Economic Delivery Quantities
For Capacitated Multi-Stage
Production Systems

by

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I. INTRODUCTION

Procurement lot sizes for externally purchased components used in final assemblies are often computed using simple rules. When the production rate for final assemblies remains relatively constant over time, some variation of the Wilson lot size formula is often used to determine component procurement lot sizes. If the demand rates for final products change over time, then some other technique might be used, such as the part period balancing method, the least unit cost approach, or the Silver-Meal heuristic. See reference 4. All of these methods compute component lot sizes so that inventory carrying costs are compared to fixed ordering costs in such a way that total component procurement costs are approximately minimized. By focusing primarily on this criterion, a number of issues of practical significance are often ignored.

We have observed in several instances that the amount ordered for a component may be delivered in several lots. For example, a contract may specify that 6000 units will be delivered during a year with 1000 units being shipped every two months. The purchasing analyst often takes the demand projection for a component for a period of time, such as a year, and solicits bids from suppliers. The supplier is selected based on timeliness of deliveries, quality of the components, and unit cost. These component order quantities are normally fixed based on demand projections for final assemblies as stated in a master production schedule; however, the delivery quantities may be smaller than the order quantities. If the production rate for the final product is approximately constant, then the order quantity is often an integer multiple of the delivery quantity. Delivery quantities, and the time between the receipt of such quantities, called the delivery cycle, are set based on other considerations such as receiving dock capacity, warehouse space, storage space on the shop floor, storage and delivery container
capacities, material handling capacities between the receiving dock and warehouse and between the warehouse and shop floor, and inventory carrying costs. The capacities we have mentioned can be related to both equipment and manpower. In addition, the selection of an optimal delivery quantity should take into account the lot splitting possibilities that can take place after a delivery is made (e.g. the quantity of a component sent from a warehouse to the shop floor can be less than that component's delivery quantity). We call the optimal delivery quantity the economic delivery quantity.

To make these ideas more clear, we will discuss the operation of a particular system. This system, which is similar to one found in a large manufacturing and assembly system that we have closely examined, is the basis for our study. The model that we will subsequently develop is founded on our observations concerning this system's behavior.

The physical system consists of a set of receiving docks, warehouses, an automatic guided vehicle system (AGVS) used to make deliveries of components from the warehouse to staging areas on the shop floor, and storage areas at line stations on the shop floor. The flow of components occurs as depicted in Figure 1. There are several systems in the plant we studied of the type we shall describe; however, they operate independently, so that we can analyze them separately.

Shipments of components arrive at the receiving dock where they are unloaded from trucks, verified according to the purchase order number, inspected for quality and count, and sent by fork truck to a warehouse. The puts and picks in some warehouses in the plant are accomplished via forklift truck while in others are done using an AS/RS system. (The model we develop can be used in either case by appropriately setting the values of the parameters.) Once components are removed from the storage racks, they are taken to a marshalling area in the warehouse. The quantities removed
Component Flow In The Assembly Plant

Figure 1.
from storage racks at any time need not be as large as a delivery quantity. Thus lot-splitting can occur. After containers of possibly several types of components are accumulated in the marshalling area that are to be sent to staging areas which are located relatively closely together, they are placed on a train. The train, which is part of an AGVS, takes the components to the proper staging area where they are unloaded and placed into temporary storage. A container of a single type of component is then taken to a work station on the production line. As was the case in the warehouse, the quantity delivered to the line station could be smaller than the lot delivered to the staging area. Thus another opportunity for lot-splitting exists. The components at a line station are used ultimately in the production of final assemblies.

The system we have described has capacity constraints in each phase of the system's operation. The total number of shipments of components that can be processed at the receiving dock is limited during any period of time. The number of puts into and picks from the warehouse's storage racks are also constrained as are the total number of separate component shipments from the marshalling area. We also have storage capacity limitations at the line station. Storage capacity limits also exist at a staging area and in the warehouse. In the model we will present all of these constraints are considered explicitly or implicitly.

Obviously not all plants fit the above description precisely. However, many of the assembly plants we have seen do operate in a relatively similar manner. Our objective is to indicate how a model of this environment can be developed that can be used to establish economic delivery quantities to each storage area, recognizing capacity limitations. It is our belief that the proposed model can be easily modified to accommodate many other situations
without significantly increasing either its complexity or the difficulty in finding an optimal or near optimal solution.

In the next section we present the assumptions and goals in detail. We then discuss our models and a basic approach for finding a solution. In Section III we present our algorithm for finding a solution to the problem. The algorithm is a heuristic approach based on the concept of marginal analysis. We conclude with a summary and some comments concerning future research efforts.
II. MODELING THE ECONOMIC DELIVERY QUANTITY PROBLEM

In this section we will develop a pair of related models that we will use to find a solution to the economic delivery quantity problem that was described in Section I. The model we will present is a planning model. It is used to find delivery quantities for each type of component. However, detailed scheduling of the exact arrival and shipping times of lot sizes from stage to stage is not accomplished using the model. Consequently only aggregate activity is considered in each stage of the system. Constraints will be expressed in terms of available hours per year. This constraint can be stated such that only some percentage, say 90%, of total annual capacity is considered available to accommodate events such as machine breakdowns, absenteeism, and scheduling conflicts. No attempt is made to relate the solution to an actual day-to-day delivery plan. Furthermore, the constraints on warehouse and staging area space will not be expressed explicitly. Rather, it is assumed that storage costs are correctly charged so that these constraints are not violated. We note that it is impossible to model these capacity constraints precisely in an aggregate model since the capacity used at any point in time depends on the detailed scheduling rule that is followed.

A. ASSUMPTIONS

The models we will develop are based on a large number of assumptions, which we will now state and discuss.

1. The assembly rate for all final products is constant and continuous. Hence the demand rate for component \( p \) (\( p = 1, \ldots, P \)) is constant and
continuous at a value of $D_p$ units per period, where $P$ represents the number of components in the system.

2. Each component is used at only one line station. This assumption is critical to the computational procedure we will present. Modifications to the algorithm would be necessary if this assumption were relaxed.

3. Each delivery lot processed at the receiving dock, marshalling area, staging areas, and warehouse requires both a fixed amount of time and an amount of time proportional to the size of the lot. (Rather than using a surrogate fixed charge for each lot that is processed, we feel strongly that the appropriate way to model these fixed times is to include them directly in the constraints.) The following parameters are all measured in hours. Let

$$a^{DK} = \text{fixed time per lot at the receiving dock},$$

$$b^{DK}_p = \text{variable time per unit of component } p \text{ in the receiving dock},$$

$$a^{MA} = \text{fixed time per lot in the marshalling area},$$

$$b^{MA}_p = \text{variable time per unit of component } p \text{ in the marshalling area},$$

$$a^{TS} = \text{fixed time per lot at a train stop},$$

$$b^{TS}_p = \text{variable time per unit of component } p \text{ at a train stop},$$

$$a^{WH,PUT} = \text{fixed time per lot put into the warehouse},$$

$$b^{WH,PUT}_p = \text{variable time per unit of component } p \text{ put into the warehouse},$$

$$a^{WH,PICK} = \text{fixed time per lot picked from the warehouse},$$
\( b_{WH, PICK}^p \) = variable time per unit of component \( p \) picked from the warehouse,

\( a_{SA, PUT}^p \) = fixed time per lot put into a staging area,

\( b_{SA, PUT}^p \) = variable time per unit of component \( p \) put into a staging area,

\( a_{SA, PICK}^p \) = fixed time per lot picked from a stopping area,

\( b_{SA, PICK}^p \) = variable time per unit of component \( p \) picked from a staging area.

