Symmetries, charges and conservation laws in General Relativity

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Symmetries provide valuable insights into theories of physics. In recent years, analyses of the symmetries of general relativity – our best tested theory of gravity – have led to remarkable advances in our understanding of the nature of gravity in our universe. It is the goal of this thesis to describe some of these advances. A technical summary of the contents of this work is as follows. In this thesis, the algebra of symmetries and the charges associated with null and spatial boundaries in four-dimensional spacetimes in general relativity are rigorously derived. All reported work herein is in the context of asymptotic boundaries in asymptotically flat spacetimes. In addition, previously conjectured "matching" relations between the symmetries and charges associated with past and future null infinity in asymptotically flat spacetimes are proven. Moreover, proposals for extensions of the algebra of asymptotic symmetries are also studied. Finally, to analyze the implications of asymptotic symmetries and the associated charges for classical scattering processes, a detailed study of the low energy dynamics of a classical (complex) scalar field coupled to electromagnetism on Minkowski spacetime is also included.

Biographical sketch

Ibrahim Shehzad received his Bachelor's degree in physics from the Lahore University of Management Sciences (LUMS) in Lahore, Pakistan, in June 2014. From July 2014 to June 2015, he worked as a research assistant at the physics department at LUMS. From August 2015 to June 2016, he attended the Perimeter Scholars International (Master's) program at the Perimeter institute for theoretical physics in Waterloo, Canada. He started his Ph.D. in theoretical physics at Cornell University in August 2016 and completed his dissertation in May 2022.

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I dedicate this thesis to my parents and grandmother, my relatives across Pakistan, in Cranbury township, Dallas and Toronto and, most of all, to the life and work of Professor Abdus Salam — my inspiration for wanting to become a physicist.

"The creation of Physics is the shared heritage of all mankind" - Professor Abdus Salam.

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Contents

Chapter 1

Introduction

General relativity is governed by a set of coupled, non-linear, second-order partial differential equations. Solving these equations analytically in generic settings is an arduous challenge and, in most cases, outright impossible.[1] An idea from a first mechanics course however suggests a possible work-around. Recall that the speed of a non-relativistic, frictionless ball (of a certain mass at a certain height relative to its initial position, after it has been released with a certain speed) does not necessarily need to be calculated by solving the second-order differential equation that governs its motion (i.e Newton's second law). It can in fact just be inferred from conservation of energy which is related to the time translation symmetry of the system. The correspondence between symmetries and conservation laws continues to similarly hold in general relativity and a detailed analysis of the symmetries of a gravitational system can be used to deduce valuable information about it by appealing to the corresponding conservation laws. A typical example of this would be a black hole whose mass or electric charge at any time can be deduced from knowledge of the values of these quantities when the black hole formed and the amount that has been radiated out by some later time.[2] These conservation laws also serve as important checks for numerical solutions (which is often the best one can do) of the field equations.

It is the goal of this thesis to analyze what the symmetries of a general relativistic gravitational

[1] At least given the existing methods for solving partial differential equations.

[2] While this could sound like an intuitive statement, proofs that statements to this effect hold in general relativity spanned years of work and were only recently completed in [7, 9–11]; see also chapter 5.

system are and what conservation laws these correspond to. A particular focus of our interest will be special kinds of gravitational systems, namely, *isolated* gravitational systems. Examples of such systems include the exteriors of stars, galaxies and merging black holes in our universe when studied in a setting where the influence of the rest of the universe is to be ignored. Such a setting is, for example, ideal for studying a burst of gravitational wave radiation in relation to the black hole merger that produced it. The notion of *asymptotic flatness* was developed by Penrose [12, 13] together with Geroch [14], Arnowitt, Deser and Misner [15], and Ashtekar and Hansen [16, 17], precisely to model such systems. This is why the framework of asymptotic flatness is used, for example, in all numerical simulations of cosmic events like merging black holes (see, e.g, [18]). Motivated by the large class of physically interesting systems in our universe that are modeled by the notion of asymptotic flatness, we will exclusively focus on asymptotically flat spacetimes (to be precisely defined in chapter 2 and chapter 3) and rigorously analyze their symmetries and conservation laws.

What makes an analysis of symmetries in general relativity particularly important is that understanding these symmetries can provide insights into quantum gravity, a thus far elusive theory which remains the holy grail of modern day theoretical physics. It was realized by Ashtekar [19–21] that a promising avenue towards quantum gravity in asymptotically flat spacetimes is to quantize the radiative modes of general relativity on the null asymptotic boundaries (or *null infinities*) of asymptotically flat spacetimes. This means that the group of symmetries defined on these boundaries provides a natural first guess for the symmetry group of the quantum theory. Promising evidence that this guess is indeed correct has in fact recently been provided in the work of Strominger [22, 23]. It has further been argued [24, 25] that the structure of the symmetry algebra on the aforementioned boundaries points to the existence of a conformal field theory that lives entirely on (cross-sections of) these boundaries (and is referred to as a *celestial* conformal field theory) as a fully quantum, holographic description of the bulk gravitational spacetime (see, e.g, [26] for a review of this topic). While there is still work to do to fully thresh out these hypotheses and prove them, they appear to be extremely promising. Their value underscores the need for a good understanding of the symmetry structure of general relativity in order to get a handle on quantum gravity.

The study of symmetries in general relativity first acquired center stage as an area of the research in the 1960's when the work of Bondi, van der Berg, and Metzner [27], and Sachs [28], aimed at rigorously analyzing the "physicality" of gravitational waves led to a somewhat surprising conclusion. They found that as one recedes infinitely far away in null directions from an (isolated) gravitating source into a "weak" gravity regime, the symmetry group of this system does *not* reduce to the Poincaré group, as one would have intuitively expected. In fact, in addition to the translations and Lorentz transformations that comprise the Poincaré group, one obtains infinitely-many more symmetries which (in an appropriate frame) correspond to arbitrarily angle-dependent translations, or *supertranslations*. It was therefore found that the symmetry group of general relativity at (each) null infinity in asymptotically flat spacetimes is an infinite dimensional group which is now referred to as the Bondi-Metzner-Sachs (BMS) group.[3] Building on this, in recent years, there has been a surge in activity surrounding the long-distance (or "infrared") dynamics of general relativity (and, more generally, of gauge theories). As a result of this activity, a remarkable web of relations has been discovered between three seemingly unrelated areas of infrared physics, namely: (i) soft theorems, (ii) symmetries associated with asymptotic boundaries[4] (or *asymptotic symmetries*) and (iii) memory effects (see e.g [21, 23] for recent reviews of this subject and a complete list of references to related papers). Soft theorems govern the universal properties of scattering amplitudes in gauge and gravitational theories in the limit where the energy of an external massless particle is taken to zero. These are known to exist not just for gravitons in general relativity [29, 30], but also for photons in electromagnetism [29, 31, 32], gluons in non-abelian Yang-Mills theory [33] and even photinos [34] in supersymmetric gauge theories. In a series of seminal papers by Strominger et al. (see [23]), these soft theorems were shown to be equivalent to the conservations laws associated with asymptotic symmetries. Fourier transforms of these soft theorems were in-turn shown to be related to memory effects which are physical effects that encode or "remember" the effect of an asymptotic symmetry transformation. An example of a memory effect is the *displacement memory effect* where a gravitational wave (whose integrated action on asymptotic data is that of an asymptotic symmetry transformation) passing between two freely falling test masses results in a permanent shift in the displacements of the masses relative to their initial displacements. Aspects of this triangular web of

[3] A detailed discussion of definition, structure and other properties of this group will be given in chapter 2.

[4] As we will elaborate below, the symmetries discussed in this thesis are *all* tied to the boundaries of the manifold on which the theory is studied.

relations have also been extended to higher dimensional spacetimes and various new kinds of soft theorems, asymptotic symmetry transformations and memory effects as new avenues continue to be explored in this rich subject (see, e.g, [35–37]).

Perhaps the most intriguing proposal to come out of this line of research in recent years has been for black hole hair. It has been argued [38–40] that black holes possess (infinitely many) *soft* (or zero-energy) "hair," in addition to their mass, angular momentum and charge. It has further been argued that the existence of these additional soft hair can be used to recover information from black holes and thus circumvent the black hole information paradox [41]. This hypothesis has led to (and in some cases borrowed from) proposals for an enlarged asymptotic symmetry algebra in asymptotically flat spacetimes. It has been argued that the asymptotic structure of asymptotically flat spacetimes in fact admits a larger symmetry algebra than the BMS algebra and that it is the charges (to be defined below) associated with these extended symmetries that endow black hole with soft hair (see, e.g Appendix D of [42] and Sec. 7.2 of [23]). Two such proposals have been to extend the infinitesimal Lorentz transformations in the BMS algebra to (i) all infinitesimal local conformal Killing transformations (also called *Virasoro* transformations) [43] and (ii) all smooth diffeomorphisms of the 2-sphere [44, 45]. While a lot of recent work has been dedicated to these two (and indeed in recent years other [46]) extensions, the question of whether these are actually genuine extensions with a well-defined physical meaning associated with them is still somewhat up for debate.[5] Nevertheless, it is clear that research on asymptotic symmetries in general relativity gets right to the heart of some of the most important problems in modern day theoretical physics.

* * *

An important part of this thesis will be calculations of symmetry algebras and their generators on phase space. A mathematically rigorous way of studying symmetries and their corresponding generators is given by the *covariant phase space formalism*, first developed in [47–51]. Since this formalism will be used in multiple places throughout this thesis, we describe it in detail here, following the pedagogical discussion given in [52].

[5] Short of resolving this question, we will discuss the second extension mentioned above at length in chapter 4 of this thesis.

A summary of the covariant phase space formalism

The essential goal of the covariant phase space formalism is to provide a covariant description of the dynamics of a field theory on a manifold without picking a preferred foliation by "constant time" slices. This is in contrast to what is generally done in "non-covariant" phase space analyses of the kind one encounters in the hamiltonian formulation of general relativity which involves making a choice of time slicing. In the covariant phase space formalism, one works with the entire set of solutions to the field equations. This set has the structure of a pre-phase space on which one can define a pre-symplectic structure as we will discuss below. Quotienting out the degeneracies of the pre-symplectic form then gives us the "true" symplectic form and the resulting space then has the structure of a "true" phase space on which Poisson brackets are well-defined. In the ensuing analysis in this thesis (unless otherwise stated), we will drop the prefix "pre-" while assuming that the aforementioned reduction procedure can be and has been carried out on our space of solutions to arrive at the "true" phase space. It is worth pointing out that in a globally hyperbolic spacetime, all solutions to the field equations can be identified, up to gauge transformations, with their respective initial data on a Cauchy slice and this identification is what provides a canonical map between the covariant phase space and the aforementioned "non-covariant" phase space.

To demonstrate how the covariant phase space formalism works, consider a field theory on a four-dimensional manifold. Use Φ to collectively denote the dynamical fields in the field theory and δ to denote a variation on phase space such that for any Φ which solves the equation of motion, $\delta\Phi$ solves the linearized equations of motion around the background field configuration given by Φ. We then have the following relation

$$\delta\mathbf{L}(\Phi) = E(\Phi)\delta\Phi + d\boldsymbol{\theta}(\delta\Phi, \Phi)\,, \tag{1.0.1}$$

where $\mathbf{L} := \mathcal{L}\,\hat{\epsilon}_4$, $\hat{\epsilon}_4$ is the spacetime volume form, $\mathcal{L} = \mathcal{L}(\Phi)$ is a choice of the Lagrangian density of the field theory,[6] and $E(\Phi)$ denotes the equation of motion. $\boldsymbol{\theta}(\delta\Phi, \Phi)$ is referred to as the *symplectic*

[6] The words "choice of" here have been added to emphasize that the Lagrangian of a field theory is not uniquely determined and can in fact be shifted by an exact form without effecting the equations of motion.

potential and is a 3-form on spacetime. Given a symplectic potential, taking a second, independent variation of it and anti-symmetrizing yields the *symplectic current*

$$\boldsymbol{\omega}(\Phi, \delta_1\Phi, \delta_2\Phi) := \delta_1\boldsymbol{\theta}(\Phi, \delta_2\Phi) - \delta_2\boldsymbol{\theta}(\Phi, \delta_1\Phi)\,, \tag{1.0.2}$$

which is an antisymmetric, bilinear functional of two field variations and a 3-form on spacetime. On shell, $d\boldsymbol{\omega}(\Phi, \delta_1\Phi, \delta_2\Phi) = 0$ where d represents the exterior derivative. This can be seen as follows

$$\begin{aligned} d\boldsymbol{\omega}(\Phi, \delta_1\Phi, \delta_2\Phi) &= \delta_1 d\boldsymbol{\theta}(\Phi, \delta_2\Phi) - \delta_2 d\boldsymbol{\theta}(\Phi, \delta_1\Phi) \\ &= \delta_1(\delta_2 L(\Phi) - E(\Phi)\delta_2\Phi) - \delta_2(\delta_1 L(\Phi) - E(\Phi)\delta_1\Phi) \\ &= -\delta_1 E(\Phi)\delta_2\Phi + \delta_2 E(\Phi)\delta_1\Phi = 0\,, \end{aligned} \tag{1.0.3}$$

where in addition to Eq. (1.0.1) and Eq. (1.0.2), we have used the fact that δ and d commute and in the last line, we have used the fact that both terms vanish identically since all $\delta\Phi$ satisfy the linearized equations of motion. Given a symplectic current, we define the *symplectic form*, $\Omega_{\mathcal{AB}}$, for some tensor indices $\mathcal{A}$, $\mathcal{B}$, on phase space, which is given by

$$\Omega_{\mathcal{AB}}(\delta_2\Phi)^{\mathcal{A}}(\delta_1\Phi)^{\mathcal{B}} = \int_\Sigma \boldsymbol{\omega}(\Phi, \delta_1\Phi, \delta_2\Phi)\,. \tag{1.0.4}$$

Here, Σ is a 3-surface in the spacetime which is usually taken to be a Cauchy surface. It is also often useful to consider the integral of the symplectic current, as above, over non-Cauchy, 3-surfaces and in which case $\Omega_{\mathcal{AB}}$ does not define the symplectic form of the full theory but can loosely be interpreted as the symplectic form of a sub-phase space.

Once we have a non-degenerate symplectic form after quotienting out its degeneracies, it can then be inverted to define Poisson brackets as follows. For any two functionals, $F(\Phi)$ and $G(\Phi)$ on phase space, the Poisson bracket of $F(\Phi)$ and $G(\Phi)$ (written with their arguments suppressed below) is defined by

$$\{F, G\}_{\text{PB}} := \Omega^{\mathcal{AB}} \frac{\delta F}{(\delta\Phi)^{\mathcal{A}}} \frac{\delta G}{(\delta\Phi)^{\mathcal{B}}}\,, \tag{1.0.5}$$

where the inverse of the symplectic form, $\Omega^{\mathcal{AB}}$, satisfies $\Omega_{\mathcal{AB}}\Omega^{\mathcal{BC}}\Omega_{\mathcal{CD}} = \Omega_{\mathcal{AD}}$ and the subscript "PB" stands for "Poisson bracket." Consider now the definition of the generator of a transformation on phase space that maps one solution to another. The generator, H for such a transformation is defined by the following relation

$$\{H, \Phi^A\}_{\mathrm{PB}} = (\delta_H \Phi)^{\mathcal{A}} = \Omega^{\mathcal{BC}} \frac{\delta H}{(\delta\Phi)^{\mathcal{B}}} \frac{(\delta\Phi)^{\mathcal{A}}}{(\delta\Phi)^{\mathcal{C}}} = \Omega^{\mathcal{BA}} \frac{\delta H}{(\delta\Phi)^{\mathcal{B}}} \,, \tag{1.0.6}$$

where the subscript H in δ_H denotes the fact that this is the transformation corresponding to the generator H. Contracting $\Omega_{\mathcal{AC}}$ into the right hand side of the above equation then implies that

$$\frac{\delta H}{(\delta\Phi)^{\mathcal{C}}} = \Omega_{\mathcal{AC}} (\delta_H \Phi)^{\mathcal{A}} \,, \tag{1.0.7}$$

from which we see that

$$\delta H = \Omega_{\mathcal{AB}} (\delta_H \Phi)^{\mathcal{A}} (\delta\Phi)^{\mathcal{B}} = \int_\Sigma \boldsymbol{\omega}(\Phi, \delta\Phi, \delta_H \Phi) \,. \tag{1.0.8}$$

This is an important expression used for calculating the generator of a transformation on phase space given a symplectic current, and one that we will come back to repeatedly in this thesis. On shell, for gauge transformations in gauge theories and diffeomorphisms in any diffeomorphism covariant theory of gravity (we will see this specifically for general relativity in chapters 2 and 3), $\boldsymbol{\omega}(\Phi, \delta\Phi, \delta_\zeta \Phi)$ is a total derivative, where ζ may parametrize the action of a gauge transformation or a diffeomorphism. As a result, the integral for the corresponding generator localizes to the boundaries of the spacetime manifold. This means that gauge transformations or diffeomorphisms that have vanishing support on the boundaries of the spacetime manifold do not generate a transformation of fields on phase space while ones that have non-trivial support on the boundary can. Such gauge transformations or diffeomorphisms would be "physical" in the sense that they map physically distinct solutions of the field equations onto each other. An important point on terminology now. Diffeomorphisms that have this property and which preserve some suitable structure on the boundary (this structure depends on the nature of the boundary and will be specified on a case-by-case basis in chapter 2 and chapter 3 for each of the boundaries considered therein) are what we will mean by *symmetries* throughout this thesis. When the boundary in question happens to be an asymptotic boundary, which will be the cases throughout this thesis, these are called *asymptotic symmetries*. The corresponding generators in each of these scenarios are referred to as the associated *charges*. Note that in situations when there is a flux of some kind or gravitational radiation, we will find that "charges" are actually not defined by Eq. (1.0.8). We will clarify exactly what we mean by "charges" in these cases as and when this issue comes up but roughly speaking, the intuition "charge" $\sim$ "generator of symmetry" will always hold.

* * *

Below, we give a brief summary of the main questions addressed in this thesis.

A summary of the questions addressed in this thesis

What is the asymptotic symmetry algebra of vacuum general relativity in asymptotically flat spacetimes and what are the associated charges?

Although the charges associated with the BMS algebra have been at the heart of the recent developments mentioned above, until recently, there had not been a systematic derivation of these expressions that made their covariance properties manifest and one that could be used as a reference point to rule out differing charge expressions that have appeared in the literature. Therefore, to put the aforementioned developments on a more solid footing, in chapter 2, building on the work of Wald and Zoupas [53], we provide this derivation using the covariant phase space formalism. This work is based on [4]. As alluded to above, there has been a lot of recent interest in exploring extensions of the BMS algebra as well, partly because it is the charges associated with these extended symmetries that have been conjectured to endow black holes with additional soft hair. In particular, it has been argued that the duality between gravitational charges and soft theorems requires the Lorentz symmetries in the BMS algebra to be extended to arbitrary diffeomorphisms on the 2-sphere. In chapter 4 (which is based on [6]), by proving that this particular extension corresponds to divergent charges that cannot be renormalized within the usual formulation of the covariant phase space formalism, it is shown that one would need to augment covariant phase space methods to incorporate this extension in a consistent manner. Although not discussed in this thesis, some follow-up work on this question was recently done in [54] where we developed a general framework that encompasses the aforementioned and various other recently proposed extensions of the BMS algebra (in particular one where the BMS algebra is extended to match the algebra of symmetries on any finite null surface which was derived in [55]). We showed that a technique called *holographic renormalization* can always be used to cure any divergences in the associated charges and therefore enables these extensions to be incorporated in a consistent manner (although requiring the introduction of extra background structures that break some level of covariance).

How do asymptotic symmetries constrain gravitational scattering?

For massless fields in asymptotically flat spacetimes, past and future null infinity are initial and final data surfaces respectively and scattering is a statement about how initial data at past null infinity evolves to final data at future null infinity. It has been an important recent question to address how asymptotic symmetries at past and future null infinity constrain scattering. It was in fact conjectured by Strominger that the BMS asymptotic symmetries and their associated charges on past and future null infinity match in the limit to spatial infinity, giving rise to infinitely many flux conservation laws that govern classical scattering in GR. This conjecture was proven for the supertranslation symmetries that form a subalgebra of the BMS algebra, for a class of asymptotically flat spacetimes, in a seminal paper by Prabhu [10]. In chapter 5, this proof is extended to the full BMS algebra thereby showing the existence of an infinite number of conservation laws that constrain classical gravitational scattering in a class of asymptotically flat spacetimes. This work is founded on a covariant phase space based analysis of asymptotic symmetries (called "Spi" symmetries) at spatial infinity and the most general expressions for the associated charges which are discussed in chapter 3. Chapter 5 is based on [7] while chapter 3 is based on [5].

Do soft charges constrain scattering in gauge theories?

While recent work on extensions of the BMS algebra has led to proposals for soft charges for black holes, the role of soft degrees of freedom in constraining scattering in gauge theories and gravity has also been called into question. In particular, it has been claimed (in, e.g, [56]) that soft degrees of freedom generically decouple from the scattering of all "hard" degrees of freedom and therefore do not constrain, for example, Hawking radiation emitted from black holes. In chapter 6 which is based on [8], we study these claims in a theory with a charged massless scalar field coupled to electromagnetism in 4-dimensional Minkowski spacetime. Our explicit calculations show that the mutually decoupled evolution of the soft and hard degrees of freedom is a feature of scattering only at low orders in the coupling constant and that at high orders in the coupling constant, the soft and hard sectors of this theory *do* mix under evolution. We therefore conclude that soft charges non-trivially constrain scattering of the hard degrees of freedom of the theory. Given the similarity between gravity and non-linear gauge theories, we expect this feature to also hold in general relativity. This would then remove an important obstruction to the role of soft charges in constraining gravitational scattering and indeed black hole evaporation.

A range of results that supplement the discussion in the main body of the thesis are included in the appendices.

Other work

Over the course of my Ph.D., I also worked on a couple of other projects that are not included in this thesis. These projects and the questions they sought to address are summarized below.

What do gravitational charges in AdS imply for quantum information via the AdS/CFT duality?
Gravitational charges in Anti de-Sitter (AdS) spacetimes have also been an important recent area of study and have lead to important insights on quantum information theoretic quantities via the AdS/CFT duality. In this regard in [57], we gave evidence for a holographic duality between gravitational charges in AdS and a quantum information theoretic quantity called the "Rényi refined relative entropy." We showed that properties of this quantity imply certain positive energy theorems in a class of AdS spacetimes.

Is there a stress-tensor that gives gravitational charges on null boundaries?
An alternative to covariant phase space methods for computing charges associated with gravitational

systems was developed by Brown and York [58] and it has so far predominantly been used to compute charges for timelike boundaries. This method involves computing the variation of the on-shell action with respect to boundary data to obtain an expression for the so-called Brown-York stress tensor. It is natural to ask how this construction generalizes to null boundaries. One reason why this is particularly interesting is that a stress tensor for null infinity in asymptotically flat spacetimes can be compared with the stress tensor of the celestial conformal field theory [59] to check for the existence of a holographic, bulk-boundary correspondence in asymptotically flat spacetimes in the same spirit as the AdS/CFT correspondence. The derivation of the Brown-York stress tensor for null boundaries and a comparison of the corresponding charges with covariant phase space charges was the main subject of [60].

Notations and conventions

Throughout this thesis, we follow the conventions of Wald [61] for the metric signature, Riemann tensor, and differential forms, and use abstract index notation with Latin letters from the beginning of the alphabet $a, b, c, \ldots$ to denote spacetime tensor indices and those from the middle of the alphabet $i, j, k, \ldots$ for tensors defined on a timelike or null bounding hypersurface. Quantities defined on the physical spacetime are denoted by a "hat," while the ones on the conformally-completed, unphysical spacetime are denoted without a "hat" (e.g., $\hat{g}_{ab}$ denotes the physical metric while g_{ab} denotes the unphysical metric). In addition, we use "$\hat{=}$" to denote equality at null infinity (denoted by $\mathscr{I}$) and (unless stated otherwise) $\underleftarrow{}$ to denote the pullback to null infinity. We also use both the indexed and index-free notations for differential forms with "$\equiv$" used to translate between indexed and index-free notation, writing, e.g., $\boldsymbol{\varepsilon}_4 \equiv \varepsilon_{abcd}$. Since "$\equiv$" is used for this translation, we use ":=" for definitions. Spatial directions at spatial infinity (denoted by i^0) are denoted by $\vec{\eta}$. In chapter 3, regular direction-dependent limits of tensor fields, which we denote to be $C^{>-1}$, are represented by a boldface symbol e.g. $\boldsymbol{C}_{abcd}(\vec{\eta})$ is the limit of the (rescaled) unphysical Weyl tensor along spatial directions at i^0.

We emphasize that unless explicitly stated otherwise, all results in this thesis will be in the context of vacuum general relativity and should not be assumed to extend to other theories of gravity.

Chapter 2

The Wald-Zoupas prescription for asymptotic charges at null infinity

(Adapted with permission from [4])

Chapter summary

In this chapter, we use the formalism developed by Wald and Zoupas to derive formulae for the charges and fluxes associated with Bondi-Metzner-Sachs symmetries at null infinity in asymptotically flat spacetimes. Generalizing older expressions in the literature, these expressions hold in radiating, non-stationary regions of null infinity, are local and covariant, conformally-invariant, and are independent of the choice of foliation of null infinity and of the chosen extension of the symmetries away from null infinity (both of which are a priori arbitrary choices). Our covariant expressions can be used to obtain charge formulae in any choice of coordinates at null infinity and, in particular, can be compared with expressions in the literature that are written in Bondi-Sachs and conformal Gaußian null coordinates. We also include comparisons of our expressions with other expressions for the charges and fluxes that have appeared in the literature: the Ashtekar-Streubel flux formula, the Komar formulae and the linkage charge, and Penrose's twistor charge formulae. Such comparisons are easier to perform using our explicit expressions, instead of those which appear in the original work by Wald and Zoupas.

2.1 | Context

We consider 4-dimensional asymptotically flat spacetimes defined in a coordinate independent manner through Penrose's conformal completion (see Def. 2.2.1). In this picture, null infinity[1] of the (physical) spacetime is represented by a smooth null surface, $\mathscr{I}$, in the conformally-completed (unphysical) spacetime. This null surface has the topology of $\mathscr{I} \cong \mathbb{R} \times \mathbb{S}^2$ where $\mathbb{R}$ represents the null directions and $\mathbb{S}^2$ the angular directions at infinity. It follows from the definition of asymptotic flatness that there is a *universal* structure associated with $\mathscr{I}$ (see Sec. 2.3). This structure is universal in the sense that it is independent of which asymptotically flat physical spacetime is considered and thus provides a "fixed background" at infinity which is common to all physical spacetimes. As we will show in detail later in this chapter, the generators of diffeomorphisms (i.e. vector fields) which preserve this universal structure form the BMS algebra (see Sec. 2.4).

Like in classical mechanics on flat manifolds, one wishes to define certain "conserved quantities" (similar to mass, energy or angular momentum) associated with any given physical spacetime and an asymptotic symmetry. In generic spacetimes in general relativity, such quantities will not be conserved due to the presence of gravitational radiation. However, one can do the following: let $S \cong \mathbb{S}^2$ be some cross-section (or "cut") of null infinity $\mathscr{I}$; the choice of S represents an "instant of time" on $\mathscr{I}$. Then, for a vector field ξ^a representing an asymptotic symmetry and on every cross-section S, we define a *charge* $\mathcal{Q}[\xi; S]$ which represents the value of the "not really conserved quantity" at that "instant of time". The change in this charge with "time", i.e. between two cross-sections, is then a *flux* $\mathcal{F}[\xi; \Delta\mathscr{I}]$ where $\Delta\mathscr{I}$ is the region of null infinity between the two cross-sections. Without a set of physical criteria, one could make up arbitrary expressions for such charge and flux formulae. Even if following the intuition from classical mechanics, one requires that any notion of charge associated with time translations and rotations coincide with the mass and angular momentum usually defined in Kerr spacetimes, it is not clear how to generalize the charge expressions to all BMS symmetries and to non-stationary spacetimes.

[1] While asymptotically flat spacetimes have a past and a future null infinity, the distinction between the two will not be important in this chapter and so we will just use null infinity (or $\mathscr{I}$) to refer to either one of them.

To get started on this problem, we list a set of criteria which all physically reasonable expressions for charges associated with asymptotic symmetries (or "asymptotic charges") and fluxes should satisfy: (1) Any charge or flux expression one defines must be independent of any choice of coordinates; for us this will be guaranteed since we will work directly with the covariant formulation of null infinity mentioned above, without fixing any coordinates. (2) The charges and fluxes should be associated with the physical spacetime and not with the choice of conformal-completion used to obtain the unphysical conformally-completed spacetime, or with any additional structure used to compute these quantities. Thus, we require that the charges and fluxes be independent of the conformal factor used to obtain the Penrose conformal-completion. Below, we will also use a foliation of $\mathscr{I}$ to simplify our computations. We will therefore also demand that the charges and fluxes be independent of this additional choice. (3) The charges and fluxes should be associated with the BMS symmetries at $\mathscr{I}$ and so we demand that they be independent of any arbitrary extension of these asymptotic symmetries into the spacetime (these extensions are considered "pure gauge" and are discussed in Sec. 2.4.1). (4) The charges and fluxes must be local and covariant in the following sense: the value of the charge $\mathcal{Q}[\xi; S]$ must be obtained as an integral over S of a 2-form which is constructed from the available fields and the BMS symmetry, and finitely-many of their derivatives at S. The flux $\mathcal{F}[\xi; \Delta\mathscr{I}]$ must be the integral over $\Delta\mathscr{I}$ of a 3-form constructed in a similar way from the fields and symmetry in the region $\Delta\mathscr{I}$. (5) Finally, since we want the fluxes $\mathcal{F}[\xi; \Delta\mathscr{I}]$ to characterize physical, dynamical processes like gravitational radiation, we also require that the flux associated with *any* BMS symmetry between *any* two cross-sections vanishes when the physical spacetime is stationary.

