Answers to three not quite straightforward questions in structural stability

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Abstract
In this contribution, the following three questions will be answered by means of both, theoretical proofs and practical examples:

- Are linear prebuckling paths and linear stability problems mutually conditional?
- Does the conversion from imperfection sensitivity into imperfection insensitivity by means of a modification of the original structural design require a symmetric postbuckling path?
- Is hilltop buckling, characterized by the coincidence of a bifurcation point and a snap-through point on a load-displacement path, necessarily imperfection sensitive?

1. Introduction
Despite the long history of structural stability as a field of great scientific relevance and practical importance, it holds a number of questions which so far were not rigorously answered. The reasons for some pieces of the structural stability landscape still being uncharted range from missing mathematical proofs to aspects that are commonly regarded as matters of course, which, at first glance, render thorough proofs dispensable. This is the motivation to ponder in this contribution over the three questions mentioned in the abstract.

They are related to the computation and study of load-displacement paths and, in particular, to loss of stability phenomena, exhibiting either imperfection sensitivity or insensitivity (Mang et al. [4]). After a brief theoretical introduction into the topic, theoretical answers to the posed questions will be given based on mathematical proofs. The lecture will supplement the theory by representative problems which were solved analytically and numerically, respectively.

2. Theoretical foundations
The behavior of a static, conservative system can be deduced from the potential energy function $V(u, \lambda): \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$. The vector $u \in \mathbb{R}^N$ contains the displacement coordinates, implying that the system has $N$ degrees of freedom. The parameter $\lambda \in \mathbb{R}$ is a load multiplier scaling a constant reference load $P \in \mathbb{R}^N$. Therefore, $G(u, \lambda) := V_u = F'(u) - \lambda P$ may be interpreted as an out-of-balance force, which vanishes along any equilibrium path in the $u$-$\lambda$-space. Here, $F'(u) \in \mathbb{R}^N$ is the vector of internal forces.

A crossing point $(u_c, \lambda_c)$ of two equilibrium paths is called a bifurcation point. The equilibrium path containing the unloaded state is the primary or prebuckling path $(\bar{u}(\lambda), \lambda)$, the other one the secondary or postbuckling path.

The differential of $G = 0$, i.e.,

$$K_T \cdot du - d \lambda P = 0,$$

(1)
with the symmetric tangent-stiffness matrix $K_s = V_{uu}(u, \lambda)$, commonly serves as the basis for the solution of nonlinear structural problems by the FEM. Specializing (1) for the primary path, using the notation $\tilde{K}_s(\lambda) = V_{uu}(\tilde{u}(\lambda), \lambda)$, and disregarding, for the time being, snap-through points characterized by $d\lambda = 0$, (1) can be expressed as

$$\tilde{K}_s \cdot \tilde{u}_\lambda - P = 0,$$

(2)

or, after differentiation with respect to $\lambda$, as

$$\tilde{K}_{s,\lambda} \cdot \tilde{u}_\lambda + \tilde{K}_s \cdot \tilde{u}_{\lambda\lambda} = 0.$$

(3)

The secondary path is parameterized by a scalar $\eta$, with $\eta = 0$ corresponding to the bifurcation point $(u_c, \lambda_c)$. The displacement offset between the primary and the secondary path is defined by the vector $\eta \in \mathbb{R}^N$. Thus, $u(\eta) = \tilde{u}(\lambda(\eta)) + \nu(\eta)$ describes the displacement along the secondary path. Single-valuedness is guaranteed for $G_0 c : \mathbb{R} \rightarrow \mathbb{R}^N$ but not for $\tilde{u}(\lambda) : \mathbb{R} \rightarrow \mathbb{R}^N$. Frequently, the coordinates are chosen such that $\eta$ is a component of $u$. Insertion of the series expansions

$$\lambda(\eta) = \bar{\lambda}_c + \lambda_\eta + \lambda_\eta^2 + \lambda_\eta^3 + \mathcal{O}(\eta^4)$$

(4)

$$\nu(\eta) = \nu_1 + \nu_\eta + \nu_\eta^2 + \nu_\eta^3 + \mathcal{O}(\eta^4)$$

(5)

into the specialization of $G$ for the secondary path, i.e., $G(\eta) = G(\tilde{u}(\lambda(\eta)) + \nu(\eta), \hat{\lambda}(\eta)) = 0$, yields the new series expansion

$$G(\eta) = G_{0c} + G_{1c} \eta + G_{2c} \eta^2 + \mathcal{O}(\eta^3) = 0$$

(6)

with $G_{nc} = G_{pp}|_{p=0}/n! \forall n \in \mathbb{N}$. Since (6) must hold for arbitrary values of $\eta$, $G_{0c} = 0 \forall n \in \mathbb{N}$. This condition paves the way for successive calculation of the unknowns $\nu_1, \lambda_1, \nu_2, \lambda_2,$ etc. To render this calculation unique, the length of $v_1$ has to be chosen (not equal to zero) and the orthogonality condition $v_i \cdot v_j = 0 \forall i > 1$, suggested in Budiansky [2] can be materialized.