Note that a unit load may be defined differently from one stage to the next. For example, a unit load may be a pallet at the receiving dock whereas it may be a small container at a line station. Note also that we have chosen to make the fixed times to be independent of the component type. This assumption has been based on our experience. The parameters could, however, be made to reflect a dependence on \( p \).

4. There are capacities on

a. The available hours per year at the receiving dock \( (B^{DK}) \).

b. The number of hours available per year to process components loaded onto trains leaving the marshalling area \( (B^{MA}) \).

c. The number of hours available per year to move components from a train stop to a staging area \( (B^{TS}_t, \ t = 1, \ldots, T, \) where \( T \) is the number of train stops and staging areas). (If labor is shared among several train stops and staging areas, then the constraints must reflect aggregate usage over the appropriate set of areas. This aggregation will not affect the form of the constraints we will present.)
d. The number of hours per year available for put and pick operations at
   i. the warehouse ($B^{WH}$)
   ii. the staging area ($B^{SA}_t$, $t = 1, \ldots, T$).

e. The maximum number of components of each type that can be stored at a line station.

5. The amount received and sent to or from each location must be the equivalent of $1, 2, 4, 8, \ldots, 2^{K-1}, \ldots$ periods of final demand for a component, where $K$ is a positive integer. A period could be a shift, day, week, or any other length of time. The restriction we have made is not crucial to the development of the model. Any rational number of periods worth of demand could be shipped from stage to stage. We have chosen to present the model with this limitation since we feel it is possible to implement this type of solution relatively easily when scheduling of the exact arrival times of each component must be taken into account to balance short run work loads. Without this assumption we will have substantial difficulty developing a period-by-period schedule corresponding to the solution since it is not easy to combine individual solutions that do not have this property into a workable schedule.

B. OBJECTIVE FUNCTION

The goal is to determine the lot sizes such that the constraints we have discussed are satisfied while minimizing average annual costs of carrying inventory and using space in the warehouse and staging areas. As mentioned earlier, space costs will be charged for use in the warehouse and staging areas rather than placing constraints on these amounts.
C. DEVELOPMENT OF MODELS

We now present two equivalent models for the economic delivery quantity problem. These models differ only in the way in which the variables are defined. The reason for presenting two different but equivalent models is that they are both used in the algorithm we will present in the next section.

The first model we present is expressed in terms of variables that measure the number of periods between the delivery of a lot of components from one location to another. Specifically, let

\[ w^D_K_p = \text{the number of periods between the delivery of lots of component } p \text{ to (and from) the receiving dock} \]

\[ (w^D_K_p \in \{2^{K-1}: K = 1, 2, 3, \ldots\}), \]

\[ w^M_A_p = \text{the number of periods between delivery of component } p \text{ from the marshalling area to the appropriate staging area} \]

\[ (w^M_A_p \in \{2^{K-1}: K = 1, 2, 3, \ldots\}), \]

and

\[ w^T_S_p = \text{the number of periods between the delivery of component } p \text{ from the staging area to the appropriate line station} \]

\[ (w^T_S_p \in \{2^{K-1}: K = 1, 2, 3, \ldots\}) \]

be the decision variables. Also let,

\[ c_p = \text{purchase cost per unit of component } p. \]

\[ h = \text{holding cost factor common to all components.} \]

\[ d = \text{conversion factor measuring periods per year.} \]

\[ S^L_S_p = \text{storage capacity for component } p \text{ at a line station.} \]
As mentioned, the constraints we consider measure the maximum activity allowed per year in the receiving dock (available time per year for processing arriving shipments), the marshalling area (available time per year for processing shipments made from the marshalling area), the train stops (the available time for processing shipments to and from a train stop), the staging areas (the available number of hours per year for puts and picks), and the warehouse (the available number of hours per year for puts and picks). That we use a year as the length of the time horizon is arbitrary; any length could be used. Also, there is a constraint on the maximum amount of component \( p \) that can be shipped at any time to a line station from a staging area due to space limitations at the line station. Expressed mathematically these constraints are as follows.

**CONSTRAINTS FOR FORMULATION 1**

**RECEIVING DOCK**

\[
\begin{align*}
\sum_p \left( \frac{a_{DK}}{w_p} + b_p \cdot D_p \right) & \leq B_{DK} \\
\text{or} & \\
\sum_p \frac{a_{DK}}{w_p} & \leq \frac{B_{DK}}{d} - \sum_p b_p \cdot D_p & \triangleq a^{DK}
\end{align*}
\]

**MARSHALLING AREA**

\[
\begin{align*}
\sum_p \frac{a_{MA}}{w_p} & \leq b_{MA} \triangleq \frac{B_{MA}}{d} - \sum_p b_{MA} \cdot D_p & \triangleq b^{MA}
\end{align*}
\]
TRAIN STOPS (one for each of \( T \) stops)

\[
\sum_{p \in P} \frac{a_{TS}^{TS}}{w_p^{TS}} \leq B_{TS}^{T} \leq \frac{B_{TS}^{T}}{d} - \sum_{p \in P} b_{TS}^{T} \cdot d_{p}^{T},
\]  \((3)\)

where \( P_t \) represents the set of components delivered to train stop \( t \);

STAGING AREAS (one for each of \( T \) staging areas)

\[
\sum_{p \in P} \left( \frac{a_{SA,PUT}^{SA,PUT}}{w_p^{MA}} + \frac{a_{SA,PICK}^{SA,PICK}}{w_p^{TS}} \right) \leq B_{SA}^{T} \leq \frac{B_{SA}^{T}}{d} - \sum_{p \in P} D_{p} \left( b_{SA,PUT}^{SA,PUT} + b_{SA,PICK}^{SA,PICK} \right); \quad (4)
\]

WAREHOUSE

\[
\sum_{p} \left( \frac{a_{WH,PUT}^{WH,PUT}}{w_p^{DK}} + \frac{a_{WH,PICK}^{WH,PICK}}{w_p^{MA}} \right) \leq B_{WH}^{T} \leq \frac{B_{WH}^{T}}{d} - \sum_{p} D_{p} \left( b_{WH,PUT}^{WH,PUT} + b_{WH,PICK}^{WH,PICK} \right); \quad (5)
\]

LINE STATION STORAGE

\[
\frac{w_p^{TS}}{s_p^{LS}} \leq B_{LS}^{T},
\]  \((6)\)

and

\[
\frac{w_p^{DK}}{w_p^{MA}} \text{ and } \frac{w_p^{TS}}{w_p^{T}} \in \{2^{K-1} : K = 1, 2, \ldots\}. \quad (7)
\]

We note that if component \( p \) could go to more than one line station, then an additional subscript must be added to represent the line station to which shipments are being made. Similarly, if component \( p \) can go to
STRUCTURE OF CONSTRAINTS FOR EXAMPLE PROBLEM WITH THREE ITEMS AND TWO STAGING AREAS: FORMULATION 1.