A prescription for obtaining asymptotic charges and fluxes, that satisfy these criteria, in any local and covariant Lagrangian-derived theory of gravity was given by Wald and Zoupas [53]. We will detail this procedure in Sec. 2.5 for our case of interest, namely, asymptotically flat spacetimes in vacuum general relativity. Using this prescription, we will compute the formulae for the charges and fluxes, written explicitly in terms of fields defined on null infinity, in full generality. We call these the "Wald-Zoupas (WZ) charge" and "Wald-Zoupas (WZ) flux" respectively. To simplify our computations, we will start by choosing an arbitrary but fixed foliation of null infinity. Later on however, we will show that our formulae are independent of any choice of foliation. This computation can be done easily for the flux since the WZ flux expression is written entirely in terms of quantities at null infinity without a dependence of any auxiliary structure. The computation for the charge, however, is more complicated: unlike the flux, a piece of the expression for the WZ charge involves the limit of the integral of a quantity that is defined in a neighborhood of null infinity. To compute this limit, we will make use of Bondi coordinates, much like [42]. However, we will show that the value of this limit is independent of the choices that are made in using Bondi coordinates, such as the choice of foliation, conformal factor, and extension of the BMS vector field off of null infinity. As a result, we can then convert the value of the limit back into a covariant form, allowing us to write an expression for the WZ charge (see Eq. (2.5.44)) in terms of covariant quantities defined on null infinity. Since this procedure is somewhat involved, we will then check explicitly that the change of the charge is consistent with the flux formula (in Eq. (2.5.33)). These explicit formulae can then be easily compared with the expressions one would obtain in *any* coordinate system; see e.g. Sec. 2.6. These expressions can also be compared with other expressions for BMS charges, some of which we will detail in Appendix A.2.

The rest of this chapter is organized as follows. We start by reviewing the definition of asymptotic-flatness at null infinity in Sec. 2.2 and summarize the various asymptotic fields arising at null infinity that we will use in subsequent computations. In Sec. 2.3, we discuss the universal structure at null infinity and the restrictions on metric perturbations which follow from the definition of asymptotic flatness. In Sec. 2.4, we review the BMS algebra and its relevant properties. In Sec. 2.5, we describe the Wald-Zoupas prescription for obtaining the charges and fluxes associated with BMS symmetries and obtain manifestly covariant expressions for these quantities. Finally, we discuss the construction of Bondi-Sachs and conformal Gaußian null coordinate frames near null infinity in Sec. 2.6 and express the BMS symmetries and the WZ charge in these coordinates. We end by discussing some future directions in Sec. 2.7. While in we work in a fixed conformal frame (where the Bondi condition (see Eq. (2.2.6)) holds) in this chapter, we include a summary of our main results in more general conformal frames in Appendix A.1. In Appendix A.2, we compare the WZ charge and flux formulae to some of the other charge and flux formulae that have been proposed for BMS symmetries, namely, the Ashtekar-Streubel flux formula, the Komar charge formulae and their linkage versions and

Penrose's charge formula. Appendix A.3 collects some useful results on symmetric tracefree tensors on a 2-sphere.

2.2 Asymptotic-flatness at null infinity

In this section, we recall the covariant definition of asymptotic flatness and define the asymptotic fields and their equations at null infinity that will be used later in our analysis.

Definition 2.2.1 (Asymptotic flatness)**.** A *physical* spacetime $(\hat{M}, \hat{g}_{ab})$, which satisfies the vacuum Einstein equation $\hat{G}_{ab} = 0$, is asymptotically flat at null infinity if there exists an *unphysical* spacetime (M, g_{ab}) with a boundary $\mathscr{I} = \partial M$ and an embedding of $\hat{M}$ into M (we use this embedding to identify $\hat{M}$ as a submanifold of M), such that

(1) There exists a smooth function Ω (the *conformal factor*) on M satisfying $\Omega \hat{=} 0$ and $\nabla_a \Omega \hat{\neq} 0$ such that $g_{ab} = \Omega^2 \hat{g}_{ab}$ is smooth on M including at $\mathscr{I}$.

(2) $\mathscr{I}$ is topologically $\mathbb{R} \times \mathbb{S}^2$.

(3) Defining $n_a := \nabla_a \Omega$, the vector field $\omega^{-1} n^a$ is complete on $\mathscr{I}$ for any smooth function ω on M such that $\omega > 0$ on M and $\nabla_a(\omega^4 n^a) \hat{=} 0$.

Detailed expositions on the motivations for this definition may be found in [61, 62]. The differentiability conditions on the unphysical spacetime can be significantly weakened, but we restrict to the smooth case for simplicity.

Using the conformal transformation relating the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see, e.g, Appendix D of [61]), the vacuum Einstein equation can be written as

$$S_{ab} = -2\Omega^{-1} \nabla_{(a} n_{b)} + \Omega^{-2} n^c n_c g_{ab} , \tag{2.2.1}$$

where S_{ab} is given by

$$S_{ab} = R_{ab} - \tfrac{1}{6} R g_{ab}. \tag{2.2.2}$$

It follows from Eq. (2.2.1) and the smoothness of Ω and the unphysical metric g_{ab} at $\mathscr{I}$ that $n_a n^a \mathrel{\widehat{=}} 0$. This implies that $\mathscr{I}$ is a smooth null hypersurface in M with normal $n_a = \nabla_a \Omega$ and that the vector field $n^a = g^{ab} n_b$ is a null geodesic generator of $\mathscr{I}$.

Further, the Bianchi identity $\nabla_{[a} R_{bc]de} = 0$ on the unphysical Riemann tensor along with Eq. (2.2.1) gives the following equations for the unphysical Weyl tensor C_{abcd} (see [62] for details)

$$\nabla_{[a}(\Omega^{-1} C_{bc]de}) = 0\,. \tag{2.2.3a}$$

$$\nabla^d C_{abcd} = -\nabla_{[a} S_{b]c}\,. \tag{2.2.3b}$$

Note that given a physical spacetime $(\hat{M}, \hat{g}_{ab})$ there is freedom in constructing its conformal-completion to obtain the unphysical spacetime (M, g_{ab}), in particular, in making a choice of the conformal factor Ω. Note that for the same physical spacetime, any other choice of the conformal factor $\Omega' = \omega\Omega$, with $\omega|_{\mathscr{I}} > 0$ also satisfies Def. 2.2.1 with a different unphysical metric $g'_{ab} = \Omega'^2 \hat{g}_{ab} = \omega^2 g_{ab}$. Since we are interested in studying the asymptotic properties of the physical spacetime — the conformal-completion is used only to bring the asymptotic boundary "at infinity" of the physical spacetime to a finite boundary $\mathscr{I}$ in the unphysical spacetime — all physical quantities (such as symmetries, charges and fluxes) must be independent of the choice of conformal factor.

All of our computations which follow can be done using an arbitrary conformal factor; however it is convenient to fix some of the conformal freedom by imposing the *Bondi condition* as follows. On $\mathscr{I}$, let Φ be defined by

$$\Phi := \tfrac{1}{4} \nabla_a n^a \big|_{\mathscr{I}}\,. \tag{2.2.4}$$

Under a change of conformal factor $\Omega \mapsto \omega\Omega$ and $g_{ab} \mapsto \omega^2 g_{ab}$, we have

$$\Phi \mapsto \omega^{-1}(\Phi + \pounds_n \ln \omega)\,. \tag{2.2.5}$$

Now, without loss of generality, we can choose ω to be a solution of $\Phi + \pounds_n \ln\omega \;\hat{=}\; 0$ to set $\Phi \;\hat{=}\; 0$ [61, 62].[2] In this choice of the conformal factor, Eq. (2.2.1) implies the *Bondi condition*

$$\nabla_a n_b \;\hat{=}\; 0, \tag{2.2.6}$$

and

$$n_a n^a = O(\Omega^2). \tag{2.2.7}$$

We will henceforth work in a conformal frame where the Bondi condition holds. Having imposed the Bondi condition, the remaining freedom in the conformal factor is of the form $\Omega \mapsto \omega\Omega$ where

$$\omega|_{\mathscr{I}} > 0\,, \quad \pounds_n \omega \;\hat{=}\; 0. \tag{2.2.8}$$

We reiterate that the choice to work in a conformal frame where the Bondi condition holds is made purely for convenience and is not essential for the calculations in this chapter. The main results of this chapter, expressed in general conformal frames where the Bondi condition is not imposed, are included in Appendix A.1. In this chapter, our statements about conformal invariance will pertain to conformal transformations that satisfy Eq. (2.2.8). More general conformal transformations for which $\pounds_n \omega \;\hat{\neq}\; 0$ will only be considered in Appendix A.1.

Finally, let q_{ab} denote the pullback of g_{ab} to $\mathscr{I}$. This defines a degenerate metric on $\mathscr{I}$ such that

$$q_{ab} n^b \;\hat{=}\; 0\,, \quad \pounds_n q_{ab} \;\hat{=}\; 0, \tag{2.2.9}$$

where the second condition follows from Eq. (2.2.6). Thus, q_{ab} defines a Riemannian metric on the space of generators of $\mathscr{I}$ which is diffeomorphic to $\mathbb{S}^2$.

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[2] Note that while this condition can always be imposed on $\mathscr{I}$, it cannot be imposed at spatial infinity where solutions, ω, to the equation $\Phi + \pounds_n \ln\omega \;\hat{=}\; 0$ diverge [61].

1 | Auxiliary foliation and null normal at $\mathscr{I}$

For carrying out explicit computations on $\mathscr{I}$, it is convenient to pick an "auxiliary" structure on $\mathscr{I}$. This is given by a choice of foliation of $\mathscr{I}$ by a one-parameter family of *cross-sections* which are diffeomorphic to $\mathbb{S}^2$. Note that such a foliation always exists since $\mathscr{I} \cong \mathbb{R} \times \mathbb{S}^2$ although there is no unique choice for this foliation. The results in this chapter can be obtained without reference to any choice of foliation but it is simpler to start with a fixed foliation and then verify that the results are independent of this choice.

For any choice of foliation, we obtain a 1-form l_a on $\mathscr{I}$ which is normal to each cross-section of the foliation. Then there exists a *unique* vector field l^a defined at $\mathscr{I}$ such that $l_a \mathrel{\widehat{=}} g_{ab} l^b$ is the normal to the chosen foliation and

$$l^a l_a \mathrel{\widehat{=}} 0\,, \quad l^a n_a \mathrel{\widehat{=}} -1. \tag{2.2.10}$$

We will call this vector field the *auxiliary normal* associated with the foliation. Note that the vector field l^a can be extended arbitrarily away from $\mathscr{I}$ and our computations will be (as they should be) independent of which extension is chosen.

Using the auxiliary null normal l^a, we now define a tensor Q_{ab} at $\mathscr{I}$ by

$$Q_{ab} := g_{ab} + 2 l_{(a} n_{b)}. \tag{2.2.11}$$

Note that $q_{ab} = \underleftarrow{Q}_{ab}$, and so Q_{ab} is a choice of pushforward of the intrinsic (degenerate) metric on $\mathscr{I}$. By the definition of Q_{ab}, we have

$$Q_{ab} l^b \mathrel{\widehat{=}} 0, \qquad Q_{ab} n^b \mathrel{\widehat{=}} 0. \tag{2.2.12}$$

For later use, we also define the symmetric trace-free part of a tensor with respect to Q_{ab} by

$$\text{STF}\, A_{ab} := \left(Q_{(a}{}^c Q_{b)}{}^d - \tfrac{1}{2} Q_{ab} Q^{cd} \right) A_{cd}. \tag{2.2.13}$$

Next, we define a volume form on $\mathscr{I}$ and on the cross-sections of the chosen foliation by

$$\varepsilon_{abc} := l^d \varepsilon_{dabc} \,, \quad \varepsilon_{ab} := \varepsilon_{abcd} l^c n^d \mathrel{\hat{=}} \varepsilon_{abc} n^c. \tag{2.2.14}$$

Remark 2.2.1 (Orientation conventions). In this chapter, the orientation of these volume forms are such that $\mathscr{I}$ is outward-facing as a manifold in M and the cross-sections have a future orientation within $\mathscr{I}$. Note that this convention for the orientations for $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{\varepsilon}_3$ is somewhat unexpected. In Minkowski spacetime, setting $l_a = -\nabla_a u$ and $n_a = \nabla_a \Omega$ (with $\Omega = 1/r$ and $u = t - r$), we have that $\boldsymbol{\varepsilon}_3 = -\sin\theta \, du \wedge d\theta \wedge d\phi$ and $\boldsymbol{\varepsilon}_2 = -\sin\theta \, d\theta \wedge d\phi$, which is the opposite of the "expected" sign for $\boldsymbol{\varepsilon}_2$. Despite this sign difference, we still have that

$$\int_S \boldsymbol{\varepsilon}_2 > 0; \tag{2.2.15}$$

which, in fact, is the *definition* of an orientation. In Minkowski, this basically means that the bounds of integration are in the "reversed" order (say, the θ integral is from π to 0, instead of 0 to π). Finally, note that in our choice of orientation, if we consider a cross-section S as a limit of spheres S' within a spacelike hypersurface Σ in the unphysical spacetime, then the orientation for S' that is compatible with that of S is the *inward*-facing one within Σ.

The *shear* of the auxiliary normal on $\mathscr{I}$ is defined by

$$\sigma_{ab} \mathrel{\hat{=}} \mathrm{STF}\, \nabla_a l_b \mathrel{\hat{=}} \tfrac{1}{2}\, \mathrm{STF}\, \pounds_l g_{ab} \,. \tag{2.2.16}$$

The twist $\varepsilon^{ab} \nabla_a l_b$ vanishes on $\mathscr{I}$ since l_a is normal to the cross-sections of a foliation of $\mathscr{I}$. For the most part, we will not need the expansion of l_a in our analysis (it is introduced in Sec. 2.6.2 where it is needed).

The change in l_a along the null generators of $\mathscr{I}$ is given by

$$\tau_a \mathrel{\hat{=}} Q_a{}^b n^c \nabla_c l_b. \tag{2.2.17}$$

By the normalization conditions for l^a and n^a and the Bondi condition, we have

$$\tau_a \hat{=} n^b \nabla_b l_a \hat{=} \pounds_n l_a, \tag{2.2.18}$$

On $\mathscr{I}$, we also can define a "sphere derivative" along the cross-sections for any tensor field that is orthogonal to l^a and n^a on all indices:

$$\mathscr{D}_a T^{b_1 \cdots b_r}{}_{c_1 \cdots c_s} := Q^d{}_a Q^{b_1}{}_{e_1} \cdots Q^{b_r}{}_{e_r} Q^{f_1}{}_{c_1} \cdots Q^{f_s}{}_{c_s} \nabla_d T^{e_1 \cdots e_r}{}_{f_1 \cdots f_s}. \tag{2.2.19}$$

One can show that this derivative operator is compatible with both Q_{ab} and ε_{ab} on $\mathscr{I}$, that is

$$\mathscr{D}_a Q_{bc} \hat{=} 0, \qquad \mathscr{D}_a \varepsilon_{bc} \hat{=} 0. \tag{2.2.20}$$

With our choice of $\boldsymbol{\varepsilon}_3$ and $\boldsymbol{\varepsilon}_2$, we can derive several useful formulas from Stokes' theorem. Consider some portion $\Delta\mathscr{I}$ of $\mathscr{I}$ whose boundary is given by two cross-sections S_1 and S_2 in our foliation, with S_2 to the future of S_1. Then, for any scalar α we have

$$\int_{\Delta\mathscr{I}} \boldsymbol{\varepsilon}_3 \ \pounds_n \alpha = \int_{S_2} \boldsymbol{\varepsilon}_2 \ \alpha - \int_{S_1} \boldsymbol{\varepsilon}_2 \ \alpha \,. \tag{2.2.21}$$

Also, for any vector v^a which is orthogonal to both n_a and l_a we have

$$\int_{\Delta\mathscr{I}} \boldsymbol{\varepsilon}_3 \ \nabla_a v^a \hat{=} \int_{\Delta\mathscr{I}} \boldsymbol{\varepsilon}_3 \ (\mathscr{D}_a + \tau_a) v^a \hat{=} 0, \tag{2.2.22}$$

where the second expression is obtained using the Bondi condition along with Eqs. (2.2.11) and (2.2.18), and the vanishing of the expression follows from the fact that $\boldsymbol{\varepsilon}_3 \nabla_a v^a$ is an exact 3-form and $l_a v^a \hat{=} 0$. Similarly, for any cross-section S of $\mathscr{I}$, we also have

$$\int_S \boldsymbol{\varepsilon}_2 \ \mathscr{D}_a v^a \hat{=} 0 \,. \tag{2.2.23}$$

The final quantities that we will need are components of the Weyl tensor at $\mathscr{I}$. By the peeling theorem, we have $C_{abcd} \hat{=} 0$ at $\mathscr{I}$, and thus $\Omega^{-1} C_{abcd}$ admits a limit to $\mathscr{I}$ (see Theorem 11 of [62]).

In a choice of the foliation of $\mathscr{I}$, we define the Weyl tensor fields

$$\mathcal{R}_{ab} := (\Omega^{-1}C_{cdef})Q_a{}^c n^d Q_b{}^e n^f, \qquad \mathcal{S}_a := (\Omega^{-1}C_{cdef})l^c n^d Q_a{}^e n^f, \tag{2.2.24a}$$

$$\mathcal{P} := (\Omega^{-1}C_{cdef})l^c n^d l^e n^f, \qquad \mathcal{P}^* := \tfrac{1}{2}(\Omega^{-1}C_{cdef})l^c n^d \varepsilon^{ef}, \tag{2.2.24b}$$

$$\mathcal{J}_a := (\Omega^{-1}C_{cdef})n^c l^d Q_a{}^e l^f, \qquad \mathcal{I}_{ab} := (\Omega^{-1}C_{cdef})Q_a{}^c l^d Q_b{}^e l^f. \tag{2.2.24c}$$

Note that due to the symmetries of the Weyl tensor, $\mathcal{R}_{ab}$ and $\mathcal{I}_{ab}$ are symmetric and traceless.

For the fields defined in Eq. (2.2.24), Eq. (2.2.3a) implies the following evolution equations along $\mathscr{I}$, which can be verified to be conformally-invariant:

$$\pounds_n \mathcal{S}_a = (\mathscr{D}^b + \tau^b)\mathcal{R}_{ab}, \tag{2.2.25a}$$

$$\pounds_n \mathcal{P} \;= (\mathscr{D}^a + 2\tau^a)\mathcal{S}_a - \sigma^{ab}\mathcal{R}_{ab}, \tag{2.2.25b}$$

$$\pounds_n \mathcal{P}^* = -\varepsilon^{ab}(\mathscr{D}_a + 2\tau_a)\mathcal{S}_b + \varepsilon_b{}^c\sigma^{ab}\mathcal{R}_{ac}, \tag{2.2.25c}$$

$$\pounds_n \mathcal{J}_a = \tfrac{1}{2}(\mathscr{D}_b + 3\tau_b)(Q_a{}^b\mathcal{P} - \varepsilon_a{}^b\mathcal{P}^*) - 2\sigma_a{}^b\mathcal{S}_b, \tag{2.2.25d}$$

$$\pounds_n \mathcal{I}_{ab} = \mathrm{STF}(\mathscr{D}_a + 4\tau_a)\mathcal{J}_b - \tfrac{3}{2}\sigma_{ac}(Q_b{}^c\mathcal{P} - \varepsilon_b{}^c\mathcal{P}^*). \tag{2.2.25e}$$

Next, we list the conformal weights of the various quantities defined so far. Any quantity α is said to have *conformal weight* w if, under $\Omega \mapsto \omega\Omega$, it transforms as $\alpha \mapsto \omega^w\alpha$. Clearly, have that

$$g_{ab} : w = 2, \qquad \varepsilon_{abcd} : w = 4. \tag{2.2.26}$$

Next, from

$$n_a \mapsto \omega n_a + \Omega\nabla_a\omega, \tag{2.2.27}$$

and $\Omega \mathrel{\hat{=}} 0$ on $\mathscr{I}$, we have that $n_a : w = 1$ on $\mathscr{I}$, and so

$$l_a : w = 1, \qquad (Q_{ab}, \varepsilon_{ab}) : w = 2, \qquad \varepsilon_{abc} : w = 3. \tag{2.2.28}$$

Moreover, we have that

$$\sigma_{ab} : w = 1, \tag{2.2.29}$$

while τ_a transforms as

$$\tau_a \mapsto \tau_a + \mathscr{D}_a \ln \omega \,, \tag{2.2.30}$$

under a conformal transformation and therefore does not have well defined conformal weight. Finally, the fields constructed from the Weyl tensor have the following conformal weights:

$$(\mathcal{R}_{ab}, \mathcal{I}_{ab}) : w = -1 \,, \quad (\mathcal{S}_a, \mathcal{J}_a) : w = -2 \,, \quad (\mathcal{P}, \mathcal{P}^*) : w = -3. \tag{2.2.31}$$

We remark that there is a particularly convenient choice of conformal factor and foliation, namely one where q_{ab} is given by the unit 2-sphere metric, and $\tau^a \mathrel{\hat{=}} 0$. This is the *Bondi frame* and, in particular, it is a further restriction of the remaining conformal freedom $\Omega \mapsto \omega\Omega$ after imposing the Bondi condition. Even though a Bondi frame can always be achieved by a conformal transformation and a change of foliation, we do not enforce a Bondi frame since we want to consider charges on arbitrary cross-sections of $\mathscr{I}$ (and fluxes between these cross-sections) and want our expressions to be manifestly conformally invariant (apart from the enforcement of the Bondi condition which, as mentioned earlier, will be relaxed in Appendix A.1).

2 News tensor

In this subsection, we define the *News tensor*, which characterizes the radiative degrees of freedom of the gravitational field at null infinity. This tensor can either be defined in terms of a foliation, or invariantly from the universal structure that exists on $\mathscr{I}$; we review both definitions and show that they are equivalent.

We define the News tensor as the (projected) Lie derivative along n^a of the shear σ_{ab}:

$$N_{ab} := 2 Q_a{}^c Q_b{}^d \pounds_n \sigma_{cd} = 2 \, \mathrm{STF} \, \pounds_n \sigma_{ab} \,. \tag{2.2.32}$$

Note that it follows from the above definition that

$$N_{ab} n^b \mathrel{\hat{=}} g^{ab} N_{ab} \mathrel{\hat{=}} Q^{ab} N_{ab} \mathrel{\hat{=}} 0 \,. \tag{2.2.33}$$

It is straightforward to verify that the News is conformally invariant.

We can also write N_{ab} in terms of S_{ab} as follows. Consider the quantity STF $\pounds_n \pounds_l g_{ab}$. Despite involving covariant derivatives of l_a, when evaluated at $\mathscr{I}$, this quantity is independent of the choice of l_a away from $\mathscr{I}$. By choosing a particular extension of l_a, one can show from Eqs. (2.2.16) and (2.2.18) that

$$\mathrm{STF}\, \pounds_n \pounds_l g_{ab} \,\hat{=}\, 2\,\mathrm{STF}(\pounds_n \sigma_{ab} - \tau_a \tau_b) \,\hat{=}\, N_{ab} - 2\,\mathrm{STF}(\tau_a \tau_b), \tag{2.2.34}$$

where the second equality follows from Eq. (2.2.32). Using $[\pounds_n, \pounds_l] = \pounds_{[n,l]}$, together with Eq. (2.2.18), we therefore find that

$$N_{ab} \,\hat{=}\, \mathrm{STF}[\pounds_l \pounds_n g_{ab} + 2(\mathscr{D}_a + \tau_a)\tau_b]. \tag{2.2.35}$$

As a result, using Eq. (2.2.1), we have

$$N_{ab} \,\hat{=}\, \mathrm{STF}[S_{ab} + 2(\mathscr{D}_a + \tau_a)\tau_b]. \tag{2.2.36}$$

Further, from Eqs. (2.2.3b) and (2.2.36), the News is related to the Weyl tensor components (defined in Eq. (2.2.24)) by

$$Q_a{}^c Q_b{}^d \pounds_n N_{cd} \,\hat{=}\, 2\mathcal{R}_{ab}\,, \quad \mathscr{D}^b N_{ab} \,\hat{=}\, 2\mathcal{S}_a\,, \tag{2.2.37}$$

and

$$\mathcal{P}^* = \varepsilon^{ab}\left[\mathscr{D}_a(\mathscr{D}_c - \tau_c)\sigma_b{}^c - \tfrac{1}{2}N_{ac}\sigma_b{}^c\right]. \tag{2.2.38}$$

As defined in Eq. (2.2.32), it is not obvious that the News tensor is independent of our choice of foliation. However, we show below that this definition of the News coincides with the covariant definition given by Geroch [62] which makes no reference to any foliation of $\mathscr{I}$. While most of the literature exclusively uses only one of these two definitions, we will find it convenient to use either interchangeably, and (since there does not appear to be any proof that we could find), we will now show that these two definitions yield the same tensor.

Consider the *Geroch News tensor* defined in [62] as

$$N_{ab} := \underset{\leftarrow}{S_{ab}} - \rho_{ab}, \tag{2.2.39}$$

where $\underset{\leftarrow}{S_{ab}}$ denotes the pullback of S_{ab} to $\mathscr{I}$ and ρ_{ab} is the *unique* symmetric tensor field on $\mathscr{I}$ constructed from the intrinsic universal structure on $\mathscr{I}$, defined in Theorem 5 of [62].

First, note that the Geroch News tensor is conformally invariant, and satisfies the conditions Eq. (2.2.33) (see [62]). Moreover, projections of Eq. 68 of [62] show that Eq. (2.2.37) also holds for the Geroch News tensor. Thus, if λ_{ab} is the difference of the Geroch News tensor and the one defined in Eq. (2.2.32), then λ_{ab} is a symmetric tensor field on $\mathscr{I}$ which satisfies

$$\lambda_{ab}n^b \mathrel{\hat{=}} Q^{ab}\lambda_{ab} \mathrel{\hat{=}} Q_a{}^c Q_b{}^d \pounds_n \lambda_{cd} \mathrel{\hat{=}} \mathscr{D}^b \lambda_{ab} \mathrel{\hat{=}} 0\,. \tag{2.2.40}$$

Thus, λ_{ab} is a tensor field on S that is symmetric, traceless and divergence-free, and therefore vanishes by Prop. A.3.2. Consequently, the News tensor defined in Eq. (2.2.32) is equivalent to the covariant definition by Geroch.

Finally, we review a key property of the News tensor, namely, that it characterizes the presence of gravitational radiation in a spacetime. This can be deduced from the following result which is due to Geroch (see pp. 53-54 of [62]): consider an asymptotically flat spacetime $(\hat{M}, \hat{g}_{ab})$, and some portion $\Delta\mathscr{I}$ of null infinity. If $\Delta\mathscr{I}$ is asymptotically stationary, in the sense that there exists a vector field t^a in a neighborhood of $\Delta\mathscr{I}$ that is a timelike Killing vector field with respect to $\hat{g}_{ab}$, then $N_{ab} \mathrel{\hat{=}} 0$ on $\Delta\mathscr{I}$. Since any notion of gravitational radiation should vanish in asymptotically stationary regions of null infinity, this motivates the use of the News tensor as an indicator of the presence of radiation. It should be noted, however, that it is not known whether the converse of this statement is true, namely, whether all regions $\Delta\mathscr{I}$ where the News tensor vanishes are asymptotically stationary.

2.3 | Universal structure and metric perturbations

In this section, we summarize the *universal structure* at null infinity. This is the structure that is common to the conformal completion of *all* spacetimes that satisfy the definition of asymptotic flatness given in Def. 2.2.1 and is thus independent of the specific physical spacetime under consideration.

If (M, g_{ab}, Ω) and (M', g'_{ab}, Ω') are the unphysical spacetimes corresponding to *any* two asymptotically-flat physical spacetimes then, a priori, M and M' are distinct manifolds each with their own boundary $\mathscr{I}$ and $\mathscr{I}'$. However, we argue below that there exists a smooth diffeomorphism from a neighbourhood of $\mathscr{I}$ in M to a neighbourhood of $\mathscr{I}'$ in M' which can be used to identify these unphysical spacetimes (in this neighbourhood) and which maps $\mathscr{I}$ to $\mathscr{I}'$. Since we are only interested in the asymptotic properties near null infinity, we can then work with just one manifold M and one null boundary $\mathscr{I}$ to represent null infinity for any two (and thus, all) asymptotically-flat spacetimes. Further, *without any loss of generality*, this diffeomorphism can also be chosen so that $\Omega' = \Omega$ in a neighborhood of $\mathscr{I}$ and $g'_{ab} \mathrel{\hat{=}} g_{ab}$.

