ON THE BEHAVIOR OF PRICE IN A SUPPLY CHAIN MARKET FOR CAPACITY

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2004

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Cornell University 2004

We are interested in the concept of dynamic pricing of production capacity in a supply chain and in particular, understanding how the supply chain structure might affect the volatility of capacity prices. We find that supply chains with high capacity costs will experience high price volatility.

We consider a continuous time market for a single homogeneous commodity. The market consists of two kinds of agents: sellers, who own capacity and incur short-term and long-term costs of updating capacity and earn their revenue through the sale of capacity to the second kind of agents, buyers, who hold inventory and satisfy the demand of end-consumers. The end-consumers and their interactions with buyers are an exogenous component of this model. We consider three different models that differ in the modelling of the end-consumer demand. In the first model, end-consumer demand is deterministic. We obtain closed form expressions for market capacity, production and equilibrium price.

We use the solution to the first model to analyze the second model in which end-consumer demand is the sum of a deterministic and a Brownian Motion component. Again, we obtain closed-form expressions for equilibrium price, production and capacity. We use the closed form solution of the equilibrium trajectories to obtain their variance and, subsequently, to analyze the impact of cost parameters on their

variance. We find that the variance of the price increases as the short-term and long-term costs of changing capacity increase relative to the holding cost. We obtain similar results using a variation of the above model in which capacity of each seller is exogenously fixed.

In the third model, we incorporate the evolution of forecasts in the end-consumer demand model but fix the capacity of each seller. We find that the early learning of end-consumer demand results in early learning in the market price forecast process. We also find that the market cost parameters affect the rate of learning of the price forecast.

BIOGRAPHICAL SKETCH

Amar Sapra was born on July 8, 1978 in the district hospital of Shahjahanpur. Being the youngest child in the family, he was showered with all kinds of affection. Surprisingly, it did not spoil him much!

He received his early education at home. At the age of 4 years and 7 months, he was admitted to Saraswati Shishu Mandir (SSM) on the birthday of Saraswati, the Goddess of Wisdom according to Hindu mythology. He sobbed for three days after being admitted but then mingled with other students.

After fifth grade, he shifted to Kendriya Vidyalaya # 1, where he rarely arrived on time for morning prayers. The lack of a boundary for the school, however, ensured that he was rarely caught and punished for being late. One of his regrets at the time of leaving school was that he never ventured into the neighboring guava gardens to steal guavas as most of his class mates did. He changed school after tenth grade to join, Kendriya Vidyalaya # 2 where he stayed until the end of his high school.

He went to the University of Roorkee (now, Indian Institute of Technology, Roorkee) to pursue his baccalaureate degree in Mechanical Engineering. The four years spent in college were fun. His experiences in machine shops with ill-formed jobs gave him an indication that Mechanical Engineering is not his forte. Taking due cognizance of this observation, he selected to major in Operations Research at Cornell University after obtaining his undergraduate degree.

To my parents

ACKNOWLEDGEMENTS

It is a daunting task to acknowledge all the wonderful people whose presence in my life over last few years made my stay at Cornell a memorable one. What follows is an attempt to implement this task.

First and foremost, I must thank my dissertation advisor, Peter Jackson, for his guidance and encouragement throughout my graduate studies. I appreciate his willingness to spare time for me despite his perennially busy schedule. I have learnt several personal and professional qualities from him over last few years. I also thank him for taking an active interest in my personal and professional development. I would also like to thank Leslie Trotter and Joseph Thomas for agreeing to serve on my committee and teaching me courses. Joseph Thomas has taught me that teaching can be fun. I am grateful to him for giving me valuable career advice.

I would also like to express my gratitude towards several other faculty members at Cornell who taught me courses and from whom I have learnt a great deal. Among them are Jack Muckstadt, Robin Roundy, Uma Prabhu, Michael Todd, Sidney Resnick, Gennady Samorodnitsky, and David Ruppert.

Jack Muckstadt has an uncanny ability to read a person and a desire to be helpful whenever and wherever he can. I have benefitted immensely, personally and professionally, from numerous discussions with him. Robin Roundy taught me the proper protocol in several confusing professional situations. His passion for research is almost infectious. Thanks are also due to Uma Prabhu, who has always been willing to discuss any issue and help. I appreciated his invitations to his home for a delicious lunch of *Dosas*.

I have been fortunate to come across several wonderful graduate students inside and outside the school. In the school, thanks in particular to Tuncay, Janis, Millie, Chek Beng, Tim Huh, Ganesh, Rommel, Trevor, Parthanil, Frans, Van Anh, and Yuriy for their friendship. Tim, Mike Freimer and Van Anh have been very understanding research colleagues. Yuriy and Chek Beng were very helpful in my transition from the world of Engineering to Mathematics in my first year.

Outside the School of OR & IE, I would like to express my gratitude to my roommate for five years, Rama, Swamy, Manpreet, Prita, Niranjan, and Ishan for their friendships. With Ishan, in particular, I have shared several competitive (life-and-death) games of squash, hours of talking nonsense, and strategies for getting free food. He has always been a helpful soul and has the quality for being an all-round advisor.

Finally, maximum credits are due to my parents and siblings whose support and love always helped me to be determined and not lose sight of my goals. It is an occasion to remember the sacrifices (by my parents and grandparents) that enabled me to reach this point in my career. From the point of a penurious existence right after the partition in India in 1947, being able to afford high education of four children, as my parents did, required a great deal of sacrifice and I owe it to my parents. My siblings instilled a competitive spirit within me and encouraged me to try my best.

I have fond memories of my nephew Dev as well who was born within months of my coming to Cornell and whose bustling childhood was cut short by brain tumor last year. His departure provoked a sad sense of loss and helplessness, and a feeling of how trifling our lives are and how naive is the importance that we attach to ourselves. If there is existence after death, I wish him peace in that existence.

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Chapter 1
Introduction

We are seeing the advent of markets in which manufacturers operate as job shops but price their services dynamically in response to the level of their capacity utilization. We are interested in the volatility of prices in such a market and we recognize that this volatility cannot be properly understood without recognizing the intertemporal nature of demand price elasticity. That is, the demand for manufacturing capacity in such markets typically comes from consumer products assemblers who have the opportunity to modulate their demand over time in response to the current price and anticipated price changes. By carrying inventories forward or by enduring backorders, these assemblers can shift their demand forward or backward in time as prices change. We investigate this phenomenon using classical economic equilibrium analysis under the assumption of perfect markets and rational expectations. We further restrict attention to the special case of quadratic cost functions. This approach allows us to completely characterize the dynamics of the price process and, from that, to investigate the sensitivity of price volatility to underlying market parameters.

We consider a continuous-time market for the capacity of a single homogeneous product with numerous agents of two types, owners and consumers of capacity. We will refer to this model as the Market model. The consumers of the capacity (whom we refer to as "buyers") form an interface between the owners of the capacity (whom we refer to as "sellers") and the end-consumers of the product. We assume that all the agents are rational, risk-neutral, and price-takers. That is, no agent is so large that the price of capacity is noticeably affected by the agent's behavior.

In the capacity market, the price at every instant is determined by the balance of supply and demand for capacity. Each buyer chooses an order quantity of the homogeneous product to minimize the sum of her purchasing cost and the inventory holding/shortage cost. We assume that the price paid by a buyer in the capacity market does not affect the price charged to the consumers in the end-product market. We assume that consumer demand in the end-product market is exogenously determined.

Similarly, each seller decides the level of capacity to be installed and the instantaneous rate of production, continuously over time. Sellers incur cost for changing capacity levels but they are permitted to make changes continuously in time. Sellers are penalized if the rate of production is not equal to the installed capacity. Production beyond installed capacity may be carried out by overtime, extra shifts, and outsourcing. Underutilization of the capacity also incurs a penalty. The sole source of revenue for sellers is the purchase of capacity made by the buyers in the capacity market. We assume that there exists a constant exogenous base price for the capacity and we consider only the premium or discount to this base price. Thus, the price in our model can be either positive, negative, or zero, depending on the balance of supply and demand.

Admittedly, the requirement that, in equilibrium, market demand equals market supply at every instant of time is an heroic assumption. This assumption is common in classical economic models but these models are typically discrete-time formulations. In such models, the time periods are assumed to be long enough for price adjustment mechanisms in the marketplace to react to new conditions and achieve equilibrium. There is a body of economics literature that explores these mechanisms for both the existence and stability of equilibria. Our assumption that equilibrium is achieved in continuous time begs the question of what price adjustment mechanism could bring this about. Though we do not develop the idea in this paper, we could, perhaps, avoid the question of a specific mechanism by imagining a series of discrete time economies in which some unspecified price adjustment takes place within the periods to achieve equilibrium by the end of each period. We then imagine a convergence of these economies, with a scaling of time, to a continuous time economy of the type we have formulated here. It is the disappearance of local time, the time during which prices adjust, in the limit that creates some of the anomalies of our model from a control theory perspective. While it appears that the price process is exogenous from the planning perspective of any agent, buyer or seller, it is, in our imagined continuous-time economy, a highly tuned process sensitive to the slightest change in state. This, combined with the equally heroic assumption of rational expectations (that all agents act in accordance with a price process they all agree would achieve equilibrium), allows us to apply the mechanics of stochastic control theory to derive insights into a price process that simultaneously satisfies all first order conditions for the optimal control of agents and the condition of equilibrium in the market.

Any buyer's decision regarding the order quantity is dependent on the anticipated future prices in the capacity market. If the buyer expects price to be high in future, she will place an order for a higher quantity. Similarly, when she expects price to drop, she will place an order for a lower quantity. The ability of the buyer to hold inventory or incur backorders temporarily permits her to shift the time of the demand for capacity. On the other hand, the seller does not hold inventory or incur backorders. This introduces an intertemporal component of demand in the market for capacity. The existing literature in Supply Chain Management typically ignores this and considers demand for capacity to be independent of price expectations.

In the existing literature, a price-demand relationship is often assumed to model the behavior of price sensitive customers. Usually this relationship is assumed to have one of the two forms - linear ($D_0 = a - bP_0$) or multiplicative ($D_0 = aP_0^{-b}$). Here D_0 is the demand, P_0 is the price and a and b are parameters. While such models can capture the price-sensitivity of demand, they fail to capture the intertemporal nature of price sensitivity in actual markets. If the price in the current period is high compared to the previous period, we do not doubt that the demand is likely be lower compared to the previous period. However, the demand is not necessarily lost permanently. The savvy customer may have just postponed the purchase until the price becomes favorable. This intertemporal component of the demand is largely ignored in the existing literature. In the multi-period, pricing literature, the demand in one period is usually considered independent of the past and future and solely dependent on the current price.

We present an approach which is based on the hypothesis that only the timing of the demand changes depending upon the price in the capacity market but the demand does not get lost permanently. We do not assume any price-demand relationship explicitly. We assume that a buyer forecasts future market price and, taking those forecast prices as given, chooses the rate of order placement to minimize her total costs. The price emerges as the equilibrium price that clears the market.

The outline for rest of this dissertation is as follows. In Chapter 2, we provide a brief review of the relevant literature. In Chapter 3, we consider a deterministic model over an infinite horizon for the market for capacity. We obtain closed-form expressions for the equilibrium price, production, and capacity. We also show the equivalence of market behavior in equilibrium to that of an integrated system.

In Chapter 4, we extend the model presented in Chapter 3 to incorporate stochasticity in consumer demand. We obtain closed-form expressions for market price, production, and capacity in equilibrium. This permits us to analyze the impact of supply chain cost parameters on the volatility of market variables in equilibrium. We also consider a model in which the capacity of each seller is fixed.

In Chapter 5, we develop a continuous time extension of the Martingale Model of Forecast Evolution (MMFE). We then use this model to analyze how the rate of learning regarding the consumer demand translates to the rate of learning of the equilibrium price. We also study the effect of supply chain cost parameters on the rate of learning of an equilibrium price.

Chapter 2
Literature Review

Papers in the existing literature that share characteristics with the models presented in this dissertation can be classified into roughly four categories. These categories are

1. the Maximum Principle and its applications to Supply Chain Management,
2. Peak-Load Pricing Theory,
3. Rational Expectations (in particular, the Food Grain Storage Problem), and
4. Supply Chain Coordination.

2.1 The Maximum Principle and Applications

In Chapter 3, we use Euler's equations to obtain optimal capacity, production and price trajectories in a market for capacity using a deterministic model. There are several papers in the production planning and inventory control literature that have applied a similar solution technique, namely, the Maximum Principle. The Maximum Principle provides necessary conditions for the optimality over infinite-dimensional spaces (Yong and Zhou [79]). Pekelman [62] uses the deterministic Maximum Principle to characterize optimal production when the price trajectory is known in a competitive industry. Pekelman [61] studies the problem of determining optimal production and price jointly when the demand is a linear function of price over a finite horizon. Feichtinger and Hartl [22] extend the same model to the case of a non-linear demand price relationship.

In this category, the paper closest to the models in Chapter 3 is Gaimon [27]. The author uses the deterministic Maximum principle to optimize production, price, capacity and inventory in a monopolist setting. Unlike our model, production is restricted to be less than capacity which, as in our model, can be updated in continuous time. Other features of this model which differ from ours, include the absence of backlogging, a linear demand-price relationship and the dependence of production cost on capacity acquisition. Other papers in the same category include Thompson et al. [73], Hwang et al. [41] and Gaimon [26]. Sethi and Thompson [68] review and present some applications of the Maximum Principle.

We model uncertainty in consumer demand through a Weiner's process in Chapter 4 and use a stochastic version of the Maximum Principle to solve the resulting model. Modeling of demand as a Weiner's process and a subsequent application of the Maximum Principle as a solution technique in a production planning problem was first used by Sethi and Thomson [67]. They consider a linear-quadratic cost model in which the objective is to find optimal production levels to minimize cost incurred due to production and inventory levels being different from factory optimal levels for both finite and infinite horizons. Allowing production to be negative (in other words, permitting disposal), they obtain closed form expressions for optimal production rates in feedback form (that is, as a function of the current inventory). Bensoussan et al. [3] consider a similar model for an infinite horizon problem in which the rate of production is constrained to be non-negative. They characterize the optimal feedback form of the solution. Fleming et al. [24] also consider an infinite horizon production planning problem but they assume demand to evolve as a continuous time Markov Chain with a finite state space. They consider a cost model involving convex holding/shortage and production costs and show the existence of a unique optimal feedback production policy. None of these papers consider the role of price in a market model.

We use several results from Yong and Zhou [79] and Cadenillas and Karatzas [9] to solve models in Chapters 4 and 5 using the stochastic version of the Maximum Principle.

2.2 Peak-Load Pricing Theory

The papers in the area of Peak-Load Pricing consider a market for a single non-storable or very-expensive-to-store commodity (such as electricity). Boiteux [4] and Steiner [72] were the pioneers in developing models to determine the optimal capacity and price to be charged in a discrete time setting when the demand for capacity is not uniform across time periods. Against the backdrop of application to utility industries, the objective in these papers is typically to maximize social welfare. Social welfare is maximized when the (possibly weighted) sum of consumers' and supplier's utility is maximized.

Some other examples of papers in this area are as follows. Williamson [77] studied this problem permitting only discrete additions of capacity. Pressman [63] introduced inter-dependence of demand over different periods. Dansby [18] considered time-varying demand within the same period and diverse technologies. Panzar [60] showed that whether or not consumers in all periods contribute towards capacity costs is dependent on the cost assumptions. Papers by Gravelle [30] and Nguyen [59] permit storage but they do not permit backlogging. Crew, Fernando, and Kleindorfer, (two books - [15] and [16], and an article - [13]), provide a comprehensive review of models in this area.

There exist several differences between the models discussed in the dissertation and the papers in this area including the mathematical formulations of the models. Other differing characteristics are continuous time setting and the possibility of storage and backordering.

The main similarity lies in the basic notion of using price to effect change in the demand schedule resulting in better utilization of capacity.

Assumption of independence of price-demand relationships over different periods is not restricted to Peak Load Pricing theory alone. Other examples in which authors assume independence include Samuelson [64] in the area of Food-grain Storage problem and Anand et al. [2] in the area of Supply Chain Coordination.

2.3 Rational Expectations

Rational expectations theory was first proposed by John Muth (Muth [58]) in 1961. Sargent [65] provides a concise but rich introduction to the theory of rational expectations. This theory lends structure to the formation of expectations by economic agents and its subsequent impact on the decision making by the agents and on the outcomes. In an uncertain environment with multiple decision makers where the outcome cannot be dictated by a single agent, this theory provides valuable foundation on which economic models can be built. An economic model based in a market setting is one example of such an application.

The theory of rational expectations has been used to analyze various economic situations, examples being the efficient markets theory of asset prices, the permanent income theory of consumption, and the price evolution of storable commodities (Sargent [65]). We will discuss the literature on commodity price evolution in detail here since the theory's models share some features with our models. The goal of such models is to study the impact of speculation on price stabilization of storable commodities such as wheat. There are three types of agents in the market - producers, speculators, and consumers. In each of the periods, producers bring the "harvest" to the market and sell it to speculators. The speculators decide how much of the supply to sell and the remainder is stored for the next period. The consumers of the commodity have price-dependent demand. The decision of the speculators regarding selling quantity affects prices. The speculators are the sole decision makers in this model. The uncertainty regarding the "harvest" in future brings risk to the speculators. From a modeling perspective, understanding the formation of the expectations regarding prices in the future is critical for modeling the behavior of speculators. The pioneering effort in applying rational expectations theory in this area was led by Samuelson [64]. Some other articles dealing with this problem are Chambers and Bailey [11], Deaton and Laroque [19], and Scheinkman and Schechtman [66]. A comprehensive review is provided in Wright and Williams [78].

Apart from the rational expectations hypothesis, our models are similar to the commodity markets literature in terms of utilizing the concept of using inventories to take advantage of prices. On the other hand, there are some differences too. While our model is based in a continuous time setting, commodity market models have been based in a discrete time setting. Our model differs also in terms of the objective, problem formulation and solution techniques employed.

2.4 Supply Chain Coordination

Finally, a detailed review of supply chain coordination models can be found in Cachon [8]. These models have focused on the design of contracts between buyers and sellers. The intertemporal nature of demand elasticity is not considered in any of the models reviewed by Cachon [8]. That is, the supply chain coordination models have not considered the ability of capacity buyers to shift their demand forward or backward in time through the use of inventory or backorders.

Chapter 3
Pricing Capacity in the Face of Intertemporal Demand: Deterministic Case

3.1 Introduction

In this chapter, we present a deterministic model, the Market model, to analyze the market for capacity in a supply chain. We also consider a cost model of a single firm that owns the supply chain, the Integrated model, and show the equivalence of optimal model variables in the Integrated model to equilibrium variables in the Market model. We begin with a description of notation for the Market model.

3.2 The Market Model

We consider a simple, continuous-time market for the commitment of production capacity to actual production of a single homogeneous product. There are S sellers in this market. These sellers own production facilities and accept production orders. They are also able, through investment and disinvestment, to adjust the capacity of their facilities over time. Let $C_k(t)$ denote the capacity of seller k at time t and let C_{0k} denote the initial capacity of seller k. The sellers can engage in overtime and outsourcing as well as undertime so it is not necessary for production to exactly equal capacity at any time. Let $Y_k(t)$ denote the cumulative production by seller k through time t. We assume $Y_k(0) = 0$. There are B buyers in the market who place production orders and satisfy demand from end-consumers. Let $F_j(t)$ denote the cumulative demand from end-consumers for the sales of buyer j through time t. This demand process is exogenous to the model. The buyers can hold inventory or incur backorders so it is not necessary for production orders to exactly equal end-consumer demand at any time. Let $X_j(t)$ denote the cumulative orders for production placed by buyer j through time t and let X_{0j} denote the initial inventory of buyer j. Let the absence of a j or k subscript indicate the total of the corresponding function over all entities in the market. For example, $C(t)$ is the total capacity in the market at time t and C_0 is the total initial capacity in the economy; similarly for $Y(t), X(t), X_0$, and $F(t)$. These market quantities will be referred to as *market* quantities (market capacity, market orders, etc.) For the evolution of capacity, cumulative production, cumulative orders, and cumulative demand, we restrict attention to functions in $\mathcal{C}^4$. Denote the ith derivative of any of these functions by a superscript (i), as in $C_k^{(i)}(t)$. In reality, most capacity investments and disinvestments are discrete decisions of large magnitude so this model has limited applicability. Our goal is to model price behavior in an idealized setting.

A market exists for buyers to place production orders and for sellers to accept them. There is no lead time between order placement and order delivery: production is instantaneously distributed from sellers to buyers. A market price $P(t)$ per unit is paid by buyers and received by sellers for each unit of production. All agents in this market are assumed to be price-takers. That is, the number of buyers, B, and the number of sellers, S, are assumed to be large and no one buyer or seller is large enough to influence price. We consider this price to be a premium or discount from some exogenously determined price that considers such factors as unit production costs and consumer demand price sensitivity for the final good. These factors are ignored in this analysis, consistent with our assumption that the end-consumer demand process is exogenous. In this way, we focus on the role of price in managing the evolution of capacity and inventory in the economy. Since it can be either a premium or a discount, we do not constrain the sign of $P(t)$.

In the models to follow, we suppress the time argument, t, unless needed for clarification. In both the buyer and seller models, we assume a quadratic cost structure in order to derive explicit solutions. The modelling and some of the analysis could proceed along similar lines using more general penalty functions, provided they are strictly convex.

In describing the behavior of buyers and sellers, we initially assume that the price process, P is given and known by both buyers and sellers. Given a price process P, seller k's problem is to choose a capacity policy, C_k, and a cumulative production policy, Y_k, to minimize the discounted cost of investment/disinvestment, and short term capacity adjustment less the revenue derived from production. The infinite horizon version of this problem with quadratic costs (Seller model) is:

$$\begin{aligned}
&\min_{C_k, Y_k} \int_0^\infty e^{-rt}\{\beta(C_k^{(1)})^2 + \kappa(C_k - Y_k^{(1)})^2 - PY_k^{(1)}\}dt \\
&\text{s.t.} \\
&Y_k^{(1)} \geq 0, C_k \geq 0; \quad \text{and} \\
&Y_k(0) = 0,\ C_k(0) = C_{0k};
\end{aligned} \tag{3.2.1}$$

where r denotes the continuous-time interest rate, β is the penalty coefficient on the rate of change of capacity (i.e. on investment/disinvestment), and κ is the penalty coefficient on the over- or under-utilization of capacity. The parameter β captures long term costs of changing capacity, such as changes in plant and equipment, and the parameter κ captures short term costs of capacity adjustments, such as changes in workforce. The constraints ensure that the production rate (the first derivative of cumulative production) and capacity are never negative (i.e. for all values $t \geq 0$ of the suppressed time index) and that initial capacity is fixed exogenously.

Given a price process $P(\cdot)$, buyer j's problem is to choose a cumulative production order policy X_j to minimize the discounted cost of production orders and inventory/shortfall costs. The infinite horizon version of this problem with quadratic costs (Buyer model) is:

$$\begin{aligned}
&\min_{X_j} \int_0^\infty e^{-rt}\{PX_j^{(1)} + \pi(X_j - F_j)^2\}dt \\
&\text{s.t.} \\
&X_j^{(1)} \geq 0;\ \text{and} \\
&X_j(0) = X_{0j};
\end{aligned} \tag{3.2.2}$$

where π denotes the net inventory penalty coefficient. Note that $X_j(t)-F_j(t)$ is net inventory at time t (on-hand inventory less backorders) so the objective penalizes any deviation of net inventory from zero. The constraints ensure that production orders (the first derivative of cumulative production orders) are never negative and that initial production orders are fixed exogenously.

Observe that the costs of production (material, labor, and capital) are ignored in the Seller model (3.2.1): only the costs of capacity adjustment, short and long term, are captured. Also observe that the revenue from consumer sales are ignored in the Buyer model (3.2.2): only the inventory/shortfall costs are relevant. As a result, the price in this market will reflect the tradeoff between the sellers' capacity adjustment costs and the buyers' inventory/shortfall costs.

Since all agents are assumed to be price takers, the cumulative production and production order policies, $Y(\cdot)$ and $X(\cdot)$, that optimize seller and buyer problems, respectively, will depend on the price process $P(\cdot)$. We assume that the market will be in equilibrium at all times. That is, the price process $P(\cdot)$ must ensure that

$$Y^{(1)}(t) = X^{(1)}(t) \text{ for all } t \geq 0. \tag{3.2.3}$$

In equilibrium, therefore, $Y(t) = X(t) - X_0$.

We assume the end-consumer demand to be deterministic and to have a seasonal component. We use the following model of cumulative demand for buyer j in order to derive explicit solutions:

$$F_j(t) \equiv D_j t + \alpha_j \sin \gamma t \tag{3.2.4}$$

and, hence,

$$F_j^{(1)}(t) \equiv D_j + \alpha_j \gamma \cos \gamma t, \tag{3.2.5}$$

where D_j is the average demand rate, α_j is the amplitude of seasonal variation for buyer j, and γ measures the frequency of seasonal demand. We assume that all buyers face the same seasonal frequency, γ, but may differ in average demand rates and seasonal amplitudes. For each buyer, j, the average demand rate D_j will be assumed to be suitably large relative to α_j and γ to ensure that cumulative demand is non-decreasing. That is, we require $D_j \geq \alpha_j \gamma$. The sign of α_j does not affect the analysis. If $\sum_j \alpha_j > 0$, then the system starts with a falling demand rate. If, on the other hand, $\sum_j \alpha_j < 0$ then the system starts with a rising demand rate.

By means of (3.2.1), (3.2.2), (3.2.3), and (3.2.4), we have described a simple market for capacity in which the demand for capacity is intertemporal in nature: if capacity prices are high, buyers can defer production orders (depleting inventory or incurring shortages) and if capacity prices are low, then buyers can advance production orders (eliminating shortages or building inventory). We proceed to solve these models and demonstrate this behavior.

3.2.1 Analysis

Consider the Buyer model (3.2.2) first. Our approach is to use Euler's equations (Gelfand and Fomin [29]) as necessary conditions for an optimum to derive the optimal production order policy, X_j^* . In applying Euler's equations, the non-negativity constraint on $X_j^{(1)}$ is temporarily ignored. The non-negativity constraint is subsequently shown to be non-binding by assuming large enough demand rates, D_j and D. To apply Euler's equations in an infinite horizon setting, some regularity conditions are needed (Hadley and Kemp [33, pp 29-37]). The regularity condition for the Buyer model (ignoring the non-negativity constraint) is:

$$\lim_{t\to\infty} \frac{\partial}{\partial X_j^{(1)}}[e^{-rt}(PX_j^{(1)} + \pi(F_j - X_j)^2)] = 0,$$

which reduces to

$$\lim_{t\to\infty} e^{-rt}P = 0. \tag{3.2.6}$$

We restrict attention to only those price functions that satisfy (3.2.6).

Proposition 3.2.1. *Ignoring the non-negativity constraint and assuming (3.2.6) holds, the solution X_j^* to (3.2.2) must satisfy the following differential equation:*

$$X_j^* = F_j + \frac{1}{2\pi}(P^{(1)} - rP). \tag{3.2.7}$$

Proof. See appendix. □

Observe that as a function of price, the demand for capacity by buyer j, X_j^*, has a linear component ($-\frac{r}{2\pi}P$) and an intertemporal component ($\frac{1}{2\pi}P^{(1)}$). It is this intertemporal component that many recent papers in capacity market analysis ignore. For any given current price of capacity, the buyer will increase his production order if he perceives capacity prices to be rising ($P^{(1)} > 0$) and correspondingly decrease the production order if capacity prices are perceived to be falling ($P^{(1)} < 0$). If prices are not changing ($P^{(1)} = 0$), then net inventory will be positive if a capacity price discount is offered ($P < 0$) and net inventory will be negative (shortages will be incurred) if a price premium is charged ($P > 0$). Also observe that the sensitivity of the net inventory ($X_j - F_j$) to price is inversely proportional to the inventory/shortage penalty coefficient, π. The more costly are the deviations in net inventory, the less sensitive is net inventory to price and price changes.

Next, consider the Seller model (3.2.1). The regularity conditions for Euler's equations to apply in the infinite horizon setting are as follows:

$$\lim_{t\to\infty} \frac{\partial}{\partial C_k^{(1)}}[e^{-rt}(\beta(C_k^{(1)})^2 + \kappa(C_k - Y_k^{(1)})^2 - PY_k^{(1)})] = 0,$$

and

$$\lim_{t\to\infty} \frac{\partial}{\partial Y_k^{(1)}}[e^{-rt}(\beta(C_k^{(1)})^2 + \kappa(C_k - Y_k^{(1)})^2 - PY_k^{(1)})] = 0.$$

Upon simplification, these become:

$$\lim_{t\to\infty} e^{-rt} C_k^{(1)} = 0, \tag{3.2.8}$$

and

$$\lim_{t\to\infty} e^{-rt}[2\kappa(Y_k^{(1)} - C_k) - P] = 0, \tag{3.2.9}$$

respectively. Summing over all the sellers, we have:

$$\lim_{t\to\infty} e^{-rt} C^{(1)} = 0, \tag{3.2.10}$$

and

$$\lim_{t\to\infty} e^{-rt}[2\kappa(Y^{(1)} - C) - SP] = 0. \tag{3.2.11}$$

We restrict attention to only those policies satisfying these infinite horizon conditions. In what follows, the superscript (*) denoting an optimal policy will be suppressed, in most cases, to avoid notational overload.

Proposition 3.2.2. *Ignoring non-negativity constraints and assuming (3.2.8) and (3.2.9) hold, the solution* (C_k^*, Y_k^*) *to (3.2.1) must satisfy the following differential equations simultaneously:*

$$C_k^{(2)} - rC_k^{(1)} - \frac{\kappa}{\beta}C_k = -\frac{\kappa}{\beta}Y_k^{(1)}, \tag{3.2.12}$$

and

$$-2\kappa(C_k^{(1)} - Y_k^{(2)}) + 2\kappa r(C_k - Y_k^{(1)}) - P^{(1)} + rP = 0. \tag{3.2.13}$$

Proof. See appendix. □

The strict convexity of the objective functions in (3.2.1) and (3.2.2) ensures that the differential equations are also sufficient conditions for the minimum. For a given price process, P, (3.2.7) and (3.2.12-3.2.13) provide the necessary and sufficient conditions for the optimality of the Seller model (3.2.1) and the Buyer model (3.2.2), respectively. The equilibrium price process is then obtained using the equilibrium condition (3.2.3).

Proposition 3.2.3. *In equilibrium, ignoring non-negativity restrictions, the market capacity* $C^*(t)$ *must satisfy the following differential equation:*

$$C^{(4)} - 2rC^{(3)} - (\frac{S\pi}{B\kappa} - r^2)C^{(2)} + r\frac{S\pi}{B\kappa}C^{(1)} + \frac{S\pi}{B\beta}C = \frac{S\pi}{B\beta}F^{(1)}. \tag{3.2.14}$$

Proof. See appendix. □

Steady State Behavior

The solution to (3.2.14) and the other differential equations above will each consist of a transient component and a steady state component. We denote the steady state component with a subscript, ∞.

Theorem 3.2.4. *In equilibrium, ignoring non-negativity restrictions, the steady state components of market capacity, $C^*_\infty(t)$, of market production rate (order rate), $Y^{*(1)}_\infty(t)(= X^{*(1)}_\infty(t))$, and of market price, $P^*_\infty(t)$ are given by:*

1. $C^*_\infty(t) = D + \zeta \frac{S\pi}{B\beta}\alpha\gamma\cos(\gamma t - \phi)$,
2. $X^{*(1)}_\infty(t) = Y^{*(1)}_\infty(t) = F^{(1)} - \zeta\alpha\gamma^3(r^2+\gamma^2)\cos(\gamma t + \theta - \phi)$, *and*
3. $P^*_\infty(t) = \frac{2\pi}{B}\zeta\alpha\gamma^2(r\sin(\gamma t + \theta - \phi) + \gamma\cos(\gamma t + \theta - \phi)$,

respectively, where

$$
\begin{aligned}
q_1 &= 2r\gamma^3 + r\gamma\frac{S}{B}\frac{\pi}{\kappa}, \\
q_2 &= \gamma^4 - (r^2 - \frac{S}{B}\frac{\pi}{\kappa})\gamma^2 + \frac{S}{B}\frac{\pi}{\beta}, \\
\zeta &= \frac{1}{\{q_1^2 + q_2^2\}^{1/2}}, \\
\phi &= \tan^{-1}(\frac{q_1}{q_2}), \\
\theta &= \tan^{-1}(\frac{2r\gamma}{\gamma^2 - r^2}).
\end{aligned}
$$

Furthermore, for sufficiently large D, $C^*_\infty(\cdot)$, $X^{*(1)}_\infty(\cdot)$, and $Y^{*(1)}_\infty(\cdot)$ are non-negative processes.

Proof. See appendix. □

Observe that in steady state, the market capacity will oscillate about the average market demand rate. The frequency of oscillation, γ, matches the seasonal frequency of demand but there is a phase shift ϕ between the market demand rate function, $F^{(1)}$, and the market capacity function. That phase difference may be attributed to the time value of money since $\phi \to 0$ as $r \to 0$.

If the rate of discounting is small enough, then the amplitude of seasonal swing of the market capacity trajectory in the steady state is smaller than the amplitude of the seasonal swing in market end-consumer demand rate. One sufficient condition to ensure this is that r be less than 1. We state this formally in the following corollary.

Corollary 3.2.5. *If $r \leq 1$ then $\zeta \frac{S\pi}{B\beta} < 1$ and therefore, the amplitude of the seasonal swing in the market capacity is strictly less than the amplitude of the seasonal swing in the market end-consumer demand.*

Proof. In Theorem 3.2.4, $q_2 > \frac{S\pi}{B\beta}$ if $r \leq 1$, which implies that $\zeta < \frac{B\beta}{S\pi}$. □

Observe that for $\gamma \geq \gamma_0$ where γ_0 is large enough, ζ decreases as γ increases. This implies that if the seasonal swings in the end-consumer demand are sufficiently frequent, the market capacity corrections reduce in magnitude even though the frequency of updates remains the same as for the end-consumer demand. The frequency of oscillation in the steady state component of the market production rate, γ, matches the seasonal frequency of end-consumer demand but there is a phase shift of $\theta - \phi$ between the demand and production. The phase difference may be attributed to the time value of money since $\theta - \phi$ approaches 0 as r approaches 0. However, the phase difference does not depend only on r but also depends on other market parameters.

The factor ζ also decreases as β and/or κ increase relative to the holding cost parameter π, implying that the market response to the seasonal end-consumer demand is subdued due to high costs of updating capacity.

The nature of the transient solution to the differential equation (3.2.14) can be determined by studying the roots of the following polynomial equation:

$$x^4 - 2rx^3 + (r^2 - \frac{S}{B}\frac{\pi}{\kappa})x^2 + r\frac{\pi}{\kappa}\frac{S}{B}x + \frac{\pi}{\beta}\frac{S}{B} = 0. \tag{3.2.15}$$

The roots of (3.2.15) are given by:

$$\frac{r \pm \sqrt{r^2 + 2\frac{\pi S}{\kappa B} \pm 4\sqrt{(\frac{\pi S}{2\kappa B})^2 - \frac{\pi S}{\beta B}}}}{2}. \tag{3.2.16}$$

Whether the roots in (3.2.16) are real or complex can have markedly different consequences on the way the market evolves initially. The transient component of market capacity and production is determined by the roots of (3.2.15) and the initial conditions. When all the roots are real, the transient component consists of exponential functions with negative arguments. As t increases, the transient component will thus have a smooth decay. However, when the roots are complex, the transient component will decay with sinusoidal oscillations. In the following lemma, we obtain conditions under which roots of the equation (3.2.15) are real or complex.