Figure 2.
more than one staging area (or train stop) from the marshalling area, then additional variables must be defined to account for this fact. Correspondingly, the constraints must be altered to account for these changes. The number of variables grows considerably with these additions.

These constraints have an interesting structure, which is illustrated using an example in Figure 2. In the example we assume there are three items and two train stop/staging areas. Items 1 and 2 go to staging area 1 while item 3 goes to staging area 2. Notice the manner in which the variables are linked to each other. This structure is exploited in the algorithm we develop in the next section. Also, observe that the constraints are not all linear even if the integer restriction on the decision variables is relaxed.

Next, let us develop the objective function for this model. The objective function, as mentioned previously, measures the inventory carrying charges and warehouse and staging area space costs. Let's first determine how inventory carrying costs can be calculated.

First consider the carrying costs for units stored in the warehouse. Assume that \( \frac{w_p^{DK}}{w_p^{MA}} \geq w_p^{MA} \) for component \( p \). Then the graph of on-hand inventory has a shape similar to the one portrayed in Figure 3. In the example shown in Figure 3, \( \frac{w_p^{MA}}{w_p} = \frac{1}{4} \frac{w_p^{DK}}{w_p} \). Since we have restricted ourselves to the case where \( w_p^{DK}, w_p^{MA} \epsilon \{2^{K-1}; K = 1,2,\ldots\} \), \( \frac{w_p^{DK}}{w_p^{MA}} \) must be a positive integer.

Furthermore, we see in this case that

\[
\text{Ave On-Hand Inv} = \frac{1}{2} D_p \left( \frac{w_p^{DK}}{w_p^{MA}} - \frac{w_p^{MA}}{w_p} \right).
\]

In the special situation where \( w_p^{DK} = w_p^{MA} \), the component lot should arrive at the receiving dock at the time it is needed in the marshalling area.
Consequently, the average amount of inventory in the warehouse for component \( p \) is 0. We have assumed that all units enter and are processed in the warehouse before moving to the marshalling area. Adjustments to the constraints would have to be made to reflect other situations.

Lastly, we assume that \( w_p^{DK} < w_p^{MA} \). In this case, due to our assumptions, \( w_p^{MA}/w_p^{DK} \) is a positive integer. A typical graph of warehouse on hand inventory for this situation is given in Figure 4. The average on-hand warehouse inventory in this case is \( \frac{1}{2} D_p (w_p^{MA} - w_p^{DK}) \). Thus combining the above observations we see that the general expression for average on-hand warehouse inventory is

\[
\frac{1}{2} D_p \left| w_p^{DK} - w_p^{MA} \right|
\]

Consequently, the average holding cost for component \( p \) in the warehouse is

\[
\frac{1}{2} h c_p D_p \left| w_p^{MA} - w_p^{DK} \right|
\]

Using the same reasoning, it is easy to see that the average carrying cost for component \( p \) in the staging area is

\[
\frac{1}{2} h c_p D_p \left| w_p^{MA} - w_p^{TS} \right|
\]

Finally, the graph of on-hand inventory at a line station for component \( p \) follows the familiar saw tooth pattern, as depicted in Figure 5.

The general expression for average carrying cost for component \( p \) at a line station is thus

\[
\frac{1}{2} h c_p D_p w_p^{TS}
\]

Upon combining the expressions we have given for each location, we see that the average inventory holding cost can be expressed as

\[
\sum_p \frac{1}{2} h c_p D_p \left\{ \left[ w_p^{DK} - w_p^{MA} \right] + \left[ w_p^{MA} - w_p^{TS} \right] + w_p^{TS} \right\}
\]
ON-HAND INVENTORY IN WAREHOUSE

Figure 3.

ON-HAND WAREHOUSE INVENTORY

Figure 4.

ON-HAND INVENTORY AT A LINE STATION

Figure 5.
Suppose there are no constraints on the problem. Then it is easy to see that \( w_p^{DK} \geq w_p^{MA} \geq w_p^{TS} \). If, for example, \( w_p^{DK} < w_p^{MA} \) for some \( p \) then it is easy to construct a lower cost solution by letting \( w_p^{DK} = w_p^{MA} \), and keeping all other variables at their same values. When \( w_p^{DK} = w_p^{MA} \), the holding costs are lowered by \( \frac{1}{2} h c_p^D (w_p^{MA} - w_p^{DK}) \). Furthermore, space costs in the warehouse must also be lower when \( w_p^{DK} = w_p^{MA} \), since no inventory of component \( p \) ever remains in the warehouse for a positive amount of time. A similar observation can be made for the relationship between \( w_p^{MA} \) and \( w_p^{TS} \).

When the constraints on the problem are taken into account, these relationships must remain true. Consider any feasible solution for which \( w_p^{DK} < w_p^{MA} \). Suppose we increase \( w_p^{DK} \) so that it assumes a value equal to \( w_p^{MA} \). We know that the objective function value is reduced. Furthermore, the left hand sides of constraints (1) and (5) are reduced so that feasibility is maintained. Hence we have shown

Theorem 1. In any optimal solution \( w_p^{DK} > w_p^{MA} > w_p^{TS} \). (8)

As a consequence of this theorem, we see that the average holding cost function is

\[
\frac{1}{2} h c_p^D w_p^{DK}.
\]

This result implies that the delivery quantities out of each inventory stocking location cannot exceed the amount in each incoming lot. This result is similar to ones developed by others for different systems (e.g. by Crowston, Wagner, and Williams [1] for certain multistage assembly systems and by Love [2] for multistage serial systems).

The second part of the objective function measures the costs of using space in the warehouse and each staging area. These costs are based upon the
warehouse and staging area capacities and the maximum inventory stored in these places for each item. Maximum storage requirements in these locations depend on the method used to assign warehouse and staging area space to components, and on the scheduling rules used to sequence the lots through each of the facilities. For example, if dedicated storage locations are assigned to each component in the warehouse, then the space requirements will be considerably greater than if a random storage location assignment rule is used. We assume that whatever storage location assignment technique is used to operate the warehouse in combination with a lot sequencing rule that we are able to estimate factors \( f_{p}^{WH} \) and \( f_{pt}^{SA} \), the fraction of the storage capacity in and warehouse and staging area required per unit of demand for component \( p \), respectively. Also, let \( g_{p}^{WH} \) and \( g_{p}^{SA} \) represent the space cost per unit of component \( p \) stored in the warehouse and the appropriate staging area, respectively. We have allowed these space factors and costs to differ by item. However, in many cases they are the same value.

