITH BOUNDARY

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Artem Dmytrovych Pulemotov

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GEOMETRIC FLOWS ON MANIFOLDS WITH BOUNDARY

Artem Dmytrovych Pulemotov, Ph.D.

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The present dissertation discusses three connected subjects. Firstly, we establish a version of the Hopf boundary point lemma for sections of a vector bundle over a manifold M with nonempty boundary. This result may be viewed as a counterpart to Richard Hamilton's tensor maximum principle. Secondly, we prove the Li-Yau-Hamilton estimate for the heat equation on M . Such estimates are typically used to derive monotonicity formulas related to geometric flows. Thirdly, we establish bounds for a solution $\nabla(t)$ of the Yang-Mills heat equation in a vector bundle over M . Our results imply that the curvature of $\nabla(t)$ does not blow up if the dimension of M is less than 4 or if the initial energy of $\nabla(t)$ is sufficiently small.

BIOGRAPHICAL SKETCH

Artem Pulemotov was born in Kyiv, Ukraine, on October 17, 1982. He attended Kyiv National T. Shevchenko University from 1999 to 2004 earning his Bachelor's degree and pursuing research under Yuri Berezansky. At that stage, he obtained several results in infinite-dimensional analysis and spectral theory. Artem entered the Ph.D. program at Cornell University in 2004 and started to conduct research under Leonard Gross. His work at the graduate school materialized in a series of results collected in the present dissertation. He considers himself an analyst because he believes that analysis is to mathematics what hard rock is to music.

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CHAPTER 1

INTRODUCTION

The present dissertation considers three related subjects. Chapter 2 establishes a version of the Hopf boundary point lemma for sections of a vector bundle. This result may be viewed as a counterpart to the so-called tensor maximum principle. Chapter 3 establishes the Li-Yau-Hamilton estimate for the heat equation on a manifold with boundary. Results of this kind are known to be useful in the study of geometric flows. Chapter 4 discusses estimates for the solutions of the Yang-Mills heat equation in a vector bundle over a manifold with boundary. The proofs in this chapter utilize a probabilistic technique. Our results imply that the curvature of a solution does not blow up if the dimension of the manifold is less than 4 or if the initial energy is sufficiently small.

The maximum principle for sections of a general vector bundle over a closed manifold was originally obtained in [21]. This statement is also known as the tensor maximum principle. It proved to be a powerful implement in the study of the Ricci flow; see [12]. In particular, it was used to establish important facts about four-manifolds with nonnegative curvature operator. Other applications were considered, as well; see, for instance, [9, 22, 6].

A specific version of the maximum principle for sections appeared in [20]. This version only applied to 2-tensors. Two important generalizations of the maximum principle for sections were suggested in [14]. We refer to [10] and [12, Chapter 10] for an overview of relevant results. We emphasize that the theory discussed above was developed largely on closed manifolds.

The maximum principle for sections may be regarded as an evolution of the

maximum principle for systems of scalar parabolic equations obtained in [50]. It must be noted that the statement in [50] has become a powerful implement in the study of parabolic systems. In particular, it was applied to the investigation of the existence and the asymptotic behavior of solutions. We refer to [44, Chapter 14] for several relevant results and a vast bibliography; some of the references not mentioned there are [40, 16, 29, 1].

An important comment should be made at this point. The maximum principles discussed above rely on the concept of an invariant set. The definition of an invariant set for a system of scalar parabolic equations can be found, for example, in [44, Chapter 14]. This definition generalizes easily to cover the case of an equation for vector bundle sections. We remark that invariant sets should be viewed from a slightly different standpoint when the boundary conditions are specified for the solutions; see, for instance, [40, 28, 29].

The paper [43], being devoted to the study of the Ricci flow on manifolds with boundary, offers a specific version of the Hopf boundary point lemma. This version applies to 2-tensors over a manifold with boundary. In essence, it is an analogue of the maximum principle for 2-tensors proved in [20]. At the same time, in spite of the fact that the universal maximum principle for sections suggested in [21] is a recognized powerful tool, no counterparts of this statement have yet been obtained in the presence of a boundary. Chapter 2 establishes a general version of the Hopf boundary point lemma. Our statement applies to sections of a general vector bundle over a manifold with boundary. It appears to constitute a comprehensive counterpart to the maximum principle suggested in [21]. A result analogous to (although not exactly the same as) ours but for parabolic systems can be found in [40].

After proving our Hopf lemma for sections, we state three of its immediate corollaries. They are all closely related to the concept of an invariant set. The first corollary may be viewed as the basic maximum principle for sections of a vector bundle over a manifold with boundary. The second corollary shows that the maximum principle of [21] holds in the presence of a boundary provided that Neumann-type boundary conditions are imposed on the sections in question. Such a result is expected to prove useful in the study of the Ricci flow; cf. [43]. The third corollary provides an explicit connection between invariant sets of an equation for vector bundle sections and the boundary conditions specified for the solutions. In certain situations, it allows to find an invariant set for a given boundary value problem. (In one form or another, this task was addressed in many works; see, for instance, [28, 29, 1] and [44, Chapter 14].) Alternatively, the third corollary may be used to construct relatively sophisticated boundary value problems with given invariant sets. We utilize it below to prove Theorem 3.6.

The Li-Yau-Hamilton estimate for the heat equation, discussed in Chapter 3, generalizes the well-known differential Harnack inequality of [31] and was originally obtained on manifolds without boundary in the paper [22]. It is typically used to prove monotonicity formulas related to various geometric evolution equations; see, for example, [23]. In their turn, such monotonicity formulas are essential for establishing the existence of solutions.

Let us mention that [13] offers a constrained version of the Li-Yau-Hamilton estimate from [22]. The paper [6] adapts the result of [22] to Kähler manifolds. We point out that an inequality similar to the Li-Yau-Hamilton estimate for the heat equation comes up in the investigation of the Ricci flow. Its precise formu-

lation and various applications are presented in [12, Chapter 15]. Analogous results hold for the Kähler-Ricci flow. Their formulations and relevant references can be found in [11, Chapter 2] and in [35].

Suppose M is a smooth compact Riemannian manifold without boundary. Consider a positive solution $p(t, x)$ to the heat equation on M such that the integral $\int_M p(t, x) dx$ does not exceed 1 for any $t \in (0, \infty)$. Then there exist constants $A > 0$ and $B > 0$ that depend only on the manifold M and satisfy

$$D_{\cdot}^2 \log p(t, x) \geq - \left(\frac{1}{2t} + A \left(1 + \log \left(\frac{B}{t^{\frac{\dim M}{2}} p(t, x)} \right) \right) \right) \langle \cdot, \cdot \rangle, \\ t \in (0, 1], x \in M. \quad (1.1)$$

In this formula, D_{\cdot}^2 is the second covariant derivative, and $\langle \cdot, \cdot \rangle$ is the Riemannian metric. The inequality is to be understood in the sense of bilinear forms. If M is Ricci parallel and has nonnegative sectional curvatures, then (1.1) holds with $A = 0$. This is the case when M is, for example, a sphere or a flat torus. Formula (1.1) constitutes the Li-Yau-Hamilton estimate for the heat equation. It was originally obtained in [22].

Suppose now that M is a smooth compact Riemannian manifold with nonempty boundary ∂M . Chapter 3 establishes formula (1.1) in this case. The solution $p(t, x)$ of the heat equation is assumed to satisfy the Neumann boundary condition. Theorem 3.1 proves (1.1) in the situation where no restrictions are imposed on the curvature of M away from ∂M . But the boundary of M must be totally geodesic for this result to hold. Moreover, several derivatives of the curvature of M have to vanish at ∂M . Theorem 3.6 deals with a more exclusive situation. It shows that inequality (1.1) holds with $A = 0$ if the manifold M is Ricci parallel and has nonnegative sectional curvatures. As before, ∂M must be totally geodesic. However, the previously mentioned derivatives of the curva-

ture of M are no longer required to vanish at ∂M . Our proofs of Theorems 3.1 and 3.6 differ considerably in their techniques.

Both incarnations of estimate (1.1) appearing in Chapter 3 play significant roles in establishing the results of Chapter 4. More precisely, they enable us to obtain a monotonicity formula related to the Yang-Mills heat equation. This formula is given by Lemma 4.9. It helps us establish an estimate for the solutions to the Yang-Mills heat equation in dimensions 5 and higher.

In order to prove Theorem 3.1, we employ the doubling method. More precisely, we consider two identical copies of M and glue them together along the boundary. This procedure produces a closed manifold \mathcal{M} . The desired estimate follows by applying the results of the paper [22] on \mathcal{M} . Of course, several technical questions need to be handled in order to make the doubling method work for our purpose.

The proof of Theorem 3.6 relies on the Hopf boundary point lemma for vector bundle sections appearing in Chapter 2. The technique we use resembles those employed in [31, 22]. One may also apply the doubling method to prove Theorem 3.6. However, the approach adopted in the present dissertation appears to be more effective. Firstly, it enables us to avoid the assumption on the curvature of M near ∂M that is required to carry out the doubling procedure. Secondly, it does not rely on the previously known versions of the Li-Yau-Hamilton estimate. Last but not least, our approach seems to be more natural and to provide a better ground for further generalizations.

Chapter 4 of the present dissertation deals with the Yang-Mills heat equation in a vector bundle over a compact Riemannian manifold M with nonempty

boundary. In order to describe our results, we need to outline the setup. Let E be a vector bundle over M . Suppose the time-dependent connection $\nabla(t)$ in E solves the Yang-Mills heat equation

$$\frac{\partial}{\partial t} \nabla(t) = -\frac{1}{2} d_{\nabla(t)}^* R^{\nabla(t)}, \quad t \in [0, T]. \quad (1.2)$$

Here and in what follows, $d_{\nabla(t)}$ is the exterior covariant derivative, $d_{\nabla(t)}^*$ is its adjoint, and $R^{\nabla(t)}$ is the curvature of $\nabla(t)$. By definition, $R^{\nabla(t)}$ is a 2-form on M with its values in the endomorphism bundle $\text{End } E$. The Yang-Mills heat equation is a potentially powerful instrument for minimizing the Yang-Mills energy functional; see, for example, [3, 38, 2]. It has a number of applications in topology and in mathematical physics. Some of these applications are comprehensively discussed in the book [17] and the dissertation [41]; see also [4]. The existence of solutions is one of the most important questions regarding the Yang-Mills heat equation.

Since ∂M is assumed to be nonempty, we have to specify the boundary conditions for the time-dependent connection $\nabla(t)$. Doing so is a delicate matter. As detailed in Remark 4.11, it is more natural for us to impose the boundary conditions on the curvature $R^{\nabla(t)}$ than on $\nabla(t)$ itself. We assume

$$\left(R^{\nabla(t)}\right)_{\text{tan}} = 0, \quad \left(d_{\nabla(t)}^* R^{\nabla(t)}\right)_{\text{tan}} = 0, \quad t \in [0, T]. \quad (1.3)$$

The subscript “tan” stands for the component of the corresponding $\text{End } E$ -valued form that is tangent to ∂M . Alternatively, we may assume

$$\left(R^{\nabla(t)}\right)_{\text{norm}} = 0, \quad \left(d_{\nabla(t)} R^{\nabla(t)}\right)_{\text{norm}} = 0, \quad t \in [0, T]. \quad (1.4)$$

(Actually, the second equality always holds due to the Bianchi identity.) The subscript “norm” signifies the component that is normal to ∂M . Conditions (1.3) and (1.4) are analogous to the relative and the absolute boundary conditions for

real-valued forms. The results in Chapter 4 prevail regardless of whether we choose (1.3) or (1.4) to hold on ∂M . Other ways to introduce the boundary conditions in the context of Yang-Mills theory were considered in several works including, for example, [32, 45, 47, 19, 7]. We should mention, however, that none of these works except [7] deals with parabolic-type equations like (1.2). The relationship between the boundary conditions utilized in the present dissertation and the boundary conditions appearing elsewhere is discussed in Remark 4.12.

Chapter 4 provides estimates for the curvature $R^{\nabla(t)}$ of the solution $\nabla(t)$ to the Yang-Mills heat equation (1.2) subject to (1.3) or (1.4). Roughly speaking, we show that $R^{\nabla(t)}$ is bounded at every point of M by expressions involving the initial energy of $\nabla(t)$. Theorem 4.1 considers the case where the dimension of M is either 2 or 3. It yields an estimate on $R^{\nabla(t)}$ and demonstrates that $R^{\nabla(t)}$ does not blow up. Theorem 4.2 deals with the case where the dimension is equal to 4. It requires that the initial energy of $\nabla(t)$ be smaller than a constant depending on M . If this assumption is satisfied, the theorem produces a bound on $R^{\nabla(t)}$. It is easy to see that $R^{\nabla(t)}$ does not blow up when this bound holds. Theorem 4.3 considers the situation where the dimension of M is greater than or equal to 5. It produces an estimate on $R^{\nabla(t)}$ under a rather sophisticated condition. The theorem implies that the curvature of a solution to Eq. (1.2) cannot blow up after time ρ if the initial energy is smaller than a number depending on ρ .

When the dimension of M equals 2, 3, or 4, the boundary ∂M has to be convex for the results in Chapter 4 to hold. No other assumptions on the geometry of M are required. However, if the dimension is 5 or higher, the situation is different. In this case, ∂M has to be totally geodesic, and restrictions have to be imposed on the curvature of M . The reason for such a phenomenon lies in the fact that,

when the dimension is 5 or higher, our arguments involve the Li-Yau-Hamilton estimate (1.1). Both Theorems 3.1 and 3.6 are exploited.

We thus observe a trichotomy in the behavior of the solution $\nabla(t)$ to Eq. (1.2). Theorems 4.1, 4.2, and 4.3 provide three different sets of conditions ensuring that $R^{\nabla(t)}$ does not blow up. Each of these sets corresponds to a certain range of dimensions of M . A similar trichotomy occurs on closed manifolds; see, for instance, [2]. However, the difference in the geometric assumptions that was discussed in the previous paragraph is not observed in this case.

Let us make a comment as to the practical importance of the results in Chapter 4. Proving that the curvature does not blow up is the principal ingredient in establishing the long-time existence of solutions to the Yang-Mills heat equation. The list of relevant references includes but is not limited to [17, 38, 46, 2, 7]. We should point out that all these works except [7] restrict their attention to manifolds without boundary.

The proofs of Theorems 4.1, 4.2, and 4.3 rely on the probabilistic technique developed in [2]. The origin of this technique lies in the theory of harmonic maps; see [48]. The pivotal stochastic process in our considerations is a reflecting Brownian motion on the manifold M . Let us mention that the probabilistic approach to Yang-Mills theory was investigated rather extensively. The paper [2] contains a series of results and a list of references on the subject.

While establishing the theorems in Chapter 4, we prove a noteworthy property of End E -valued forms on M . The precise phrasing of this property is given by Lemma 4.5. Roughly speaking, it states that, if ∂M is convex and an End E -valued form ϕ satisfies (1.3) or (1.4), then the derivative of the squared absolute

value of ϕ in the direction of the outward normal to ∂M must be nonpositive. A simpler version was established in [7].

The results of Chapter 2 of the present dissertation originally appeared in the paper [36], and the results of Chapters 3 and 4 in the paper [37].

CHAPTER 2

THE HOPF LEMMA FOR VECTOR BUNDLE SECTIONS

Consider a smooth, compact, connected, oriented Riemannian manifold M with nonempty boundary ∂M . We use the notation $\nu(x)$ for the outward unit normal to ∂M at the point $x \in \partial M$. The differentiation of real-valued functions in the direction of $\nu(x)$ will be designated by $\frac{\partial}{\partial \nu}$. Let V be a vector bundle over M . The fiber of V over $x \in M$ will be denoted by V_x . The designation $\pi(v)$ refers to the projection of $v \in V$ onto M . We suppose V is equipped with a fiber metric $\langle \cdot, \cdot \rangle_V$. Let $\| \cdot \|_V$ stand for the corresponding norm.

