Answers to three not quite straightforward questions in structural stability

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May 29, 2008
modification of an imperfection-sensitive structure such that it becomes imperfection-insensitive
modification of an imperfection-sensitive structure such that it becomes imperfection-insensitive
**Question I**

**Agenda**

Are linear prebuckling paths and linear stability problems mutually conditional?

**Literature**

Question II

Does the conversion from imperfection sensitivity into imperfection insensitivity require a symmetric postbuckling path?

Literature

Is hilltop buckling necessarily imperfection sensitive?

Literature

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Are linear prebuckling paths and linear stability problems mutually conditional?

- **Motivation**
Are linear prebuckling paths and linear stability problems mutually conditional?

**Linear stability problem**

\[
\det(\tilde{K}_T(\lambda)) = \det(K_0 + \lambda K_1) = 0
\]

- load factor small
- geometric stiffness
- displacement stiffness

- no prebuckling rotations allowed
- simplifies computation of critical load \(\lambda_C\)

**Motivation**

**Theory**

**Examples**

**Conclusions**

**Literature**


### Linear stability problem

\[
\det(\tilde{K}_T(\lambda)) = \det(K_0 + \lambda K_1) = 0
\]

- load factor small
- geometric stiffness
- displacement
- stiffness

- no prebuckling rotations allowed
- simplifies computation of the critical load \(\lambda_C\)

\[
\tilde{u}_{\lambda} = k = \text{const.} \quad \rightarrow \quad \tilde{u}(\lambda) = u_0 + \lambda k
\]
Conditions for a linear stability problem in the prebuckling domain
- negligible change of material tangent stiffness matrix
- small displacements
- linear stress-load relation
- loads do not depend on the displacements

Sources of nonlinearity
- geometric nonlinearity
- material behavior
- boundary conditions
Are linear prebuckling paths and linear stability problems mutually conditional?

**Theory**

potential energy $V$

displacements $u$

out-of-balance-force $G$

$$G := \frac{\partial V}{\partial u} = F^I(u) - \lambda P$$

- internal forces
- reference load
- load factor

$V_{,u} = G = 0 \ldots$ equilibrium condition

$V_{,uu} = K_T \ldots$ tangent stiffness matrix
Are linear prebuckling paths and linear stability problems mutually conditional?

\[ V_u = G = F^I(u) - \lambda P = 0 \quad \ldots \text{equilibrium condition} \]

\[ \frac{d}{d\lambda} \text{ along primary path} \]

\[ \tilde{K}_T \cdot \tilde{u},_\lambda - P = 0 \]

\[ \frac{d}{d\lambda} \text{ along primary path} \]

\[ \tilde{K}_T,\lambda \cdot \tilde{u},_\lambda + \tilde{K}_T \cdot \tilde{u},_{\lambda\lambda} = 0 \]

\[ \tilde{u},_\lambda = k \quad \ldots \text{linear prebuckling path} \]

\[ \tilde{K}_T,\lambda \cdot k = 0 \]

\[ \text{not sufficient for a linear stability problem} \quad \tilde{K}_T(\lambda) = K_0 + \lambda K_1 \]
Are linear prebuckling paths and linear stability problems mutually conditional?

\[ V, u = G = F^I(u) - \lambda P = 0 \quad \text{... equilibrium condition} \]

\[ \frac{d}{d\lambda} \text{ along primary path} \]

\[ \tilde{K}_T \cdot \tilde{u},\lambda - P = 0 \]

\[ \tilde{K}_T(\lambda) = K_0 + \lambda K_1 \quad \text{... linear stability problem} \]

\[ (K_0 + \lambda K_1) \cdot \tilde{u},\lambda - P = 0 \]

\textbf{not} sufficient for a linear prebuckling path \( \tilde{u}(\lambda) = u_0 + \lambda k \)
Are linear prebuckling paths and linear stability problems mutually conditional?

Example I

A linear stability problem

2 DOF, static, conservative, contour (a nonlinear spring)

\[ f(x) = \frac{k_0 P}{k_1 F} \left( \frac{P}{k_1} (e^{xk_1/P} - 1) - x \right) \]
Are linear prebuckling paths and linear stability problems mutually conditional?