Lemma 3.2.6. *If $\beta \geq \frac{4\kappa^2 B}{\pi S}$, then all roots of (3.2.15) are real. If $\beta < \frac{4\kappa^2 B}{\pi S}$, then all roots of (3.2.15) are complex.*

Proof. Clearly, if $\beta < \frac{4\kappa^2 B}{\pi S}$, then $\sqrt{r^2 + 2\frac{\pi S}{\kappa B} \pm 4\sqrt{(\frac{\pi S}{2\kappa B})^2 - \frac{\pi S}{\beta B}}}$ will be complex and, hence, all four roots will be complex. Suppose $\beta \geq \frac{4\kappa^2 B}{\pi S}$. It is enough to show that $r^2 + 2\frac{\pi S}{\kappa B} \geq 4\sqrt{(\frac{\pi S}{2\kappa B})^2 - \frac{\pi S}{\beta B}}$. Suppose, to the contrary, that $r^2 + 2\frac{\pi S}{\kappa B} < 4\sqrt{(\frac{\pi S}{2\kappa B})^2 - \frac{\pi S}{\beta B}}$. Squaring both sides and simplifying, we get:

$$r^4 + 4\frac{\pi S}{\kappa B}r^2 < -16\frac{\pi S}{\beta B}$$

which cannot be true since parameters π, β, S, and B are positive. These results hold even when $r = 0$. □

In the following proposition, we derive another structural result regarding roots of equation (3.2.15).

Proposition 3.2.7. *Either*

(a) all roots of (3.2.15) are real, in which case two roots (possibly identical) will be strictly negative and two roots (possibly identical) will be strictly positive; or,

(b) all roots of (3.2.15) are complex in which case two roots will have strictly negative real parts and two roots will have strictly positive real parts and the same holds for the complex parts.

Proof. By Lemma 3.2.6, either all roots are real or all roots are complex.

a) Suppose all roots are real. In this case, $r^2 + 2\frac{S\pi}{B\kappa} - 4\sqrt{(\frac{S\pi}{2B\kappa})^2 - \frac{S\pi}{B\beta}} > r^2$ since $2\frac{S\pi}{B\kappa} > 4\sqrt{(\frac{S\pi}{2B\kappa})^2 - \frac{S\pi}{B\beta}}$ and so,

$$r - \sqrt{r^2 + 2\frac{S\pi}{B\kappa} \pm 4\sqrt{(\frac{S\pi}{2B\kappa})^2 - \frac{S\pi}{B\beta}}} < 0.$$

The other two roots,

$$r + \sqrt{r^2 + 2\frac{S\pi}{B\kappa} \pm 4\sqrt{(\frac{S\pi}{2B\kappa})^2 - \frac{S\pi}{B\beta}}}$$

are easily seen to be positive.

b) Suppose all roots are complex. In this case, the four roots will be two pairs of conjugates (since the product of roots is $\frac{S\pi}{B\beta}$, which is real). Further, since the sum of roots is $2r > 0$, at most two roots can have negative real parts. Now consider sum of products of any three of them. Let the roots be $u_1 \pm iv_1$ and $u_2 \pm iv_2$. The sum of products of any three of them equals $(u_1^2 + v_1^2)(2u_2) + (u_2^2 + v_2^2)(2u_1)$ and from the polynomial equation (3.2.15), we know this equals $-r\frac{S\pi}{B\kappa}$. Hence either u_1 or $u_2 < 0$ which proves the claim. □

In the following theorem, we obtain equilibrium market variables when both the negative roots of (3.2.15) are real and distinct. Each market variable is obtained by adding a steady state component derived in Theorem 3.2.4 and a transient component that depends on the roots of (3.2.15).

Theorem 3.2.8. *If* $\beta > \frac{4\kappa^2 B}{\pi S}$ *then, in equilibrium and ignoring non-negativity constraints,*

1. $C^*(t) = C^*_\infty(t) + c_1 e^{u_1 t} + c_2 e^{u_2 t}$,
2. $X^{*(1)}(t) = Y^{*(1)}(t) = Y^{*(1)}_\infty(t) - \frac{B\beta}{S\pi}\{c_1 u_1^2 (r-u_1)^2 e^{u_1 t} + c_2 u_2^2 (r-u_2)^2 e^{u_2 t}\}$, *and*
3. $P^*(t) = P^*_\infty(t) + \frac{2\beta}{S}\{c_1 u_1 (r-u_1) e^{u_1 t} + c_2 u_2 (r-u_2) e^{u_2 t}\}$

where ζ, θ, *and* ϕ *are as defined in Theorem 3.2.4,* u_1, u_2 *are the negative real roots of (3.2.15),*

$$
\begin{aligned}
u_{1,2} &= \frac{r - \sqrt{r^2 + 2\frac{S\pi}{B\kappa} \pm 4\sqrt{(\frac{S\pi}{2B\kappa})^2 - \frac{S\pi}{B\beta}}}}{2}, \text{ and} \\
c_1 &= \frac{u_2(r-u_2)^2(C_0 - D - \zeta\frac{S\pi}{B\beta}\alpha\gamma\cos\phi) + \frac{S\pi}{B\beta}(X_0 + \zeta\alpha\gamma^2(r^2+\gamma^2)\sin(\theta-\phi))}{u_2(r-u_2)^2 - u_1(r-u_1)^2},
\end{aligned}
$$

and

$$
c_2 = (C_0 - D - \zeta\frac{S\pi}{B\beta}\alpha\gamma\cos\phi) - c_1.
$$

Furthermore, for sufficiently large D, $C^*(\cdot)$, $X^{*(1)}(\cdot)$, *and* $Y^{*(1)}(\cdot)$ *are non-negative processes.*

Proof. See appendix. □

The transient component of $C^*(t)$ consists of exponential terms with negative exponents. The constant coefficients of these exponential terms, c_1 and c_2, sum to the deviation between initial capacity, C_0, and the steady state target, $D + \zeta\cos\phi$. As r increases, u_1 and u_2 decrease in absolute magnitude.

Theorem 3.2.9. *If $\beta < \frac{4\kappa^2 B}{\pi S}$, then, in equilibrium and ignoring non-negativity constraints,*

1. $C^*(t) = C^*_\infty(t) + e^{ut}(d_1 \cos vt + d_2 \sin vt)$,

2. $X^{*(1)}(t) = Y^{*(1)}(t) = Y^{*(1)}_\infty(t) + \frac{B\beta}{S\pi} e^{ut} U\{(ud_1 + vd_2)\sin(vt+\varphi) + (vd_1 - ud_2)\cos(vt+\varphi)\}$, *and*

3. $P^*(t) = P^*_\infty(t) + e^{ut}(f_1 \cos(vt+\varphi) + g_1 \sin(vt+\varphi))$,

where ζ, θ and ϕ are as defined in Theorem 3.2.4, $u \pm iv$ is a conjugate pair of roots of (3.2.15) such that u is negative, i.e.,

$$
\begin{aligned}
u &= \frac{r}{2} - \sqrt{\frac{r^2}{8} + \frac{\pi S}{4\kappa B} + \sqrt{\frac{r^4}{64} + \frac{\pi S}{8\kappa B} r^2 + \frac{\pi S}{4\beta B}}},\\
v &= \sqrt{u^2 - ru - \frac{S\pi}{2B\kappa}},\\
U &= \{(u(u-r)^2 + 2rv^2 - 3uv^2)^2 + v^2((3u-r)(u-r) - v^2)^2\}^{1/2},\\
d_1 &= C_0 - D - \zeta \frac{S\pi}{B\beta} \alpha\gamma \cos\phi,\\
d_2 &= \frac{U \sin\varphi (C_0 - D - \zeta \frac{S\pi}{B\beta}\alpha\gamma\cos\phi) - \frac{S\pi}{B\beta}\zeta\alpha\gamma^2(r^2+\gamma^2)\sin(\theta - \phi) - \frac{S\pi}{B\beta} X_0}{U\cos\varphi},\\
\varphi &= \tan^{-1}\left(\frac{-(u(u-r)^2 + 2rv^2 - 3uv^2)}{v((3u-r)(u-r) - v^2)}\right),\\
f_1 &= -\frac{2\beta}{S} U \frac{d_2(u-r) + d_1 v}{(u-r)^2 + v^2}, \text{ and}\\
g_1 &= \frac{2\beta}{S} U \frac{d_1(u-r) - d_2 v}{(u-r)^2 + v^2}.
\end{aligned}
$$

Furthermore, for sufficiently large D, $C^*(\cdot)$, $X^{*(1)}(\cdot)$, and $Y^{*(1)}(\cdot)$ are non-negative processes.

Proof. See appendix. □

The rate of decay to steady state, u, increases in magnitude as π increases and decreases in magnitude as κ and β increase.

Theorem 3.2.10. *If* $\beta = \frac{4\kappa^2 B}{\pi S}$*, then, in equilibrium and ignoring non-negativity constraints,*

1. $C^*(t) = C^*_\infty(t) + e^{ut}(h_1 t + h_2)$

2. $X^{*(1)}(t) = Y^{*(1)}(t) = Y^{*(1)}_\infty(t) - \frac{B\beta}{S\pi} e^{ut}\{2h_1 u(2u-r)(u-r) + (h_1 t + h_2)u^2(u-r)^2\}$, *and*

3. $P^*(t) = P^*_\infty(t) + e^{ut}(f_2 t + g_2)$

where ζ, θ*, and* ϕ *are as defined in Theorem 3.2.4;*

$$
\begin{aligned}
h_1 &= \frac{-u(u-r)^2(C_0 - D - \frac{S\pi}{B\beta}\zeta\alpha\gamma\cos\phi) - \frac{S\pi}{B\beta}\zeta\alpha\gamma^2(r^2+\gamma^2)\sin(\theta-\phi) - \frac{S\pi}{B\beta}X_0}{(3u-r)(u-r)}, \\
h_2 &= C_0 - D - \zeta\frac{S\pi}{B\beta}\alpha\gamma\cos\phi, \\
u &= \frac{r - \sqrt{r^2 + 2\frac{\pi S}{\kappa B}}}{2} < 0, \\
f_2 &= -\frac{2\beta}{S}h_1 u(u-r), \text{ and} \\
g_2 &= -\frac{2\beta}{S}\{(2u-r)h_1 + u(u-r)h_2\}.
\end{aligned}
$$

Furthermore, for sufficiently large D*,* $C^*(\cdot)$*,* $X^{*(1)}(\cdot)$*, and* $Y^{*(1)}(\cdot)$ *are non-negative processes.*

Proof. See appendix. □

When $\beta = \frac{4\kappa^2 B}{\pi S}$, the decay of the transient components of C^*, Y^* and P^* is exponential as in Theorem 3.2.8. In addition, the rate of decay of the transient component, u, increases in magnitude with π and decreases as κ increases.

Theorem 3.2.11. *In equilibrium, ignoring non-negativity restrictions, the optimal capacity, production, and orders by individual sellers and buyers are given by*

1. $C^*_k = \frac{1}{S}C^* + c_k$*, where* $c_k = C^*_{0k} - \frac{1}{S}C_0$*, for each seller* k*;*

2. $Y_k^* = \frac{1}{S}Y^* + c_k t$, *for each seller* k*; and*

3. $X_j^* = F_j + \frac{1}{B}(X^* - F)$, *for each buyer* j,

respectively. For sufficiently large D *and* D_j*, all non-negativity restrictions are satisfied.*

Proof. See appendix. □

3.2.2 Example

In this section, we illustrate the evolution of various market variables over time. The specifications of parameters are as follows.

Table 3.1: Data for Figures 3.1-3.4

β	κ	π	D	α	γ	r	S, B	C_0	X_0
125	5	1	6	-2	1	0.25	5	4	4

Note that $\beta > \frac{4\kappa^2 B}{\pi S}$ in the above example, hence the results will correspond to results stated in Theorem 3.2.8. The values of selected constants are as follows:

Table 3.2: Values of Constants in Figures 3.1-3.4

u_1	u_2	c_1	c_2	θ	ϕ	ζ
-0.27	-0.14	1.37	-3.36	0.49	0.45	-0.013

Figure 3.1 shows evolution of cumulative market production and cumulative market demand over time. Note that the cumulative market demand curve is above the cumulative market production curve. This phenomenon is explained by the presence of the initial inventory.

The graph in Figure 3.2 shows the relationship between the current demand and the steady state component of price. Observe that the troughs and crests of the

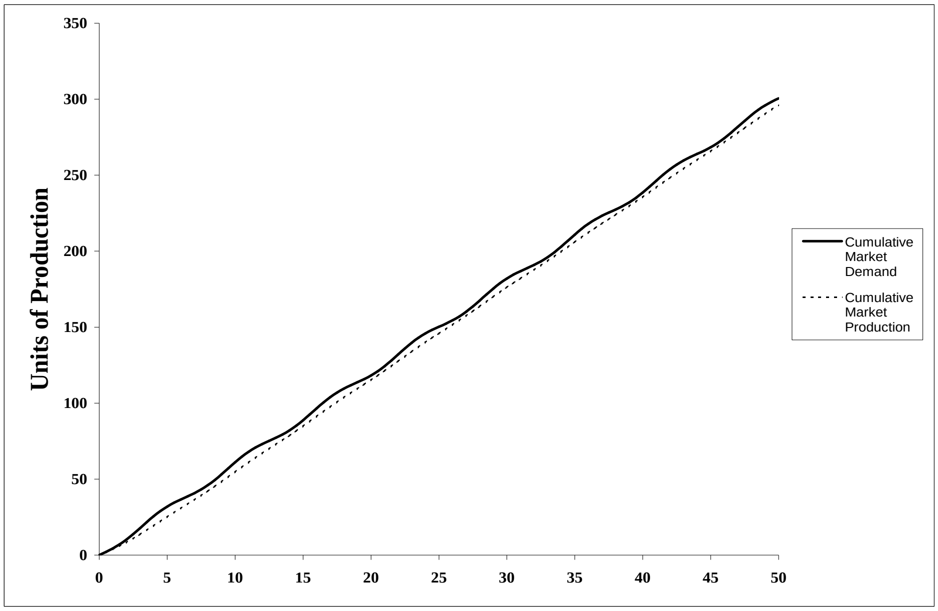

Figure 3.1: Cumulative Demand and Cumulative Production

two curves are almost in synchronization with each other. Perfect synchronization does not occur due to the discounting factor. Intuitively, this implies that near the times of peak demand, the price is at its highest level, too. Similarly, the price is at its lowest level, when the demand is at its lowest level.

The graph in Figure 3.3 shows the relationship between the current capacity, production and the demand levels in steady state. Observe that the amplitude of the capacity trajectory is very small compared to the production and the demand amplitudes. Even the amplitude of current production is much smaller when compared to the amplitude of current demand. Also the three curves have a phase difference among each other which again can be attributed to the discounting factor. It may also be observed that every cycle may be divided into two subcycles. In

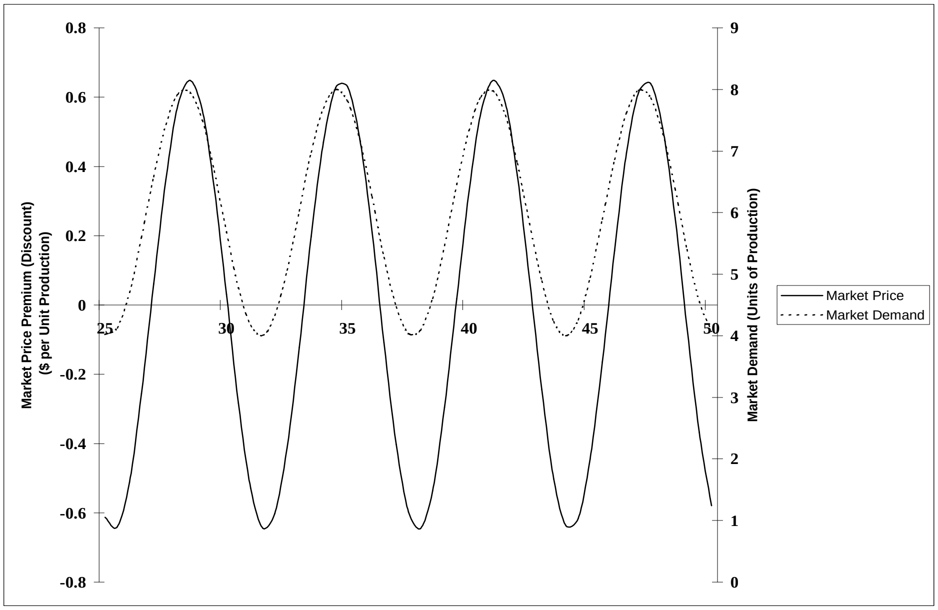

Figure 3.2: Evolution of Steady State Price and Current Demand

the first subcycle, buyers hold inventory, whereas in the second one, buyers incur backorders.

The graph in Figure 3.4 depicts the smooth decaying evolution of the capacity trajectory. This graph shows the transient component of capacity which will eventually converge to zero. The capacity at any instant is equal to this component added to the steady state component. The sign of this component is driven by the relative value of the steady state component to the initial capacity level. For this example, the initial capacity is lower than the steady state component of the capacity at time 0, thus resulting in negative transient component.

Figure 3.5 shows evolution of transient component of capacity when $\beta < \frac{4\kappa^2 B}{\pi S}$. For this graph, $\beta = 10$ has been used. The values of other parameters remain

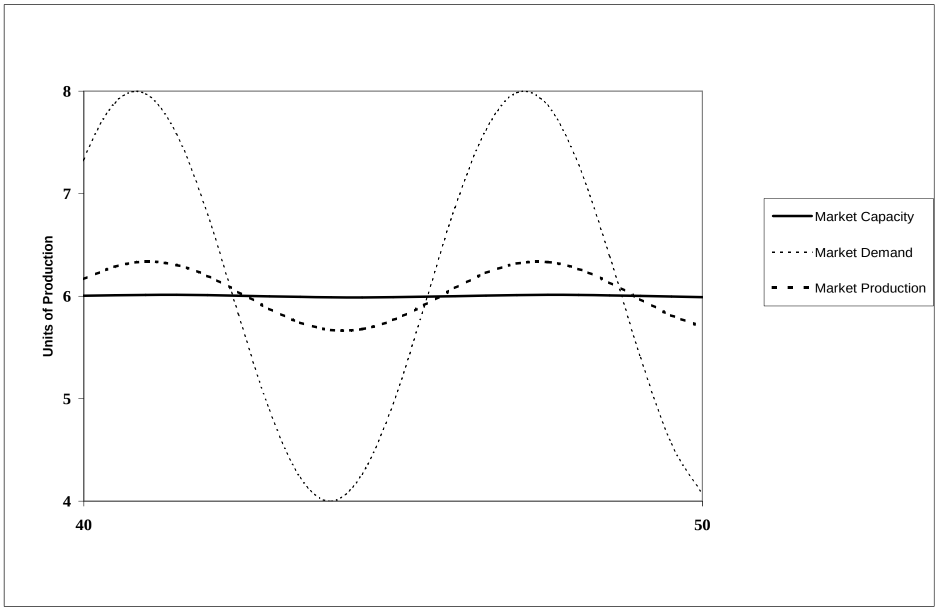

Figure 3.3: Capacity, Current Demand and Production in Steady State

same. The values of some of the derived constants are as follows: The trajectory

Table 3.3: Values of Constants in Figure 3.5

u	v	d_1	d_2	θ	ϕ	ζ	φ
-0.15	0.25	-1.98	-1.28	0.49	1.33	-0.1	0.09

corresponds to results stated in Theorem 3.2.9. It may be seen that the trajectory varies sinusoidally but decays rapidly.

3.3 Integrated Model

We consider an integrated supply chain setting in this model. There is only one agent in the market who not only controls the capacity but also holds the inventory

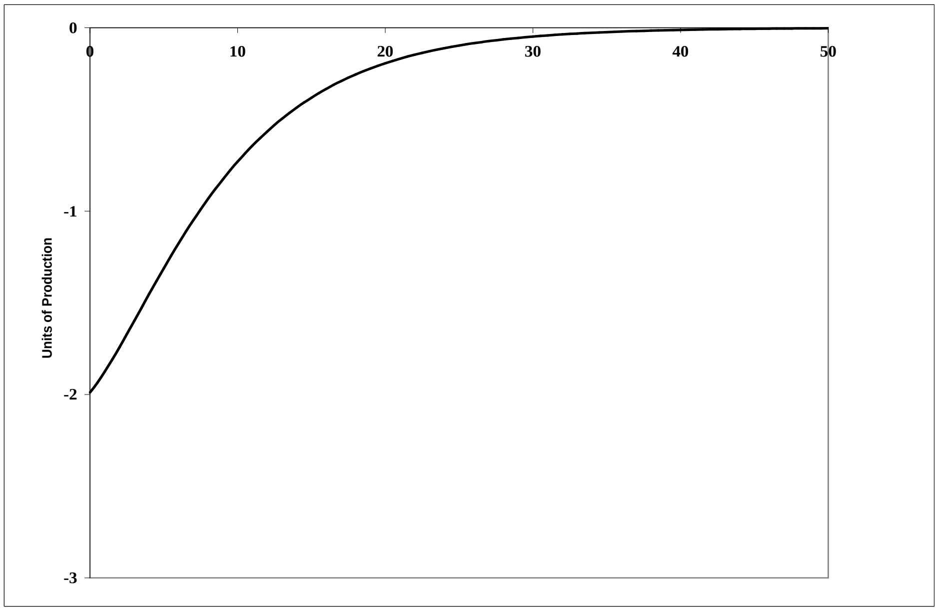

Figure 3.4: Evolution of Transient Aggregate Capacity

and satisfies consumer demand. Just as in the Market model, costs are incurred when the capacity is adjusted, when the rate of production is not equal to the capacity and when (positive or negative) inventory is held. The seller is interested in determining the optimal capacity and production level so as to minimize the cost of the whole supply chain.

3.3.1 Notation

The notation for the Integrated model is similar to the Market model, with one difference, a subscript ι. Thus, $F_\iota(t)$ is used to denote the cumulative consumer demand by time t, $Y_\iota(t)$ is the cumulative production by time t, and $C_\iota(t)$ is the capacity at time t. We add a superscript (i) in each of these variables to denote the

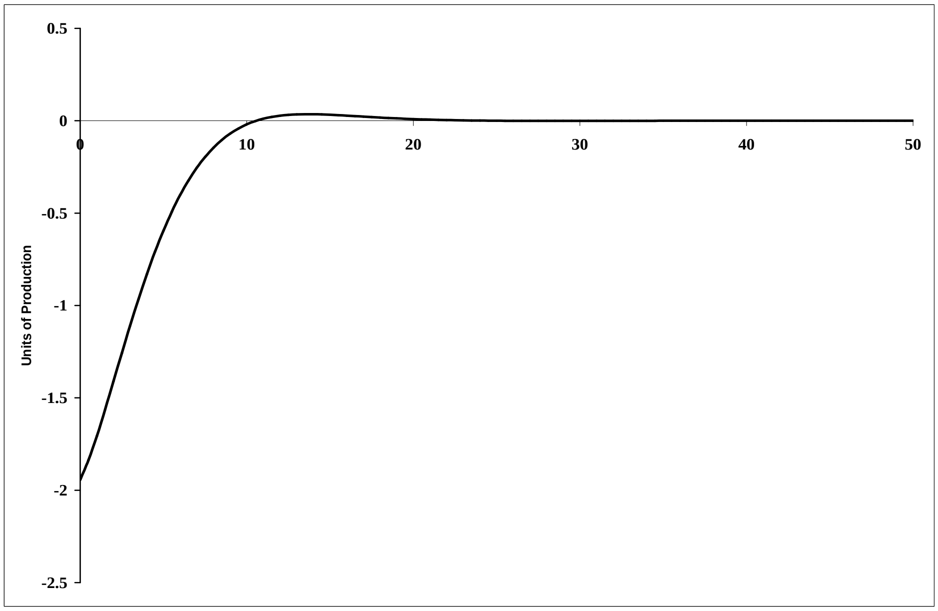

Figure 3.5: Evolution of Transient Aggregate Capacity

ith derivative, for example, $F_\iota^{(i)}(t)$ is the ith derivative of $F_\iota(t)$ at time t. Finally, $C_{0,\iota}$ and $Y_{0,\iota}$ are used to denote the initial capacity and net inventory, respectively.

In the following subsection, we present the optimization model for the Integrated model.

3.3.2 Model

The optimization problem is formulated as follows:

$$\begin{gathered}\min_{C_\iota, Y_\iota} \int_0^\infty e^{-rt}\{\beta(C_\iota^{(1)})^2 + \kappa(C_\iota - Y_\iota^{(1)})^2 + \pi_\iota(F_\iota - Y_\iota)^2\}dt \\ \text{s.t.} Y_\iota^{(1)} \geq 0,\ C_\iota \geq 0; \\ C_\iota(0) = C_{0,\iota}, Y_\iota(0) = Y_{0,\iota}; \\ F_\iota(t) = D_\iota t + \alpha_\iota \sin(\gamma_\iota t).\end{gathered} \tag{3.3.1}$$

In the above formulation, D_ι, α_ι, and γ_ι are positive constants, representing long-run average of demand, amplitude, and frequency of a seasonal swing, respectively. The parameter r denotes the continuous-time interest rate. The cost parameter β, κ, and π_ι represent the penalty coefficients for long-term cost of changing capacity, capacity-production mismatch, and holding inventory or incurring backorders. We assume that the cost parameters β, κ, and the discount factor r are equal to the corresponding parameters in the Market model.

For the sake of clarity, the dependence of $C_\iota(t)$, $Y_\iota(t)$, and $F_\iota(t)$ on t is omitted in the above formulation. The minimizing variables, here, are Y_ι and C_ι which are functions in themselves. Similar to the Market model, the permissible Y_ι and C_ι must satisfy the following two conditions:

$$\lim_{t\to\infty} \frac{\partial}{\partial C_\iota^{(1)}}[e^{-rt}\{\beta(C_\iota^{(1)})^2 + \kappa(C_\iota - Y_\iota^{(1)})^2 + \pi_\iota(F_\iota - Y_\iota)^2\}] = 0$$

which simplifies to

$$\lim_{t\to\infty} e^{-rt}C_\iota^{(1)}(t) = 0, \tag{3.3.2}$$

and

$$\lim_{t\to\infty} \frac{\partial}{\partial Y_\iota^{(1)}}[e^{-rt}\{\beta(C_\iota^{(1)})^2 + \kappa(C_\iota - Y_\iota^{(1)})^2 + \pi_\iota(F_\iota - Y_\iota)^2\}] = 0$$

which simplifies to

$$\lim_{t\to\infty} e^{-rt}(C_\iota(t) - Y_\iota^{(1)}(t)) = 0. \tag{3.3.3}$$

Proposition 3.3.1. *Ignoring non-negativity constraints and assuming (3.3.2) and (3.3.3) hold, the solution (C_ι^*, Y_ι^*) to (3.3.1) must solve the following differential equations simultaneously:*

$$Y_\iota^{(2)} - rY_\iota^{(1)} - \frac{\pi_\iota}{\kappa}Y_\iota = -\frac{\pi_\iota}{\kappa}F_\iota + C_\iota^{(1)} - rC_\iota \qquad (3.3.4)$$

and

$$C_\iota^{(2)} - rC_\iota^{(1)} - \frac{\kappa}{\beta}C_\iota = -\frac{\kappa}{\beta}Y_\iota^{(1)} \qquad (3.3.5)$$

with initial conditions $Y_\iota(0) = Y_{0,\iota}$ and $C_\iota(0) = C_{0,\iota}$.

Proof. See appendix. □

Corollary 3.3.2. *The optimal capacity C_ι^* must solve the following fourth order differential equation with initial condition $C_\iota(0) = C_{0,\iota}$.*

$$C_\iota^{(4)} - 2rC_\iota^{(3)} + (r^2 - \frac{\pi_\iota}{\kappa})C_\iota^{(2)} + r\frac{\pi_\iota}{\kappa}C_\iota^{(1)} + \frac{\pi_\iota}{\beta}C_\iota = \frac{\pi_\iota}{\beta}F_\iota^{(1)} \qquad (3.3.6)$$

Proof. Taking derivative of both sides of (3.3.4), yields an equation in terms of $Y_\iota^{(1)}, Y_\iota^{(2)}$ and $Y_\iota^{(3)}$. Then the LHS of (3.3.5) yields an expression for $Y_\iota^{(1)}$; the derivative of the LHS of (3.3.5) gives an expression for $Y_\iota^{(2)}$ and finally, the second derivative of the LHS of (3.3.5) provides an expression for $Y_\iota^{(3)}$. Substituting these back into the derivative of (3.3.4) results in (3.3.6). □

Observe that for $S = B$, the differential equation in (3.2.14) has the same functional form as the one in (3.3.6). Since S and B are parameters, the solution to (3.3.6) can be derived from the solution to the Market model. This gives the following theorem.

Theorem 3.3.3. *Suppose $F_\iota(t) = \sum_j F_j(t), \forall t, C_{0,\iota} = \sum_j C_{0j}$, and $Y_{0,\iota} = \sum_j X_{0j}$. Further, $\gamma_\iota = \gamma$ and $\pi_\iota = \frac{S}{B}\pi$. Then, the optimal capacity and production for an integrated supply chain are equal to the optimal capacity and production orders for* *a market, that is,*

$$\begin{aligned} Y_\iota^*(t) &= X^*(t), \\ C_\iota^*(t) &= C^*(t). \end{aligned}$$

3.4 Conclusion

We analyzed two models in this chapter to study the evolution of capacity prices in the face of deterministic demand with seasonal swing. In the Market model, the evolution of capacity prices reflects the seasonality: the peaks and troughs of the price trajectory roughly follow the peaks and troughs of the consumer demand. We find that the buyers take advantage of the low prices and buy in advance to store for the time when the consumer demand and the prices would be high. We also show that the optimal solution to an integrated supply chain model can be derived from the solution to the Market model.

3.5 Appendix

3.5.1 Proof of Proposition 3.2.1

Ignore the non-negativity constraint in (3.2.2). Later, we will discuss conditions under which the order process always remains non-negative. To obtain a necessary condition for an extremum, we apply Euler's conditions. Let

$$G(X_j^{(1)}, X_j) = e^{-rt}\{PX_j^{(1)} + \pi(F_j - X_j)^2\}.$$

Application of the Euler's equation yields:

$$\frac{\partial G}{\partial X_j} - \frac{d}{dt}(\frac{\partial G}{\partial X_j^{(1)}}) = 0 = e^{-rt}\{-2\pi(F_j - X_j) - P^{(1)} + rP\}$$

which upon simplification yields (3.2.7).

3.5.2 Proof of Proposition 3.2.2

Let

$$G_1(C_k^{(1)}, C_k, Y_k^{(1)}, Y_k) = e^{-rt}\{\beta(C_k^{(1)})^2 + \kappa(C_k - Y_k^{(1)})^2 - PY_k^{(1)}\}.$$

Ignore non-negativity constraints as before. Application of Euler's equation to $G_1(C_k^{(1)}, C_k, Y_k^{(1)}, Y_k)$ for state vector (C_k, Y_k) results in (3.2.12) and (3.2.13), respectively.

3.5.3 Proof of Proposition 3.2.3

The market equilibrium condition yields

$$\sum_k Y_k^{(1)} = \sum_j X_j^{(1)}. \tag{3.5.1}$$

Since $\sum_k Y_k(0) = 0$ and $\sum_j X_j(0) = X_0$, (3.5.1) implies $Y = X - X_0$. Aggregating (3.2.12) over all suppliers, we get:

$$C^{(2)} - rC^{(1)} - \frac{\kappa}{\beta}C = -\frac{\kappa}{\beta}Y^{(1)}. \tag{3.5.2}$$

Similar operation on (3.2.13) gives

$$-2\kappa(C^{(1)} - Y^{(2)}) + 2\kappa r(C - Y^{(1)}) - S(P^{(1)} - rP) = 0. \tag{3.5.3}$$

Aggregating (3.2.7) over all buyers yields:

$$X = F + \frac{B}{2\pi}(P^{(1)} - rP). \tag{3.5.4}$$

Combining (3.5.2), (3.5.3), and (3.5.4) to eliminate $Y^{(1)}$ yields:

$$-2\beta(C^{(3)} - rC^{(2)}) + 2\beta r(C^{(2)} - rC^{(1)}) - 2\pi\frac{S}{B}(X - F) = 0.$$

Therefore,

$$Y + X_0 = X = F - \frac{B\beta}{S\pi}(C^{(3)} - 2rC^{(2)} + r^2C^{(1)}). \tag{3.5.5}$$

Taking derivative and rearranging terms gives:

$$-\beta(C^{(4)} - rC^{(3)}) + \beta r(C^{(3)} - rC^{(2)}) - \pi\frac{S}{B}(Y^{(1)} - F^{(1)}) = 0,$$

which, using (3.5.2) and simplification, results in (3.2.14).

3.5.4 Proof of Theorem 3.2.4

Since $F = Dt + \alpha \sin\gamma t$, the steady state solution of (3.2.14) is given by $D + \frac{S\pi}{B\beta}\zeta\alpha\gamma\cos(\gamma t - \phi)$ where

$$\zeta = \frac{1}{\{(\gamma^4 - (r^2 - \frac{S}{B}\frac{\pi}{\kappa})\gamma^2 + \frac{S\pi}{B\beta})^2 + (2r\gamma^3 + r\gamma\frac{S}{B}\frac{\pi}{\kappa})^2\}^{1/2}}$$

and

$$\tan\phi = \frac{2r\gamma^3 + r\gamma\frac{S}{B}\frac{\pi}{\kappa}}{\gamma^4 - (r^2 - \frac{S}{B}\frac{\pi}{\kappa})\gamma^2 + \frac{S\pi}{B\beta}}.$$

Using (3.5.5),

$$X^*_\infty(t) = Y^*_\infty(t) + X_0 = Dt + \alpha\sin\gamma t - \zeta\alpha\gamma^2(r^2 + \gamma^2)\sin(\gamma t + \theta - \phi)$$

where $\theta = \tan^{-1}\left(\frac{2r\gamma}{\gamma^2 - r^2}\right)$. To compute the component of the price trajectory corresponding to the steady state solution of (3.2.14), using (3.5.4):

$$P^{(1)} - rP = -\frac{2\pi}{B}\zeta\alpha\gamma^2(r^2 + \gamma^2)\sin(\gamma t + \theta - \phi).$$

The steady state component of above differential equation is equal to

$$\frac{2\pi}{B}\zeta\alpha\gamma^2\left(r\sin(\gamma t + \theta - \phi) + \gamma\cos(\gamma t + \theta - \phi)\right)$$

and the homogeneous solution is given by c_3e^{rt} but c_3 must be set to 0 to satisfy (3.2.6).

3.5.5 Proof of Theorem 3.2.8

The optimal capacity trajectory is sum of the transient component and the steady state component (computed in Theorem 3.2.4). The transient component of optimal capacity trajectory is equal to $\sum_{i=1}^{4} c_i e^{u_i t}$, where u_3 and u_4 are the positive roots of (3.2.15). The necessary condition (3.2.10) is equivalent to $\lim_{t\to\infty} \sum_{i=1}^{4} c_i u_i e^{(u_i - r)t} = 0$. This condition can be true only if $c_3 = c_4 = 0$ since u_3 and u_4 are greater than r. Hence

$$C^*(t) = C^*_\infty(t) + c_1 e^{u_1 t} + c_2 e^{u_2 t}. \tag{3.5.6}$$

Similarly, Y^* is equal to sum of terms corresponding to steady state and transient components of C^*. Y^* can be computed using (3.5.5) and (3.5.6) and is given by:

$$Y^*(t) + X_0 = X^*(t) = Y^*_\infty(t) - \frac{B\beta}{S\pi}\{c_1 e^{u_1 t} u_1 (r-u_1)^2 + c_2 e^{u_2 t} u_2 (r-u_2)^2\} \tag{3.5.7}$$

Using initial conditions, $Y^*(0) = 0$ and $C^*(0) = C_0$, the values of c_1 and c_2 are obtained as:

$$\begin{aligned} c_1 &= \frac{\frac{S\pi}{B\beta} X_0 + u_2(r-u_2)^2 (C_0 - D - \zeta \frac{S\pi}{B\beta}\alpha\gamma\cos\phi) + \zeta\frac{S\pi}{B\beta}\alpha\gamma^2(r^2+\gamma^2)\sin(\theta-\phi)}{u_2(r-u_2)^2 - u_1(r-u_1)^2} \\ c_2 &= C_0 - D - \zeta\frac{S\pi}{B\beta}\alpha\gamma\cos\phi - c_1. \end{aligned}$$

Similarly, to compute terms in optimal price trajectory corresponding to transient components of C^*, we use (3.5.4) and (3.5.7) to get:

$$P^{(1)} - rP = -\frac{2\beta}{S}\{c_1 u_1 (r-u_1)^2 e^{u_1 t} + c_2 u_2 (r-u_2)^2 e^{u_2 t}\}.$$

The solution to above differential equation is equal to

$$\frac{2\beta}{S}\left(c_1 u_1 (r-u_1) e^{u_1 t} + c_2 u_2 (r-u_2) e^{u_2 t}\right).$$

Therefore:

$$P^*(t) = P^*_\infty(t) + \frac{2\beta}{S}\left(c_1u_1(r-u_1)e^{u_1t} + c_2u_2(r-u_2)e^{u_2t}\right).$$

It may be noted that since the necessary condition (3.2.6) is satisfied by the optimal price function hence, (3.2.11) is also satisfied by the optimal trajectories.