Consider a time-dependent section $f(t, x)$ of the vector bundle V . In this chapter, the time parameter t varies through the interval $[0, T]$ with a fixed $T > 0$. Choose a connection ∇ in V compatible with $\langle \cdot, \cdot \rangle_V$. We understand ∇ as a mapping that takes a section τ of V to a section $\nabla\tau$ of the bundle $T^*M \otimes V$. It is customary to interpret $\nabla\tau$ as a V -valued 1-form on the manifold M . Consider a vector field X on M . We write $\nabla_X\tau$ to indicate the application of $\nabla\tau$ to X . Given a smooth real-valued function $h(x)$ on M , the formula

$$\nabla_X(h\tau) = (Xh)\tau + h\nabla_X\tau$$

must be satisfied.

Employing the connection ∇ in V and the Levi-Civita connection in the cotangent bundle T^*M , one can define the second covariant derivative $\nabla^2\tau$. We write $\nabla_{\chi_1, \chi_2}^2\tau$ to indicate the application of $\nabla^2\tau$ to the vectors $\chi_1, \chi_2 \in T_xM$. The Laplacian Δ_V acts on the section τ by taking the trace of $\nabla^2\tau$.

Let $\phi(t, v)$ be a time-dependent mapping of V into itself such that $\phi(t, v) \in V_{\pi(v)}$ for any $(t, v) \in [0, T] \times V$. Suppose every compact set $U \subset V$ admits a constant

$C_\phi(U) > 0$ satisfying

$$\|\phi(t, v_1) - \phi(t, v_2)\|_V \leq C_\phi(U) \|v_1 - v_2\|_V. \quad (2.1)$$

The estimate must hold for any $t \in (0, T)$, and any $v_1, v_2 \in U$ subject to $\pi(v_1) = \pi(v_2)$. Let $\zeta(t, x)$ be a time-dependent vector field on M . Suppose $f(t, x)$ solves the second-order equation

$$\frac{\partial}{\partial t} f(t, x) = \Delta_V f(t, x) + \nabla_{\zeta(t, x)} f(t, x) + \phi(t, f(t, x)) \quad (2.2)$$

on $(0, T) \times M$. In particular, $f(t, x)$ must be continuous in $t \in [0, T]$ and C^1 -differentiable in $t \in (0, T)$.

Consider a nonempty set $W \subset V$. We assume W is invariant under the parallel translation with respect to the connection ∇ fixed in V . The set $W_x = W \cap V_x$ must be closed and convex in the fiber V_x for every $x \in M$. When writing ∂W_x , we refer to the boundary of W_x in V_x . It should be noted that ∂W_x is not required to be smooth for any $x \in M$. Given a point $\omega \in W$ subject to $\omega \in \partial W_{\pi(\omega)}$, we call $\lambda \in V_{\pi(\omega)}$ a *supporting vector* for W at ω if $\|\lambda\|_V = 1$ and the inequality $\langle \lambda, \sigma \rangle_V \leq \langle \lambda, \omega \rangle_V$ holds for all $\sigma \in W_{\pi(\omega)}$. The set of all the supporting vectors for W at ω will be denoted by $S_\omega W$. In a sense, the elements of $S_\omega W$ are outward unit normals to $\partial W_{\pi(\omega)}$ at ω .

Introduce the notation

$$\text{dist}_W v = \inf_{\omega \in W_{\pi(v)}} \|v - \omega\|_V$$

for $v \in V$. Let $\omega(v)$ be the unique point in $W_{\pi(v)}$ such that $\text{dist}_W v = \|v - \omega(v)\|_V$. Obviously, $\text{dist}_W v$ represents the distance between $v \in V$ and $W_{\pi(v)}$, while $\omega(v)$ is the unique point in $W_{\pi(v)}$ closest to v . We call $(t, x) \in [0, T] \times M$ a *maximal distance pair* if

$$\text{dist}_W f(t, x) = \sup_{y \in M} \text{dist}_W f(t, y) > 0.$$

Let $\lambda(v)$ denote the difference $v - \omega(v)$ for $v \in V$.

We are now ready to formulate our Hopf lemma for sections. It should be remarked that the assumption on the mapping $\phi(t, v)$ in our statement is quite standard. Roughly speaking, we demand that $\phi(t, v)$ point into W when v is subject to $v \in \partial W_{\pi(v)}$. This is equivalent to the “ordinary differential equation assumption” employed in [21]; see Lemma 4.1 in that paper.

Theorem 2.1. *Suppose the solution $f(t, x)$ of Eq. (2.2) and the mapping $\phi(t, v)$ appearing in the right-hand side of Eq. (2.2) meet the following requirements:*

1. *The initial value $f(0, x)$ lies in W for all $x \in M$.*
2. *The estimate $\langle \lambda, \phi(t, \omega) \rangle_V \leq 0$ holds for any $t \in (0, T)$, any $\omega \in W$ subject to $\omega \in \partial W_{\pi(\omega)}$, and any supporting vector $\lambda \in S_{\omega}W$.*

If the value $f(t, x)$ lies outside of W for some $(t, x) \in (0, T] \times M$, then there exists a maximal distance pair $(t_{\text{pos}}, x_{\text{pos}}) \in (0, T) \times \partial M$ such that the formula

$$\left\langle \lambda(f(t_{\text{pos}}, x_{\text{pos}})), \nabla_{v(x_{\text{pos}})} f(t_{\text{pos}}, x_{\text{pos}}) \right\rangle_V > 0 \quad (2.3)$$

holds true.

Before proving the theorem, we need to make some preliminary arrangements. Given a real-valued function $\theta(t)$ on $[0, T)$, define

$$\dot{\theta}^+(t) = \limsup_{h \rightarrow 0^+} \frac{\theta(t+h) - \theta(t)}{h}$$

for $t \in [0, T)$. The following lemma will be required; cf. Lemma 3.1 and Corollary 3.3 in [21], or Lemma 7 in [14].

Lemma 2.2. *Suppose $\theta(t)$ is a nonnegative continuous function on $[0, T)$ with $\theta(0) = 0$. Suppose also $\theta(t)$ is not identically 0 on $[0, T)$. Given a constant $C > 0$, there exists a point $t_C \in (0, T)$ such that $\dot{\theta}^+(t_C) > C\theta(t_C)$ and $\theta(t_C) > 0$.*

Proof. Assume the existence of $C > 0$ satisfying the estimate $\dot{\theta}^+(t) \leq C\theta(t)$ whenever $\theta(t) > 0$. Introduce a new nonnegative continuous function $\eta(t) = e^{-Ct}\theta(t)$. Clearly, the equality $\eta(0) = 0$ holds, and $\dot{\eta}^+(t) \leq 0$ whenever $\eta(t) > 0$.

Fix $\epsilon_1, \epsilon_2 > 0$. We will now prove that $\eta(t) \leq \epsilon_1 t + \epsilon_2$ for all $t \in [0, T)$. Let a be the largest possible number in $(0, T]$ such that the inequality $\eta(t) \leq \epsilon_1 t + \epsilon_2$ holds on $[0, a)$. (Since $\eta(0) = 0 < \epsilon_2$, the set of such numbers is not empty, and a is well defined.) We claim that $a = T$. Indeed, if $a < T$, then $\eta(a) = \epsilon_1 a + \epsilon_2 > 0$ by continuity and

$$\limsup_{h \rightarrow 0^+} \frac{\eta(a+h) - \eta(a)}{h} \leq 0.$$

But this implies $\eta(t) \leq \epsilon_1 t + \epsilon_2$ on $[0, a + \delta)$ for some $\delta > 0$, which contradicts the definition of a .

Thus $\eta(t) \leq \epsilon_1 t + \epsilon_2$ for all $t \in [0, T)$. Since this inequality holds for any $\epsilon_1, \epsilon_2 > 0$, we can conclude that $\eta(t)$ is identically 0. Hence $\theta(t)$ is identically 0, which contradicts the suppositions of the lemma. \square

Proof of Theorem 2.1. It suffices to carry out the proof assuming W is compact. In order to justify this statement, fix a number $R > 0$ large enough to ensure that $\|f(t, x)\|_V < R$ and $\|\omega(f(t, x))\|_V < R$ for any $(t, x) \in [0, T] \times M$. Introduce the set $\hat{W} = \{w \in W \mid \|w\|_V \leq R\}$. One can verify that \hat{W} is compact. Clearly, it is invariant under the parallel translation with respect to ∇ , and its intersection with the fiber V_x is closed and convex in V_x for every $x \in M$. Let $\kappa(v)$ be a smooth function

acting from V to the interval $[0, 1]$. We choose $\kappa(v)$ demanding that $\kappa(v) = 1$ when $\|v\|_V \leq R$ and $\kappa(v) = 0$ when $\|v\|_V \geq 2R$. Define the time-dependent mapping $\hat{\phi}(t, v)$ of V into itself by the formula $\hat{\phi}(t, v) = \kappa(v)\phi(t, v)$. Estimate (2.2) is obviously satisfied for $\hat{\phi}(t, v)$ with the constant $C_{\hat{\phi}}(U) = C_{\phi}(U)$ when the compact set U is equal to $f([0, T] \times M) \cup \hat{W}$. (We note that the proof of the theorem will not require estimate (2.2) to hold when U is other than $f([0, T] \times M) \cup W$.) The section $f(t, x)$ would remain a solution of Eq. (2.2) if the mapping $\hat{\phi}(t, v)$ appeared in the right-hand side of this equation instead of the mapping $\phi(t, v)$. A straightforward argument demonstrates that it suffices to prove the theorem with W and $\phi(t, v)$ replaced by \hat{W} and $\hat{\phi}(t, v)$. Therefore, supposing W is compact does not lead to a loss of generality.

Introduce the function

$$s(t) = \sup_{x \in M} \text{dist}_W f(t, x)$$

for $t \in [0, T]$. Evidently, it is nonnegative. One can show that $s(t)$ is continuous. Our requirement 1 implies that $s(0) = 0$. If $f(t, x)$ lies outside of W for some $(t, x) \in (0, T] \times M$, then $s(t)$ is not identically 0 on $[0, T]$. Assuming the assertion of the theorem fails to hold, we will prove the estimate $s^+(t) \leq Cs(t)$ for a fixed constant $C > 0$ and an arbitrary $t \in (0, T)$ such that $s(t) > 0$. Lemma 2.2 would then provide a contradiction.

Fix a point $t \in (0, T)$ satisfying $s(t) > 0$. When $x \in M$ is subject to $\text{dist}_W f(t, x) > 0$, the equality

$$\text{dist}_W f(t, x) = \sup_{\omega \in \partial W_x} \sup_{\lambda \in S_{\omega} W} \langle \lambda, f(t, x) - \omega \rangle_V$$

holds true. This implies

$$s(t) = \sup_{(\omega, \lambda) \in \Omega} \langle \lambda, f(t, \pi(\omega)) - \omega \rangle_V,$$

$$\Omega = \{(\omega, \lambda) \in V \times V \mid \omega \in \partial W_{\pi(\omega)}, \lambda \in S_{\omega} W\}.$$

The set Ω is compact in $V \times V$. Therefore, we can apply Lemma 9 in [14], see also Lemma 3.5 in [21], to conclude

$$\dot{s}^+(t) \leq \sup_{(\omega, \lambda) \in \Omega'} \frac{\partial}{\partial r} \langle \lambda, f(r, \pi(\omega)) - \omega \rangle_V|_{r=t},$$

$$\Omega' = \{(\omega, \lambda) \in \Omega \mid s(t) = \langle \lambda, f(t, \pi(\omega)) - \omega \rangle_V\}.$$

Fix a pair $(\omega, \lambda) \in \Omega'$. For brevity, we write x instead of $\pi(\omega)$. The point $x \in M$ is thus fixed from now on. Assuming the assertion of the theorem fails to hold, we will show that $\frac{\partial}{\partial r} \langle \lambda, f(r, x) - \omega \rangle_V|_{r=t} \leq Cs(t)$ for a constant $C > 0$ independent of t . This would yield the desired estimate $\dot{s}^+(t) \leq Cs(t)$.

Eq. (2.2) yields

$$\begin{aligned} & \frac{\partial}{\partial r} \langle \lambda, f(r, x) - \omega \rangle_V|_{r=t} \\ &= \langle \lambda, \Delta_V f(t, x) \rangle_V + \langle \lambda, \nabla_{\zeta(t, x)} f(t, x) \rangle_V + \langle \lambda, \phi(t, f(t, x)) \rangle_V. \end{aligned} \quad (2.4)$$

The inclusion $(\omega, \lambda) \in \Omega'$ implies that (t, x) is a maximal distance pair and the vector λ coincides with $\frac{\lambda(f(t, x))}{\|\lambda(f(t, x))\|_V}$. If the assertion of the theorem were incorrect, then either x would be in the interior of M or $\langle \lambda, \nabla_{v(x)} f(t, x) \rangle_V$ would be non-positive. Assuming this alternative, we will estimate each of the three terms in the right-hand side of Eq. (2.4).

Let us establish the equality $\langle \lambda, \nabla_{\chi} f(t, x) \rangle_V = 0$ for an arbitrary $\chi \in T_x M$. Obviously, it would imply

$$\langle \lambda, \nabla_{\zeta(t, x)} f(t, x) \rangle_V = 0. \quad (2.5)$$

At the first step, we consider a vector $\chi \in T_x M$ admitting a geodesic segment $\gamma_\chi(u)$ defined for $u \in [0, \epsilon_\chi]$ in such a way that $\gamma_\chi(0) = x$ and $\frac{d\gamma_\chi}{du}(0) = \chi$. The number ϵ_χ should be chosen small enough to ensure the geodesic segment's not intersecting itself. The initial goal is to show that $\langle \lambda, \nabla_\chi f(t, x) \rangle_V \leq 0$.

For the sake of brevity, we write $\gamma(u)$ instead of $\gamma_\chi(u)$ and ϵ instead of ϵ_χ . One can extend the vectors λ and ω to parallel (with respect to the connection ∇) sections $\lambda'(\gamma(u))$ and $\omega'(\gamma(u))$ of the bundle V defined along $\gamma(u)$. The covariant derivatives of $\lambda'(\gamma(u))$ and $\omega'(\gamma(u))$ with respect to ∇ at the point $x = \gamma(0)$ exist in the direction of χ . Writing $\nabla_\chi \lambda'(x)$ and $\nabla_\chi \omega'(x)$ for these covariant derivatives, we can easily see that $\nabla_\chi \lambda'(x) = 0$ and $\nabla_\chi \omega'(x) = 0$.

Introduce the function $g(u) = \langle \lambda'(\gamma(u)), f(t, \gamma(u)) - \omega'(\gamma(u)) \rangle_V$ on $[0, \epsilon]$. Obviously, $g(0) = s(t)$. Using the fact that the parallel transport is an isometry of the fibers, one proves $\omega'(\gamma(u)) \in \partial W_{\gamma(u)}$ and $\lambda'(\gamma(u)) \in S_{\omega'(\gamma(u))} W$ for any $u \in [0, \epsilon]$. These inclusions imply the inequality

$$g(0) = s(t) \geq \langle \lambda'(\gamma(u)), f(t, \gamma(u)) - \omega'(\gamma(u)) \rangle_V = g(u)$$

for any $u \in [0, \epsilon]$. As a consequence, the function $g(u)$ has a maximum at 0, and the one-sided derivative $\frac{dg}{du}(0)$ is non-positive. Since the connection ∇ is compatible with the fiber metric, we have the formula

$$\begin{aligned} \langle \lambda, \nabla_\chi f(t, x) \rangle_V &= \langle \nabla_\chi \lambda'(x), f(t, x) \rangle_V + \langle \lambda, \nabla_\chi f(t, x) \rangle_V \\ &= \frac{\partial}{\partial u} \langle \lambda'(\gamma(u)), f(t, \gamma(u)) \rangle_V \Big|_{u=0} = \frac{dg}{du}(0). \end{aligned}$$

Hence $\langle \lambda, \nabla_\chi f(t, x) \rangle_V \leq 0$.

Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $T_x M$. We will show that $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V = 0$ for any $k = 1, \dots, n$. Suppose x lies in the interior of M . Then a geodesic segment $\gamma_{e_k}(u)$, the parameter u varying through $[0, \epsilon_{e_k}]$,

subject to $\gamma_{e_k}(0) = x$ and $\frac{d\gamma_{e_k}}{du}(0) = e_k$ exists for any $k = 1, \dots, n$. As a consequence, the scalar products $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V$ are non-positive. Substituting $-e_k$ for e_k and repeating the argument, we conclude that the scalar products $\langle \lambda, \nabla_{-e_k} f(t, x) \rangle_V$ are also nonnegative. Thus $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V = 0$ for any $k = 1, \dots, n$.

Suppose x lies in the boundary of M . Without loss of generality, we assume e_n coincides with the inward normal to the boundary of M . It is easy to verify the existence of a geodesic segment $\gamma_{e_n}(u)$ defined for $u \in [0, \epsilon_{e_n}]$ in such a way that $\gamma_{e_n}(0) = x$ and $\frac{d\gamma_{e_n}}{du}(0) = e_n$. Consequently, the scalar product $\langle \lambda, \nabla_{e_n} f(t, x) \rangle_V$ is non-positive. At the same time, our hypothesis implies that $\langle \lambda, \nabla_{e_n} f(t, x) \rangle_V = -\langle \lambda, \nabla_{\nu(x)} f(t, x) \rangle_V$ is nonnegative. Thus $\langle \lambda, \nabla_{e_n} f(t, x) \rangle_V = 0$. Provided $n \geq 2$, we now prove that $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V = 0$ for $k = 1, \dots, n-1$. The situation is slightly more complicated here because a geodesic emanating from x in the direction of e_k does not necessarily exist. In order to overcome this problem, we will carry out an approximation procedure. Namely, fix a sequence $(e_k^m)_{m=1}^\infty$ converging to e_k for every $k = 1, \dots, n-1$. We choose the vectors e_k^m demanding that the scalar product of e_k^m and e_n with respect to the Riemannian metric in M be strictly positive. Given k and m , it is easy to verify the existence of a geodesic segment $\gamma_{e_k^m}(u)$, the parameter u varying through $[0, \epsilon_{e_k^m}]$, subject to $\gamma_{e_k^m}(0) = x$ and $\frac{d\gamma_{e_k^m}}{du}(0) = e_k^m$. As a consequence, $\langle \lambda, \nabla_{e_k^m} f(t, x) \rangle_V \leq 0$. The convergence of $(e_k^m)_{m=1}^\infty$ to e_k then implies $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V \leq 0$. Substituting $-e_k$ for e_k and repeating the argument, we conclude that $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V \geq 0$. Thus $\langle \lambda, \nabla_{e_k} f(t, x) \rangle_V = 0$ for $k = 1, \dots, n-1$.

By virtue of the established equalities, $\langle \lambda, \nabla_\chi f(t, x) \rangle_V = 0$ for an arbitrary $\chi \in T_x M$. This clearly proves formula (2.5).

Our next goal is to obtain the estimate

$$\langle \lambda, \Delta_V f(t, x) \rangle_V \leq 0. \quad (2.6)$$

As before, consider a vector $\chi \in T_x M$ admitting a geodesic segment $\gamma_\chi(u)$ defined for $u \in [0, \epsilon_\chi]$ in such a way that $\gamma_\chi(0) = x$ and $\frac{d\gamma_\chi}{du}(0) = \chi$. The number ϵ_χ should be small enough to ensure the absence of self-intersections. We now show that $\langle \lambda, \nabla_{\chi, \chi}^2 f(t, x) \rangle_V \leq 0$. This would provide us with a basis for the proof of estimate (2.6).

Again, we write $\gamma(u)$ instead of $\gamma_\chi(u)$ and ϵ instead of ϵ_χ . It will be convenient to use the notation $\gamma'(u)$ for $\frac{d\gamma}{du}(u)$. A parallel section $\lambda'(\gamma(u))$ of the bundle V along $\gamma(u)$ has been introduced above. The covariant derivative of this section with respect to the connection ∇ at the point $\gamma(u)$ exists in the direction of $\gamma'(u)$ for any $u \in [0, \epsilon]$. Writing $\nabla_{\gamma'(u)} \lambda'(\gamma(u))$ for this covariant derivative, we can easily see that $\nabla_{\gamma'(u)} \lambda'(\gamma(u)) = 0$ for any $u \in [0, \epsilon]$.

Since ∇ is compatible with the fiber metric, the equality

$$\begin{aligned} \langle \lambda, \nabla_{\chi, \chi}^2 f(t, x) \rangle_V &= \langle \nabla_\chi \lambda'(x), \nabla_\chi f(t, x) \rangle_V + \langle \lambda, \nabla_{\chi, \chi}^2 f(t, x) \rangle_V \\ &= \frac{\partial}{\partial u} \langle \lambda'(\gamma(u)), \nabla_{\gamma'(u)} f(t, \gamma(u)) \rangle_V \Big|_{u=0} \\ &= \frac{\partial}{\partial u} \left(\langle \nabla_{\gamma'(u)} \lambda'(\gamma(u)), f(t, \gamma(u)) \rangle_V \right. \\ &\quad \left. + \langle \lambda'(\gamma(u)), \nabla_{\gamma'(u)} f(t, \gamma(u)) \rangle_V \right) \Big|_{u=0} \\ &= \frac{\partial^2}{\partial u^2} \langle \lambda'(\gamma(u)), f(t, \gamma(u)) \rangle_V \Big|_{u=0} \\ &= \frac{d^2 g}{du^2}(0) \end{aligned}$$

holds true. The introduced above function $g(u)$ has a maximum at 0. It has been proven that $\frac{dg}{du}(0) = \langle \lambda, \nabla_\chi f(t, x) \rangle_V = 0$. Hence $\frac{d^2 g}{du^2}(0) \leq 0$, which yields $\langle \lambda, \nabla_{\chi, \chi}^2 f(t, x) \rangle_V \leq 0$.

Suppose x lies in the interior of M . Then every vector from the chosen above basis $\{e_1, \dots, e_n\}$ appears as a tangent vector for a certain geodesic segment emanating from x . As a consequence, $\langle \lambda, \nabla_{e_k, e_k}^2 f(t, x) \rangle_V \leq 0$ for every $k = 1, \dots, n$.

Suppose x lies in the boundary of M . Recall that e_n is assumed to coincide with the inward normal to the boundary of M . As mentioned before, e_n appears as a tangent vector for a certain geodesic segment emanating from x . Therefore, $\langle \lambda, \nabla_{e_n, e_n}^2 f(t, x) \rangle_V \leq 0$. Provided $n \geq 2$, we can approximate the other basis vectors with the previously fixed sequences $(e_k^m)_{m=1}^\infty$ to conclude that $\langle \lambda, \nabla_{e_k, e_k}^2 f(t, x) \rangle_V \leq 0$ for every $k = 1, \dots, n - 1$.

According to the definition of the Laplacian,

$$\langle \lambda, \Delta_V f(t, x) \rangle_V = \sum_{k=1}^n \langle \lambda, \nabla_{e_k, e_k}^2 f(t, x) \rangle_V.$$

By virtue of the established inequalities, all the terms in the right-hand side are non-positive. This clearly implies formula (2.6).

Finally, let us prove the estimate

$$\langle \lambda, \phi(t, f(t, x)) \rangle_V \leq Cs(t) \tag{2.7}$$

with a constant $C > 0$ independent of t . The vector λ belongs to $S_\omega W$. It must also belong to $S_{\omega(f(t, x))} W$, although ω does not necessarily coincide with $\omega(f(t, x))$. (Recall that $\omega(f(t, x))$ stands for the unique point in W_x closest to $f(t, x)$.) In accordance with our requirement 2, $\langle \lambda, \phi(t, \omega(f(t, x))) \rangle_V \leq 0$. Hence the estimate

$$\begin{aligned} \langle \lambda, \phi(t, f(t, x)) \rangle_V &\leq \langle \lambda, \phi(t, f(t, x)) \rangle_V - \langle \lambda, \phi(t, \omega(f(t, x))) \rangle_V \\ &\leq \|\phi(t, f(t, x)) - \phi(t, \omega(f(t, x)))\|_V \\ &\leq C\|f(t, x) - \omega(f(t, x))\|_V = Cs(t) \end{aligned}$$

holds with the constant $C > 0$ equal to the constant $C_\phi(f([0, T] \times M) \cup W) > 0$ given by formula (2.1). This concludes the proof of (2.7). Remark that the argument we used does not depend on whether x is in the boundary of M or in the interior of M .

Eq. (2.4) now provides $\frac{\partial}{\partial r} \langle \lambda, f(r, x) - \omega \rangle_V|_{r=t} \leq Cs(t)$. As mentioned before, this inequality implies $s^+(t) \leq Cs(t)$, which is impossible in view of Lemma 2.2. \square

Remark 2.3. The assumption on the mapping $\phi(t, v)$ imposed by the theorem may be slightly refined. Namely, it suffices to demand that the estimate $\langle \lambda, \phi(t, \omega) \rangle_V \leq 0$ hold when ω is equal to $\omega(f(t, x))$ and λ is equal to $\lambda(f(t, x))$ for all the maximal distance pairs $(t, x) \in (0, T) \times M$.

Remark 2.4. If the boundary of M were empty, then the suppositions of the theorem could not be satisfied simultaneously. In this case, requirements 1 and 2 ensure that $f(t, x)$ cannot lie outside of W . This fact is essentially equivalent to the maximum principle obtained in [21].

Remark 2.5. The theorem would prevail if the Riemannian metric in M and the connection ∇ fixed in V depended on the time parameter $t \in [0, T]$. Of course, then we would have to modify some of the assumptions imposed above. Firstly, the connection $\nabla(t)$ fixed in V at time t would be required to be compatible with the fiber metric $\langle \cdot, \cdot \rangle_V$ for all $t \in (0, T)$. Secondly, the set W would have to be invariant under the parallel translation with respect to $\nabla(t)$ for all $t \in (0, T)$. The details of defining the Laplacian and writing down Eq. (2.2) in the situation under discussion can be found in [12, Chapter 10]. The covariant derivative and the outward normal in formula (2.3) would have to be computed with respect to the connection $\nabla(t_{\text{pos}})$ and the Riemannian metric in M at time t_{pos} .

We will now formulate three immediate corollaries of Theorem 2.1. The fol-

lowing statement may be viewed as the basic maximum principle for sections of a vector bundle over a manifold with boundary.

Corollary 2.6. *Suppose the solution $f(t, x)$ and the mapping $\phi(t, v)$ meet requirements 1 and 2 of Theorem 2.1. If $f(t, x)$ lies in W for all $(t, x) \in (0, T) \times \partial M$, then $f(t, x)$ lies in W for all $(t, x) \in [0, T] \times M$.*

The following statement shows that the maximum principle of [21] holds for $f(t, x)$ provided that Neumann-type boundary conditions are imposed.

Corollary 2.7. *Suppose the solution $f(t, x)$ and the mapping $\phi(t, v)$ meet requirements 1 and 2 of Theorem 2.1. If the boundary condition*

$$\nabla_{v(x)} f(t, x) = 0$$

is satisfied for all $(t, x) \in (0, T) \times \partial M$, then $f(t, x)$ lies in W for all $(t, x) \in [0, T] \times M$.

Let $\bar{\lambda}(v)$ be a mapping of V into itself such that $\bar{\lambda}(v) \in V_{\pi(v)}$ for any $v \in V$. The following statement establishes an explicit connection between invariant sets of Eq. (2.2) and the boundary conditions specified for the solutions.

Corollary 2.8. *Suppose the solution $f(t, x)$ and the mapping $\phi(t, v)$ meet requirements 1 and 2 of Theorem 2.1. Suppose also $\bar{\lambda}(v) = \lambda(v)$ for any $v \in V$ lying outside of W . If the boundary condition*

$$\langle \bar{\lambda}(f(t, x)), \nabla_{v(x)} f(t, x) \rangle_V = 0$$

is satisfied for all $(t, x) \in (0, T) \times \partial M$, then $f(t, x)$ lies in W for all $(t, x) \in [0, T] \times M$.

It should be noted that both Corollary 2.6 and Corollary 2.7 can be deduced from Corollary 2.8.

CHAPTER 3

THE LI-YAU-HAMILTON INEQUALITY

As before, we consider a smooth, compact, connected, oriented, n -dimensional Riemannian manifold M with nonempty boundary ∂M . We suppose $n \geq 2$. This chapter aims to study the solutions of the heat equation on M with the Neumann boundary condition. More precisely, we will obtain two versions of the Li-Yau-Hamilton estimate for such solutions.

The Riemannian curvature tensor will be designated by $R(X, Y)Z$ when applied to the vectors X, Y , and Z from the tangent space $T_x M$ at the point $x \in M$. We use the usual notation

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle, \quad X, Y, Z, W \in T_x M.$$

The angular brackets with no lower index refer to the scalar product in the space $T_x M$ given by the Riemannian metric. The Ricci tensor will be written as $\text{Ric}(X, Y)$ when applied to $X, Y \in T_x M$. We will impose substantial assumptions on the curvature of M in Theorem 3.6 below.

The Levi-Civita connection D in the tangent bundle TM induces connections in the tensor bundles over M . We preserve the notation D for all of them. Our further arguments require introducing higher-order differential operators. Let us describe the corresponding procedure. Fix a tensor field T and two or more vector fields Y_1, \dots, Y_k on M . Set $D_{Y_1}^1 T$ equal to $D_{Y_1} T$. We define the k th covariant derivative $D_{Y_1, \dots, Y_k}^k T$ inductively by the formula

$$D_{Y_1, \dots, Y_k}^k T = D_{Y_k} \left(D_{Y_1, \dots, Y_{k-1}}^{k-1} T \right) - \sum_{i=1}^{k-1} D_{Y_1, \dots, Y_{i-1}, D_{Y_k} Y_i, Y_{i+1}, \dots, Y_{k-1}}^{k-1} T.$$

One can verify that the value of $D_{Y_1, \dots, Y_k}^k T$ at $x \in M$ does not depend on the values of Y_1, \dots, Y_k away from x .

If the point x lies in ∂M , then the space $T_x M$ contains the subspace $T_x \partial M$ tangent to ∂M . We write $\text{II}(X, Y)$ for the second fundamental form of ∂M applied to $X, Y \in T_x \partial M$. By definition, $\text{II}(X, Y) = \langle D_X \nu, Y \rangle$. Some of the statements below require that ∂M be totally geodesic. In this case, $\text{II}(X, Y) = 0$ for all $X, Y \in T_x \partial M$ at every point $x \in \partial M$.

Suppose the smooth positive function $p(t, x)$ defined on $(0, \infty) \times M$ solves the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) p(t, x) = 0, \quad t \in (0, \infty), \quad x \in M, \quad (3.1)$$

with the Neumann boundary condition

$$\frac{\partial}{\partial \nu} p(t, x) = 0, \quad t \in (0, \infty), \quad x \in \partial M. \quad (3.2)$$

The notation Δ_M represents the Laplace-Beltrami operator on M . It should be mentioned that Theorem 3.1 and Remark 3.5 below assume the inequality $\int_M p(t, x) dx \leq 1$ for all $t \in (0, \infty)$. Here and in what follows, the integration over a Riemannian manifold is to be carried out with respect to the Riemannian volume measure on the manifold.