Example I  A linear stability problem

• potential energy
  \[ V = -\lambda P (u_1 + L (1 - \cos(u_2))) + F f(u_1) + \frac{k_2}{2} L^2 \sin^2(u_2) \]

• nonlinear prebuckling path, load-displacement relation
  \[ \lambda = \frac{k_0}{k_1} (e^{u_1 k_1/P} - 1), \ u_2 = 0 \]

• linear stability problem, tangent stiffness matrix
  \[ \tilde{K}_T = \begin{bmatrix} k_0 + k_1 \lambda & 0 \\ 0 & L(k_2 L - \lambda P) \end{bmatrix} \]
### Question I

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### Example II

**A linear prebuckling path**

Are linear prebuckling paths and linear stability problems mutually conditional?

2 DOF, static, conservative
Are linear prebuckling paths and linear stability problems mutually conditional?

### Example II

**A linear prebuckling path**

- **potential energy**
  \[
  V = -\lambda P u_1 + \frac{1}{2} c u_2^2 + \frac{1}{2} k \left( L - \frac{L - u_1}{\cos(u_2)} \right)^2
  \]

- **linear prebuckling path, load-displacement relation**
  \[
  \lambda = u_1 k/P, \quad u_2 = 0
  \]

- **nonlinear stability problem, tangent stiffness matrix**
  \[
  \tilde{K}_T = \begin{bmatrix}
  k & 0 \\
  0 & c - \lambda P (L - \lambda P/k)
  \end{bmatrix}
  \]
Example II

A linear prebuckling path

- effect is of higher order
- usually negligible
- e.g. Euler buckling cases

Are linear prebuckling paths and linear stability problems mutually conditional?

2 DOF, static, conservative
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<td>A linear prebuckling path is</td>
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<td></td>
<td>• neither necessary</td>
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<td></td>
<td>• nor sufficient</td>
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<td>for a stability problem to be linear.</td>
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**Literature**

Question II

Does the conversion from imperfection sensitivity into imperfection insensitivity require a symmetric postbuckling path?
Motivation

- general research interest: conversion from imperfection sensitivity structures into imperfection insensitive structures
- What are the conditions for this conversion?
- Is symmetry required?
- *ab initio* design for imperfection insensitivity
- existence of qualitative properties which influence imperfection insensitivity
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| Motivation  | **Koiter’s initial postbuckling analysis**  
Idea: expansion of the out-of-balance force  
\[ V_u' = G = F^I(u) - \lambda P \]  
into a Taylor series at the bifurcation point \( C \) |
| Theory      |  
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Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Koiter’s initial postbuckling analysis**

Idea: expansion of the out-of-balance force

\[ V'_u = G = F^I(u) - \lambda P \]

into a Taylor series at the bifurcation point \( C \)
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Koiter’s initial postbuckling analysis
Idea: expansion of the out-of-balance force
\[ V_u = G = F^I(u) - \lambda P \]
into a Taylor series at the bifurcation point \( C \)

\[ \lambda(\eta) = \left[ 1, \tilde{u}_{\lambda}(\lambda(\eta_B)) \right]^T \]

primary path

\[ \nu(\eta) \]

secondary path

\[ [\lambda, \lambda \tilde{u}_{\lambda}^T + \nu^T]^T \]
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Question II**

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**Koiter’s initial postbuckling analysis**

Idea: expansion of the out-of-balance force

\[
V_u = G = F^I(u) - \lambda P
\]

into a Taylor series at the bifurcation point \( C \)

- coordinate transformation for the secondary path

\[
(v, \eta) \mapsto (u, \lambda) = (\tilde{u}(\lambda(\eta)) + v, \lambda(\eta))
\]

- series expansion of coordinates

\[
\begin{align*}
\nu(\eta) &= \nu_1 \eta + \nu_2 \eta^2 + \nu_3 \eta^3 + \ldots \\
\lambda(\eta) &= \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \lambda_3 \eta^3 + \ldots
\end{align*}
\]
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

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|             |            | • transformed out-of-balance force

\[
G(v, \eta) := G(\tilde{u}(\lambda(\eta)) + v, \lambda(\eta))
\]

• series expansion of the out-of-balance force at \( C \)

\[
G(v, \eta) = G_{0C} + G_{1C} \eta + G_{2C} \eta^2 + G_{3C} \eta^3 + \ldots = 0
\]

must hold for arbitrary values of \( \eta \)

\[
G_{nC} = 0 \quad \forall n \in \mathbb{Q}
\]

allows computation

\((\lambda_1, v_1), (\lambda_2, v_2), (\lambda_3, v_3), \ldots\)
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