The aforementioned identification can be made by setting up a suitable, geometrically-defined coordinate system in a neighborhood of $\mathscr{I}$ and identifying the two unphysical spacetimes in these coordinates, as we now explain (see also the argument on p. 22 in [62]). On the null infinity of any asymptotically-flat spacetime, we define a parameter u along the null generators of $\mathscr{I}$ such that $n^a \nabla_a u \mathrel{\hat{=}} 1$. We then pick some cross-section $S_0 \cong \mathbb{S}^2$ with constant $u = u_0$. On S_0 define a coordinate system x^A (with $A = 1, 2$), and extend these coordinates to all of $\mathscr{I}$ by parallel transport along n^a:

$$n^a \nabla_a x^A \mathrel{\hat{=}} 0 \,. \tag{2.3.1}$$

This gives us a coordinate system (u, x^A) on $\mathscr{I}$. Next, since $\Omega \mathrel{\hat{=}} 0$ and $n_a \mathrel{\hat{=}} \nabla_a \Omega \mathrel{\hat{\neq}} 0$, we can use Ω as a coordinate transverse to $\mathscr{I}$. As discussed above, there is considerable freedom in the choice of the conformal factor at $\mathscr{I}$ which we need to fix to specify this coordinate. We pick the conformal factor so that the Bondi condition (Eq. (2.2.6)) is satisfied, which still leaves us freedom to change the conformal factor on the cross-sections of $\mathscr{I}$ or according to Eq. (2.2.8). To fix this freedom we proceed as follows. Consider the induced metric q_{ab} on the cross-section S_0 chosen above. It follows from the *uniformization theorem* (for instance see Ch. 8 of [63]) that any metric on S_0 is conformal to the unit round metric on $\mathbb{S}^2$ (that is, the metric with the Ricci scalar equaling 2). Thus, we can always choose the conformal factor so that the metric q_{ab} on this cross-section S_0 is also the unit round metric on $\mathbb{S}^2$ and from Eq. (2.2.9), this holds on any cross-section. Thus, we choose as our transverse coordinate the choice of conformal factor Ω which satisfies the Bondi condition and makes the metric on the cross-sections of $\mathscr{I}$ the unit round metric on $\mathbb{S}^2$. This completely fixes the freedom in the choice of Ω on $\mathscr{I}$. One can then choose to extend Ω, u and x^A into a neigborhood of $\mathscr{I}$ in the unphysical spacetime (using, e.g, the geometric construction that leads to the conformal Bondi-Sachs coordinate system, discussed in 2.6). This gives us a coordinate system (Ω, u, x^A) near $\mathscr{I}$ for *any* asymptotically flat spacetime. Since this construction can be done for any asymptotically flat spacetime, we can, without any loss of generality, identify the null infinities of all asymptotically flat spacetimes by identifying their points in these coordinates. This shows that there exists a diffeomorphism between the null infinities of different spacetimes such that we can identify their boundaries with one "abstract" manifold $\mathscr{I}$ and also that we can choose the same conformal factor Ω for each of them.

Next, we show that the unphysical metric at $\mathscr{I}$ can be chosen to be the same for the conformal completion of any physical spacetime. Consider a foliation of $\mathscr{I}$ by cross-sections S_u of constant u. Then, by Eq. (2.3.1) the null generator n^a, in these coordinates, can be written as $n^a \mathrel{\widehat{=}} \partial/\partial u$. The 1-form on $\mathscr{I}$ normal to this foliation is given by $l_a \mathrel{\widehat{=}} -\nabla_a u$. Using $n_a = \nabla_a \Omega$, and $n^a l_a \mathrel{\widehat{=}} -1$, we obtain the following expression for the line element of the unphysical metric on $\mathscr{I}$ in the coordinate system (Ω, u, x^A)

$$ds^2 \mathrel{\widehat{=}} 2d\Omega du + s_{AB} dx^A dx^B \,, \tag{2.3.2}$$

where s_{AB} is the unit round metric on $\mathbb{S}^2$ in the chosen coordinates x^A.[3] Note that the form of the unphysical metric Eq. (2.3.2) is completely independent of which physical spacetime is under

[3] The precise choice of coordinates x^A on the cross-sections is irrelevant; one could pick polar coordinates $x^A = (\theta, \phi)$ to put the unit round sphere metric in the standard form $s_{AB} dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$, but any other coordinate system is just as good.

consideration, that is, *all* asymptotically-flat spacetimes have the same universal unphysical metric at null infinity. Different choices of the physical spacetime are only reflected in the unphysical metric *away* from $\mathscr{I}$. Note that the definition of asymptotic-flatness (Def. 2.2.1) includes an embedding map from the physical spacetime manifold $\hat{M}$ into the unphysical manifold M. The existence of the universal structure at $\mathscr{I}$ described above implies that one can embed *any* physical spacetime into an unphysical spacetime by identifying some physical spacetime coordinates with the coordinates (Ω, u, x^A) constructed above so that the unphysical metric g_{ab} takes the form Eq. (2.3.2). In the conformal-completion framework, this corresponds to the statement that "all asymptotically flat spacetimes behave like the Minkowski spacetime to leading order at infinity" and the difference between two physical spacetimes only shows up at "sub-leading order".

Note that since the manifold $\mathscr{I}$ is universal, the choice of the foliation can also be made independently of the physical spacetime. It follows that the auxiliary normal $l_a \hat{=} -\nabla_a u$ is also universal, and (from Eq. (2.3.2)) we have that $l^a \hat{=} -\partial/\partial\Omega$ is also universal.

It is worth pointing out that some of the choices made in constructing the coordinates discussed above were not essential to the argument and were made just for convenience. For instance, the choice of the unit-metric s_{AB} is irrelevant. In *any* asymptotically-flat spacetime, we can instead choose the freedom in the conformal factor at $\mathscr{I}$ so that the induced metric on the cross-sections $q_{ab} = q^{(0)}_{ab}$ where $q^{(0)}_{ab}$ is *any fixed* metric on $\mathbb{S}^2$. Similarly, one could have chosen a different foliation of $\mathscr{I}$ if one wishes. These choices simply correspond to the freedom of choosing the embedding map from the physical spacetime into the unphysical spacetime. Then the rest of the construction proceeds as before and $g_{ab}|_{\mathscr{I}}$ is universal.

1 Metric perturbations near $\mathscr{I}$

Next, we consider linearized perturbations of the physical metric and derive conditions on them that arise from requiring asymptotic flatness. We consider a one-parameter family of physical metrics $\hat{g}_{ab}(\lambda)$ with $\hat{g}_{ab} = \hat{g}_{ab}(\lambda = 0)$ being any chosen background spacetime metric and define the physical metric perturbation by

$$\delta\hat{g}_{ab} := \frac{d}{d\lambda}\hat{g}_{ab}(\lambda)\Big|_{\lambda=0} . \tag{2.3.3}$$

We also use the notation δ to denote perturbations of other quantities defined in a similar way. We emphasize that the δ denotes changes of quantities when the physical metric is varied; the quantities appearing in the universal structure described above including the unphysical metric at $\mathscr{I}$, the conformal factor and the choice of foliation do not vary under δ. This does not mean that the conformal factor and foliation cannot be changed — our expressions for the charges and fluxes are independent of the conformal factor and foliation — it just means that these quantities can always be held fixed when the physical metric is varied.

Taking the one-parameter family of physical metrics $\hat{g}_{ab}(\lambda)$ to all be asymptotically flat, we now consider the behaviour of the metric perturbations at null infinity. Let $g_{ab}(\lambda)$ be the one-parameter family of unphysical metrics obtained by the conformal completion of $\hat{g}_{ab}(\lambda)$. As discussed above, without loss of generality, we can take all the unphysical metrics $g_{ab}(\lambda)$ to be defined on the same manifold M, with a common boundary $\mathscr{I}$ describing null infinity. Further, the conformal factor Ω can also be chosen to be independent of the parameter λ. Thus, we get

$$g_{ab}(\lambda) := \Omega^2\hat{g}_{ab}(\lambda)\,, \quad \delta g_{ab} = \Omega^2\delta\hat{g}_{ab}\,, \tag{2.3.4}$$

where δg_{ab} is the perturbation of the unphysical metric. Moreover, since the unphysical metric at $\mathscr{I}$ is universal, we have that $\delta g_{ab} \mathrel{\hat{=}} 0$ and thus

$$\delta g_{ab} = \Omega\gamma_{ab}, \tag{2.3.5}$$

for some γ_{ab} which is smooth at $\mathscr{I}$. Since the conformal factor is chosen to satisfy the Bondi condition Eq. (2.2.6) in *any* spacetime, varying the Bondi condition we get

$$\delta(\nabla_a n_b) \mathrel{\hat{=}} 0 \implies \gamma_{ab}n^b \mathrel{\hat{=}} 0\,. \tag{2.3.6}$$

Thus, since γ_{ab} is smooth at $\mathscr{I}$, there exists a smooth γ_a such that

$$\gamma_{ab}n^b = \Omega\gamma_a\,. \tag{2.3.7}$$

Hence, the perturbations δg_{ab} of the unphysical metric are given by the tensor fields γ_{ab} and γ_a with

$$\gamma_{ab} = \Omega^{-1}\delta g_{ab}\,, \quad \gamma_a = \Omega^{-2}\delta g_{ab}n^b = \Omega^{-1}\gamma_{ab}n^b\,. \tag{2.3.8}$$

These tensor fields are constrained by the linearized vacuum Einstein equations. A particularly important component of these equations is given by Eq. 68 of [53]

$$\nabla^b\gamma_{ab} - 3\gamma_a - \nabla_a\gamma^b{}_b \mathrel{\widehat{=}} 0. \tag{2.3.9}$$

Further, using the auxiliary foliation of $\mathscr{I}$, we can relate the unphysical metric perturbation characterized by γ_{ab} to the perturbation of the shear σ_{ab} as follows. Varying the definition of the shear (Eq. (2.2.16)), noting that the foliation is kept fixed and that $\delta g_{ab} \mathrel{\widehat{=}} 0$, from Eq. (2.3.8) it is straightforward to compute that

$$\delta\sigma_{ab} = -\tfrac{1}{2}\,\mathrm{STF}\,\gamma_{ab}. \tag{2.3.10}$$

2.4 | Asymptotic symmetries at $\mathscr{I}$: the BMS Lie algebra

Consider a smooth vector field ξ^a in the physical spacetime $\hat{M}$ and let $\hat{g}_{ab}(\lambda)$ be the one-parameter family of physical metrics generated by diffeomorphims along ξ^a. The physical metric perturbation corresponding to this family of metrics is given by

$$\delta_\xi\hat{g}_{ab} = \pounds_\xi\hat{g}_{ab}. \tag{2.4.1}$$

The corresponding perturbation of the unphysical metric is

$$\delta_\xi g_{ab} = \Omega^2\pounds_\xi\hat{g}_{ab} = \pounds_\xi g_{ab} - 2\Omega^{-1}\xi^c n_c g_{ab}. \tag{2.4.2}$$

For ξ^a to be an asymptotic symmetry, the infinitesimal diffeomorphism generated by ξ^a must preserve the universal structure discussed in Sec. 2.3. We now obtain the conditions on ξ^a for it to be an asymptotic symmetry.

Firstly, since the unphysical spacetime M is smooth up to and including $\mathscr{I}$, ξ^a must extend to a smooth vector field on M including at $\mathscr{I}$. Secondly, since the unphysical metric is smooth at $\mathscr{I}$, the perturbation $\delta_\xi g_{ab}$ in Eq. (2.4.2) is also smooth at $\mathscr{I}$. This condition implies that

$$\xi^a n_a \mathrel{\hat{=}} 0. \tag{2.4.3}$$

That is, as expected, an asymptotic symmetry ξ^a must be tangent to $\mathscr{I}$ and thus preserves the asymptotic boundary. For convenience we define

$$\alpha_{(\xi)} := \Omega^{-1}\xi^a n_a\,, \tag{2.4.4}$$

which is smooth at $\mathscr{I}$.

Next, δ_ξ preserves the universal structure, which implies that (Eq. (2.3.8))

$$\gamma^{(\xi)}_{ab} := \Omega^{-1}\delta_\xi g_{ab}\,, \quad \gamma^{(\xi)}_a := \Omega^{-1}\gamma^{(\xi)}_{ab} n^b\,, \tag{2.4.5}$$

must be smooth at $\mathscr{I}$. Using Eqs. (2.4.2) and (2.4.4), the smoothness of $\gamma^{(\xi)}_{ab}$ implies

$$\pounds_\xi g_{ab} \mathrel{\hat{=}} 2\alpha_{(\xi)} g_{ab}, \tag{2.4.6}$$

while the smoothness of $\gamma^{(\xi)}_a$ implies

$$\pounds_\xi n^a \mathrel{\hat{=}} -\alpha_{(\xi)} n^a\,, \quad \pounds_n \alpha_{(\xi)} \mathrel{\hat{=}} 0. \tag{2.4.7}$$

The pullback of Eq. (2.4.6) gives

$$\pounds_\xi q_{ab} \mathrel{\hat{=}} 2\alpha_{(\xi)} q_{ab}. \tag{2.4.8}$$

Intrinsically on $\mathscr{I}$, an asymptotic symmetry is, thus, given by a vector field ξ^a tangent to $\mathscr{I}$ which satisfies Eqs. (2.4.7) and (2.4.8) for some smooth function $\alpha_{(\xi)}$. Such vector fields correspond to the BMS symmetries on null infinity.

In any choice of foliation of $\mathscr{I}$ we can further characterize the BMS symmetries as follows. Since ξ^a is tangent to $\mathscr{I}$, we can write

$$\xi^a \mathrel{\hat{=}} \beta n^a + X^a, \tag{2.4.9}$$

where

$$\beta := -l_a \xi^a, \qquad X^a := Q^a{}_b \xi^b, \tag{2.4.10}$$

We note their conformal weights:

$$\beta : w = 1\,, \quad X^a : w = 0\,. \tag{2.4.11}$$

That is, β is a smooth function of conformal weight 1 and X^a is a smooth vector field tangent to the cross-sections of the chosen foliation on $\mathscr{I}$. The conditions on β and X^a follow from Eqs. (2.4.6) and (2.4.7) as we derive next. Note that Eqs. (2.4.6) and (2.4.7) only depend on the vector field ξ^a at $\mathscr{I}$ and are independent of how this vector field is extended away from $\mathscr{I}$.

The only non-trivial component of Eq. (2.4.6) is given by its projection tangent to the cross-sections of the foliation which gives

$$Q_a{}^c Q_b{}^d \pounds_\xi g_{cd} \mathrel{\hat{=}} 2\mathscr{D}_{(a} X_{b)} \mathrel{\hat{=}} 2\alpha_{(\xi)} Q_{ab}, \tag{2.4.12}$$

which shows both that

$$\text{STF}\, \mathscr{D}_a X_b \mathrel{\hat{=}} 0, \tag{2.4.13}$$

and that

$$\alpha_{(\xi)} \mathrel{\hat{=}} \tfrac{1}{2} \mathscr{D}_a X^a. \tag{2.4.14}$$

Next, we consider Eq. (2.4.7) on $\mathscr{I}$ using Eq. (2.4.9) to get

$$\pounds_\xi n^a = -\pounds_n \xi^a \mathrel{\hat{=}} -n^a \pounds_n \beta - \pounds_n X^a \mathrel{\hat{=}} -\alpha_{(\xi)} n^a. \tag{2.4.15}$$

Projecting along the cross-sections we get

$$Q^a{}_b \pounds_n X^b \mathrel{\hat{=}} 0, \tag{2.4.16}$$

whereas $X^a l_a \mathrel{\hat{=}} 0$, the Bondi condition, and Eq. (2.2.18) imply that

$$l_a \pounds_n X^a \mathrel{\hat{=}} -X_a \pounds_n l^a \mathrel{\hat{=}} -X_a \tau^a, \qquad n_a \pounds_n X^a \mathrel{\hat{=}} 0, \tag{2.4.17}$$

so that

$$\pounds_n X^a \mathrel{\hat{=}} n^a X_b \tau^b. \tag{2.4.18}$$

Contracting Eq. (2.4.15) with l_a, we find that

$$\pounds_n \beta + l_a \pounds_X n^a \mathrel{\hat{=}} \pounds_n \beta - X_a \tau^a \mathrel{\hat{=}} \alpha_{(\xi)}, \tag{2.4.19}$$

where the second equality follows from Eq. (2.4.18). We therefore find that Eq. (2.4.19) becomes

$$\pounds_n \beta \mathrel{\hat{=}} \alpha_{(\xi)} - X_a \tau^a \mathrel{\hat{=}} \tfrac{1}{2}(\mathscr{D}_a - 2\tau_a) X^a. \tag{2.4.20}$$

In summary, in a chosen foliation of $\mathscr{I}$ any BMS symmetry can be written as $\xi^a \mathrel{\hat{=}} \beta n^a + X^a$ where X^a is tangent to the cross-sections of the foliation and the following conditions are satisfied:

$$\pounds_n \beta \mathrel{\hat{=}} \tfrac{1}{2}(\mathscr{D}_a - 2\tau_a) X^a, \tag{2.4.21a}$$

$$\pounds_n X^a \mathrel{\hat{=}} n^a X_b \tau^b, \tag{2.4.21b}$$

$$\text{STF}\, \mathscr{D}_a X_b \mathrel{\hat{=}} 0, \tag{2.4.21c}$$

$$\alpha_{(\xi)} \mathrel{\hat{=}} \tfrac{1}{2} \mathscr{D}_a X^a. \tag{2.4.21d}$$

Note that, using Eqs. (2.2.8), (2.2.27), (2.2.30), (2.4.4) and (2.4.11), one can show that these equations are invariant under conformal transformations that preserve Bondi condition, that is, with $\pounds_n \omega \mathrel{\hat{=}} 0$ (for the form of these equations when Bondi condition does not hold, see Eq. (A.1.7)).

Let $\xi_1^a \mathrel{\hat{=}} \beta_1 n^a + X_1^a$ and $\xi_2^a \mathrel{\hat{=}} \beta_2 n^a + X_2^a$ be two BMS symmetries (with Eq. (2.4.21) holding for each) then their Lie bracket can be computed to give

$$\begin{aligned} &\xi^a \mathrel{\hat{=}} [\xi_1, \xi_2]^a \mathrel{\hat{=}} \beta n^a + X^a \quad \text{with} \\ &\beta \mathrel{\hat{=}} \pounds_{X_1}\beta_2 - \tfrac{1}{2}\beta_2 \mathscr{D}_a X_1^a - (1 \leftrightarrow 2)\,, \quad X^a \mathrel{\hat{=}} \pounds_{X_1} X_2^a\,. \end{aligned} \tag{2.4.22}$$

It can be checked that ξ^a is also a BMS symmetry, i.e. the β and X^a in Eq. (2.4.22) also satisfy the conditions Eq. (2.4.21). Thus, the BMS symmetries form a Lie algebra $\mathfrak{b}$.

The structure of the BMS algebra can be analyzed using Eq. (2.4.22). Consider a BMS symmetry of the form $\xi_1^a \mathrel{\hat{=}} f_1 n^a$ where f_1 is a smooth function on $\mathscr{I}$ satisfying $\pounds_n f_1 \mathrel{\hat{=}} 0$ (from Eq. (2.4.21a)). Then, from Eq. (2.4.22) we see that the Lie bracket of $\xi_1^a \mathrel{\hat{=}} f_1 n^a$ with any other BMS symmetry is also of the form $\xi^a \mathrel{\hat{=}} f n^a$ with $\pounds_n f \mathrel{\hat{=}} 0$, that is, the set of such vector fields is invariant under the Lie bracket. Further, the Lie bracket of any two such symmetries vanishes. Thus, BMS symmetries of the form $f n^a$ form a preferred infinite-dimensional abelian subalgebra $\mathfrak{s}$ which is a *Lie ideal* of the BMS algebra $\mathfrak{b}$ consisting of *supertranslations*. The quotient algebra $\mathfrak{b}/\mathfrak{s}$ can then be parameterized by X^a. From Eq. (2.4.21c) we see that this consists of conformal Killing fields on the cross-sections of $\mathscr{I}$. Since the cross-sections are diffeomorphic to $\mathbb{S}^2$ we get that $\mathfrak{b}/\mathfrak{s}$ is isomorphic to the Lorentz algebra $\mathfrak{so}(1,3)$. Note that the Lorentz algebra is only identified as a quotient algebra — since the Lie bracket of a supertranslation and X^a is a supertranslation, there is no invariant choice of Lorentz subalgebra within the BMS algebra $\mathfrak{b}$. Thus the BMS algebra is the semi-direct sum

$$\mathfrak{b} \cong \mathfrak{s} \ltimes \mathfrak{so}(1,3)\,, \tag{2.4.23}$$

of the Lie ideal $\mathfrak{s}$ of supertranslations with the Lorentz algebra; the $\ltimes$ indicates the non-trivial Lie bracket between the two factors.

There is another finite-dimensional Lie ideal within the BMS algebra given by supertranslations $f n^a \in \mathfrak{s}$ which satisfy the additional condition

$$\mathrm{STF}(\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b) f \mathrel{\hat{=}} 0\,. \tag{2.4.24}$$

It can be checked that this equation is conformally-invariant. Further, the space of solutions to this equation is 4-dimensional and is preserved under the Lie bracket of the BMS algebra (see Remark 2.4.1 below). This 4-dimensional Lie ideal $\mathfrak{t}$ can be viewed as the space of *translations*. In fact, if the physical spacetime $(\hat{M}, \hat{g}_{ab})$ possesses any Killing vectors ξ^a, then they can be extended to $\mathscr{I}$; moreover, if the Killing field is of the form $\xi^a \mathrel{\hat{=}} fn^a$, then ξ^a must be a translation in the sense of f obeying Eq. (2.4.24) (see [64]).

A special case of the translations are those that are given by a time-translation in some conformal frame. A time-translation in a given conformal frame is given by $f = 1$, a definition that is motivated by the behavior of the time-translation vector field in Minkowski (and, similarly, the time-translation vector field in Kerr). Since f transforms by $f \mapsto \omega f$ under a conformal transformation, a conformally-invariant notion of a time-translation is given by $f > 0$, since $\omega > 0$.

Remark 2.4.1 (Characterization of BMS symmetries through spherical harmonics). Consider the Bondi frame — where the conformal factor is chosen so that the metric on the cross-sections is the unit 2-sphere metric and the foliation is chosen so that $\tau_a \mathrel{\hat{=}} 0$. A general supertranslation can then be expanded in spherical harmonics on $\mathbb{S}^2$. The condition Eq. (2.4.24) implies that the translations are spanned by the $\ell = 0, 1$ harmonics which is indeed a 4-dimensional space. The fact that the translations are preserved under Lie brackets can also be shown using some spherical harmonics technology (see Appendix A.9). Note that since supertranslations have conformal weight 1, this spherical harmonic decomposition only holds in the Bondi frame. Similarly, the Lorentz vector fields satisfying

$$\text{STF}\, \mathscr{D}_a X_b \mathrel{\hat{=}} 0\,, \tag{2.4.25}$$

are spanned by vector spherical harmonics with $\ell = 1$ (see, e.g. [42]) which is the 6-dimensional space of the Lorentz algebra $\mathfrak{so}(1,3)$.

Remark 2.4.2 (Supertranslation "ambiguity" in the Lorentz algebra). As noted above, while the supertranslations form a subalgebra of the BMS algebra, there is no preferred Lorentz subalgebra. Instead, the Lorentz algebra arises as the quotient algebra $\mathfrak{b}/\mathfrak{s}$: the set of equivalence classes $[\xi^a]$ of $\mathfrak{b}$, where ξ^a_1 and ξ^a_2 are members of the same equivalence class if $\xi^a_1 - \xi^a_2 \mathrel{\hat{=}} fn^a \in \mathfrak{s}$.

For a given cross-section S, one can consider the algebra $\mathfrak{l}_S$ of BMS vector fields of the form $\xi^a|_S = X^a$ (that is, vector fields tangent to S). This is a subalgebra of $\mathfrak{b}$, and in fact it is straightforward to show that $\mathfrak{l}_S \cong \mathfrak{b}/\mathfrak{s}$. The issue, therefore, is not that there is no Lorentz subalgebra of $\mathfrak{b}$ but that there is an uncountably infinite number of them, one per cross-section of $\mathscr{I}$.

The fact that the members of the quotient algebra $\mathfrak{b}/\mathfrak{s}$ are only defined up to supertranslations is known as the *supertranslation ambiguity*. A similar situation occurs with the Poincaré algebra, which, while possessing a Lie ideal in the form of the algebra of translations, possesses no unique Lorentz subalgebra. For each origin in Minkowski spacetime, there is a Lorentz subalgebra that consists of infinitesimal rotations and boosts that fix that point which is analogous to the Lorentz subalgebra $\mathfrak{l}_S$ associated with some cross-section S of $\mathscr{I}$.

Associated with each cross-section of $\mathscr{I}$, there is a Poincaré subalgebra, much like how there is a Lorentz subalgebra. This is given by the semi-direct sum of the Lorentz subalgebra associated with the cross section with the translation subalgebra picked out by Eq. (2.4.24). This Poincaré subalgebra can be shown to contain any *exact* Killing vector field that might exist in the physical spacetime (see Theorem 1 of [64]). It is important to note there is no invariant notion of a set of charges conjugate to Lorentz vector fields. Such a set of charges only exists for a given cross-section, so that (in particular) one cannot have an invariant notion of the flux of angular momentum between two cross-sections. This is similar to how the notion of angular momentum in flat space is origin-dependent.

Remark 2.4.3 (Extended BMS algebra). As discussed in chapter 1, there have been various recent proposals for extending the BMS algebra. These include proposals to extend the Lorentz quotient algebra, $\mathfrak{b}/\mathfrak{s}$, to the Virasoro algebra [65,66] and to the algebra of all diffeomorphisms of $\mathbb{S}^2$ [37,45,67]. The Virasoro vector fields are however singular at isolated points of $\mathbb{S}^2$ and one would have to modify the definition of asymptotic flatness to incorporate them as symmetries. Similarly, as detailed in Sec. 2.3, it follows from the definition of asymptotic flatness that the conformal class of induced metrics on cross-sections of $\mathscr{I}$ is always universal. As a result, arbitrary diffeomorphisms of $\mathbb{S}^2$ do not arise as symmetries in this context. We will also show in chapter 4 below (see also [6]) that the extension to all diffeomorphisms of $\mathbb{S}^2$ cannot be implemented in a fully covariant manner. In the rest of this chapter, we will not discuss either of these proposals further.

1 **Extensions of BMS symmetries away from $\mathscr{I}$**

Although we started this discussion with vector fields ξ^a that were defined throughout the unphysical spacetime, up to this point, we have mostly worked with their restriction to $\mathscr{I}$. The extension of the BMS symmetries away from $\mathscr{I}$ into the spacetime is arbitrary — one can choose different coordinate systems or gauge conditions to obtain various extensions. As emphasized at the start of this chapter, all physical quantities (like charges and fluxes) associated with the BMS symmetries must be independent of such a choice. Below, we collect some important results on the extensions of BMS symmetries away from $\mathscr{I}$ that will be useful for proving this independence later on.

First, we show that the extension of ξ^a away from $\mathscr{I}$ is determined up to $O(\Omega^2)$:

Proposition 2.4.1 (Equivalent representatives of a BMS symmetry)**.** *If ξ^a and ξ'^a are vector fields in M which represent the same BMS symmetry, i.e., $\xi^a \hat{=} \xi'^a \in \mathfrak{b}$ then $\xi'^a = \xi^a + O(\Omega^2)$.*

Proof. Since $\xi'^a \hat{=} \xi^a$, let $\xi'^a = \xi^a + \Omega Z^a$ from which we obtain

$$\alpha_{(\xi')} - \alpha_{(\xi)} = \Omega^{-1} n_a(\xi'^a - \xi^a) = n_a Z^a \,. \tag{2.4.26}$$

Since $\pounds_\xi g_{ab} - 2\alpha_{(\xi)} g_{ab} \hat{=} 0$ for any BMS vector field, we have

$$0 \hat{=} \left(\pounds_{\xi'} g_{ab} - 2\alpha_{(\xi')} g_{ab}\right) - \left(\pounds_\xi g_{ab} - 2\alpha_{(\xi)} g_{ab}\right) \hat{=} 2 n_{(a} Z_{b)} - 2 n_c Z^c g_{ab} \,. \tag{2.4.27}$$

Taking the trace gives $n_a Z^a \hat{=} 0$. Then we have $n_{(a} Z_{b)} \hat{=} 0$ which implies $Z^a \hat{=} 0$. Thus $Z^a = O(\Omega)$ and $\xi'^a = \xi^a + O(\Omega^2)$. □

The perturbation $\gamma_{ab}^{(\xi)}$ generated by a BMS symmetry $\xi^a|_{\mathscr{I}}$ will, in general, depend on the extension of the symmetry away from $\mathscr{I}$ and hence is *not* well-defined for the BMS symmetries. However, using the lemma above, it can be shown that STF $\gamma_{ab}^{(\xi)}$ on $\mathscr{I}$ is in fact independent of the extension of the BMS symmetry ξ^a away from $\mathscr{I}$ and thus is well-defined for any BMS symmetry.

Corollary 2.4.1. *If ξ^a and ξ'^a are any two extensions of a given BMS symmetry $\xi^a|_{\mathscr{I}}$ away from $\mathscr{I}$ then*

$$\mathrm{STF}\,\gamma_{ab}^{(\xi)} \mathrel{\hat=} \mathrm{STF}\,\gamma_{ab}^{(\xi')}. \tag{2.4.28}$$

Proof. From Prop. 2.4.1, we have $\xi'^a = \xi^a + \Omega^2 W^a$, from which (using Eq. (2.4.4)) we compute

$$\alpha_{(\xi')} = \Omega^{-1} n_a \xi'^a = \Omega^{-1} n_a \xi^a + \Omega n_a W^a = \alpha_{(\xi)} + \Omega n_a W^a\,. \tag{2.4.29}$$

Similarly

$$\pounds_{\xi'} g_{ab} = \pounds_{\xi} g_{ab} + 4\Omega n_{(a} W_{b)} + O(\Omega^2)\,. \tag{2.4.30}$$

Thus, we find that

$$\begin{aligned} \Omega\gamma_{ab}^{(\xi')} &= \pounds_{\xi'} g_{ab} - 2\alpha_{(\xi')} g_{ab} = \pounds_{\xi} g_{ab} - 2\alpha_{(\xi)} g_{ab} + 4\Omega\left(n_{(a}W_{b)} - \tfrac{1}{2} n_c W^c g_{ab}\right) + O(\Omega^2) \\ \implies \gamma_{ab}^{(\xi')} &= \gamma_{ab}^{(\xi)} + 4\left(n_{(a}W_{b)} - \tfrac{1}{2} n_c W^c g_{ab}\right) + O(\Omega)\,. \end{aligned} \tag{2.4.31}$$

Evaluating the STF on $\mathscr{I}$ of both sides of the above equation we find the desired result. □

For later computations it will be useful to have an explicit expression for $\mathrm{STF}\,\gamma_{ab}^{(\xi)}$ at $\mathscr{I}$ for any BMS symmetry ξ^a in terms of the fields defined on $\mathscr{I}$. We show below that

$$\mathrm{STF}\,\gamma_{ab}^{(\xi)} \mathrel{\hat=} -2\,\mathrm{STF}\left[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta + \pounds_X \sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_c X^c)\sigma_{ab}\right]. \tag{2.4.32}$$

Comparing the above formula to Eq. (2.3.10) we find that under a BMS symmetry the shear transforms as

$$\delta_\xi \sigma_{ab} = \mathrm{STF}\left[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta + \pounds_X \sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_c X^c)\sigma_{ab}\right]. \tag{2.4.33}$$

This is the infinitesimal version of the transformation of the shear found by Sachs [28]. It is also the same as that given in e.g. Eq. 2.18b of [42] for the transformation of the shear C_{AB} in Bondi coordinates (which is related to σ_{ab} by Eq. (2.6.10)).