We are now in a position to formulate the first result of this chapter. It establishes a general version of the Li-Yau-Hamilton estimate for the function $p(t, x)$.

Theorem 3.1. *Let the boundary ∂M be totally geodesic. Suppose the following statements hold:*

1. *The covariant derivative $(D_{\nu, \dots, \nu}^k R)(\nu, X, \nu, Y)$ is equal to 0 for all positive odd k and all $X, Y \in T_x M$ at every point $x \in \partial M$.*
2. *The integral $\int_M p(t, x) dx$ of the solution $p(t, x)$ to the boundary value problem (3.1)–(3.2) does not exceed 1 at any $t \in (0, \infty)$.*

Then there exist constants $A > 0$ and $B > 0$ independent of $p(t, x)$ such that the estimate

$$D_{x,x}^2 \log p(t, x) \geq - \left(\frac{1}{2t} + A \left(1 + \log \left(\frac{B}{t^{\frac{n}{2}} p(t, x)} \right) \right) \right) \langle X, X \rangle \quad (3.3)$$

holds for every $t \in (0, 1]$, $x \in M$, and $X \in T_x M$. (Recall that n is the dimension of M .)

Conceptually, the proof consists in doubling M to get a manifold without boundary and exploiting the results of [22]. A few technical aspects need to be handled. The most essential problem is to make sure the function to which we apply the theorem in [22] possesses the necessary differentiability properties.

Proof. Let \mathcal{M} be the double of M . More precisely, \mathcal{M} appears as the quotient $(M \times \{1, 2\}) / \sim$. The equivalence relation \sim is given as follows: Two distinct pairs, (x, i) and (y, j) , satisfy $(x, i) \sim (y, j)$ if and only if x coincides with y and lies in ∂M . We preserve the notation (x, i) for the equivalence class of $(x, i) \in M \times \{1, 2\}$. As described in [34], \mathcal{M} carries the canonical smooth structure. One may also obtain this structure by using Theorem 5.77 in [51] and the diffeomorphism $\mu(r, x)$ defined below. We explain further in the proof how to introduce a local coordinate system around $(x, i) \in \mathcal{M}$ when $x \in \partial M$. Note that \mathcal{M} is a manifold without boundary. The map $\mathcal{E}_i(x)$ taking $x \in M$ to $(x, i) \in \mathcal{M}$ is an embedding for both $i = 1$ and $i = 2$.

The Riemannian metric on M induces a Riemannian metric on \mathcal{M} in a natural fashion. More precisely, the scalar product $\langle X, Y \rangle_{\mathcal{M}}$ of the vectors $X, Y \in T_{(x,i)} \mathcal{M}$ is given by the formula $\langle X, Y \rangle_{\mathcal{M}} = \langle (d\mathcal{E}_i)^{-1} X, (d\mathcal{E}_i)^{-1} Y \rangle$. It is not difficult to verify that $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is well-defined at every $(x, i) \in \mathcal{M}$. The proposition in [34], along with Assumption 1 of our theorem, implies that $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ depends smoothly on $(x, i) \in \mathcal{M}$.

Introduce a positive function $\tilde{p}(t, z)$ on $(0, \infty) \times \mathcal{M}$ by setting $\tilde{p}(t, (x, i)) = \frac{1}{2}p(t, x)$. Its integral over the manifold \mathcal{M} is bounded by 1. Our next goal is to demonstrate that $\tilde{p}(t, z)$ solves the heat equation on \mathcal{M} . This would allow us to apply the results of [22] and obtain estimate (3.3) for this function. Theorem 3.1 would then follow as a direct consequence.

First and foremost, we need to prove that $\tilde{p}(t, z)$ is twice continuously differentiable in the second variable. Consider the set $\mathcal{M}^\partial \subset \mathcal{M}$ equal to $\mathcal{E}_1(\partial M)$. Of course, this set is also equal to $\mathcal{E}_2(\partial M)$. Using the smoothness of the function $p(t, x)$ on M , one can easily establish the smoothness of $\tilde{p}(t, z)$ outside of \mathcal{M}^∂ . In consequence, it suffices to show that $\tilde{p}(t, z)$ is twice continuously differentiable in a neighborhood of an arbitrarily picked point $\tilde{z} \in \mathcal{M}^\partial$.

There exists a unique $\tilde{x} \in \partial M$ satisfying $\tilde{z} = \mathcal{E}_1(\tilde{x}) = \mathcal{E}_2(\tilde{x})$. We need to introduce local coordinates in M around \tilde{x} . Suppose $\epsilon > 0$ is small enough to ensure that the mapping $\mu(r, x)$ defined on $[0, \epsilon) \times \partial M$ by the formula $\mu(r, x) = \exp_x(-r\nu)$ is a diffeomorphism onto its image. The existence of such an $\epsilon > 0$ is justified in [33, Chapter 11]. Fix a coordinate neighborhood U^∂ of \tilde{x} in the boundary ∂M with a local coordinate system y_1, \dots, y_{n-1} in U^∂ centered at \tilde{x} . Define the set U as the image of $[0, \epsilon) \times U^\partial$ under $\mu(r, x)$. Clearly, U is a neighborhood of \tilde{x} in M . We extend y_1, \dots, y_{n-1} to a coordinate system x_1, \dots, x_n in U by demanding that the equalities

$$\begin{aligned} x_k(\mu(r, x)) &= y_k(x), \quad x_n(\mu(r, x)) = r, \\ r &\in [0, \epsilon), \quad x \in U^\partial, \quad k = 1, \dots, n-1, \end{aligned}$$

hold true; cf. [34]. Importantly, $\frac{\partial}{\partial x_i}$ is tangent to the boundary on U^∂ for every $i = 1, \dots, n-1$. The vector field $\frac{\partial}{\partial x_n}$ coincides with $-\nu$ on this set.

The coordinate system x_1, \dots, x_n in U gives rise to a coordinate system z_1, \dots, z_n in the neighborhood $\mathcal{U} = \mathcal{E}_1(U) \cup \mathcal{E}_2(U)$ of \check{z} . Namely, suppose $z \in \mathcal{U}$ equals $\mathcal{E}_i(x)$ with $x \in U$. Define $z_k(z) = x_k(x)$ when $k = 1, \dots, n-1$ and $z_n(z) = (-1)^{i+1} x_n(x)$. We will now analyze the partial derivatives of $\tilde{p}(t, z)$ with respect to the newly introduced local coordinates. By doing so, we will establish the desired differentiability properties of this function.

It is easy to understand that $\frac{\partial}{\partial z_k} \tilde{p}(t, z)$ exists and coincides with $\frac{1}{2} \frac{\partial}{\partial x_k} p(t, x)$ if $z = (x, i) \in \mathcal{U}$ and $k = 1, \dots, n-1$. Furthermore, $\frac{\partial}{\partial z_k} \tilde{p}(t, z)$ is continuous on \mathcal{U} for these k . The situation is slightly more complicated when we differentiate with respect to the last coordinate. A straightforward argument shows

$$\frac{\partial}{\partial z_n} \tilde{p}(t, z) = \frac{(-1)^{i+1}}{2} \frac{\partial}{\partial x_n} p(t, x)$$

when $z = (x, i) \in \mathcal{U} \setminus \mathcal{M}^\partial$. The one-sided derivatives $\frac{\partial^+}{\partial z_n} \tilde{p}(t, z)$ and $\frac{\partial^-}{\partial z_n} \tilde{p}(t, z)$ coincide with $\frac{1}{2} \frac{\partial}{\partial x_n} p(t, x)$ and $-\frac{1}{2} \frac{\partial}{\partial x_n} p(t, x)$, respectively, if $z = (x, i) \in \mathcal{M}^\partial$. The boundary condition (3.2) ensures that $\frac{\partial}{\partial z_n} \tilde{p}(t, z)$ is well-defined and equal to 0 on \mathcal{M}^∂ . We conclude that $\frac{\partial}{\partial z_n} \tilde{p}(t, z)$ exists in \mathcal{U} . Furthermore, it is continuous on \mathcal{U} .

Let us turn our attention to the second derivatives. Analogous reasoning can be used here. The existence and the continuity of $\frac{\partial^2}{\partial z_k \partial z_l} \tilde{p}(t, z)$ on \mathcal{U} are clear for $k = 1, \dots, n-1$ and $l = 1, \dots, n$. In order to analyze $\frac{\partial^2}{\partial z_n \partial z_k} \tilde{p}(t, z)$ with $k = 1, \dots, n-1$, observe that the formula

$$\frac{\partial^+}{\partial z_n} \frac{\partial}{\partial z_k} \tilde{p}(t, z) = \frac{1}{2} \frac{\partial^2}{\partial x_n \partial x_k} p(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_n} p(t, x) = 0$$

holds when $z = (x, i) \in \mathcal{M}^\partial$. A similar calculation suggests the equality $\frac{\partial^-}{\partial z_n} \frac{\partial}{\partial z_k} \tilde{p}(t, z) = 0$ on \mathcal{M}^∂ . As a consequence, $\frac{\partial^2}{\partial z_n \partial z_k} \tilde{p}(t, z)$ is well-defined and continuous on \mathcal{U} . The same can be said about $\frac{\partial^2}{\partial z_n^2} \tilde{p}(t, z)$. Indeed, the formula

$$\frac{\partial^2}{\partial z_n^2} \tilde{p}(t, z) = \frac{(-1)^{2i+2}}{2} \frac{\partial^2}{\partial x_n^2} p(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x_n^2} p(t, x)$$

holds when $z = (x, i) \in \mathcal{U}$.

Summarizing the arguments above, we arrive at the following verdict: The function $\tilde{p}(t, z)$ is twice continuously differentiable in z on the manifold \mathcal{M} . The smoothness of $\tilde{p}(t, z)$ in t is evident. With this in mind, one can readily verify that the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta_{\mathcal{M}}\right)\tilde{p}(t, z) = 0, \quad t \in (0, \infty), z \in \mathcal{M}, \quad (3.4)$$

is satisfied ($\Delta_{\mathcal{M}}$ denoting the Laplace-Beltrami operator on \mathcal{M}). In addition, the integral of $\tilde{p}(t, z)$ over \mathcal{M} is bounded by 1. These observations enable us to apply Theorem 4.3 of [22]. As a result, we get the existence of constants $\tilde{A} > 0$ and $\tilde{B} > 0$ such that

$$\tilde{D}_{X,X}^2 \log \tilde{p}(t, z) \geq -\left(\frac{1}{2t} + \tilde{A} \left(1 + \log \left(\frac{\tilde{B}}{t^{\frac{n}{2}} \tilde{p}(t, z)}\right)\right)\right) \langle X, X \rangle$$

for every $t \in (0, 1]$, $z \in \mathcal{M}$, and $X \in T_z \mathcal{M}$. Here, $\tilde{D}_{X,X}^2$ refers to the second covariant derivative given by the Levi-Civita connection in $T\mathcal{M}$. Inequality (3.3) follows immediately with $A = \tilde{A}$ and $B = 2\tilde{B}$. \square

Remark 3.2. As in the proof of Theorem 3.1, let \mathcal{M} be the double of the manifold M . Given $z \in \mathcal{M}$, the tangent space $T_z \mathcal{M}$ carries a natural scalar product induced by the Riemannian metric on M . This scalar product depends smoothly on $z \in \mathcal{M}$ if and only if the boundary ∂M is totally geodesic and Assumption 1 of Theorem 3.1 is fulfilled. The justification of this fact can be found in [34].

Remark 3.3. Since the function $\tilde{p}(t, z)$ appearing in the proof satisfies (3.4), it must be smooth on $(0, \infty) \times \mathcal{M}$. In order to verify this, one may use the uniqueness and the integral representation of solutions to the heat equation; see, e.g., [25, Proposition 4.1.2].

Remark 3.4. Estimate (3.3) means that $D_{\cdot}^2 \log p(t, x)$ is greater than or equal to

$$-\left(\frac{1}{2t} + A \left(1 + \log \left(\frac{B}{t^{\frac{n}{2}} p(t, x)}\right)\right)\right) \langle \cdot, \cdot \rangle$$

in the sense of bilinear forms for every $t \in (0, 1]$ and $x \in M$.

Remark 3.5. If Assumption 2 of Theorem 3.1 is fulfilled, then there exists a constant $C > 0$ independent of $p(t, x)$ such that

$$p(t, x) \leq Ct^{-\frac{n}{2}}, \quad t \in (0, 1], x \in M. \quad (3.5)$$

Note that ∂M does not have to be totally geodesic for this to hold. In the case where $p(t, x)$ tends to a delta function as t tends to 0, formula (3.5) follows from the parametrix construction for the Neumann heat kernel. This observation was made in [24, Proof of Lemma 3.2]. We also refer to [49] for relevant results. In the general case, formula (3.5) can be established by using the integral representation of the solution to the heat equation; see, e.g., [25, Proposition 4.1.2]. Importantly, if all the assumptions of Theorem 3.1 are fulfilled and C satisfies (3.5), then there exists a constant $A_C > 0$ such that (3.3) holds with $A = A_C$ and $B = C$.

We now state a more specific version of the Li-Yau-Hamilton estimate for the function $p(t, x)$. It shows how (3.3) simplifies when the appropriate curvature restrictions are imposed on M away from the boundary. Note that the inequality $\int_M p(t, x) dx \leq 1$ is no longer required for our arguments.

Theorem 3.6. *Let the boundary ∂M be totally geodesic. Suppose the following statements hold at every point $x \in M$:*

1. *The covariant derivative $(D_X \text{Ric})(Y, Z)$ is equal to 0 for all $X, Y, Z \in T_x M$.*
2. *The sectional curvature of every plane in $T_x M$ is nonnegative. That is, $R(X, Y, Y, X) \geq 0$ for all $X, Y \in T_x M$.*

Then the solution $p(t, x)$ of the boundary value problem (3.1)–(3.2) satisfies the inequality

$$D_{X,X}^2 \log p(t, x) \geq -\frac{1}{2t} \langle X, X \rangle \quad (3.6)$$

for every $t \in (0, \infty)$, $x \in M$, and $X \in T_x M$.

In many situations, estimate (3.6) can be established by the same technique we used to establish Theorem 3.1. One just has to exploit Corollary 4.4 in [22] instead of Theorem 4.3 in [22]. However, we prefer to adduce a direct method of proving (3.6) here based on the Hopf lemma for vector bundle sections. Firstly, because this method does not require the equality $(D_{\nu, \dots, \nu}^k R)(\nu, X, \nu, Y) = 0$ to hold on ∂M . Secondly, because it avoids using the results of [22]. Last but not least, we believe the direct method is more illuminating and gives a more fertile ground for generalizations.

Proof. Take a number $\epsilon > 0$. Given $t \in [0, \infty)$, introduce the two times covariant tensor field L_t^ϵ by the formula

$$L_t^\epsilon(X, Y) = (t + \epsilon)D_{X, Y}^2 \log p(t + \epsilon, x) + \frac{1}{2}\langle X, Y \rangle, \quad X, Y \in T_x M.$$

Our plan is to use the Hopf boundary point lemma of Chapter 2 for showing that L_t^ϵ is positive semidefinite at every point of M . The theorem will then be proved by taking the limit as ϵ goes to 0.

In what follows, we assume $p(t, x)$ is defined and smooth on $[0, \infty) \times M$. This does not lead to any loss of generality. Indeed, we can always establish the desired estimate for the function $p_\delta(t, x) = p(t + \delta, x)$, $\delta > 0$, and pass to the limit as δ tends to 0.