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\[ \lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots \]

- the first non-vanishing coefficient must have an even subscript and must be **positive**
- \( \lambda_1 = 0 \) … horizontal tangent is necessary

**Literature**
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Symmetric** load-displacement paths

- linear mapping $\mathbf{T} : \mathbb{U}^N \rightarrow \mathbb{U}^N$ (symmetry group)

example: $\mathbf{T} = \begin{bmatrix} 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}$
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Symmetric** load-displacement paths

- linear mapping $T : \mathbb{U}^N \rightarrow \mathbb{U}^N$ (symmetry group)
- symmetry requires $V(u, \lambda) = V(T \cdot u, \lambda)$
- mirror symmetry w.r.t. $\eta$

$$V(\tilde{u}(\lambda(\eta)) + v(\eta), \lambda(\eta)) = V(\tilde{u}(\lambda(-\eta)) + v(-\eta), \lambda(-\eta)) \quad \forall \eta \in \mathbb{U}$$

- consequences:

  uniqueness of the primary path requires

  ① $\tilde{u}(\lambda) = T \cdot \tilde{u}(\lambda) \quad \forall \lambda \in \mathbb{U}$
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Symmetric** load-displacement paths

- linear mapping \( T : \mathbb{U}^N \rightarrow \mathbb{U}^N \) (symmetry group)
- symmetry requires \( V(u, \lambda) = V(T \cdot u, \lambda) \)
- mirror symmetry w.r.t. \( \eta \)

\[
V(\tilde{u}(\lambda(\eta)) + v(\eta), \lambda(\eta)) = V(\tilde{u}(\lambda(-\eta)) + v(-\eta), \lambda(-\eta)) \quad \forall \eta \in \mathbb{U}
\]

- consequences:
  - secondary path
    - \( 2 \) \( v(\eta) = T \cdot v(-\eta) \quad \forall \eta \in \mathbb{U} \)
    - \( 3 \) \( \lambda(\eta) = \lambda(-\eta) \quad \forall \eta \in \mathbb{U} \)
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Imperfection insensitivity**

\[ \lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots \]

the first non-vanishing coefficient must have an even subscript and must be positive

**Symmetry**

1. \[ \tilde{u}(\lambda) = T \cdot \tilde{u}(\lambda) \quad \forall \lambda \in \mathbb{U} \]
2. \[ \nu(\eta) = T \cdot \nu(-\eta) \quad \forall \eta \in \mathbb{U} \]
3. \[ \lambda(\eta) = \lambda(-\eta) \quad \forall \eta \in \mathbb{U} \quad \Rightarrow \quad \lambda_1 = \lambda_3 = \ldots = 0 \]

symmetry is not necessary for the conversion from imperfection sensitivity into insensitivity
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Example

2 DOF
static, conservative
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Example

2 DOF static, conservative

\[ V(u_1, u_2, \lambda) \neq V(u_1, -u_2, \lambda) \]

non-symmetric bifurcation
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Example**

- Design parameters

\[ \gamma/\chi \text{ and } \mu \text{ are chosen such that } \lambda_1 = 0 \land \lambda_3 = 0 \]

but \( \lambda_5 \neq 0 \),

\( k \) is modified
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?
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Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

\[ \kappa = 0.275 \]

\[ \kappa = 0.2 \]

\[ \kappa = 0.0454 \]

\[ \kappa = 0 \]

\[ \kappa = 0.3 \]

\[ \kappa = 0.5 \]
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

\[ \kappa = 0 \]
\[ \kappa = 0.0454 \]
\[ \kappa = 0.2 \]
\[ \kappa = 0.4 \]

region of imperfection insensitivity
Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

**Conclusions**

- Conversion into *imperfection insensitivity* requires $\lambda_1 = 0$, which holds automatically for symmetric bifurcation.
- Symmetric bifurcation is not necessary.
- Conversion is possible *without* change of the prebuckling behavior and *without* change of the buckling load.
- Increasing the stiffness
  - can lead to conversion into *imperfection insensitivity*
  - may result in *qualitative* changes of the secondary path
Is hilltop buckling necessarily imperfection sensitive?