Space costs are assumed to be charged proportional to the space required. Since \( w_{p}^{DK} \geq w_{p}^{MA} > w_{p}^{TS} \), the space costs for the warehouse are

\[
\sum_{p} f_{p}^{WH} \cdot g_{p}^{WH} \cdot d_{p} \cdot w_{p}^{DK},
\]

and for staging area \( t \) are

\[
\sum_{p \in P_{t}} f_{pt}^{SA} \cdot g_{p}^{SA} \cdot d_{p} \cdot w_{p}^{MA}.
\]

Thus storage costs corresponding to a solution are

\[
\sum_{p} f_{p}^{WH} \cdot g_{p}^{WH} \cdot d_{p} \cdot w_{p}^{DK} + \sum_{t \in P_{t}} f_{tp}^{SA} \cdot g_{p}^{SA} \cdot d_{p} \cdot w_{p}^{MA}.\]

We are now able to write the objective function corresponding to the first formulation of the economic delivery quantity problem. To do so, let
\[ c_{p}^{DK} = \frac{1}{2} h c_{p} D_{p} + f_{p}^{WH} g_{p}^{WH} D_{p}, \quad p = 1, \ldots, P, \text{ and} \]

\[ c_{p}^{MA} = f_{p}^{SA} g_{p}^{SA} D_{p}, \quad \text{for } p \in P_{t}, \text{ and } t = 1, \ldots, T. \]

Thus the desired expression is

\[ \sum_{p} \left( c_{p}^{DK} w_{p}^{DK} + c_{p}^{MA} w_{p}^{MA} \right). \]

Note that \( w_{p}^{TS} \) has a coefficient of zero in the objective function. We also observe that in order for this objective function to hold, we must also enforce the restriction that

\[ w_{p}^{DK} \geq w_{p}^{MA} \geq w_{p}^{TS}. \]

Otherwise, the holding cost portion of the cost coefficients will not be expressed correctly.

Then the first formulation of the problem, which we call \( Fl \), is

\[ \min \sum_{p} \left( c_{p}^{DK} w_{p}^{DK} + c_{p}^{MA} w_{p}^{MA} \right) \]

subject to (1) - (8).

As mentioned earlier, we will find it useful to have a second, but equivalent, formulation of the problem. In this alternative formulation we let the variables be of the 0-1 type. The problem will be stated as a selection problem in which the variables indicate whether or not a particular cycle is selected for each part. Specifically, the second formulation has the following decision variables.

Let

\[ x_{p k}^{DK} = \begin{cases} 1, & \text{if component } p \text{ is sent to/from the dock every } k \text{ periods} \\ 0, & \text{otherwise} \end{cases} \]

\[ x_{p k}^{MA} = \begin{cases} 1, & \text{if component } p \text{ is sent from the marshalling area every} \\ k \text{ periods} \\ 0, & \text{otherwise} \end{cases} \]
\[ x_{pk}^{TS} = \begin{cases} 1 & \text{, if component } p \text{ is sent from the staging area to a line station every } k \text{ periods} \\ 0 & \text{, otherwise} \end{cases} \]

Thus, for example, if \( x_{pk}^{DK} = 1 \), then \( w_{p}^{DK} = k \).

Correspondingly, let

\[ a_{k}^{DK} = \frac{a_{k}^{DK}}{k} = \text{fixed time required at the dock per period if a component is on cycle } k \text{ (i.e. } x_{pk}^{DK} = 1) \]  

\[ a_{k}^{MA} = \frac{a_{k}^{MA}}{k} = \text{fixed time required in the marshalling area per period if a component is on cycle } k \]  

\[ a_{k}^{TS} = \frac{a_{k}^{TS}}{k} = \text{fixed time required per period at a train stop if a component is on cycle } k \]  

\[ a_{k}^{WH,PUT} = \frac{a_{k}^{WH,PUT}}{k} = \text{fixed time required per period in the warehouse for component puts when the component is on cycle } k \]  

\[ a_{k}^{WH,PICK} = \frac{a_{k}^{WH,PICK}}{k} = \text{fixed time needed per period for picking component } p \text{ in the warehouse when component } p \text{ is on cycle } k \]  

\[ a_{k}^{SA,PUT} = \frac{a_{k}^{SA,PUT}}{k} = \text{fixed time needed per period for put operations for component } p \text{ in a staging area when component } p \text{ is on cycle } k \]  

\[ a_{k}^{SA,PICK} = \frac{a_{k}^{SA,PICK}}{k} = \text{fixed time needed per period for picking component } p \text{ from a staging area when component } p \text{ is on cycle } k \]  

Using these definitions we can restate constraints (1) - (6) as follows.

**CONSTRAINTS FOR FORMULATION 2**

**RECEIVING DOCK**

\[ \sum_{p} \sum_{k} a_{k}^{DK} x_{pk}^{DK} \leq b^{DK}, \quad (1') \]
MARSHALLING AREA

\[ \sum_p \sum_k a_k^M A x_{pk}^{MA} \leq B^{MA}, \quad (2') \]

TRAIN STOPS

\[ \sum_{p \in P_t} \sum_k a_k^{TS} x_{pk}^{TS} \leq B_t^{TS}, \quad (t = 1, \ldots, T), \quad (3') \]

STAGING AREAS

\[ \sum_{p \in P_t} \sum_k \left( a_k^{SA, PUT} x_{pk}^{MA} + a_k^{SA, PICK} x_{pk}^{TS} \right) \leq B_{t}^{SA}, \quad (t = 1, \ldots, T), \quad (4') \]

WAREHOUSE

\[ \sum_{p} \sum_k \left( a_k^{WH, PUT} x_{pk}^{DK} + a_k^{WH, PICK} x_{pk}^{MA} \right) \leq B^{WH}, \quad (5') \]

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\[ \sum_{k} X_{pk}^{TS} \leq S_{p}^{LS}. \quad (6') \]

Observe that (6') is not necessary since we can include \( x_{pk}^{TS} \) in the formulation only if \( k \leq S_{p}^{LS} \). Thus the constraints (6') need not be considered explicitly.

Different, but equivalent, logic constraints replace (7). They indicate that one cycle must be selected for each operation for each component and that variables must take on values of 0, 0, 1.

\[ \sum_k x_{pk}^{DK} = 1, \sum_k x_{pk}^{MA} = 1, \sum_k x_{pk}^{TS} = 1, \quad (7') \]

\[ x_{pk}^{DK}, x_{pk}^{MA}, x_{pk}^{TX} \in \{0,1\} \quad \text{for} \quad p = 1, \ldots, P. \]
The coefficient array for the example problem given earlier using the second formulation is found in Figure 6. In addition we assume that \( k \) can only assume values of 1, 2, or 4. Since the constraints are equivalent to ones corresponding to the first formulation, it is not surprising that the structure is similar to that displayed in Figure 6. Note that generalized upper bound type constraints have been added and that if the integrality restriction is dropped the problem's constraints are all linear.