3. Are linear prebuckling paths and linear stability problems mutually conditional?

A primary path is linear if

$$\tilde{u}_\lambda = k = \text{const.} \quad \forall \lambda \in \mathbb{R}.$$

(7)

Thus, $\tilde{u}(\lambda) = \tilde{u}(0) + \lambda k$ with a constant (non-zero) vector $k$. A stability problem is considered as linear if the tangent stiffness matrix specialized for the primary path can be written as

$$\tilde{K}_s = K_0 + \lambda K_1,$$

(8)

with constant matrices $K_0$ and $K_1$ (cf. Zienkiewicz and Taylor [7]). Provided that the unloaded state $(\lambda = 0)$ is stable, $K_0$ is positive definite. $K_1 = K_1^T$ may be any constant non-zero matrix. Consequently, $\det(\tilde{K}_s(\lambda)) = 0$, i.e., the condition for loss of stability, is a scalar algebraic equation in $\lambda$, which facilitates the computation of the critical load level $\bar{\lambda}_c$.

3.1 A linear prebuckling path is not sufficient for a linear stability problem

Utilization of (7) in (3) yields

$$\tilde{K}_{s,\lambda} \cdot k = 0 \quad \forall \lambda \in \mathbb{R}.$$

(9)

Clearly, for any value of $\lambda, k$ is a zero eigenvector of $\tilde{K}_{s,\lambda}$. Yet, this is not sufficient for (8), i.e., (7) $\neq$ (8).

3.2 A linear stability problem is not sufficient for a linear prebuckling path

Substitution of (8) into (2) shows that for a linear stability problem, $\tilde{u}$ is defined by the differential equation

$$(K_0 + \lambda K_1) \cdot \tilde{u}_\lambda - P = 0.$$

(10)
The existence of an appropriate vector $k$ such that (7) is a solution of (10) is not guaranteed. Thus, (8) \( \Rightarrow \) (7), which is explained in more detail in Steinboeck and Mang [6]. Moreover, it follows from (10) that a linear stability problem entails a linear prebuckling path only if in addition $K_1 \ddot{u}_c = 0 \ \forall \lambda$.

4. Does the conversion from imperfection sensitivity into imperfection insensitivity require a symmetric postbuckling path?

To achieve a conversion from imperfection sensitivity into insensitivity, the original structure must be modified. The degree of such a modification may be parameterized by means of a scalar $\kappa$. In many cases, it is of interest to find values of $\kappa$ for which the system is imperfection insensitive.

4.1 Conditions for symmetric load-displacement paths

For a definition of symmetric load-displacement paths, it is reasonable to start out from the potential energy function. As suggested in Steinboeck et al. [5], symmetry requires

$$V(u, \lambda) = V(T(u), \lambda) \ \forall (u, \lambda) \in \mathbb{R}^N \times \mathbb{R},$$

(11)

where the linear mapping $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an element of a symmetry group. Moreover, symmetry of the secondary path with respect to $\eta$ requires

$$V(\ddot{u}(\lambda(\eta)) + v(\eta), \lambda(\eta)) = V(\ddot{u}(\lambda(-\eta)) + v(-\eta), \lambda(-\eta)) \ \forall \eta \in \mathbb{R}.$$  

(12)

Condition (11) must hold along the primary path, and the uniqueness of this path implies $\ddot{u}(\lambda(\eta)) = T(\ddot{u}(\lambda(\eta)))$. Moreover, a comparison of (11) and (12) reveals that $\lambda(\eta) = \lambda(-\eta)$ and $v(\eta) = T(v(-\eta))$. These conditions may be summarized as follows:

A postbuckling path is said to be symmetrical with respect to $\eta$ if it obeys the definitions:

$$\lambda(\eta) = \lambda(-\eta) \ \land$$

(13)

$$v(\eta) = T(v(-\eta)) \ \land$$

(14)

$$\ddot{u}(\lambda(\eta)) = T(\ddot{u}(\lambda(\eta))).$$

(15)

It follows (trivially) from (4) that $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = 0$ is a necessary condition for symmetry.