In the remainder of this section we detail the computations which lead to Eq. (2.4.32). Let $\xi^a|_{\mathscr{I}} = \beta n^a + X^a$ be a BMS symmetry on $\mathscr{I}$. In the following, let l^a be *any* vector field in a neighbourhood of $\mathscr{I}$ which coincides with the chosen auxiliary normal at $\mathscr{I}$ — the result of the computation can be checked to be independent of how the auxiliary normal is extended away from $\mathscr{I}$.

As shown above in Cor. 2.4.1, STF $\gamma^{(\xi)}_{ab}|_{\mathscr{I}}$ is independent of how the BMS symmetry $\xi^a|_{\mathscr{I}}$ is extended away from $\mathscr{I}$. Thus, we can choose to extend the BMS symmetry away from $\mathscr{I}$ as follows. We first extend β and X^a away from $\mathscr{I}$ to satisfy

$$\pounds_l \beta \mathrel{\hat{=}} 0, \qquad \pounds_l X^a \mathrel{\hat{=}} 0. \tag{2.4.34}$$

With this choice, any extension of the BMS symmetry $\xi^a|_{\mathscr{I}}$ can be written as

$$\xi^a = \beta n^a + X^a + \Omega Z^a. \tag{2.4.35}$$

where Z^a is some smooth vector field. Next, from Prop. 2.4.1 we see that $Z^a|_{\mathscr{I}}$ can be determined in terms of β and X^a. To do this, first note that, by Eq. (2.4.4),

$$\alpha_{(\xi)} \mathrel{\hat{=}} -\pounds_l(\xi^a n_a) \mathrel{\hat{=}} -n_a \pounds_l \xi^a \mathrel{\hat{=}} n_a Z^a. \tag{2.4.36}$$

Next, we compute $l^b \pounds_\xi g_{ab}$, using Eq. (2.4.6), together with Eq. (2.4.34):

$$l^b \pounds_\xi g_{ab} \mathrel{\hat{=}} -\nabla_a \beta + \pounds_X l_a - Z_a + n_a l^b Z_b \mathrel{\hat{=}} 2\alpha_{(\xi)} l_a. \tag{2.4.37}$$

Contracting this equation with l^a, using Eq. (2.4.34) and $l^a \pounds_X l_a \mathrel{\hat{=}} -l_a \pounds_X l^a \mathrel{\hat{=}} 0$, we find that $l_a Z^a \mathrel{\hat{=}} 0$. Therefore, we can rearrange Eq. (2.4.37) to solve for Z_a:

$$Z_a \mathrel{\hat{=}} -\nabla_a \beta - 2\alpha_{(\xi)} l_a + \pounds_X l_a. \tag{2.4.38}$$

Further, since l_a is the normal and X^a is tangent to the cross-sections of the chosen foliation, we have $Q_a{}^b \pounds_X l_b \mathrel{\hat{=}} 0$. So

$$Z_a \mathrel{\hat{=}} -\mathscr{D}_a\beta + l_a(\pounds_n\beta - 2\alpha_{(\xi)} + l_b\pounds_X n^b) \mathrel{\hat{=}} -\mathscr{D}_a\beta - \alpha_{(\xi)}l_a, \tag{2.4.39}$$

where we have used Eqs. (2.4.21a) and (2.4.21b). In summary, with the choice Eq. (2.4.34), we can write any extension of a given BMS symmetry $\xi^a|_{\mathscr{I}}$ as

$$\xi^a = \beta n^a + X^a - \Omega(\mathscr{D}^a\beta + \alpha_{(\xi)}l^a) + \Omega^2 W^a, \tag{2.4.40}$$

for some smooth W^a.

With this choice of extension of the BMS symmetry, we compute STF $\gamma^{(\xi)}_{ab}|_{\mathscr{I}}$ in terms of β and X^a. Using the expansion in Eq. (2.4.40) we have

$$\begin{aligned}\Omega\gamma^{(\xi)}_{ab} &= \pounds_\xi g_{ab} - 2\alpha_{(\xi)}g_{ab}\\ &= \beta\pounds_n g_{ab} + \pounds_X g_{ab} - 2\alpha_{(\xi)}g_{ab} - 2n_{(a}[\mathscr{D}_{b)}\beta + \alpha_{(\xi)}l_{b)}]\\ &\quad - \Omega[2\nabla_a\nabla_b\beta + \alpha_{(\xi)}\pounds_l g_{ab} + 2l_{(a}\nabla_{b)}\alpha_{(\xi)} - 4n_{(a}W_{b)}] + \Omega^2\pounds_W g_{ab}\end{aligned} \tag{2.4.41}$$

We now want to solve this equation for STF $\gamma^{(\xi)}_{ab}|_{\mathscr{I}}$. This can be done by taking the $\pounds_l$ of the above equation, evaluating on $\mathscr{I}$, and then taking the STF. Using $\pounds_l\Omega \mathrel{\hat{=}} l^a n_a = -1$ and that $\pounds_l n_a \propto n_a$ at $\mathscr{I}$, a long but straightforward computation gives

$$\mathrm{STF}\,\gamma^{(\xi)}_{ab} \mathrel{\hat{=}} -\mathrm{STF}[\beta\pounds_l\pounds_n g_{ab} + \pounds_l\pounds_X g_{ab} + 2\mathscr{D}_a\mathscr{D}_b\beta - \alpha_{(\xi)}\pounds_l g_{ab}], \tag{2.4.42}$$

where we have used Eq. (2.4.34). Note that by Eq. (2.4.34), we find that

$$\mathrm{STF}\,\pounds_l\pounds_X g_{ab} \mathrel{\hat{=}} \mathrm{STF}\,\pounds_X\pounds_l g_{ab} \mathrel{\hat{=}} 2\,\mathrm{STF}\,\pounds_X\sigma_{ab}, \tag{2.4.43}$$

where the final equality can be shown using Eqs. (2.2.16) and (2.4.21c), together with the fact that $X^a n_a \mathrel{\hat{=}} X^a l_a \mathrel{\hat{=}} 0$. Combining Eqs. (2.4.42) and (2.4.43), together with Eqs. (2.2.16), (2.2.35) and (2.4.21d), we find that

$$\text{STF}\,\gamma^{(\xi)}_{ab} = -2\,\text{STF}\left[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta + \pounds_X \sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_c X^c)\sigma_{ab}\right], \tag{2.4.44}$$

as claimed in Eq. (2.4.32).

2.5 | Asymptotic charges and fluxes: The Wald-Zoupas prescription

The prescription of Wald and Zoupas [53] provides a method of determining charges and fluxes at null infinity, and can be applied to any local and covariant theory. It relies on the covariant phase space formalism which was discussed for a general theory in chapter 1. Here, we repeat that discussion while specializing to the case of vacuum general relativity. Note that the addition of matter, for example, does not significantly complicate this discussion (see for example, [53, 68]).

We start with the Einstein-Hilbert Lagrangian 4-form $\boldsymbol{L}$:

$$\boldsymbol{L} = \frac{1}{16\pi}\hat{R}\;\hat{\boldsymbol{\varepsilon}}_4. \tag{2.5.1}$$

where $\hat{R}$ is the Ricci scalar and $\hat{\boldsymbol{\varepsilon}}_4$ is the 4-form volume element of the physical metric. The dynamical field is the physical metric $\hat{g}_{ab}$, and varying this dynamical field we obtain

$$\delta\boldsymbol{L} = \boldsymbol{E}^{ab}\delta\hat{g}_{ab} + d\boldsymbol{\theta}(\delta\hat{g}), \tag{2.5.2}$$

where the 3-form $\boldsymbol{\theta}$ is the symplectic potential, and $\boldsymbol{E}^{ab}$ is a tensor-valued 4-form which gives the equations of motion in the form $\boldsymbol{E}^{ab} = 0$. Here and throughout the rest of this analysis, we suppress the dependence of $\boldsymbol{\theta}$ on the background physical metric, $\hat{g}_{ab}$, but it is understood to exist. For general relativity, we have [69]

$$\boldsymbol{E}^{ab} = -\frac{1}{16\pi}\hat{\boldsymbol{\varepsilon}}_4\hat{G}^{ab}, \tag{2.5.3a}$$

$$\boldsymbol{\theta}(\delta\hat{g}) \equiv -\frac{1}{8\pi}\hat{\varepsilon}_{abc}{}^{[d}\hat{g}^{e]f}\hat{\nabla}_e\delta\hat{g}_{df}, \tag{2.5.3b}$$

The symplectic current is defined by taking a second, independent variation of the symplectic potential and anti-symmetrizing in the perturbations:

$$\boldsymbol{\omega}(\delta_1\hat{g},\delta_2\hat{g}) := \delta_1\boldsymbol{\theta}(\delta_2\hat{g}) - \delta_2\boldsymbol{\theta}(\delta_1\hat{g}). \tag{2.5.4}$$

Computing $d\boldsymbol{\omega}$ from this equation, using the fact that d and δ commute, using the second variation of Eq. (2.5.2), and using the fact that δ_1 and δ_2 commute, one finds that

$$d\boldsymbol{\omega}(\delta_1\hat{g},\delta_2\hat{g}) = \delta_2\boldsymbol{E}^{ab}\delta_1\hat{g}_{ab} - \delta_1\boldsymbol{E}^{ab}\delta_2\hat{g}_{ab}, \tag{2.5.5}$$

which vanishes whenever the perturbations satisfy the linearized equations of motion $\delta\boldsymbol{E}^{ab}=0$. When the dynamical fields $\hat{g}_{ab}$ satisfy the equations of motion, and $\delta\hat{g}_{ab}$ satisfy the linearized equations of motion, one can show that (see, e.g, [48, 49, 70])

$$\boldsymbol{\omega}(\delta\hat{g},\pounds_\xi\hat{g}) = d\left[\delta\boldsymbol{Q}_\xi - \xi\cdot\boldsymbol{\theta}(\delta\hat{g})\right], \tag{2.5.6}$$

for all diffeomorphisms generated by ξ^a, where the 2-form $\boldsymbol{Q}_\xi$ is the *Noether charge* associated with ξ^a. In general relativity, we have that [53, 69]

$$\boldsymbol{\omega}(\delta_1\hat{g},\delta_2\hat{g}) \equiv \frac{1}{16\pi}\hat{\varepsilon}_{dabc}\hat{P}^{defghi}\left[\delta_1\hat{g}_{ef}\hat{\nabla}_g\delta_2\hat{g}_{hi} - (1\leftrightarrow 2)\right], \tag{2.5.7}$$

where

$$\hat{P}^{abcdef} := \hat{g}^{ae}\hat{g}^{fb}\hat{g}^{cd} - \tfrac{1}{2}\hat{g}^{ad}\hat{g}^{be}\hat{g}^{fc} - \tfrac{1}{2}\hat{g}^{ab}\hat{g}^{cd}\hat{g}^{ef} - \tfrac{1}{2}\hat{g}^{bc}\hat{g}^{ae}\hat{g}^{fd} + \tfrac{1}{2}\hat{g}^{bc}\hat{g}^{ad}\hat{g}^{ef}. \tag{2.5.8}$$

Moreover, the Noether charge is given by [48]

$$\boldsymbol{Q}_\xi \equiv -\frac{1}{16\pi}\hat{\varepsilon}_{cdab}\hat{\nabla}^c\xi^d. \tag{2.5.9}$$

As discussed in chapter 1, the symplectic current, when integrated over some hypersurface Σ, provides a symplectic product on phase space. Moreover, the perturbed Hamiltonian corresponding to a diffeomorphism generated by ξ^a is given by the symplectic product of an arbitrary perturbation

δg_{ab} with $\delta_\xi \hat{g}_{ab} = \pounds_\xi \hat{g}_{ab}$:

$$\delta H_\xi = \int_\Sigma \boldsymbol{\omega}(\delta\hat{g}, \pounds_\xi \hat{g}) = \int_\Sigma d\left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] = \int_{\partial\Sigma}\left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right], \tag{2.5.10}$$

where the second equality follows by Eq. (2.5.6) and $\partial\Sigma$ is the boundary of Σ.

Now consider the case where the hypersurface Σ extends as a smooth surface to $\mathscr{I}$ in the unphysical spacetime which intersects $\mathscr{I}$ at a cross-section S. We take Eq. (2.5.10), rewritten in terms of the unphysical fields which are smooth at $\mathscr{I}$. In general relativity, using the behaviour of the unphysical metric perturbations detailed in Sec. 2.3.1, the symplectic current has a finite limit to $\mathscr{I}$; as shown in [53]. However, one should not conclude from Eq. (2.5.6) that $\delta\boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})$ also has a limit to $\mathscr{I}$; in general relativity it can be shown that $\boldsymbol{\theta}(\delta\hat{g})$ again has a finite limit to $\mathscr{I}$ (see [53]), but $\delta\boldsymbol{Q}_\xi$ diverges in this limit. Note that any procedure to "subtract out the diverging part" is highly non-unique. Fortunately, there is no need to resort to any such ad hoc procedure. Instead we proceed as follows.

Lemma 2.5.1. *Let S' be some sequence of 2-spheres in unphysical spacetime which limits (continuously) to a chosen cross-section S of $\mathscr{I}$. Then, the limiting integral*

$$\lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] \tag{2.5.11}$$

defined by first *integrating $\delta\boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})$ over the sequence of 2-spheres S' and then taking the limit as the sequence tends to $S \subset \mathscr{I}$ exists and is independent of the chosen sequence of 2-spheres used in the limit.*

Proof. Let S_0 be some 2-surface in the unphysical spacetime and let Σ be a smooth 3-surface which extends from S_0 and intersects $\mathscr{I}$ at a cross-section S. Note that this surface Σ is a *compact* 3-manifold in the unphysical spacetime. The integral of $\boldsymbol{\omega}(\delta\hat{g}, \pounds_\xi \hat{g})$ over Σ is necessarily finite since $\boldsymbol{\omega}(\delta\hat{g}, \pounds_\xi \hat{g})$ is a continuous 3-form on Σ (including at the "boundary" at S). Integrating Eq. (2.5.6) over Σ we obtain[4]

$$\int_\Sigma \boldsymbol{\omega}(\delta\hat{g}, \pounds_\xi \hat{g}) = \int_{S_0} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] - \lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right], \tag{2.5.12}$$

where the last expression on the right-hand side means "integrate over some 2-sphere $S' \subset \Sigma$, and then take the limit of this 2-sphere to the boundary S". This limiting procedure is necessary because, although the integral on the left-hand side of Eq. (2.5.12) is always finite, the 2-form integrand on the right-hand side need not have a finite limit to $\mathscr{I}$ in general. It follows that the limit in the last expression on the right-hand side of Eq. (2.5.12) exists. The limit is also independent of the choice of sequence of S' that is used in its definition due to Eq. (2.5.6) and the fact that Σ is compact and $\boldsymbol{\omega}(\delta\hat{g}, \pounds_\xi \hat{g})$ is continuous on Σ. Moreover, by Stokes' theorem, this expression is the same for *any* choice of the 3-surface Σ whose boundary is S, since $d\boldsymbol{\omega}(\delta\hat{g}, \delta_\xi \hat{g}) = 0$ by Eq. (2.5.5) and since $\boldsymbol{\omega}(\delta\hat{g}, \delta_\xi \hat{g})$ extends continuously to $\mathscr{I}$. $\square$

Remark 2.5.1 (Necessity of $\boldsymbol{\omega}$ extending continuously to $\mathscr{I}$). We emphasize that the condition that $\boldsymbol{\omega}(\delta\hat{g}, \delta_\xi \hat{g})$ extend continuously (as a 3-form in M) to $\mathscr{I}$ is necessary in the above argument. For instance, if we instead only assume that $\int_\Sigma \boldsymbol{\omega}(\delta\hat{g}, \delta_\xi \hat{g})$ is finite on *every* surface Σ (or that the pullback of $\boldsymbol{\omega}(\delta\hat{g}, \delta_\xi \hat{g})$ to *every* Σ is continuous within Σ), then $d\boldsymbol{\omega} = 0$ does *not* imply that the limit of $\int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right]$ is well-defined since it can depend on the choice of surface Σ used to define the limiting integral. As this point is often overlooked in the application of Stokes' theorem at $\mathscr{I}$, we provide a simple example on the Euclidean plane below.

On $\mathbb{R}^2$, let (x, y) be the usual Cartesian coordinates. Consider the 1-form

$$\boldsymbol{\omega} \equiv \frac{1}{(x^2 + y^2)^{3/2}} \left[y^2 \; dx - xy \; dy\right]. \tag{2.5.13}$$

[4] We note that on the left-hand side, we have taken Σ to be future-oriented while on the right-hand side, the 2-spheres *do not* have the usual outward-facing orientation within Σ, but the opposite. This choice of orientation is more natural, since (as mentioned in Remark 2.2.1), the limit of the orientation as $S' \to S$ is the future-directed orientation of S within $\mathscr{I}$ as specified by $\boldsymbol{\varepsilon}_2$. This choice of orientation is the opposite of the one that is used by Wald and Zoupas (see footnotes 2, 3, and 8 of [53]), and so some of our equations have the opposite sign compared to theirs.

This 1-form can be written as the exterior derivative of a 0-form (i.e., a function) as[5]

$$\boldsymbol{\omega} = dQ\,, \quad Q = \frac{x}{(x^2+y^2)^{1/2}}\,. \tag{2.5.14}$$

Note that $\boldsymbol{\omega}$ does not extend continuously to the origin $(x, y) = (0, 0)$, but $d\boldsymbol{\omega} = 0$ everywhere else and extends continuously to the origin. Further, it can be checked that the pullback of this $\boldsymbol{\omega}$ to any smooth curve Σ through the origin is continuous at the origin *within* this curve. However, we *cannot* use Stokes' theorem to conclude that $\int_\Sigma \boldsymbol{\omega}$ is independent of the choice of curve Σ joining the origin to some other point, since $\boldsymbol{\omega}$ is not continuous at the origin as a 1-form in $\mathbb{R}^2$. A direct computation shows that for any curve Σ from the origin to $(1, 1)$, we have

$$\int_\Sigma \boldsymbol{\omega} = \tfrac{1}{\sqrt{2}} - \cos\theta\,, \tag{2.5.15}$$

where θ is the angle with the x-axis of the tangent of Σ at the origin. So the integral of $\boldsymbol{\omega}$ along any curve through the origin is finite but depends on the curve Σ. Thus, we cannot define the value of Q at the origin by taking such integrals over curves Σ since it depends on the curve used in the limiting procedure. This can be explicitly checked from the expression for Q given above.

Next, we show that in general relativity, the limiting integral of $\left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right]$ is independent of the choice of extension of the BMS symmetry away from $\mathscr{I}$.

Lemma 2.5.2. *If ξ^a and ξ'^a are equivalent representatives of a BMS symmetry on $\mathscr{I}$ in general relativity then*

$$\lim_{S'\to S}\int_{S'}\left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] = \lim_{S'\to S}\int_{S'}\left[\delta \boldsymbol{Q}_{\xi'} - \xi' \cdot \boldsymbol{\theta}(\delta\hat{g})\right], \tag{2.5.16}$$

for all background spacetimes, all perturbations $\delta\hat{g}_{ab}$ and all cross-sections S of $\mathscr{I}$.

Proof. From Prop. 2.4.1, two equivalent representatives of a BMS symmetry are related by $\xi'^a = \xi^a + \Omega^2 W^a$ for some smooth W^a. Hence to prove the desired result we only need to show that the above integral computed for the vector field $\Omega^2 W^a$ vanishes. In general relativity it can be shown

[5] The choice of the constant in the function Q is irrelevant for our argument.

that the 3-form $\boldsymbol{\theta}(\delta\hat{g})$ is finite at $\mathscr{I}$ [53] and thus $(\Omega^2 W)\cdot\boldsymbol{\theta}(\delta\hat{g}) \hat{=} 0$. So we only need to the compute the Noether charge term which can be written as

$$\boldsymbol{Q}_{\Omega^2 W} \equiv -\frac{1}{16\pi}\hat{\varepsilon}_{abcd}\hat{\nabla}^c(\Omega^2 W^d) = -\frac{1}{16\pi}\varepsilon_{abcd}\nabla^c W^d\,. \tag{2.5.17}$$

Note that this is manifestly finite at $\mathscr{I}$ and so we can dispense with the limiting procedure used in the integral and evaluate the variation of the above expression directly at $\mathscr{I}$. Using the asymptotic conditions on the metric perturbations (Eq. (2.3.8)), we get (we ignore the overall signs and factors)

$$\delta\boldsymbol{Q}_{\Omega^2 W}\big|_{\mathscr{I}} \propto \varepsilon_{abcd}n^c\gamma^d{}_e W^e\,. \tag{2.5.18}$$

Using the definition of the volume-forms (Eq. (2.2.14)) and $n_a l^a \hat{=} -1$, the integral over the cross-section S is then (using $\gamma_{ab}n^b \hat{=} \Omega\gamma_a$)

$$\int_S \delta\boldsymbol{Q}_{\Omega^2 W} \propto \int_S \boldsymbol{\varepsilon}_2\; n^a\gamma_{ab}W^b \hat{=} \int_S \boldsymbol{\varepsilon}_2\; \Omega\gamma_a W^a \hat{=} 0\,. \tag{2.5.19}$$

□

The results of Lemmas 2.5.1 and 2.5.2 prove that the limiting integral of $\left[\delta\boldsymbol{Q}_\xi - \xi\cdot\boldsymbol{\theta}(\delta\hat{g})\right]$ is well-defined on BMS symmetries at $\mathscr{I}$. Thus, from Eq. (2.5.12), it would be natural to define a charge associated with the asymptotic symmetry ξ^a at S as a function $Q[\xi;S]$ in the phase space of the theory such that

$$\delta Q[\xi;S] := \lim_{S'\to S}\int_{S'}\left[\delta\boldsymbol{Q}_\xi - \xi\cdot\boldsymbol{\theta}(\delta\hat{g})\right], \tag{2.5.20}$$

for all backgrounds $\hat{g}_{ab}$, all perturbations $\delta\hat{g}_{ab}$, and all cross-sections S. However, in general, no such function $Q[\xi;S]$ exists, since the right-hand side is not integrable in phase space; that is, it cannot be written as the variation of some quantity for *all* perturbations. To see this, suppose that the charge defined in Eq. (2.5.20) does exist. Then, one must have $(\delta_1\delta_2 - \delta_2\delta_1)Q[\xi;S] = 0$ for *all* backgrounds $\hat{g}_{ab}$ and *all* perturbations $\delta_1\hat{g}_{ab}$, and $\delta_2\hat{g}_{ab}$ (satisfying the corresponding equations of motion). However, it follows from Eqs. (2.5.4) and (2.5.20) and the commutativity of δ_1 and δ_2 that

$$(\delta_1\delta_2 - \delta_2\delta_1)Q[\xi, S] = -\int_S \xi \cdot \boldsymbol{\omega}(\delta_1\hat{g}, \delta_2\hat{g}). \tag{2.5.21}$$

Thus, a charge defined by Eq. (2.5.20) will exist if the right-hand side of the above equation vanishes. This is the case if ξ^a is tangent to S. However, in general, the right-hand side is non-vanishing, and so one cannot define any charge $Q[\xi; S]$ using Eq. (2.5.20).

Remark 2.5.2 (Subtracting the "non-integrable part"). It might be tempting to simply stare at some explicit expression for the right-hand-side of Eq. (2.5.20) and then subtract off the "non-integrable part" of the expression to obtain an expression which is manifestly integrable to define the charge at S. But this "procedure" is very ad hoc; for instance suppose one manages to write the right-hand side of Eq. (2.5.20) as $\delta A + B$ where δA is a manifestly integrable expression and B is not. However, one can trivially write this in the alternative form $\delta A + B = \delta(A + C) + (B - \delta C)$ where C is some tensorial expression in terms of the available fields. Obviously, $\delta(A + C)$ is integrable while $(B - \delta C)$ is non-integrable for any choice of C. Thus the procedure to "subtract off the non-integrable part" is highly ambiguous — without any additional criteria, one cannot know which "non-integrable part" B or $B - \delta C$ should be "subtracted off".

The obstruction to the non-integrability of Eq. (2.5.20) was resolved by the rather general prescription of Wald and Zoupas [53]. Their procedure for defining integrable charges associated with asymptotic symmetries can be summarized as follows: let $\boldsymbol{\Theta}$ be a symplectic potential for the pullback of the symplectic current to $\mathscr{I}$; that is,

$$\underleftarrow{\boldsymbol{\omega}}(\delta_1\hat{g}, \delta_2\hat{g}) = \delta_1\boldsymbol{\Theta}(\delta_2\hat{g}) - \delta_2\boldsymbol{\Theta}(\delta_1\hat{g}), \tag{2.5.22}$$

for *all* backgrounds and *all* perturbations, with appropriate asymptotic conditions and equations of motion imposed. We then require that the choice of $\boldsymbol{\Theta}$ satisfies the following properties:

(1) $\boldsymbol{\Theta}$ must be locally and covariantly constructed out of the dynamical fields g_{ab}, the perturbations δg_{ab}, and finitely many of their derivatives, along with any fields in the "universal background

structure" present at $\mathscr{I}$ described in Sec. 2.3;

(2) $\boldsymbol{\Theta}$ must be independent of any arbitrary choices made in specifying the background structure; that is, $\boldsymbol{\Theta}$ must be conformally invariant and independent of the choice of the auxiliary normal l^a; and

(3) if g_{ab} is a stationary background solution, then $\boldsymbol{\Theta}(\hat{g}; \delta\hat{g}) = 0$, for *all* (not necessarily stationary) perturbations δg_{ab}.

The first of these conditions is motivated by the fact that, as given in our list of criteria at the start of this chapter, the prescription used to define charges should only depend on the tensor fields that exist on $\mathscr{I}$, and should not depend on additional structure (for example, some choice of coordinates). The second condition is required to ensure that the charges that are defined by this prescription are associated with the physical spacetime and do not depend that the particular choices that go into the conformal-completion (such as the conformal factor) or other additional choices like the foliation of $\mathscr{I}$. The final criterion plays an important role in showing that the flux of these charges vanishes for stationary spacetimes as we shall see below.

If such a symplectic potential $\boldsymbol{\Theta}$ can be found, define $\mathcal{Q}[\xi; S]$ to be a function on the phase space at $\mathscr{I}$ by

$$\delta\mathcal{Q}[\xi; S] := \lim_{S' \to S} \int_{S'} \left[\delta\boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] + \int_S \xi \cdot \boldsymbol{\Theta}(\delta\hat{g}). \tag{2.5.23}$$

It can easily be checked (using Eqs. (2.5.20)–(2.5.22)) that this expression is integrable in phase space; that is, $(\delta_1\delta_2 - \delta_2\delta_1)\mathcal{Q}[\xi; S] = 0$. Together with some choice of reference solution g_0 on which $\mathcal{Q}[\xi; S] = 0$ for all asymptotic symmetries ξ^a and all cross-sections S, Eq. (2.5.23) can be integrated in phase space to define the *Wald-Zoupas (WZ) charge* $\mathcal{Q}[\xi; S]$ associated with the asymptotic BMS symmetry ξ^a at S. We note the following properties of the WZ charge defined by the above procedure:

(1) Since $\boldsymbol{\Theta}$ is locally and covariantly constructed from the physical metric and the universal structure of $\mathscr{I}$, the charge can similarly be written in terms of quantities that are defined at $\mathscr{I}$. In particular the charge only depends on the BMS symmetry at $\mathscr{I}$ and not on any choice of its extension into the spacetime. While this is clear for the term that is an integral of $\xi \cdot \boldsymbol{\Theta}$,

that this is also true for the remaining terms, which are defined by a limit in the unphysical spacetime, follows by Lemmas 2.5.1 and 2.5.2.

(2) Moreover, since $\boldsymbol{\Theta}$ is chosen to be conformally invariant and independent of the choice of the auxiliary foliation of $\mathscr{I}$ and the choice of the reference solution is to be specified without any particular choice of its conformal completion or the choice of auxiliary foliation, the WZ charge is also conformally invariant and independent of the choice of the auxiliary foliation. Note, however that the WZ charge does depend on the chosen cross-section S on which it is evaluated and hence can depend on the auxiliary normal l_a at S (but does not depend on the choice of l_a away from S).