At this point we will take advantage of the assumption that the fixed time required per lot moved into or out of each area is independent of the component type. Hence, we can rewrite constraints \((1') - (3')\) as

\[
\sum_p \sum_k \frac{x_{dk}^{DK}}{k} \leq \frac{B^{DK}}{a_k} \equiv B^{DK}, \\
\sum_p \sum_k \frac{x_{pk}^{MA}}{k} \leq \frac{B^{MA}}{a_k} \equiv B^{MA},
\]

and

\[
\sum_{p \in P_t} \sum_k \frac{x_{tk}^{TS}}{k} \leq \frac{B^{TS}}{a_k} \equiv B^{TS}.
\]

We can also rewrite constraints \((4') \) and \((5')\) as

\[
\sum_{p \in P_t} \sum_k \left\{ \frac{x_{pk}^{MA}}{k} + \frac{\gamma_{SA,t}}{a_k} x_{pk}^{TS} \right\} \leq \frac{B^{SA,t}}{a_k,PUT} \equiv B^{SA,t}, \quad t = 1, \ldots, T,
\]

and

\[
\sum_p \sum_k \left\{ \frac{x_{dk}^{DK}}{k} + \frac{\gamma_{WH,t}}{a_k} x_{pk}^{MA} \right\} \leq \frac{B^{WH,t}}{a_k,PUT} \equiv B^{WH},
\]

respectively, where \( \gamma_{SA,t} = \frac{a_k,SA,PICK}{a_k,PUT} \) and \( \gamma_{WH,t} = \frac{a_k,WH,PICK}{a_k,WH,PUT} \).
The objective function is written as we did for Formulation 1; namely, we write cost coefficients for each \( x_{DK}^{pk} \), \( x_{MA}^{pk} \), and \( x_{TS}^{pk} \) variable to reflect the holding and space costs corresponding to the particular cycle that is selected. Let \( \bar{c}_{DK}^{pk} \), \( \bar{c}_{MA}^{pk} \), and \( \bar{c}_{TS}^{pk} \) represent these coefficients. We see that
\[
\bar{c}_{DK}^{pk} = c_{DK}^{p} \cdot k
\]
\[
\bar{c}_{MA}^{pk} = c_{MA}^{p} \cdot k
\]
and
\[
\bar{c}_{TS}^{pk} = 0.
\]

In order for these cost coefficients to hold, we must enforce additional constraints that insure that cycle consistency is maintained. That is, we must enforce constraints equivalent to those of type (8). These constraints can be written as
\[
x_{DK}^{pk} \leq \sum_{j=1}^{k} x_{MA}^{pj}
\]
and
\[
x_{MA}^{pk} \leq \sum_{j=1}^{k} x_{TS}^{pj},
\]
that is, for example, component \( p \) cannot be on cycle \( k \) in the receiving dock unless it is on cycle \( k \) or less in the marshalling area. If we denote the set of cycle consistent solutions by \( \mathcal{C} \), then a solution \( X = (x_{DK}^{1}, \ldots, x_{TS}^{pk}) \) is optimal only if \( X \in \mathcal{C} \), where \( k_p = \max k : k \leq S_{LS}^{p} \).

Then the second formulation of the economic delivery quantity problem, denoted by \( P2 \), is
\[
\text{minimize } \sum_{p} \left( \sum_{k} \bar{c}_{DK}^{pk} x_{DK}^{pk} + \sum_{k} \bar{c}_{MA}^{pk} x_{MA}^{pk} \right) \quad (P2)
\]
subject to \( X \in \mathcal{C} \) and constraints (1") - (5") and (7').
An example of the coefficient array corresponding to the constraints (1") - (5") and (7") for Formulation 2 is given in Figure 7. In that example we have three items. Items 1 and 2 use staging area 1 and item 3 uses staging area 2. Also, items 1 and 3 are limited to $k \leq 4$ and item 2 must have $k \leq 2$ due to warehouse restrictions.
III. FINDING ECONOMIC DELIVERY QUANTITIES

In this section we present a method for calculating economic delivery quantities. The real environments we have examined have several hundred to 10,000 components, even when ignoring simple hardware components such as fasteners, screws, nuts, and bolts. Consequently the number of integer variables in a typical problem could range from over 1,000 to 30,000 for Formulation 1 and from several thousand to 70,000 or more for Formulation 2. Applying a branch and bound method even when based on the problem's structure requires too much computational effort. Optimal solutions are not usually needed, however. Rather, solutions whose objective function values are relatively close to the optimal value can be determined with relatively little difficulty. We will develop a heuristic that can be used to find such solutions.

The heuristic we will describe is used to solve Formulation P2. It systematically builds a solution using a myopic greedy algorithm that eventually satisfies all the problem's constraints. The starting point for this heuristic is a solution obtained from a relaxation of problem P1. We begin our presentation of the heuristic by discussing this relaxation of P1.

A. RELAXATION OF P1

In this section we will show how to determine an optimal solution to a particular relaxation of problem P1. Rather than finding the optimal solution to P1, we will find what we call the "natural cycles" for deliveries to and from the warehouse. By a natural cycle we mean the cycle having lowest cost that satisfies only the constraint pertaining to a particular physical area in the system and that ignores all other constraints
For example, to find the natural cycle for the receiving dock variables \( w_p^{DK} \), we consider only the constraint associated with the receiving dock -- constraint (1) -- and ignore constraints (2) - (6). Furthermore, we relax constraint (7) and permit \( w_p^{DK} \) to assume any non-negative value. We also temporarily ignore the cycle consistency relationships. Thus, to find the natural cycle associated with the \( w_p^{DK} \) variables we solve

\[
\min \sum_p c_p^{DK} w_p^{DK}
\]

subject to

\[
\sum_p \frac{1}{w_p^{DK}} \leq \frac{B^{DK}}{a} = B^{DK} \quad (R1)
\]

\[w_p^{DK} \geq 0.
\]

We call this relaxed problem \( R1 \).

Similarly, we can find the natural cycle for the \( w_p^{MA} \) variables by solving the following relaxation of \( R1 \).

\[
\min \sum_p c_p^{MA} w_p^{MA}
\]

subject to

\[
\sum_p \frac{1}{w_p^{MA}} \leq \frac{B^{MA}}{a} = B^{MA}, \quad (R2)
\]

\[w_p^{MA} \geq 0.
\]

This relaxation is denoted by \( R2 \).

Optimal solutions to both \( R1 \) and \( R2 \) can be found in a straight-forward manner. We will show how this method can be developed for \( R1 \).