In the following, we shall show that for large enough D, non-negativity of Y^* and C^* is satisfied. Discussion of non-negativity constraints for the individual agents will be completed in the proof of Theorem 3.2.11.

For large enough D, the transient components of optimal market capacity and optimal market instantaneous rate of production are also non-negative. Considering first the transient component of capacity, WLOG assume that $|u_2| > |u_1|$. Change in C^* at time t as D to $D+\delta$, where $\delta > 0$:

$$\Delta C^* = \delta + \frac{\delta}{u_2(r-u_2)^2 - u_1(r-u_1)^2}(-u_2(r-u_2)^2e^{u_1t} + u_1(r-u_1)^2e^{u_2t}).$$

Since $u_2 < u_1 < 0$, the expression $u_2(r-u_2)^2 - u_1(r-u_1)^2$ is negative. Further, for all positive t, $e^{u_2t} < e^{u_1t}$ and therefore, the numerator in the second term in the above expression is positive. Further, as t increases, the numerator decreases. Hence, the value of the second term is minimized at $t = 0$ and as t increases, the value of the second term always remains greater than $-\delta$, its value at $t = 0$. Thus, for large enough D, the transient component will be non-negative.

The proof of current production also follows in a similar way. Again, WLOG assume that $|u_2| > |u_1|$. Change in $Y^{*(1)}$ as D is increased by δ, where $\delta > 0$:

$$\Delta Y^{*(1)} = \delta - \delta\frac{B\beta}{S\pi}\left(\frac{-e^{u_1t}u_1^2(r-u_1)^2u_2(r-u_2)^2 + e^{u_2t}u_2^2(r-u_2)^2u_1(r-u_1)^2}{u_2(r-u_2)^2 - u_1(r-u_1)^2}\right).$$

Now, $u_1u_2(r-u_1)(r-u_2)$ represents the product of roots of (3.2.15) which is equal to $\frac{S\pi}{B\beta}$. Therefore, the RHS of the above equation becomes:

$$\frac{\delta[u_1\left(e^{u_1 t}(r-u_1)(r-u_2)-(r-u_1)^2\right)+u_2\left(-e^{u_2 t}(r-u_1)(r-u_2)+(r-u_2)^2\right)]}{u_2(r-u_2)^2-u_1(r-u_1)^2} \tag{3.5.8}$$

At $t = 0$, the numerator of the above expression is negative. Since the denominator is negative, the overall expression is positive. It is enough to show that the numerator remains negative as t increases. The time dependent subexpression in the numerator,

$$\delta(r-u_1)(r-u_2)\left(u_1 e^{u_1 t}-u_2 e^{u_2 t}\right),$$

is initially positive. As t increases, the negative term $u_1 e^{u_1 t}$ decays at a slower rate compared to the positive term $-u_2 e^{u_2 t}$. Therefore, for any $t > 0$, the above expression is less than $\delta(r-u_1)(r-u_2)(u_1-u_2)$ and may even become negative. Since the time independent subexpression in the numerator of (3.5.8), $u_2(r-u_2)^2-u_1(r-u_1)^2$ is negative, the numerator remains negative for all values of t. Hence for large enough $\delta, Y^{*(1)} \geq 0$.

3.5.6 Proof of Theorem 3.2.9

To compute transient component of C^*, define $u_1 \pm iv_1$ as the other two roots of (3.2.15) in addition to $u \pm iv$ such that $u_1 > 0$. Then, the transient component of C^* is given by

$$e^{ut}(d_1 \cos vt + d_2 \sin vt) + e^{u_1 t}(d_3 \cos v_1 t + d_4 \sin v_1 t).$$

Since $u_1 = \frac{r}{2} + \sqrt{v_1^2 + \frac{r^2+2\frac{\pi S}{\kappa B}}{4}} > r$, so condition (3.2.10) is satisfied only if $d_3 = d_4 = 0$. Therefore, optimal market capacity trajectory is given by:

$$C^*(t) = C^*_\infty(t) + e^{ut}(d_1 \cos vt + d_2 \sin vt). \tag{3.5.9}$$

As before using (3.5.5) and (3.5.9):

$$Y^*(t) + X_0 = X^*(t) = Y^*_\infty(t) + \frac{B\beta}{S\pi} e^{ut} U\{d_1 \sin(vt + \varphi) - d_2 \cos(vt + \varphi)\} \quad (3.5.10)$$

where

$$\tan\varphi = \frac{-(u(u-r)^2 + 2rv^2 - 3uv^2)}{v\{(3u-r)(u-r) - v^2\}}$$

and

$$U = [(u(u-r)^2 + 2rv^2 - 3uv^2)^2 + v^2((3u-r)(u-r) - v^2)^2]^{1/2}.$$

Using initial conditions $Y(0) = 0$ and $C(0) = C_0$, d_1 and d_2 are computed to be:

$$\begin{aligned} d_1 &= C_0 - D - \zeta\frac{S\pi}{B\beta}\alpha\gamma\cos\phi, \\ d_2 &= \frac{U\sin\varphi(C_0 - D - \frac{S\pi}{B\beta}\zeta\alpha\gamma\cos\phi) - \frac{S\pi}{B\beta}\zeta\alpha\gamma^2(r^2+\gamma^2)\sin(\theta-\phi) - \frac{S\pi}{B\beta}X_0}{U\cos\varphi}. \end{aligned}$$

Next we show that for large enough D, the transient component of the optimal market capacity becomes non-negative. For this, assume D is increased to $D + \delta, \delta > 0$. Change in C^* will be $\delta - \delta e^{ut}(\cos vt + \tan\varphi \sin vt)$ which can be simplified to:

$$\Delta C^* = \delta(1 - \frac{e^{ut}\cos(vt - \varphi)}{\cos\varphi}).$$

Its value at stationary points is given by:

$$\Delta C^* = \delta(1 \pm e^{ut^*}\frac{v}{\sqrt{u^2+v^2}\cos\varphi})$$

where t^* is a stationary point. Thus it is enough to show that $|\cos\varphi| > \frac{v}{\sqrt{u^2+v^2}}$. Now, using (3.2.16):

$$u \pm iv = \frac{r - \sqrt{r^2 + 2\frac{\pi S}{\kappa B} \pm 4i\sqrt{-(\frac{\pi S}{2\kappa B})^2 + \frac{\pi S}{\beta B}}}}{2}$$

which implies that

$$v^2 < u^2 - ur. \tag{3.5.11}$$

Showing $\cos\varphi > \frac{v}{\sqrt{u^2+v^2}}$ is equivalent to proving:

$$((3u-r)(u-r)-v^2)^2(u^2+v^2) > U^2$$

which after simplification becomes:

$$((3u-r)(u-r)-v^2)^2u^2 > (u(u-r)^2+2rv^2-3uv^2)^2.$$

The above inequality may be simplified as

$$u^3(u-r)^2(2u-r)-v^4(u-r)(2u-r)+v^2ur(u-r)(2u-r) > 0,$$

which further reduces to,

$$(u-r)(2u-r)(u^2+v^2)(u^2-v^2-ru) > 0,$$

which holds due to (3.5.11).

Now, as D is increased to $D+\delta, \delta > 0$, change in $Y^{*(1)}$ becomes

$$\delta(1-\frac{B\beta}{S\pi}U\frac{e^{ut}}{\cos\varphi}(u\sin vt+v\cos vt))$$

Now, $\frac{\pi S}{\beta B}$ being equal to product of roots of (3.2.15), equals $(u^2+v^2)((u-r)^2+v^2)$. Further, the value of $e^{ut}(u\sin vt+v\cos vt)$ at stationary points is equal to $\pm e^{ut^*}v$ where t^* is a stationary point. Therefore, it is enough to show

$$(u^2+v^2)((u-r)^2+v^2)\cos\varphi > U,$$

or, equivalently,

$$(u^2+v^2)((u-r)^2+v^2)((3u-r)(u-r)-v^2) > U^2.$$

The above inequality simplifies to

$$\begin{aligned} & v^2(u-r)^2(-u^2-2ur-v^2)+v^2(3u^2+r^2-4ru)(u^2+3v^2) \\ + & v^2(-2v^4-10u^2v^2-4r^2v^2+12urv^2)+2u^3(u-r)^3>0, \end{aligned}$$

or, equivalently,

$$v^2(4u^2r^2-2ur^3+2u^4-4ru^3-2v^4-2u^2v^2+2urv^2-2r^2v^2)+2u^3(u-r)^3>0.$$

The LHS of the above inequality may be written as,

$$\begin{aligned} & v^2[2r^2(u^2-ur-v^2)+2u^2(u^2-ur-v^2)-2ur(u^2-ur-v^2)] \\ + & 2(u^3(u-r)^3-v^6) \end{aligned}$$

which is clearly positive using (3.5.11).

To compute the term corresponding to transient component in C^* in optimal price trajectory, using (3.5.4) and (3.5.10), we get:

$$P^{(1)}-rP=\frac{2\beta}{S}e^{ut}U\{d_1\sin(vt+\varphi)-d_2\cos(vt+\varphi)\}.$$

Hence,

$$P^*(t)=P^*_\infty(t)+e^{ut}(f_1\cos(vt+\varphi)+g_1\sin(vt+\varphi))$$

where

$$\begin{aligned} f_1 &= -\frac{2\beta}{S}U\frac{d_2(u-r)+d_1v}{(u-r)^2+v^2} \\ g_1 &= \frac{2\beta}{S}U\frac{d_1(u-r)-d_2v}{(u-r)^2+v^2}. \end{aligned}$$

Again, condition (3.2.6) is satisfied here too and (3.2.11) is also satisfied by the optimal trajectories.

3.5.7 Proof of Theorem 3.2.10

To compute the transient component of optimal capacity trajectory, define u_1 as the other root of (3.2.15) such that $u_1 > 0$ in addition to u. Then, the transient component in the solution to (3.2.14) is given by

$$e^{ut}(h_1 t + h_2) + e^{u_1 t}(h_3 t + h_4).$$

Since $u_1 = \frac{r}{2} + \sqrt{\frac{r^2 + 2\frac{\pi S}{\kappa B}}{4}} > r$ so condition (3.2.10) is satisfied only if $h_3 = h_4 = 0$. Therefore, optimal market capacity trajectory is given by:

$$C^*(t) = C^*_\infty(t) + e^{ut}(h_1 t + h_2). \tag{3.5.12}$$

As before using (3.5.5) and (3.5.12):

$$\begin{aligned} X^*(t) = Y^*(t) + X_0 &= Y^*_\infty(t) - \frac{B\beta}{S\pi} e^{ut}\{h_1(3u - r)(u - r) \\ &+ (h_1 t + h_2) u (u - r)^2\}. \end{aligned} \tag{3.5.13}$$

Using initial conditions $Y(0) = 0$ and $C(0) = C_0$, values of h_1 and h_2 can be computed to be

$$\begin{aligned} h_2 &= C_0 - D - \frac{S\pi}{B\beta}\zeta\alpha\gamma\cos\phi, \\ h_1 &= \frac{-u(u-r)^2(C_0 - D - \zeta\cos\phi) - \frac{S\pi}{B\beta}\zeta\alpha\gamma^2(r^2+\gamma^2)\sin(\theta-\phi) - \frac{S\pi}{B\beta}X_0}{(3u-r)(u-r)}. \end{aligned}$$

To compute terms in optimal price trajectory corresponding to transient component of C^*, we use (3.5.4) and (3.5.13) to get:

$$P^{(1)} - rP = -\frac{2\beta}{S} e^{ut}\{h_1(3u - r)(u - r) + (h_1 t + h_2) u (u - r)^2\}.$$

Hence,

$$P^*(t) = P^*_\infty(t) - \frac{2\beta}{S} e^{ut}\left(u(u - r)h_1 t + (2u - r)h_1 + u(u - r)h_2\right).$$

Again, condition (3.2.6) is satisfied here too and (3.2.11) is also satisfied by the optimal trajectories.

Next, we want to show that $Y^{*(1)}$ and the transient component of C^* become non-negative for large enough D. The change in C^* at time t as D is increased to $D+\delta$, where $\delta > 0$:

$$\Delta C^* = e^{ut}(\frac{u(u-r)}{3u-r}t - 1)\delta + \delta.$$

A function of the form $e^{at}(bt+c)$, where $a < 0$, is max(min)imized at $t^* = \frac{b+ac}{-ab}$. If $b > 0$, this function is maximized and when $b < 0$, it is minimized. The value of the function at t^* is equal to $-\frac{b}{a}e^{at^*}$.

In our case $b = \frac{u(u-r)\delta}{3u-r} < 0$ as $u < 0$. Thus, the minimum possible value of change in C^* as D is increased by δ, is $(\frac{-(u-r)}{3u-r}e^{ut} + 1)\delta > 0$. Therefore, for large enough δ, the capacity trajectory will remain non-negative.

The proof for $Y^{*(1)}$ also follows in a similar way. Upon increasing D to $D+\delta$, $\delta > 0$, change in $Y^{*(1)}$ becomes:

$$\Delta Y^{*(1)} = \delta - \delta \frac{B}{S}\frac{\beta}{\pi}e^{ut}\left(\frac{u^3(u-r)^3}{(3u-r)}t + \frac{u^2(u-r)^3}{(3u-r)}\right).$$

The product of roots of (3.2.15) in this case is given by $u^2(r-u)^2$ which also equals $\frac{S\pi}{B\beta}$.

The maximum possible value of $\frac{B\beta}{S\pi}e^{ut}\left(\frac{u^3(u-r)^3}{(3u-r)}t + \frac{u^2(u-r)^3}{(3u-r)}\right)\delta$ occurs at $t = 0$ which equals $\frac{B\beta}{S\pi}\frac{u^2(u-r)^3}{(3u-r)}\delta$ which upon substitution of $\frac{1}{u^2(u-r)^2}$ for $\frac{B\beta}{S\pi}$ becomes $\frac{u-r}{3u-r}\delta$ which is less than δ. Hence, $Y^{(1)}$ remains non-negative for large enough D.

3.5.8 Proof of Theorem 3.2.11

1. Combining equations (3.2.12) and (3.2.13) together to eliminate derivatives of Y_k:

$$2\beta\{-C_k^{(3)} + 2rC_k^{(2)} - r^2 C_k^{(1)}\} = P^{(1)} - rP. \tag{3.5.14}$$

Combining (3.5.2) and (3.5.3) to eliminate derivatives of Y yields:

$$\frac{2\beta}{S}\{-C^{(3)} + 2rC^{(2)} - r^2 C^{(1)}\} = P^{(1)} - rP. \tag{3.5.15}$$

Combining (3.5.14) and (3.5.15):

$$-(C_k^{(3)} - \frac{1}{S}C^{(3)}) + 2r(C_k^{(2)} - \frac{1}{S}C^{(2)}) - r^2(C_k^{(1)} - \frac{1}{S}C^{(1)}) = 0.$$

$C_k = \frac{1}{S}C + c_k$ solves the last equation where c_k is a constant to be determined using the initial conditions. Therefore, optimal capacity trajectory for seller k is given by

$$C_k^* = \frac{1}{S}C^* + c_k.$$

To determine c_k, recall $C_k^*(0) = C_{0k}$ and $C^*(0) = C_0$. Hence $c_k = C_{0k} - \frac{1}{S}C_0$. We know from the proofs of Theorems 3.2.8, 3.2.9 and 3.2.10 that C^* is an increasing function of D. Therefore, for sufficiently large D, C_k^* will satisfy the non-negativity constraint as well since for sufficiently large $D, |C^*| > S|c_k|$ for all t.

2. Replacing C_k in equation (3.2.12) by its value above, we obtain:

$$-\frac{\kappa}{\beta}Y_k^{*(1)} = \frac{1}{S}(C^{*(2)} - rC^{*(1)} - \frac{\kappa}{\beta}C^*) - \frac{\kappa}{\beta}c_k$$

which is equal to

$$\frac{\kappa}{\beta}(-\frac{Y^{*(1)}}{S} - c_k)$$

by equation (3.5.2). Hence

$$Y_k^* = \frac{Y^*}{S} + c_k t + y_k$$

where y_k is determined using initial conditions. Recall $Y_k(0) = 0$ and $Y^*(0) = 0$. Hence $y_k = 0$.

Regarding non-negativity of instantaneous rate of production, $Y_k^{*(1)}$, we know from the proofs of Theorems 3.2.8, 3.2.9 and 3.2.10 that $Y^{*(1)}$ is an increasing function of D. Therefore, for sufficiently large D, $Y_k^{*(1)}$ will satisfy the non-negativity constraint since for sufficiently large D, $|Y^{*(1)}(t)| > S|c_k|$ at all t.

3. Eliminating terms relating to P from (3.2.7) and (3.5.4), we get:

$$X_j^* = F_j + \frac{1}{B}(X^* - F)$$

Taking derivative with respect to time, we get:

$$X_j^{*(1)} = F_j^{(1)} + \frac{1}{B}(X^{*(1)} - F^{(1)})$$

Recall from Theorems 3.2.8, 3.2.9 and 3.2.10 that as D is increased by 1 unit, $X^{*(1)}(t)$ increases by $x(t) \geq 0$ units. Now, if D_j is increased by 1 unit, $F_j^{(1)}$ will go up by one unit and $\frac{1}{B}(X^{*(1)} - F^{(1)})$ will decrease by $\frac{1-x(t)}{B}$ units. Therefore, with $B \geq 2$ and for sufficiently large D_j and D, $X_j^{*(1)}$ will remain non-negative at all times.

3.5.9 Proof of Theorem 3.2.11

Let G be the integrand in the objective function of (3.3.1). Ignore the constraints. The necessary conditions for extremum are given by Euler's equations which yield:

$$Y_\iota^{(2)} - rY_\iota^{(1)} - \frac{\pi_\iota}{\kappa} Y_\iota = -\frac{\pi_\iota}{\kappa} F_\iota + C_\iota^{(1)} - rC_\iota$$

and

$$C_\iota^{(2)} - rC_\iota^{(1)} - \frac{\kappa}{\beta}C_\iota = -\frac{\kappa}{\beta}Y_\iota^{(1)}.$$

Chapter 4
Evolution of Price Uncertainty in a Market for Supply Chain Capacity

4.1 Introduction

We extend the deterministic case presented in the previous chapter to incorporate randomness in consumer demand for every buyer. The stochastic component of the demand in this model evolves as a Weiner's process. In addition to the stochastic extensions of the Market model and Integrated model presented in previous chapter, we also analyze a special case of the Market model in which capacity is fixed. We obtain closed-form expression for the equilibrium market price and use that to understand the relationship between the supply chain cost parameters and price volatility.

We invoke the hypothesis of Rational Expectations (Muth [58]) in order to model how prices evolve in the capacity market. Under this approach, each agent views the evolution of prices as an exogenous stochastic process and plans his or her actions relative to a particular specification of that process. For a rational agent to adopt or select a particular specification of the price process, that specification must be consistent with market equilibrium. In particular, we assume that each agent possesses perfect information and is capable of deducing what actions all other agents would plan if those agents planned according to the same price process specification that he or she has chosen. From this, each agent could deduce whether the specified process process will result in supply-demand imbalances in the present or at any time in the future. No rational agent will adopt a price process specification that can be deduced to result in dis-equilibrium if adopted by all. Under the rational expectations hypothesis, therefore, we restrict attention to those price processes which will result in equilibrium, if adopted by all. Furthermore, no rational agent will adopt a price process specification that is different from that adopted by the majority of other agents since the realized price process will tend to follow that adopted by the majority. Based on this argument, we assume that all agents base their plans on the same price process specification and that this specification has the property that it clears the market, in the sense of aggregated planned actions of supply and demand, at all instances in the present and future. We also assume the existence of powerful, rapid, market mechanisms such as arbitrage that force the realized price process to follow the universally adopted price process specification, Clearly, this is an idealized view of a market economy but it brings the analysis of price behavior within the scope of the tools of stochastic control.

A summary of the remaining sections is as follows. In Section 4.2, we present and discuss the Market model for the capacity. We derive necessary and sufficient conditions for optimality of the Market model and use them to obtain the equilibrium market trajectories. We compute the variance of the equilibrium trajectories and study the relationship between the cost parameters and the variance of the equilibrium trajectories. In Section 4.3, we introduce the Integrated model and show the equivalence of its optimal solution to the equilibrium solution of the Market model. In Section 4.4, we examine a special case of the Market model in which capacity is fixed exogenously. We obtain equilibrium trajectories for this model as well. Finally, we conclude in Section 4.5 with a summary of the results achieved.

4.2 The Market Model

We use the following model of instantaneous demand for buyer j:

$$dF_j(t) = D_j + \alpha_j \gamma \cos \gamma t + \sigma_j(t)\mathbf{dW}(t) \tag{4.2.1}$$

and, hence, the cumulative demand by time t is given by

$$F_j(t) = D_j t + \alpha_j \sin \gamma t + \int_0^t \sigma_j(s)\mathbf{dW}(s), \tag{4.2.2}$$

where D_j is the expected average demand rate, α_j is the amplitude of seasonal variation, γ is the seasonal frequency (assumed identical for all the agents in the market), and $\sigma_j(\cdot) := \frac{1}{B}\sigma(\cdot)$ is the diffusion coefficient which is identical across all the buyers. The market diffusion coefficient $\sigma : \mathcal{R} \to \mathcal{R}^n$ is assumed to be square-integrable, that is, $E\int_0^T \sigma^2(t)dt < \infty$. $\mathbf{W}(\cdot)$ is an $n-$ dimensional Weiner's process defined on $(\Omega, \mathcal{F}, \mathcal{P})$, a complete probability space. Define the filtration $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. We assume $n = 1$ for the sake of simplicity and without loss in generality, through the rest of this chapter. Note that the same Weiner's process is used to drive the demand process for all the buyers.

For each buyer j, the expected average demand rate D_j will be assumed to be suitably large relative to the seasonal amplitude, α_j, and seasonal frequency, γ, to ensure that the deterministic component of cumulative demand is non-decreasing. That is, we require $D_j \geq |\alpha_j \gamma|$. We assume that all buyers face the same seasonal frequency γ but may differ in expected average demand rates and seasonal amplitudes.

Let $P(t)$ denote the price of capacity, the homogeneous good, bought and sold at time t. We assume that this process is of the form:

$$P(t) = a(t) + \int_0^t b(s,t)dW(s) \tag{4.2.3}$$

for suitable value of $a : \mathcal{R} \to \mathcal{R}$ and $b : \mathcal{R}^2 \to \mathcal{R}$ where the Weiner's process $W(\cdot)$ is the same process that underlies the demand model. We assume $a(\cdot)$ and $b(\cdot,\cdot)$ are such that the price process is square-integrable. We formally state this assumption as follows:

Assumption 4.2.1. *$E\int_0^T P(t)^2 dt < \infty$ and $P(t)$ is $\mathcal{F}_t$-adapted.*

We are unable to prove that the equilibrium price process has this form but we can derive values of $a(\cdot)$ and $b(\cdot,\cdot)$ that are consistent with equilibrium conditions and this assumption. Provided that $a(\cdot)$ and $b(\cdot,\cdot)$ satisfy certain equilibrium conditions described below, we assume that each agent in the economy plans and implements production and procurement decisions according to this specific price process. Hence, the sellers' production plans and the buyers' procurement plans will be seen to be stochastic functions of $a(\cdot)$ and $b(\cdot,\cdot)$.

In the models to follow, we suppress the time argument t unless needed for clarification. In both the buyer and seller models, we assume a quadratic cost structure in order to derive explicit solutions.

Each seller faces the following stochastic control problem. Given a price process $P(\cdot)$, seller k's problem is to choose a capacity policy, $c_k(\cdot)$, and a production policy, $y_k(\cdot)$, to minimize the expected total cost of investment/disinvestment, and the short-term capacity adjustment less the revenue derived from production. The finite horizon version of this problem with quadratic costs is:

$$\begin{aligned}
&\inf_{c_k, y_k \in \mathcal{U}_k[0,T]} E \int_0^T \{\beta c_k^2 + \kappa (C_k - y_k)^2 - P y_k\} dt \\
&\text{s.t.} \\
&dC_k(t) = c_k(t) dt, \\
&dY_k(t) = y_k(t) dt, \\
&Y_k(0) = Y_{0k} = 0, \ C_k(0) = C_{0k};
\end{aligned} \tag{4.2.4}$$

where $T > 0$ denotes the length of the horizon, β is the penalty coefficient on the rate of change of capacity (i.e. on investment/disinvestment), and κ is the penalty coefficient on the over- or under-utilization of capacity. The parameter β captures long-term costs of changing capacity, such as changes in plant and equipment, and the parameter κ captures short-term costs of production adjustments, such as changes in workforce. The first two constraints describe the evolution over time of capacity and cumulative production, respectively. The remaining constraints state that the initial production and capacity are fixed exogenously.

In order to solve the model using the Stochastic Maximum Principle (Cadenillas and Karatzas [9]), we impose following assumptions on the control pair (y_k, c_k).

Assumption 4.2.2. *The set of controls* $\mathcal{U}_k[0,T]$ *consists of all* $y_k : [0,T] \times \Omega \to \mathcal{R}^1$ *and* $c_k : [0,T] \times \Omega \to \mathcal{R}^1$ *such that* $y_k(\cdot)$ *and* $c_k(\cdot)$ *are measurable,* $\mathcal{F}_t$*-adapted,* $E \int_0^T y_k(t)^2 dt < \infty$*, and* $E \int_0^T c_k(t)^2 dt < \infty$*.*

Assumption 4.2.3. *For any* $(y_k^1, c_k^1), (y_k^2, c_k^2) \in \mathcal{U}_k[0,T]$*, and* $\rho \in [0,1]$*, the following holds:*

$$E\left[\int_0^T |2\beta(c_k^1 + \rho c_k^2) + 2\kappa(C_k^1 + \rho C_k^2 - y_k^1 - \rho y_k^2) - P|^2 dt\right] < \infty,$$

where C_k^1 *and* C_k^2 *are states of the systems controlled by* c_k^1 *and* c_k^2*, respectively.*

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 4.2.2 along with the linearity of the state equations for capacity and cumulative production in the Seller model (4.2.4), ensure a unique solution to the state equations. Assumption 4.2.3 is satisfied due to the square-integrability of P and control pair (y_k, c_k).

Similarly, each buyer faces the following stochastic control problem. Given a price process $P(\cdot)$, buyer j's problem is to choose a production order policy $x_j(\cdot)$ to minimize the expected total cost of production orders and inventory/shortfall costs. The finite horizon version of this problem with quadratic costs is:

$$\begin{aligned}
&\inf_{x_j \in \mathcal{U}_j} E \int_0^T \{P x_j + \pi I_j^2\} dt \\
&\text{s.t.} \\
&dI_j(t) = (x_j(t) - D_j - \alpha_j \gamma \cos \gamma t) dt - \tfrac{1}{B} \sigma(t) dW(t), \\
&X_j(0) = X_{0j};
\end{aligned} \tag{4.2.5}$$

where π denotes the net inventory penalty coefficient and $I_j(t)$ is the net inventory at time t (on hand inventory less backorders) so the objective function penalizes any deviation of net inventory from zero. The first constraint in the Buyer model (4.2.5) describes the evolution of net inventory over time. The second constraint reflects that the initial cumulative production orders are fixed exogenously. We assume that all the buyers start with the same level of inventory, that is, $X_{0j} = \frac{X(0)}{B}$.

In order to solve the Buyer model using the Stochastic Maximum Principle (Cadenillas and Karatzas [9]), we impose the following conditions on the control variable x_j.

Assumption 4.2.4. *The set of controls $\mathcal{U}_j[0,T]$ consists of all $x_j : [0,T] \times \Omega \to \mathcal{R}^1$ such that $x_j(\cdot)$ is measurable, $\mathcal{F}_t$-adapted and $E \int_0^T x_j(t)^2 dt < \infty$.*

Assumption 4.2.5. *For any $x_j^1, x_j^2 \in \mathcal{U}_j[0,T]$, and $\rho \in [0,1]$ the following holds:*

$$E\left[\int_0^T |I_j^1 + \rho I_j^2|^2 dt\right] < \infty,$$

where I_j^1 and I_j^2 are states of the system controlled by x_j^1 and x_j^2, respectively.

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 4.2.4 along with the linear nature of the state equation for the evolution of net inventory in the Buyer model (4.2.5) ensure a unique solution to the state equation. Assumption 4.2.5 is satisfied due to the square-integrability of the control variable x_j.

Note that we are not restricting the production rate and capacity variables in the Seller model and order rate in the Buyer model to be non-negative. Although such constraints are desirable from a practical point of view, imposition of those constraints makes it impossible to derive optimal trajectories in closed form. The closed form expressions we derive are essential in our approach to obtain the variance of the optimal trajectories. In the previous chapter, we show that production rates, order rates, and capacity are positive for large enough rate of demand D_j when end-consumer demand is deterministic. When the end-consumer demand has an element of randomness as in (4.2.1), we conjecture that the linear component of demand D_j can be made large enough so that production rates, order rates, and capacity are positive a.e. with high probability.

The state equation for the net inventory can be written as:

$$dI_j(t) = dX_j(t) - dF_j(t),$$

and hence, the net inventory at time t is equal to

$$I_j(t) = X_j(t) - F_j(t), \tag{4.2.6}$$

where $X_j(t)$ is the cumulative quantity ordered by time t. It should be noted that an alternate but equivalent approach to state the Buyer model would be by using

$X_j(t)$ as a state variable in place of $I_j(t)$. Indeed, in the rest of the chapter, we shall switch between presenting results using $I_j(t)$ and $X_j(t)$, respectively, depending upon the ease of the exposition. By (4.2.6), no confusion should arise in general.

Throughout this chapter, we assume that all the sellers face the same penalty parameters β and κ for change of capacity and production-capacity mismatch, respectively. Similarly, all the buyers face the same inventory/shortage penalty parameter π.

Observe that the costs of production (material, labor, and capital) are ignored in the Seller model (4.2.4): only the costs of capacity adjustment, short and long-term, are captured. Also observe that the revenue from consumer sales are ignored in the Buyer model (4.2.5): only the inventory/shortfall costs are relevant. As a result, the price in this market will reflect only the trade-off between the sellers' capacity adjustment costs and the buyers' inventory/shortfall costs. The price $P(\cdot)$ could then be interpreted as a premium (if positive) or a discount (if negative) on another price (not modelled) that captures the trade-off we have ignored.

Since all agents are assumed to be price takers, the production and production order policies, $y_k(\cdot)$ and $x_j(\cdot)$, that optimize seller and buyer problems, respectively, will depend on the price process $P(\cdot)$. We assume that the market will be in equilibrium at all times. That is, the price process $P(\cdot)$ must ensure that

$$y(t) = x(t) \text{ for all } t \geq 0. \tag{4.2.7}$$

In equilibrium, therefore,

$$Y(t) = X(t) - X_0.$$

By means of (4.2.2), (4.2.4), (4.2.5), and (4.2.7), we have described a simple market for capacity in which the demand for capacity is intertemporal in nature:

if capacity prices are high, buyers can defer production orders (depleting inventory or incurring shortages) and if capacity prices are low, then buyers can advance production orders in time (eliminating shortages or building inventory). We proceed to solve these models and to demonstrate this behavior.

4.2.1 Necessary and Sufficient Conditions for the Optimality of the Market Model

Buyer Model

As before, we suppress the argument t unless needed for clarity. While applying the Stochastic Maximum Principle, we treat the price as a square-integrable and $\mathcal{F}_t-$ adapted random coefficient of the order quantity in the Buyer model.

Define the Hamiltonian function for the Buyer model as:

$$H_j(x_j, I_j, p_{1,j}, q_{1,j}) = p_{1,j}(x_j - D_j - \alpha_j \gamma \cos \gamma t) + q_{1,j}\sigma_j - Px_j - \pi I_j^2,$$

where I_j is the state of the system controlled by x_j. The pair of adjoint variables $(p_{1,j}, q_{1,j})$, where $p_{1,j} : \Omega \times [0,T] \to \mathcal{R}$, $q_{1,j} : \Omega \times [0,T] \to \mathcal{R}$, is measurable, adapted and is defined by the following stochastic differential equation:

$$\begin{aligned} dp_{1,j}(t) &= 2\pi I_j(t)dt + q_{1,j}(t)dW(t), \\ p_{1,j}(T) &= 0. \end{aligned}$$

The adjoint variable $p_{1,j}$ can be interpreted as the shadow price corresponding to the net inventory resource. Observe that the coefficient of $p_{1,j}$ in the Hamiltonian function H_j corresponds to the drift term in the state equation for $I_j(t)$ in (4.2.5). Similarly, the coefficient of $q_{1,j}$ corresponds to the diffusion term in the state equation for net inventory in (4.2.5). The Hamiltonian function can be interpreted as the infinite-dimensional-space counterpart of the Lagrangian function.

Using Proposition 1.2 in Cadenillas and Karatzas [9], a necessary and sufficient condition for $\bar{x}_j$ to be optimal for the Buyer model (4.2.5) is that $\forall x_j \in \mathcal{U}_j$:

$$E\left(\int_0^T (P(t) - \bar{p}_{1,j}(t))(x_j(t) - \bar{x}_j(t))\right) \geq 0,$$

where the adjoint variable pair $(\bar{p}_{1,j}, \bar{q}_{1,j})$ corresponds to a system controlled by $\bar{x}_j$. The above condition is satisfied if and only if

$$P(t) = \bar{p}_{1,j}(t), a.e.(t, \omega) \in [0, T] \times \Omega. \tag{4.2.8}$$

Therefore, optimality of the Buyer model requires that the price in the market be exactly equal to the value of the resource I_j to buyer j. In fact, if $P \neq \bar{p}_{1,j}$ over any set of time t of positive measure then the Hamiltonian function is unbounded over that set. That is, (4.2.8) imposes a condition on the functions $a(\cdot)$ and $b(\cdot, \cdot)$ for equilibrium behavior.

Summing (4.2.8) over all the buyers:

$$P(t) = \frac{\bar{p}_1(t)}{B}, a.e.(t, \omega) \in [0, T] \times \Omega. \tag{4.2.9}$$

where $\bar{p}_1(t) = \sum_j \bar{p}_{1,j}(t)$. We next apply the Stochastic Maximum Principle to the Seller model.

Seller Model

As before, we treat the price at time t as a square-integrable and $\mathcal{F}_t-$adapted coefficient of the rate of production $y_k(t)$ in the objective function. Define the Hamiltonian function for the Seller model as:

$$H_k(y_k, c_k, Y_k, C_k, p_{2,k}, q_{2,k}, p_{3,k}, q_{3,k}) = p_{2,k}c_k + p_{3,k}y_k - \beta c_k^2 - \kappa(C_k - y_k)^2 + Py_k$$

where Y_k and C_k are the states of the system controlled by y_k and c_k. The pairs of adjoint variables $(p_{2,k}, q_{2,k}), (p_{3,k}, q_{3,k})$ are measurable, adapted and defined by the following stochastic differential equations:

$$\begin{aligned} dp_{2,k}(t) &= 2\kappa(C_k(t) - y_k(t))dt + q_{2,k}(t)dW(t), \qquad (4.2.10) \\ dp_{3,k}(t) &= q_{3,k}(t)dW(t), \\ p_{2,k}(T) &= p_{3,k}(T) = 0. \end{aligned}$$

The adjoint variables, $p_{2,k}$ and $p_{3,k}$, can be interpreted as the shadow prices corresponding to the resources, C_k and Y_k, respectively. Even though the adjoint variables, $q_{2,k}$ and $q_{3,k}$, are not required to satisfy any differential equations, they cannot be set identically to zero. The solution of the stochastic differential equation (4.2.10) with $q_{2,k} = q_{3,k} \equiv 0$, may not be $\mathcal{F}_t$-adapted.