4.2 Conditions for imperfection insensitivity

According to Bochenek [1], a symmetric load-displacement behavior in the vicinity of $(u_c, \lambda_c)$ and satisfaction of the inequality $\lambda_{\lambda}(\eta) \text{sign}(\eta) \geq 0$ in an open local domain around $(u_c, \lambda_c)$ are necessary and sufficient for imperfection insensitivity. In fact, $\lambda_{\lambda}(\eta) \text{sign}(\eta)$ must not vanish in this local domain except at $(u_c, \lambda_c)$. Therefore, with the help of $m_{\min} = \min\{m | m \in \mathbb{N} \setminus \{0\}, \lambda_m \neq 0\}$, a necessary and sufficient condition for imperfection insensitivity is found as

$$m_{\min} \text{ is even} \ \land \ \lambda_{m_{\min}} > 0.$$  

(16)

If this condition is not satisfied, the system is imperfection sensitive.

4.3 A symmetric postbuckling path is not necessary for the conversion from imperfection sensitivity into imperfection insensitivity

A comparison of (13)-(15) with (16) shows that imperfection insensitivity is independent of (14) and (15). Thus, if the coefficients $\lambda_i$ are computed up to an index $i = m$ such that $\lambda_m \neq 0$, (16) facilitates a decision about imperfection sensitivity or insensitivity. From these considerations it follows that a conversion from imperfection sensitivity into imperfection insensitivity is characterized by a sign reversal of $\lambda_{m_{\min}}$, which does not require symmetry.

Evidently, the choice of $\eta$ is not unique, i.e. $\eta$ may be replaced by means of a bijective coordinate transformation $\eta \mapsto \bar{\eta}(\theta)$ obeying $\bar{\eta}(0) = 0$. As it can be shown, satisfaction of (16) in the original system ensures that $\tilde{m}_{\min}$ is even $\land \tilde{\lambda}_{\tilde{m}_{\min}} > 0$ in the transformed system. As expected from a physical viewpoint, the (intrinsic) property of imperfection insensitivity (sensitivity) is invariant with respect to coordinate changes.
5. Is hilltop buckling necessarily imperfection sensitive?

Hilltop buckling is characterized by the coincidence of a bifurcation point and a snap-through point of a load-displacement path (Fujii and Noguchi [3]). The following gives a brief outline of determining whether the secondary path emerging from a hilltop buckling point is necessarily imperfection sensitive. For simplicity, the discussion is restricted to cases where \( \lambda_i = 0 \), which is a necessary condition for imperfection insensitivity (cf. section 4).

In Mang et al. [4], the pivotal equation \( \lambda_2 = a_2 \lambda_2^2 + b_2 \lambda_2 + d_2 \) describing the relationship between \( \lambda_2 \) and \( \lambda_4 \) in terms of scalar coefficients \( a_2 \), \( b_2 \), and \( d_2 \) is deduced. In fact, if the structure under consideration is modified, a scalar design parameter \( \kappa \) defining the degree of the modification may be introduced (cf. section 4). Hence, the aforementioned equation turns out to be parameter-dependent, and the solution for \( \lambda_2(\kappa) \) is obtained as

\[
\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2a_2(\kappa)} \pm \sqrt{\frac{b_2^2(\kappa) - 4a_2(\kappa)(d_2(\kappa) - \lambda_2(\kappa))}{2a_2(\kappa)}}.
\]

Equation (17) allows distinguishing between two characteristic classes of hilltop buckling problems. For the first class, \( a_2 = -\infty \) and \( b_2 = +\infty \). In the limit, \( b_2/a_2 = 0 \), however the second addend of (17), with \( d_2 - \lambda_2 = +\infty \), is negative in sign. Thus, for this class of problems, all load-displacement paths crossing the hilltop buckling point are imperfection sensitive.

The second class of hilltop buckling problems belongs to a category of buckling problems characterized by a vanishing discriminant for any value of \( \kappa \), i.e. \( b_2^2(\kappa) - 4a_2(\kappa)(d_2(\kappa) - \lambda_2(\kappa)) = 0 \). Hence, \( \lambda_2(\kappa)_{1,2} = -b_2(\kappa)/(2a_2(\kappa)) \). For this class, \( a_2 = -\infty \), \( b_2 = -\infty \). In the limit, \( -b_2/a_2 \) is negative in sign. Thus, as for the first class of problems, all load-displacement paths crossing the hilltop buckling point are imperfection sensitive.

For both classes of hilltop buckling, \( -\infty < \lambda_2(\kappa) < 0 \) \( \forall \kappa \). Departing from hilltop buckling by increasing the value of \( \kappa \) it is seen that \( \kappa \rightarrow +\infty \) corresponds, for the first class, with \( \lambda_2(\kappa) \rightarrow +\infty \) and \( \lambda_4(\kappa) \rightarrow +\infty \), whereas, for the second class, with \( \lambda_2(\kappa) \rightarrow +\infty \) and \( \lambda_4(\kappa) \rightarrow -\infty \), indicating a worse quality of asymptotic initial postbuckling behavior, generally preceded by a worse quality of transition from imperfection sensitivity to insensitivity.

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References