First, observe that in any optimal solution to \( R1 \) the constraint:

\[
\sum_p \frac{1}{w_p^{DK}} \leq B^{DK}
\]

must hold as a strict equality.
Second, the optimal solution must also satisfy the Kuhn-Tucker conditions. Let $\Theta^{DK}_p$ be the multiplier for this first constraint in R1. Then these Kuhn-Tucker conditions are:

(i) \[ c^{DK}_p - \frac{\Theta^{DK}_p}{(w^{DK}_p)^2} > 0 , \]

(ii) \[ w^{DK}_p \left( c^{DK}_p - \Theta^{DK}_p/(w^{DK}_p)^2 \right) = 0 , \]

(iii) \[ \Theta^{DK}_p \left( \sum_p \frac{1}{w^{DK}_p} - \lambda^{DK} \right) = 0 , \]

(iv) \[ \sum_p \frac{1}{w^{DK}_p} \leq \lambda^{DK} , \]

and \[ \Theta^{DK}_p > 0 . \]

Clearly in any optimal solution $w^{DK}_p > 0$. Thus, from (ii),

\[ c^{DK}_p - \frac{\Theta^{DK}_p}{(w^{DK}_p)^2} = 0 , \] or

\[ w^{DK}_p = \sqrt{\frac{\Theta^{DK}_p}{c^{DK}_p}} . \] (9)

Furthermore, as we observed, (iv) must hold as a strict equality. Hence

\[ \sum_p \frac{1}{\sqrt{\Theta^{DK}_p/c^{DK}_p}} = \lambda^{DK} , \]

and consequently
\[ \theta_{DK} = \left( \frac{a_{DK}}{p} \sum_{j} \sqrt{c_j} \right) \left( \frac{1}{B^{DK}} \right) \]

Combining the above we see that

\[ w_{DK}^p = \frac{1}{B^{DK}} \left\{ \sum_{j} \sqrt{\frac{c_j}{c_{DK}}} \right\} \]  \hspace{1cm} (10)

It is also easy to see that the natural cycle for the variable \( w_{MA}^p \) satisfies

\[ w_{MA}^p = \frac{1}{B^{MA}} \left\{ \sum_{j} \sqrt{\frac{c_j}{c_{MA}}} \right\} \]  \hspace{1cm} (11)

Observe that the solution (9) is the well known Wilson lot size formula in disguise. \( r_{DK} \) is equivalent to 1/2 times the unit holding cost and \( \theta_{DK} \) represents the fixed order cost.

B. STATEMENT OF THE HEURISTIC

Our heuristic for solving F2 begins using the natural cycles found for the \( w_{DK}^p \) and \( w_{MA}^p \) variables. These values are systematically adjusted to obtain a feasible solution to F2. The greedy method for making these needed changes begins by considering only constraints (1") - (3") and (7"), including the integrality restrictions. This relaxation yields a problem that is separable by the DK, MA, and TS superscripted variables. This can be seen by examining Figure 7. The solution found using the myopic greedy algorithm that satisfies these constraints is then modified to satisfy constraints (4") and (5") again using a greedy type of algorithm.

The heuristic is stated below.
Heuristic For Finding
Economic Delivery Quantities

Step 1: Find the natural cycles for all components, that is, find

\[ w_p^{DK} = \frac{1}{B^{DK}} \left\{ \sum_j \sqrt{c_{DJ}^{DK}} \right\} \]

and

\[ w_p^{MA} = \frac{1}{B^{MA}} \left\{ \sum_j \sqrt{c_{DJ}^{MA}} \right\} \]

If \( w_p^{DK} \notin \{2^{K-1} : k = 1, 2, \ldots\} \equiv A \), set \( w_p^{DK} \) to the element of \( A \) that is closest from a cost viewpoint to that of the original value of the natural cycle.

If \( w_p^{MA} \notin A \), set \( w_p^{MA} \) to the element of \( A \) closest in a cost sense to the original value of the natural cycle.

Step 2: Cycle Consistency Check

a) If \( w_p^{DK} < w_p^{MA} \), set \( w_p^{MA} \) to the value of \( w_p^{DK} \). Repeat this step for all components \( p = 1, \ldots, P \).
b) Set $X_{pk}^{DK} = \begin{cases} 1, & \text{if } w_p^{DK} = k \\ 0, & \text{otherwise} \end{cases}$, $X_{pk}^{MA} = \begin{cases} 1, & \text{if } w_p^{MA} = k \\ 0, & \text{otherwise} \end{cases}$

and $X_{pk}^{TS} = 1$ if $X_{pk}^{MA} = 1$ and $0$ otherwise, when $k \leq K_p$; otherwise, set $X_{pk}^{MA} = 1$ and $X_{pk}^{MA} = 0$ for $k \neq K_p$.

Step 3: Obtain a solution that satisfies constraints (1") - (3") and maintains cycle consistency.

a) Begin with constraints of type (3").

For each $t' = 1, \ldots, T$:

Determine whether the type (3") constraint is feasible for $t = t'$. If not, let $\ell_p$ be the cycle for component $p$ such that $X_{p\ell_p}^{TS} = 1$. Find the value of $\ell_p$ for which the

\[
\text{increase in cost of changing the cycle from } \ell_p \text{ to } 2\ell_p = \frac{1}{2} K_p
\]

is minimized and $2\ell_p \leq K_p$. (The increase in cost of changing from cycle of length $\ell_p$ to a cycle of length $2\ell_p$ must consider the possible increases in cost due to changing cycles for corresponding $X_p^{MA}$ and $X_p^{DK}$ variables required to maintain cycle consistency.) Call this value $p'$. Set $X_{p'\ell_p}^{TS} = 0$, $X_{p'}^{TS} = 1$, and adjust $X_{p'k}^{DK}$ and $X_{p'k}^{MA}$ as necessary to maintain cycle consistency. Set
\( \ell_p' \leftarrow 2 \cdot \ell_p' \). Repeat for \( t = t' \) until feasibility is obtained; then increment \( t \) and repeat until all \( T \) constraints are examined.

b) Constraint (2'')

While (2'') is not satisfied, find the value of \( p' \), call it \( p' \), for which the increase in cost of changing the cycle

\[
\frac{1}{2 \cdot \ell_p'}
\]

is minimized. (Initially in this step \( \ell_p' \) is the value for component \( p \) for which \( x^{MA}_{p} = 1 \) following Step 3 a.) Set

\( x^{MA}_{p'} p' = 0 \), \( x^{MA}_{p'} , 2 \cdot \ell_p' = 1 \), and adjust the value of

\( x^{DK}_{p'} k \) as necessary to maintain cycle consistency.

Set \( \ell_{p} \leftarrow 2 \cdot \ell_p' \).

c) Constraint (1'')

While (1'') is not satisfied, find the value of \( p' \), call it \( p' \), for which the increase in cost of increasing the cycle

\[
\frac{1}{2 \cdot \ell_p'}
\]

is minimized. (Initially in this step \( \ell_p' \) is the value for which \( x^{DK}_{p} = 1 \) for component \( p \) following Step 3 b.) Set

\( x^{DK}_{p'} p' = 0 \) and \( x^{DK}_{p'} , 2 \cdot \ell_p' = 1 \). Set \( \ell_{p} \leftarrow 2 \cdot \ell_p' \).
Step 4: Check constraints (4")

Beginning with \( t = 1 \), determine whether or not constraint \( t \) of type (4") is satisfied. Let \( X_{p,m}^{TS} \) and \( X_{p,m}^{MA} \) be the variables found at the end of Step 3, whose values equal 1.