Observe that the terminal condition for $p_{3,k}(\cdot)$ is satisfied if and only if $p_{3,k} = q_{3,k} \equiv 0$. As a result, we can ignore $p_{3,k}$ and $q_{3,k}$ in the subsequent analysis. It is appropriate that $p_{3,k}$ be zero since the cumulative production variable Y_k does not appear in the objective function of the Seller model.

Setting $p_{3,k}$ to zero in the expression for the Hamiltonian function for the Seller model yields:

$$H_k(y_k, c_k, Y_k, C_k, p_{2,k}, q_{2,k}) = p_{2,k}c_k - \beta c_k^2 - \kappa(C_k - y_k)^2 + Py_k.$$

According to Theorem 3.2, Cadenillas and Karatzas [9], if the objective function is convex in the state and control variables and is (possibly) random, then $(\bar{y}_k, \bar{c}_k)$ is a pair of optimal control variables if and only if

$$\max_{(y_k,c_k)\in\mathcal{U}_k} H_k(y_k, c_k, \bar{Y}_k, \bar{C}_k, \bar{p}_{2,k}, \bar{q}_{2,k}) = H_k(\bar{y}_k, \bar{c}_k, \bar{Y}_k, \bar{C}_k, \bar{p}_{2,k}, \bar{q}_{2,k}),$$
$$a.e.(t,\omega) \in [0,T] \times \Omega.$$

where $(\bar{Y}_k, \bar{C}_k)$ and $(\bar{p}_{2,k}, \bar{q}_{2,k})$ are the state variable and adjoint variable pairs corresponding to the system controlled by $(\bar{y}_k, \bar{c}_k)$. The above equation yields the following two necessary and sufficient conditions:

$$
\begin{aligned}
P &= 2\kappa(\bar{y}_k - \bar{C}_k), \\
\bar{p}_{2,k} - 2\beta\bar{c}_k &= 0, a.e.(t,\omega) \in [0,T] \times \Omega.
\end{aligned}
$$

Summing the two equations over all the sellers:

$$
\begin{aligned}
SP &= 2\kappa(\bar{y} - \bar{C}), \qquad (4.2.11) \\
\bar{p}_2 &= 2\beta\bar{c}, a.e.(t,\omega) \in [0,T] \times \Omega,
\end{aligned}
$$

where $\bar{p}_2 = \sum_k \bar{p}_{2,k}$. Substituting for $P(\cdot)$ in (4.2.11) using (4.2.9):

$$
\begin{aligned}
\frac{S\bar{p}_1}{B} + 2\kappa(\bar{C} - \bar{y}) &= 0, \qquad (4.2.12) \\
\bar{p}_2 - 2\beta\bar{c} &= 0, \qquad (4.2.13)
\end{aligned}
$$

$$a.e.(t,\omega) \in [0,T] \times \Omega.$$

Using the equilibrium condition (4.2.7), we combine the necessary and sufficient conditions for the optimality of the Buyer and Seller models and obtain the necessary and sufficient conditions for the optimality of the Market model. The results are stated formally in the following proposition.

Proposition 4.2.6. *Let* $\bar{q}_1(t) = \sum_j \bar{q}_{1,j}(t)$ *and* $\bar{q}_2(t) = \sum_k \bar{q}_{2,k}(t)$. *Then, the vector of market variables,* $(\bar{Y}, \bar{C}, \bar{y}(= \bar{x}), \bar{c}, \bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2)$ *is in equilibrium if and only if it* *satisfies the following system of equations:*

$$
\begin{aligned}
\bar{c}(t) &= \frac{1}{2\beta}\bar{p}_2(t), \qquad (4.2.14)\\
\bar{x}(t) &= \bar{y}(t) = \bar{C}(t) + \frac{S}{2\kappa B}\bar{p}_1(t),\\
d\bar{I}(t) &= (\bar{y}(t) - D - \alpha\gamma\cos\gamma t)dt - \sigma(t)dW(t),\\
d\bar{C}(t) &= \bar{c}(t)dt,\\
d\bar{p}_1(t) &= 2\pi\bar{I}(t)dt + \bar{q}_1(t)dW(t),\\
d\bar{p}_2(t) &= 2\kappa(\bar{C}(t) - \bar{y}(t))dt + \bar{q}_2(t)dW(t),\\
\bar{p}_1(T) &= \bar{p}_2(T) = 0.
\end{aligned}
$$

Proof. The "if" part is clear. To show the "only if" part, it is enough to find a disaggregated solution for each buyer and seller, given an aggregated solution of the above equations, that satisfies the necessary and sufficient conditions for the Buyer and Seller models. Consider the following disaggregated solution for the Buyer model,

$$
\begin{aligned}
(\bar{x}_j(t), \bar{I}_j(t), \bar{p}_{1,j}(t), \bar{q}_{1,j}(t)) &= (\frac{\bar{x}(t) - D - \alpha\gamma\cos\gamma t}{B} + D_j + \alpha_j\gamma\cos\gamma t, \frac{\bar{I}(t)}{B},\\
&\quad \frac{\bar{p}_1(t)}{B}, \frac{\bar{q}_1(t)}{B}),
\end{aligned}
$$

and the Seller model,

$$
\begin{aligned}
(\bar{y}_k(t), \bar{Y}_k(t), \bar{c}_k(t), \bar{C}_k(t), \bar{p}_{2,k}(t), \bar{q}_{2,k}(t)) &= (\frac{\bar{y}(t) - C_0}{S} + C_{0,k},\\
&\quad \frac{\bar{Y} - Y_0 - C_0 t}{S} + C_{0,k}t + Y_{0,k}, \frac{\bar{c}(t)}{S},\\
&\quad \frac{\bar{C}(t) - C_0}{S} + C_{0,k}, \frac{\bar{p}_2(t)}{S}, \frac{\bar{q}_2(t)}{S}).
\end{aligned}
$$

where the equality holds componentwise. Clearly, the above solution satisfies the necessary and sufficient conditions for the Buyer and Seller models. The proof is completed by noting the uniqueness of solution to (4.2.14) as demonstrated in Proposition 4.2.7. □

As the proof of above result shows, the market distributes the equilibrium capacity and rate of production among the sellers equally (save for the correction due to initial values). Similarly, all the buyers place divide the market rate of order placement equally among themselves though the difference in the deterministic component of the end-consumer demand is taken into account. This result is hardly surprising in that the sellers and buyers are identical, with regard to their cost parameters.

So far, we have derived the necessary and sufficient conditions for optimality of the Seller and the Buyer models and then used the market equilibrium condition to obtain conditions which the equilibrium market variables must satisfy. In the following subsection, we obtain the equilibrium market solution using the necessary and sufficient conditions (4.2.14).

4.2.2 Optimal Solution to the Market Model

To obtain the solution to the necessary and sufficient conditions for optimality of the Market model (4.2.14), we hypothesize the following relationship between the adjoint variable vector $[\bar{p}_1(t), \bar{p}_2(t)]^T$ and the corresponding state vector $[\bar{I}(t), \bar{C}(t)]^T$:

$$\begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = -Z(t) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} - \varphi(t),$$

where $Z(\cdot) \in \mathcal{C}^1([0,T]; \mathcal{S}^2)$ and $\varphi(\cdot) \in \mathcal{C}^1([0,T]; \mathcal{R}^2)$. Using the above hypothesis, we obtain the optimal feedback solution to the Market model. In the feedback solution, the control and the adjoint variables are expressed as functions of the state variables. Thus, the instantaneous rate of production and capacity update at any instant are functions of the net inventory and the capacity at that instant in the optimal feedback solution.

Before we state the optimal feedback solution, we define shorthand notation. Let

$$Q = \begin{bmatrix} 2\pi' & 0 \\ 0 & 2\kappa \end{bmatrix}, U = \begin{bmatrix} 0 & -2\kappa \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 2\kappa & 0 \\ 0 & 2\beta \end{bmatrix},$$

and $$b = \begin{pmatrix} -D - \alpha\gamma\cos\gamma t \\ 0 \end{pmatrix},$$

where

$$\pi' = \frac{S}{B}\pi.$$

In the following proposition, we state the optimal feedback solution to the Market model.

Proposition 4.2.7. *Let $Z(\cdot) \in \mathcal{C}^1([0,T];\mathcal{S}^2)$ and $\varphi(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^2)$ be the solutions of the following ordinary differential equations:*

$$\begin{aligned} \dot{Z} + Q - (Z+U)^T R^{-1}(Z+U) &= 0, \\ Z(T) &= 0, \end{aligned} \tag{4.2.15}$$

and

$$\begin{aligned} \dot{\varphi} - (R^{-1}(Z+U))^T\varphi + Zb &= 0, \\ \varphi(T) &= 0, \end{aligned} \tag{4.2.16}$$

respectively. The unique, adapted and square-integrable solution to (4.2.14) is given *by:*

$$\begin{bmatrix} \bar{y}(t) \\ \bar{c}(t) \end{bmatrix} = -R^{-1}\left((Z(t)+U)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} + \varphi(t)\right), \tag{4.2.17}$$

$$\begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = -Z(t)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} - \varphi(t), \tag{4.2.18}$$

$$\begin{bmatrix} \frac{S}{B}\bar{q}_1(t) \\ \bar{q}_2(t) \end{bmatrix} = Z(t)\begin{bmatrix} \sigma(t) \\ 0 \end{bmatrix}. \tag{4.2.19}$$

Proof. See appendix. □

Using the adaptiveness and square-integrability of $\bar{p}_1$ and the relationship (4.2.9), the square-integrability and adaptiveness of the equilibrium price also follow.

Computation of the solution to the differential equations (4.2.15)-(4.2.16) is considered in the next subsection.

To compute the variance of the equilibrium price process, we will need to express the optimal production and capacity trajectories, $(\bar{Y}(t), \bar{C}(t))$ in closed form. We will exploit stochastic differential equation solution tools to obtain the optimal production and capacity trajectories in closed form. Let

$$Z(t) := \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}(t) & Z_{22}(t) \end{bmatrix} \tag{4.2.20}$$

where the anti-diagonal entries are equal due to the symmetry of $Z(t)$. Formally, we multiply both sides of equation (4.2.17) by dt and substitute for $(\bar{y}(t), \bar{c}(t))$ in terms of $(\bar{Y}(t), \bar{C}(t))$ on the LHS. Using the relationship (4.2.6), equation (4.2.17) can be rewritten for $t \in [0,T]$ as:

$$\begin{aligned}
\begin{bmatrix} d\bar{Y}(t) \\ d\bar{C}(t) \end{bmatrix} &= -R^{-1}(Z(t)+U)\begin{bmatrix} \bar{Y}(t) \\ \bar{C}(t) \end{bmatrix} dt \\
&+ R^{-1}\left(Z(t)\begin{bmatrix} F(t)-X_0+Y_0 \\ 0 \end{bmatrix} - \varphi(t)\right) dt, \qquad (4.2.21) \\
\bar{Y}(0) &= Y_0, \bar{C}(0) = C_0,
\end{aligned}$$

where

$$F(t) = Dt + \alpha \sin \gamma t + \int_0^t \sigma(s)dW(s).$$

According to Theorem 6.16, pp 49, Yong and Zhou [79], the above differential equation has a unique solution. Using an approach similar to the Variation of Constants method, we obtain the solution to the above differential equation. We state the solution in the following proposition.

Proposition 4.2.8. *The unique solution to (4.2.21) is given by:*

$$\begin{aligned}
\begin{bmatrix} \bar{Y}(v) \\ \bar{C}(v) \end{bmatrix} &= \phi(v)\int_0^v \phi^{-1}(t)R^{-1}\left(Z(t)\begin{bmatrix} F(t)-X_0+Y_0 \\ 0 \end{bmatrix} - \varphi(t)\right) dt \\
&+ \phi(v)\begin{bmatrix} Y_0 \\ C_0 \end{bmatrix}, \qquad (4.2.22)
\end{aligned}$$

where $\phi(\cdot)$ is the unique solution to the following system of differential equations:

$$\begin{aligned}
d\phi(t) &= -R^{-1}(Z(t)+U)\phi(t)dt, \qquad (4.2.23) \\
\phi(0) &= I_{22},
\end{aligned}$$

where I_{22} is a 2×2 identity matrix. Further, the equilibrium market price is given by:

$$\bar{P}(v) = -\frac{Z_{11}(v)\bar{I}(v) + Z_{12}(v)\bar{C}(v) + \varphi_1(v)}{S}, \qquad (4.2.24)$$

where $\varphi_1(\cdot)$ *is the first component of* $\varphi(\cdot)$*.*

Proof. Derivation of equation (4.2.22) is provided in the appendix. Equation (4.2.24) is obtained by substituting for $\bar{p}_1(t)$ in (4.2.18) by $P(t)$ using (4.2.9). $\square$

Note that with an interchange of integrals, it would be possible to write (4.2.24) in the form (4.2.3). At this point, we have achieved a major goal of this chapter by expressing the equilibrium trajectories of $Y(\cdot), C(\cdot)$, and $P(\cdot)$ as stochastic integrals of real matrix valued functions $Z(\cdot), \varphi(\cdot)$, and $\phi(\cdot)$ that are determined implicitly from a system of first order differential equations with known boundary conditions.

A complete specification of the optimal production and capacity trajectories requires computation of $Z(\cdot)$, $\varphi(\cdot)$ and $\phi(\cdot)$. The differential equation (4.2.15) defining $Z(\cdot)$ is a matrix-Riccatti differential equation. Riccatti differential equations are typically not amenable to analytical solutions, especially when they are based in a matrix of dimension higher than 1. Observe, however, that (4.2.15) and (4.2.16) do not involve $\sigma(\cdot)$. Consequently, the solution to these differential equations are common to a family of equilibrium models including the deterministic case, $\sigma(\cdot) \equiv 0$. We exploit this fact in the next subsection using an indirect approach to compute the solution to this system of differential equations. We also show how to obtain the solution to (4.2.23) in the following subsection.

Computation of $Z(\cdot)$ and $\varphi(\cdot)$

We define a deterministic class of problems to obtain $Z(\cdot)$ and $\varphi(\cdot)$. These problems are indexed by the parameter $s \in [0, T)$ and a generic problem is denoted by $\mathcal{D}(s)$. The parameter s specifies the beginning of the problem horizon. The problem $\mathcal{D}(s)$

for some $s \in [0, T)$ may be stated as:

$$\begin{aligned}
&\inf_{c^s, y^s \in \mathcal{U}^s} \int_s^T \{\beta c^s(t)^2 + \kappa(C^s(t) - y^s(t))^2 + \pi' I^s(t)^2\} dt \\
&\text{s. t.} \\
&dI^s(t) = (y^s(t) - D - \alpha\gamma \cos\gamma t)dt, \qquad (4.2.25) \\
&dC^s(t) = c^s(t)dt, \\
&I^s(0) = Y_s, C^s(0) = C_s,
\end{aligned}$$

where

$$\mathcal{U}^s := \left\{ (y^s, c^s) : [s, T] \to \mathcal{R}^2, \int_s^T y^s(t)^2 dt < \infty \text{ and } \int_s^T c^s(t)^2 dt < \infty \right\}.$$

All the variables and parameters in the above formulation have an interpretation similar to the Market model (4.2.4-4.2.5).

For any $s \in [0, T)$, $\mathcal{D}(s)$ is a deterministic Linear-Quadratic problem and is called *solvable* if for any $t \geq s$, an optimal feedback control exists. We apply Corollary 2.10, pp 297, Yong and Zhou [79] to obtain the optimal feedback solution for $\mathcal{D}(s)$. We state the results in the following proposition.

Proposition 4.2.9. *Let $Z(\cdot) \in \mathcal{C}^1([0, T]; \mathcal{S}^2)$ and $\varphi(\cdot) \in \mathcal{C}^1([0, T]; \mathcal{R}^2)$ be the solutions of the following differential equations:*

$$\begin{aligned}
\dot{Z} + Q - (Z + U)^T R^{-1}(Z + U) &= 0, \\
Z(T) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\dot{\varphi} - (R^{-1}(Z + U))^T \varphi + Zb &= 0, \qquad (4.2.26) \\
\varphi(T) &= 0,
\end{aligned}$$

respectively. Then $\mathcal{D}(s)$ is solvable with the optimal control pair $(\bar{y}^s, \bar{c}^s)$ being of *the following form:*

$$\begin{bmatrix} \bar{y}^s(t) \\ \bar{c}^s(t) \end{bmatrix} = -R^{-1}\left((Z(t)+U) \begin{bmatrix} \bar{I}^s(t) \\ \bar{C}^s(t) \end{bmatrix} + \varphi(t) \right), t \in [s,T].$$

Further, let

$$V^s(Y_s, C_s) := \inf_{(c^s, y^s) \in \mathcal{U}^s[s,T]} \int_s^T \{\beta c^s(t)^2 + \kappa(C^s(t) - y^s(t))^2 + \pi' I^s(t)^2\} dt$$

then

$$V^s(Y_s, C_s) = \frac{1}{2} z^T Z(s) z + \varphi(s)^T z + \frac{1}{2}\int_s^T [2\varphi(t)^T b(t) - |R^{-\frac{1}{2}}\varphi(t)|^2] dt,$$
$$\forall z := (Y_s, C_s) \in \mathcal{R}^2. \quad (4.2.27)$$

Proof. The proof follows directly from Theorem 2.8, pp 294, Yong and Zhou [79]. □

Observe that the differential equations for $Z(\cdot)$ and $\varphi(\cdot)$ in the above proposition are the same as in Proposition 4.2.7. Using (4.2.27), the second derivative of $V^s(Y_s, C_s)$ with respect to initial state values, if it exists, yields $Z(s)$. We state this result formally in the following corollary.

Corollary 4.2.10. *Let $V^s(Y_s, C_s)$ be as defined in Proposition 4.2.9. Assuming $V^s(Y_s, C_s) \in \mathcal{C}^2(\mathcal{R}^2; \mathcal{R})$:*

$$Z(s) = \frac{\partial^2 V^s(Y_s, C_s)}{\partial z^2} \quad (4.2.28)$$

where $z = (Y_s, C_s)$ is the vector of initial state values in (4.2.25).

Proof. The proof is immediate from Proposition 4.2.9. □

Given $Z(\cdot)$, we could easily compute $\varphi(\cdot)$ using the optimal feedback solution to $\mathcal{D}(0)$. Another approach involves using the expression for $V^s(Y_s, C_s)$ stated in (4.2.27) in Proposition 4.2.9. The following corollary states the second approach formally.

Corollary 4.2.11. *Let $V^s(Y_s, C_s)$ be as defined in Proposition 4.2.9. Assuming $V^s(Y_s, C_s) \in \mathcal{C}^2(\mathcal{R}^2; \mathcal{R})$:*

$$\varphi(s) = \frac{\partial V^s(Y_s, C_s)}{\partial z} - Z(s)z$$

where $z = (Y_s, C_s)$ is the vector of initial state values in (4.2.25).

Proof. The proof is immediate from Proposition 4.2.9. □

Before we can use Corollaries 4.2.10 and 4.2.11 to obtain $Z(\cdot)$ and $\varphi(\cdot)$, respectively, we require a closed form expression for $V^s(Y_s, C_s)$. Fortunately, each of the problems in $\mathcal{D}(s)$ can be solved directly and a closed form expression for $V^s(Y_s, C_s)$ with desirable degree of smoothness, can indeed be obtained. We are aware of at least two techniques to solve the class of problems in $\mathcal{D}(s)$. A demonstration of the application of "Euler's Equations" is provided in the previous chapter. Another approach is the Deterministic Maximum principle.

We do not provide the complete solution to the problems in $\mathcal{D}(s)$ here. The interested reader is referred to the previous chapter, in which the complete solutions (optimal controls and the state trajectories) are derived in the case of an infinite horizon. For any s, the functional form of the solution remains the same in the case of a finite or an infinite horizon. For an infinite horizon, the functional form of the solution depends on whether $\beta >, =,$ or $< \frac{4\kappa^2}{\pi'}$ and the same is true for all the problems in $\mathcal{D}(s)$ as well. For the sake of illustration, however, we provide an example in which we state optimal control and state trajectories when $\beta > \frac{4\kappa^2}{\pi'}$.

Example 4.2.12. *Let $\beta > \frac{4\kappa^2}{\pi'}$. The optimal solution to the problem $\mathcal{D}(s)$ at*

$t \in [s, T]$ is given by:

$$\begin{aligned}
\bar{y}^s(t) &= D + \alpha\gamma\cos\gamma t - \frac{\alpha\gamma^5\cos\gamma t}{\gamma^4 + \frac{\pi'}{\kappa}\gamma^2 + \frac{\pi'}{\beta}} \\
&- \frac{\beta}{\pi'}\left(c_1 d_1^4 e^{d_1 t} + c_2 d_1^4 e^{-d_1 t} + c_3 d_2^4 e^{d_2 t} + c_4 d_2^4 e^{-d_2 t}\right) \qquad (4.2.29) \\
\bar{c}^s(t) &= -\frac{\alpha\gamma^2}{\gamma^4 + \frac{\pi'}{\kappa}\gamma^2 + \frac{\pi'}{\beta}}\frac{\pi'}{\beta}\sin\gamma t \\
&+ c_1 d_1 e^{d_1 t} - c_2 d_1 e^{-d_1 t} + c_3 d_2 e^{d_2 t} - c_4 d_2 e^{-d_2 t} \qquad (4.2.30) \\
\bar{I}^s(t) &= -\frac{\alpha\gamma^4}{\gamma^4 + \frac{\pi'}{\kappa}\gamma^2 + \frac{\pi'}{\beta}}\sin\gamma t \\
&- \frac{\beta}{\pi'}\left(c_1 d_1^3 e^{d_1 t} - c_2 d_1^3 e^{-d_1 t} + c_3 d_2^3 e^{d_2 t} - c_4 d_2^3 e^{-d_2 t}\right) \qquad (4.2.31) \\
\bar{C}^s(t) &= D + \frac{\alpha\gamma}{\gamma^4 + \frac{\pi'}{\kappa}\gamma^2 + \frac{\pi'}{\beta}}\frac{\pi'}{\beta}\cos\gamma t \\
&+ c_1 e^{d_1 t} + c_2 e^{-d_1 t} + c_3 e^{d_2 t} + c_4 e^{-d_2 t} \qquad (4.2.32)
\end{aligned}$$

where

$$d_1, d_2 = \frac{\sqrt{2\frac{\pi'}{\kappa} \pm 4\sqrt{(\frac{\pi'}{2\kappa})^2 - \frac{\pi'}{\beta}}}}{2}$$

and $c_i, 1 \leq i \leq 4$, are constants determined by solving the following system of equations:

$$\begin{aligned}
\bar{I}^s(s) &= Y_s, \\
\bar{C}^s(s) &= C_s, \\
\bar{c}^s(T) &= 0, \textit{and} \\
\bar{C}^s(T) &= \bar{y}(T)
\end{aligned}$$

arising out of boundary conditions in an application of the Maximum Principle. Further, the value function $V^s(Y_s, C_s)$ can be computed using:

$$V^s(Y_s, C_s) = \int_s^T \left\{\beta\bar{c}^s(t)^2 + \kappa(\bar{C}^s(t) - \bar{Y}^s(t))^2 + \pi'(\bar{Y}^s(t) - Dt - \alpha\sin\gamma t)^2\right\} dt.$$

The closed form expression for $V^s(Y_s, C_s)$ is unwieldy, containing dozens of terms, and hence is not reproduced here. Similarly, closed form expressions for $Z(\cdot)$ and $\varphi(\cdot)$ also contain hundreds of terms and are not stated here. We have derived and programmed all of these expressions to obtain the numerical results in Section 4.2.4.

Having obtained $Z(\cdot)$ and $\varphi(\cdot)$, we next focus on obtaining $\phi(\cdot)$ in the following subsection.

Computation of $\phi(\cdot)$

The system of differential equations (4.2.23) is matrix-based and the coefficient-matrix of $\phi(t)$ on the RHS is not symmetric. To our knowledge, there do not exist any differential equation solution tools that provide a closed form solution to such a system of differential equations. Once again, we will resort to an indirect approach in order to obtain $\phi(\cdot)$.

Define

$$\begin{bmatrix} Y_1(t) \\ C_1(t) \end{bmatrix} = \phi(t) \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix}.$$

Then,

$$\begin{aligned} \begin{bmatrix} dY_1(t) \\ dC_1(t) \end{bmatrix} &= -R^{-1}(Z(t) + U) \begin{bmatrix} Y_1(t) \\ C_1(t) \end{bmatrix} dt, t \in [0, T], \\ Y_1(0) &= Y_0, C_1(0) = C_0. \end{aligned} \quad (4.2.33)$$

Observe (using Proposition 4.2.9) that the above system characterizes the optimal feedback solution to the special case of the problem $\mathcal{D}(0)$ in which the demand is uniformly zero. Thus, if we set $D = \alpha = 0$ in the solution to $\mathcal{D}(0)$, then the resulting optimal capacity and cumulative production trajectories would satisfy the above system of differential equations. In other words, the solution to the above system of differential equations can be obtained by setting $D = \alpha = 0$ in the solution to $\mathcal{D}(0)$. The coefficients of Y_0 and C_0 in the optimal solution to $\mathcal{D}(0)$ when demand is uniformly zero, provide elements of $\phi(\cdot)$. We state this formally in the following corollary.

Corollary 4.2.13. *Let $D = \alpha = 0$ in $\mathcal{D}(0)$. The optimal production and capacity to $\mathcal{D}(0)$ is of the following functional form:*

$$\begin{bmatrix} \bar{Y}^0(t) \\ \bar{C}^0(t) \end{bmatrix} = \eta(t) \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix}, t \in [0, T]. \tag{4.2.34}$$

Further, $\eta(\cdot)$ defined in the above equation is a unique solution to the differential equation (4.2.23).

Proof. See appendix. □

To see what functional form $\phi(\cdot)$ may take, we provide an example below.

Example 4.2.14. *Let $\beta > \frac{4\kappa^2}{\pi'}$. The optimal solution to $\mathcal{D}(0)$ is given by (after setting $D = \alpha = 0$) in (4.2.31-4.2.32) in Example 4.2.12:*

$$\begin{aligned} Y_1(t) &= -\frac{\beta}{\pi'}\left(c_1 d_1^3 e^{d_1 t} - c_2 d_1^3 e^{-d_1 t} + c_3 d_2^3 e^{d_2 t} - c_4 d_2^3 e^{-d_2 t}\right) \\ C_1(t) &= c_1 e^{d_1 t} + c_2 e^{-d_1 t} + c_3 e^{d_2 t} + c_4 e^{-d_2 t} \end{aligned}$$

where d_1, d_2 are specified and $c_i, 1 \le i \le 4$ are characterized in Example 4.2.12. It is easily seen that in this case:

$$c_i = a_i C_0 + b_i Y_0, 1 \le i \le 4.$$

Therefore,

$$\begin{aligned} Y_1(t) &= -\frac{\beta}{\pi'}\left(a_1 d_1^3 e^{d_1 t} - a_2 d_1^3 e^{-d_1 t} + a_3 d_2^3 e^{d_2 t} - a_4 d_2^3 e^{-d_2 t}\right) C_0 \\ &- \frac{\beta}{\pi'}\left(b_1 d_1^3 e^{d_1 t} - b_2 d_1^3 e^{-d_1 t} + b_3 d_2^3 e^{d_2 t} - b_4 d_2^3 e^{-d_2 t}\right) Y_0, \\ C_1(t) &= \left(a_1 e^{d_1 t} + a_2 e^{-d_1 t} + a_3 e^{d_2 t} + a_4 e^{-d_2 t}\right) C_0 \\ &+ \left(b_1 e^{d_1 t} + b_2 e^{-d_1 t} + b_3 e^{d_2 t} + b_4 e^{-d_2 t}\right) Y_0. \end{aligned}$$

Thus,

$$\begin{aligned} \phi_{11}(t) &= -\frac{\beta}{\pi'}\left(b_1 d_1^3 e^{d_1 t} - b_2 d_1^3 e^{-d_1 t} + b_3 d_2^3 e^{d_2 t} - b_4 d_2^3 e^{-d_2 t}\right) \\ \phi_{21}(t) &= b_1 e^{d_1 t} + b_2 e^{-d_1 t} + b_3 e^{d_2 t} + b_4 e^{-d_2 t} \\ \phi_{12}(t) &= -\frac{\beta}{\pi'}\left(a_1 d_1^3 e^{d_1 t} - a_2 d_1^3 e^{-d_1 t} + a_3 d_2^3 e^{d_2 t} - a_4 d_2^3 e^{-d_2 t}\right) \\ \phi_{22}(t) &= a_1 e^{d_1 t} + a_2 e^{-d_1 t} + a_3 e^{d_2 t} + a_4 e^{-d_2 t} \end{aligned}$$

where ϕ_{ij} is the (i, j)th element of ϕ.

In a similar way, $\phi(\cdot)$ can be computed for the cases when $\beta < \frac{4\kappa^2}{\pi'}$ and $\beta = \frac{4\kappa^2}{\pi'}$. At this point, we have derived closed form expressions for the optimal market production and capacity and the equilibrium market price. We have also outlined an approach to determine unknown coefficients, $Z(\cdot)$, $\varphi(\cdot)$, and $\phi(\cdot)$ in the optimal trajectories. In the next section, we compute the variance of the optimal market trajectories.

4.2.3 Variance of the Equilibrium Price

In the expressions for the optimal market trajectories (4.2.22) and (4.2.24), the stochastic integrals are inner integrals. Therefore, the computation of the variance of the optimal market trajectories is not straightforward. In order to interchange the order of the integrals and hence be able to obtain the variance of the optimal trajectories, we need the following variant of Fubini's Theorem.

Proposition 4.2.15. *(A special case of Lemma 4.1, Ikeda and Watanabe [43], pp116): Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space and let $W(\cdot)$ be a Weiner's process defined on it. Let $Q_1, Q_2 \in \mathcal{C}([0,T]; \mathcal{R})$. Then:*

$$\int_0^u \int_0^t Q_1(s)Q_2(t)dW(s)dt = \int_0^u \int_s^u Q_1(s)Q_2(t)dtdW(s). \tag{4.2.35}$$

Proof. See appendix. □

Let

$$\phi(\cdot) := \begin{bmatrix} \phi_{11}(\cdot) & \phi_{12}(\cdot) \\ \phi_{21}(\cdot) & \phi_{22}(\cdot) \end{bmatrix}$$

and

$$\phi^{-1}(\cdot) = \psi(\cdot) := \begin{bmatrix} \psi_{11}(\cdot) & \psi_{12}(\cdot) \\ \psi_{21}(\cdot) & \psi_{22}(\cdot) \end{bmatrix}.$$

In the following corollary, we provide the variance-covariance matrix of the optimal production and capacity vector. We also provide the variance of the equilibrium market price.

Corollary 4.2.16. *1. The variance-covariance matrix of the optimal market production and optimal market capacity at time $v \in [0,T]$ is given by:*

$$Var\begin{pmatrix} \bar{Y}(v) \\ \bar{C}(v) \end{pmatrix} = \phi(v)\begin{bmatrix} A_{11}(v) & A_{12}(v) \\ A_{12}(v) & A_{22}(v) \end{bmatrix}\phi^T(v)$$

where

$$\begin{aligned} A_{11}(v) &= \int_0^v \sigma^2(s)\left(\int_s^v \left(\frac{\psi_{11}(t)Z_{11}(t)}{2\kappa} + \frac{\psi_{12}(t)Z_{12}(t)}{2\beta}\right)dt\right)^2 ds, \\ A_{12}(v) &= \int_0^v \sigma^2(s)\int_s^v \left(\frac{\psi_{11}(t)Z_{11}(t)}{2\kappa} + \frac{\psi_{12}(t)Z_{12}(t)}{2\beta}\right)dt \\ &\quad \cdot \int_s^v \left(\frac{\psi_{21}(z)Z_{11}(z)}{2\kappa} + \frac{\psi_{22}(z)Z_{12}(z)}{2\beta}\right)dzds, \\ A_{22}(v) &= \int_0^v \sigma(s)^2\left(\int_s^v \left(\frac{\psi_{21}(t)Z_{11}(t)}{2\kappa} + \frac{\psi_{22}(t)Z_{12}(t)}{2\beta}\right)dt\right)^2 ds. \end{aligned}$$

2. The variance of the equilibrium market price is given by:

$$
\begin{aligned}
Var(\bar{P}(v)) &= \frac{1}{S^2}(Z_{11}^2(v)Var(\bar{I}(v)) + Z_{12}^2(v)A_{22}(v) \\
&+ 2Z_{11}(v)Z_{12}(v)Cov(\bar{I}(v), \bar{C}(v)))
\end{aligned}
$$

where

$$
\begin{aligned}
Var(\bar{I}(v)) &= A_{11}(v) + \int_0^v \sigma^2(s)ds \\
&- 2\int_0^v \sigma^2(s) \int_s^v \left(\frac{\psi_{11}(t)Z_{11}(t)}{2\kappa} + \frac{\psi_{12}(t)Z_{12}(t)}{2\beta}\right) dtds, \\
Cov(\bar{I}(v), \bar{C}(v)) &= \int_0^v \sigma^2(s) \left[\int_s^v \left(\frac{\psi_{11}(t)Z_{11}(t)}{2\kappa} + \frac{\psi_{12}(t)Z_{12}(t)}{2\beta}\right) dt - 1\right] \\
&\cdot \int_s^v \left(\frac{\psi_{21}(z)Z_{11}(z)}{2\kappa} + \frac{\psi_{22}(z)Z_{12}(z)}{2\beta}\right) dzds.
\end{aligned}
$$

Proof. There are two ways in which the variance of the optimal production and capacity can be obtained. Firstly, using Proposition 4.2.15, we change the order of the integrals in the optimal market trajectories (4.2.22) in Proposition 4.2.8. The variance can be computed easily once the stochastic integral is outside. For a second approach using "time substitution" (McKean [55]), see the appendix.

The variance of the market price can be obtained easily using (4.2.24) once the variance-covariance matrix of the optimal production and capacity is determined.

□

In the following section, we numerically study the impact of the supply chain cost parameters, β, κ, and π on the variance of the optimal market production and capacity trajectories.

4.2.4 Numerical Results

In this subsection, we study the impact of the supply chain cost parameters on the variance of the optimal capacity, rate of production, and equilibrium price at a fixed time in the Market model. We fit curves on the variance of the optimal trajectories as a function of the cost parameters at a fixed instant. We also study how the variance of the equilibrium price evolves over time.

To see the impact of the cost parameters on the variance of optimal trajectories, we fix a time instant $t_0 < T$. At that instant, we compute numerically the variance of the optimal trajectories for several β, κ, and π combinations. Observe that it is the ratios of the cost parameters and not their absolute values that affect the variance. Therefore, we compute variance as a function of $\frac{\kappa}{\pi}$ and $\frac{\beta}{\pi}$ only. In a similar way, it is not the absolute number of buyers or sellers but rather the relative number of buyers and sellers that affects the variance of optimal market trajectories. In fact, the ratio of number of sellers to number of buyers, $\frac{S}{B}$, always appears multiplied by π in the expressions for the optimal market trajectories. Hence, we do not analyze the impact of the relative number of sellers to buyers here as it can be indirectly analyzed by altering π. We also assume that $\sigma(\cdot)$ is constant.