The flux of the perturbed WZ charge, through a portion $\Delta\mathscr{I}$ of $\mathscr{I}$ whose boundary is given by two cross-sections S_1 and S_2, is given by (see Eqs. 28 and 29 of [53])

$$\delta\mathcal{F}[\xi;\Delta\mathscr{I}] := \delta\mathcal{Q}[\xi;S_2] - \delta\mathcal{Q}[\xi;S_1] = \int_{\Delta\mathscr{I}} \Big[\underleftarrow{\boldsymbol{\omega}}(\delta\hat{g}, \pounds_\xi\hat{g}) + d\{\xi\cdot\boldsymbol{\Theta}(\delta\hat{g})\}\Big]\,. \tag{2.5.24}$$

The last term of this equation can also be written as

$$d[\xi\cdot\boldsymbol{\Theta}(\delta\hat{g})] = \pounds_\xi\boldsymbol{\Theta}(\delta\hat{g}) = -\underleftarrow{\boldsymbol{\omega}}(\delta\hat{g}, \pounds_\xi\hat{g}) + \delta\boldsymbol{\Theta}(\pounds_\xi\hat{g})\,, \tag{2.5.25}$$

where in the first equality, we have used the fact that $\boldsymbol{\Theta}$ is a 3-form intrinsic to $\mathscr{I}$ and so its exterior derivative is zero while second equality follows from the definition of $\boldsymbol{\Theta}$ as a symplectic potential for $\underleftarrow{\boldsymbol{\omega}}$ (Eq. (2.5.22)). The flux of the perturbed WZ charge is therefore simply given by

$$\delta\mathcal{F}[\xi;\Delta\mathscr{I}] = \int_{\Delta\mathscr{I}} \delta\boldsymbol{\Theta}(\pounds_\xi\hat{g}). \tag{2.5.26}$$

To get the unperturbed charge and flux, we have to choose a reference solution g_0 on which the charges are required to vanish. Since the symplectic potential $\boldsymbol{\Theta}$ is required to vanish on stationary backgrounds, we choose the reference solution g_0 to also be stationary. For our concrete case of general relativity, we will pick g_0 to be (any conformal completion of) Minkowski spacetime. Then, the flux of the WZ charge is given by

$$\mathcal{F}[\xi;\Delta\mathscr{I}] = \mathcal{Q}[\xi;S_2] - \mathcal{Q}[\xi;S_1] = \int_{\Delta\mathscr{I}} \boldsymbol{\Theta}(\pounds_\xi \hat{g}). \tag{2.5.27}$$

There are two important properties that follow from this expression for the flux. The first, which the flux inherits from $\boldsymbol{\Theta}$, is that, for any stationary background, the flux will vanish. This property captures the fact that, if there is no radiation, there should be no flux and the charges should be conserved quantities. The second is that, if ξ^a is an exact Killing vector field in the physical spacetime then the flux will vanish as well. This follows from the vanishing of $\pounds_\xi \hat{g}_{ab}$ for such vector fields. This property is reminiscent of Noether's theorem [71]: if there is an exact symmetry, then there should be a quantity related to that symmetry (in this case, the charge) that is conserved. In the next two sections, we will find expressions for the WZ flux and the WZ charge in vacuum general relativity. Since the charge and flux calculations will be performed in different ways, we will check these calculations by showing that they agree with Eq. (2.5.27).

Finally, we remark that the Wald-Zoupas prescription has certain ambiguities related to the choices of the symplectic potential $\boldsymbol{\theta}$ and the choice of $\boldsymbol{\Theta}$. However, it was argued in [53] that these ambiguities do not affect the final results for the WZ charge and flux and so we will not discuss them here.

1 Wald-Zoupas flux in general relativity

We first consider the flux of the Wald-Zoupas charge. From Eq. (2.5.27), it is apparent that determining this flux requires finding $\boldsymbol{\Theta}$. First, a lengthy calculation starting with Eq. (2.5.7), then using Eq. (2.3.8) for the unphysical metric perturbation δg_{ab}, along with the variation of the vacuum Einstein equations (Eq. (2.2.1)), shows that [53]

$$\underleftarrow{\boldsymbol{\omega}}(\delta_1\hat{g},\delta_2\hat{g}) \mathrel{\hat{=}} -\frac{1}{32\pi}\left(\delta_1 S_{ab}\gamma_2^{ab} - \delta_2 S_{ab}\gamma_1^{ab}\right)\boldsymbol{\varepsilon}_3. \tag{2.5.28}$$

Since $\gamma^{ab} n_b \mathrel{\hat{=}} 0$ by Eq. (2.3.8), δS_{ab} in this expression can be replaced with $\delta \underleftarrow{S}_{ab}$. Moreover, ρ_{ab} is universal, and so $\delta \rho_{ab} = 0$; as such, Eq. (2.2.39) implies that we can replace $\delta \underleftarrow{S}_{ab}$ with δN_{ab}, yielding

$$\underleftarrow{\boldsymbol{\omega}}(\delta_1 \hat{g}, \delta_2 \hat{g}) \mathrel{\hat{=}} -\frac{1}{32\pi} \left(\delta_1 N_{ab} \gamma_2^{ab} - \delta_2 N_{ab} \gamma_1^{ab} \right) \boldsymbol{\varepsilon}_3 \mathrel{\hat{=}} \frac{1}{16\pi} \left(\delta_1 N^{ab} \delta_1 \sigma_{ab} - \delta_2 N^{ab} \delta_1 \sigma_{ab} \right) \boldsymbol{\varepsilon}_3, \tag{2.5.29}$$

where the final equality uses the fact that the News tensor is traceless and Eq. (2.3.10).

A symplectic potential for $\underleftarrow{\boldsymbol{\omega}}$ is therefore given by

$$\boldsymbol{\Theta}(\delta \hat{g}) \mathrel{\hat{=}} -\frac{1}{32\pi} N^{ab} \gamma_{ab} \boldsymbol{\varepsilon}_3 \mathrel{\hat{=}} \frac{1}{16\pi} N^{ab} \delta \sigma_{ab} \boldsymbol{\varepsilon}_3. \tag{2.5.30}$$

One must check that this symplectic potential satisfies the requirements given above. First, it is constructed from g_{ab}, δg_{ab}, their derivatives (such as S_{ab}), and fields that are part of the universal structure at $\mathscr{I}$ (such as n^a and, less obviously, ρ_{ab} [53]). Moreover, it is independent of the choice of conformal factor Ω and auxiliary normal l^a. To see this, first note that the only piece that depends on l^a is $\boldsymbol{\varepsilon}_3 = l \cdot \boldsymbol{\varepsilon}_4$, but upon taking the pullback, this dependence drops out. To see that it is conformally invariant, use the fact that $\gamma_{ab} = \Omega \delta \hat{g}_{ab}$, and so under a conformal transformation $\Omega \mapsto \omega \Omega$,

$$N^{ab} \mapsto \omega^{-4} N^{ab}, \quad \gamma_{ab} \mapsto \omega \gamma_{ab}, \quad \boldsymbol{\varepsilon}_3 \mapsto \omega^3 \boldsymbol{\varepsilon}_3, \tag{2.5.31}$$

and so we find that $\boldsymbol{\Theta}$ is conformally invariant. Finally, note that this choice of $\boldsymbol{\Theta}$ vanishes on stationary solutions g_{ab}, for *any* perturbation δg_{ab}, by the argument on pp. 53–54 of [62] (which shows that N_{ab} vanishes for stationary vacuum spacetimes).

Using a chosen foliation of $\mathscr{I}$, we now provide a more explicit expression for the WZ flux in terms of fields defined on $\mathscr{I}$. Note that, since the News tensor is traceless, we have that

$$N^{ab} \gamma_{ab}^{(\xi)} \mathrel{\hat{=}} N^{ab} \, \mathrm{STF} \, \gamma_{ab}^{(\xi)}. \tag{2.5.32}$$

As such, from Cor. 2.4.1, we have that our flux is independent of the extension of ξ^a off of $\mathscr{I}$.

Plugging in our expression for STF $\gamma^{(\xi)}_{ab}$ from Eq. (2.4.32), we therefore find that[6]

$$\begin{aligned}\mathcal{F}[\xi;\Delta\mathscr{I}] &\mathrel{\hat{=}} -\frac{1}{16\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\varepsilon}_3 N^{ab}\gamma^{(\xi)}_{ab} \\ &\mathrel{\hat{=}} -\frac{1}{16\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\varepsilon}_3 N^{ab}[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta + \pounds_X\sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_c X^c)\sigma_{ab}].\end{aligned} \tag{2.5.33}$$

Note that the first expression is manifestly independent of any choice of foliation of $\mathscr{I}$, a property it inherits from $\mathbf{\Theta}$. This is not obviously true for the second, which instead is more useful for explicit computations (such as comparing with expressions in Bondi coordinates). Both, however, are clearly local and covariant, and can be easily shown to be independent of the conformal factor. We reiterate that these properties of the flux are motivated by the desire that these expressions not be dependent on any arbitrary choices that we could make, such as a coordinate system, conformal factor etc. As mentioned below Eq. (2.5.27), the flux in Eq. (2.5.33) has the property that it vanishes for both stationary backgrounds as well as when ξ^a is an exact Killing vector field of the physical spacetime. The first of these properties is evident both in the first and second expressions in Eq. (2.5.33), as the integrands are both proportional to the News. That the flux vanishes when ξ^a is an exact Killing vector field follows immediately from the fact that $\gamma^{(\xi)}_{ab}$ vanishes for such vector fields.

In the case where ξ^a is a translation, namely $\xi^a = f n^a$, where f obeys Eq. (2.4.24), it is the case that

$$\mathcal{F}[fn;\Delta\mathscr{I}] = -\frac{1}{32\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\varepsilon}_3\, f N^{ab}N_{ab}. \tag{2.5.34}$$

In the case where f is everywhere positive, ξ^a corresponds to a time-translation Eq. (2.4.24). In this case, the flux is negative, corresponding to the *loss* of mass/energy during the emission of gravitational waves.

Remark 2.5.3 (BMS fluxes and Hamiltonians). On any region $\Delta\mathscr{I}$ of null infinity we can take the integral of the symplectic current $\int_{\Delta\mathscr{I}}\underleftarrow{\boldsymbol{\omega}}(\delta_1\hat{g},\delta_2\hat{g})$ to define a *symplectic form* on the radiative phase on $\Delta\mathscr{I}$. Then, Eq. (2.5.24) implies that

$$\delta\mathcal{F}[\xi;\Delta\mathscr{I}] = \int_{\Delta\mathscr{I}}\underleftarrow{\boldsymbol{\omega}}(\delta\hat{g},\pounds_\xi\hat{g}) + \int_{S_2}\xi\cdot\mathbf{\Theta}(\delta\hat{g}) - \int_{S_1}\xi\cdot\mathbf{\Theta}(\delta\hat{g}), \tag{2.5.35}$$

[6] The first form of the WZ flux in Eq. (2.5.33) was anticipated by Geroch and Winicour long before Wald and Zoupas (see Eq. 28 of [72]).

Note that if the boundary terms at S_1 and S_2 vanish for all perturbations δg_{ab} and all background solutions g_{ab} then the WZ flux $\mathcal{F}[\xi; \Delta\mathscr{I}]$ would define a *Hamiltonian generator* corresponding to the BMS symmetry ξ^a, as defined by [49, 53]. In general, these boundary terms do not and the WZ flux does not define a Hamiltonian. However, in the case where we consider all of null infinity, instead of some finite portion, and use appropriate boundary conditions at timelike and spatial infinity, then $\mathcal{F}[\xi; \mathscr{I}]$ is a Hamiltonian generator on the full radiative phase space on $\mathscr{I}$. To see this, let u be a parameter along the null generators of $\mathscr{I}$ such that $n^a \nabla_a u \hat{=} 1$, and impose, as $u \to \pm\infty$, the following boundary conditions:

$$N_{ab} = O(1/|u|^{1+\epsilon}), \quad \gamma_{ab} = O(1), \tag{2.5.36}$$

for some $\epsilon > 0$. These conditions ensure that the integral of the symplectic current over all of $\mathscr{I}$, as given in Eq. (2.5.29), is finite. Further, Eq. (2.4.21a) implies that $\xi^a = O(|u|)$, and so it follows from Eqs. (2.5.30) and (2.5.36) that

$$\lim_{u \to \pm\infty} \xi \cdot \boldsymbol{\Theta}(\delta\hat{g}) = 0. \tag{2.5.37}$$

Note that the conditions Eq. (2.5.36) are preserved by all BMS symmetries, and thus the total flux $\mathcal{F}[\xi; \mathscr{I}]$ defines a Hamiltonian generator for any BMS symmetry ξ^a on the radiative phase space on $\mathscr{I}$.

2 | Wald-Zoupas charge in general relativity

Having obtained the WZ flux, we now wish to find an expression for the WZ charge in terms of fields on $\mathscr{I}$. From Eq. (2.5.23), the WZ charge $\mathcal{Q}[\xi; S]$ is determined by

$$\delta\mathcal{Q}[\xi; S] = \lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta\hat{g})\right] + \int_S \xi \cdot \boldsymbol{\Theta}(\delta\hat{g}) \,, \tag{2.5.38}$$

along with the requirement that $\mathcal{Q}[\xi; S]$ vanish on Minkowski spacetime for all BMS symmetries ξ^a and all cross-sections S. The main difficulty in carrying out this computation directly is that the 2-form $\delta\boldsymbol{Q}_\xi$ does not have a limit to $\mathscr{I}$. To compute the right-hand-side of Eq. (2.5.38), one must first choose some family of 2-spheres inside the spacetime, evaluate the integral and then take the limit as these 2-spheres tend to the chosen cross-section S of $\mathscr{I}$.

To compute this term, we note that by the general arguments in Lemmas 2.5.1 and 2.5.2,

$$\lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g}) \right] , \tag{2.5.39}$$

is guaranteed to exist, is independent of the family of 2-spheres chosen to take the limit and is independent of how the BMS symmetry is extended into the spacetime away from $\mathscr{I}$. Moreover, it is manifestly independent of the choice of the conformal factor and the choice of the foliation of $\mathscr{I}$. Thus, we can compute Eq. (2.5.39) in the choices given by the conformal Bondi-Sachs coordinates in a neighbourhood of $\mathscr{I}$. The detailed construction of these coordinates and the form of the unphysical and physical metrics in these coordinates is given in Sec. 2.6.1. The conformal Bondi-Sachs coordinates specify a family of null surfaces $\mathcal{N}_u$ labeled by a coordinate u and a family of 2-spheres S' labeled by the coordinates Ω and u along each null surface $\mathcal{N}_u$ such that as $\Omega \to 0$ the 2-spheres limit to a cross-section S of $\mathscr{I}$. We use this family of 2-spheres to evaluate Eq. (2.5.39) and then take the limit $\Omega \to 0$ along this family.

With this setup, we take the form of the unphysical metric in Bondi-Sachs coordinates (given by Eqs. (2.6.2) and (2.6.5)), the corresponding expressions for the unphysical metric perturbations along with the expression for ξ^a derived in Eq. (2.6.19). We use this to evaluate Eq. (2.5.39) using Eqs. (2.5.3b) and (2.5.9), after converting the physical metric and physical metric perturbations to their unphysical counterparts. The resulting expression for the 2-form $\delta \boldsymbol{Q}_\xi$ has a term that diverges as $\Omega \to 0$, which is given by

$$\boldsymbol{\varepsilon}_2 \; \Omega^{-1} \left(2 X^A \delta U_A^{(2)} \right) = - \boldsymbol{\varepsilon}_2 \; \Omega^{-1} X^A \mathscr{D}^B \delta C_{AB} \, . \tag{2.5.40}$$

The indices A, B are abstract indices for tensor fields on the chosen family of 2-spheres, and $\mathscr{D}_A$ is the covariant derivative on these 2-spheres. The equality Eq. (2.5.40) follows from Eq. (2.6.7) which is a consequence of the (linearized) Einstein equation in the Bondi-Sachs coordinates. Now, since X^A is a Lorentz vector field (satisfying Eq. (2.6.14) in the conformal Bondi-Sachs coordinates) and

δC_{AB} is traceless, this term vanishes when integrated over the 2-spheres. As a result, the limit of $\int_{S'} \delta \boldsymbol{Q}_\xi$ as $S' \to S$, i.e. as $\Omega \to 0$, is finite. Then, computing the remaining terms in Eq. (2.5.39) (which are manifestly finite as $\Omega \to 0$) we obtain

$$\begin{aligned}
&\lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})\right] \\
&= -\frac{1}{8\pi} \delta \int_S \boldsymbol{\varepsilon}_2 \left[\beta \left(\mathcal{P} + \tfrac{1}{2}\sigma_{ab} N^{ab}\right) + X^a \mathcal{J}_a + X^a \sigma_{ab} \mathscr{D}_c \sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab} \mathscr{D}_c X^c\right] \\
&\quad - \frac{1}{16\pi} \int_S \boldsymbol{\varepsilon}_2 \; \beta \, N^{ab} \delta \sigma_{ab} \,.
\end{aligned} \tag{2.5.41}$$

where we have used Eq. (2.6.10) to convert the metric components in the conformal Bondi-Sachs coordinates to covariant quantities defined on $\mathscr{I}$. We can immediately see this expression is non-integrable due to the last term.[7]

Using Eqs. (2.2.14), (2.5.30) and (2.6.19), we have

$$\int_S \xi \cdot \boldsymbol{\Theta}(\delta \hat{g}) = \tfrac{1}{16\pi} \int_S \boldsymbol{\varepsilon}_2 \; \beta N^{ab} \delta \sigma_{ab} \,. \tag{2.5.42}$$

Thus, the perturbed WZ charge is given by

$$\begin{aligned}
\delta \mathcal{Q}[\xi; S] &= \lim_{S' \to S} \int_{S'} \left[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})\right] + \int_S \xi \cdot \boldsymbol{\Theta}(\delta \hat{g}) \\
&= -\frac{1}{8\pi} \delta \int_S \boldsymbol{\varepsilon}_2 \left[\beta \left(\mathcal{P} + \tfrac{1}{2}\sigma_{ab} N^{ab}\right) + X^a \mathcal{J}_a + X^a \sigma_{ab} \mathscr{D}_c \sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab} \mathscr{D}_c X^c\right] .
\end{aligned} \tag{2.5.43}$$

Therefore, the expression for the WZ charge is given by

$$\mathcal{Q}[\xi; S] \mathrel{\hat{=}} -\frac{1}{8\pi} \int_S \boldsymbol{\varepsilon}_2 \left[\beta(\mathcal{P} + \tfrac{1}{2}\sigma^{ab} N_{ab}) + X^a \mathcal{J}_a + X^a \sigma_{ab} (\mathscr{D}_c - \tau_c) \sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab}(\mathscr{D}_c - 2\tau_c) X^c\right] . \tag{2.5.44}$$

Note that in Eq. (2.5.44) we have added terms which depend on τ_a; these terms cancel amongst each other using the identity Eq. (A.3.2). These additional terms make each term in the integrand of conformal weight -2 (which can be verified using the conformal weights given in Eqs. (2.2.29)–(2.2.31)

[7] As emphasized in Remark 2.5.2 above, this *should not* be taken to mean that the last line above is the "non-integrable part" of the expression which should be "simply subtracted away" without specifying any additional criteria.

and (2.4.11)) which, along with the conformal weight $+2$ of ε_2, makes the charge conformally-invariant. Further, since these τ_a terms cancel, the charge is independent of the choice of the foliation and only depends on the chosen cross-section S. We will verify below that the flux of this charge across any region $\Delta\mathscr{I}$ is given by Eq. (2.5.33). Finally, since the flux vanishes in Minkowski spacetime we can compute the charge on any cross-section of $\mathscr{I}$. Using a shear-free cross-section we see that the charge also vanishes on *any* cross-section of $\mathscr{I}$ in Minkowski spacetime. Thus, the charge formula in Eq. (2.5.44) satisfies all the properties required by the WZ charge.

For a supertranslation symmetry with $\beta \mathrel{\hat{=}} f$ satisfying $\pounds_n f \mathrel{\hat{=}} 0$ it is straightforward to verify that Eq. (2.5.44) reproduces Geroch's supermomentum [53, 62]. Further, if at some chosen cross-section S we pick $\beta|_S \mathrel{\hat{=}} 0$ then the resulting formula is equal to the linkage charge defined by Geroch and Winicour [53, 72]; this was proven in [53] and was shown more explicitly in [4]. Now, while on any fixed cross-section S we can decompose a general BMS symmetry into a supertranslation part and a part tangent to S, this decomposition is not preserved along $\mathscr{I}$ (see Remark 2.4.2). Thus, in general the WZ charge is *not* the sum of Geroch's supermomentum with the Geroch-Winicour charge. In Appendix A.2.2, we will also include a brief comparison of the WZ charge with the Komar formula and the linkage charge in addition to a formula due to Penrose.

By the general arguments following Eq. (2.5.24), the change in the charge Eq. (2.5.44) between two cross-sections is given by the flux formula Eq. (2.5.33). However, showing this explicitly is a non-trivial computation, which we detail in the remainder of this section.

Let S_2 and S_1 be *any* two cross-sections of $\mathscr{I}$, with S_2 to the future of S_1, and let $\Delta\mathscr{I}$ be the portion of $\mathscr{I}$ bounded by these cross-sections. Then, the change in the charge is given by

$$\begin{aligned}
&\mathcal{F}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} \mathcal{Q}[\xi;S_2] - \mathcal{Q}[\xi;S_1] \\
&\mathrel{\hat{=}} -\frac{1}{8\pi}\int_{\Delta\mathscr{I}} \varepsilon_3 \pounds_n \left[\beta(\mathcal{P} + \tfrac{1}{2}\sigma^{ab}N_{ab}) + X^a\mathcal{J}_a + X^a\sigma_{ab}\mathscr{D}_c\sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab}\mathscr{D}_c X^c\right] .
\end{aligned} \tag{2.5.45}$$

Note that we have dropped the τ_a terms from the expression since they do not contribute to the charge as explained above.

Now we simplify Eq. (2.5.45) term-by-term starting with the first and second terms. Using

$\mathcal{J}_a n^a \mathrel{\hat{=}} 0$, as well as Eqs. (2.2.25b), (2.2.25d), (2.2.37), (2.4.21a) and (2.4.21b) and the integration-by-parts formula Eq. (2.2.22), we see that the first two terms in Eq. (2.5.45) contribute

$$-\frac{1}{8\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\varepsilon}_3 \Big[N^{ab}\Big(\tfrac{1}{4}\beta N_{ab} + \tfrac{1}{2}(\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta + (\mathscr{D}_a + \tau_a)(X^c\sigma_{bc}) + \tfrac{1}{4}\sigma_{ab}(\mathscr{D}_c - \tau_c)X^c\Big) \\ - \tfrac{1}{2}\varepsilon_a{}^b X^a(\mathscr{D}_b + 3\,\tau_b)\mathcal{P}^*\Big], \tag{2.5.46}$$

to the flux. Now consider the contribution of the third term in Eq. (2.5.45):

$$\pounds_n(X^a\sigma_{ab}\mathscr{D}_c\sigma^{bc}) \mathrel{\hat{=}} \tfrac{1}{2}X^a N_{ab}\mathscr{D}_c\sigma^{bc} + X^a\sigma_{ab}\pounds_n\mathscr{D}_c\sigma^{bc} \tag{2.5.47}$$

where we have used Eq. (2.4.21b) and Eq. (2.2.32). For the last term above we have

$$\begin{aligned}\pounds_n\mathscr{D}_c\sigma^{bc} &\mathrel{\hat{=}} \pounds_n(Q^a{}_c Q^b{}_d)\nabla_a\sigma^{cd} + Q^a{}_c Q^b{}_d \pounds_n\nabla_a\sigma^{cd} \\ &\mathrel{\hat{=}} (Q^a{}_c n^b\tau_d + Q^b{}_d n^a\tau_c)\nabla_a\sigma^{cd} + \tfrac{1}{2}\mathscr{D}_a N^{ab} \\ &\mathrel{\hat{=}} (Q^a{}_c n^b\tau_d)\nabla_a\sigma^{cd} + \tfrac{1}{2}(\mathscr{D}_a + \tau_a)N^{ab}\,,\end{aligned} \tag{2.5.48}$$

where we have made liberal use of the Bondi condition (Eq. (2.2.6)) along with Eqs. (2.2.11) and (2.2.18), and commuted the $\pounds_n$ past the ∇_a using $\sigma^{ab}n_a \mathrel{\hat{=}} Q_{ab}n^a \mathrel{\hat{=}} 0$. Using the above expression in Eq. (2.5.47) the first term vanishes by $\sigma_{ab}n^b \mathrel{\hat{=}} 0$ and we get

$$\pounds_n(X^a\sigma_{ab}\mathscr{D}_c\sigma^{bc}) \mathrel{\hat{=}} \tfrac{1}{2}X^a N_{ab}\mathscr{D}_c\sigma^{bc} + \tfrac{1}{2}(\mathscr{D}_c + \tau_c)(X^a\sigma_{ab}N^{bc}) - \tfrac{1}{2}N_{bc}\mathscr{D}^b(X_a\sigma^{ac})\,. \tag{2.5.49}$$

Note that the second term on the right-hand side above drops out of the flux formula using Eq. (2.2.22).

Consider now the fourth term in Eq. (2.5.45). Since $\alpha_{(\xi)} \mathrel{\hat{=}} \frac{1}{2}\mathscr{D}_a X^a$ and $\pounds_n\alpha_{(\xi)} \mathrel{\hat{=}} 0$ (Eq. (2.4.7)), using Eq. (2.2.32) we get

$$-\tfrac{1}{4}\pounds_n(\sigma_{ab}\sigma^{ab}\mathscr{D}_c X^c) \mathrel{\hat{=}} -\tfrac{1}{4}\sigma^{ab}N_{ab}\mathscr{D}_c X^c\,. \tag{2.5.50}$$

Putting together Eqs. (2.5.46), (2.5.49) and (2.5.50), we see that the flux is given by

$$\begin{aligned}\mathcal{F}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} -\frac{1}{8\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\varepsilon}_3\Big[N^{ab}\Big(\tfrac{1}{4}\beta N_{ab}+\tfrac{1}{2}(\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta+\tfrac{1}{2}(\mathscr{D}_a+2\tau_a)(X^c\sigma_{bc})\\ -\tfrac{1}{2}\sigma_{ab}\tau_c X^c+\tfrac{1}{2}X^a\mathscr{D}_c\sigma^{bc}\Big)-\tfrac{1}{2}\varepsilon_a{}^b X^a(\mathscr{D}_b+3\tau_b)\mathcal{P}^*\Big]\,.\end{aligned} \tag{2.5.51}$$

Next we simplify the last term using the identity Eq. (2.2.38) for $\mathcal{P}^*$. The term arising from this identity which involves derivatives of the shear reads $\varepsilon_3\,\varepsilon^{de}\varepsilon_a{}^b X^a(\mathscr{D}_b+3\tau_b)[\mathscr{D}_d(\mathscr{D}_c-\tau_c)\sigma_e{}^c]$. This term can be shown to vanish upon integrating over the cross-sections as follows. Note that this term is conformally-invariant (which can be seen using Eq. (2.2.38) and the conformal weights given in Eqs. (2.2.28), (2.2.29), (2.2.31) and (2.4.11)). Therefore, we can evaluate this term in the Bondi frame where the metric on the cross-sections is chosen to be the unit round metric and $\tau_a \mathrel{\hat{=}} 0$. This allows us to make use of spherical harmonics — σ_{ab} is a symmetric and trace-free tensor and thus is supported on $\ell \geq 2$ tensor spherical harmonics while X^a is supported only on $\ell = 1$ vector spherical harmonics (see Remark 2.4.1). Using the orthogonality of the spherical harmonics, the integral of this term over the cross-sections vanishes. The remaining term arising from Eq. (2.2.38) contains the News tensor which, after an integration-by-parts using Eq. (2.2.22), becomes

$$-\tfrac{1}{4}\varepsilon^{ab}\varepsilon_e{}^d N_{ac}\sigma_b{}^c(\mathscr{D}_d-2\tau_d)X^e \mathrel{\hat{=}} \tfrac{1}{2}N_{ab}\sigma^b{}_c\mathscr{D}^{[a}X^{c]}-N_{ab}\sigma^b{}_c X^{[c}\tau^{a]}\,. \tag{2.5.52}$$

Replacing the above in the last term in Eq. (2.5.51) we get

$$\mathcal{F}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} -\frac{1}{8\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\varepsilon}_3 N^{ab}\Big[\tfrac{1}{4}\beta N_{ab}+\tfrac{1}{2}(\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta+\tfrac{1}{2}(\mathscr{D}_a+2\tau_a)(X^c\sigma_{bc}) \tag{2.5.53}$$

$$-\tfrac{1}{2}\sigma_{ab}\tau_c X^c+\tfrac{1}{2}X_a\mathscr{D}^c\sigma_{bc}+\tfrac{1}{2}\sigma_b{}^c\mathscr{D}_{[a}X_{c]}-\sigma_b{}^c X_{[c}\tau_{a]}\Big]\,, \tag{2.5.54}$$

The last term on the first line and first term on the second line in the expression above can together be written as

$$N^{ab}(X^c\sigma_{bc}\tau_a-\tfrac{1}{2}\sigma_{ab}\tau_c X^c) \mathrel{\hat{=}} N^{ab}\sigma_b{}^c(X_c\tau_a-\tfrac{1}{2}Q_{ca}X_d\tau^d)\,. \tag{2.5.55}$$

Moreover, it follows from Eq. (A.3.2) that $N_{ab}\sigma^b{}_c \mathrel{\hat{=}}$ pure trace term $+\, N_{b[a}\sigma^b{}_{c]}$ and so Eq. (2.5.55) becomes $N^{ab}\sigma_b{}^c X_{[c}\tau_{a]}$. This exactly cancels the last term in Eq. (2.5.53), and therefore the flux formula simplifies to

$$\mathcal{F}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} -\frac{1}{8\pi}\int_{\Delta\mathscr{I}} \varepsilon_3 N^{ab}\Big[\tfrac{1}{4}\beta N_{ab} + \tfrac{1}{2}(\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta \\ + \tfrac{1}{2}\mathscr{D}_a(X^c\sigma_{bc}) + \tfrac{1}{2}X_a\mathscr{D}^c\sigma_{bc} + \tfrac{1}{2}\sigma_b{}^c\mathscr{D}_{[a}X_{c]}\Big]. \tag{2.5.56}$$

The second line above can be simplified using Eq. (A.3.4) to get

$$\mathcal{F}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} -\frac{1}{16\pi}\int_{\Delta\mathscr{I}} \varepsilon_3 N^{ab}[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta + \pounds_X\sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_cX^c)\sigma_{ab}]. \tag{2.5.57}$$

which matches the WZ flux derived in Eq. (2.5.33).