For all \( p \in P_t \) for which \( m \neq l \), set \( X_{p,m}^{TS} = 1 \) and \( X_{p,m}^{MA} = 0 \). If the constraint \( t \) remains infeasible, find the value of \( p \) for which the increase in total of changing the cycle for component \( p \) in MA and TS (and possibly the DK) from \( l_p \) to \( 2l_p \)

\[
\frac{1}{2l_p} + \begin{cases} 
S_{SA,PICK} \cdot \frac{1}{2l_p}, & \text{if } 2l_p < S_{LS} \\
S_{SA,PUT} \cdot \frac{1}{2l_p}, & \text{otherwise} 
\end{cases}
\]

is minimized. Set \( X_{p,l_p}^{MA} = 0 \), and \( X_{p,2l_p}^{MA} = 1 \), and, if \( 2l_p < S_{LS} \), \( X_{p,l_p}^{TS} = 0 \), and \( X_{p,2l_p}^{TS} = 1 \). Let \( l_p \rightarrow 2l_p \).

Repeat until constraint \( t \) is feasible. Once \( t \) is feasible increment \( t \) and repeat until all \( T \) constraints have been satisfied.

Step 5: Check constraints (5")

Let \( X_{p,m}^{DK} \) and \( X_{p,m}^{MA} \) be the variables for component \( p \) that equal one at the end of Step 4. If constraint (5") is satisfied, then stop. While (5") is not satisfied, find

\[
\begin{align*}
\mu &= \min \left\{ \frac{\text{increase in cost of changing the cycle from } l_p \text{ to } 2l_p \text{ in DK over components }}{2l_p} : p \text{ for which } m = l_p \right\} \\
&= \min \left\{ \frac{1}{2l_p} : p \text{ for which } m = l_p \right\}
\end{align*}
\]
\[ v = \min \left\{ \text{increase in cost of changing the cycle from } m_p \text{ to } 2m_p \text{ in MA over components} \right\} \]

\[ \frac{1}{2^p} \frac{a}{\text{WH,PUT}} \frac{1}{a} \frac{\text{WH, PICK}}{p} \]

Let \( w = \min \{u, v\} \) and component \( p' \) be a value of \( p \) that yields the value of \( w \). If \( u \leq v \), set \( X_{p'}^{DK} = 0 \), \( X_{p'} = 1 \), and \( m_{p'} = 2 \). Otherwise, set \( X_{p'}^{MA} = 0 \), \( X_{p'}^{MA} = 1 \) and \( m_{p'} = 2 \). Once the constraint is satisfied, stop.

We now briefly comment about the various steps of the algorithm. In Step 1 we find the natural cycle for each item using Formulation 1. Recall that these natural cycles are found considering only the first two constraints of F1. To find \( w_p^{DK} \) and \( w_p^{MA} \) the cycle restrictions were also ignored. Once these values are calculated, cycle consistency is achieved in Step 2.

In Step 2a we reset the values for variables \( w_p^{DK} \) and \( w_p^{MA} \) if \( w_p^{DK} < w_p^{MA} \). These variables are set to the same value. The reason we do this is based on the observations developed in the Appendix.

Step 3 involves taking the solution found in the previous steps and finding a solution that satisfies constraints \( 1'' \) - \( 3'' \) of Formulation 2 and maintains cycle consistency. We begin Step 3 by first examining the type \( 3'' \) constraints. We choose to begin with these constraints since adjustments to the values of the \( X_{pk}^{TS} \) variables also can affect feasibility in the type \( 1'' \) and type \( 2'' \) constraints. Let's see why this is the case.

Suppose \( X_{p'}^{TS} = 1 \), \( X_{p'}^{MA} = 1 \) and \( X_{p'}^{DK} = 1 \) in the solution found in Step 2. Furthermore, suppose component \( p' \) is selected as the component to adjust (we'll discuss the rule for finding the variable to adjust later on), and suppose \( l_{p'} = m_{p'} = n_{p'} \). Then not only will \( X_{p'}^{TS} = 1 \), but the receiving dock and marshalling area variables must be changed to maintain
cycle consistency, that is, \( X^{DK}_{p',2 \cdot n_p} = 1 \) and \( X^{MA}_{p',2 \cdot m_p} = 1 \). Note that these changes lower the value of the left hand sides of constraints (1") and (2") thereby moving these constraints closer to feasibility (if they were not already satisfied).

Beginning with constraints (1") or (2") would not produce this same beneficial affect. For example, if we started with the type (1") constraint, then when we examined constraints (2") and (3") we would already have a feasible solution for (1"). Thus the effect of reversing the order in which we examined the constraints could increase the cost of the final solution.

The component \( p' \) that is selected for adjustment is determined based on a myopic greedy algorithm. Constraints (1"), (2"), (4") and (5") are not directly examined in the calculation. We again note that any changes made in this phase have a positive affect on the chance of observing feasibility at a subsequent step. However, only constraints of type (3") and the cycle consistency constraints are considered. To find the component to adjust, calculate the cost of increasing the current cycle for component \( p \), call it \( \ell_p \), and measure the effect on feasibility. The cost calculation is made as follows:

\[
\Delta C(p) = C^{TS}_{p,2 \cdot \ell_p} - C^{TS}_{p,\ell_p} + \begin{cases} 
C^{MA}_{p,2 \cdot \ell_p} - C^{MA}_{p,\ell_p} & \text{if } 2 \cdot \ell_p > m_p \\
0 & \text{otherwise}
\end{cases}
\]

\[
+ \begin{cases} 
C^{DK}_{p,2 \cdot \ell_p} - C^{DK}_{p,\ell_p} & \text{if } 2 \cdot \ell_p > n_p \\
0 & \text{otherwise}
\end{cases}
\]

Thus if the corresponding MA and DK variables must be adjusted to maintain cycle consistency, the the cost effect is taken into account. The effect on the constraint of increasing the cycle length from \( \ell_p \) to \( 2 \cdot \ell_p \) is
Thus the rule for selecting the component to adjust is to determine
the smallest value of

\[
\frac{1}{p} - \frac{1}{2p} = \frac{1}{2p}
\]

Subsequent steps in the algorithm are all based on applying the same
logic we have discussed. The reason for examining constraint (4") before
(5") is the same as that for examining (3") before (1") or (2").

In summary, the algorithm is an exceedingly simple one to apply since
it is a single pass procedure. The computational burden increases approxi-
mately linearly as the number of components is increased. Since the under-
lying approach is based on a greedy type of algorithm, it has a strong
intuitive appeal. Finally, since each step of the heuristics actually
benefits future ones, the solution found using the myopic procedure, that
is, using a method that does not simultaneously consider all constraints,
should be close to optimal.
IV. SUMMARY AND CONCLUDING COMMENTS

In this paper we have developed two models for the multi-stage economic delivery quantity problem, which is a natural extension of the classic economic lot size problem. We showed that lot sizes should be established considering not only procurement costs, but also the effects on workload in different areas of the assembly plant. Specifically, constraints on available time or space limitations in the receiving dock, warehouse, marshalling area, staging areas, train stops, and line stations must all be taken into account when setting supplier delivery quantities and transfer lots from one storage location to another. The preferred lot sizes were determined based on minimizing inventory holding costs and space costs.