Question III

Is \textit{hilltop} buckling necessarily \textit{imperfection sensitive}?
Motivation

- divide and conquer - the category of hilltop buckling
- hilltop buckling as a starting point of sensitivity analysis
- Is hilltop buckling a “worst case“ buckling scenario?
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<td>• Assertion II: Hilltop buckling is necessarily imperfection sensitive.</td>
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**Literature**


Is hilltop buckling necessarily imperfection sensitive?

**Proof**

- path parameter $\xi$ referring to the primary path
- at the snap-through point $\rightarrow \lambda,\xi = 0 \land \lambda,\xi\xi < 0$ (local maximum of $\lambda$)
- coefficient $a_1$ occurs in some expressions $G_{nC}$

$$a_1 = -\frac{1}{2} \frac{\mathbf{v}_1^T \cdot \mathbf{\tilde{K}}_{T,\lambda\lambda} \cdot \mathbf{v}_1}{\mathbf{v}_1^T \cdot \mathbf{\tilde{K}}_{T,\lambda} \cdot \mathbf{v}_1} = -\frac{1}{2 \lambda,\xi} \left( \frac{\mathbf{v}_1^T \cdot \mathbf{\tilde{K}}_{T,\xi\xi} \cdot \mathbf{v}_1}{\mathbf{v}_1^T \cdot \mathbf{\tilde{K}}_{T,\xi} \cdot \mathbf{v}_1} - \frac{\lambda}{\lambda,\xi} \right)$$

- $\mathbf{v}_1^T \cdot \mathbf{\tilde{K}}_{T,\xi} \cdot \mathbf{v}_1 \neq 0$ is known from the consistently linearized eigenproblem

$$a_1 = \frac{1}{2} \frac{\lambda,\xi\xi}{\lambda,\xi} = -\infty \quad \text{... pole of } 2^{\text{nd}} \text{ order}$$
Is hilltop buckling necessarily imperfection sensitive?

**Proof**

- path parameter $\eta$ referring to the secondary path
- $\lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots$
- $\eta=0$ refers to the stability limit $\lambda_C$

- coefficient $a_1 = -\frac{1}{2\lambda,\eta} \left( \frac{v_1^T \tilde{K}_{T,\eta} \cdot v_1}{v_1^T \tilde{K}_{T,\eta} \cdot v_1 - \frac{\lambda,\eta}{\lambda,\eta}} \right) \bigg|_{\eta=0}$

\[
\lambda,\eta \bigg|_{\eta=0} = \lambda_1, \quad \lambda,\eta,\eta \bigg|_{\eta=0} = 2\lambda_2
\]

\[
\tilde{K}_{T,\eta} / \lambda,\eta = \tilde{K}_{T,\xi} / \lambda,\xi
\]

\[
a_1 = -\frac{1}{2\lambda_1^2} \left( \frac{v_1^T \tilde{K}_{T,\eta} \bigg|_{\eta=0} \cdot v_1}{v_1^T \tilde{K}_{T,\xi} \cdot v_1} \lambda,\xi - 2\lambda_2 \right)
\]
Is hilltop buckling necessarily imperfection sensitive?

**Proof**

\[ a_1 = -\frac{1}{2\lambda_1^2} \left( v_1^T \tilde{K}_{T,\eta\eta} v_1 \right) \eta=0 \cdot \nu_1 \begin{vmatrix} \nu_1^T \tilde{K}_{T,\xi} \nu_1 \end{vmatrix} \lambda_{,\xi} - 2\lambda_2 \]

**No hilltop buckling**

\[ \lambda_1 = 0, \quad \lambda_{,\xi} \neq 0 \]

\[ \downarrow \]

\[ a_1 \neq 0 \]

\[ \downarrow \]

\[ v_1^T \tilde{K}_{T,\eta\eta} v_1 \bigg|_{\eta=0} \nu_1 \begin{vmatrix} \nu_1^T \tilde{K}_{T,\xi} \nu_1 \end{vmatrix} \lambda_{,\xi} - 2\lambda_2 = 0 \]
Is hilltop buckling necessarily imperfection sensitive?