2.6 | Expressions in some coordinate systems

Our entire preceding analysis was completely covariant and was done without referring to any particular coordinate systems. In this section, we consider two examples of coordinate systems on the unphysical spacetime in a neighbourhood of null infinity. These coordinates can be used to also obtain asymptotic coordinates on the physical spacetime. We will find the asymptotic form of the metric, both physical and unphysical, and derive the expressions for the BMS symmetries and their charges in these coordinates.

As described in Sec. 2.3, one can construct a geometrically defined coordinate system (Ω, u, x^A) at $\mathscr{I}$ where x^A are coordinates on the cross-sections of $\mathscr{I}$, u satisfies $n^a\nabla_a u \mathrel{\hat{=}} 1$ and the conformal factor Ω is chosen so that the Bondi condition is satisfied and so that the induced metric on the cross-sections is the unit round metric s_{AB} on $\mathbb{S}^2$. Then for any asymptotically flat spacetime the line element of the unphysical metric at $\mathscr{I}$ is

$$ds^2 \mathrel{\hat{=}} 2d\Omega du + s_{AB}dx^A dx^B\,. \tag{2.6.1}$$

These coordinates can be extended away from $\mathscr{I}$ in different ways, and these give rise to the different coordinates that are often used in the analysis of asymptotic symmetries and their associated charges.

Two commonly used coordinates are the Bondi-Sachs coordinates and conformal Gaußian null coordinates, and we will focus on these in the remainder of this section. We emphasize that the form of the (physical or unphysical) metric in these coordinates follows directly from the geometric construction of the coordinate systems and the covariant definition of asymptotic flatness without any additional assumptions.

1 Bondi-Sachs coordinates

One way to extend the coordinates described above away from $\mathscr{I}$ is as follows. Let S_u be cross-sections of $\mathscr{I}$ with u = constant, and consider a family of null surfaces $\mathcal{N}_u$ which intersect $\mathscr{I}$ transversely in the cross-sections S_u. These surfaces $\mathcal{N}_u$ foliate a neighbourhood of $\mathscr{I}$. We first extend the coordinate u away from $\mathscr{I}$ so that it is constant along each null surface $\mathcal{N}_u$. Then $l_a := -\nabla_a u$ is the null normal to each $\mathcal{N}_u$ with $l^a l_a = 0$ in addition to $l^a n_a \mathrel{\hat{=}} -1$. Next, we extend the angular coordinates x^A on each cross-section S_u by parallel transport i.e. $l^a \nabla_a x^A = 0$.

Fixing the induced metric on cross sections of $\mathscr{I}$ to be the unit round sphere metric fixes the conformal factor, Ω, on $\mathscr{I}$. To extend Ω farther away from $\mathscr{I}$, we use the remaining freedom in the conformal factor to demand that the 2-spheres at constant u and Ω have the same area element as the unit sphere, that is, if h_{AB} is the 2-metric on the surfaces of constant u and Ω then we demand that $\det h = \det s$ in the x^A-coordinates. This fixes Ω uniquely away from $\mathscr{I}$. Thus we have set up the conformal Bondi-Sachs coordinate system (Ω, u, θ^A) in a neighborhood of $\mathscr{I}$.

The most general form of the unphysical metric in conformal Bondi-Sachs coordinates is given by[8]

$$ds^2 \equiv -We^{2B}du^2 + 2e^{2B}d\Omega du + h_{AB}(dx^A - U^A du)(dx^B - U^B du)\,, \tag{2.6.2}$$

where the metric components $g_{\Omega\Omega}$ and $g_{\Omega A}$ vanish everywhere due to the conditions $l^a l_a = l^a \nabla_a x^A = 0$, and W, B, h_{AB}, and U^A are smooth functions of the coordinates (Ω, u, x^A). Since the metric at $\mathscr{I}$

[8] Note that the function we denote by B is conventionally denoted by β, but we use a different symbol to avoid conflict with the n^a-component of a BMS vector field.

is given by Eq. (2.6.1), we also have

$$W = O(\Omega)\,, \quad B = O(\Omega)\,, \quad U^A = O(\Omega)\,, \quad h_{AB} = s_{AB} + O(\Omega)\,. \tag{2.6.3}$$

Further, evaluating the Bondi condition $\nabla_a n_b \,\hat{=}\, 0$ (Eq. (2.2.6)) gives

$$W = O(\Omega^2)\,, \quad B = O(\Omega^2)\,, \quad U^A = O(\Omega^2)\,. \tag{2.6.4}$$

We therefore consider the following expansion of the metric components

$$\begin{gathered} W = \Omega^2 W^{(2)} - 2\Omega^3 M + O(\Omega^4)\,, \quad U^A = \Omega^2 U^{(2)A} + 2\Omega^3 L^A + O(\Omega^4)\,, \\ B = \Omega^2 B^{(2)} + O(\Omega^3)\,, \quad h_{AB} = s_{AB} + \Omega C_{AB} + \Omega^2 d_{AB} + O(\Omega^3)\,. \end{gathered} \tag{2.6.5}$$

Next, imposing $\det h = \det s$, we get

$$s^{AB} C_{AB} = 0\,, \quad s^{AB} d_{AB} = \tfrac{1}{2} C^{AB} C_{AB}\,. \tag{2.6.6}$$

We then impose the Einstein equation, Eq. (2.2.1), order by order in Ω. At $O(\Omega^0)$, it gives

$$W^{(2)} = 1\,, \quad B^{(2)} = -\tfrac{1}{32} C^{AB} C_{AB}\,, \quad U^{(2)}_A = -\tfrac{1}{2} \mathscr{D}^B C_{AB}\,, \tag{2.6.7}$$

while at $O(\Omega)$ we get

$$\partial_u \,\mathrm{STF}\, d_{AB} = \mathscr{D}^A \,\mathrm{STF}\, d_{AB} = 0\,. \tag{2.6.8}$$

Since d_{AB} is a smooth tensor on a 2-sphere, this implies that $\mathrm{STF}\, d_{AB} = 0$ and thus (from Eq. (2.6.6))

$$d_{AB} = \tfrac{1}{4} C^{CD} C_{CD} s_{AB}\,. \tag{2.6.9}$$

The metric component C_{AB} is related to the shear and the News tensor while M and L_A are related to the Weyl tensor components (Eq. (2.2.24)) through

$$\begin{aligned} \sigma_{AB} = -\tfrac{1}{2} C_{AB}\,, \quad & N_{AB} = -\partial_u C_{AB}\,, \\ \mathcal{P} = -2M + \tfrac{1}{4} C_{AB} N^{AB}\,, \quad & \mathcal{J}_A = 3L_A - \tfrac{3}{32} \mathscr{D}_A (C_{BC} C^{BC}) - \tfrac{3}{4} C_A{}^B \mathscr{D}^C C_{BC}\,. \end{aligned} \tag{2.6.10}$$

Imposing the Einstein equation to higher order in Ω either relates the higher order metric components to the lower order ones or gives evolution equations along u — for instance, one gets equations for $\partial_u M$ and $\partial_u L_A$ which are equivalent to Eq. (2.2.25) using Eq. (2.6.10). We will not need the explicit form of these higher order relations in our analysis.

The conformal Bondi-Sachs coordinates defined above can be used to define the *physical Bondi-Sachs coordinates* (r, u, x^A) which are often used in asymptotic analyses near $\mathscr{I}$.[9] Defining the physical "radial coordinate" $r := \Omega^{-1}$, and using (r, u, x^A) as coordinates, the physical metric $\hat{g}_{ab} = \Omega^{-2} g_{ab} = r^2 g_{ab}$ has the line element

$$d\hat{s}^2 \equiv -U e^{2B} du^2 - 2e^{2B} du dr + r^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du)\,, \tag{2.6.11}$$

where, from Eqs. (2.6.2), (2.6.5), (2.6.7) and (2.6.9), we have the asymptotic expansions

$$\begin{aligned} U = r^2 W = 1 - \tfrac{2}{r} M + O(1/r^2)\,, \\ B = -\tfrac{1}{32 r^2} C^{AB} C_{AB} + O(1/r^3)\,, \\ U^A = -\tfrac{1}{2r^2} \mathscr{D}_B C^{AB} + \tfrac{2}{r^3} L^A + O(1/r^4)\,, \\ h_{AB} = s_{AB} + \tfrac{1}{r} C_{AB} + \tfrac{1}{4r^2} s_{AB} C^{CD} C_{CD} + O(1/r^3)\,. \end{aligned} \tag{2.6.12}$$

Note that since Ω, and hence r, is chosen so that $\det h = \det s$, the area (in the physical metric) of the 2-spheres of constant u and r is precisely $4\pi r^2$. Thus r is a "radial coordinate" along the outgoing null surfaces $\mathcal{N}_u$ as constructed by Bondi and van der Burg [27].

In the conformal Bondi-Sachs coordinate system, the asymptotic BMS symmetries are coordinate transformations which preserve the Bondi-Sachs form of the unphysical metric. We shall only

[9] We present these equations in the two-sphere covariant form appearing in [73], and in a more modern form in [42, 74], instead of the original notation of [27].

consider the infinitesimal coordinate transformations, i.e., we take (Ω, u, x^A) and (Ω', u', x'^A) to be two conformal Bondi-Sachs coordinates, as constructed above, to be related by an infinitesimal coordinate transformation parametrized by a vector field ξ^a. Next, we obtain an expression for this vector field in the coordinate system (Ω, u, x^A).

Since in both coordinates (Ω, u, x^A) and (Ω', u', x'^A), null infinity $\mathscr{I}$ lies at $\Omega = \Omega' = 0$, the component ξ^Ω vanishes at $\Omega = 0$, i.e., ξ^a must be tangent to $\mathscr{I}$. Thus, ξ^a can be written in the coordinate system (Ω, u, x^A) as

$$\xi^a \equiv \beta \partial_u + X^A \partial_A + \Omega Z^a \partial_a + \Omega^2 W^a \partial_a + O(\Omega^3)\,, \tag{2.6.13}$$

where each of β, X^A, Z^a and W^a are (thus far) arbitrary functions of (u, x^A).

Next we note that since we are using the conformal factor itself as a coordinate, an infinitesimal change in the coordinate system is accompanied by an infinitesimal change of the conformal factor parametrized by the component ξ^Ω. Thus, when changing the coordinate system, the unphysical metric changes infinitesimally by $\pounds_\xi g_{ab} - 2\Omega^{-1}\xi^\Omega g_{ab}$; note that this is finite at $\mathscr{I}$ since ξ^Ω vanishes there. We now require that this change preserve the Bondi-Sachs form of the metric obtained in Eq. (2.6.2) along with Eq. (2.6.5) and the equations below it.

As discussed above, the metric on $\mathscr{I}$, given by Eq. (2.6.1), is universal and so we require that at $O(\Omega^0)$, $\pounds_\xi g_{ab} - 2\Omega^{-1}\xi^\Omega g_{ab} \hat{=} 0$. This gives us the following conditions: X^A is constant along u and satisfies the conformal Killing equation on the cross-sections, that is,

$$\mathscr{D}_{(A} X_{B)} = \tfrac{1}{2} q_{AB} \mathscr{D}_C X^C\,, \quad \partial_u X^A = 0\,, \tag{2.6.14}$$

while the components of Z^a satisfy

$$Z^\Omega = \partial_u \beta = \tfrac{1}{2} \mathscr{D}_A X^A\,, \quad Z^u = 0\,, \quad Z_A = -\mathscr{D}_A \beta\,. \tag{2.6.15}$$

The first condition in Eq. (2.6.15) allows us to solve for the u-dependence of β to get

$$\beta = f + \tfrac{1}{2}(u - u_0)\mathscr{D}_A X^A \,, \tag{2.6.16}$$

where f is an arbitrary function of x^A which denotes the value of β at some choice of cross-section with $u = u_0$.

At $O(\Omega)$, requiring that the form of the unphysical metric be preserved, we further obtain

$$W^u = 0 \,, \quad W_A = \tfrac{1}{2} C_{AB} \mathscr{D}^B \beta \,, \tag{2.6.17}$$

and the requirement that the metric component C_{AB} remain trace-free with respect to the unit round sphere metric gives us

$$W^\Omega = \tfrac{1}{2}\mathscr{D}_A Z^A = -\tfrac{1}{2}\mathscr{D}^2 \beta \,. \tag{2.6.18}$$

Putting all of this together, we obtain

$$\xi^a \equiv \beta \partial_u + X^A \partial_A + \Omega(-\mathscr{D}^A \beta \, \partial_A + \tfrac{1}{2}\mathscr{D}_A X^A \partial_\Omega) + \tfrac{1}{2}\Omega^2(-\mathscr{D}^2 \beta \, \partial_\Omega + C^{AB}\mathscr{D}_B \beta \, \partial_A) + O(\Omega^3) \,. \tag{2.6.19}$$

Up to $O(\Omega)$, this expression agrees with the covariant expression derived in Sec. 2.4. The form of the $O(\Omega^2)$ terms here is fixed by the choice of the conformal Bondi-Sachs coordinates and that also matches Eq. 2.16 of [42]. One could continue this computation to higher orders in Ω to obtain expressions for higher order terms in the components of ξ^a (which appear in Eqs.III.5-7 of [75]) as well as the transformation laws for the various metric components. However, these do not add anything to the discussion here and so we skip writing them.

Using the relations Eq. (2.6.10) in Eq. (2.5.44) the WZ charge on *any* cross-section S of $\mathscr{I}$ can be written in terms of the metric components in the Bondi-Sachs form of the metric to get

$$\mathcal{Q}[\xi, S] \mathrel{\hat{=}} -\frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2 \Big[-2M\beta + X^A\big(3L_A - \tfrac{1}{32}\mathscr{D}_A(C_{BC}C^{BC}) - \tfrac{1}{2}C_A{}^B \mathscr{D}^C C_{BC}\big)\Big] \,. \tag{2.6.20}$$

The above charge expression matches the charge expression written by Flanagan and Nichols in

Eq. 3.5 of [42]. Their analysis was specialized to cross sections of $\mathscr{I}$ with vanishing N_{ab} and we see here that the final charge expression remains unchanged even when N_{ab} is non-vanishing. A similar expression was also obtained by Barnich and Troessaert [76] with non-vanishing N_{ab} but they do not obtain an integrable expression.

From the above expression we also see that the function M determines the Bondi mass at any cross-section of $\mathscr{I}$ and can be called the *mass aspect*. In the Bondi-Sachs coordinates it coincides with the (constant) mass parameter of Kerr spacetimes. Similarly, in the usual choice of Bondi-Sachs coordinates in Kerr spacetime, the angular momentum parameter a appears in the metric component L_A [77]. Note however that on cross-sections with shear, there are other terms containing C_{AB} in the charge formula.

2 Conformal Gaußian null coordinates

Instead of extending the coordinates (Ω, u, x^A) away from $\mathscr{I}$ along null hypersurfaces, we can extend them into the (unphysical) spacetime along affine null geodesics, transverse to $\mathscr{I}$. We recall this construction below which leads to the *conformal Gaußian null coordinates* in a neighborhood of $\mathscr{I}$.

We fix the conformal factor away from $\mathscr{I}$ as follows. Consider the *expansion* of l^a at $\mathscr{I}$ defined by

$$\vartheta := Q^{ab}\nabla_a l_b \mathrel{\hat{=}} \tfrac{1}{2} Q^{ab} \pounds_l g_{ab} \,. \tag{2.6.21}$$

We can set this expansion to vanish by suitably choosing the conformal factor (infinitesimally) away from $\mathscr{I}$ as follows. Note that the conformal factor on $\mathscr{I}$ has already been fixed so that the induced metric on cross-sections of $\mathscr{I}$ is the unit-sphere metric. Consider a new conformal factor $\tilde{\Omega} = \omega\Omega$ (with $\omega \mathrel{\hat{=}} 1$) so that $\tilde{g}_{ab} = \omega^2 g_{ab}$. Then, the expansion of the new auxiliary normal $\tilde{l}^a \mathrel{\hat{=}} l^a$ can be computed to be (the behaviour of the auxiliary normal away from $\mathscr{I}$ is not relevant here)

$$\tilde{\vartheta} \mathrel{\hat{=}} \tfrac{1}{2} \tilde{Q}^{ab} \pounds_{\tilde{l}} \tilde{g}_{ab} \mathrel{\hat{=}} \vartheta + 2\pounds_l \omega \,. \tag{2.6.22}$$

Then, choosing ω to be any solution of $\vartheta + 2\pounds_l\omega = 0$ with $\omega \mathrel{\hat{=}} 1$, we can set $\tilde{\vartheta} \mathrel{\hat{=}} 0$. In the rest of this section we work with the choice of conformal factor where the auxiliary normal is expansion-free at $\mathscr{I}$ — for simplicity, we drop the "tilde" from the notation from hereon.

Having made this choice, we then extend the vector field $l^a \mathrel{\hat{=}} -\partial/\partial\Omega$ away from $\mathscr{I}$ such that it is the generator of affine null geodesics so that $l^b \nabla_b l^a = 0$. We can further use the remaining freedom in the conformal factor Ω to pick Ω to be the affine parameter along the null geodesics generated by l^a. To summarize, we can always choose the conformal factor Ω and extend the auxiliary normal away from $\mathscr{I}$ so that in a neighbourhood of $\mathscr{I}$ we have

$$l^a = -\frac{\partial}{\partial\Omega}\,, \quad \vartheta \mathrel{\hat{=}} 0\,, \quad l^a l_a = 0\,, \quad l^b \nabla_b l^a = 0. \tag{2.6.23}$$

Finally, we extend (u, x^A) into the spacetime by parallel-transport along l^a, that is, we require

$$l^a \nabla_a u = l^a \nabla_a x^A = 0\,. \tag{2.6.24}$$

This construction gives us the conformal Gaußian null coordinates in a neighbourhood of $\mathscr{I}$.

The most general form of the unphysical metric in these coordinates is given by

$$ds^2 = 2du(d\Omega - \alpha du - \beta_A dx^A) + h_{AB} dx^A dx^B\,, \tag{2.6.25}$$

To see why this is the most general form, note that $g_{\Omega\Omega} = 0$ by $l^a l_a = 0$ and $g_{\Omega A} = 0$ by $l^a \nabla_a x^A = 0$. Then, $l^b \nabla_b l^a = 0$ gives $\frac{\partial g_{u\Omega}}{\partial\Omega} = 0$ which implies $g_{u\Omega} = 1$ by the condition $l^a n_a \mathrel{\hat{=}} -1$. Further,

$$\alpha = O(\Omega)\,, \quad \beta_A = O(\Omega)\,, \quad h_{AB} = s_{AB} + O(\Omega)\,, \tag{2.6.26}$$

since the metric at $\mathscr{I}$ is given by Eq. (2.6.1). Imposing the Bondi condition $\nabla_a n_b \mathrel{\hat{=}} 0$ leads to the following conditions

$$\alpha = O(\Omega^2)\,, \quad \beta_A = O(\Omega^2)\,. \tag{2.6.27}$$

We therefore consider the asymptotic expansions

$$\alpha = \Omega^2\alpha^{(2)}+\Omega^3\alpha^{(3)}+O(\Omega^4)\,, \quad \beta_A = \Omega^2\beta^{(2)}_A+\Omega^3\beta^{(3)}_A+O(\Omega^4) h_{AB} = s_{AB}+\Omega C_{AB}+\Omega^2 h^{(2)}_{AB}+O(\Omega^3)\,. \tag{2.6.28}$$

The condition that the expansion of l^a vanishes on $\mathscr{I}$ gives us

$$s^{AB}C_{AB} = 0\,. \tag{2.6.29}$$

We then impose the Einstein equation Eq. (2.2.1). At $O(\Omega^0)$, this gives the conditions

$$\alpha^{(2)} = \tfrac{1}{2}\,, \quad \beta^{(2)}_A = -\tfrac{1}{2}\mathscr{D}^B C_{AB}\,, \quad s^{AB}h^{(2)}_{AB} = \tfrac{1}{4}C^{AB}C_{AB}\,, \tag{2.6.30}$$

while at $O(\Omega)$, it implies

$$\mathscr{D}^B h^{(2)}_{AB} = \tfrac{1}{8}\mathscr{D}_A(C_{BC}C^{BC})\,, \quad \partial_u h^{(2)}_{AB} = \tfrac{1}{8}s_{AB}\partial_u(C_{CD}C^{CD})\,. \tag{2.6.31}$$

As in the Bondi-Sachs case, this implies that $h^{(2)}_{AB}$ is pure trace.

The metric coefficients C_{AB}, $\alpha^{(3)}$ and $\beta^{(3)}_A$ are directly related to the shear and the Weyl tensor components by

$$\sigma_{AB} = -\tfrac{1}{2}C_{AB}\,, \quad \mathcal{P} = 2\alpha^{(3)}\,, \quad \mathcal{J}_A = \tfrac{3}{2}\beta^{(3)}_A\,. \tag{2.6.32}$$

To write an expression for the physical metric, define $\lambda := \Omega^{-1}$ so that in the coordinates (λ, u, x^A), the physical metric $\hat{g}_{ab} = \Omega^{-2}g_{ab} = \lambda^2 g_{ab}$ has the components

$$\begin{aligned}
&\hat{g}_{\lambda\lambda} = 0\,, \quad \hat{g}_{A\lambda} = 0\,, \quad \hat{g}_{u\lambda} = -1\,, \quad \hat{g}_{uu} = -1 - \frac{1}{\lambda}\mathcal{P} + O(1/\lambda^2)\,,\\
&\hat{g}_{uA} = \tfrac{1}{2}\mathscr{D}^B C_{AB} - \tfrac{2}{3\lambda}\mathcal{J}_A + O(1/\lambda^2)\,, \quad \hat{g}_{AB} = \lambda^2 s_{AB} + \lambda C_{AB} + \tfrac{1}{8}s_{AB}C_{CD}C^{CD} + O(1/\lambda)\,.
\end{aligned} \tag{2.6.33}$$

Note that the vector field

$$\hat{l}^a \equiv \frac{\partial}{\partial\lambda} = -\Omega^2\frac{\partial}{\partial\Omega} = \Omega^2 l^a\,, \tag{2.6.34}$$

generates outgoing null geodesics which are affinely parametrized with respect to the physical metric $\hat{g}_{ab}$ with the affine parameter being λ. Eq. (2.6.33) is consistent with Eqs. 5 and 58 of [78] put together (with the additional condition that our C_{AB} is tracefree since we picked l^a to be expansion-free; see Remark 2.6.1). This also gives the physical metric in the coordinates used in the *affine-null* form used in [79, 80], as well as in Newman-Unti coordinates [81].

We can also derive the form of the BMS vector fields by considering infinitesimal coordinate transformations between two conformal Gaußian null coordinate systems and demanding that the conditions on the unphysical metric derived above be preserved. Since the computation proceeds exactly as in the case of Bondi-Sachs coordinates detailed above, we skip the details. The end result is that the BMS vector field in conformal Gaussian null coordinates takes the same form as Eq. (2.6.19) above — the difference in the form of the BMS vector fields written in conformal Gaußian null coordinates and conformal Bondi-Sachs coordinates only appears at $O(\Omega^3)$ and higher. Note that this form is different than the one obtained by [78] since our coordinates differ slightly from theirs as explained in Remark 2.6.1 below. The WZ charge (Eq. (2.5.44)) can also be straightforwardly written in these coordinates using Eq. (2.6.32).

Remark 2.6.1 (Comparison of different conformal Gaussian null coordinates). The conformal Gaußian null coordinates constructed in this section are closely related to, but not the same as, the ones used in [78, 82, 83]. Note that these references use the freedom in the conformal factor to set the metric coefficient $\alpha^{(2)} = 1/2$ (as in Eq. (2.6.30)) and then the Einstein equations imply that the expansion ϑ of the auxiliary normal l^a is constant along n^a on $\mathscr{I}$, i.e., $\pounds_n \vartheta \hat{=} 0$. In contrast, we used the conformal freedom to set $\vartheta \hat{=} 0$ and then $\alpha^{(2)} = 1/2$ followed from the Einstein equation.

2.7 | Future directions

While the analysis presented here was limited to null infinity in asymptotically flat spacetimes in vacuum general relativity, the Wald-Zoupas prescription is in fact much more general and can be used to obtain local and covariant charges for arbitrary diffeomorphism covariant Lagrangian theories of gravity including gravity coupled to electromagnetism [68] and Brans-Dicke theory [84].

The Wald-Zoupas prescription has also been applied to the context of symmetries and charges associated with finite null surfaces and horizons [55,85] and spatial infinity [5] (discussed in chapter 3) in asymptotically flat spacetimes in vacuum general relativity. We anticipate that the explicit computations presented here will be useful for future similar analyses in other contexts, for instance, in spacetimes with compact extra dimensions [86].

Chapter 3

Asymptotic symmetries and charges at spatial infinity in general relativity

(Adapted with permission from [5])

Chapter summary

In this chapter, we analyze the asymptotic symmetries and their associated charges at spatial infinity in 4-dimensional asymptotically flat spacetimes. We use the covariant formalism of Ashtekar and Hansen where the asymptotic fields and symmetries live on the 3-manifold of spatial directions at spatial infinity represented by a timelike unit-hyperboloid (or de Sitter space). Using the covariant phase space formalism, we derive formulae for the charges corresponding to asymptotic supertranslations and Lorentz symmetries at spatial infinity. To keep our results as general as possible, we do not impose any restrictions on the choice of conformal factor in contrast to previous work on this problem. Several results derived in this chapter will be useful for the calculations of chapter 5.

3.1 Context

As described in the introduction section of this thesis, there have a series of recent developments in the study of the asymptotic symmetries and charges in asymptotically flat spacetimes in recent in years. In one such development, it was conjectured by Strominger [22] that the (a priori independent) BMS groups at past and future null infinity are related via an antipodal reflection near spatial infinity. This matching relation gives a *global* "diagonal" asymptotic symmetry group for general relativity which was subsequently cojectured to be the symmetry group of the quantum gravity S-matrix in asymptotically flat spacetimes. If similar matching conditions relate the gravitational fields, then there exist infinitely many conservation laws in classical gravitational scattering between the incoming fluxes associated with the BMS group at past null infinity and the outgoing fluxes of the corresponding (antipodally identified) BMS group at future null infinity.

These matching conditions on the asymptotic symmetries and fields have been proven in Maxwell theory on Minkowski spacetime [87] as well as more generally in general asymptotically-flat spacetimes [9]. In the gravitational case, the matching of the supertranslation symmetries and the corresponding (supermomentum) charges has also been proven for linearized perturbations on a Minkowski background [88] as well as in general asymptotically flat spacetimes [10]. For the translation symmetries, the proof in [10] reduces to the older result of [11] which shows that the Bondi 4-momentum on future and past null infinity matches the ADM 4-momentum at spatial infinity.

The main technique used in [9, 10, 87, 88] to prove these matching conditions is to "interpolate" between the symmetries and charges at past and future null infinities using the field equations and the asymptotic symmetries and charges defined near spatial infinity. In a background Minkowski spacetime this analysis can be done using asymptotic Bondi-Sachs coordinates near each null infinity and asymptotic Beig-Schmidt coordinates (discussed in Appendix A.4) near spatial infinity. Using the explicit transformations between these coordinate systems the matching conditions can be shown to hold for Maxwell fields and linearized gravity on Minkowski spacetime [87, 88]. However, in general asymptotically flat spacetimes, the transformations between the asymptotic coordinates is not known explicitly. In this case the covariant formulation of asymptotic-flatness given by Ashtekar and Hansen [16], which treats both null and spatial infinities in a unified spacetime-covariant manner, has proven fruitful to analyze the matching of the symmetries and charges [9, 10].

In this chapter, with an eye towards establishing these conjectured matching conditions for Lorentz symmetries and charges which we will discuss at length in chapter 5, we revisit the formulation of the asymptotic symmetries and charges at spatial infinity. The asymptotic behaviour at spatial infinity can be studied using many different (but related) formalisms. Since our primary motivation is to make contact with null infinity, it will be useful to use a spacetime covariant formalism without using a $(3+1)$ decomposition of the spacetime by spacelike hypersurfaces [14, 15, 89]. Such a 4-dimensional formulation of asymptotic-flatness at spatial infinity can be be given using suitable asymptotic coordinates as formulated by Beig and Schmidt [90]. The asymptotic symmetries and charges using the asymptotic expansion of the metric in these coordinates have been worked out in detail in [90–92]. However, as mentioned above, the relation between the Beig-Schmidt coordinates and the coordinates adapted to null infinity (like the Bondi-Sachs coordinates) is not known in general spacetimes. Thus, we will use the coordinate independent formalism of Ashtekar and Hansen [16, 17] (Def. 3.2.1) to investigate the symmetries and their associated charges at spatial infinity.[1]

The asymptotic behaviour of the gravitational field for any asymptotically flat spacetime is most conveniently described in a conformally-related unphysical spacetime given by Penrose's conformal completion. In the unphysical spacetime, null infinities $\mathscr{I}^{\pm}$ are smooth null boundaries while spatial infinity is a boundary point i^0 which is the vertex of "the light cone at infinity" formed by $\mathscr{I}^{\pm}$. For Minkowski spacetime, the unphysical spacetime is smooth (and in fact, analytic) at i^0. However, in more general spacetimes, the unphysical metric is not even once-differentiable at spatial infinity unless the ADM mass of the spacetime vanishes [16], and the unphysical spacetime manifold *does not* have a smooth differential structure at i^0. In the Ashtekar-Hansen formalism, instead of working directly at the point i^0 where sufficiently smooth structure is unavailable, one works on a "blowup" — the space of spatial directions at i^0 — given by a timelike-unit-hyperboloid $\mathscr{H}$ in the tangent space at i^0. Suitably conformally rescaled fields, whose limits to i^0 depend on the direction of approach, induce *smooth* fields on $\mathscr{H}$ and we can study these smooth limiting fields using standard differential calculus on $\mathscr{H}$. For instance, (as we will show below) in Maxwell theory the rescaled field tensor ΩF_{ab} and in general relativity the rescaled (unphysical) Weyl tensor $\Omega^{1/2} C_{abcd}$ (where Ω is the conformal factor used in the Penrose conformal completion) admit regular direction-dependent limits to i^0, and these fields induce smooth tensor fields on $\mathscr{H}$. Similarly, the Maxwell gauge transformations

[1] The relation between the Ashtekar-Hansen formalism and the Beig-Schmidt coordinates is summarized in Appendix A.4.

and vector fields in the physical spacetime (suitably rescaled) admit regular direction-dependent limits which generate the asymptotic symmetries at i^0.