Recall that our goal was to establish lot sizes that reflect the relationships among the various activities while considering inventory and storage costs. We did not attempt to schedule the delivery times of specific lots from one stage to the next. This is the subject of future research. However, note that by fixing delivery quantities as suggested in this paper to be a multiple of 2 times the demand rate per period for the final assembly, the difficulty in determining a schedule is reduced significantly. See reference 3 for a discussion of why this is the case.

The two models that were discussed were large scale integer programming models. Since standard branch and bound procedures are impossible to use for realistic situations in which there are thousands of items, a simple one-pass heuristic was developed which takes advantage of the problem's structure. A solution is constructed by systematically considering only a portion of the problem's constraints. In each step of
the heuristic, a greedy type algorithm was used to determine which variables should be adjusted so that a particular constraint could be satisfied in a cost effective manner. As was shown, the sequence in which the constraints are considered can substantially affect the solution and cost that is obtained. We feel that the solution found using the heuristic should be relatively close to the optimal solution due to the manner in which it has been constructed.
REFERENCES


APPENDIX

In this section we present some observations that substantiate Step 2a of the heuristic.

Suppose we solve

$$\min \sum_p \left( c_p w_p + \frac{c_p}{w_p} + \frac{c_p}{w_p} \right)$$

subject to

$$\sum_p 1/w_p \leq \hat{B}/\hat{A}$$

(R3)

$$\sum_p 1/w_p \leq \hat{B}/\hat{A}$$

$$w_p > 0$$ for all $p$.

This problem R3 is problems R1 and R2 with the addition of the constraints $w_p \geq w_p$. The optimal solutions for R1 and R2 are also optimal for R3 if these final constraints are satisfied. Let $\hat{w}_p$ and $\hat{w}_p$ designate the optimal solutions to problems R1 and R2, respectively, and let $\hat{w}_p$ and $\hat{w}_p$ the optimal solution for R3.

Theorem 2. If $\hat{w}_p = w_p$, then $\hat{w}_p = w_p$.

Proof: Suppose $\hat{w}_p$ and $\hat{w}_p$ represent the optimal multiplier values for the solutions to R1 and R2. The Kuhn-Tucker conditions for R3 are

$$-\lambda_p + c_p - \frac{\hat{\theta}}{w_p^2} \geq 0, \lambda_p + c_p - \frac{\hat{\theta}}{w_p^2} \geq 0$$

for all $p$

$$w_p \left( \lambda_p + c_p - \hat{\theta}_p \right) = 0$$

$$w_p \left( \lambda_p + c_p - \hat{\theta}_p \right) = 0$$

for all $p$. 


\[ \theta^{DK} \left( \sum_p \frac{1}{w_p^{DK}} - \frac{B^{DK}}{a^{DK}} \right) = 0, \quad \theta^{MA} \left( \sum_p \frac{1}{w_p^{MA}} - \frac{B^{MA}}{a^{MA}} \right) = 0 \]

\[ \lambda_p \left( \frac{w_p^{MA}}{w_p^{DK}} - \frac{w_p^{DK}}{w_p^{MA}} \right) = 0 \text{ for all } p \]

\[ \sum_p \frac{1}{w_p^{DK}} \leq \frac{B^{DK}}{a^{DK}}, \quad \sum_p \frac{1}{w_p^{MA}} \leq \frac{B^{MA}}{a^{MA}} \]

and \( w_p^{DK} > w_p^{MA} \geq 0 \) for all \( p \), where \( \lambda_p \) is the multiplier for the cycle consistency constraint for component \( p \), and \( \lambda_p, \theta^{DK}, \theta^{MA} \) are non-negative.

Observe that

\[ \hat{w}_p^{DK} = \sqrt{\frac{\theta^{DK}}{c^{DK} - \lambda_p}} \quad \text{and} \quad \hat{w}_p^{MA} = \sqrt{\frac{\theta^{MA}}{c^{MA} + \lambda_p}} \]

where \( \theta^{DK} \) and \( \theta^{MA} \) represent the optimal multiplier values for the receiving dock and marshalling area constraints, respectively.

We will prove the result using a proof by contradiction. Suppose that \( \hat{w}_p^{DK} > \hat{w}_p^{MA} \) when \( w_p^{DK} < w_p^{MA} \). Note that in any optimal solution that \( \hat{w}_p^{DK} > 0 \) and \( \hat{w}_p^{MA} > 0 \). Otherwise the first two constraints in R3 would not be satisfied. For this to happen \( \theta^{DK} > 0 \) and \( \theta^{MA} > 0 \). Thus, from the Kuhn-Tucker conditions, \( \sum_p \frac{1}{w_p^{DK}} = \frac{B^{DK}}{a^{DK}} \) and \( \sum_p \frac{1}{w_p^{MA}} = \frac{B^{MA}}{a^{MA}} \).

Furthermore, assume that \( \hat{\theta}^{DK} > \theta^{DK} \). Then

\[ \hat{w}_p^{DK} = \sqrt{\frac{\theta^{DK}}{c^{DK} - \lambda_p}} > \sqrt{\frac{\theta^{DK}}{c^{DK} - \hat{\theta}^{DK}} > \sqrt{\frac{\theta^{DK}}{c^{DK} - \hat{\theta}^{DK}}} = \hat{w}_p^{DK} \]. Consequently,
\[ \hat{\beta}_{DK} = \sum_p \frac{1}{p} \hat{w}_{DK}^p < \sum_p \frac{1}{p} \hat{w}_{MA}^p = \hat{\beta}_{MA}, \] which is impossible. Hence \( \hat{\beta}_{DK} < \hat{\beta}_{MA}. \)

Similarly we can show that \( \hat{\Theta}_{MA} > \hat{\Theta}_{MA}. \) As a result, \( \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{DK}} \leq |1| \leq \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} = \hat{\Theta}_{MA} \) or \( \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{DK}} < \hat{\Theta}_{MA}. \)

Since we have assumed that \( \hat{w}_{DK}^p < \hat{w}_{MA}^p, \) \( \frac{\hat{\Theta}_{DK}}{\hat{\Theta}_{MA}} < \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} \) or \( \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} > \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}}. \)

Also, if \( \hat{w}_{DK}^p > \hat{w}_{MA}^p, \) then \( \lambda_p = 0 \) and \( \frac{\hat{\Theta}_{DK}}{\hat{\Theta}_{MA}} > \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} \) or \( \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} < \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}}. \)

Thus \( \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{DK}} < \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{DK}} < \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}} < \frac{\hat{\Theta}_{MA}}{\hat{\Theta}_{MA}}. \) (The last inequality is due to the result found in the preceding paragraph). Q.E.D.