**Proof**

\[
a_1 = -\frac{1}{2\lambda_1^2} \left( \frac{v_1^T \cdot \tilde{K}_{T,\eta\eta}|_{\eta=0} \cdot v_1}{v_1^T \cdot \tilde{K}_{T,\xi} \cdot v_1} \lambda_{,\xi} - 2\lambda_2 \right)
\]

**no hilltop buckling**

\[
\lambda_1 = 0, \quad \lambda_{,\xi} \neq 0
\]

\[
downarrow
\]

\[
a_1 \neq 0
\]

\[
\left( \frac{\theta^T \cdot \tilde{K}_{T,\eta\eta}|_{\eta=0} \cdot \theta_1}{\theta^T \cdot \tilde{K}_{T,\xi} \cdot \theta_1} \lambda_{,\xi} - 2\lambda_2 \right) = 0
\]

**hilltop buckling**

\[
\lambda_{,\xi} = 0, \quad v_1^T \cdot \tilde{K}_{T,\xi} \cdot v_1 \neq 0
\]

\[
downarrow
\]

\[
a_1 = \frac{\lambda_2}{\lambda_1^2}
\]

\[
downarrow
\]

\[
a_1 = -\infty
\]

\[
\lambda_1 = 0, \quad -\infty < \lambda_2 < 0
\]

q.e.d.
Is hilltop buckling necessarily imperfection sensitive?

**Motivation**

Motivation for the question of hilltop buckling.

**Theory**

Proof:

\[ a_1 = -\frac{1}{2\lambda_1^2} \left( \nu_1^T \cdot \tilde{K}_{T,\eta\eta} \bigg|_{\eta=0} \cdot \nu_1 \right) \lambda,_{\xi} - 2\lambda_2 \]

**Example**

- **no hilltop buckling**
  \[ \lambda_1 = 0, \; \lambda,_{\xi} \neq 0 \]
  \[ a_1 \neq 0 \]

- **hilltop buckling**
  \[ \lambda,_{\xi} = 0, \; \nu_1^T \cdot \tilde{K}_{T,\xi} \cdot \nu_1 \neq 0 \]
  \[ a_1 = \frac{\lambda_2}{\lambda_1^2} \]
  \[ a_1 = -\infty \]

**Conclusions**

Hilltop buckling is necessarily imperfection sensitive.
Is hilltop buckling necessarily imperfection sensitive?

**Theory (cont.)**

- design parameter $\kappa$

- series expansion

\[
G(\nu, \eta) = G_{0C} + G_{1C} \eta + G_{2C} \eta^2 + \ldots = 0
\]

yields parameter-dependent equation

\[
\lambda_4(\kappa) = a_1(\kappa) \lambda_2^2(\kappa) + b_2(\kappa) \lambda_2(\kappa) + d_3(\kappa)
\]

"solution"

\[
\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2a_1(\kappa)} \pm \sqrt{\frac{b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}{2a_1(\kappa)}}
\]

- allows differentiation between two characteristic classes of the relation $\lambda_4 = \lambda_4(\lambda_2(\kappa))$
Is hilltop buckling necessarily imperfection sensitive?

\[ \lambda_{2(\kappa)}_{1,2} = -\frac{b_2(\kappa)}{2 a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4 a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2 a_1(\kappa)} \]

**Class ①**

\[ d_3(\lambda_2), \lambda_4(\lambda_2) \]

need not be negative

(qualitative plot)
Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2a_1(\kappa)}$$

**Class 1**

- generally, $$b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa)) > 0$$
  - double roots are **not** possible
- at the hilltop buckling point
  - $$b_2 = +\infty \ldots$$ pole of 1\(^{\text{st}}\) order
  - $$a_1 = -\infty \ldots$$ pole of 2\(^{\text{nd}}\) order
  - $$\frac{b_2}{a_1} = 0$$
- $$\lambda_2 = 0$$ is **not** possible because it would be a double root
- $$\lambda_2 < 0 \rightarrow \lambda_2(\kappa) = -\sqrt{\lambda_4(\kappa) - d_3(\kappa)} / a_1(\kappa)$$
Is hilltop buckling necessarily imperfection sensitive?

\[
\lambda_{2}(\kappa)_{1,2} = -\frac{b_{2}(\kappa)}{2a_{1}(\kappa)} \pm \frac{\sqrt{b_{2}^{2}(\kappa) - 4a_{1}(\kappa)(d_{3}(\kappa) - \lambda_{4}(\kappa))}}{2a_{1}(\kappa)}
\]

Class ② \[d_{3}(\lambda_{2}), \lambda_{4}(\lambda_{2})\]

must be negative

(qualitative plot)
Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2a_1(\kappa)}$$