We use the following data for the numerical computation:

Table 4.1: Data for Figures 4.1-3.3

T	t_0	$\frac{S}{B}$	α	γ	D	$\sigma(\cdot)$
30	15	1	2	0.1	6	1

We are interested in studying the relationship between the variance of the optimal trajectories and the cost parameters when the system is in steady state. Based on numerical experimentation, we find that if

$$\frac{\beta/\pi}{\kappa/\pi} = \frac{\beta}{\kappa} \leq 25,$$

then the transient component of deterministic part of the optimal capacity and

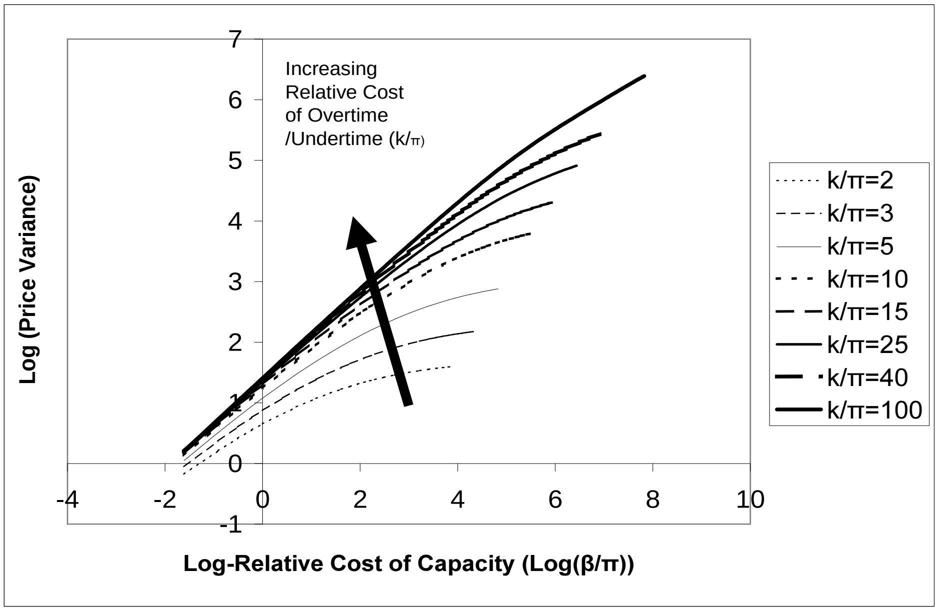

Figure 4.1: Dependence of Price Variability on Relative Cost Parameters

production trajectories at t_0 is less than 5% (usually 2%) of the value of the optimal trajectories at that point. Therefore, while computing variance of the market trajectories, we restrict $\frac{\beta}{\kappa}$ to be at most 25, in order to focus on steady state behavior.

In Figure 4.1, we plot the log of the variance of the market price versus the log of $\frac{\beta}{\pi}$ for different values of $\frac{\kappa}{\pi}$. We observe that the price variance is non-decreasing in the relative cost of capacity ($\frac{\beta}{\pi}$) and the relative cost of overtime/undertime production ($\frac{\kappa}{\pi}$). That is, we can anticipate higher volatility of price for capacity in supply chains characterized by high relative long- and short-term costs of changing capacity. For the above data, we also fit a polynomial function of the cost parameters of order three to the variance of price. For each value of $\frac{\kappa}{\pi}$, we assume the following relationship between the variance of price and $\frac{\beta}{\pi}$:

$$Var(P) = a_1 \left(\frac{\beta}{\pi}\right)^3 + a_2 \left(\frac{\beta}{\pi}\right)^2 + a_3 \left(\frac{\beta}{\pi}\right) + a_4$$

where, for $i = 1, 2, 3,$ and 4 :

$$a_i = a_{i1} \left(\frac{\kappa}{\pi}\right)^3 + a_{i2} \left(\frac{\kappa}{\pi}\right)^2 + a_{i3} \left(\frac{\kappa}{\pi}\right) + a_{i4}.$$

For the given data, values of $\{a_{i1}, a_{i2}, a_{i3}, a_{i4} : 1 \leq i \leq 4\}$ are given in the following table. The value of the statistical parameter R^2 for this fit is 0.99 indicating the tightness of the fit.

Table 4.2: Coefficients of the Polynomial Approximation of Price-Variance Curve

	a_{i1}	a_{i2}	a_{i3}	a_{i4}
a_1	-4×10^{-5}	0.0009	-0.0046	0.0034
a_2	-0.0015	0.009	0.0027	-0.0578
a_3	0.0046	-0.0636	0.3024	0.2507
a_4	0.0188	-0.2191	0.8844	0.1439

In Figure 4.2, we plot the log of the variance of the market capacity versus the log of $\frac{\beta}{\pi}$ for different values of $\frac{\kappa}{\pi}$. We find that the variability in the capacity decreases as $\frac{\beta}{\pi}$ increases for a fixed $\frac{\kappa}{\pi}$. On the other hand, the variability in the capacity increases as $\frac{\kappa}{\pi}$ increases for a fixed $\frac{\beta}{\pi}$. This implies that as the cost of overtime/undertime increases relative to the cost of changing capacity and the cost of holding inventory/shortage, the variability in capacity increases.

In Figure 4.3, we plot the variability in the instantaneous rate of production. In contrast with the market price, the variability of the instantaneous rate of production decreases as the relative cost of changing capacity as well as the relative cost of overtime/undertime increases.

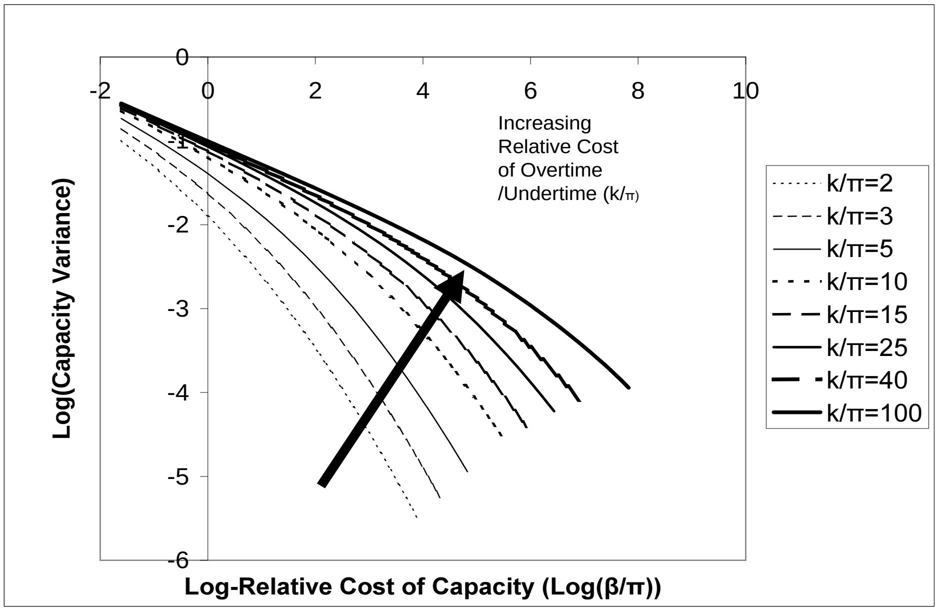

Figure 4.2: Dependence of Capacity Variability on Relative Cost Parameters

Observations from the above three plots provide a glimpse of how the market for capacity as a system handles the exogenous uncertainty which is realized through the end-consumer demand. The market has three variables at its disposal to tackle the exogenous uncertainty: capacity, production, and inventory. In a perfect market, either the sellers may update capacity and/or rate of production keeping pace with the uncertainty as it unfolds itself or, buyers may hold (incur) excessive inventory (backorders). A combination of the three is also possible. If the market handles uncertainty through the inventory/backorders then the buyers must be given incentives in the form of the price discounts/premiums which increases the variability in price. The relative values of the cost parameters β, κ, and π determine the share of each variable in handling uncertainty.

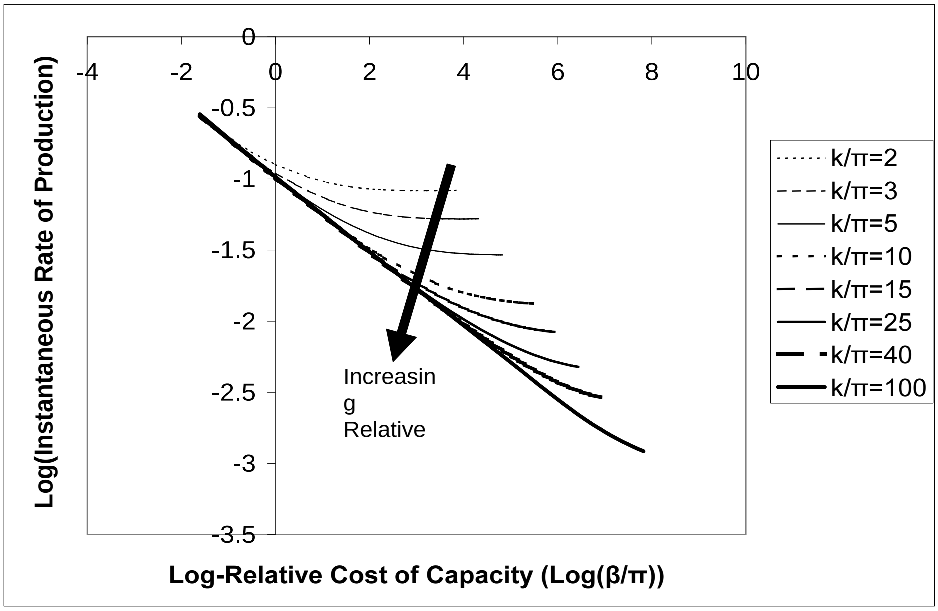

Figure 4.3: Dependence of Production Variability on Relative Cost Parameters

Indeed, observations from Figures 4.1, 4.2, and 4.3 provide evidence for the above surmise. When both the costs of changing capacity β and κ are relatively high compared to the holding cost π, varying capacity and rate of production is expensive and the market handles the exogenous uncertainty through the inventory which is reflected in increased price variability (Figure 4.1). If κ is high relative to β and π, inventory as well as capacity share the uncertainty (Figure 4.2). When β is high relative to κ and π, both the capacity and rate of production are relatively stable leaving inventory to handle uncertainty alone. Using the same assertion, the involvement of the capacity and rate of production increases as the cost parameters β and κ come relatively closer to π.

Note that the variance plots of the capacity and rate of production in Figures

4.2 and 4.3, respectively, can also be fitted well by polynomial functions of the cost parameters.

Next, we examine the evolution of variability of price over time in Figure 4.4. For this plot, we use the following data:

Table 4.3: Data for Figure 4.4

T	$\frac{S}{B}$	α	γ	D	$\sigma(\cdot)$
30	1	2	0.1	6	1

The evolution of the variance of price can be divided into three stages. In the first stage, the variance increases over time. This implies that less information is available farther down the future during the first stage. However, the variance stops increasing beyond some threshold and it becomes roughly constant which constitutes the second stage. This implies that the information in the market regarding the price is the same during the second stage. We conjecture that the reason for the flat nature of the price-variance curve lies in the modelling of the instantaneous rate of demand. We assume the following model of instantaneous rate of market demand:

$$dF(t) = (D + \alpha\gamma \cos \gamma t)dt + \sigma(t)dW(t).$$

If $\sigma(\cdot) \equiv \sigma$ then we have the same information regarding the instantaneous rate of demand at any instant in the future. We believe that this behavior translates to the market price and therefore, after an initial period of increasing variance, the price variance curve becomes flat indicating that we have the same information regarding the equilibrium price. In the third stage, the variance of price decreases down to zero. This behavior can be attributed to the end-of-horizon effect.

In the following section, we introduce the cost model of a firm that owns the

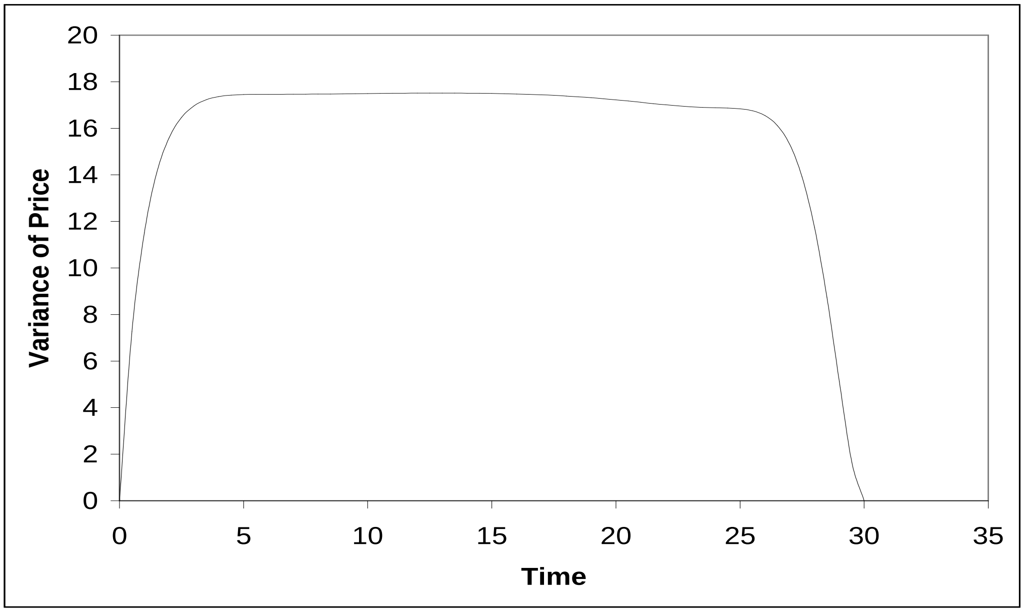

Figure 4.4: Evolution of Price Variability over Time

supply chain and show that the optimal solution for the firm is the same as that of the Market model for the same model parameters.

4.3 Integrated Model

The Integrated model corresponds to the cost model of a firm that owns the whole supply chain. Not only this firm owns the capacity for production but also satisfies the end-consumer demand directly. At every instant, the goal of the firm is to choose the optimal capacity and instantaneous rate of production. The firm incurs costs for changing capacity, for producing at a rate not equal to the capacity and for holding inventory or shortage. The functional form of these cost components remains the same as in the Market model.

We show that the optimal solutions to the Market model and the Integrated model are the same, provided the market demand is equal to the demand in the Integrated model and all cost parameters are the same except for the shortage/holding cost parameter which is defined as

$$\pi' = \frac{S}{B}\pi.$$

This will imply that the optimal trajectories in the Market model are exactly equal to those in the Integrated model with appropriate selection of parameters.

We use the same notation as in the Market model the only difference being an additional subscript, ι. Therefore, in the Integrated model $C_\iota(t)$ is used to denote capacity at time t, $y_\iota(t)$ is the instantaneous rate of production at time t, etc. Similar to the Market model, $W(\cdot)$ is a one-dimensional standard Weiner's process defined on a *complete* probability space, $(\Omega, \mathcal{F}, \mathcal{P})$. Define $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{W(s) : 0 \leq s \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.

We next state the Integrated model:

$$\begin{aligned}
&\inf_{y_\iota, c_\iota \in \mathcal{U}_\iota[0,T]} E \int_0^T \{\beta c_\iota(t)^2 + \kappa(C_\iota(t) - y_\iota(t))^2 + \pi' I_\iota(t)^2\} dt \\
&\text{s.t.} \\
&dC_\iota(t) = c_\iota(t)dt, \qquad (4.3.1) \\
&dI_\iota(t) = (y_\iota(t) - D_\iota - \alpha_\iota \gamma_\iota \cos \gamma_\iota t)dt - \sigma_\iota(t)dW(t), \\
&C_\iota(0) = C_{0,\iota}, I_\iota(0) = Y_{0,\iota}.
\end{aligned}$$

There are four constraints in the Integrated model. The first two constraints correspond to state equations for the evolution of capacity and net inventory over time, respectively. The remaining constraints reflect that the initial capacity and initial production orders are fixed exogenously.

In order to apply the Stochastic Maximum Principle (Cadenillas and Karatzas [9]) to solve the Integrated model, we assume the following.

Assumption 4.3.1. *The set of controls* $\mathcal{U}_\iota[0,T]$ *consists of all* $y_\iota : [0,T] \times \Omega \to \mathcal{R}^1$ *and* $c_\iota : [0,T] \times \Omega \to \mathcal{R}^1$ *such that* $y_\iota(\cdot)$ *and* $c_\iota(\cdot)$ *are measurable,* $\mathcal{F}_t$*-adapted,* $E\int_0^T y_\iota(t)^2 dt < \infty$*, and* $E\int_0^T c_\iota(t)^2 dt < \infty$*.*

Assumption 4.3.2. *For any* $(y_\iota^1, c_\iota^1), (y_\iota^2, c_\iota^2) \in \mathcal{U}_\iota[0,T]$*, and* $\rho \in [0,1]$*, the following holds:*

$$E\left[\int_0^T |\beta(c_\iota^1 + \rho c_\iota^2) + \kappa(C_\iota^1 + \rho C_\iota^2 - y_\iota^1 - \rho y_\iota^2) + \pi'(I_\iota^1 + \rho I_\iota^2)|^2 dt\right] < \infty,$$

where (I_ι^1, C_ι^1) *and* (I_ι^2, C_ι^2) *are states of the systems controlled by* (y_ι^1, c_ι^1) *and* (y_ι^2, c_ι^2)*, respectively.*

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 4.3.1 along with the linearity of the state equations for net inventory and capacity in the Integrated model (4.3.1) ensure a unique solution to the state equations. Assumption 4.3.2 is satisfied due to the square-integrability of control pair (y_ι, c_ι).

Just as for the Market model, we do not impose non-negativity constraints on the capacity or rate of production. As before, we assume that the expected rate of demand D_ι is large enough so as to result in positive production rate and capacity with high probability.

Note that the deterministic version of the Integrated model (that is, $\sigma_\iota(\cdot) \equiv 0$) is the same as $\mathcal{D}(0)$. In the following subsection, we establish a connection between the Integrated model and the Market model.

4.3.1 Relationship Between the Market Model and the Integrated Model

In this section, we first derive the necessary and sufficient conditions for optimality of the Integrated model. We next show the equivalence of the set of solutions satisfying the necessary and sufficient conditions for the Integrated model to that for the Market model.

Necessary and Sufficient Conditions for Optimality of the Integrated Model

In order to derive the necessary and sufficient conditions for optimality, we follow an approach similar to the Market model. Define the Hamiltonian function for the Integrated model as:

$$\begin{aligned} H_\iota &= p_{1,\iota}(y_\iota - D_\iota - \alpha_\iota \gamma_\iota \cos \gamma_\iota t) + p_{2,\iota} c_\iota - q_{1,\iota}\sigma_\iota \\ &- \beta c_\iota^2 - \kappa (C_\iota - y_\iota)^2 - \pi' I_\iota^2, \end{aligned}$$

where $(p_{1,\iota}, q_{1,\iota})$ and $(p_{2,\iota}, q_{2,\iota})$ are pairs of adjoint variables defined by the following backward stochastic differential equations:

$$\begin{aligned} dp_{1,\iota}(t) &= 2\pi' I_\iota(t)dt + q_{1,\iota}(t)dW(t), \\ dp_{2,\iota}(t) &= 2\kappa(C_\iota(t) - y_\iota(t))dt + q_{2,\iota}(t)dW(t), \\ p_{1,\iota}(T) &= p_{2,\iota}(T) = 0, \end{aligned}$$

and where (I_ι, C_ι) are the state variables in the system controlled by (y_ι, c_ι). The adjoint variables, $p_{1,\iota}$ and $p_{2,\iota}$ can be interpreted as the "shadow prices" corresponding to the net inventory (I_ι) and the capacity (C_ι) resources, respectively.

Using Theorem 3.2, Cadenillas and Karatzas ([9]), a necessary and sufficient condition for the optimality of the optimal control pair $(\bar{y}_\iota, \bar{c}_\iota)$ is that it maximizes the Hamiltonian function. In other words,

$$\begin{aligned} \max_{(y_\iota, c_\iota) \in \mathcal{U}_\iota} H_\iota(y_\iota, c_\iota, \bar{Y}_\iota, \bar{C}_\iota, \bar{p}_{2,\iota}, \bar{q}_{2,\iota}) &= H_\iota(\bar{y}_\iota, \bar{c}_\iota, \bar{Y}_\iota, \bar{C}_\iota, \bar{p}_{2,\iota}, \bar{q}_{2,\iota}), \\ & a.e. (t, \omega) \in [0, T] \times \Omega, \end{aligned}$$

where $(\bar{Y}_\iota, \bar{C}_\iota)$ and $(\bar{p}_{2,\iota}, \bar{q}_{2,\iota})$ are the state variable and adjoint variable pairs corresponding to the system controlled by $(\bar{y}_\iota, \bar{c}_\iota)$. The above equation yields the following two equations:

$$\begin{aligned} \bar{p}_{2,\iota} - 2\beta\bar{c}_\iota &= 0, \\ \bar{p}_{1,\iota} + 2\kappa(\bar{C}_\iota - \bar{y}_\iota) &= 0. \end{aligned}$$

We formally state the necessary and sufficient conditions for optimality of the Integrated model in the following corollary.

Corollary 4.3.3. *The vector $(\bar{I}_\iota, \bar{C}_\iota, \bar{y}_\iota, \bar{c}_\iota, \bar{p}_{1,\iota}, \bar{p}_{2,\iota}, \bar{q}_{1,\iota}, \bar{q}_{2,\iota})$ is optimal to the Integrated model if and only if it satisfies the following system of equations:*

$$\begin{aligned} d\bar{C}_\iota(t) &= \bar{c}_\iota(t)dt, \qquad (4.3.2)\\ d\bar{I}_\iota(t)dt &= (\bar{y}_\iota - D_\iota - \alpha_\iota\gamma_\iota \cos\gamma t)dt - \sigma_\iota(t)dW(t), \\ \bar{y}_\iota(t) &= \bar{C}_\iota(t) + \frac{1}{2\kappa}\bar{p}_{1,\iota}(t), \\ \bar{c}_\iota(t) &= \frac{1}{2\beta}\bar{p}_{2,\iota}(t), \\ d\bar{p}_{1,\iota}(t) &= 2\pi'\bar{I}_\iota(t)dt + \bar{q}_{1,\iota}(t)dW(t), \\ d\bar{p}_{2,\iota}(t) &= 2\kappa(\bar{C}_\iota(t) - \bar{y}_\iota(t))dt + \bar{q}_{2,\iota}(t)dW(t), \\ \bar{p}_{1,\iota}(T) &= \bar{p}_{2,\iota}(T) = 0. \end{aligned}$$

$a.e. t \in [0, T], \mathcal{P}-a.s.$

The necessary and sufficient conditions for optimality to the Integrated model are similar to those for the Market model. We establish a connection between the solution to the Integrated model and that of the Market model in the following proposition.

Proposition 4.3.4. *Assume $\pi' = \frac{S}{B}\pi$ and that the cost parameters β and κ are the same for both the Integrated and the Market models. Further, let $D_\iota =$*

$D, \alpha_\iota = \alpha, \gamma_\iota = \gamma$, and $\sigma_\iota(\cdot) = \sigma(\cdot)$. Then, the optimal vector of market variables, $(\bar{I}, \bar{C}, \bar{y}(= \bar{x}), \bar{c}, \bar{p}_1, \bar{q}_1, \bar{p}_2, \bar{q}_2)$ and the optimal vector of the Integrated model, $(\bar{I}_\iota, \bar{C}_\iota, \bar{y}_\iota, , \bar{c}_\iota, \bar{p}_{1,\iota}, \bar{q}_{1,\iota}, \bar{p}_{2,\iota}, \bar{q}_{2,\iota})$ are related by the following set of equations:

$$\begin{aligned} (\bar{I}_\iota, \bar{C}_\iota, \bar{y}_\iota, \bar{c}_\iota, \bar{p}_{2,\iota}, \bar{q}_{2,\iota}) &= (\bar{I}, \bar{C}, \bar{y}, \bar{c}, \bar{p}_2, \bar{q}_2) \\ (\bar{p}_{1,\iota}, \bar{q}_{1,\iota}) &= \frac{S}{B}(\bar{p}_1, \bar{q}_1) \end{aligned} \tag{4.3.3}$$

where equality is componentwise.

Proof. Substitute the optimal Integrated model variables in (4.3.2) by the optimal market variables using (4.3.3). The resulting set of equations is same as the necessary and sufficient conditions for optimality to the Market model (4.2.14). The proof then follows by the uniqueness of the solution to (4.2.14). □

We derived the closed form expressions for the equilibrium market price and other market trajectories for the Market model in Section 4.2. However, the solution to the Market model is too complicated to explicitly express the variance of the optimal trajectories as a function of the cost parameters. In the following section, we introduce and analyze a simpler version of the Market model in which capacity of each seller is exogenously fixed.

4.4 Constant Capacity Model

In this section, we consider a special case of the model presented in Section 4.2 in which capacity is not a variable. Therefore, each seller controls only the instantaneous rate of production. We shall refer to this model as the Constant Capacity Market model. The necessary and sufficient conditions for the optimality of this model are identical to those for the Market model with the rate of change of capacity identically set to zero. For compactness, we use the same notation as the

Market model but no confusion should arise in general. By simplifying the model in this way, we are able to derive in closed form, an expression for the variance of the equilibrium price process.

In this model, seller k chooses optimal production quantity $y_k(t)$ at time t to minimize the sum of production-capacity mismatch cost less the revenue earned over a finite horizon $[0, T]$. Seller k's optimization problem can be stated as:

$$\begin{aligned} & \min_{y_k \in \mathcal{U}_k^{CC}[0,T]} E \int_0^T \left\{ \kappa(C_k - y_k(t))^2 - P(t) y_k(t) \right\} dt \qquad (4.4.1) \\ & \text{s.t.} \\ dY_k(t) &= y_k(t)dt, t \in [0, T], \\ Y_k(0) &= Y_{0k}, \end{aligned}$$

where capacity C_k is exogenously fixed. In order to apply the Stochastic Maximum Principle (Cadenillas and Karatzas [9]) to obtain the necessary and sufficient conditions for optimality of the Constant Capacity Seller model, we make the following assumptions.

Assumption 4.4.1. *The set of controls* $\mathcal{U}_k^{CC}[0, T]$ *consists of all* $y_k : [0, T] \times \Omega \to \mathcal{R}^1$ *such that* $y_k(\cdot)$ *is measurable,* $\mathcal{F}_t$*-adapted and* $E \int_0^T y_k(t)^2 dt < \infty$.

Assumption 4.4.2. *For any* $y_k^1, y_k^2 \in \mathcal{U}_k^{CC}[0, T]$*, and* $\rho \in [0, 1]$*, the following holds:*

$$E \left[\int_0^T |2\kappa(C^k - y_k^1 - \rho y_k^2) - P|^2 dt \right] < \infty.$$

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 4.4.1 along with the linearity of the state equation for cumulative production in the Constant Capacity Seller model (4.4.1) ensure a unique solution to the state equation. Assumption 4.4.2 is satisfied due to the square-integrability of P and y_k.

The Constant Capacity Buyer model remains the same as in the Market model (4.2.5). We first state the necessary and sufficient conditions for optimality of the Constant Capacity Market model (4.4.1) in the following proposition.

Proposition 4.4.3. *The vector of market variables, $(\bar{I}, \bar{y}(=\bar{x}), \bar{p}_1, \bar{q}_1)$ is optimal if and only if it satisfies the following system of equations in equilibrium :*

$$\begin{aligned} d\bar{I}(t) &= (\bar{y}(t) - D - \alpha\gamma\cos\gamma t)dt - \sigma(t)dW(t), \\ \bar{x}(t) &= \bar{y}(t) = C + \frac{1}{2\kappa}\frac{S}{B}\bar{p}_1(t), \\ d\bar{p}_1(t) &= 2\pi\bar{I}(t)dt + \bar{q}_1(t)dW(t), \\ \bar{p}_1(T) &= 0, \end{aligned} \tag{4.4.2}$$

a.e. $t \in [0,T], \mathcal{P} - a.s.$ *where* $q_1(t) = \sum_j q_{1,j}(t)$ *and* $C = \sum_k C_k$.

Proof. We set $\bar{c} = 0$ in the necessary and sufficient conditions for optimality of the Market model in Proposition 4.2.6 to obtain the above system of equations. The proof for the "only if" part is obtained in a way similar to Proposition 4.2.6. □

Next, we obtain the optimal feedback solution to the Constant Capacity Market model. As before, we hypothesize the following relationship between the adjoint variable $\bar{p}(\cdot)$ and the state variable $\bar{I}(\cdot)$ in order to obtain the optimal feedback solution:

$$\frac{S}{B}\bar{p}_1(t) = -(Z(t)\bar{I}(t) + \varphi(t))$$

where $Z(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^1)$ and $\varphi(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^1)$. The solution parameter pair, $(Z(\cdot), \varphi(\cdot))$, satisfies differential equations whose form is determined using (4.4.2). We state the optimal feedback solution to the Constant Capacity Market model in the following proposition.

Proposition 4.4.4. *Let $Z(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^1)$ be the solution of the following differential equation:*

$$\begin{aligned} \dot{Z}(t) + 2\pi' - \frac{1}{2\kappa} Z(t)^2 &= 0, \\ Z(T) &= 0, \end{aligned} \tag{4.4.3}$$

and let $\varphi(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^1)$ be the solution of the following differential equation:

$$\begin{aligned} \dot{\varphi}(t) - \frac{1}{2\kappa} Z(t)\varphi(t) + Z(t)(C - D - \alpha\gamma\cos\gamma t) &= 0, \\ \varphi(T) &= 0. \end{aligned} \tag{4.4.4}$$

Then, the optimal rate of instantaneous production $\bar{y}(\cdot)$ is given by:

$$\bar{y}(t) = C - \frac{1}{2\kappa}\left(Z(t)\bar{I}(t) + \varphi(t)\right), t \in [0,T]. \tag{4.4.5}$$

Proof. See appendix. □

The expression for the optimal rate of production provides some insights regarding the behavior of the model. The rate of production at time t depends on the capacity as well as the net inventory at time t. A marginal change in the capacity is completely transmitted to the optimal rate of production. However, a marginal change in the net inventory produces less effect on the rate of production as κ increases relative to π' since $Z(t)/2\kappa$ decreases as κ increases (see Corollary 4.4.5 below) for any fixed t. In other words, net inventory becomes a relatively less important factor as the cost of overtime/undertime increases with respect to holding/shortage cost.

In the following corollary, we state the solution to the differential equation (4.4.3) for $Z(\cdot)$. This differential equation is a standard one-dimensional Riccati-differential equation and its solution is easily obtained. See Boyce [5] for the solution method.

Corollary 4.4.5. *The solution to the differential equation (4.4.3) is given by:*

$$Z(t) = 2\sqrt{\pi'\kappa}\left(\frac{1-e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1+e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right). \tag{4.4.6}$$

Given $Z(\cdot)$, the differential equation (4.4.4) can be solved using the *Variation of Constants* method to obtain $\varphi(\cdot)$.

Next, we obtain a closed form expression for the optimal cumulative production, $\bar{Y}(\cdot)$, using the optimal feedback solution (4.4.5). Multiplying by dt on both sides and substituting $\bar{I}(t)$ by $\bar{Y}(t) - F(t) - Y_0 + X_0$ on the RHS and $\bar{y}(t)dt$ by $d\bar{Y}(t)$ on the LHS in (4.4.5):

$$d\bar{Y}(t) = \left(C - \frac{1}{2\kappa}\left(Z(t)(\bar{Y}(t) - F(t) - Y_0 + X_0) + \varphi(t)\right)\right)dt.$$

The above stochastic differential equation can be solved using an approach similar to the *Variation of Constants* method. We provide the solution in the following proposition.

Proposition 4.4.6. *Let $Z(\cdot)$ and $\varphi(\cdot)$ be as defined in Proposition 4.4.4. The optimal production trajectory in the Constant Capacity Market model is given by:*

$$\begin{aligned}\bar{Y}(v) &= \cosh(\sqrt{\frac{\pi'}{\kappa}}(v-T))\int_0^v \left(\frac{\frac{Z(t)}{2\kappa}(F(t)+Y_0-X_0)+C-\frac{\varphi(t)}{2\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}\right)dt \\ &+ Y_0\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(v-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}T}, v \in [0,T].\end{aligned} \tag{4.4.7}$$

Further, equilibrium market price is given by:

$$P(v) = -\frac{Z(v)\bar{I}(v)+\varphi(v)}{S},$$

where $v \in [0,T]$.

Proof. See appendix. ∎

In the following corollary, we provide the variance of the optimal production and market price. In order to compute the variance, we interchange the order of the stochastic and the Lebesgue integrals in the expression for the optimal cumulative production (4.4.7).

Corollary 4.4.7. *The variance of the optimal production trajectory and market price at time $v \in [0, T]$ is given by:*

$$Var(\bar{Y}(v)) = \int_0^v \left(1 - \frac{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} + e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}}{e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} + e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}}\right)^2 \sigma^2(s)ds, \tag{4.4.8}$$

$$Var(P(v)) = 4\left(\frac{\pi\kappa}{SB}\right) \int_0^v \left\{\frac{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}}{e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} + e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}}\right\}^2 \sigma^2(s)ds. \tag{4.4.9}$$

Proof. See appendix. □

If $\sigma^2(t)$ is constant for all t, then the expression for the variance of the market price can be simplified by computation of the integral in (4.4.9). We state the simplified expression in the following corollary.

Corollary 4.4.8. *Let $\sigma^2(\cdot) \equiv \sigma^2$. Then the variance of the market equilibrium price at time $v \in [0, T]$ is given by:*

$$\begin{aligned} Var(P(v)) = {} & \left(\frac{\pi\kappa}{SB}\right)\sqrt{\frac{\kappa}{\pi'}}\left\{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}\right\}^2 \\ & \cdot \left\{\frac{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}}{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} + e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}} - \frac{e^{-\sqrt{\frac{\pi'}{\kappa}}T} - e^{\sqrt{\frac{\pi'}{\kappa}}T}}{e^{-\sqrt{\frac{\pi'}{\kappa}}T} + e^{\sqrt{\frac{\pi'}{\kappa}}T}}\right\}\sigma^2. \end{aligned}$$

In the following subsection, we numerically examine the impact of the cost parameters on the variance of the market price.

4.4.1 Numerical Results

In this subsection, we numerically analyze the relationship between the cost of overtime/undertime, κ, relative to the cost of holding inventory/shortage, π, and the price variability. We also study the evolution of the variance of price over time for a fixed ratio of cost parameters, $\frac{\kappa}{\pi}$.

Similar to the Market model, the effect of the change in relative number of sellers to buyers can be analyzed indirectly by redefining the holding cost parameter. Also, the variability of price is *not* affected by the absolute values of the cost parameters, κ and π but rather by their relative values captured by the ratio, $\frac{\kappa}{\pi}$.

In Figure 4.5, we plot the variability of price at a fixed time instant t_0 over different values of $\frac{\kappa}{\pi}$. We use the following data for Figure 4.5.

Table 4.4: Data for Figure 4.5

T	t_0	$\frac{S}{B}$	α	γ	D	$\sigma(\cdot)$
30	15	1	2	0.1	6	1

We observe that as the cost of overtime/undertime increases relative to the holding/shortage cost (that is, as $\frac{\kappa}{\pi}$ increases), the variability in price increases. Similar to the Market model, we can explain this observation in terms of how the market handles exogenous uncertainty. Either the sellers update the rate of production or buyers hold inventory/incur backorders to grapple with the end-consumer demand uncertainty. The relative values of the cost parameters κ and π determines which variable will be more volatile. When κ is high relative to π, updating rate of production is expensive and therefore, inventory acts as a tool to counter the uncertainty which is reflected in high price variance. The variance of price increases because price (through premium or discount) acts as an indirect mechanism to make buyers hold inventory or incur backorders. To summarize, we should anticipate high variability of price in a supply chain in which the capacity is exogenously fixed and the short-term cost of changing capacity is high.