There have already been other other studies of asymptotic symmetries at spatial infinity using the Ashtekar-Hansen formalism by Ashtekar and Hansen [16, 17]. However in deriving the charges associated with these symmetries, they reduced the asymptotic symmetry algebra from the infinite-dimensional $\mathfrak{spi}$ algebra to the Poincaré algebra consisting only of translations and Lorentz transformations. This reduction was accomplished by demanding that the "leading order" magnetic part of the Weyl tensor, given by a tensor $\boldsymbol{B}_{ab}$ on $\mathscr{H}$ (see Eq. (3.4.5)), vanish and, additionally, choosing the conformal factor near i^0 so that the tensor potential $\boldsymbol{K}_{ab}$ for $\boldsymbol{B}_{ab}$ also vanishes (see Remark 3.6.3). This restriction was also imposed in [91, 93]. In the work of Compère and Dehouck in [92], the condition $\boldsymbol{B}_{ab} = 0$ was not imposed however, they also specialized to a conformal factor where the trace $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab}$ (where $\boldsymbol{h}^{ab}$ denotes the inverse of the metric on $\mathscr{H}$) was set to vanish. As we will show below (see Sec. 3.7.3) the charges of the Lorentz symmetries at spatial infinity are not conformally-invariant but shift by the charge of a supertranslation. This is entirely analogous to the supertranslation ambiguities in the Lorentz charges at null infinity (recall Remark 2.4.2). To keep our expressions as general as possible, we will not impose any such restrictions on the conformal factor or impose any conditions on $\boldsymbol{K}_{ab}$ (apart from its equations of motion arising from the Einstein equation) in our analysis. As we will show, one peculiar consequence of keeping a completely unrestricted conformal factor will be that our charges will not be exactly conserved but will have a non-vanishing flux through regions of $\mathscr{H}$ (except for pure translations). This means that these charges are not associated with the point i^0 at spatial infinity, but with cross-sections of the "blowup" $\mathscr{H}$. This is not a serious drawback; as shown in [9, 10], for matching the symmetries and charges at null infinity, one only requires that the *total* flux of the charges through all of $\mathscr{H}$ vanish —there can be a non-vanishing flux through local regions of $\mathscr{H}$.

* * *

In our analysis of the asymptotic charges we will use the covariant phase space formalism which was described in chapter 1. Since the relevant quantities in the covariant phase space are defined in terms of the physical metric and their perturbations, we first analyze the conditions on the corresponding unphysical quantities so that they preserve the asymptotic-flatness conditions and the universal structure at i^0 (Sec. 3.5). To derive the asymptotic symmetry algebra we then consider a physical metric perturbation $\pounds_\xi \hat{g}_{ab}$ generated by an infinitesimal diffeomorphism and demand that it preserve the asymptotic conditions in the unphysical spacetime in the limit to i^0. This will provide us with the following description of the asymptotic symmetries at i^0 (Sec. 3.6). The asymptotic symmetry algebra, called the $\mathfrak{spi}$ algebra, is parametrized by a pair $(\boldsymbol{f}, \boldsymbol{X}^a)$ where $\boldsymbol{f}$ is any smooth function and $\boldsymbol{X}^a$ is a Killing field on $\mathscr{H}$. The function $\boldsymbol{f}$ parametrizes the supertranslations [2] and $\boldsymbol{X}^a$ parametrize the Lorentz symmetries. The $\mathfrak{spi}$ algebra is then a semi-direct sum of the Lorentz algebra with the infinite-dimensional abelian subalgebra of supertranslations. Note that this is the same as the asymptotic symmetry algebra derived in [16, 17]. The main difference in our analysis is that we obtain the symmetries by analyzing the conditions on diffeomorphisms in the physical spacetime instead of using the unphysical spacetime directly as in [16, 17].

To obtain the charges associated with these symmetries, the primary quantity of interest is the symplectic current derived from the Lagrangian of a theory. Recall from chapters 1 and 2 that the symplectic current [3] $\omega(\hat{g}; \delta_1 \hat{g}, \delta_2 \hat{g})$, is a local and covariant 3-form and is an antisymmetric bilinear in two metric perturbations, $\delta\hat{g}$ of the physical spacetime metric. It can be shown that when the second perturbation $\delta\hat{g}_{ab} = \pounds_\xi \hat{g}_{ab}$ is the perturbation corresponding to an infinitesimal diffeomorphism generated by a vector field ξ^a, we have

$$\omega(\hat{g}; \delta\hat{g}, \pounds_\xi \hat{g}) = d[\delta Q_\xi - \xi \cdot \theta(\delta\hat{g})] \,, \tag{3.1.1}$$

where we have assumed that $\hat{g}_{ab}$ satisfies the equations of motion and $\delta\hat{g}_{ab}$ satisfies the linearized equations of motion. The 2-form Q_ξ is the Noether charge associated with the vector field ξ^a and the 3-form $\theta(\delta\hat{g})$ is the symplectic potential. If we integrate Eq. (3.1.1) over a 3-dimensional surface

[2] Note that the supertranslations and Lorentz symmetries discussed in this chapter are distinct from the ones discussed in chapter 2. They only become related in a certain limit as we will discuss in chapter 5; see also [10].

[3] To avoid conflict with our bold-faced notation for regular direction-dependent tensor fields, we drop the bold face on ω, θ and Q_ξ in this chapter.

Σ with boundary $\partial\Sigma$ we get

$$\int_\Sigma \omega[\hat{g}; \delta\hat{g}, \pounds_\xi \hat{g}] = \int_{\partial\Sigma} \delta Q_\xi - \xi \cdot \theta(\delta\hat{g}) \,. \tag{3.1.2}$$

To define the asymptotic charges at spatial infinity, we would like to evaluate Eq. (3.1.2) when the surface Σ extends to a suitably regular 3-surface at i^0 in the unphysical spacetime. Given the low amount of differentiability at i^0 the appropriate condition is that Σ extends to a $C^{>1}$ surface at i^0. The limit of the boundary $\partial\Sigma$ to i^0 corresponds to a 2-sphere cross-section S of the unit-hyperboloid $\mathscr{H}$ in the Ashtekar-Hansen formalism. Then, the limiting integral on the right-hand-side of Eq. (3.1.2) (with the asymptotic conditions imposed on the metric perturbations as well as the symmetries) will define a perturbed charge on S associated with the asymptotic symmetry generated by ξ^a. However, even though the explicit expressions for the integrand on the right-hand-side of Eq. (3.1.2) are well-known (see for instance [53]), computing this limiting integral is difficult. Therefore, we will use an alternative strategy which we describe next.

We will show that with the appropriate asymptotic-flatness conditions at i^0, the symplectic current 3-form $\omega \equiv \omega_{abc}$ is such that $\Omega^{3/2}\omega_{abc}$ has a direction-dependent limit to i^0. The pullback of this limit to $\mathscr{H}$, which we denote by $\underleftarrow{\boldsymbol{\omega}}$, defines a symplectic current on $\mathscr{H}$. We will show that when one of the perturbations in this symplectic current is generated by an asymptotic $\mathfrak{spi}$ symmetry $(\boldsymbol{f}, \boldsymbol{X}^a)$, we have

$$\underleftarrow{\boldsymbol{\omega}}(g; \delta g, \delta_{(\boldsymbol{f},\boldsymbol{X})} g) = -\boldsymbol{\varepsilon}_3 \boldsymbol{D}^a \boldsymbol{Q}_a(g; \delta g, (\boldsymbol{f}, \boldsymbol{X}))\,, \tag{3.1.3}$$

where $\boldsymbol{\varepsilon}_3$ and $\boldsymbol{D}$ are the volume element and covariant derivative on $\mathscr{H}$. The covector $\boldsymbol{Q}_a(g; \delta g, (\boldsymbol{f}, \boldsymbol{X}))$ is a local and covariant functional of the background fields corresponding to the asymptotic (unphysical) metric g_{ab}, and is linear in the asymptotic (unphysical) metric perturbations δg_{ab} and the asymptotic symmetry parametrized by $(\boldsymbol{f}, \boldsymbol{X}^a)$. Thus, we can write the symplectic current, with one perturbation generated by an asymptotic symmetry, as a total derivative on $\mathscr{H}$. Then, in analogy with Eq. (3.1.2), we define the perturbed charge on a cross-section S of $\mathscr{H}$ by the integral

$$\int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{Q}_a(g; \delta g, (\boldsymbol{f}, \boldsymbol{X})\,, \tag{3.1.4}$$

where $\boldsymbol{\varepsilon}_2$ is the area element and $\boldsymbol{u}^a$ is a unit-timelike normal to the cross-section S within $\mathscr{H}$. We then show that when the asymptotic symmetry is a supertranslation $\boldsymbol{f}$, the quantity $\boldsymbol{Q}_a(g;\delta g,\boldsymbol{f})$ is integrable, i.e, it can be written as the δ of some covector which is itself a local and covariant functional of the asymptotic fields and supertranslation symmetries. Then "integrating" Eq. (3.1.4) in the space of asymptotic fields, we can define a charge associated with supertranslations on any cross-section S of $\mathscr{H}$ (see Sec. 3.7.1). When the asymptotic symmetry is a Lorentz symmetry parameterized by a Killing vector field $\boldsymbol{X}^a$ on $\mathscr{H}$, Eq. (3.1.4) *cannot* be written as the δ of some quantity (unless we restrict to the choice of conformal factor where $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab} = 0$ as described above). In this case, we will adapt the prescription by Wald and Zoupas [53] (discussed in Sec. 2.5 as well) to define an integrable charge for Lorentz symmetries (Sec. 3.7.2). Then, the change of these charges over a region $\Delta\mathscr{H}$ bounded by two cross-sections provides a flux formula for these charges. In general, these fluxes will be non-vanishing (except for translation symmetries) unless we again restrict to the conformal factor where $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab} = 0$. However, as discussed above, this is not a problem.

* * *

The rest of this chapter is organized as follows. In Sec. 3.2 we recall the definition of asymptotic-flatness at spatial infinity in terms of an Ashtekar-Hansen structure. To illustrate the approach outlined above, we first study the simpler case of Maxwell fields at spatial infinity, and derive the associated symmetries and charges in Sec. 3.3. In Sec. 3.4 we then consider the asymptotic gravitational fields and Einstein equations at spatial infinity. We also describe the universal structure, that is the structure that is common to all spacetimes which are asymptotically-flat at i^0, in Sec. 3.4.1. In Sec. 3.5 we analyze the conditions on metric perturbations which preserve asymptotic flatness and obtain the limiting form of the symplectic current of general relativity on $\mathscr{H}$. In Sec. 5.3.2, using the analysis of the preceding section, we derive the asymptotic symmetry algebra (the $\mathfrak{spi}$ algebra) by considering infinitesimal metric perturbations generated by diffeomorphisms which preserve the asymptotic flatness conditions. In Sec. 3.7 we derive the charges and fluxes corresponding to the $\mathfrak{spi}$ symmetries. We end by describing a few possible future directions in Sec. 3.8. The appendices contain some useful results that supplement the discussion in this chapter. In Appendix A.4, we construct a suitable coordinate system near i^0 using the asymptotic flatness conditions on the unphysical metric and relate it to the Beig-Schmidt coordinate system in the physical spacetime. Appendix A.5 collects useful results on the unit-hyperboloid $\mathscr{H}$ on Killing vector fields, symmetric tensor fields and a theorem by Wald showing that (with suitable conditions) closed differential forms are exact. Computations detailing the change in the Lorentz charge under conformal transformations are presented in Appendix A.6. In Appendix A.7, we show that our charges are unambiguously defined by the the symplectic current of vacuum general relativity. In Appendix A.8 we generalize the Lorentz charges derived in Sec. 3.7.2 to include spacetimes where the "leading order" magnetic part of the Weyl tensor $\boldsymbol{B}_{ab}$ is allowed to be non-vanishing.

3.2 | Asymptotic-flatness at spatial infinity: Ashtekar-Hansen structure

We define spacetimes which are asymptotically-flat at null and spatial infinity using an Ashtekar-Hansen structure [16, 17]. We use the following the notation for causal structures from [94]: $J(i^0)$ is the causal future of a point i^0 in M, $\overline{J}(i^0)$ is its closure, $\dot{J}(i^0)$ is its boundary and $\mathscr{I} := \dot{J}(i^0) - i^0$. We also use the definition and notation for direction-dependent tensors from [95], see also Appendix B of [10].

Definition 3.2.1 (Ashtekar-Hansen structure [17]). A *physical* spacetime $(\hat{M}, \hat{g}_{ab})$ has an *Ashtekar-Hansen structure* if there exists another *unphysical* spacetime (M, g_{ab}), such that

(1) M is C^∞ everywhere except at a point i^0 where it is $C^{>1}$,

(2) the metric g_{ab} is C^∞ on $M - i^0$, and C^0 at i^0 and $C^{>0}$ along spatial directions at i^0,

(3) there is an embedding of $\hat{M}$ into M such that $\overline{J}(i^0) = M - \hat{M}$,

(4) there exists a function Ω on M, which is C^∞ on $M - i^0$ and C^2 at i^0 so that $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$ and

   (a) $\Omega = 0$ on $\dot{J}(i^0)$,

   (b) $\nabla_a \Omega \neq 0$ on $\mathscr{I}$,

   (c) at i^0, $\nabla_a \Omega = 0$, $\nabla_a \nabla_b \Omega = 2 g_{ab}$.

(5) There exists a neighbourhood N of $\dot{J}(i^0)$ such that (N, g_{ab}) is strongly causal and time orientable, and in $N \cap \hat{M}$ the physical metric $\hat{g}_{ab}$ satisfies the vacuum Einstein equation $\hat{R}_{ab} = 0$,

(6) The space of integral curves of $n^a = g^{ab}\nabla_b\Omega$ on $\dot{J}(i^0)$ is diffeomorphic to the space of null directions at i^0,

(7) The vector field $\varpi^{-1}n^a$ is complete on $\mathscr{I}$ for any smooth function ϖ on $M - i^0$ such that $\varpi > 0$ on $\hat{M} \cup \mathscr{I}$ and $\nabla_a(\varpi^4 n^a) = 0$ on $\mathscr{I}$.

The physical role of the conditions in Def. 3.2.1 is to ensure that the point i^0 is spacelike related to all points in the physical spacetime $\hat{M}$, and represents *spatial infinity*, and that null infinity $\mathscr{I} := \dot{J}(i^0) - i^0$ has the usual structure. Note that the metric g_{ab} is only $C^{>0}$ at i^0 along spatial directions, that is, the metric is continuous but the metric connection is allowed to have limits which depend on the direction of approach to i^0. This low differentiability structure is essential to allow spacetimes with non-vanishing ADM mass [16, 17]. In what follows, we will only consider the behaviour of the spacetime approaching i^0 along spatial directions, and we will not need the conditions corresponding to null infinity.

* * *

For spacetimes satisfying Def. 3.2.1 we have the following limiting structures at i^0 when approached along spatial directions.

Along spatial directions $\eta_a := \nabla_a\Omega^{1/2}$ is $C^{>-1}$ at i^0 and

$$\boldsymbol{\eta}^a := \lim_{\to i^0} \nabla^a\Omega^{1/2}\,, \tag{3.2.1}$$

determines a $C^{>-1}$ spatial unit vector field at i^0 representing the spatial directions $\vec{\eta}$ at i^0. The space of directions $\vec{\eta}$ in Ti^0 is a unit-hyperboloid $\mathscr{H}$.

If $T^{a\dots}{}_{b\dots}$ is a $C^{>-1}$ tensor field at i^0 in spatial directions then, $\lim_{\to i^0} T^{a\dots}{}_{b\dots} = \boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta})$ is a smooth tensor field on $\mathscr{H}$. Further, the derivatives of $\boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta})$ to all orders with respect to the direction $\vec{\eta}$ satisfy[4]

$$\boldsymbol{\partial}_c \cdots \boldsymbol{\partial}_d \boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta}) = \lim_{\to i^0} \Omega^{1/2} \nabla_c \cdots \Omega^{1/2} \nabla_d T^{a\dots}{}_{b\dots}\,, \tag{3.2.2}$$

where $\boldsymbol{\partial}_a$ is the derivative with respect to the directions $\vec{\eta}$ defined by

$$\begin{aligned} &\boldsymbol{v}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta} + \epsilon \vec{v}) - \boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta}) \right] \quad \text{for all } \boldsymbol{v}^a \in T\mathscr{H}\,, \\ &\boldsymbol{\eta}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\dots}{}_{b\dots}(\vec{\eta}) := 0\,. \end{aligned} \tag{3.2.3}$$

The metric $\boldsymbol{h}_{ab}$ induced on $\mathscr{H}$ by the universal metric $\boldsymbol{g}_{ab}$ at i^0, satisfies

$$\boldsymbol{h}_{ab} := \boldsymbol{g}_{ab} - \boldsymbol{\eta}_a \boldsymbol{\eta}_b = \boldsymbol{\partial}_a \boldsymbol{\eta}_b\,. \tag{3.2.4}$$

Further, if $\boldsymbol{T}^{a\cdots}{}_{b\dots}(\vec{\eta})$ is orthogonal to $\boldsymbol{\eta}^a$ in all its indices then it defines a tensor field $\boldsymbol{T}^{a\cdots}{}_{b\dots}$ intrinsic to $\mathscr{H}$. In this case, it follows from Eq. (3.2.4) and $\boldsymbol{\partial}_c \boldsymbol{g}_{ab} = 0$ (since $\boldsymbol{g}_{ab}$ is direction-independent at i^0) that projecting *all* the indices in Eq. (3.2.2) using $\boldsymbol{h}_{ab}$ defines a derivative operator $\boldsymbol{D}_a$ intrinsic to $\mathscr{H}$ which is also the covariant derivative operator associated with $\boldsymbol{h}_{ab}$. We also define

$$\boldsymbol{\varepsilon}_{abc} := -\boldsymbol{\eta}^d \boldsymbol{\varepsilon}_{dabc}\,, \quad \boldsymbol{\varepsilon}_{ab} := \boldsymbol{u}^c \boldsymbol{\varepsilon}_{cab}\,, \tag{3.2.5}$$

where $\boldsymbol{\varepsilon}_{abcd}$ is volume element at i^0 corresponding to the metric $\boldsymbol{g}_{ab}$, $\boldsymbol{\varepsilon}_{abc}$ is the induced volume element on $\mathscr{H}$, and $\boldsymbol{\varepsilon}_{ab}$ is the induced area element on some cross-section S of $\mathscr{H}$ with a future-pointing timelike normal $\boldsymbol{u}^a$ such that $\boldsymbol{h}_{ab}\boldsymbol{u}^a\boldsymbol{u}^b = -1$.

Note that $\mathscr{H}$ admits a reflection isometry which can be seen as follows. We introduce coordinates (τ, θ^A) on $\mathscr{H}$— where $\tau \in (-\infty, \infty)$, and $\theta^A = (\theta, \phi)$ are the usual spherical coordinates on $\mathbb{S}^2$— such that in these coordinates, the metric on $\mathscr{H}$ is

$$\boldsymbol{h}_{ab} \equiv -d\tau^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2)\,. \tag{3.2.6}$$

[4] The factors of $\Omega^{1/2}$ on the right hand side of Eq. (3.2.2) convert between ∇_a and the derivatives with respect to the directions; see [17, 62].

Using $\Upsilon\circ$ to denote the action of the reflection map Υ on tensor fields on $\mathscr{H}$, we see that

$$\begin{aligned} &\Upsilon : \mathscr{H} \to \mathscr{H} : (\tau, \theta^A) \mapsto (-\tau, -\theta^A) \\ &\text{with } \Upsilon \circ \boldsymbol{h}_{ab} = \boldsymbol{h}_{ab} \,, \end{aligned} \tag{3.2.7}$$

where $\theta^A = (\theta, \phi) \mapsto -\theta^A = (\pi - \theta, \phi \pm \pi)$ is the antipodal reflection on $\mathbb{S}^2$; the sign is chosen so that $\phi \pm \pi \in [0, 2\pi)$.

Remark 3.2.1 (Conformal freedom). It follows from the conditions in Def. 3.2.1 that the allowed conformal freedom $\Omega \mapsto \omega\Omega$ is such that $\omega > 0$ is smooth in $M - i^0$, is $C^{>0}$ at i^0 and $\omega|_{i^0} = 1$. From these conditions it follows that

$$\omega = 1 + \Omega^{1/2} \alpha \,, \tag{3.2.8}$$

where α is $C^{>-1}$ at i^0. Let $\boldsymbol{\alpha}(\vec{\eta}) := \lim_{\to i^0} \alpha$, then from Eq. (3.2.8) we also get

$$\lim_{\to i^0} \nabla_a \omega = \boldsymbol{\alpha} \boldsymbol{\eta}_a + \boldsymbol{D}_a \boldsymbol{\alpha} \,. \tag{3.2.9}$$

Note in particular, that the unphysical metric $\boldsymbol{g}_{ab}$ at i^0 is invariant under conformal transformations. While

$$\eta^a \mapsto \omega^{-2} [\omega^{1/2} \eta^a + \tfrac{1}{2} \omega^{-1/2} \Omega^{1/2} \nabla^a \omega] \implies \boldsymbol{\eta}^a \mapsto \boldsymbol{\eta}^a \,. \tag{3.2.10}$$

Thus, unit spatial directions $\vec{\eta}$, the space of directions $\mathscr{H}$, and the induced metric on it $\boldsymbol{h}_{ab}$ are also invariant under conformal transformations.

3.3 | Maxwell fields: symmetries and charges at i^0

To illustrate our general strategy, we first consider the simpler case of Maxwell fields on any fixed background spacetime satisfying Def. 3.2.1.

In the physical spacetime $\hat{M}$, let $\hat{F}_{ab}$ be the Maxwell field tensor satisfying the Maxwell equations

$$\hat{g}^{ac} \hat{g}^{bd} \hat{\nabla}_b \hat{F}_{dc} = 0 \,, \quad \hat{\nabla}_{[a} \hat{F}_{bc]} = 0 \,. \tag{3.3.1}$$

In the unphysical spacetime M with $F_{ab} := \hat{F}_{ab}$ we have

$$\nabla_b F^{ba} = 0\,, \quad \nabla_{[a} F_{bc]} = 0\,. \tag{3.3.2}$$

The Maxwell tensor F_{ab} is smooth everywhere in the unphysical spacetime except at i^0. Analyzing the behaviour of F_{ab} in the simple case of a static point charge in Minkowski spacetime, it can be seen that F_{ab} diverges in the limit to i^0 but that ΩF_{ab} admits a direction-dependent limit.[5] Hence we assume as our asymptotic condition that

$$\lim_{\to i^0} \Omega F_{ab} = \boldsymbol{F}_{ab}(\vec{\eta}) \text{ is } C^{>-1}\,. \tag{3.3.3}$$

The direction-dependent limit of the Maxwell tensor, $\boldsymbol{F}_{ab}$, induces smooth tensor fields on $\mathscr{H}$. These are given by the "electric" and "magnetic" parts of the Maxwell tensor defined by

$$\boldsymbol{E}_a(\vec{\eta}) = \boldsymbol{F}_{ab}(\vec{\eta})\boldsymbol{\eta}^b\,, \quad \boldsymbol{B}_a(\vec{\eta}) = *\boldsymbol{F}_{ab}(\vec{\eta})\boldsymbol{\eta}^b\,. \tag{3.3.4}$$

where $*\boldsymbol{F}_{ab}(\vec{\eta}) := \frac{1}{2}\boldsymbol{\varepsilon}_{ab}{}^{cd}\boldsymbol{F}_{cd}(\vec{\eta})$ is the Hodge dual with respect to the unphysical volume element $\boldsymbol{\varepsilon}_{abcd}$ at i^0. The electric and magnetic fields are orthogonal to $\boldsymbol{\eta}^a$ and thus induce intrinsic fields $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ on $\mathscr{H}$. Note that $\boldsymbol{F}_{ab}$ can be reconstructed from $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ using

$$\boldsymbol{F}_{ab} = 2\boldsymbol{E}_{[a}\boldsymbol{\eta}_{b]} + \boldsymbol{\varepsilon}_{abcd}\boldsymbol{\eta}^c\boldsymbol{B}^d\,. \tag{3.3.5}$$

The asymptotic Maxwell equations are obtained by multiplying Eq. (3.3.2) by $\Omega^{3/2}$ and taking the limit to i^0 in spatial directions (see [16] for details)

$$\begin{aligned} \boldsymbol{D}^a\boldsymbol{E}_a = 0\,, &\quad \boldsymbol{D}_{[a}\boldsymbol{E}_{b]} = 0\,, \\ \boldsymbol{D}^a\boldsymbol{B}_a = 0\,, &\quad \boldsymbol{D}_{[a}\boldsymbol{B}_{b]} = 0\,. \end{aligned} \tag{3.3.6}$$

[5] Note that this diverging behaviour of F_{ab} refers to the tensor in the unphysical spacetime with the chosen $C^{>1}$ differential structure at i^0. In an asymptotically Cartesian coordinate system of the physical spacetime, this behaviour reproduces the standard $1/r^2$ falloff for F_{ab} and $\boldsymbol{F}_{ab}(\vec{\eta})$ is the "leading order" piece at $O(1/r^2)$.

To use the covariant phase space formalism for Maxwell theory, we will need to introduce the vector potential as the basic dynamical field. Let $\hat{A}_a$ be a vector potential for $\hat{F}_{ab}$ so that $\hat{F}_{ab} = 2\hat{\nabla}_{[a}\hat{A}_{b]}$ in the physical spacetime. Then, $A_a := \hat{A}_a$ is a vector potential for F_{ab} in the unphysical spacetime. We further assume that the vector potential A_a for F_{ab} is chosen so that $\Omega^{1/2} A_a$ is $C^{>-1}$ at i^0. Define the asymptotic potentials

$$\boldsymbol{V}(\vec{\eta}) := \boldsymbol{\eta}^a \lim_{\to i^0} \Omega^{1/2} A_a\,, \quad \boldsymbol{A}_a(\vec{\eta}) := \boldsymbol{h}_a{}^b \lim_{\to i^0} \Omega^{1/2} A_b\,. \tag{3.3.7}$$

Then the corresponding smooth fields $\boldsymbol{V}$ and $\boldsymbol{A}_a$ induced on $\mathscr{H}$ act as potentials for the electric and magnetic field through

$$\boldsymbol{E}_a = \boldsymbol{D}_a \boldsymbol{V}\,, \quad \boldsymbol{B}_a = \tfrac{1}{2}\boldsymbol{\varepsilon}_a{}^{bc}\boldsymbol{D}_b\boldsymbol{A}_c\,. \tag{3.3.8}$$

Even though we do not need this form, for completeness, we note that the Maxwell equations on $\mathscr{H}$ (Eq. (3.3.6)) can be written in terms of the potentials $\boldsymbol{V}$ and $\boldsymbol{A}_a$ as

$$\boldsymbol{D}^2\boldsymbol{V} = 0\,, \quad \boldsymbol{D}^2\boldsymbol{A}_a = \boldsymbol{D}_a\boldsymbol{D}^b\boldsymbol{A}_b + 2\boldsymbol{A}_a\,. \tag{3.3.9}$$

Now consider a gauge transformation of the vector potential

$$A_a \mapsto A_a + \nabla_a \lambda\,, \tag{3.3.10}$$

where λ is $C^{>-1}$ at i^0. Then with $\boldsymbol{\lambda}(\vec{\eta}) := \lim_{\to i^0} \lambda$, the gauge transformations of the asymptotic potentials (Eq. (3.3.7)) on $\mathscr{H}$ is given by

$$\boldsymbol{V} \mapsto \boldsymbol{V}\,, \quad \boldsymbol{A}_a \mapsto \boldsymbol{A}_a + \boldsymbol{D}_a\boldsymbol{\lambda}\,. \tag{3.3.11}$$

Thus, the asymptotic symmetries of Maxwell fields at i^0 are given by the functions $\boldsymbol{\lambda}$ on $\mathscr{H}$.

Remark 3.3.1 (Special choices of gauge). The gauge freedom in the Maxwell vector potential can be used to impose further restrictions on the potential $\boldsymbol{A}_a$ on $\mathscr{H}$. We illustrate the following two gauge conditions which will have analogues in the gravitational case (see Remark 3.6.3).

(1) Consider the Lorenz gauge condition $\hat{g}^{ab}\hat{\nabla}_a\hat{A}_b = 0$ on the physical vector potential $\hat{A}_a$ in the physical spacetime as used in [87, 96]. Multiplying this condition by Ω^{-1} and taking the limit to i^0, using Eq. (3.3.7) we get the asymptotic gauge condition

$$\boldsymbol{D}^a\boldsymbol{A}_a = 2\boldsymbol{V}\,. \tag{3.3.12}$$

Alternatively, from Eq. (3.3.11), we see that

$$\boldsymbol{D}^a\boldsymbol{A}_a \mapsto \boldsymbol{D}^a\boldsymbol{A}_a + \boldsymbol{D}^2\boldsymbol{\lambda}\,. \tag{3.3.13}$$

By solving a linear hyperbolic equation for $\boldsymbol{\lambda}$ we can choose a new gauge in which

$$\boldsymbol{D}^a\boldsymbol{A}_a = 0\,. \tag{3.3.14}$$

Both these gauge conditions reduce the allowed asymptotic symmetries to

$$\boldsymbol{D}^2\boldsymbol{\lambda} = 0\,. \tag{3.3.15}$$

(2) If we impose the restriction $\boldsymbol{B}_a = 0$ then $\boldsymbol{D}_{[a}\boldsymbol{A}_{b]} = 0$ and thus there exists a function $\boldsymbol{A}$ so that $\boldsymbol{A}_a = \boldsymbol{D}_a\boldsymbol{A}$.[6] Then, using the transformation Eq. (3.3.11), we can set $\boldsymbol{A}_a = 0$. The remaining asymptotic symmetries are just the Coulomb symmetries $\boldsymbol{\lambda} =$ constant. This is analogous to the condition used by Ashtekar and Hansen in the gravitational case to reduce the asymptotic symmetries to the Poincaré algebra [16].