Firstly, let us compute $(\frac{\partial}{\partial t} - \Delta_{\text{tens}})L_t^\epsilon$. The Laplacian Δ_{tens} in this expression appears as the trace of the second covariant derivative D^2 in the bundle $T^*M \otimes T^*M$. Recall that the connection in this bundle is induced by the Levi-Civita connection in TM .

The Riemannian metric on M yields a scalar product of tensors over a point $x \in M$. The notation $\langle \cdot, \cdot \rangle$ is preserved for this scalar product. Set $P^\epsilon(t, x) = \text{grad log } p(t + \epsilon, x)$. We omit the (t, x) at $P^\epsilon(t, x)$ when this does not lead to ambiguity. Introduce the mapping $\Phi(t, w)$ acting from $[0, \infty) \times (T_x^*M \otimes T_x^*M)$ to $T_x^*M \otimes T_x^*M$ by the equality

$$\begin{aligned} \Phi(t, w)(X, Y) &= 2\langle \mathcal{R}_{X,Y}, w \rangle - \langle \iota_X \text{Ric}, \iota_Y w \rangle - \langle \iota_Y \text{Ric}, \iota_X w \rangle \\ &\quad + \frac{2}{t + \epsilon} \langle \iota_X w, \iota_Y w \rangle + 2(t + \epsilon)R(X, P^\epsilon, P^\epsilon, Y) \\ &\quad - \frac{1}{t + \epsilon} w(X, Y), \quad X, Y \in T_x M. \end{aligned}$$

Here, the tensor $\mathcal{R}_{X,Y}$ is defined as $\mathcal{R}_{X,Y}(Z, W) = R(X, Z, W, Y)$ for $Z, W \in T_x M$, and ι denotes the interior product. A standard calculation, together with Assumption 1 of our theorem, shows that

$$\left(\frac{\partial}{\partial t} - \Delta_{\text{tens}} \right) L_t^\epsilon = D_{2p^\epsilon} L_t^\epsilon + \Phi(t, L_t^\epsilon), \quad t \in [0, \infty),$$

at every $x \in M$. For relevant arguments, see [22, 13, 6] and [15, Section 2.5].

Let $W \subset T^*M \otimes T^*M$ be the set of two times covariant, symmetric, positive semidefinite tensors. Suppose ϵ is chosen sufficiently small to ensure that L_t^ϵ belongs to W at every point of M when $t = 0$. The existence of such an ϵ follows from the smoothness of $p(t, x)$ on $[0, \infty) \times M$. Fixing $T > 0$, we will apply Theorem 2.1 (the Hopf lemma) to demonstrate that L_t^ϵ must belong to W at every point of M for all $t \in [0, T]$.

Some more notation has to be introduced here. Given $x \in M$, define the set W_x as the intersection of W with $T_x^*M \otimes T_x^*M$. Evidently, W_x is closed and convex in $T_x^*M \otimes T_x^*M$. Let $\omega(w)$ stand for the point in W_x nearest to $w \in T_x^*M \otimes T_x^*M$. More precisely, the minimum of the scalar product $\langle w - v, w - v \rangle$ over $v \in W_x$ must be attained at $v = \omega(w)$. Denote $\lambda(w) = w - \omega(w)$.

We now verify the assumptions of Theorem 2.1. It was already noted that $L_t^\epsilon \in W$ at every point of M when $t = 0$ and that W_x was closed and convex for all $x \in M$. The set W is invariant under the parallel translation in $T^*M \otimes T^*M$; see [12, The arguments preceding Corollary 10.12]. The mapping $\Phi(t, w)$, obviously, satisfies inequality (2.1). Thus, Requirement 2 of Theorem 2.1 remains the only statement to be checked. Considering Remark 2.3, it suffices to prove the inequality

$$\langle \Phi(t, \omega(L_t^\epsilon)), \lambda(L_t^\epsilon) \rangle \leq 0, \quad t \in [0, T], \quad (3.7)$$

over every point of M .

Fix $t \in [0, T]$. We omit the subscript t at L_t^ϵ in order to simplify the notation. Pick an orthonormal basis $\{e_1, \dots, e_n\}$ of the space $T_x M$ for some $x \in M$. Without loss of generality, suppose this basis diagonalizes L^ϵ at x . One can easily understand that

$$\begin{aligned} \omega(L^\epsilon)(e_i, e_j) &= \max\{L^\epsilon(e_i, e_j), 0\}, \\ \lambda(L^\epsilon)(e_i, e_j) &= \min\{L^\epsilon(e_i, e_j), 0\}, \quad i, j = 1, \dots, n. \end{aligned}$$

Hence

$$\langle \Phi(t, \omega(L^\epsilon)), \lambda(L^\epsilon) \rangle = \sum_{i=1}^n \Phi(t, \omega(L^\epsilon))(e_i, e_i) \min\{L^\epsilon(e_i, e_i), 0\}.$$

If $L^\epsilon(e_i, e_i) < 0$, then $\omega(L^\epsilon)(e_i, e_j) = 0$ for all $j = 1 \dots, n$. Using this fact along with our Assumption 2, one can readily prove that

$$\Phi(t, \omega(L^\epsilon))(e_i, e_i) \geq 0$$

when $L^\epsilon(e_i, e_i) < 0$. Thus, estimate (3.7) holds true.

We are now in a position to apply Theorem 2.1. More precisely, we apply Corollary 2.8 of that theorem. Let us establish the equality $\langle \lambda(L_t^\epsilon), D_\nu L_t^\epsilon \rangle = 0$ over

an arbitrarily chosen point $x \in \partial M$ for all $t \in [0, T]$. This would lead us to the conclusion that L_t^ϵ is always positive semidefinite.

As before, we fix $t \in [0, T]$ and write L^ϵ instead of L_t^ϵ . Pick an orthonormal basis $\{v_1, \dots, v_{n-1}\}$ of the space $T_x \partial M$ tangent to the boundary. Suppose this basis diagonalizes the restriction of L^ϵ to $T_x \partial M \otimes T_x \partial M$. A straightforward verification shows

$$L^\epsilon(v_i, v) = -(t + \epsilon) \text{II}(v_i, P^\epsilon), \quad i = 1, \dots, n-1.$$

(Remark that P^ϵ is tangent to ∂M due to the Neumann boundary condition (3.2).) The right-hand side of the above formula is equal to 0 because ∂M is totally geodesic. Hence $L^\epsilon(v_i, v) = 0$ for $i = 1, \dots, n-1$. We conclude that the orthonormal basis $\{v_1, \dots, v_{n-1}, v\}$ diagonalizes L^ϵ at x and

$$\begin{aligned} \langle \lambda(L^\epsilon), D_\nu L^\epsilon \rangle &= \sum_{i=1}^{n-1} \min\{L^\epsilon(v_i, v_i), 0\} (D_\nu L^\epsilon)(v_i, v_i) \\ &\quad + \min\{L^\epsilon(v, v), 0\} (D_\nu L^\epsilon)(v, v). \end{aligned} \quad (3.8)$$

Each of the summands on the right-hand side of (3.8) is 0. Indeed, since ∂M is totally geodesic, we can introduce the normal coordinates x_1, \dots, x_n around x so that $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_n}$ coincide with v_i and $-v$, respectively, at the origin. A calculation in these coordinates yields

$$\begin{aligned} (D_\nu L^\epsilon)(v_i, v_i) &= -(t + \epsilon)(D_{v_i}(t P^\epsilon \text{II}))(v_i) \\ &\quad - (t + \epsilon) \text{II}(v_i, D_{v_i} P^\epsilon) \\ &\quad + (t + \epsilon) R(v_i, P^\epsilon, v_i, v), \quad i = 1, \dots, n-1. \end{aligned} \quad (3.9)$$

(The vector $D_{v_i} P^\epsilon$ is tangent to the boundary because $\langle D_{v_i} P^\epsilon, v \rangle = \frac{1}{t+\epsilon} L^\epsilon(v_i, v) = 0$.) The second fundamental form II vanishes identically. Therefore, the first two terms in (3.9) equal 0. Given $X, Y, Z \in T_x \partial M$, it is easy to see that $R(X, Y)Z$

coincides with the Riemannian curvature tensor of ∂M applied to these vectors. Hence $R(v_i, P^\epsilon)v_i$ is tangent to ∂M , and the third term in (3.9) equals 0, as well. As a result, $(D_\nu L^\epsilon)(v_i, v_i) = 0$ for $i = 1 \dots, n - 1$.

Another calculation (cf. [31]) yields

$$\begin{aligned} (D_\nu L^\epsilon)(\nu, \nu) &= (t + \epsilon) \frac{\partial}{\partial \nu} \Delta_M \log p(t + \epsilon, x) - \sum_{i=1}^{n-1} (D_\nu L^\epsilon)(v_i, v_i) \\ &= (t + \epsilon) \frac{\partial}{\partial \nu} \Delta_M \log p(t + \epsilon, x) = 2(t + \epsilon) \Pi(P^\epsilon, P^\epsilon). \end{aligned}$$

Since Π vanishes identically, the above implies $(D_\nu L^\epsilon)(\nu, \nu) = 0$. In view of (3.8), we conclude $\langle \lambda(L^\epsilon), D_\nu L^\epsilon \rangle$ equals 0 over our arbitrarily chosen $x \in \partial M$.

Corollary 2.8 of Theorem 2.1 now suggests that L_t^ϵ is positive semidefinite at every point of M for all $t \in [0, T]$. Since no restrictions were imposed on the number T , this tensor field must be positive semidefinite at every point for all $t \in [0, \infty)$. Taking the limit as ϵ tends to 0 proves (3.6). □

CHAPTER 4
THE YANG-MILLS HEAT EQUATION

This chapter aims to study the solutions to the Yang-Mills heat equation in a vector bundle over the manifold M . Roughly speaking, we show that the curvature of such a solution is bounded if the dimension of M is less than 4 or if the initial energy is sufficiently small. The proofs utilize a probabilistic method. When the dimension of M is greater than or equal to 5, our technique requires the Li-Yau-Hamilton estimate established in Chapter 3. Notably, this reflects on the assumptions we impose on the geometry of M .

Many statements below demand that the boundary ∂M be convex. The concept of convexity is quite delicate for Riemannian manifolds. Different definitions and the relations between them are surveyed in [42]. In what follows, when saying ∂M is convex, we mean that the formula

$$\text{II}(X, X) \geq 0, \quad X \in T\partial M, \quad (4.1)$$

must hold for the second fundamental form of ∂M .

The next few paragraphs provide a description of the structure required to formulate the Yang-Mills heat equation. Additional references on the background material include [5, 30, 27, 17, 18].

Recall that the manifold M is assumed to be compact. Let E be a vector bundle over M with the standard fiber \mathbb{R}^d and the structure group G . We suppose G appears as a Lie subgroup of $O(d)$ and acts naturally on \mathbb{R}^d . The symbol \mathfrak{g} stands for the Lie algebra of G . In what follows, we assume \mathbb{R}^d is equipped with the standard scalar product. Every element of \mathfrak{g} appears as a skew-symmetric en-

isomorphism of \mathbb{R}^d . Define the scalar product in this Lie algebra by the formula

$$\langle A, B \rangle_{\mathfrak{g}} = -\text{trace } AB, \quad A, B \in \mathfrak{g}.$$

The adjoint bundle $\text{Ad } E$, whose standard fiber is equal to \mathfrak{g} , carries the fiber metric induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Let ∇ be a connection in E . We suppose ∇ is compatible with the structure group G . The curvature of ∇ will be denoted by R^∇ . Let us mention that R^∇ appears as a 2-form on M with its values in the bundle $\text{Ad } E$. Our goal is to write down the Yang-Mills heat equation. In order to do this, we need to introduce the operators of covariant exterior differentiation corresponding to a connection in E .

Consider the bundle $\Lambda^p T^*M \otimes \text{Ad } E$ for a nonnegative integer p . Its sections are interpreted as $\text{Ad } E$ -valued p -forms on the manifold M . The set of all these sections will be designated by $\Omega^p(\text{Ad } E)$. The Riemannian metric on M and the fiber metric in $\text{Ad } E$ give rise to a scalar product in the fibers of $\Lambda^p T^*M \otimes \text{Ad } E$. We use the notation $\langle \cdot, \cdot \rangle_E$ for this scalar product and the notation $|\cdot|_E$ for the corresponding norm.

The connections D in TM and ∇ in E induce a connection in the bundle $\Lambda^p T^*M \otimes \text{Ad } E$. It appears as a mapping from $\Omega^p(\text{Ad } E)$ to the set of sections of $T^*M \otimes \Lambda^p T^*M \otimes \text{Ad } E$. We preserve the notation ∇ for this connection in $\Lambda^p T^*M \otimes \text{Ad } E$. Define the operator d_∇ acting from $\Omega^p(\text{Ad } E)$ to $\Omega^{p+1}(\text{Ad } E)$ by the formula

$$\begin{aligned} (d_\nabla \phi)(X_1, \dots, X_{p+1}) \\ = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i} \phi)(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{p+1}). \end{aligned}$$

Here, ϕ belongs to $\Omega^p(\text{Ad } E)$, and X_1, \dots, X_{p+1} belong to $T_x M$ for some $x \in M$. It is easy to understand that d_{∇} plays the role of the covariant exterior derivative corresponding to ∇ . The operator d_{∇}^* acting from $\Omega^{p+1}(\text{Ad } E)$ to $\Omega^p(\text{Ad } E)$ is defined by the equality

$$(d_{\nabla}^* \psi)(X_1, \dots, X_p) = - \sum_{i=1}^n (\nabla_{e_i} \psi)(e_i, X_1, \dots, X_p).$$

Here, ψ belongs to $\Omega^{p+1}(\text{Ad } E)$, the vectors X_1, \dots, X_p belong to $T_x M$ for some $x \in M$, and $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x M$. We set d_{∇}^* to be equal to zero on $\Omega^0(\text{Ad } E)$. In view of Lemma 4.4 below, this operator may be understood as the formal adjoint of d_{∇} .

Fix a number $T > 0$. Consider a connection $\nabla(t)$ in E depending on $t \in [0, T)$. The parameter t will be interpreted as time. We require that $\nabla(t)$ be compatible with the structure group G for all $t \in [0, T)$. Suppose $\nabla(t)$ satisfies the Yang-Mills heat equation

$$\frac{\partial}{\partial t} \nabla(t) = -\frac{1}{2} d_{\nabla(t)}^* R^{\nabla(t)}, \quad t \in [0, T). \quad (4.2)$$

In particular, this connection must be once continuously differentiable in $t \in [0, T)$. The factor $\frac{1}{2}$ appears in the right-hand side because we want to achieve maximum conformity with the probabilistic results employed below. In interpreting $\frac{\partial}{\partial t} \nabla(t)$, one should remember that $\nabla(t)$ lies, for each $t \in [0, T)$, in the linear space of mappings taking sections of E to sections of $T^* M \otimes E$. Our next step is to specify the boundary conditions for $\nabla(t)$. Doing this is quite a delicate matter. We discuss some of the nuances in Remarks 4.11 and 4.12 in the end of this chapter.