In Figure 4.6, we plot the variability of price over the horizon for a fixed value

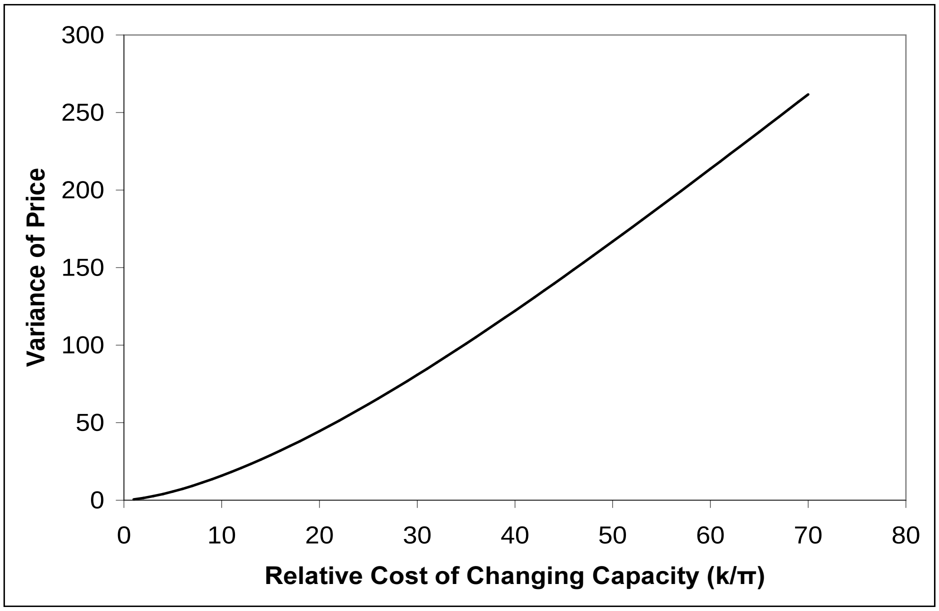

Figure 4.5: Dependence of Price Variability on Relative Cost Parameters

of $\frac{\kappa}{\pi} = 10$ and $S = B = 5$. The remaining data remain the same as Figure 4.5. We observe that the variability of price increases first and then stabilizes after a certain threshold of time before decreasing down to zero towards the end of horizon. The explanation for this behavior remains the same as for the Market model.

4.5 Conclusion

We consider a continuous time market for the capacity of a single homogeneous product with stochastic demand and derive closed form expressions for the optimal capacity, production and the equilibrium price using the Stochastic Maximum Principle. We obtain the variance of the optimal trajectories as a function of the supply chain cost parameters. We find that in a supply chain with high short

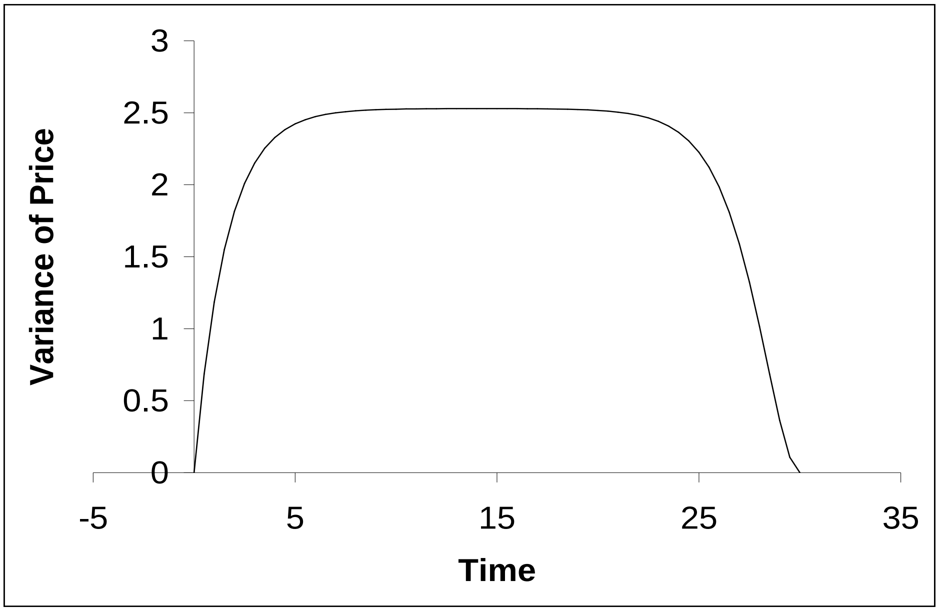

Figure 4.6: Evolution of Price Variability over Time

and/or long-term costs of changing capacity, the variability of the market price is also high. On the other hand, the variability of the rate of production is low in the same environment. The variability of capacity is low in a supply chain with high long-term cost of changing capacity but the variability increases as the short-term cost of changing capacity increases. We obtain similar insights from a simpler model in which capacity is exogenously fixed.

We also consider the cost model of a firm that owns the whole supply chain and show that the optimal solution for this model remains the same as for the capacity market with the same cost parameters and after incorporating the effect of the size of the market.

4.6 Appendix

4.6.1 Proof of Proposition 4.2.7

Following an approach outlined in Chapter 6 in Yong and Zhou [79], we begin with hypothesizing the relationship between $\begin{bmatrix} \bar{p}_1, & \bar{p}_2 \end{bmatrix}^T$ and $\begin{bmatrix} \bar{I}(t), & \bar{C}(t) \end{bmatrix}^T$:

$$\begin{aligned} \begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} &= -G(t)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} - H(t), \\ G(T) &= H(T) = 0 \end{aligned} \tag{4.6.1}$$

where $G(\cdot) \in \mathcal{C}^1([0,T];\mathcal{S}^2)$ and $H(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R}^2)$ are to be determined. Using Itô's formula:

$$\begin{bmatrix} \frac{S}{B}d\bar{p}_1(t) \\ d\bar{p}_2(t) \end{bmatrix} = -\dot{G}(t)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt - \dot{H}(t)dt - G(t)\begin{bmatrix} d\bar{I}(t) \\ d\bar{C}(t) \end{bmatrix}.$$

Using (4.2.14) to substitute for $d\bar{p}_1(t)$ and $d\bar{p}_2(t)$:

$$\begin{aligned} \begin{bmatrix} \frac{S}{B}(2\pi\bar{I}(t)dt + \bar{q}_1(t))dW(t) \\ 2\kappa(\bar{C}(t) - \bar{y}(t))dt + \bar{q}_2(t)dW(t) \end{bmatrix} &= -\dot{G}(t)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt - \dot{H}(t)dt \\ - \quad G(t)&\begin{bmatrix} (\bar{y}(t) - D - \alpha\gamma\cos\gamma t)dt - \sigma(t)dW(t) \\ \bar{c}(t)dt \end{bmatrix}. \end{aligned}$$

Next, we substitute for $\bar{y}(t)$ and $\bar{c}(t)$ in terms of $\bar{p}_1(t)$ and $\bar{p}_2(t)$ using (4.2.14):

$$\begin{aligned} \begin{bmatrix} \frac{S}{B}(2\pi\bar{I}(t)dt + \bar{q}_1(t))dW(t) \\ -\frac{S}{B}\bar{p}_1(t)dt + \bar{q}_2(t)dW(t) \end{bmatrix} &= -\dot{G}(t)\begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt - \dot{H}(t)dt \\ - \quad G(t)&\begin{bmatrix} (\bar{C}(t) + \frac{1}{2\kappa}\frac{S}{B}\bar{p}_1(t) - D - \alpha\gamma\cos\gamma t)dt - \sigma(t)dW(t) \\ \frac{1}{2\beta}\bar{p}_2(t)dt \end{bmatrix}. \end{aligned}$$

Using matrix notation introduced in Section 4.2.2 preceding Proposition 4.2.7, the above equation may be written as:

$$\begin{aligned}
& (Q - U^T R^{-1} U) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt + \begin{bmatrix} \frac{S}{B}\bar{q}_1(t) \\ \bar{q}_2(t) \end{bmatrix} dW(t) + (R^{-1}U)^T \begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} dt \\
= & -\dot{G}(t) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt - \dot{H}(t) dt \\
- & G(t) \left[\left(R^{-1} \left(\begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} - U \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} \right) + b \right) dt - \begin{bmatrix} \sigma(t) \\ 0 \end{bmatrix} dW(t) \right].
\end{aligned}$$

Now substituting for $\begin{bmatrix} p_1(t), & p_2(t) \end{bmatrix}^T$ in the above equation using (4.6.1) results in:

$$\begin{aligned}
& (Q - U^T R^{-1} U) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt + \begin{bmatrix} \frac{S}{B}\bar{q}_1(t) \\ \bar{q}_2(t) \end{bmatrix} dW(t) \\
+ & (R^{-1}U)^T \left[-G(t) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} - H(t) \right] dt \\
= & -\dot{G}(t) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} dt - \dot{H}(t) dt \\
- & G(t) \left(R^{-1} \left(\left[-G(t) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} - H(t) \right] - U \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} \right) + b \right) dt \\
- & G(t) \begin{bmatrix} \sigma(t) \\ 0 \end{bmatrix} dW(t).
\end{aligned}$$

Comparing coefficients for $\begin{bmatrix} \bar{I}(t) & \bar{C}(t) \end{bmatrix}^T$:

$$\begin{aligned}
\dot{G}(t) + Q - (G(t) + U)^T R^{-1} (G(t) + U) &= 0, \\
G(T) &= 0.
\end{aligned}$$

which is the same as (4.2.15). Comparing diffusion terms:

$$\begin{bmatrix} \frac{S}{B}\bar{q}_1(t) \\ \bar{q}_2(t) \end{bmatrix} = -G(t) \begin{bmatrix} \sigma(t) \\ 0 \end{bmatrix}.$$

The remaining terms yield:

$$\begin{aligned} \dot{H}(t) - (R^{-1}(G(t)+U))^T H(t) + G(t)b &= 0, \\ H(T) &= 0. \end{aligned}$$

Using Proposition 4.2.6 and (4.6.1):

$$\begin{aligned} \begin{bmatrix} \bar{y}(t) \\ \bar{c}(t) \end{bmatrix} &= R^{-1}\left(\begin{bmatrix} \frac{S}{B}\bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} - U \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} \right) \\ &= -R^{-1}\left((G(t)+U) \begin{bmatrix} \bar{I}(t) \\ \bar{C}(t) \end{bmatrix} + H(t) \right). \end{aligned}$$

According to Corollary 5.7, Yong and Zhou [79], there exists a unique adapted and square-integrable solution to the system of equations that define the optimal solution to the Integrated model. Using the equivalence of the Market model and Integrated model in Proposition 4.3.4, we claim the uniqueness, square-integrability and adaptiveness of the solution to (4.2.14).

4.6.2 Proof of Proposition 4.2.8

Let $\psi(t) \in \mathcal{C}^1([0,T];\mathcal{R}^{2\times 2})$ be such that for $t \in [0,T]$:

$$d\left(\psi(t)\begin{bmatrix} \bar{Y}(t) \\ \bar{C}(t) \end{bmatrix}\right) = \psi(t)R^{-1}\left(Z(t)\begin{bmatrix} F(t) - X_0 + Y_0 \\ 0 \end{bmatrix} - \varphi(t) \right) dt.$$

Then, $\psi(t)$ must satisfy,

$$\begin{aligned} d\psi(t) &= \psi(t)(R^{-1}(Z(t)+U))dt, \\ \psi(0) &= I_{22}, \end{aligned}$$

where I_{22} is a 2×2 identity matrix. Using Theorem 6.14, pp 47, Yong and Zhou [79], a unique $\psi(\cdot)$ exists and has an inverse $\phi(\cdot)$ which is a unique solution to the following differential equation:

$$\begin{aligned} d\phi(t) &= -R^{-1}(Z(t)+U)\phi(t)dt, \\ \phi(0) &= I_{22}. \end{aligned}$$

The solution to (4.2.21) is given by:

$$\begin{aligned} \begin{bmatrix} \bar{Y}(v) \\ \bar{C}(v) \end{bmatrix} &= \phi(v)\int_0^v \phi^{-1}(t)R^{-1}\left(Z(t)\begin{bmatrix} F(t)-X_0+Y_0 \\ 0 \end{bmatrix} - \varphi(t)\right)dt \\ &+ \phi(v)\begin{bmatrix} Y_0 \\ C_0 \end{bmatrix}. \end{aligned}$$

4.6.3 Proof of Corollary 4.2.13

When $D = \alpha = 0$, the solution to the differential equation (4.2.26) is $\varphi(\cdot) \equiv 0$. Therefore, the optimal feedback solution to $\mathcal{D}(0)$ when $D = \alpha = 0$ is equal to:

$$\begin{bmatrix} \bar{y}^0(t) \\ \bar{c}^0(t) \end{bmatrix} = -R^{-1}(Z(t)+U)\begin{bmatrix} \bar{Y}^0(t) \\ \bar{C}^0(t) \end{bmatrix}, t \in [0,T]$$

using Proposition 4.2.9. Multiplying both sides by dt and substituting for $\bar{y}^0(t)dt$ and $\bar{c}^0(t)dt$ by $d\bar{Y}^0$ and $d\bar{C}^0$, respectively, on the LHS:

$$\begin{aligned} \begin{bmatrix} d\bar{Y}^0(t) \\ d\bar{C}^0(t) \end{bmatrix} &= -R^{-1}(Z(t)+U)\begin{bmatrix} \bar{Y}^0(t) \\ \bar{C}^0(t) \end{bmatrix} dt, t \in [0,T] \\ \bar{Y}^0(0) &= Y_0, \bar{C}^0(0) = C_0. \end{aligned}$$

The solution to the above differential equation is equal to:

$$\begin{bmatrix} \bar{Y}^0(t) \\ \bar{C}^0(t) \end{bmatrix} = \eta(t)\begin{bmatrix} Y_0 \\ C_0 \end{bmatrix}, t \in [0,T]$$

where $\eta(\cdot)$ is called the fundamental solution matrix (Boyci [5]), such that $\eta(0) = I_{22}$ where I_{22} is the 2×2 identity matrix.

Now,

$$\begin{aligned}
\begin{bmatrix} d\bar{Y}^0(t) \\ d\bar{C}^0(t) \end{bmatrix} &= d\eta(t) \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix} = -R^{-1}(Z(t)+U) \begin{bmatrix} \bar{Y}^0(t) \\ \bar{C}^0(t) \end{bmatrix} dt \\
\Rightarrow d\eta(t) \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix} &= -R^{-1}(Z(t)+U)\eta(t) \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix} dt.
\end{aligned}$$

Since the last equation holds for any Y_0 and C_0, therefore, $\eta(\cdot)$ must satisfy:

$$d\eta(t) = -R^{-1}(Z(t)+U)\eta(t)dt$$

which along with the boundary condition, $\eta(0) = I_{22}$, is the same as (4.2.23).

4.6.4 Proof of Proposition 4.2.15

Our goal is to show that the four conditions specified in the proof of Lemma 4.1 in Ikeda and Watanabe [43] are satisfied when the integrand has the form stated in Proposition 4.2.15. Let $\Psi(s,t,\omega) = Q_1(s)Q_2(t)1_{[s,u]}(t)$. As a result, we can write the LHS of (4.2.35) as:

$$\int_0^u \int_0^t Q_1(s)Q_2(t)dW(s)dt = \int_{\mathcal{R}^1} \int_0^t Q_1(s)Q_2(t)1_{[s,u]}dW(s)dt$$

Note that we have specified ω as an argument to Ψ even though the RHS is independent of ω. This is done so that we can apply the four conditions specified in Lemma 4.1 in Ikeda and Watanabe [43] directly.

Condition 1: *$((s,\omega),t) \in ([0,\infty),\Omega) \times \mathcal{R}^1 \to \Psi(s,t,\omega)$ is $\mathcal{S} \times \mathcal{B}(\mathcal{R}^1)$ - measurable where $\mathcal{S}$ is the smallest $\sigma-$field on $[0,\infty) \times \Omega$ s.t. all left continuous $\mathcal{F}_s-$ adapted processes $Z : [0,\infty) \times \Omega \to Z_s(\omega)$ are measurable.*

Consider the following functions:

$$\begin{aligned} Z_n(s,t) &= Z(s,\frac{i}{n}), 0 < s \leq t, 0 \leq \frac{i-1}{n}u < t \leq \frac{i}{n}u, i \in \{1,..,n\}, \\ Z_n(0,0) &= Z(0,0) \\ &= 0; s > t, t \in (u,\infty) \cup (-\infty, 0], \end{aligned}$$

where

$$Z(s,\frac{i}{n}u) = Q_2(\frac{i}{n}u)Q_1(s).$$

Clearly, $Z_n(s,\cdot)$ is left continuous for each fixed s. Consider any set $A \in \mathcal{B}(\mathcal{R}^1)$. For $t \in B_i := (\frac{i}{n}u, \frac{i+1}{n}u]$ for some $i < n$, $Z_n^{-1}(\cdot,t)(A) =: C_{s,i} \in \mathcal{S}$. By definition of product spaces $(C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Therefore $Z_n^{-1}(\cdot,\cdot)(A) = \cup_i (C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$ proving the measurability of $Z_n(\cdot,\cdot)$ with respect to $\mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Now, due to the left continuity of $Z_n(s,t)$ in t, $Z_n(s,t) \to Z(s,t)$. Therefore, $Z(s,t) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$.

Condition 2: *There exists a non-negative Borel-measurable function $f(t)$ such that*

$$|\Psi(s,t,\omega)| \leq f(t)$$

for every s,t,ω. Follows immediately from the continuity of $Q_1(\cdot)$ and $Q_2(\cdot)$.

Condition 3: $(t,\omega) \to \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega)$ *is $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}-$measurable for each $t_1 \geq 0$.*

To see this,

$$\begin{aligned} \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega) &= \int_0^{t_1} Q_2(t)1_{(0,u]}(t)1_{(0,t]}(s)Q_1(s)dW(s,\omega) \\ &= Q_2(t)1_{(0,u]}(t)\int_0^{t_1} 1_{(0,t]}(s)Q_1(s)dW(s,\omega) \\ &= Q_2(t)1_{(0,u]}(t)W(H(t_1 \wedge t),\omega) \end{aligned}$$

where

$$H(\cdot) = \int_0^{\cdot} Q_1^2(s)ds.$$

The last step follows from the "time substitution" (McKean [55]). For any ω , define:

$$\begin{aligned} Z_n(t,\omega) &= Q_2(\frac{p}{n}u)W(H(t_1 \wedge \frac{p}{n}u),\omega), p \in \{1,.,n\}, t \in (\frac{p-1}{n}u, \frac{p}{n}u] \\ &= 0, \text{ otherwise.} \end{aligned}$$

For each n, and for any $\omega, t \to Z_n(t,\omega)$, is left continuous. Further, for any t, $Z_n(t,\omega)$, is measurable with respect to $\mathcal{F}_{H(t_1 \wedge \frac{p}{n}u)} \subset \mathcal{F}$. In a way similar to above (see Condition 1), it can be shown that $Z_n(t,\omega) \in \mathcal{B}(\mathcal{R}^1) \times \mathcal{F}$.

Due to the left continuity of $Z_n(t,\omega)$ in t,

$$\lim_{n\to\infty} Z_n(t,\omega) = Z(t,\omega) = Q_2(t)W(H(t_1 \wedge t),\omega)1_{(0,u]}(t),$$

will also be measurable with respect to $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}$.

Condition 4: $\int_{\mathcal{R}^1} f(t)dt < \infty$.

Take

$$f(t) := \sup_{s\in[0,u]} |Q_1(s)| \sup_{t\in[0,u]} |Q_2(t)| < \infty.$$

Then

$$\int_{\mathcal{R}^1} f(t)dt \le u(\sup_{s\in[0,u]} |Q_1(s)| \sup_{t\in[0,u]} |Q_2(t)|) < \infty.$$

4.6.5 Proof of Corollary 4.2.16

In the following, we present another technique to compute variance of the optimal trajectories using the "time substitution" technique. For more details regarding this technique, see McKean [55]. We present here the derivation of the variance-covariance matrix of the optimal production and capacity only. Given the variance-covariance matrix of the optimal production and capacity, the variance of price can be obtained easily.

$$Var \begin{bmatrix} \bar{Y}(v) \\ \bar{C}(v) \end{bmatrix} = \phi(v) Var \begin{pmatrix} \int_0^v a_1(t) \int_0^t \sigma(s) dW(s) dt \\ \int_0^v a_2(t) \int_0^t \sigma(s) dW(s) dt \end{pmatrix} \phi^T(v)$$

where

$$\begin{aligned} a_1(t) &= \frac{\psi_{11}(t) Z_{11}(t)}{2\kappa} + \frac{\psi_{12}(t) Z_{12}(t)}{2\beta}, \\ a_2(t) &= \frac{\psi_{21}(t) Z_{11}(t)}{2\kappa} + \frac{\psi_{22}(t) Z_{12}(t)}{2\beta}. \end{aligned}$$

Applying "time-substitution", the variance-covariance matrix of the optimal production and capacity trajectories becomes

$$= \phi(v) Var \begin{pmatrix} \int_0^v a_1(t) W(H(t)) dt \\ \int_0^v a_2(t) W(H(t)) dt \end{pmatrix} \phi^T(v)$$

where

$$H(t) = \int_0^t \sigma^2(u) du.$$

Next, we compute the elements of the variance-covariance matrix of the vector in the last equation.

$$\begin{aligned} E\left(\int_0^v a_1(t) W(H(t)) dt\right)^2 &= E\left(\int_0^v a_1(t) W(H(t)) dt\right)\left(\int_0^v a_1(z) W(H(z)) dz\right) \\ &= \int_0^v \int_0^v a_1(t) a_1(z) E(W(H(t)) W(H(z))) dz dt \quad (4.6.2) \end{aligned}$$

Now,

$$E(W(H(t))W(H(z))) = E\int_0^t \sigma(s) dW(s) \int_0^z \sigma(s) dW(s) = \int_0^{t \wedge z} \sigma^2(s) ds.$$

Equation (4.6.2) may be written as:

$$\begin{aligned}
&= \int_0^v \int_{0<z<t} a_1(t)a_1(z) \int_0^z \sigma^2(s)dsdzdt + \int_0^v \int_{v>z>t} a_1(t)a_1(z) \int_0^t \sigma^2(s)dsdzdt \\
&= \int_0^v \int_0^t \int_s^t a_1(t)a_1(z)\sigma^2(s)dzdsdt + \int_0^v \int_0^t \int_t^v a_1(t)a_1(z)\sigma^2(s)dzdsdt \\
&= \int_0^v \int_s^v a_1(t)^2\sigma^2(s)dtds
\end{aligned}$$

where the last step follows by changing the order of the integrals. In a way similar to above, it can be shown that:

$$E\left(\int_0^v a_2(t)\int_0^t \sigma(s)dW(s)dt\right)^2 = \int_0^v \int_s^v a_2(t)^2\sigma^2(s)dtds.$$

Next, we consider

$$\begin{aligned}
&E\left(\int_0^v a_1(t)\int_0^t \sigma(s)dW(s)dt\right)\left(\int_0^v a_2(z)\int_0^z \sigma(s)dW(s)dz\right) \\
= \; &E\left(\int_0^v a_1(t)W(H(t))dt\right)\left(\int_0^v a_2(z)W(H(z))dz\right) \qquad (4.6.3)
\end{aligned}$$

where, as before

$$H(\cdot) = \int_0^{\cdot} \sigma^2(u)du.$$

Equation (4.6.3) may be written as:

$$\begin{aligned}
&= E\int_0^v \int_0^v a_1(t)a_2(z)W(H(t))W(H(z))dzdt \\
&= \int_0^v \int_0^v a_1(t)a_2(z)G(t,z)dzdt
\end{aligned}$$

where

$$G(t,z) = \int_0^{t\wedge z} \sigma^2(s)ds,$$

$$\begin{aligned}
&= \int_0^v \int_0^t a_1(t)a_2(z) \int_0^z \sigma^2(s)dsdzdt + \int_0^v \int_t^v a_1(t)a_2(z) \int_0^t \sigma^2(s)dsdzdt \\
&= \int_0^v \int_0^t \int_s^t a_1(t)a_2(z)\sigma^2(s)dzdsdt + \int_0^v \int_0^t \int_t^v a_1(t)a_2(z)\sigma^2(s)dzdsdt \\
&= \int_0^v \int_s^v a_1(t)dt \int_s^v a_2(z)dz\sigma^2(s)ds
\end{aligned}$$

where the last two equalities are obtained by changing the order of the integrals.

4.6.6 Proof of Proposition 4.4.4

Define

$$y'(t) = C - y(t)$$

and hypothesize:

$$\frac{S}{B}\bar{p_1}(t) = -Z(t)\bar{I}(t) - \varphi(t).$$

Following the same sequence of steps as in the proof of Proposition 4.2.7, we find that the optimal feedback solution to the Constant Capacity Market model is given by:

$$\bar{y}'(t) = \frac{1}{2\kappa}\big(Z(t)\bar{I}(t) + \varphi(t)\big) \tag{4.6.4}$$

where $Z(\cdot), \varphi(\cdot) \in \mathcal{C}^1([0,T];\mathcal{R})$ are solutions of the following differential equations, respectively:

$$\begin{aligned} \dot{Z}(t) + 2\pi' - \frac{1}{2\kappa}Z(t)^2 &= 0, \\ Z(T) &= 0, \end{aligned} \tag{4.6.5}$$

and

$$\begin{aligned} \dot{\varphi}(t) - \frac{1}{2\kappa}Z(t)\varphi(t) + Z(t)(C - D - \alpha\gamma\cos\gamma t) &= 0, \\ \varphi(T) &= 0. \end{aligned}$$

The solution to (4.6.5) is given in (4.4.6). Substituting for $\bar{y}'(\cdot)$ by $C - \bar{y}(\cdot)$ in equation (4.6.4) results in (4.4.5).

4.6.7 Proof of Proposition 4.4.6

From Proposition 4.4.4,

$$\bar{y}(t) = C - \frac{1}{2\kappa}\big(Z(t)\bar{I}(t) + \varphi(t)\big)$$

which may be written as:

$$\begin{aligned} d\bar{Y}(t) &= -\frac{1}{2\kappa}\left(Z(t)\left(\bar{Y}(t) - F(t)\right) - 2\kappa C + \varphi(t)\right)dt, \\ \bar{Y}(0) &= Y_0. \end{aligned}$$

The solution to the above stochastic differential equation can be obtained using an approach similar to the *Variation of Constants* method as demonstrated in the proof of Proposition 4.2.8 and is given by:

$$\begin{aligned} \bar{Y}(v) &= Y_0 \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(v-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(T)} \\ &+ \cosh\sqrt{\frac{\pi'}{\kappa}}(v-T)\int_0^v \sec h\sqrt{\frac{\pi'}{\kappa}}(t-T)\left(\frac{Z(t)}{2\kappa}F(t) + C - \frac{\varphi(t)}{2\kappa}\right)dt. \end{aligned}$$

Next, using the equations:

$$P(\cdot) = \frac{\bar{p}_1(\cdot)}{B}.$$

and

$$\bar{p}_1(\cdot) = -\frac{B}{S}(Z(\cdot)\bar{I}(\cdot) + \varphi(\cdot))$$

we obtain the expression for the equilibrium price.

4.6.8 Proof of Corollary 4.4.7

The variance of optimal production is given by:

$$Var(\bar{Y}(v)) = Var\left(\cosh(\sqrt{\frac{\pi'}{\kappa}}(v-T))A(v)\right)$$

where

$$A(v) = \int_0^v \sec h(\sqrt{\frac{\pi'}{\kappa}}(t-T))\left(\frac{Z(t)}{2\kappa}\int_0^t \sigma(s)dW(s)\right)dt.$$

Using Fubini's Theorem (Proposition 4.2.15) and that $\frac{Z(t)}{2\kappa} = -\sqrt{\frac{\pi'}{\kappa}}\tanh(\sqrt{\frac{\pi'}{\kappa}}(t - T))$,

$$A(v) = \int_0^v \int_s^v \sqrt{\frac{\pi'}{\kappa}} \frac{\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}{\cosh^2(\sqrt{\frac{\pi'}{\kappa}}(t-T))} \sigma(s) dt dW(s).$$

Simplifying $A(v)$ by solving the inner integral,

$$\begin{aligned} Var(\bar{Y}(v)) &= Var(\int_0^v \sigma(s) \left(1 - \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(v-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(s-T))}\right) dW(s)) \\ &= \int_0^v \sigma^2(s) \left(1 - \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(v-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(s-T))}\right)^2 ds. \end{aligned}$$

The variance of the price is equal to:

$$\begin{aligned} Var(P(v)) &= \frac{Z^2(v)}{S^2} Var\left(\int_0^v \sigma(s) \left(\frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(v-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(s-T))}\right) dW(s)\right) \\ &= \frac{4\pi\kappa \sinh^2(\sqrt{\frac{\pi'}{\kappa}}(v-T))}{SB} \int_0^v \sigma^2(s) \left(\frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(s-T))}\right)^2 ds \\ &= \frac{4\pi\kappa}{SB} \int_0^v \sigma^2(s) \left(\frac{e^{\sqrt{\frac{\pi'}{\kappa}}(v-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(v-T)}}{e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} + e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}}\right)^2 ds. \end{aligned}$$

Chapter 5
The Martingale Evolution of Price Forecasts in a Market for Supply Chain Capacity

5.1 Introduction

The Martingale Model of Forecast Evolution (MMFE), developed by Graves et al. [31] and Heath and Jackson [37], provides a framework to model evolution of forecasts in a discrete time setting. Heath and Jackson [37] assume that the forecast update in any period for demand in a future period is a normally distributed zero-mean random variable which is independent of all previous forecast updates. This assumption implies that the forecasts evolve as a Martingale process, thus lending the model its name. Several research models in the supply chain management have been built upon the MMFE. (See Iida and Zipkin [42], Lu et al. [54], Dong and Lee [20], Gullu [32], Toktay and Wein [74].) In this chapter, we apply the MMFE within the context of a market for supply chain capacity.

In the first half of this chapter, we develop the continuous time analog of the additive MMFE model for demand forecasts. In our model, we assume that the forecast for the rate of demand at any instant $t, 0 < t < \infty$, evolves as a continuous Martingale process over $[0, t]$. The forecast at any instant s of the rate of demand for any $t \geq s$, is thus equal to the conditional expectation of the rate of demand given all the available information until s.

In the second half of the chapter, we apply the continuous-time MMFE model to study the evolution of the equilibrium price in a market for capacity. Our goal is to study how the resolution of exogenous uncertainty translates to the resolution of uncertainty of the equilibrium market price. Using the Stochastic Maximum Principle, we obtain closed form solutions for optimal production and market price trajectories and show that they evolve as Martingales. We also derive the forecasting process corresponding to the optimal variables. This allows us to observe how the rate of resolution in exogenous uncertainty, which affects the market through the end-consumer demand process, filters through to the optimal variables. We also study the impact of supply chain cost parameters on the rate of resolving price uncertainty.

The rest of the chapter is structured as follows. In Section 5.2, we develop the continuous time analog of the additive Martingale Model of Forecast Evolution. In Section 5.3, we present an application of the continuous time MMFE to a market for capacity. In Section 5.4, we discuss the timing of resolution of uncertainty in price and conclude in Section 5.5.

5.2 Continuous Time Martingale Model of Forecast Evolution

We develop a Weiner's process model for the continuous resolution of forecasts for the exogenous demand of a generic good in a generic market. Let $f(s,t)$ denote the forecast at time s of the rate of demand of that good at time t and, let $d_s f(s,t)$ denote the forecast update at time s. Thus $f(t,t)$ represents the actual demand at time t. Let $\mathbf{W}(\cdot)(\equiv (W_1(\cdot), W_2(\cdot), ..., W_n(\cdot))$ be an $n-$dimensional Weiner's process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\mathcal{F}_t$ be the filtration.

Assumption 5.2.1. $\mathcal{F}_t = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$. *Further,* $\mathcal{F}_0$ *is* $\mathcal{P}-$*complete and* $\mathcal{P}-$ *degenerate.*

Compared to the assumption on the filtration in Heath and Jackson [37], Assumption 5.2.1 is more restrictive. As we shall see, the specification of filtration automatically ensures that $d_s f(s,t)$ is uncorrelated with $d_{s_1} f(s_1,t)$, for $s_1 < s$.

We assume that the forecasting process, $f(\cdot,t)$ evolves as a Martingale. Therefore, the forecast at time s of the rate of demand at t is the conditional expectation of $f(t,t)$ given the information in $\mathcal{F}_s$. We formally state the second assumption as follows.

Assumption 5.2.2. *$f(\cdot,t) : \Omega \times [0,t] \rightarrow \mathcal{R}$ is a square-integrable Martingale for any given $t \in [0,\infty)$.*

Using Theorem 4.15, Karatzas and Shreve [47]), if $f(s,t)$ is in $\mathcal{L}^2$ for every $s \leq t$ and is measurable with respect to $\mathcal{F}_s$, then there exist progressively measurable $\sigma_i \in \mathcal{L}^2, i = 1,..,n$ such that:

$$\begin{aligned} f(t,t) &= f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t) dW_i(s) \\ &= f(0,t) + \int_0^t \sigma(s,t) \mathbf{dW}(s)^T, \end{aligned}$$

where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$. Hence,

$$f(s,t) := E(f(t,t)|\mathcal{F}_s) = f(0,t) + \int_0^s \sigma(u,t) \mathbf{dW}(u)^T,$$

which implies:

$$d_s f(s,t) = \sigma(s,t) \mathbf{dW}(s)^T = \sum_{i=1}^{n} \sigma_i(s,t) dW_i(s).$$

Remark 1: Observe that at this stage, unlike Heath and Jackson [37], we do not assume that $d_s f(s,t)$ is stationary in $t-s$. We shall, however, make this assumption later when specifying $\sigma(s,t)$.

Remark 2: It is easily seen that $d_s f(s,t)$ is uncorrelated with $d_{s_1} f(s_1,t)$ for $s_1 < s$.

5.2.1 Additive Model

For the rest of this section, we shall assume $\sigma(\cdot, t)$ is independent of ω. Let $F(s,t)$ be the forecast at s of the cumulative demand until $t \geq s$. Thus $F(t,t)$ represents the actual cumulative demand until t. Define the realized cumulative demand by,

$$dF(t,t) = f(t,t)dt + \sigma(t,t)\mathbf{dW}(t)^T.$$

In the following lemma, we show that the cumulative demand process as defined above also evolves as a Martingale.

Corollary 5.2.3. *$F(\cdot, t) : \Omega \times [0,t] \to \mathcal{R}$ is a Martingale Process.*

Proof.

$$\begin{aligned}
F(s,t) &= F(s,s) + \int_s^t f(s,v)dv \\
&= F(s,s) + \int_s^t f(0,v)dv + \int_s^t \int_0^s d_u f(u,v)dv \\
&= \int_0^s f(v,v)dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_s^t f(0,v)dv \\
&+ \int_s^t \int_0^s d_u f(u,v)dv,
\end{aligned}$$

where the last step follows by noting that

$$F(s,s) = F(0,0) + \int_0^s f(v,v)dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T,$$

and by assuming $F(0,0) = 0$.

$$\begin{aligned}
F(s,t) &= \int_0^s \left(f(0,v) + \int_0^v d_u f(u,v) \right) dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T \\
&+ \int_s^t f(0,v)dv + \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv \\
&= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&+ \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv.
\end{aligned}$$

Therefore,

$$F(t,t) = \int_0^t \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^t \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv.$$

Now,

$$\begin{aligned}
E(F(t,t)|\mathcal{F}_s) &= E\left(\left(\int_0^t \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^t \sigma(v,v)\mathbf{dW}(v)^T\right)|\mathcal{F}_s\right) \\
&+ \int_0^t f(0,v)dv \\
&= E\left(\int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv|\mathcal{F}_s\right) \\
&+ \int_s^t \int_0^v \left(\sigma(u,v)\mathbf{dW}(u)^T dv|\mathcal{F}_s\right) \\
&+ \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv \\
&+ \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&= F(s,t).
\end{aligned}$$

□

In the following lemma, we present a result which will help us in comparing our model to the discrete time MMFE model.