**Class 2**

- \(b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa)) = 0 \quad \forall \kappa \in \mathbb{U} \)

  double root \(\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2a_1(\kappa)} = -\frac{2(d_3(\kappa) - \lambda_4(\kappa))}{b_2(\kappa)}\)

- \(2a_1(\kappa)\lambda_2(\kappa)_{1,2} + b_2(\kappa) = 0\)

  \(\lambda_2 = 0\) corresponds to \(a_1 = 0 \quad \Rightarrow \quad b_2 = 0 \quad \land \quad d_3 - \lambda_4 = 0\)

- but hilltop buckling requires \(a_1 = -\infty, \lambda_2 < 0\)

  \(b_2 = -\infty \quad \ldots \) pole of 2\(^{nd}\) order

  \(d_3 - \lambda_4 = -\infty \quad \ldots \) pole of 2\(^{nd}\) order
Is hilltop buckling necessarily imperfection sensitive?

**Example of class ①**

2 DOF
static, conservative
symmetric bifurcation
Is hilltop buckling necessarily imperfection sensitive?

**Example of class ①**

2 DOF
static, conservative
symmetric bifurcation

**Motivation**

**Theory**

**Example**

**Conclusions**
Is hilltop buckling necessarily imperfection sensitive?

Load-displacement path - hilltop buckling

\[
\frac{\lambda P}{kL} = 0
\]

\[\kappa = 0\]

\[\lambda_2 < 0\]

imperfection sensitive
Is hilltop buckling necessarily imperfection sensitive?

**Load-displacement path** - zero-stiffness postbuckling

\[ \lambda \frac{P}{kL} \]

\[ \kappa = \frac{\mu}{4} \]

\[ \lambda_2 = 0 \]

\[ \lambda_4 = 0 \]

\[ \vdots \]
Is hilltop buckling necessarily imperfection sensitive?

Load-displacement path - zero-stiffness postbuckling

\[
\frac{\lambda P}{kL} = \kappa = \frac{\mu}{4}
\]

\[
\lambda_2 = 0
\]

\[
\lambda_4 = 0
\]

Is zero-stiffness postbuckling predictable?

Yes, it occurs if

\[
\kappa \rightarrow \infty \quad b_2 \rightarrow -\infty \quad \text{pole of 1}\text{st order}
\]

\[
d_3 \rightarrow +\infty \quad \text{pole of 2}\text{nd order}
\]
Is hilltop buckling necessarily imperfection sensitive?

**Load-displacement path** - saddle point

\[ \frac{\lambda P}{kL} \]

Motivation

\[ \kappa = 1 - \cos(u_{10}) \]

Theory

\[ \lambda_2 > 0 \]

Example

Conclusions

imperfection insensitive
Is hilltop buckling necessarily imperfection sensitive?

Question III

Motivation

Theory

Example

Conclusions

Coefficients of quadratic equation

\[ a_1 \]

\[ \lambda \]

\[ \infty \]

\[ 0 \]

\[ 5 \]

\[ -5 \]

\[ -10 \]

\[ a_1 \]

\[ \lambda_2 \]
Is hilltop buckling necessarily imperfection sensitive?

**Question III**

Motivation

Theory

Example

Conclusions

**Coefficients of quadratic equation**

$a_1, b_2$

indicator for zero-stiffness postbuckling
Is hilltop buckling necessarily imperfection sensitive?

**Question III**

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Coefficients of quadratic equation:

\[ a_1, b_2, d_3 - \lambda_4 \]

- \( +\infty \) indicator for zero-stiffness postbuckling
- \( -\infty \)

Graph showing the behavior of \( a_1, b_2, d_3 - \lambda_4 \) with \( \lambda_2 \).
Is hilltop buckling necessarily imperfection sensitive?

\[ b_2^2(\kappa) - 4a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa)) \]
Conclusions

- two characteristic classes of hilltop buckling
- different behavior for $\kappa \rightarrow +\infty$
- Hilltop buckling is necessarily imperfection sensitive.
Further research

- FEM analysis of multiple-DOF systems
- Identification of qualitative properties pivotal for the conversion from imperfection sensitivity into insensitivity
- Investigation of the reasons for the initial postbuckling behavior according to class ① and ②, respectively
- Proof of the a priori predictability of zero-stiffness postbuckling behavior