Every $\text{Ad } E$ -valued p -form $\phi \in \Omega^p(\text{Ad } E)$ can be decomposed into the sum of its tangential component ϕ_{tan} and its normal component ϕ_{norm} on the boundary of M . Roughly speaking, ϕ_{tan} coincides with the restriction of ϕ to the vectors

from $T\partial M$. If ϕ lies in $\Omega^0(\text{Ad } E)$, then ϕ_{tan} equals ϕ on ∂M . We are now ready to impose the boundary conditions on $\nabla(t)$. Assume the equalities

$$\left(R^{\nabla(t)}\right)_{\text{tan}} = 0, \quad \left(d_{\nabla(t)}^* R^{\nabla(t)}\right)_{\text{tan}} = 0 \quad (4.3)$$

hold on ∂M for all $t \in [0, T)$. One should view (4.3) as a version of the relative boundary conditions on real-valued forms; see, for example, [39]. Alternatively, we may assume the formulas

$$\left(R^{\nabla(t)}\right)_{\text{norm}} = 0, \quad \left(d_{\nabla(t)} R^{\nabla(t)}\right)_{\text{norm}} = 0 \quad (4.4)$$

hold on ∂M for all $t \in [0, T)$. (Actually, the second one is always satisfied due to the Bianchi identity.) These should be viewed as a version of the absolute boundary conditions; again, [39] is a good reference. The arguments in this chapter will prevail regardless of whether we choose Eqs. (4.3) or Eqs. (4.4) to hold on ∂M . For other problems and techniques, however, only one of the choices may be appropriate.

We should make an important comment at this point. In essence, Eqs. (4.3) and (4.4) are restrictions on the curvature form $R^{\nabla(t)}$. Another possible strategy is to impose the boundary conditions directly on the connection $\nabla(t)$. We postpone a discussion of this issue until after the proofs of our results; see Remarks 4.11 and 4.12.

Introduce the function

$$\text{YM}(t) = \int_M |R^{\nabla(t)}|_E^2 dx$$

for $t \in [0, T)$. In accordance with the conventions of Chapter 3, the integration is to be carried out with respect to the Riemannian volume measure on M . It is reasonable to call $\text{YM}(t)$ the energy at time t . A standard argument involving

Lemma 4.4 below shows that $\text{YM}(t)$ is non-increasing in $t \in [0, T]$; see [7] and also, for example, [27, 38, 8].

We now state the main results of Chapter 4. Our first theorem concerns the lower-dimensional case. It offers a bound for $R^{\nabla(t)}$ in terms of the initial energy $\text{YM}(0)$ and demonstrates that $R^{\nabla(t)}$ does not blow up at time T . In what follows, the notation $R^{\nabla(t)}(x)$ refers to the curvature of $\nabla(t)$ at the point $x \in M$.

Theorem 4.1. *Let the dimension $\dim M$ equal 2 or 3. Suppose ∂M is convex in the sense of (4.1). Then the solution $\nabla(t)$ of Eq. (4.2), subject to the boundary conditions (4.3) or (4.4), satisfies the estimate*

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{4 \text{YM}(0)}{\rho^2}, \theta_1 e^{\theta_2 \sqrt{\text{YM}(0)}} \text{YM}(0) \right\} \quad (4.5)$$

for all $\rho \in (0, T)$. Here, $\theta_1 > 0$ and $\theta_2 > 0$ are constants depending only on the manifold M .

A similar result can be obtained in dimension 4 provided that the initial energy $\text{YM}(0)$ is smaller than a certain value ξ . We emphasize that ξ depends on nothing but M .

Theorem 4.2. *Let the dimension $\dim M$ equal 4. Suppose the boundary ∂M is convex in the sense of (4.1). Then there exists a constant $\xi > 0$ depending only on the manifold M and satisfying the following statement: The solution $\nabla(t)$ of Eq. (4.2) with the boundary conditions (4.3) or (4.4) obeys the estimate*

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{4 \sqrt{\text{YM}(0)}}{\rho^2}, \sqrt{\text{YM}(0)} \right\}, \quad \rho \in (0, T), \quad (4.6)$$

if the initial energy $\text{YM}(0)$ is smaller than ξ .

We turn our attention to dimensions 5 and higher. In this case, the proof of

the result will require the Li-Yau-Hamilton estimate established in Chapter 3. This forces us to impose stronger geometric assumptions on the manifold M .

The following theorem yields a bound on $R^{\nabla(\rho)}$ provided $\text{YM}(0)$ is smaller than a certain value $\xi(\rho)$ depending on $\rho \in [0, T)$. This result implies that the curvature of a solution to Eq. (4.2) cannot blow up after time ρ if the initial energy does not exceed $\xi(\rho)$. In the above setting, the connection $\nabla(t)$ is defined for each $t \in [0, T)$ and depends differentiably on t on this interval. Therefore, $R^{\nabla(t)}$ does not blow up at time T if $\text{YM}(0) < \xi(\rho)$ for some $\rho \in (0, T)$.

Theorem 4.3. *Let the dimension $\dim M$ be greater than or equal to 5. Suppose the boundary ∂M is totally geodesic. Moreover, suppose either Assumption 1 of Theorem 3.1 or Assumptions 1 and 2 of Theorem 3.6 are fulfilled for M . Then there exists a positive non-decreasing function $\xi(s)$ on $(0, \infty)$ that depends on nothing but M and satisfies the following statement: Given $\rho \in (0, T)$, the solution $\nabla(t)$ of Eq. (4.2) with the boundary conditions (4.3) or (4.4) obeys the estimate*

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{16 \sqrt{\text{YM}(0)}}{\rho^2}, \sqrt{\text{YM}(0)} \right\} \quad (4.7)$$

if the initial energy $\text{YM}(0)$ is smaller than $\xi(\rho)$.

The assertions of Theorems 4.1, 4.2, and 4.3 may be refined. We present them here in the less general form in order to ensure that the technical details do not obscure the qualitative meaning. The possible refinements are explained in Remarks 4.6, 4.7, and 4.10.

To prove the three theorems above, we employ the probabilistic technique developed in [2]. The main stochastic process to be used for our arguments is a reflecting Brownian motion on the manifold M . Its transition density is the

Neumann heat kernel on M . Before introducing the probabilistic machinery, we need to state two geometric results.

First of all, it is necessary to formulate a version of the integration by parts formula. Let us recollect some conventions and notation. The boundary of M carries a natural Riemannian metric inherited from M . The orientation of ∂M is induced by that of M . The integration over ∂M is to be carried out with respect to the Riemannian volume measure on ∂M . We write ν for the outward unit normal vector field on the boundary. The letter ι stands for the interior product.

We are now ready to lay down integration by parts formula. Our source for this result is the paper [7].

Lemma 4.4. *Let ∇ be a connection in E compatible with the structure group G . Consider $\text{Ad } E$ -valued forms $\phi \in \Omega^p(\text{Ad } E)$ and $\psi \in \Omega^{p+1}(\text{Ad } E)$ with $p = 0, \dots, \dim M - 1$. The equality*

$$\int_M (\langle d_\nabla \phi, \psi \rangle_E - \langle \phi, d_\nabla^* \psi \rangle_E) dx = \int_{\partial M} \langle \phi, \iota_\nu \psi \rangle_E dx$$

holds true.

As mentioned above, an argument involving Lemma 4.4 proves that $\text{YM}(t)$ is non-increasing in $t \in [0, T)$; see, for instance, [38, 7]. This fact is crucial for our further considerations.

The next step is to understand what Eqs. (4.3) and (4.4) can tell us about the behavior of $|R^{\nabla(t)}(x)|_E^2$ near the boundary of M . In order to do this, we present the following result. It may be viewed as a variant of Lemma 3.1¹ in [7] for manifolds with convex boundary. The proof utilizes a computation carried out

¹This statement was labeled Lemma 3.1 in a preliminary version of [7]. It may appear under a different tag in the final manuscript.

in [7]. Given $\phi \in \Omega^p(\text{Ad } E)$ and $x \in M$, the notation $\phi(x)$ refers to the restriction of ϕ to $(T_x M)^p$.

Lemma 4.5. *Let the boundary ∂M be convex in the sense of (4.1). Suppose ∇ is a connection in E compatible with the structure group G . Consider an $\text{Ad } E$ -valued p -form $\phi \in \Omega^p(\text{Ad } E)$ with $p = 0, \dots, \dim M$. If either the equations*

$$\phi_{\text{tan}} = 0, \quad (d_{\nabla}^* \phi)_{\text{tan}} = 0 \quad (4.8)$$

or the equations

$$\phi_{\text{norm}} = 0, \quad (d_{\nabla} \phi)_{\text{norm}} = 0 \quad (4.9)$$

are satisfied on ∂M , then the formula

$$\frac{\partial}{\partial \nu} |\phi(x)|_E^2 \leq 0, \quad x \in \partial M, \quad (4.10)$$

holds true.

Proof. We begin by selecting a local coordinate system on M convenient for our arguments. Choose a point $\tilde{x} \in \partial M$. Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of the space $T_{\tilde{x}} \partial M$ such that

$$\Pi(e_i, e_j) = \delta_i^j \lambda_i, \quad i, j = 1, \dots, n-1.$$

In this formula, δ_i^j is the Kronecker symbol, and λ_i are the principal curvatures at \tilde{x} . Since ∂M is convex, λ_i must be nonnegative for all $i = 1, \dots, n-1$. Take a coordinate neighborhood U^∂ of \tilde{x} in ∂M with a coordinate system y_1, \dots, y_{n-1} in U^∂ centered at \tilde{x} . We assume $\frac{\partial}{\partial y_i}$ coincides with e_i at \tilde{x} for each $i = 1, \dots, n-1$. As in the proof of Theorem 3.1, consider the mapping $\mu(r, x)$ defined on $[0, \epsilon) \times \partial M$ by the formula $\mu(r, x) = \exp_x(-r\nu)$. The number $\epsilon > 0$ is chosen small enough for $\mu(r, x)$ to be a diffeomorphism onto its image. The set $U = \mu([0, \epsilon) \times U^\partial)$ is

a neighborhood of \tilde{x} in the manifold M . We extend y_1, \dots, y_{n-1} to a coordinate system x_1, \dots, x_n in U by demanding that the equalities

$$\begin{aligned} x_k(\mu(r, x)) &= y_k(x), \quad x_n(\mu(r, x)) = r, \\ r &\in [0, \epsilon), \quad x \in U^\partial, \quad k = 1, \dots, n-1, \end{aligned}$$

hold true; cf. [34]. The vector $\frac{\partial}{\partial x_i}$ coincides with e_i at \tilde{x} for each $i = 1, \dots, n-1$. It is easy to see that $\frac{\partial}{\partial x_i}$ is tangent to the boundary on the set U^∂ for $i = 1, \dots, n-1$. The vector field $\frac{\partial}{\partial x_n}$ coincides with $-v$ at every point of U^∂ .

Having fixed a suitable local coordinate system on M , we now proceed to the actual proof of the lemma. Without loss of generality, suppose Eqs. (4.9) hold for ϕ on ∂M . If this is not the case and Eqs. (4.8) hold instead, we can replace ϕ with the form $*\phi$ satisfying (4.9). (The symbol $*$ denotes the Hodge star operator.) Since $|\phi(x)|_E$ equals $|*\phi(x)|_E$ for all $x \in M$, proving the lemma for $*\phi$ would suffice.

From the technical point of view, it is convenient for us to assume that ϕ belongs to $\Omega^p(\text{Ad } E)$ with p between 1 and $\dim M$. This restriction is not significant. Indeed, if ϕ is an $\text{Ad } E$ -valued 0-form on M , then estimate (4.10) follows directly from the second formula in (4.9).

Our next step is to write down an expression for the derivative $\frac{\partial}{\partial v} |\phi(x)|_E^2$ using the coordinate system introduced above. Observe that, in the neighborhood U of the point \tilde{x} , one can represent ϕ by the equality

$$\phi(x) = \alpha(x) \wedge dx_n + \beta(x).$$

Here, α and β are $\text{Ad } E$ -valued forms defined on U and given by the formulas

$$\alpha(x) = \sum \alpha_I(x) dx^I, \quad \beta(x) = \sum \beta_J(x) dx^J.$$

The sums are taken over all the multi-indices $I = (i_1, \dots, i_{p-1})$ and $J = (j_1, \dots, j_p)$ with $1 \leq i_1 < \dots < i_{p-1} < n$ and $1 \leq j_1 < \dots < j_p < n$. The mappings $\alpha_I(x)$ and $\beta_J(x)$ defined on U are local sections of the bundle $\text{Ad } E$. The notations dx^I and dx^J refer to $dx_{i_1} \wedge \dots \wedge dx_{i_{p-1}}$ and $dx_{j_1} \wedge \dots \wedge dx_{j_p}$. If $p = 1$, then α should be interpreted as an $\text{Ad } E$ -valued 0-form on U . If $p = n$, then β equals zero.

Following the computation from [7, Proof of Lemma 3.1], we arrive at the formula

$$\frac{1}{2} \frac{\partial}{\partial v} |\phi(x)|_E^2 = \sum_{x \in U \cap \partial M} \langle \beta_J(x), \beta_K(x) \rangle_E \langle D_v dx^J, dx^K \rangle_\Lambda, \quad (4.11)$$

The summation is now carried out over all $J = (j_1, \dots, j_p)$ and $K = (k_1, \dots, k_p)$ with $1 \leq j_1 < \dots < j_p < n$ and $1 \leq k_1 < \dots < k_p < n$. The angular brackets with the lower index Λ stand for the scalar product in ΛT^*M induced by the Riemannian metric on M . If $p = n$, then the sum in (4.11) should be interpreted as 0.

We have thus laid down an expression for $\frac{\partial}{\partial v} |\phi(x)|_E^2$ in our local coordinates. The next step is to establish estimate (4.10) at the point \tilde{x} using formula (4.11). The argument will rely on the properties of the coordinate system fixed in U . Remark that \tilde{x} was originally chosen as an arbitrary point in ∂M . Therefore, establishing (4.10) at this point would suffice to prove the lemma.

Let us take a closer look at the scalar product $\langle D_v dx^J, dx^K \rangle_\Lambda$ in the right-hand

side of (4.11). The formula

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = \sum_{l=1}^p \det \begin{pmatrix} \langle dx_{j_1}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_1}, dx_{k_p} \rangle_\Lambda \\ \vdots & & \vdots \\ \langle dx_{j_{l-1}}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_{l-1}}, dx_{k_p} \rangle_\Lambda \\ \langle D_\nu dx_{j_l}, dx_{k_1} \rangle_\Lambda & \cdots & \langle D_\nu dx_{j_l}, dx_{k_p} \rangle_\Lambda \\ \vdots & & \vdots \\ \langle dx_{j_{l+1}}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_{l+1}}, dx_{k_p} \rangle_\Lambda \\ \vdots & & \vdots \\ \langle dx_{j_p}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_p}, dx_{k_p} \rangle_\Lambda \end{pmatrix}$$

holds on $U \cap \partial M$. Our choice of the coordinate system provides the identities

$$\begin{aligned} \langle dx_l, dx_m \rangle_\Lambda &= \delta_l^m, \\ \langle D_\nu dx_l, dx_m \rangle_\Lambda &= -\Pi \left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m} \right) = -\delta_l^m \lambda_l, \quad l, m = 1, \dots, n-1, \end{aligned}$$

at the point \tilde{x} . (Recall that δ_l^m is the Kronecker symbol, and λ_l are the principal curvatures.) As a consequence,

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = -(\lambda_{j_1} + \cdots + \lambda_{j_p})$$

at \tilde{x} when J coincides with K , and

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = 0$$

at \tilde{x} when J differs from K .