Lemma 5.2.4. *$Cov(f(s,t), f(s,t_1)) = \int_0^s \sigma(u,t)\sigma^T(u,t_1)du, s < t \leq t_1$.*

Proof.

$$\begin{aligned}
Cov(f(s,t), f(s,t_1)) &= E\left(\int_0^s \sigma(u,t)\mathbf{dW}(u)^T\right)\left(\int_0^s \sigma(u,t_1)\mathbf{dW}(u)^T\right) \\
&= \int_0^s \sigma(u,t)\sigma^T(u,t_1)du.
\end{aligned}$$

□

So, we are led to describe the model of forecast evolution as,

$$\begin{aligned} F(s,t) &= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\ &+ \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv. \end{aligned}$$

We refer to this model as the Continuous-Time Martingale Model of Forecast Evolution (CTMMFE).

5.2.2 Relationship with the Discrete Time Model

In this subsection, we establish the relationship between CTMMFE and the discrete time MMFE model. In particular, we show that the model presented in Heath and Jackson [37] is a special case of CTMMFE. To obtain results in this section, we need the following result in order to be able to interchange the stochastic and the Lebesgue integral. Let $\mathcal{C}([0,T]\times[0,T];\mathcal{R}^n)$ be the set of all continuous functions $\Phi : [0,T]\times[0,T] \to \mathcal{R}^n$. We state the result for a one-dimensional Weiner's process only, but the result can be easily extended to an n - dimensional Weiner's process.

Proposition 5.2.5. *(A special case of Lemma 4.1, Ikeda and Watanabe [43], pp116): Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space and let $W(\cdot)$ be Weiner's process defined on it. Let $Q_1 \in \mathcal{C}([0,T]\times[0,T];\mathcal{R}^1)$. Then:*

$$\int_0^u \int_0^t Q_1(s,t)dW(s)dt = \int_0^u \int_s^u Q_1(s,t)dtdW(s). \tag{5.2.1}$$

Proof. See appendix. □

For ease of exposition only, the lower bound of the integrals is 0 on the LHS in (5.2.1). The proof can be extended easily to the case when the lower bound of the integrals is strictly positive. Similarly, the above lemma can be extended easily to the case when the upper limits of the stochastic and Lebesgue integrals are unrelated.

For any $s, t_1, t_2 \in \mathcal{N}$ such that $s+1 \le t_1 \le t_2$, let δ_{s,t_1} be the forecast at s of the cumulative demand occurring in $(t_1 - 1, t_1]$, and let δ_{s,t_2} be the forecast at s of cumulative demand occurring in $(t_2 - 1, t_2]$. Therefore,

$$\begin{aligned}
\delta_{s,t_1} &= F(s,t_1) - F(s,t_1-1) = \int_{t_1-1}^{t_1} f(0,v)dv + \int_{t_1-1}^{t_1}\int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv, \\
\delta_{s,t_2} &= F(s,t_2) - F(s,t_2-1) = \int_{t_2-1}^{t_2} f(0,v)dv + \int_{t_2-1}^{t_2}\int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv.
\end{aligned}$$

The variable δ_{s,t_j} represents the forecast of demand in (discrete) time period j made at time s. In the following lemma, we obtain the covariance of forecasts for two periods in the future.

Lemma 5.2.6. *Let $\sigma(\cdot,\cdot) \in \mathcal{C}([0,T]\times[0,T];\mathcal{R}^n)$. Then,*

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \int_0^s \left(\int_{t_1-1}^{t_1} \sigma(u,z)dz\right)\left(\int_{t_2-1}^{t_2} \sigma(u,z)dz\right)^T du.$$

Proof.

$$\begin{aligned}
Cov(\delta_{s,t_1}, \delta_{s,t_2}) &= Cov\left(\int_{t_1-1}^{t_1}\int_0^s \sigma(u,z)\mathbf{dW}(u)^T dz, \int_{t_2-1}^{t_2}\int_0^s \sigma(u,z)\mathbf{dW}(u)^T dz\right) \\
&= Cov\left(\int_0^s\int_{t_1-1}^{t_1} \sigma(u,z)dz\mathbf{dW}(u)^T, \int_0^s\int_{t_2-1}^{t_2} \sigma(u,z)dz\mathbf{dW}(u)^T\right),
\end{aligned}$$

where

$$\int_0^s\int_{t_1-1}^{t_1} \sigma(u,z)dz\mathbf{dW}(u)^T = \sum_{i=1}^n \int_0^s\int_{t_1-1}^{t_1} \sigma_i(u,z)dzdW_i(u)^T.$$

The last step follows using Proposition 5.2.5. Therefore,

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \int_0^s \left(\int_{t_1-1}^{t_1} \sigma(u,z)dz\right)\left(\int_{t_2-1}^{t_2} \sigma(u,z)dz\right)^T du.$$

□

The above result illustrates that the forecasts of demand in different discrete time periods have a well-defined (deterministic) correlation structure.

Exponential Models

In this subsection, we assume that $\sigma_i(u,t)$ is an exponential function of $t-u$. The model in this subsection is identical to the exponential model considered by Heath and Jara [38]. According to the Stone-Weierstrass Theorem, the algebra of functions $\{e^{-\lambda v} : \lambda \geq 0\}$ can approximate, as closely as desired, any continuous function defined on $[a,b]$ for any $0 \leq a < b$. Therefore, any continuous σ_i can be approximated arbitrarily closely by

$$\sigma_i(u,t) \approx \sum_{l=1}^{m} \xi_{il} \exp(\lambda_l(u-t)), \tag{5.2.2}$$

for suitably large m. Note that the same set of $\{\lambda_l, 1 \leq l \leq m\}$ can be used to approximate all the σ_i, for $i = 1 \cdots n$.

Corollary 5.2.7. *Let $\sigma_i(u,t)$ be approximated as in (5.2.2). Then, for $s, t_1, t_2 \in \mathcal{N}$, $s+1 \leq t_1 \leq t_2$:*

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \sum_{i=1}^{n} \sum_{1 \leq l \leq p \leq m} \frac{\xi_{il}\xi_{ip}e^{-(\lambda_l t_1 + \lambda_p t_2)}(e^{(\lambda_l+\lambda_p)s} - 1)(1-e^{\lambda_l})(1-e^{\lambda_p})}{\lambda_l \lambda_p (\lambda_l + \lambda_p)}.$$

Next, we show that the forecast update over a discrete length of time can be written as a sum of normally distributed random variables. Following the same notation as in Heath and Jackson [37], let $\varepsilon_{s,t_1} = \delta_{s,t_1} - \delta_{s-1,t_1}, s \leq t_1 - 1$ be the forecast update during $(s-1, s]$ for demand in $(t_1 - 1, t_1]$ where $s+1 \leq t_1$. Then:

$$\begin{aligned} \varepsilon_{s,t_1} &= \int_{t_1-1}^{t_1} \int_{s-1}^{s} \sigma(u,z) \mathbf{dW}(u)^T dz \\ &= \int_{s-1}^{s} \left(\int_{t_1-1}^{t_1} \sigma(u,z) dz \right) \mathbf{dW}(u)^T \end{aligned}$$

where the last step follows using Proposition 5.2.5. Now, assuming σ_i is approximated as in (5.2.2):

$$\begin{aligned}\varepsilon_{s,t_1} &= \sum_{i=1}^{n}\int_{s-1}^{s}\int_{t_1-1}^{t_1}\sum_{l=1}^{m}\xi_{il}e^{\lambda_l(u-z)}dzdW_i(u)\\ &= \sum_{i=1}^{n}\int_{s-1}^{s}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(e^{\lambda_l(u-t_1+1)}-e^{\lambda_l(u-t_1)})dW_i(u)\\ &= \int_{s-1}^{s}\mathbf{c}_{t_1}(u)\mathbf{dW}(u)^T\end{aligned}$$

where $\mathbf{c}_{t_1}(u)$ is a $1\times n$ vector whose i^{th} element is given by:

$$\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(\exp(\lambda_l(u-t_1+1))-\exp(\lambda_l(u-t_1)))$$

which is of similar nature as in Heath and Jackson [37]. Using “time-substitution” (McKean [55]),

$$\varepsilon_{s,t_1}=\sum_{i=1}^{n}W_i(H_i(s,t_1))$$

where

$$H_i(s,t_1)=\int_{s-1}^{s}\left(\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(\exp(\lambda_l(u-t_1+1))-\exp(\lambda_l(u-t_1)))\right)^2du,$$

implying that ε_{s,t_1} is normally distributed with mean 0 and variance $\sum_{i=1}^{n}H_i(s,t_1)$.

It is easily seen that any finite-dimensional variance-covariance matrix for discrete-time can be approximated by a suitable choice of the parameters $\{\xi_{il},\lambda_l\}$. Following Heath and Jackson [37], let Σ denote the variance-covariance matrix of forecast updates at time s. The following corollary provides the entries of Σ.

Corollary 5.2.8. *Let $\sigma_i(u,t)$ be approximated as in (5.2.2). Then, for $s,t_1,t_2\in\mathcal{N}$, $s+1\le t_1\le t_2$, the covariance of forecast updates for t_1 and t_2 is equal to:*

$$Cov(\varepsilon_{s,t_1},\varepsilon_{s,t_2})=\sum_{i=1}^{n}\sum_{1\le l\le p\le m}\frac{\xi_{il}\xi_{ip}e^{\lambda_l(s-t_1)+\lambda_p(s-t_2)}\begin{pmatrix}e^{-(\lambda_l+\lambda_p)}-e^{\lambda_p}+e^{-\lambda_l}\\ -e^{\lambda_l}+e^{-\lambda_p}+e^{\lambda_l+\lambda_p}\end{pmatrix}}{\lambda_l\lambda_p(\lambda_l+\lambda_p)}.$$

The above corollary illustrates that the covariance of forecast updates at $t_1, t_2 \geq s+1$ depends only on "time-to-go", that is, $t_1 - s$ and $t_2 - s$, which is in alignment with the stationary nature of approximation (5.2.2) for σ. The link between the CTMMFE and the discrete time MMFE of Heath and Jackson [37] should now be clear, with the addition of assumptions of the exponential structure of $\sigma_i, i = 1 \cdots n$, and the stationarity of this structure, we can derive the discrete time MMFE as a special case of CTMMFE.

5.3 Application of MMFE: The Market Model for Capacity

In this section, we present an application of the continuous time MMFE model. We apply the continuous-time MMFE model to study the evolution of the equilibrium price in a market for capacity. Our goal is to study how the resolution of exogenous uncertainty translates to the resolution of uncertainty of the equilibrium market price.

5.3.1 Notation

We formulate a market model in which sellers produce and sell a homogenous good to buyers who store it and re-sell it to end-consumers. The homogeneous good represents production capacity of the sellers stored in the form of buyer-specific (make-to-order) products for sale to consumers. The underlying uncertainty in the model is the demand process of the end-consumers. The stochastic process of interest is the price which sellers receive, and buyers pay for the homogeneous good at any point in time, that is, the market price of capacity.

The economy consists of S sellers, indexed by k, and B buyers, indexed by j. Each agent, buyer or seller, is assumed to be a price-taker in the market for the homogeneous good, called *capacity*. The underlying uncertainty in the economy is represented by $\mathbf{W}(\cdot)$, an $n-$dimensional standard Weiner's process defined on $(\Omega, \mathcal{F}, \mathcal{P})$ which we assume to be a complete probability space. Define $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.

The notation for this model remains the same as in Section 3.2. Let $F_j(t,t)$ denote the cumulative demand from end-consumers for the sales of buyer j through time t. This demand process is exogenous to the model. Presumably, buyer j's product is sufficiently differentiated from that of other buyers' products that we can ignore competition among buyers for shares of end-consumer demand. We assume the end-consumer demand process to consist of a linear, a seasonal, a forecast update, and a Weiner's process component. We use the following model of instantaneous demand for buyer j:

$$dF_j(t,t) \equiv \left(D_j + \alpha_j \gamma \cos \gamma t + \int_0^t \sigma(u,t) \mathbf{dW}(u)^T \right) dt + \sigma(t,t) \mathbf{dW}(t)^T \quad (5.3.1)$$

and, hence,

$$\begin{aligned} F_j(t,t) &= D_j t + \alpha_j \sin \gamma t + \int_0^t \int_0^s \sigma(u,s) \mathbf{dW}(u)^T ds \\ &+ \int_0^t \sigma(s,s) \mathbf{dW}(s)^T. \end{aligned} \quad (5.3.2)$$

We assume that all buyers face the same seasonal period γ and forecast update coefficient $\sigma(\cdot,\cdot)$, but may differ in average demand rates and seasonal amplitudes. For each buyer, j, the expected average demand rate D_j will be assumed to be suitably large relative to α_j and γ to ensure that cumulative demand is non-decreasing with high-probability. In particular, we require $D_j \geq \alpha_j \gamma$. Note that we assume that the same Weiner's process drives all demand.

Let $P(t)$ denote the price of capacity, the homogeneous good, bought and sold at time t. We assume that this process is of the form:

$$P(t) = a(t) + \int_0^t \mathbf{b}(s,t)\mathbf{dW}^T(s) \tag{5.3.3}$$

for suitable value of $a : \mathcal{R} \to \mathcal{R}$ and $\mathbf{b} : \mathcal{R}^2 \to \mathcal{R}^n$ where the $n-$ dimensional Weiner's process $\mathbf{W}(\cdot)$ is the same process as underlies the demand model. We assume $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are such that the price process is square-integrable. We formally state this assumption as follows:

Assumption 5.3.1. *$E\int_0^T P(t)^2 dt < \infty$ and $P(t)$ is $\mathcal{F}_t$-adapted.*

We are unable to prove that the equilibrium price process has this form but we can derive values of $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ that are consistent with equilibrium conditions and this assumption.

Provided that $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ satisfy certain equilibrium conditions described below, we assume that each agent in the economy plans and implements production and procurement decisions according to this specific price process. Hence, the sellers' production plans and the buyers' procurement plans will be seen to be stochastic functions of $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$.

In the models to follow, we suppress the time argument t unless needed for clarification. In both the buyer and seller models, we assume a quadratic cost structure in order to derive explicit solutions.

Given a price process $P(\cdot)$, buyer j's problem is to choose a production order policy $x_j(\cdot)$ to minimize the total expected cost of production orders and inventory/shortfall costs. The finite horizon version of the Buyer problem (Buyer model) with quadratic costs is:

$$\begin{aligned}
& \min_{x_j \in \mathcal{U}_j[0,T]} E \int_0^T \left\{ \pi I_j(t)^2 + P(t) x_j(t) \right\} dt \\
& \text{s.t.} \\
dI_j(t) &= \left(x_j(t) - D_j - \alpha_j \gamma \cos \gamma t - \int_0^t \sigma(s,t) \mathbf{dW}(s)^T \right) dt \\
&- \sigma(t,t) \mathbf{dW}(t)^T, t \in [0,T] \qquad (5.3.4) \\
X_j(0) &= X_{0j},
\end{aligned}$$

where $I_j(t)$ is the net inventory at time t (on hand inventory less backorders), so the objective function penalizes any deviation of net inventory from zero. The buyers can hold inventory or incur backorders, so it is not necessary for production orders to exactly equal consumer demand at any time. We assume that all the buyers start with the same level of inventory, that is, $X_{0j} = \frac{X(0)}{B}$.

This model of buyer behavior can be criticized for the symmetry of the cost function with respect to the net inventory. In practice, backorders are penalized at a higher rate than on-hand inventory. Our focus, however, is on a higher-order behavior, that of market equilibrium price for capacity. The general trade-off considered in this chapter is between the cost of adjusting production versus the cost of inventory and backorders mismatches, and on how a market price can serve to equilibrate this general trade-off. In order to solve the Buyer model, we make the following assumptions on the control variable x_j and state variable I_j.

Assumption 5.3.2. *The set of controls* $\mathcal{U}_j[0,T]$ *consists of all* $x_j : [0,T] \times \Omega \to \mathcal{R}^1$ *such that* $x_j(\cdot)$ *is* $\mathcal{F}_t$*-adapted and* $E \int_0^T x_j(t)^2 dt < \infty$.

Assumption 5.3.3. *For any* $x_j^1, x_j^2 \in \mathcal{U}_j[0,T]$*, and* $\rho \in [0,1]$ *the following holds:*

$$E \left[\int_0^T |I_j^1 + \rho I_j^2|^2 dt \right] < \infty,$$

where I_j^1 and I_j^2 are states of the system controlled by x_j^1 and x_j^2, respectively.

Observe that price $P(\cdot)$ is treated as a random coefficient in the objective function of the Buyer model (5.3.4). According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 5.3.2 along with the linear nature of the state equation for the net inventory in the Buyer model (5.3.4) ensure a unique solution to the state equation. Assumption 5.3.3 is satisfied due to the square-integrability of the control variable x_j.

Similarly, each seller faces the following stochastic control problem. Given a price process $P(\cdot)$, seller k's problem is to choose a production rate policy y_k, to minimize the total expected cost of short term capacity adjustment less the revenue derived from production. The finite horizon version of the Seller problem (Seller model) with quadratic costs is:

$$\begin{aligned} & \min_{y_k \in \mathcal{U}_k[0,T]} E \int_0^T \left\{ \kappa (C_k - y_k(t))^2 - P(t) y_k(t) \right\} dt \\ & \text{s.t.} \\ dY_k(t) &= y_k(t)dt, t \in [0,T] \qquad (5.3.5) \\ Y_k(0) &= Y_{0k}. \end{aligned}$$

Since sellers do not hold inventory, Y_{0k} should be equal to 0 in practice. In order to solve the Seller model, we make the following assumptions on the control variable y_k.

Assumption 5.3.4. *The set of controls $\mathcal{U}_k[0,T]$ consists of all $y_k : [0,T] \times \Omega \to \mathcal{R}^1$ such that $y_k(\cdot)$ is $\mathcal{F}_t$-adapted and $E \int_0^T y_k(t)^2 dt < \infty$.*

Assumption 5.3.5. *For any $y_k^1, y_k^2 \in \mathcal{U}_k[0,T]$, and $\rho \in [0,1]$, the following holds:*

$$E \left[\int_0^T |2\kappa (C_k - y_k^1 - \rho y_k^2) - P|^2 dt \right] < \infty.$$

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 5.3.4 along with the linearity of the state equation for the cumulative production in the Seller model (5.3.5) ensure a unique solution to the state equation. Assumption 5.3.5 is satisfied due to the square-integrability of P and y_k.

Observe that we do not impose non-negativity constraints on the instantaneous rate of order process or on the instantaneous rate of production process. The imposition of those constraints would have made it impossible to obtain closed form solutions for the optimal production or the equilibrium market price. However, we assume that the value of D_j is relatively large compared to the randomness in the demand and, therefore, the probability of occurrence of negative order rate or production rate is negligible. The same assumption ensures that the demand rate is non-negative with a high probability.

Observe that the costs of production (material, labor, and capital) are ignored in the Seller model (5.3.5): only the short term costs of production adjustment are captured. Also observe that the revenue from consumer sales are ignored in the Buyer model (5.3.4): only the inventory/shortfall costs are relevant. As a result, the price in this market will reflect the trade-off between the sellers' production adjustment costs and the buyers' inventory/shortfall costs. There is no constraint on the sign of $P(t)$. If positive it can be interpreted as a premium or, if negative, as a discount on some exogenously determined price that consider the production costs and consumer revenues that we have omitted from this model.

Since all agents are assumed to be price takers, the production and production order policies $y_k(\cdot)$ and $x_j(\cdot)$ that optimize seller and buyer problems, respectively, will depend on the price process $P(\cdot)$. We assume that the market will be in equilibrium at all times. That is, the price process $P(\cdot)$ must ensure that

$$y(t) = x(t) \text{ for all } t \geq 0. \tag{5.3.6}$$

In equilibrium, therefore, $Y(t) = X(t) - X_0 + Y_0$.

By means of (5.3.2), (5.3.4), (5.3.5), and (5.3.6), we have described a simple market for capacity in which the demand for capacity is intertemporal in nature: if capacity prices are high, buyers can defer production orders (depleting inventory or incurring shortages) while if capacity prices are low, then buyers can advance production orders (eliminating shortages or building inventory). We proceed to solve these models and to demonstrate this behavior.

Throughout this section, we make the following approximation:

$$\sigma_i(s,t) = \sum_{l=1}^{m} \xi_{il} \exp(\lambda_l(s-t)) \tag{5.3.7}$$

where $\sigma_i(s,t)$ is the i^{th} component of $\sigma(s,t)$.

5.3.2 Solution to the Market Model for Capacity

In this subsection, we present and discuss the solution to the Market model for capacity. First, we sketch the necessary and sufficient conditions for equilibrium solution to the Market model. Observe that the drift term in the state equation for the net inventory is stochastic, since we take into account the forecast updates during the horizon. To obtain necessary and sufficient conditions for optimality, we apply the version of the Stochastic Maximum Principle for a Linear Quadratic problem with random objective function and random state equation coefficients (for details, see Cadenillas and Karatzas [9]).

Optimal Control of the Buyer Model

Define the Hamiltonian of the Buyer model as:

$$\begin{aligned} H_b(x_j, I_j, p_{1,j}, \mathbf{q}_{1,j}) &= -\pi I_j^2 - Px_j + \mathbf{q}_{1,j}\sigma(t,t)^T \\ &+ p_{1,j}(x_j - D_j - \alpha_j\gamma\cos\gamma t - \int_0^t \sigma(u,t)\mathbf{dW}(u)^T) \end{aligned} \quad (5.3.8)$$

Observe that the coefficient of the adjoint variable $p_{1,j}$ in the Hamiltonian function is equal to the drift term in state equation for $I_j(t)$. Similarly, the coefficient of $\mathbf{q}_{1,j}$ is equal to the diffusion term in the state equation for $I_j(t)$. The adjoint variable pair $p_{1,j} : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{q}_{1,j} : [0,T] \times \Omega \to \mathcal{R}^n$ is measurable, adapted, and is defined by the following stochastic differential equation:

$$\begin{aligned} dp_{1,j}(t) &= 2\pi I_j(t)dt + \mathbf{q}_{1,j}(t)\mathbf{dW}(t)^T, \\ p_{1,j}(T) &= 0. \end{aligned}$$

The adjoint variable $p_{1,j}$ can be interpreted as the shadow price corresponding to the net inventory “resource”. At each time instant, $p_{1,j}$ is random. In the deterministic case, if the value function is sufficiently smooth, then the time rate of change of the adjoint variable $p_{1,j}$ is equal to the negative of the partial derivative of the Hamiltonian with respect to the state variable I_j. That is, $\frac{\partial p_{1,j}(t)}{\partial t} = -\frac{\partial H}{\partial I_j}$, and $\mathbf{q}_{1,j} = 0$, in the deterministic case.

The second adjoint variable vector $\mathbf{q}_{1,j}$ is not constrained to satisfy any differential equation. However, it cannot be set to zero everywhere. The boundary condition for the stochastic differential condition for $p_{1,j}$ is specified at the end of the horizon. Therefore, if $\mathbf{q}_{1,j}$ is identically set to zero, the resulting solution for $p_{1,j}$ may not be $\mathcal{F}_t$-adapted.

Using Proposition 1.2 in Cadenillas and Karatzas [9], a necessary and sufficient condition for $\bar{x}_j$ to be optimal for the Buyer's model(5.3.4) is that $\forall x_j \in \mathcal{U}_j$:

$$E\left(\int_0^T (P(t) - \bar{p}_{1,j}(t))(x_j(t) - \bar{x}_j(t))\right) \geq 0.$$

where $(\bar{p}_{1,j}, \bar{\mathbf{q}}_{1,j})$ is the adjoint variable pair that corresponds to the system controlled by $\bar{x}_j$. The above condition is satisfied if and only if

$$P(t) = \bar{p}_{1,j}(t), a.e.(t, \omega) \in [0, T] \times \Omega. \tag{5.3.9}$$

The last equation implies that, in equilibrium, all the buyers must have the same shadow price at all instants. Summing the last equation over all the buyers gives:

$$P = \frac{\bar{p}_1}{B}, a.e.(t, \omega) \in [0, T] \times \Omega. \tag{5.3.10}$$

where $\bar{p}_1 = \sum_j \bar{p}_{1,j}$.

Optimal Control of the Seller's Model

Similarly, we define the Hamiltonian for the seller k's model as:

$$H_s(t, Y_k, y_k, p_{2,k}, \mathbf{q}_{2,k}) = -\kappa(C_k - y_k)^2 + Py_k + p_{2,k}y_k.$$

The adjoint variable pair $p_{2,k} : [0, T] \times \Omega \rightarrow \mathcal{R}$, and $\mathbf{q}_{2,k} : [0, T] \times \Omega \rightarrow \mathcal{R}^n$ is measurable, adapted and is defined by the following backward stochastic differential equation:

$$\begin{aligned} dp_{2,k} &= \mathbf{q}_{2,k}\mathbf{dW}(t)^T, \\ p_{2,k}(T) &= 0. \end{aligned}$$

It is easily seen that $p_{2,k} = \mathbf{q}_{2,k} \equiv 0$. The adjoint variable $p_{2,k}$ can be interpreted as the shadow price corresponding to Y_k. Since Y_k does not appear in the objective function for the Seller model, it is appropriate that $p_{2,k}$ be uniformly zero.

According to Theorem 3.2, Cadenillas and Karatzas [9], if the objective function is convex in the state and control variables and is allowed to be random, then $\bar{y}_k$ is an optimal control variable if and only if

$$\max_{y_k \in \mathcal{U}_k} H_s(y_k, \bar{Y}_k, \bar{p}_{2,k}, \bar{q}_{2,k}) = H_s(\bar{y}_k, \bar{Y}_k, \bar{p}_{2,k}, \bar{q}_{2,k}), a.e.(t, \omega) \in [0, T] \times \Omega,$$

where $\bar{Y}_k$ and $(\bar{p}_{2,k}, \bar{q}_{2,k})$ are the state variable and adjoint variable pair, respectively, corresponding to the system controlled by $\bar{y}_k$. The above equation yields:

$$2\kappa(\bar{y}_k - \bar{C}_k) = P, a.e.(t, \omega) \in [0, T] \times \Omega.$$

That is, in equilibrium, all sellers must have the same production-capacity mismatch. Summing the last equation over all sellers results in:

$$SP + 2\kappa(\bar{C} - \bar{y}) = \frac{S}{B}\bar{p}_1 + 2\kappa(\bar{C} - \bar{y}) = 0, a.e.(t, \omega) \in [0, T] \times \Omega \qquad (5.3.11)$$

where the RHS is obtained by substituting for P from (5.3.10).

To obtain the optimal solution to the Market model, we use the equilibrium condition (5.3.6) to link the buyers' models and the sellers' models. We are now ready to state the necessary and sufficient conditions for the optimality of the Market model in the following proposition:

Proposition 5.3.6. *Under Assumption 5.3.1, the vector of market variables, $(\bar{I}, \bar{y}(= \bar{x}), \bar{p}_1, \bar{q}_1)$ is optimal if and only if it satisfies the following system of equations in equilibrium at time $t \in [0, T]$:*

$$\begin{aligned}
d\bar{I}(t) &= (\bar{y}(t) - D - \alpha\gamma\cos\gamma t - B\sigma(s,t)\mathbf{dW}(s)^T)dt \\
&- B\sigma(t,t)\mathbf{dW}(t)^T, \qquad (5.3.12) \\
d\bar{p}_1(t) &= 2\pi\bar{I}(t)dt + \bar{\mathbf{q}}_1(t)\mathbf{dW}(t)^T, \\
\bar{p}_1(T) &= 0, \\
\bar{x}(t) &= \bar{y}(t) = C + \frac{1}{2\kappa}\frac{S}{B}\bar{p}_1(t),
\end{aligned}$$

where $\bar{\mathbf{q}}_1(t) = \sum_j \bar{\mathbf{q}}_{1,j}(t)$ *and* $C = \sum_k C_k$.

Proof. Necessity: The first equation is obtained by summing the state equation for net inventory over all buyers. Similarly, the second and third equations are obtained by adding the differential equations and the terminal conditions, respectively, for adjoint variable $p_{1,j}$ over all j. The last equation is obtained by combining (5.3.11) with the equilibrium condition (5.3.6).

Sufficiency: It is enough to find a disaggregated solution for each buyer and seller, given an aggregated solution of the above equations (5.3.12), that satisfies the necessary and sufficient conditions for the Buyer and Seller models. Consider the following disaggregated solution for the Buyer model,

$$(\bar{X}_j(t), \bar{I}_j(t), \bar{p}_{1,j}(t), \bar{\mathbf{q}}_{1,j}(t)) = (\frac{\bar{X}(t) - F(t,t)}{\sum_i \frac{\pi_j}{\pi_i}} + F_j(t,t), \frac{\bar{I}(t)}{\sum_i \frac{\pi_j}{\pi_i}}, \frac{\bar{p}_1(t)}{B}, \frac{\bar{\mathbf{q}}_1(t)}{B}),$$

and the Seller model,

$$(\bar{y}_k(t), \bar{Y}_k(t)) = (\frac{\bar{y}(t) - C}{\sum_i \frac{\kappa_k}{\kappa_i}} + C_k, \frac{\bar{Y}(t) - Ct - Y_0}{\sum_i \frac{\kappa_k}{\kappa_i}} + C_k t + Y_{0,k}),$$

where the equality holds componentwise. Clearly, the above solution satisfies the necessary and sufficient conditions for the Buyer and Seller models. The proof is completed by noting the uniqueness of the solution to (5.3.12) (see Proposition 5.3.11). □

The above result shows the market distributes the equilibrium capacity and rate of production among the sellers equally (save for the correction due to initial values). Similarly, all the buyers place orders at the same rate. This result is hardly surprising as all the sellers and buyers are identical, respectively, in cost parameters.

In order to solve for the optimal variables, we define another model which we refer to as the *Integrated model*. The structure of the Integrated model is similar to a well-known problem called the Linear Regulator problem whose solution is provided by Cadenillas and Karatzas [9]. Our approach involves obtaining the necessary and sufficient conditions for optimality in the Integrated model and showing them to be equivalent to the necessary and sufficient conditions for optimality in the Market model. Thus, the optimal solution to the Integrated model can be used to obtain the optimal solution to the Market model.

As we shall see later, the Integrated model corresponds to the integrated supply chain (hence the name) in which the supply chain is owned wholly by a single agent. Thus, the owner of the supply chain not only owns the capacity but also satisfies the end-consumer demand. In the following section, we state and solve the Integrated model.

5.3.3 Integrated Model

The goal of the Integrated model is to choose a production policy y_ι to minimize the total expected cost of overtime/undertime and inventory/shortfall costs. The finite horizon version of this problem with quadratic costs is:

$$\begin{aligned}
& \min_{y_\iota \in \mathcal{U}_\iota[0,T]} E \int_0^T \left\{ \kappa(C_\iota - y_\iota(t))^2 + \pi' I_\iota(t)^2 \right\} dt \\
& \text{s.t.} \\
dI_\iota(t) &= (y_\iota(t) - D_\iota - \alpha_\iota \gamma_\iota \cos \gamma t - \int_0^t \sigma_\iota(u,t) \mathbf{dW}(u)^T) dt \\
&- \sigma_\iota(t,t) \mathbf{dW}(t)^T, t \in [0,T] \qquad (5.3.13) \\
I_\iota(0) &= Y_{0\iota}.
\end{aligned}$$

where all the notation remains the same as before except for the distinguishing mark, subscript ι, to indicate the Integrated model and the net inventory penalty cost π' where π' is defined as:

$$\pi' = \frac{S}{B} \pi.$$

As before, $\mathbf{W}(\cdot)$ is an $n-$dimensional standard Weiner's process defined on $(\Omega, \mathcal{F}, \mathcal{P})$, a complete probability space. Define $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.

Assumption 5.3.7. *The set of controls* $\mathcal{U}_\iota[0,T]$ *consists of all* $y_\iota : [0,T]\times\Omega \to \mathcal{R}^1$ *such that* $y_\iota(\cdot)$ *is* $\mathcal{F}_t$*-adapted, and* $E\int_0^T y_\iota(t)^2 dt < \infty$.

Assumption 5.3.8. *For any* $y_\iota^1, y_\iota^2 \in \mathcal{U}_\iota[0,T]$*, and* $\rho \in [0,1]$*, the following holds:*

$$E\left[\int_0^T |\kappa(C_\iota - y_\iota^1 - \rho y_\iota^2) + \pi'(I_\iota^1 + \rho I_\iota^2)|^2 dt\right] < \infty,$$

where I_ι^1 *and* I_ι^2 *are states of the systems controlled by* y_ι^1 *and* y_ι^2*, respectively.*

According to Theorem 6.16, pp 49, Yong and Zhou [79], Assumption 5.3.7 along with the linearity of the state equation for the net inventory in the Integrated model (5.3.13) ensure a unique solution to the state equation. Assumption 5.3.8 is satisfied due to the square-integrability of y_ι.

Next, we derive the necessary and the sufficient conditions for optimality of the Integrated model.

Optimal Control of the Integrated Model

Define the Hamiltonian as:

$$\begin{aligned} H_\iota &= -\pi' I_\iota^2 - \kappa(C_\iota - y_\iota)^2 + \mathbf{q}_{1,\iota}\sigma_\iota(t,t) \\ &+ p_{1,\iota}\left(y_\iota - D_\iota - \alpha_\iota\gamma_\iota\cos\gamma_\iota t - \int_0^t \sigma_\iota(u,t)\mathbf{dW}(u)^T\right). \end{aligned} \tag{5.3.14}$$

The adjoint variable pair $p_{1,\iota} : [0,T]\times\Omega \to \mathcal{R}$, and $\mathbf{q}_{1,\iota} : [0,T]\times\Omega \to \mathcal{R}^n$ is measurable, adapted, and is defined by the following stochastic differential equation:

$$\begin{aligned} dp_{1,\iota}(t) &= 2\pi' I_\iota(t)dt + \mathbf{q}_{1,\iota}(t)\mathbf{dW}(t)^T, \\ p_{1,\iota}(T) &= 0. \end{aligned}$$

According to Theorem 3.2, Cadenillas and Karatzas [9], if the objective function is convex in the state and control variables and is allowed to be random, then $\bar{y}_\iota$ is an optimal control variable if and only if

$$\max_{y_\iota \in \mathcal{U}_\iota} H_\iota(y_\iota, \bar{I}_\iota, \bar{p}_{1,\iota}, \bar{q}_{1,\iota}) = H_\iota(\bar{y}_\iota, \bar{I}_\iota, \bar{p}_{1,\iota}, \bar{q}_{1,\iota}), a.e.(t,\omega) \in [0,T] \times \Omega,$$

where $\bar{I}_\iota$ and $(\bar{p}_{1,\iota}, \bar{q}_{1,\iota})$ are the state variable and adjoint variable pair, respectively, corresponding to the system controlled by $\bar{y}_\iota$. Optimizing the Hamiltonian yields,

$$\bar{y}_\iota = \bar{C}_\iota + \frac{1}{2\kappa}\bar{p}_{1,\iota}.$$

The necessary and the sufficient conditions for optimality of the Integrated model are very similar to those for the Market model. Indeed, if the optimal solution to one is known, the optimal solution to the other can be easily obtained. We formally state this in the following corollary.

Corollary 5.3.9. *Assume that* $\pi' = \frac{S}{B}\pi$ *and the cost parameter value* κ *is the same for the two models. Further assume* $C_\iota = C, D_\iota = D, \alpha_\iota = \alpha, \gamma_\iota = \gamma, Y_{0\iota} = X_0 - Y_0$, *and* $\sigma_\iota = B\sigma$. *Then, the optimal vector of market variables,* $(\bar{I}, \bar{y}(=\bar{x}), \bar{p}_1, \bar{\mathbf{q}}_1)$, *and the optimal vector,* $(\bar{I}_\iota, \bar{y}_\iota, \bar{p}_{1,\iota}, \bar{\mathbf{q}}_{1,\iota})$, *of the Integrated model are related, as follows:*

$$\begin{aligned} \bar{p}_{1,\iota} &= \frac{S}{B}\bar{p}_1, \\ \bar{\mathbf{q}}_{1,\iota} &= \frac{S}{B}\bar{\mathbf{q}}_1, \\ \bar{y}_\iota &= \bar{y}, \\ \bar{I}_\iota &= \bar{I}, a.e.(t,\omega) \in [0,T] \times \Omega. \end{aligned}$$

Next, we obtain the solution to the Integrated model.

Solution to the Integrated Model

Define $y'_\iota = C_\iota - y_\iota$. With this substitution, the Integrated model becomes identical to the well-known Linear Regulator problem with a random drift term in the state equation. Cadenillas and Karatzas [9] state the solution to the Linear Regulator problem. The optimal solution to the Linear-Quadratic problem is obtained by hypothesizing a linear relationship between the adjoint variables and the state variables. As an example, in the case of the Integrated model, this hypothesis would take the following form:

$$\begin{aligned}\bar{p}_{1,\iota}(t) &= -Z(t)\bar{I}_\iota(t) - \varphi(t), \\ \bar{\mathbf{q}}_{1,\iota}(t) &= -\Theta(t)\bar{I}_\iota(t) + Z(t)\sigma_\iota(t,t) - \Lambda(t),\end{aligned}$$

where $Z(\cdot), \Theta(\cdot), \varphi(\cdot)$, and $\Lambda(\cdot)$ are measurable and adapted processes. We use the solution provided by Cadenillas and Karatzas [9] to solve for the Integrated model which we state in the following proposition.

Proposition 5.3.10. *Let $Z : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{\Theta} : [0,T] \times \Omega \to \mathcal{R}^n$ be a pair of measurable, adapted processes that solve the backward stochastic differential equation:*

$$\begin{aligned}dZ(t) &= -\left(2\pi' - \frac{1}{2\kappa}Z(t)^2\right)dt + \mathbf{\Theta}(t)\mathbf{dW}(t), \\ Z(T) &= 0.\end{aligned} \tag{5.3.15}$$

Further, let $\varphi : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{\Lambda}[0,T] \times \Omega \to \mathcal{R}^n$ be a pair of measurable, adapted processes that solve the following backward stochastic differential equation:

$$\begin{aligned}&d\varphi(t) + Z(t)(-\frac{\varphi(t)}{\kappa} + C_\iota - D_\iota - \alpha_\iota\gamma_\iota\cos\gamma_\iota t - \int_0^t \sigma_\iota(z,t)\mathbf{dW}(z))dt \\ &-\mathbf{\Theta}(t)\sigma_\iota(t,t))dt + \mathbf{\Lambda}(t)\mathbf{dW}(t) = 0, \\ &\varphi(T) = 0,\end{aligned} \tag{5.3.16}$$

(where $\sigma(\cdot,t)$ is given by (5.3.7)) such that $\varphi \in \mathcal{L}^2(0,T;\mathcal{R})$. Then the square-integrable and adapted optimal control $\bar{y}_\iota(\cdot)$ of the Integrated model (5.3.13) is

given by:

$$\bar{y}_\iota(t) = C_\iota - \frac{1}{\kappa}\left(Z(t)\bar{I}_\iota(t) + \varphi(t)\right), t \in [0,T], \tag{5.3.17}$$

and the optimal adjoint processes are given by:

$$\begin{aligned} p_{1,\iota}(t) &= -Z(t)\bar{I}_\iota(t) - \varphi(t), \\ \mathbf{q}_{1,\iota}(t) &= Z(t)\sigma(t,t) - \mathbf{\Lambda}(t). \end{aligned}$$

Further, the optimal control is unique.

Proof. The Integrated model is a special case of the Linear-Regulator model solved by Cadenillas and Karatzas [9], Section 3.6.1. The uniqueness of the optimal control follows from Theorem 1.6, Cadenillas and Karatzas [9]. □

The vectors $\mathbf{\Theta}(\cdot)$ and $\mathbf{\Lambda}(\cdot)$ are not required to satisfy any additional differential equations. However, their presence is necessary to ensure measurability of $Z(\cdot)$ and $\varphi(\cdot)$ with respect to $\mathcal{F}_t$.

We combine the results of the last proposition and Corollary 5.3.9 to obtain the equilibrium solution to the Market model in the following proposition.

Proposition 5.3.11. *The unique solution to (5.3.12) is given by:*

$$\begin{aligned} \bar{y}(t) &= C - \frac{1}{\kappa}\left(Z(t)\bar{I}(t) + \varphi(t)\right), \qquad (5.3.18) \\ \frac{S}{B}p_1(t) &= -Z(t)\bar{I}(t) - \varphi(t), \\ \frac{S}{B}\mathbf{q}_1(t) &= -Z(t)\sum_{l=1}^{m}\xi_l - \mathbf{\Lambda}(t). \end{aligned}$$

The expression for the optimal rate of production provides some insights regarding the behavior of the model. The rate of production at time t depends on the capacity as well as the net inventory at time t. A marginal exogenous change in the capacity is transmitted directly to the optimal rate of production. However, a marginal change in net inventory produces less effect on the rate of production as κ increases relative to π since $Z(t)/2\kappa$ (see the following proposition) decreases as κ increases for any fixed t. In other words, net inventory becomes a relatively less important factor as the cost of capacity-production mismatch increases with respect to holding/shortage cost.

Proposition 5.3.12. *i) The solution to the stochastic differential equation (5.3.15) is given by:*

$$\begin{aligned} Z(t) &= 2\sqrt{\pi'\kappa}\left(\frac{1-e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1+e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right), \\ \mathbf{\Theta}(t) &= 0 \end{aligned} \tag{5.3.19}$$

ii) The solution to the stochastic differential equation (5.3.16) is given by:

$$\varphi(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) \tag{5.3.20}$$

where

$$\begin{aligned} \varphi_1(t) &= \frac{2(C-D)\kappa\left(\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))-1\right)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}, \\ \varphi_2(t) &= \frac{2\alpha\gamma\kappa}{\left(1+\frac{\kappa}{\pi}\gamma^2\right)\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}\left(\cos\gamma T - \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t\right) \\ &- \frac{2\alpha\gamma^2\kappa^{3/2}}{\left(\sqrt{\pi}\left(1+\frac{\kappa}{\pi}\gamma^2\right)\right)\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t, \\ \varphi_3(t) &= B\sum_{i=1}^{n}\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}(A_i(u,T)-A_i(u,t))dW_i(u), \end{aligned}$$

where

$$\begin{aligned} A_i(u,t) &= \sum_{l=1}^{m}\xi_{il}\left(\frac{e^{\lambda_l(u-t)+\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l}\right) \\ &+ \sum_{l=1}^{m}\xi_{il}\left(\frac{e^{\lambda_l(u-t)-\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right) \end{aligned}$$

and, using the definition of $A_i(\cdot,\cdot)$*,*

$$\Lambda_i(t) = \frac{B\sqrt{\pi'\kappa}A_i(t,T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}$$

where Λ_i *is the* i^{th} *component of* $\mathbf{\Lambda}$*.*

Proof. See appendix. □

The optimal solution in the feedback form stated in the Proposition 5.3.11 can be used to obtain optimal cumulative production $\bar{Y}$: multiply the equation for the optimal rate of production in the Market model (5.3.18) by dt on both sides and write it as

$$d\bar{Y}(t) = (C - \frac{1}{2\kappa}\big(Z(t)(\bar{Y}(t) - F(t,t) - Y_0 + X_0) + \varphi(t)\big))dt,$$

by substitution of $\bar{I}(t)$ by $\bar{Y}(t) - F(t,t) - Y_0 + X_0$ on RHS and of $\bar{y}(t)dt$ by $d\bar{Y}(t)$ on LHS. This stochastic differential equation can be solved using an approach similar to the *Variation of Constants* method as demonstrated in Proposition 4.2.8. We state the optimal market production and equilibrium market price in the following corollary.

Corollary 5.3.13. *The optimal market cumulative production trajectory* $\bar{Y}(t)$ *at time* $t \in [0,T]$ *is given by:*

$$\begin{aligned}\bar{Y}(t) &= \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\int_0^t \left(\frac{\frac{Z(u)}{2\kappa}(F(u,u)+Y_0-X_0)+C-\frac{\varphi(u)}{2\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(u-T))}\right)du \\ &+ Y_0\frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}, \qquad (5.3.21)\end{aligned}$$

where

$$F(u,u) = Du + \alpha\gamma\cos\gamma u + B\sum_{i=1}^{n}\sum_{l=1}^{m}\int_0^u \frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)dW_i(s)$$

and $\varphi(\cdot)$ is the solution to the differential equation given by (5.3.20). Trivially, $\bar{I}(t) = \bar{Y}(t) - F(t,t) - Y_0 + X_0$. Further, the equilibrium market price is given by:

$$P(t) = -\frac{Z(t)\bar{I}(t) + \varphi(t)}{S}.$$

Proof. The equilibrium market price is obtained by combining (5.3.10) and (5.3.18). □

In the following corollary, we show that the equilibrium price function P can indeed be written in the form as hypothesized in (5.3.3). We use closed form expressions for $\bar{I}(\cdot)$ and $\varphi(\cdot)$ from Proposition 5.3.12 and Corollary 5.3.13 to obtain this result.

Corollary 5.3.14. *The expression for equilibrium price process in closed form is given by:*

$$P(t) = a(t) + \int_0^t \mathbf{b}(s,t)\mathbf{dW}(s)$$

where

$$\begin{aligned} a(t) &= -\frac{1}{S}\left(\varphi_1(t) + \varphi_2(t) + Z(t)\left(Y_0 \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(T)} - Dt - \alpha\sin\gamma t\right)\right) \\ &- \frac{Z(t)}{S}\int_0^t \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(u-T))}\left(\frac{Z(u)}{2\kappa}(Du + \alpha\sin\gamma u - X_0 + Y_0) + C\right)du \\ &+ \frac{Z(t)}{S}\int_0^t \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(u-T))}\left(\frac{\varphi_1(u) + \varphi_2(u)}{2\kappa}\right)du \end{aligned}$$

where φ_1 and φ_2 are defined in Proposition 5.3.12. With $A_i(\cdot,\cdot)$ as defined in

Proposition 5.3.12, the i^{th} component, $b_i(s,t)$, of function $\mathbf{b}(s,t)$ is equal to:

$$
\begin{aligned}
b_i(s,t) &= \frac{BZ(t)}{S}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l) \\
&- \frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{Z(u)}{2\kappa}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)\right)du \\
&+ \frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{\sqrt{\pi'}(A_i(s,T)-A_i(s,u))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\right)du \\
&+ \frac{B\sqrt{\pi'\kappa}}{S}\left(\frac{(A_i(s,T)-A_i(s,t))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}\right).
\end{aligned}
$$

In the next section, we use the closed form optimal solution of the optimal production and equilibrium market price to show that they evolve as martingales.

5.4 Evolution of Market Price as a Martingale

In a way similar to the end-consumer demand forecasts, the forecasting processes of optimal production, optimal net inventory, and equilibrium market price can also be defined. Denote the forecast at s of cumulative production at t by $Y(s,t)$. Then $Y(s,t)$ can be defined as the expectation of $Y(t)$ conditional on all the information in $\mathcal{F}_s$. By definition, $Y(\cdot,t)$ is a Martingale. Define:

$$
\begin{aligned}
\varphi_3(s,t) &= E(\varphi_3(t)|\mathcal{F}_s) \\
&= B\sum_{i=1}^{n}\int_0^s\sum_{l=1}^{m}\xi_{il}\sqrt{\pi'\kappa}\left(\frac{e^{\lambda_l(u-T)}-e^{\sqrt{\frac{\pi'}{\kappa}}(t-T)+\lambda_l(u-t)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l}\right)dW_i(u) \\
&+ B\sum_{i=1}^{n}\int_0^s\sum_{l=1}^{m}\xi_{il}\sqrt{\pi'\kappa}\left(\frac{e^{\lambda_l(u-T)}-e^{-\sqrt{\frac{\pi'}{\kappa}}(t-T)+\lambda_l(u-t)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right)dW_i(u).
\end{aligned}
$$

Then, it can be easily seen that:

$$
\begin{aligned}
Y(s,t) &= Y_0 \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)} \\
&+ \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)) \int_0^t \left(\frac{\frac{Z(z)}{2\kappa}(F(s,z)+Y_0-X_0)+C-\frac{\varphi(s,z)}{\kappa}}{\cosh(\sqrt{\frac{\pi'}{2\kappa}}(z-T))} \right) dz.
\end{aligned}
$$

Similarly, define the forecast at s of net inventory and price at t by $I(s,t)$ and $P(s,t)$, respectively. Then:

$$
\begin{aligned}
I(s,t) &= E(I(t)|\mathcal{F}_s) \\
&= Y(s,t) - F(s,t) - Y_0 + X_0.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
P(s,t) &= E(P(t)|\mathcal{F}_s) = -\frac{Z(t)\bar{I}(s,t)+\varphi_1(t)+\varphi_2(t)+\varphi_3(s,t)}{S} \\
&= a(t) + \int_0^s \mathbf{b}(u,t)\mathbf{dW}(u). \qquad (5.4.1)
\end{aligned}
$$

The forecast update at s, $d_sP(s,t)$, of the market price at t is given by:

$$
d_sP(s,t) = -\frac{Z(t)d_s\bar{I}(s,t) + d_s\varphi_3(s,t)}{S} = \mathbf{b}(s,t)\mathbf{dW}(s).
$$

where $\mathbf{b}(s,t)$ is as defined in Corollary 5.3.14.

The expression for the price forecast in a market model for capacity (5.4.1) is the major contribution of this chapter. To gain some insight into (5.4.1), we explore the impact of various parameters on the quantity $d_sP(s,t)$ for a special case, $m=1, n=1$.

5.4.1 Resolution of Price Uncertainty

In this subsection, we numerically analyze the impact of the rate of resolution of the demand uncertainty and the supply chain cost parameters on the rate of resolution of price uncertainty. We consider the special case in which $m = 1$ and $n = 1$.

Recall that the forecast update at s for the rate of demand at $t, d_s f(s,t)$ is given by:

$$d_s f(s,t) = \left(\xi_{11} e^{\lambda_1(s-t)}\right) dW_1(s).$$

The parameter λ_1 in the last equation defines the curvature of the forecast update curve of the rate of demand at t and can be interpreted as the *inverse rate* of the resolution of the uncertainty of the instantaneous rate of demand at t. A high value of λ_1 implies that most of the uncertainty in the rate of demand is resolved just before t. On the other hand, a low value of λ_1 implies that the resolution of the uncertainty in the rate of demand occurs more uniformly in the time before t.

We refer to the diffusion coefficient (that is, the coefficient of $dW_1(\cdot)$) in the expressions for the forecast updates of a variable as the forecast update coefficient for that variable. For example, the forecast update coefficient of the instantaneous rate of demand at t is equal to $\xi_{11} e^{\lambda_1(s-t)}$ at s. The forecast update at s, of the market price at t, is given by:

$$d_s P(s,t) = -\frac{Z(t) d_s \bar{I}(s,t) + d_s \varphi_3(s,t)}{S} = b_1(s,t) dW_1(s).$$

where $b_1(s,t)$, the price forecast update coefficient, is as defined in Corollary 5.3.14. Observe that the mean of $d_s P(s,t)$ is 0 and its variance is equal to $b(s,t)^2 ds$. Therefore, the variance of cumulative forecast update until time s is equal to $\int_0^s b(u,t)^2 du$. Define the fraction of variability resolved by time s for price at time t, $FVR(s,t)$, as:

$$FVR(s,t) = \frac{\int_0^s b(u,t)^2 du}{\int_0^t b(u,t)^2 du}. \tag{5.4.2}$$

We use the following data for the numerical experiments:

Table 5.1: Data for Numerical Example

t	T	S/B	ξ_{11}
15	30	1	1

In Figures 5.1 and 5.2, we plot the forecast update coefficient of the market price over time $[0, t)$ and the fraction of variability resolved in the realization of price for different values of λ_1 and a fixed value of $\frac{\kappa}{\pi'}$ equal to 10. Figures 5.1 and 5.2 show

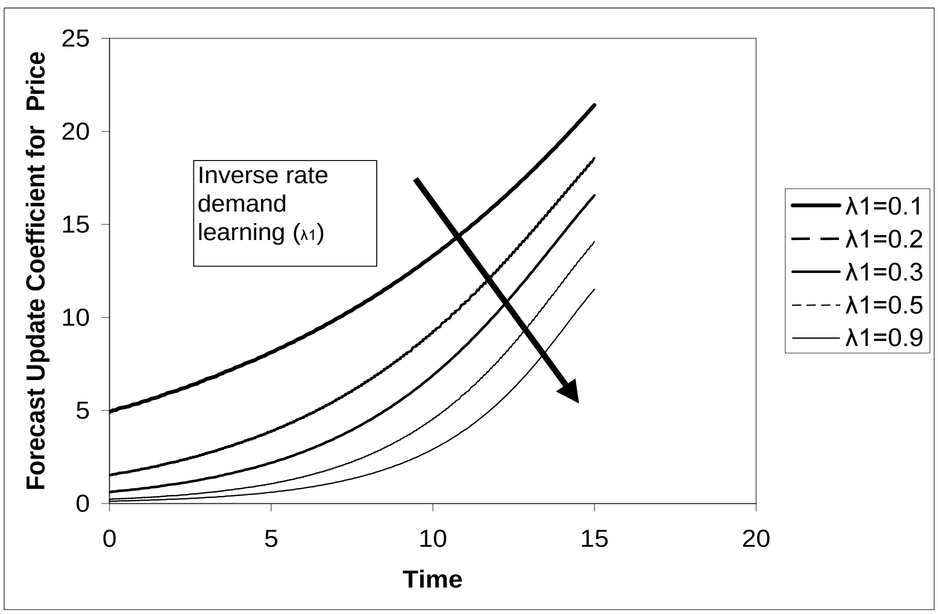

Figure 5.1: Evolution of Forecast Update Coefficient for Price

that as the value of λ_1 increases, the resolution of price uncertainty is delayed and occurs increasingly just before t. In addition, the cumulative resolution of variability at any instant, defined by the numerator of the RHS of (5.4.2), reduces

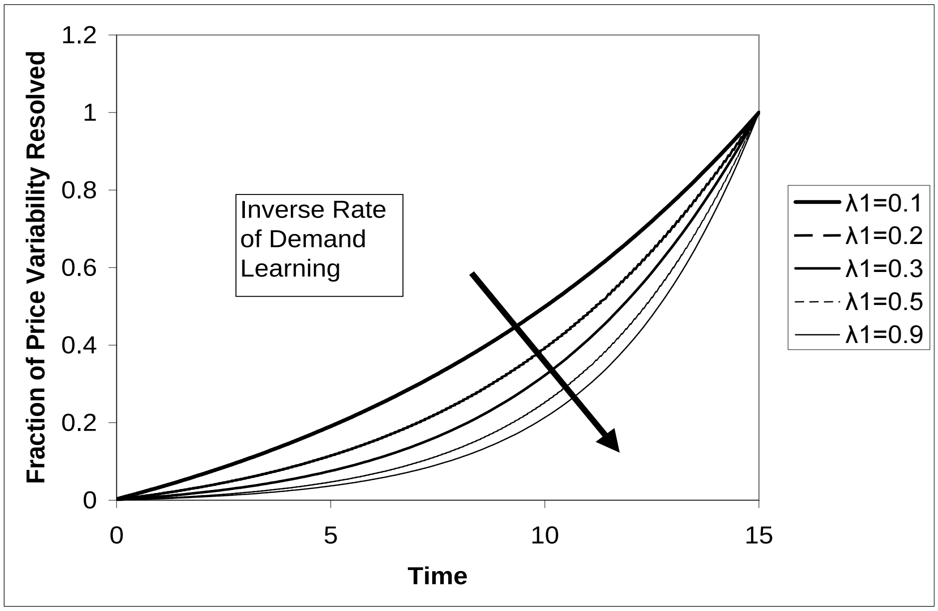

Figure 5.2: Fraction of Forecast Variability Resolved for Price

as λ_1 increases. This behavior is consistent with the resolution of the demand uncertainty as a function of λ_1. That is, the rate of resolving price uncertainty follows the same pattern as the rate of resolving demand uncertainty.

In Figure 5.3, we plot the forecast update coefficient for the market price over time in the realization of price for different values of $\frac{\kappa}{\pi'}$ and for a fixed value λ_1 equal to 1. Contrary to our own intuition, we find that the supply chain cost parameters do affect the resolution of the price uncertainty. Figure 5.3 shows that as $\frac{\kappa}{\pi'}$ (that is, the relative cost of changing production with respect to holding inventory) increases, the resolution of price uncertainty occurs more uniformly in time. On the other hand, for low values of $\frac{\kappa}{\pi'}$, a greater portion of the resolution of price uncertainty occurs over a shorter duration of time before t.

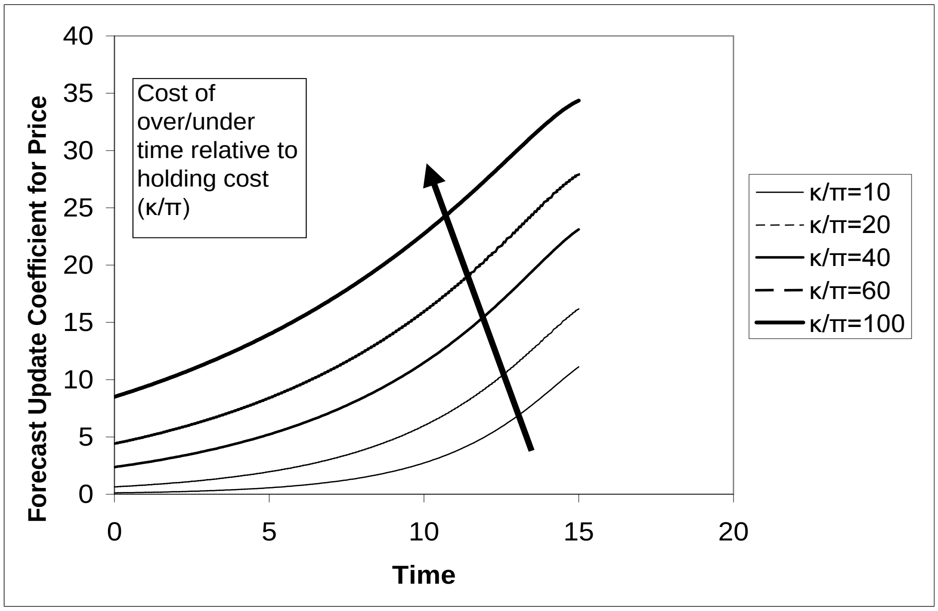

Figure 5.3: Evolution of Price Forecast Update Coefficient for Different Cost Parameters

The total variability resolved over $[0, t)$ (i.e. the numerator of RHS in (5.4.2)) may also be construed as a measure of variance of price at time t. Indeed, a price process with highly volatile forecasting process is likely be highly volatile in nature. With this understanding, we observe in Figure 5.3 that the forecast update coefficient for price at t increases with $\frac{\kappa}{\pi'}$ for any fixed instant $s < t$, implying that the variance of price increases with $\frac{\kappa}{\pi'}$. It follows that the market for supply chain capacity of a capital intensive firm with high $\frac{\kappa}{\pi'}$ ratio is likely to be highly volatile. On the other hand, the price volatility is likely to be low in a capacity market for which the relative cost of production-capacity mismatch is low compared to the holding cost.

This finding, that the cost structure of the supply chain has a direct impact on the ability of participants to forecast the price of capacity, is a new result in supply chain analysis.

Observations from Figure 5.3 provide a glimpse of how the market for capacity as a system handles the exogenous uncertainty which is realized through the end-consumer demand. The market has two variables at its disposal to tackle the exogenous uncertainty: rate of production and inventory. In a perfect market, either the sellers may update the rate of production keeping pace with the uncertainty as it unfolds itself or, buyers may hold (incur) excessive inventory (backorders). A combination of the two is also possible. If the market handles uncertainty through the inventory/backorders then the buyers must be given incentives in the form of the price discounts/premiums which increases the variability in price. The relative values of the cost parameters κ and π determine the share of each variable in handling uncertainty.

Indeed, observations from Figure 5.3 provide evidence for the above surmise. When the cost of overtime/undertime κ is relatively high compared to the holding cost π, varying the rate of production is expensive and the market handles the exogenous uncertainty through inventory/backorders. This results in increased price variability. Using the same assertion, the involvement of the rate of production increases as the cost parameter κ comes relatively closer to π.

5.5 Conclusion

In this chapter, we have generalized the existing discrete time MMFE model to a continuous-time model. We also considered a market for production capacity in which the buyers of production capacity take into account the forecast updates in the future while making production decisions. We obtained expressions for forecasts of the market price and the optimal production rate and analyzed the impact of the rate of resolving demand uncertainty and of the supply chain cost parameters on the rate of resolving price uncertainty. We found, surprisingly, that uncertainty in the price of capacity is resolved later in a market characterized by low production capacity mismatch costs, relative to holding/backorder costs.

5.6 Appendix

5.6.1 Proof of Proposition 5.2.5

Our goal is to show that the four conditions specified in the proof of Lemma 4.1 in Ikeda and Watanabe [43] are satisfied when the integrand has the form stated in Proposition 5.2.5. Let $\Psi(s,t,\omega) = Q_1(s,t)1_{[s,u]}(t)$. As a result, we can write the LHS of (5.2.1), equivalently, as:

$$\int_0^u \int_0^t Q_1(s,t)dW(s)dt = \int_{\mathcal{R}^1} \int_0^t Q_1(s,t)1_{[s,u]}(t)dW(s)dt$$

Note that we have specified ω as an argument to Ψ even though the integrand is independent of ω. This is done so that we can apply the four conditions specified in Lemma 4.1 in Ikeda and Watanabe [43] directly.

Condition 1: $((s,\omega),t) \in ([0,\infty),\Omega) \times \mathcal{R}^1 \to \Psi(s,t,\omega)$ *is* $\mathcal{S} \times \mathcal{B}(\mathcal{R}^1)$ *- measurable where* $\mathcal{S}$ *is the smallest* $\sigma-$*field on* $[0,\infty) \times \Omega$ *s.t. all left continuous* $\mathcal{F}_s-$*adapted processes* $Z : [0,\infty) \times \Omega \to Z_s(\omega)$ *are measurable.*

Consider the following functions:

$$\begin{aligned} Z_n(s,t) &= Z(s,\frac{i}{n}u), 0 < s \le t, 0 \le \frac{i-1}{n}u < t \le \frac{i}{n}u, i \in \{1,..,n\} \\ Z_n(0,0) &= Z(0,0) \\ &= 0; s > t, t \in (u,\infty) \cup (-\infty,0], \end{aligned}$$

where

$$Z(s,\frac{i}{n}u) = Q_1(s,\frac{i}{n}u).$$

Clearly, $Z_n(s,\cdot)$ is left continuous for each fixed s. Consider any set $A \in \mathcal{B}(\mathcal{R}^1)$. For any $t \in B_i = (\frac{i}{n}u, \frac{i+1}{n}u]$ for any $i < n$, $Z_n^{-1}(\cdot,t)(A) =: C_{s,i} \in \mathcal{S}$. By definition of product spaces, $(C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Therefore $Z_n^{-1}(\cdot,\cdot)(A) = \cup_i (C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$ proving the measurability of $Z_n(\cdot,\cdot)$ with respect to $\mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Now, due to the left continuity of $Z_n(s,t)$ in t, $Z_n(s,t) \to Z(s,t)$, we have that $Z(s,t) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$.

Condition 2: *There exists a non-negative Borel-measurable function $f(t)$ such that*

$$|\Psi(s,t,\omega)| \le f(t)$$

for every s,t,ω. This follows immediately from the continuity of $Q_1(\cdot,\cdot)$.

Condition 3: $(t,\omega) \to \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega)$ *is $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}-$measurable for each $t_1 \ge 0$.*

To see this,

$$\begin{aligned} \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega) &= \int_0^{t_1} Q_1(s,t)1_{(0,u]}(t)1_{(0,t]}(s)dW(s,\omega) \\ &= 1_{(0,u]}(t)\int_0^{t_1\wedge t} Q_1(s,t)dW(s,\omega) \\ &= 1_{(0,u]}(t)W(H(t,t_1\wedge t),\omega) \end{aligned}$$

where

$$H(t,\cdot) = \int_0^{\cdot} Q_1^2(s,t)ds.$$

The last step follows from "time substitution" (McKean [55]). For any ω, define:

$$\begin{aligned} Z_n(t,\omega) &= W(H(\frac{p}{n}u, t_1 \wedge \frac{p}{n}u), \omega), p \in \{1,.,n\}, t \in (\frac{p-1}{n}u, \frac{p}{n}u] \\ &= 0, \text{ otherwise.} \end{aligned}$$

For each n, and for any $\omega, t \to Z_n(t,\omega)$ is left continuous. Further, for any t, $Z_n(t,\omega)$ is measurable with respect to $\mathcal{F}_{H(t_1 \wedge \frac{p}{n}u)} \subset \mathcal{F}$. In a way similar to that used to establish Condition 1, it can be shown that

$$Z_n(t,\omega) \in \mathcal{B}(\mathcal{R}^1) \times \mathcal{F}.$$

Due to the left continuity of $Z_n(t,\omega)$ in t,

$$\lim_{n\to\infty} Z_n(t,\omega) = Z(t,\omega) = B(H(t, t_1 \wedge t), \omega)1_{(0,u]}(t),$$

which will also be measurable with respect to $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}$.

Condition 4: $\int_{\mathcal{R}^1} f(t)dt < \infty$.

Taking $f(t) := 1_{[s,u](t)} \sup_{0<s<t\leq u} |Q_1(s,t)|$, we have

$$\int_{\mathcal{R}^1} f(t)dt \leq u(\sup_{0<s<t\leq u} |Q_1(s,t)|) < \infty.$$

5.6.2 Proof of Proposition 5.3.12

The differential equation involving $Z(\cdot)$ does not have any stochastic terms other than $\mathbf{\Lambda}(t)\mathbf{dW}(t)$. Setting $\mathbf{\Lambda}(\cdot)$ to zero results in a standard one-dimensional Riccatti differential equation which is easily solvable using standard techniques.

We follow closely the approach outlined in Yong and Zhou [79] for the solution of the following backward stochastic differential equation. We drop the subscripts from $C_\iota, D_\iota, \alpha_\iota,$ and γ_ι for clarity. Also, we substitute $\sigma_\iota(\cdot,\cdot)$ by $B\sigma(\cdot,\cdot)$.

$$\begin{aligned} d\varphi(t) &+ Z(t)(-\frac{\varphi(t)}{2\kappa} + C - D - \alpha\gamma\cos\gamma t - B\int_0^t \sigma(u,t)\mathbf{dW}(u))dt \\ &+ \mathbf{\Lambda}(t)\mathbf{dW}(t) = 0 \\ & \varphi(T) = 0 \end{aligned}$$

where

$$Z(t) = 2\sqrt{\pi'\kappa}\left(\frac{1 - e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1 + e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right).$$

Consider the following stochastic differential equation:

$$\begin{aligned} d\chi(t) + \left(\frac{1}{2\kappa}Z(t)\chi(t)\right)dt &= 0, \\ \chi(0) &= 1. \end{aligned}$$

The solution to the differential equation for $\chi(\cdot)$ is given by:

$$\begin{aligned} c_1 e^{\int -\sqrt{\frac{\pi'}{\kappa}}\left(\frac{1-e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1+e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right)dt} &= c_1 e^{\int \sqrt{\frac{\pi'}{\kappa}}\tanh(\sqrt{\frac{\pi'}{\kappa}}(t-T))dt} \\ &= c_1\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)) \end{aligned}$$

where, using the boundary condition, c_1 is determined to be:

$$c_1 = \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}.$$

Applying Ito's formula to $\chi(t)\varphi(t)$:

$$\begin{aligned} d[\chi(t)\varphi(t)] &= -\chi(t)Z(t)(C - D - \alpha\gamma\cos\gamma t - B\int_0^t \sigma(u,t)\mathbf{dW}(u))dt \\ &- \chi(t)\mathbf{\Lambda}(t)\mathbf{dW}(t). \end{aligned}$$

Integrating both sides:

$$\begin{aligned}
\chi(t)\varphi(t) - \chi(T)\varphi(T) &= \int_t^T \chi(s)Z(s)\left(C - D - \alpha\gamma\cos\gamma s - B\sigma(u,s)\mathbf{dW}(u)\right)ds \\
&+ \int_t^T \chi(s)\mathbf{\Lambda}(s)\mathbf{dW}(s) \\
\chi(t)\varphi(t) &= \theta + B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s)ds + \int_t^T \chi(s)\mathbf{\Lambda}(s)\mathbf{dW}(s)
\end{aligned}$$

where

$$\theta = -B\int_0^T \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$:

$$\begin{aligned}
E(\chi(t)\varphi(t)|\mathcal{F}_t) &= E(\theta|\mathcal{F}_t) + B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s)ds \\
\chi(t)\varphi(t) &= -E\left(\left(B\int_0^T \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds\right)|\mathcal{F}_t\right) \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s) \\
&+ B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds.
\end{aligned}$$

Now, simplifying the RHS:

$$\begin{aligned}
\varphi_1'(t) &= \int_t^T \chi(s)Z(s)(C - D)ds = -\int_t^T \frac{2\sqrt{\pi'\kappa}\sinh(\sqrt{\frac{\pi'}{\kappa}}(s - T))\,(C - D)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}ds \\
&= \frac{2(C - D)\kappa}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\cosh(\sqrt{\frac{\pi'}{\kappa}}(t - T)) - 1\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi_2'(t) &= -\int_t^T \chi(s)Z(s)\alpha\gamma\cos\gamma s ds \\
&= \int_t^T \frac{2\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\sinh(\sqrt{\frac{\pi'}{\kappa}}(s-T))\right)\alpha\gamma\cos\gamma s ds \\
&= \frac{2\alpha\gamma\kappa}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\cos\gamma T - \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t\right) \\
&- \frac{2\alpha\gamma^2\kappa^{3/2}}{\sqrt{\pi'}\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t\right) - \frac{\kappa}{\pi}\gamma^2\varphi_2'(t) \\
\varphi_2'(t) &= \frac{2\alpha\gamma}{1+\frac{\kappa}{\pi}\gamma^2}\left(\frac{\kappa\cos\gamma T - \kappa\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\right) \\
&- \frac{2\alpha\gamma^2}{\left(1+\frac{\kappa}{\pi}\gamma^2\right)\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\frac{\kappa^{3/2}}{\pi^{1/2}}\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t\right)
\end{aligned}$$

and finally:

$$\begin{aligned}
\varphi_3''(t) &= -B\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&= -B\int_0^t\int_u^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\sigma(u,s)ds\mathbf{dW}(u)
\end{aligned}$$

using Proposition 5.2.5. Assuming σ is approximated by (5.3.7),

$$\varphi_3''(t) = -B\sum_{i=1}^n\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}A_i(u,t)dW_i(u)$$

where

$$\begin{aligned}
A_i(u,t) = A_i(u,t) &= \sum_{l=1}^m \xi_{il}\left(\frac{e^{\lambda_l(u-t)+\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l}\right) \\
&+ \sum_{l=1}^m \xi_{il}\left(\frac{e^{\lambda_l(u-t)-\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right). \qquad (5.6.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\theta &= B\int_0^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&= B\int_0^T\int_u^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\sigma(u,s)ds\mathbf{dW}(u) \\
&= B\sum_{i=1}^n \int_0^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}A_i(u,T)dW_i(u),
\end{aligned}$$

where $A_i(u,\cdot)$ is defined in (5.6.1).

Now, for $t < T$,

$$E(\theta|\mathcal{F}_t) = B\sum_{i=1}^n \int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}A_i(u,T)dW_i(u).$$

Therefore,

$$\begin{aligned}
\varphi_3''(t) \quad &+ \quad E(\theta|\mathcal{F}_t) \\
&= B\sum_{i=1}^n \int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}(A_i(u,T) - A_i(u,t))dW_i(u) \\
&=: \varphi_3'(t)
\end{aligned}$$

which is measurable with respect to $\mathcal{F}_t$. Therefore,

$$\varphi(t) = \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)}(\varphi_1'(t) + \varphi_2'(t) + \varphi_3'(t)).$$

Using equation (2.25), pp 352, Yong and Zhou [79]:

$$\Lambda_i(t) = \frac{B\sqrt{\pi'\kappa}A_i(t,T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}.$$

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