Let us substitute the obtained equalities into (4.11). We conclude that

$$\frac{1}{2} \frac{\partial}{\partial \nu} |\phi(x)|_E^2 = - \sum \langle \beta_J(x), \beta_J(x) \rangle_E (\lambda_{j_1} + \cdots + \lambda_{j_p}).$$

at the point \tilde{x} . The summation is carried out over all the multi-indices J as described above. The scalar product $\langle \beta_J(x), \beta_J(x) \rangle_E$ is greater than or equal to 0

for every J . The principal curvatures $\lambda_{j_1}, \dots, \lambda_{j_p}$ are all nonnegative because ∂M is convex. As a result, estimate (4.10) holds at the point \tilde{x} . This proves the lemma because \tilde{x} can be chosen arbitrarily. \square

Our intention is to employ the technique developed in [2] for establishing Theorems 4.1, 4.2, and 4.3. We now introduce the required probabilistic machinery. Consider the bundle $O(M)$ of orthonormal frames over M . The letter π denotes the projection in this bundle. Let u_t^Y be a horizontal reflecting Brownian motion on $O(M)$ starting at the frame $Y \in O(M)$. We assume u_t^Y is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ satisfying the “usual hypotheses.” The symbol \mathbb{E} will be used for the expectation. The rigorous definition of a horizontal reflecting Brownian motion on the bundle of orthonormal frames can be found in [26, Chapter V] and in [24].

Introduce the process $X_t^y = \pi(u_t^Y)$. Here, we denote $y = \pi(Y)$. It is well-known that X_t^y is a reflecting Brownian motion on M starting at the point y . Details can be found in [26, Chapter V].

By definition, the process u_t^Y satisfies the equation

$$\begin{aligned} df(t, u_t^Y) &= \sum_{i=1}^n (\mathcal{H}_i f)(t, u_t^Y) dB_t^i \\ &\quad + \left(\frac{\partial}{\partial t} + \frac{1}{2} \Delta_{O(M)} \right) f(t, u_t^Y) dt - (\mathcal{N}f)(t, u_t^Y) dL_t \end{aligned} \quad (4.12)$$

for every smooth real-valued function $f(t, u)$ on $[0, \infty) \times O(M)$. Let us describe the objects occurring in the right-hand side. As before, $n \geq 2$ is the dimension of M . The notation \mathcal{H}_i refers to the canonical horizontal vector fields on $O(M)$. The process (B_t^1, \dots, B_t^n) is an n -dimensional Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$. The operator $\Delta_{O(M)}$ is Bochner’s horizontal Laplacian. It

appears as the sum of \mathcal{H}_i^2 with $i = 1, \dots, n$. The symbol \mathcal{N} stands for the horizontal lift of the vector field ν on ∂M . The non-decreasing process L_t is the boundary local time. It only increases when $\pi(u_t^y)$ belongs to ∂M .

Consider a smooth real-valued function $h(t, x)$ on $[0, \infty) \times M$. Applying (4.12) with $f(t, u) = h(t, \pi(u))$, we obtain an equation for the process $h(t, X_t^y)$. This simple observation is important to the proofs of Theorems 4.1, 4.2, and 4.3. When $f(t, u) = h(t, \pi(u))$, the formulas

$$\begin{aligned} \Delta_{O(M)} f(t, u) &= \Delta_M h(t, x)|_{x=\pi(u)}, \\ (\mathcal{N}f)(t, u) &= \frac{\partial}{\partial \nu} h(t, x)|_{x=\pi(u)}, \quad t \in [0, \infty), u \in O(M), \end{aligned} \quad (4.13)$$

hold true.

Let $g(t, x, y)$ denote the transition density of the reflecting Brownian motion X_t^y . The function $\tilde{g}_y(t, x) = g(2t, x, y)$ is a smooth positive solution to the heat equation (3.1) with the Neumann boundary condition (3.2). Note that the density $g(t, x, y)$ will be playing a significant role in our further considerations. The estimates required to establish Theorems 4.1, 4.2, and 4.3 rely on those known for $g(t, x, y)$.

All the probabilistic objects we will need are now at hand. Introduce the notation

$$q(t, x) = |R^{\nabla(t)}(x)|_E^2, \quad t \in [0, T), x \in M.$$

Given $r \in (0, T)$, define

$$\zeta^{r,y}(t) = \int_M q(r-t, x) g(t, x, y) dx, \quad t \in (0, r].$$

The quantity $\zeta^{r,y}(t)$ may be interpreted as $\mathbb{E}(q(r-t, X_t^y))$. Applying Remark 3.5 to the function $\tilde{g}_y(t, x)$ and taking the monotonicity of $YM(t)$ into account, one

concludes that

$$\zeta^{r,y}(t) \leq C_1 t^{-\frac{\dim M}{2}} \text{YM}(0), \quad t \in (0, \min\{r, 1\}], \quad (4.14)$$

with $C_1 > 0$ determined by (3.5). We are now in a position to prove Theorems 4.1 and 4.2. Two more lemmas are required to consider the case where $\dim M$ is 5 or higher. We will state them afterwards.

Proof of Theorem 4.1. Fix $\rho \in (0, T)$. Our goal is to obtain a bound on $\sup_{x \in M} q(\rho, x)$. Choose $\alpha \in (0, 1)$ and denote $\rho_0 = \max\{0, \rho - \frac{1}{\alpha}\}$. Let the number $\sigma_0 \in (0, \rho - \rho_0]$ satisfy the equality

$$\sigma_0^2 \sup_{t \in [\rho_0 + \sigma_0, \rho]} \sup_{x \in M} q(t, x) = \sup_{\sigma \in [0, \rho - \rho_0]} \left(\sigma^2 \sup_{t \in [\rho_0 + \sigma, \rho]} \sup_{x \in M} q(t, x) \right). \quad (4.15)$$

There exist $t_* \in [\rho_0 + \sigma_0, \rho]$ and $x_* \in M$ such that

$$q(t_*, x_*) = \sup_{t \in [\rho_0 + \sigma_0, \rho]} \sup_{x \in M} q(t, x). \quad (4.16)$$

It is convenient for us to write q_0 instead of $q(t_*, x_*)$. Our next step is to estimate the number q_0 . The desired bound on $\sup_{x \in M} q(\rho, x)$ will follow therefrom.

Using the heat equation (4.2) and the Bochner-Weitzenböck formula, we can prove the existence of a constant $C_2 > 0$ such that

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_M \right) q(t, x) \leq C_2 \left(1 + \sqrt{q(t, x)} \right) q(t, x) \quad (4.17)$$

for $t \in [0, T)$ and $x \in M$; see [8, Lemma 2.2]. The definition of σ_0 implies

$$\begin{aligned} \sup_{t \in [t_*, \rho_0 + \sigma_0]} \sup_{x \in M} q(t, x) &\leq \sup_{t \in [\rho_0 + (1-\alpha)\sigma_0, \rho]} \sup_{x \in M} q(t, x) \\ &\leq \frac{\sigma_0^2}{(1-\alpha)^2 \sigma_0^2} \sup_{t \in [\rho_0 + \sigma_0, \rho]} \sup_{x \in M} q(t, x) = \tilde{\alpha}^2 q_0 \end{aligned} \quad (4.18)$$

with $\tilde{\alpha} = \frac{1}{1-\alpha}$. Inequalities (4.17) and (4.18) will play an essential role in estimating the number q_0 . Let u_t^y be a horizontal reflecting Brownian motion in the

bundle $O(M)$. We suppose u_t^Y starts at a frame Y satisfying $\pi(Y) = x_*$. Define $X_t^{x_*} = \pi(u_t^Y)$ and consider the process

$$Z_t = e^{C_2(1+\tilde{\alpha}\sqrt{q_0})t} q(t_* - t, X_t^{x_*})$$

for $t \in [0, \alpha\sigma_0)$. Formulas (4.12) and (4.13) yield

$$\begin{aligned} q_0 = Z_0 &= \mathbb{E}(Z_t) \\ &- \mathbb{E} \left(\int_0^t \left(-\frac{\partial}{\partial r} + \frac{1}{2} \Delta_M \right) e^{C_2(1+\tilde{\alpha}\sqrt{q_0})(t_*-r)} q(r, X_{t_*-r}^{x_*}) \Big|_{r=t_*-s} ds \right) \\ &+ \mathbb{E} \left(\int_0^t e^{C_2(1+\tilde{\alpha}\sqrt{q_0})s} \frac{\partial}{\partial v} q(t_* - s, X_s^{x_*}) dL_s \right). \end{aligned}$$

In view of (4.17), (4.18), and Lemma 4.5, this implies $q_0 \leq \mathbb{E}(Z_t)$ for $t \in [0, \alpha\sigma_0)$.

As a consequence, the formula

$$q_0 \leq e^{C_2(1+\tilde{\alpha}\sqrt{q_0})t} \zeta^{t_*, x_*}(t), \quad t \in [0, \alpha\sigma_0), \quad (4.19)$$

holds true. We will now use it to prove that

$$\sup_{x \in M} q(\rho, x) \leq \max \left\{ \frac{\text{YM}(0)}{\alpha^2 \rho^2}, \theta_1 e^{\theta_{2,\alpha} \sqrt{\text{YM}(0)}} \text{YM}(0) \right\} \quad (4.20)$$

with $\theta_1 > 0$ and $\theta_{2,\alpha} > 0$. Estimate (4.5) will follow by looking at the case where $\alpha = \frac{1}{2}$.

Let us assume $q_0 > 0$ and $\text{YM}(0) > 0$. This does not lead to any loss of generality. Indeed, if $q_0 = 0$, then the supremum $\sup_{x \in M} q(\rho, x)$ is equal to 0 and (4.20) holds for any θ_1 and $\theta_{2,\alpha}$. When $\text{YM}(0) = 0$, we have $\text{YM}(t_*) = 0$ due to the fact that $\text{YM}(t)$ is non-increasing in $t \in [0, T)$. In this case, q_0 equals 0, and (4.20) is again satisfied for any θ_1 and $\theta_{2,\alpha}$.

Denote $t_0 = \sqrt{\frac{\text{YM}(0)}{q_0}}$. If $t_0 \geq \alpha\sigma_0$, then

$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0 \leq \frac{\text{YM}(0)}{\alpha^2}$$

by virtue of the definitions of σ_0 and t_0 . In this case, the estimate

$$\begin{aligned} \sup_{x \in M} q(\rho, x) &\leq \frac{\text{YM}(0)}{\alpha^2(\rho - \rho_0)^2} \\ &= \frac{\text{YM}(0)}{\alpha^2 \left(\min \left\{ \rho, \frac{1}{\alpha} \right\} \right)^2} = \max \left\{ \frac{\text{YM}(0)}{\alpha^2 \rho^2}, \text{YM}(0) \right\} \end{aligned} \quad (4.21)$$

holds true, which means (4.20) is satisfied for all $\theta_1 \geq 1$ and $\theta_{2,\alpha} > 0$. If $t_0 < \alpha\sigma_0$ (note that $\alpha\sigma_0 \leq \alpha(\rho - \rho_0) \leq 1$), then formulas (4.19) and (4.14) yield

$$q_0 \leq e^{C_2(1+\tilde{\alpha})\sqrt{q_0}t_0} \zeta^{t_*, x_*}(t_0) \leq e^{C_2\tilde{\alpha}\sqrt{\text{YM}(0)}} \tilde{C} q_0^{\frac{\dim M}{4}} \text{YM}(0)^{\frac{4-\dim M}{4}}$$

with $\tilde{C} = e^{C_2} C_1$. Hence

$$\begin{aligned} \sup_{x \in M} q(\rho, x) &\leq q_0 \leq \left(e^{C_2\tilde{\alpha}\sqrt{\text{YM}(0)}} \tilde{C} \text{YM}(0)^{\frac{4-\dim M}{4}} \right)^{\frac{4}{4-\dim M}} \\ &= \left(e^{C_2\tilde{\alpha}\sqrt{\text{YM}(0)}} \tilde{C} \right)^{\frac{4}{4-\dim M}} \text{YM}(0). \end{aligned}$$

Combined with (4.21), this estimate shows that (4.20) holds for

$$\theta_1 = \max \left\{ \tilde{C}^{\frac{4}{4-\dim M}}, 1 \right\}, \quad \theta_{2,\alpha} = \frac{4}{4-\dim M} C_2 \tilde{\alpha}.$$

We now assume $\alpha = \frac{1}{2}$. The desired result follows at once. The role of the constant θ_2 is to be played by $\theta_{2,\frac{1}{2}}$. \square

Remark 4.6. While proving the theorem, we have actually established a stronger result. Namely, take a number α from the interval $(0, 1)$. Suppose the conditions of Theorem 4.1 are satisfied. Then the estimate

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{\text{YM}(0)}{\alpha^2 \rho^2}, \theta_1 e^{\theta_{2,\alpha} \sqrt{\text{YM}(0)}} \text{YM}(0) \right\}, \quad \rho \in (0, T),$$

holds true. In the right-hand side, $\theta_1 > 0$ is a constant depending only on M , whereas $\theta_{2,\alpha} > 0$ is determined by α and M . When formulating Theorem 4.1, we restricted our attention to the case where $\alpha = \frac{1}{2}$. This was done for the sake of simplicity and understandability.

Proof of Theorem 4.2. Fix $\rho \in (0, T)$, $\alpha \in (0, 1)$, and $\beta \in (0, 1)$. Denote $\rho_0 = \max\{0, \rho - \frac{1}{\alpha}\}$. Let $\sigma_0 \in (0, \rho - \rho_0]$, $t_* \in [\rho_0 + \sigma_0, \rho]$, and $x_* \in M$ satisfy Eqs. (4.15) and (4.16). We write q_0 instead of $q(t_*, x_*)$. Our next step is to demonstrate that

$$\sigma_0^2 q_0 \leq \frac{\text{YM}(0)^\beta}{\alpha^2} \quad (4.22)$$

provided $\text{YM}(0)$ is smaller than a number $\xi_{\alpha, \beta} > 0$ depending only on α, β , and the manifold M . The assertion of the theorem will be deduced from this estimate.

Suppose $\text{YM}(0) = 0$. Then $\text{YM}(t_*) = 0$ due to the monotonicity of $\text{YM}(t)$ in $t \in [0, T)$. Ergo, q_0 is equal to 0. It becomes evident that $\sigma_0^2 q_0 = \frac{\text{YM}(0)^\beta}{\alpha^2}$.

We have thus proved (4.22) in the case where $\text{YM}(0) = 0$. Let us consider the general situation. Assume (4.22) fails to hold. Then $q_0 > 0$, $\text{YM}(0) > 0$, and the number $t' = \sqrt{\frac{\text{YM}(0)^\beta}{q_0}}$ lies in the interval $(0, \alpha\sigma_0) \subset (0, 1)$. Repeating the arguments from the proof of Theorem 4.1 and using (4.14), we conclude that the inequality

$$q_0 \leq e^{C_2(1+\tilde{\alpha}\sqrt{q_0})t'} \zeta^{t_*, x_*}(t') \leq e^{C_2\tilde{\alpha}\sqrt{\text{YM}(0)^\beta}} \tilde{C} q_0 \text{YM}(0)^{1-\beta}$$

must be satisfied. Here, $\tilde{\alpha}$ stands for $\frac{1}{1-\alpha}$. The constant \tilde{C} appears as $e^{C_2} C_1$. It is easy to see, however, that the above inequality fails when

$$\text{YM}(0) < \xi_{\alpha, \beta} = \min\left\{\left(e^{C_2\tilde{\alpha}\tilde{C}}\right)^{-\frac{1}{1-\beta}}, 1\right\}.$$

This contradiction establishes (4.22) under the condition $\text{YM}(0) < \xi_{\alpha, \beta}$.

In order to complete the proof of the theorem, we estimate $\sup_{x \in M} q(\rho, x)$. The definition of σ_0 suggests that

$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0.$$

In view of (4.22), this implies

$$\begin{aligned} \sup_{x \in M} q(\rho, x) &\leq \frac{\text{YM}(0)^\beta}{\alpha^2(\rho - \rho_0)^2} \\ &= \frac{\text{YM}(0)^\beta}{\alpha^2 \left(\min \left\{ \rho, \frac{1}{\alpha} \right\} \right)^2} = \max \left\{ \frac{\text{YM}(0)^\beta}{\alpha^2 \rho^2}, \text{YM}(0)^\beta \right\} \end{aligned}$$

provided $\text{YM}(0) < \xi_{\alpha, \beta}$. The assertion of the theorem follows by assuming $\alpha = \beta = \frac{1}{2}$. Inequality (4.6) holds when $\text{YM}(0) < \xi = \xi_{\frac{1}{2}, \frac{1}{2}}$. \square

Remark 4.7. In the course of the proof, we have actually established a result stronger than Theorem 4.2. Namely, fix $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. Suppose the conditions of Theorem 4.2 are satisfied. If $\text{YM}(0)$ is smaller than $\xi_{\alpha, \beta}$, then the estimate

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{\text{YM}(0)^\beta}{\alpha^2 \rho^2}, \text{YM}(0)^\beta \right\}, \quad \rho \in (0, T),$$

holds true. Here, $\xi_{\alpha, \beta}$ is a number depending on α, β , and M . When formulating Theorem 4.2, we restricted our attention to $\alpha = \beta = \frac{1}{2}$. This was done in order to make the statement more understandable.

Let us concentrate on the case where $\dim M$ is 5 or higher. First of all, we need a few auxiliary identities. Their purpose is to help us obtain a monotonicity formula related to the Yang-Mills heat equation (4.2). We establish these identities in Lemma 4.8 below. The proof is quite transparent yet worthy of attention. It demonstrates vividly how the boundary conditions imposed on $R^{\nabla(t)}$ interact with those satisfied by $g(t, x, y)$. In a way, this interplay of boundary conditions explains why the Brownian motion used to implement the probabilistic technique in our context should be reflected at ∂M .

Desiring to remain at the higher level of abstraction, we state Lemma 4.8 for a generic $\text{Ad } E$ -valued form ϕ and a generic function $f(x)$ on M . In our further

arguments, it will be applied with ϕ equal to the curvature $R^{\nabla(r-t)}$ and $f(x)$ equal to the density $g(t, x, y)$.

Lemma 4.8. *Let ∇ be a connection in E compatible with the structure group G . Suppose $f(x)$ is a real-valued function on M such that $\frac{\partial}{\partial \nu} f(x) = 0$ on ∂M . Consider an $\text{Ad } E$ -valued p -form $\phi \in \Omega^p(\text{Ad } E)$ with $p = 1, \dots, \dim M$. If either Eqs. (4.8) or Eqs. (4.9) are satisfied for ϕ on ∂M , then the following formulas hold true:*

$$\begin{aligned} \int_M |\phi|_E \Delta_M f \, dx &= - \int_M \langle \text{grad } |\phi|_E, \text{grad } f \rangle \, dx, \\ \int_M \langle d_\nabla d_\nabla^* \phi, f \phi \rangle_E \, dx &= \int_M \langle d_\nabla^* \phi, d_\nabla^*(f \phi) \rangle_E \, dx, \\ \int_M \langle d_\nabla (\iota_{\text{grad } \log f} \phi), f \phi \rangle_E \, dx &= \int_M \langle \iota_{\text{grad } \log f} \phi, d_\nabla^*(f \phi) \rangle_E \, dx. \end{aligned} \quad (4.23)$$

Proof. The first identity in (4.23) is a direct consequence of the Stokes theorem and the fact that $\frac{\partial}{\partial \nu} f(x) = 0$. The second one can be deduced from Lemma 4.4 in a straightforward fashion. Notably, the same argument has to be used when proving $\text{YM}(t)$ is non-increasing in $t \in [0, T]$; see [7]. We will now establish the third identity in (4.23).

Let us assume Eqs. (4.8) are satisfied for ϕ . The case where Eqs. (4.9) are satisfied instead can be treated similarly. We will show that the scalar product $\langle \iota_{\text{grad } \log f} \phi, \iota_\nu (f \phi) \rangle_E$ vanishes on ∂M . In view of Lemma 4.4, the third identity in (4.23) would follow from this fact as an immediate consequence.

Observe that the formula $\frac{\partial}{\partial \nu} f(x) = 0$ implies $\frac{\partial}{\partial \nu} \log f(x) = 0$. Accordingly, the gradient $\text{grad } \log f$ is tangent to ∂M at every point of ∂M . This allows us to assume ϕ belongs to $\Omega^p(\text{Ad } E)$ with p between 2 and $\dim M$. Indeed, if ϕ is an $\text{Ad } E$ -valued 1-form on M , then $\iota_{\text{grad } \log f} \phi = 0$ due to the first formula in (4.8).

Take a point $\tilde{x} \in \partial M$. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent

space $T_{\tilde{x}}M$ demanding that e_n coincide with ν . The equality

$$\begin{aligned} & \langle \iota_{\text{grad log } f} \phi, \iota_\nu(f\phi) \rangle_E \\ &= \sum \left\langle \phi \left(\text{grad log } f, e_{i_1}, \dots, e_{i_{p-1}} \right), f\phi \left(\nu, e_{i_1}, \dots, e_{i_{p-1}} \right) \right\rangle_E \end{aligned}$$

holds at \tilde{x} . The summation is to be carried out over all the arrays (i_1, \dots, i_{p-1}) with $1 \leq i_1 < \dots < i_{p-1} \leq n$. It is easy to see that $f\phi(\nu, e_{i_1}, \dots, e_{i_{p-1}})$ vanishes when $i_{p-1} = n$. At the same time, $\phi(\text{grad log } f, e_{i_1}, \dots, e_{i_{p-1}})$ vanishes when $i_{p-1} < n$ because $\text{grad log } f$ is tangent to ∂M and $\phi_{\text{tan}} = 0$. We conclude that the scalar product $\langle \iota_{\text{grad log } f} \phi, \iota_\nu(f\phi) \rangle_E$ equals 0 at \tilde{x} . Hence the third identity in (4.23). \square

The following lemma states a monotonicity formula related to the Yang-Mills heat equation (4.2). It is an important step in establishing Theorem 4.3 by means of the probabilistic technique. We emphasize that the proof of the lemma requires the Li-Yau-Hamilton estimate obtained in Chapter 3. For relevant results, see [2] and also [23, 8].

Lemma 4.9. *Let the boundary ∂M be totally geodesic. Suppose either Assumption 1 of Theorem 3.1 or Assumptions 1 and 2 of Theorem 3.6 are fulfilled for M . Given $r \in (0, T)$ along with $y \in M$, the formula*

$$\zeta^{r,y}(t_1) \leq \frac{1}{t_1^2} \left(t_2^2 e^{u(t_2)} \zeta^{r,y}(t_2) + C_3(t_2 - t_1) \text{YM}(0) \right)$$

holds for all $t_1, t_2 \in (0, \min\{r, 1\})$ satisfying $t_1 < t_2$. Here, $u(t)$ is a positive increasing function on $(0, 1]$ such that $\lim_{t \rightarrow 0} u(t) = 0$, and $C_3 > 0$ is a constant. Both $u(t)$ and C_3 are determined solely by the manifold M .

Proof. First, suppose Assumption 1 of Theorem 3.1 is satisfied. We will consider the other case later. Proposition 3.4 and Theorem 3.7 in [2] prove the assertion of the lemma on closed manifolds. The same line of reasoning works in our situation. However, two points need to be clarified:

- The equality between expressions (3.10) and (3.11) of [2] holds in our setting due to Lemma 4.8. The same can be said about expressions (3.14) and (3.15) of that paper.
- In order to obtain estimate (3.22) of [2] for the Neumann heat kernel $g(t, x, y)$, one should apply formula (3.3) above to the function $\tilde{g}_y(t, x) = g(2t, x, y)$.

The other arguments from the proofs of Proposition 3.4 and Theorem 3.7 in [2] work in our situation without significant modifications.

We now consider the case when there are curvature restrictions imposed on M away from the boundary. More specifically, suppose Assumptions 1 and 2 of Theorem 3.6 are satisfied. Then the assertion of the lemma can be established by repeating the arguments from the proofs of Proposition 3.4 and Theorem 3.7 in [2]. The required estimate on $g(t, x, y)$ comes from formula (3.6) in the present dissertation applied to the function $\tilde{g}_y(t, x)$. \square

We are now ready to prove Theorem 4.3. Afterwards, three important remarks will be made.

Proof of Theorem 4.3. Fix $\rho \in (0, T)$, $\alpha \in (0, 1)$, and $\beta \in (0, 1)$. We denote $\rho_0 = \max\left\{(1 - \alpha)\rho, \rho - \frac{1}{\alpha}\right\}$. Let $\sigma_0 \in (0, \rho - \rho_0]$, $t_* \in [\rho_0 + \sigma_0, \rho]$, and $x_* \in M$ obey Eqs. (4.15) and (4.16). Set $q_0 = q(t_*, x_*)$. We will show that

$$\sigma_0^2 q_0 \leq \frac{\text{YM}(0)^\beta}{\alpha^2} \tag{4.24}$$

provided $\text{YM}(0)$ is smaller than a certain value $\xi_{\alpha, \beta}(\rho)$ depending on ρ as a non-decreasing function. The assertion of the theorem will be deduced from this

estimate. Note that, aside from ρ , the value $\xi_{\alpha,\beta}(\rho)$ only depends on α, β , and the manifold M .

Suppose $YM(0) = 0$. Then $YM(t_*) = 0$ due to the monotonicity of $YM(t)$ in $t \in [0, T)$. As a consequence, q_0 is equal to 0. We conclude that (4.24) is satisfied when $YM(0) = 0$.

Denote $T_0 = \min\{\rho_0 + \alpha\sigma_0, 1\}$. Observe that $\alpha\sigma_0 \leq T_0 < t_*$. This fact is essential because it will allow us to apply Lemma 4.9 further in the proof. Assume estimate (4.24) fails to hold. Then $q_0 > 0$, $YM(0) > 0$, and the number $t' = \sqrt{\frac{YM(0)^\beta}{q_0}}$ lies in the interval $(0, \alpha\sigma_0) \subset (0, 1)$. The arguments from the proof of Theorem 4.1 yield

$$q_0 \leq e^{C_2(1+\tilde{\alpha}\sqrt{q_0})t'} \zeta^{t_*,x_*}(t') \leq e^{C_2(1+\tilde{\alpha}\sqrt{YM(0)^\beta})} \zeta^{t_*,x_*}(t').$$

Here, the number $\tilde{\alpha}$ equals $\frac{1}{1-\alpha}$. Lemma 4.9 implies

$$\zeta^{t_*,x_*}(t') \leq C' \frac{q_0}{YM(0)^\beta} (T_0^2 \zeta^{t_*,x_*}(T_0) + T_0 YM(0))$$

with $C' = \max\{e^{t(1)}, C_3\}$. (Note that Theorems 3.1 and 3.6 are being used at this point. More precisely, the proof of Lemma 4.9 relies on them.) Formula (4.14) and the definition of T_0 enable us to conclude that

$$\begin{aligned} q_0 &\leq e^{C_2(1+\tilde{\alpha}\sqrt{YM(0)^\beta})} C' \frac{q_0}{YM(0)^\beta} (T_0^2 \zeta^{t_*,x_*}(T_0) + T_0 YM(0)) \\ &\leq e^{C_2\tilde{\alpha}\sqrt{YM(0)^\beta}} C'' q_0 \left(C_1 T_0^{2-\frac{\dim M}{2}} YM(0)^{1-\beta} + YM(0)^{1-\beta} \right) \\ &\leq e^{C_2\tilde{\alpha}\sqrt{YM(0)^\beta}} C'' q_0 YM(0)^{1-\beta} \left(C_1 (\min\{(1-\alpha)\rho, 1\})^{2-\frac{\dim M}{2}} + 1 \right) \end{aligned}$$

with $C'' = e^{C_2} C'$. However, this is impossible when

$$\begin{aligned} YM(0) &< \xi_{\alpha,\beta}(\rho) = \min\{\xi_{\alpha,\beta}^1(\rho), \xi_{\alpha,\beta}^2(\rho), 1\}, \\ \xi_{\alpha,\beta}^1(\rho) &= \left(2e^{C_2\tilde{\alpha}} C'' C_1 (\min\{(1-\alpha)\rho, 1\})^{2-\frac{\dim M}{2}} \right)^{-\frac{1}{1-\beta}}, \\ \xi_{\alpha,\beta}^2(\rho) &= \left(2e^{C_2\tilde{\alpha}} C'' \right)^{-\frac{1}{1-\beta}}. \end{aligned}$$

The present contradiction establishes (4.24) under the condition $\text{YM}(0) < \xi_{\alpha,\beta}(\rho)$.

To complete the proof of the theorem, we need to estimate $\sup_{x \in M} q(\rho, x)$. The definition of σ_0 suggests that

$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0.$$

According to formula (4.24), this implies

$$\begin{aligned} \sup_{x \in M} q(\rho, x) &\leq \frac{\text{YM}(0)^\beta}{\alpha^2(\rho - \rho_0)^2} \\ &= \frac{\text{YM}(0)^\beta}{\alpha^2 \left(\min \left\{ \alpha\rho, \frac{1}{\alpha} \right\} \right)^2} = \max \left\{ \frac{\text{YM}(0)^\beta}{\alpha^4 \rho^2}, \text{YM}(0)^\beta \right\} \end{aligned}$$

provided $\text{YM}(0) < \xi_{\alpha,\beta}(\rho)$. We now assume $\alpha = \beta = \frac{1}{2}$. The assertion of the theorem follows at once. Inequality (4.7) holds when $\text{YM}(0) < \xi(\rho) = \xi_{\frac{1}{2}, \frac{1}{2}}(\rho)$. \square

Remark 4.10. While proving the theorem, we have really established a stronger result. That is, suppose $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. Let the conditions of Theorem 4.3 be fulfilled. Given $\rho \in (0, T)$, if $\text{YM}(0)$ is smaller than $\xi_{\alpha,\beta}(\rho)$, then the estimate

$$\sup_{x \in M} |R^{\nabla(\rho)}(x)|_E^2 \leq \max \left\{ \frac{\text{YM}(0)^\beta}{\alpha^4 \rho^2}, \text{YM}(0)^\beta \right\}$$

is satisfied. Here, $\xi_{\alpha,\beta}(s)$ is a positive non-decreasing function on $(0, \infty)$ entirely determined by α, β , and M . In the formulation of Theorem 4.3, we only dealt with the case where $\alpha = \beta = \frac{1}{2}$. This specific framework was meant to make the statement more understandable.

Remark 4.11. In the beginning of Chapter 4, we imposed the boundary conditions (4.3) or (4.4) on the curvature form $R^{\nabla(t)}$. Another approach is feasible. Namely, one may formulate the boundary conditions for the connection $\nabla(t)$ directly. The paper [7] takes this particular standpoint; see also [32, 19]. It may or may not be more natural to impose the boundary conditions on $\nabla(t)$ than to

impose ones on $R^{\nabla(t)}$ depending on the considered problem and the chosen perspective. However, the approach adopted in the present dissertation seems to be technically simpler. The reason for this lies in the fact that, unlike $\nabla(t)$, the curvature form $R^{\nabla(t)}$ transforms as a tensor under changes of coordinates. In particular, it is meaningful to talk about the tangential and the normal components of $R^{\nabla(t)}$.

Remark 4.12. In several situations, imposing the boundary conditions on the connection is virtually equivalent to imposing ones on its curvature form. Let us present an example. If a time-dependent connection satisfies the heat equation (4.2) and the conductor boundary condition in the sense of [7], then formulas (4.3) can be proved for its curvature. The converse statement holds with an adjustment. Roughly speaking, the first formula in (4.3) ensures that $\nabla(t)$ can be gauge transformed locally into a connection satisfying the conductor boundary condition. We refer to [7] for further details.

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