In what follows we will not need to impose any gauge condition on the potential $\boldsymbol{A}_a$ and our analysis will be completely gauge invariant.

Remark 3.3.2 (Logarithmic gauge transformations). Note that above, we only considered gauge transformations Eq. (3.3.10) where the gauge parameter λ was $C^{>-1}$ at i^0. However, there is an additional ambiguity in the choice of gauge given by the *logarithmic gauge transformations* of the

[6] This follows from the fact that every 1-loop in $\mathscr{H}$ is contractible to a point and hence the first de Rahm cohomology group of $\mathscr{H}$ is trivial.

form

$$A_a \mapsto A_a + \nabla_a(\ln \Omega^{1/2} \Lambda)\,, \tag{3.3.16}$$

where Λ is $C^{>0}$ at i^0. Under this gauge transformation $\Omega^{1/2} A_a$ is still $C^{>-1}$ at i^0 and, from Eq. (3.3.7), we have the transformations

$$\boldsymbol{V} \mapsto \boldsymbol{V} + \boldsymbol{\Lambda}\,, \quad \boldsymbol{A}_a \mapsto \boldsymbol{A}_a\,, \tag{3.3.17}$$

where $\boldsymbol{\Lambda} := \lim_{\to i^0} \Lambda$ which is direction-independent at i^0 and induces a constant function on $\mathscr{H}$. From Eq. (3.3.8), we see that the fields $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ are invariant under this transformation. Since our charges and fluxes (derived below) will be expressed in terms of $\boldsymbol{E}_a$, we will not need to fix this logarithmic gauge ambiguity in the potentials for electromagnetism. However, there is an analogous logarithmic translation ambiguity in the gravitational case which we will need to fix (see Remark 3.4.2). To show how both these cases are similar, we illustrate how this logarithmic gauge ambiguity can be fixed even in electromagnetism.

Since the metric $\boldsymbol{g}_{ab}$ in the tangent space Ti^0 is universal and isometric to the Minkowski metric, it is invariant under the reflection of the spatial directions $\vec{\eta} \mapsto -\vec{\eta}$. This gives rise to a reflection isometry of the metric $\boldsymbol{h}_{ab}$ on the space of directions $\mathscr{H}$. It was shown in [9] that the Maxwell fields on $\mathscr{H}$ which "match" on to asymptotically flat Maxwell fields on null infinity are the ones where the electric field $\boldsymbol{E}_a$ is reflection-odd i.e.

$$\boldsymbol{E}_a(\vec{\eta}) = -\boldsymbol{E}_a(-\vec{\eta})\,. \tag{3.3.18}$$

Further, since the logarithmic gauge parameter $\boldsymbol{\Lambda}$ is *direction-independent*, we have that $\boldsymbol{\Lambda}$ is reflection-even

$$\boldsymbol{\Lambda}(\vec{\eta}) = \boldsymbol{\Lambda}(-\vec{\eta})\,. \tag{3.3.19}$$

Using a reflection-odd $\boldsymbol{E}_a$ in Eq. (3.3.8), we see that using a logarithmic gauge transformation we can demand that the potential $\boldsymbol{V}$ also be reflection-odd so that

$$\boldsymbol{V}(\vec{\eta}) = -\boldsymbol{V}(-\vec{\eta})\,. \tag{3.3.20}$$

This fixes the logarithmic gauge ambiguity in the potentials.

* * *

Let us now analyze the charges and fluxes for this theory. To do this, we start by studying the symplectic current. In vacuum electromagnetism, this is given by:

$$\omega_{abc}(\delta_1 A, \delta_2 A) = \hat{\varepsilon}_{abcd}\left(\delta_1 \hat{F}^{de}\delta_2 \hat{A}_e - \delta_2 \hat{F}^{de}\delta_1 \hat{A}_e\right), \tag{3.3.21}$$

where the indices on $\delta \hat{F}_{ab}$ have been raised with the physical metric $\hat{g}^{ab}$. In terms of quantities in the unphysical spacetime, we have

$$\omega_{abc}(\delta_1 A, \delta_2 A) = \varepsilon_{abcd}\left(\delta_1 F^{de}\delta_2 A_e - \delta_2 F^{de}\delta_1 A_e\right), \tag{3.3.22}$$

where we have used $\hat{\varepsilon}_{abcd} = \Omega^{-4}\varepsilon_{abcd}$, and $\hat{g}^{ab} = \Omega^2 g^{ab}$.

To obtain the limit to i^0, we rewrite this in terms of direction-dependent quantities from Eqs. (3.3.3) and (3.3.7). We see that $\Omega^{3/2}\omega_{abc}$ is $C^{>-1}$ at i^0. The pullback of this direction-dependent limit to $\mathscr{H}$ is then given by

$$\underleftarrow{\boldsymbol{\omega}}(\delta_1 A, \delta_2 A) = -\boldsymbol{\varepsilon}_3\left(\delta_1 \boldsymbol{E}^a \delta_2 \boldsymbol{A}_a - \delta_2 \boldsymbol{E}^a \delta_1 \boldsymbol{A}_a\right), \tag{3.3.23}$$

where $\boldsymbol{\varepsilon}_3 = \boldsymbol{\varepsilon}_{abc}$ is the volume element on $\mathscr{H}$.

We now take δ_2 to correspond to a gauge transformation as in Eq. (3.3.11) to get

$$\underleftarrow{\boldsymbol{\omega}}(\delta A, \delta_{\boldsymbol{\lambda}} A) = -\boldsymbol{\varepsilon}_3 \delta \boldsymbol{E}^a \boldsymbol{D}_a \boldsymbol{\lambda} = -\boldsymbol{\varepsilon}_3 \boldsymbol{D}^a(\delta \boldsymbol{E}_a \boldsymbol{\lambda}), \tag{3.3.24}$$

where in the last step, we have used the linearized Maxwell equation $\boldsymbol{D}_a \delta \boldsymbol{E}^a = 0$ (see Eq. (3.3.6)). That is, the symplectic current (with one of the perturbations being generated by a gauge transformation) can be written as a total derivative of $\delta \boldsymbol{E}_a \boldsymbol{\lambda}$. Thus we define the perturbed charge $\delta \mathcal{Q}[\boldsymbol{\lambda}; S]$

on a cross-section S of $\mathscr{H}$ by

$$\delta \mathcal{Q}[\boldsymbol{\lambda}; S] = \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \delta \boldsymbol{E}_a \boldsymbol{\lambda}\,, \tag{3.3.25}$$

where $\boldsymbol{\varepsilon}_2 \equiv \boldsymbol{\varepsilon}_{ab}$ is the area element on S and $\boldsymbol{u}^a$ is the future-directed normal to it. Note that this expression is manifestly integrable and defines the unperturbed charge once we choose a reference solution on which $\mathcal{Q}[\boldsymbol{\lambda}; S] = 0$ for all $\boldsymbol{\lambda}$ and all S. For the reference solution we choose the trivial solution $F_{ab} = 0$ so that $\boldsymbol{E}_a = 0$. Then the unperturbed charge is given by

$$\mathcal{Q}[\boldsymbol{\lambda}; S] = \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{E}_a \boldsymbol{\lambda}\,, \tag{3.3.26}$$

Let $\Delta\mathscr{H}$ be any region of $\mathscr{H}$ bounded by the cross-sections S_2 and S_1 (with S_2 in the future of S_1), then the flux of the charge Eq. (3.3.26) through $\Delta\mathscr{H}$ is given by

$$\mathcal{F}[\boldsymbol{\lambda}; \Delta\mathcal{H}] = -\int_{\Delta\mathscr{H}} \boldsymbol{\varepsilon}_3 \boldsymbol{E}_a \boldsymbol{D}^a \boldsymbol{\lambda}\,. \tag{3.3.27}$$

Note that the flux of the charge vanishes for $\boldsymbol{\lambda} =$ constant in which case Eq. (3.3.26) is the Coulomb charge. The charges associated with a general smooth $\boldsymbol{\lambda}$ are only associated with the blowup $\mathscr{H}$ and not with i^0 itself. These additional charges are nevertheless useful for relating the charges defined on past and future null infinity and derive the resulting conservation laws for their fluxes in a scattering process; see [9].

3.4 | Gravitational fields and Einstein equations at i^0

Now we turn to a similar analysis of symmetries, charges and fluxes for general relativity. To set the stage, in this section we analyze the consequences of Einstein equations and the universal structure common to all spacetimes satisfying Def. 3.2.1.

Using the conformal transformation relating the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see Appendix D of [61]), the vacuum Einstein equation $\hat{R}_{ab} = 0$ can be written as

$$\begin{aligned} S_{ab} &= -2\Omega^{-1}\nabla_a\nabla_b\Omega + \Omega^{-2}\nabla^c\Omega\nabla_c\Omega g_{ab}\,, \\ \Omega^{1/2} S_{ab} &= -4\nabla_a\eta_b + 4\Omega^{-1/2}\left(g_{ab} - \tfrac{1}{\eta^2}\eta_a\eta_b\right)\eta_c\eta^c\,, \end{aligned} \tag{3.4.1}$$

where, as before, $\eta_a = \nabla_a\Omega^{1/2}$, and S_{ab} is given by

$$S_{ab} := R_{ab} - \tfrac{1}{6}Rg_{ab}\,. \tag{3.4.2}$$

Further, the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ on the unphysical Riemann tensor along with Eq. (3.4.1) gives the following equations for the unphysical Weyl tensor C_{abcd} (see [62] for details).

$$\nabla_{[e}(\Omega^{-1}C_{ab]cd}) = 0\,, \tag{3.4.3a}$$

$$\nabla^d C_{abcd} = -\nabla_{[a}S_{b]c}\,. \tag{3.4.3b}$$

Since the physical Ricci tensor $\hat{R}_{ab}$ vanishes, the gravitational field is completely described by the physical Weyl tensor $\hat{C}_{abcd}$. The unphysical Weyl tensor is then $C_{abcd} = \Omega^2\hat{C}_{abcd}$. Since the unphysical metric g_{ab} is $C^{>0}$ at i^0, $\Omega^{1/2}C_{abcd}$ is $C^{>-1}$ at i^0 [16], and let

$$\boldsymbol{C}_{abcd}(\vec{\eta}) := \lim_{\to i^0}\Omega^{1/2}C_{abcd}\,. \tag{3.4.4}$$

The *electric* and *magnetic* parts of $\boldsymbol{C}_{abcd}(\vec{\eta})$ are, respectively, defined by

$$\boldsymbol{E}_{ab}(\vec{\eta}) := \boldsymbol{C}_{acbd}(\vec{\eta})\boldsymbol{\eta}^c\boldsymbol{\eta}^d\,, \quad \boldsymbol{B}_{ab}(\vec{\eta}) := *\boldsymbol{C}_{acbd}(\vec{\eta})\boldsymbol{\eta}^c\boldsymbol{\eta}^d\,. \tag{3.4.5}$$

where $*\boldsymbol{C}_{abcd}(\vec{\eta}) := \frac{1}{2}\boldsymbol{\varepsilon}_{ab}{}^{ef}\boldsymbol{C}_{efcd}(\vec{\eta})$. It follows from the symmetries of the Weyl tensor that both $\boldsymbol{E}_{ab}(\vec{\eta})$ and $\boldsymbol{B}_{ab}(\vec{\eta})$ are orthogonal to $\boldsymbol{\eta}^a$, symmetric and traceless with the respect to the metric $\boldsymbol{h}_{ab}$ on $\mathscr{H}$, and thus define smooth tensor fields $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ on $\mathscr{H}$, respectively. The limiting Weyl tensor can be obtained from these fields using

$$\boldsymbol{C}^{ab}{}_{cd}(\vec{\eta}) = 4\boldsymbol{\eta}^{[a}\boldsymbol{\eta}_{[c}\boldsymbol{E}^{b]}{}_{d]} - 4\boldsymbol{h}^{[a}{}_{[c}\boldsymbol{E}^{b]}{}_{d]} + 2\boldsymbol{\varepsilon}^{abe}\boldsymbol{\eta}_{[c}\boldsymbol{B}_{d]e} + 2\boldsymbol{\varepsilon}_{cde}\boldsymbol{\eta}^{[a}\boldsymbol{B}^{b]e}\,. \tag{3.4.6}$$

Further, as shown in [16], multiplying Eq. (3.4.3a) by Ω and taking the limit to i^0 gives the equations of motion

$$\boldsymbol{D}_{[a}\boldsymbol{E}_{b]c} = 0\,, \quad \boldsymbol{D}_{[a}\boldsymbol{B}_{b]c} = 0\,. \tag{3.4.7}$$

These are the asymptotic Einstein equations at spatial infinity. Taking the trace over the indices a and c and using the fact that $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ are traceless, it also follows that

$$\boldsymbol{D}^b\boldsymbol{E}_{ab} = \boldsymbol{D}^b\boldsymbol{B}_{ab} = 0\,. \tag{3.4.8}$$

To apply the covariant phase formalism in this context, we will need to consider metric perturbations instead of just perturbations of the Weyl tensor. As we will show below (Eq. (3.5.8)), suitably rescaled limits of the unphysical metric perturbations can be expressed in terms of perturbations of certain potentials for $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ provided by the tensor S_{ab} in Eq. (3.4.2). These potentials are obtained as follows: Since g_{ab} is $C^{>0}$, $\Omega^{1/2}S_{ab}$ is $C^{>-1}$ and let $\boldsymbol{S}_{ab}(\vec{\eta}) := \lim_{\to i^0} \Omega^{1/2}S_{ab}$. Define

$$\boldsymbol{E}(\vec{\eta}) := \boldsymbol{S}_{ab}(\vec{\eta})\boldsymbol{\eta}^a\boldsymbol{\eta}^b\,, \quad \boldsymbol{K}_{ab}(\vec{\eta}) := \boldsymbol{h}_a{}^c\boldsymbol{h}_b{}^d\boldsymbol{S}_{cd}(\vec{\eta}) - \boldsymbol{h}_{ab}\boldsymbol{E}(\vec{\eta})\,, \tag{3.4.9}$$

which induce the fields $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ intrinsic to $\mathscr{H}$. Following [16], multiplying Eq. (3.4.3b) by Ω and taking the limit to i^0, along with Eq. (3.4.7) implies that

$$\boldsymbol{h}_a{}^b\boldsymbol{\eta}^c\boldsymbol{S}_{bc}(\vec{\eta}) = \boldsymbol{D}_a\boldsymbol{E}\,, \tag{3.4.10}$$

and

$$\boldsymbol{E}_{ab} = -\tfrac{1}{4}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{E} + \boldsymbol{h}_{ab}\boldsymbol{E})\,, \quad \boldsymbol{B}_{ab} = -\tfrac{1}{4}\boldsymbol{\varepsilon}_{cda}\boldsymbol{D}^c\boldsymbol{K}^d{}_b\,. \tag{3.4.11}$$

Thus, $\boldsymbol{E}$ is a scalar potential for $\boldsymbol{E}_{ab}$ while $\boldsymbol{K}_{ab}$ is a tensor potential for $\boldsymbol{B}_{ab}$.[7]

[7] Since $\boldsymbol{B}_{ab}$ is curl-free (Eq. (3.4.7)), there also exists a scalar potential for $\boldsymbol{B}_{ab}$ (see Appendix. B.2 of [5]). However this scalar potential cannot be obtained as the limit of a tensor field on spacetime.

The potentials $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ are not free fields on $\mathscr{H}$. Suitably commuting the derivatives and using Eq. (A.5.1) one can verify that $\boldsymbol{E}_{ab}$ identically satisfies Eq. (3.4.7) when written in terms of the potential $\boldsymbol{E}$ while $\boldsymbol{h}^{ab}\boldsymbol{E}_{ab} = 0$ gives

$$\boldsymbol{D}^2\boldsymbol{E} + 3\boldsymbol{E} = 0\,. \tag{3.4.12}$$

On the other hand, since $\boldsymbol{K}_{ab}$ is symmetric, the magnetic field $\boldsymbol{B}_{ab}$ in Eq. (3.4.11) is identically traceless. Since $\boldsymbol{B}_{ab}$ is symmetric and satisfies Eq. (3.4.7), we get that

$$\boldsymbol{\varepsilon}_a{}^{bc}\boldsymbol{B}_{bc} = 0 \implies \boldsymbol{D}^b\boldsymbol{K}_{ab} = \boldsymbol{D}_a\boldsymbol{K}\,, \tag{3.4.13a}$$

$$\boldsymbol{\varepsilon}_a{}^{cd}\boldsymbol{D}_c\boldsymbol{B}_{db} = 0 \implies \boldsymbol{D}^2\boldsymbol{K}_{ab} = \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{K} + 3\boldsymbol{K}_{ab} - \boldsymbol{h}_{ab}\boldsymbol{K}\,, \tag{3.4.13b}$$

where $\boldsymbol{K} := \boldsymbol{h}^{ab}\boldsymbol{K}_{ab}$, and to get Eq. (3.4.13b), we have commuted derivatives using Eq. (A.5.1) and used Eq. (3.4.13a). Considering the potentials $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ as the basic fields, the asymptotic Einstein equations are given by Eqs. (3.4.12) and (3.4.13), while $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ are derived quantities through Eq. (3.4.11).

To define the charge for asymptotic Lorentz symmetries, e.g. angular momentum in Sec. 3.7.2, we will need the "subleading" part of the magnetic Weyl tensor. Following Ashtekar and Hansen [16], we will restrict to the class of spacetimes satisfying the additional condition $\boldsymbol{B}_{ab} = 0$. We also require that the "subleading" magnetic field defined by

$$\boldsymbol{\beta}_{ab} := \lim_{\to i^0} {*C}_{acbd}\eta^c\eta^d\,, \tag{3.4.14}$$

exists as a $C^{>-1}$ tensor field at i^0. The condition $\boldsymbol{B}_{ab} = 0$ is satisfied in any spacetime which is *either* stationary *or* axisymmetric [97] (although the recent results of [98] suggest that it might hold more generally). In Appendix A.8 we show, how one can define a "subleading" magnetic Weyl tensor and the Lorentz charges even when $\boldsymbol{B}_{ab} \neq 0$. Since those computations are more tedious, we impose the above restriction in this chapter.

The consequences of this restriction are as follows. Since $\boldsymbol{B}_{ab} = 0$, from Eq. (3.4.11) the "curl" of

$\boldsymbol{K}_{ab}$ vanishes

$$\boldsymbol{D}_{[a}\boldsymbol{K}_{b]c} = 0\,. \tag{3.4.15}$$

It follows from **??** that there exists a scalar potential $\boldsymbol{k}$ such that

$$\boldsymbol{K}_{ab} = \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{k} + \boldsymbol{h}_{ab}\boldsymbol{k}\,. \tag{3.4.16}$$

The scalar potential $\boldsymbol{k}$ is a free function on $\mathscr{H}$ since the equations of motion Eq. (3.4.13) are identically satisfied after using Eq. (3.4.16). Using the freedom in the choice of the conformal factor, one can now set $\boldsymbol{K}_{ab} = 0$ (see [16] and Remark 3.6.3). Since we do not wish to impose any restrictions on the conformal factor, we will *not* demand that $\boldsymbol{K}_{ab}$ vanishes.

Note that it follows from Eq. (3.4.14) that $\boldsymbol{\beta}_{ab}$ is symmetric, tangent to $\mathscr{H}$ and traceless. In what follows, we will also need an equation of motion for $\boldsymbol{\beta}_{ab}$ which is obtained as follows: Contracting the indices e and d in Eq. (3.4.3a) and multiplying by 3Ω, we obtain

$$\nabla^d C_{abcd} = \Omega^{-1} C_{abcd}\nabla^d\Omega = 2\Omega^{-1/2} C_{abcd}\eta^d\,. \tag{3.4.17}$$

This leads to

$$\Omega^{1/2}\nabla^b(*C_{acbd}\eta^c\eta^d) = -2 * C_{acbd}\eta^b\eta^c\eta^d + 2\Omega^{1/2} * C_{acbd}\nabla^b\eta^{(c}\eta^{d)}\,, \tag{3.4.18}$$

where $*$ denotes the Hodge dual. The first term on the right-hand-side vanishes due to the symmetries of the Weyl tensor. In the second term on the right hand side, we substitute for the derivative of η_a using Eq. (3.4.1) to get

$$\Omega^{1/2}\nabla^b(*C_{acbd}\eta^c\eta^d) = -\tfrac{1}{4}(\Omega^{1/2} * C_{acbd})(\Omega^{1/2} S^{bc})\eta^d\,. \tag{3.4.19}$$

Taking the limit to i^0, writing the tensor $\boldsymbol{S}_{ab}$ in terms of the gravitational potentials through Eqs. (3.4.9) and (3.4.10), and using $\boldsymbol{B}_{ab} = 0$ along with Eq. (3.4.6), we get the equation of motion

$$\boldsymbol{D}^b\boldsymbol{\beta}_{ab} = \tfrac{1}{4}\boldsymbol{\varepsilon}_{cda}\boldsymbol{E}^c{}_b\boldsymbol{K}^{bd}\,. \tag{3.4.20}$$

Remark 3.4.1 (Conformal transformations of the asymptotic fields). Under changes of the conformal factor $\Omega \mapsto \omega\Omega$, we have

$$\begin{aligned} S_{ab} &\mapsto S_{ab} - 2\omega^{-1}\nabla_a\nabla_b\omega + 4\omega^{-2}\nabla_a\omega\nabla_b\omega - \omega^{-2}g_{ab}\nabla^c\omega\nabla_c\omega\,, \\ C_{abcd} &\mapsto \omega^2 C_{abcd}\,. \end{aligned} \tag{3.4.21}$$

From the conditions in Remark 3.2.1 it follows that $\boldsymbol{E}_{ab}$, $\boldsymbol{B}_{ab}$ and $\boldsymbol{E}$ are invariant under conformal transformations while

$$\boldsymbol{K}_{ab} \mapsto \boldsymbol{K}_{ab} - 2(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\alpha} + \boldsymbol{h}_{ab}\boldsymbol{\alpha})\,. \tag{3.4.22}$$

Further, when $\boldsymbol{B}_{ab} = 0$ we also have the transformation of the "subleading" magnetic Weyl tensor $\boldsymbol{\beta}_{ab}$ which is given by

$$\boldsymbol{\beta}_{ab} \mapsto \boldsymbol{\beta}_{ab} - \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{E}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{\alpha}\,. \tag{3.4.23}$$

1 | The universal structure at i^0

In this section we summarize the *universal structure* at i^0, that is, the structure common to all spacetimes which are asymptotically flat in the sense of Def. 3.2.1 and thus is independent of the choice of the physical spacetime under consideration.

Consider any two unphysical spacetimes (M, g_{ab}, Ω) and (M', g'_{ab}, Ω') with their respective $C^{>1}$ differential structures at their spatial infinities corresponding to two different physical spacetimes. Using a C^1 diffeomorphism we can identify the points representing the spatial infinities and their tangent spaces without any loss of generality. Each of the metrics g_{ab} and g'_{ab} induces a metric in the tangent space Ti^0 which is isometric to the Minkowski metric. Thus, the metric $\boldsymbol{g}_{ab}$ at i^0 is also universal. This also implies that the spatial directions $\vec{\eta}$, the space of directions $\mathscr{H}$ and the induced metric $\boldsymbol{h}_{ab}$ are universal.

So far we have only used the C^1 differential structure. However since the differential structure at i^0 is slightly better and is$C^{>1}$, we can identify the spacetimes at the "next order" as well. In [16], this structure was imposed by suitably identifying spacelike geodesics in the *physical* spacetimes. However, as pointed out by [99], this identification cannot be performed except in very special cases.

Below we argue that a similar identification of the spacetimes can be done using equivalence classes of $C^{>1}$ curves in the unphysical spacetimes. The proof is based on constructing a suitable $C^{>1}$ coordinate system at i^0which is done in Appendix A.4. We summarize the main construction below.

Consider the unphysical spacetime (M, g_{ab}, Ω), and a spacelike $C^{>1}$ curve Γ_v in M passing through i^0 with tangent v^a. Since the curve is $C^{>1}$ its tangent vector v^a is $C^{>0}$. Using the universal metric $\boldsymbol{g}_{ab}$ at i^0, we can then demand that v^a be unit-normalized at i^0 and thus along the curve Γ_v

$$\lim_{\to i^0} v^a = \boldsymbol{\eta}^a \,, \tag{3.4.24}$$

that is the curve Γ_v points in some spatial direction $\vec{\eta}$ at i^0. Further, since Γ_v is $C^{>1}$, $v^b \nabla_b v^a$ is a $C^{>-1}$ vector. We define the *acceleration* of Γ_v at i^0 by the projection of this vector on to $\mathscr{H}$

$$\boldsymbol{A}^a[\Gamma_v] := \boldsymbol{h}^a{}_b \lim_{\to i^0} v^c \nabla_c v^b \,. \tag{3.4.25}$$

Now we define the curves Γ_v (with tangent v^a) and Γ_η (with tangent η^a) to be equivalent if their accelerations are equal at i^0. To see what this entails, note that since v^a is $C^{>0}$ and equals $\boldsymbol{\eta}^a$ in the limit to i^0 we have that $v^a = \eta^a + \Omega^{1/2} w^a$ for some w^a which is $C^{>-1}$ at i^0. Then, from Eq. (3.4.25) we have

$$\boldsymbol{A}^a[\Gamma_v] = \boldsymbol{A}^a[\Gamma_\eta] \iff \boldsymbol{h}_{ab} \lim_{\to i^0} w^b = 0 \,. \tag{3.4.26}$$

Thus, we have an equivalence class of curves through i^0 pointing in each direction $\vec{\eta}$ defined by[8]

$$\Gamma_v \sim \Gamma_\eta \iff \boldsymbol{h}_{ab} \lim_{\to i^0} \Omega^{-1/2} (v^b - \eta^b) = 0 \,. \tag{3.4.27}$$

We will show in Appendix A.4 that using a $C^{>1}$ diffeomorphism, one can identify these equivalence classes of curves between any any two spacetimes (M, g_{ab}, Ω) and (M', g'_{ab}, Ω'). Further, we show that the conformal factors Ω and Ω' can also be identified in a neighbourhood of i^0.

To summarize, the universal structure at i^0 consists of the point i^0, the tangent space Ti^0, the

[8] These equivalence classes of curves form a principal bundle over $\mathscr{H}$ called *Spi* in [16].

metric $\boldsymbol{g}_{ab}$ at i^0 and the equivalence classes of $C^{>1}$ curves given by Eq. (3.4.27). In addition, the conformal factor Ω can also be chosen to be universal.

Remark 3.4.2 (Logarithmic translations). So far we have worked with a fixed $C^{>1}$ differential structure in the unphysical spacetime at i^0. However, given a physical spacetime the unphysical spacetime is ambiguous up to a 4-parameter family of *logarithmic translations* at i^0 which simultaneously change the $C^{>1}$ differential structure and the conformal factor at i^0; see [100] or Remark B.1 of [10] for details. The logarithmic translations at i^0 are parameterized by a *direction-independent* vector $\boldsymbol{\Lambda}^a$ at i^0. Any such vector can be written as

$$\boldsymbol{\Lambda}^a = \boldsymbol{\Lambda}\boldsymbol{\eta}^a + \boldsymbol{D}^a\boldsymbol{\Lambda}\,, \tag{3.4.28}$$

where $\boldsymbol{\Lambda}(\vec{\eta}) = \boldsymbol{\eta}_a\boldsymbol{\Lambda}^a$ is a function on $\mathscr{H}$ satisfying

$$\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\Lambda} + \boldsymbol{h}_{ab}\boldsymbol{\Lambda} = 0\,. \tag{3.4.29}$$

Under such logarithmic translations the potentials Eq. (3.4.9) transform as [100]

$$\boldsymbol{E} \mapsto \boldsymbol{E} + 4\boldsymbol{\Lambda}\,, \quad \boldsymbol{K}_{ab} \mapsto \boldsymbol{K}_{ab}\,, \tag{3.4.30}$$

while $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ are invariant. These logarithmic translations will lead to the following issue when we define the charges for supertranslations in Sec. 3.7.1. For general supertranslations (which are not translations), our charges will depend on the potential $\boldsymbol{E}$ instead of just the electric field $\boldsymbol{E}_{ab}$. Thus, even if we take the physical spacetime to be the Minkowski spacetime, our charges will not vanish due to the logarithmic translation ambiguity Eq. (3.4.30) in $\boldsymbol{E}$. Since these charges ought to vanish in Minkowski, we will now fix these logarithmic translations following the argument in [100].

Since the metric $\boldsymbol{g}_{ab}$ in the tangent space Ti^0 is universal and isometric to the Minkowski metric, it is invariant under the reflection of the spatial directions $\vec{\eta} \mapsto -\vec{\eta}$. This gives rise to a reflection isometry of the metric $\boldsymbol{h}_{ab}$ on the space of directions $\mathscr{H}$. It was shown in [10] that the only spacetimes which are asymptotically-flat at spatial infinity and which "match" on to asymptotically-flat spacetimes on null infinity are the ones where $\boldsymbol{E}_{ab}$ is reflection-even, i.e.

$$\boldsymbol{E}_{ab}(\vec{\eta}) = \boldsymbol{E}_{ab}(-\vec{\eta})\,. \tag{3.4.31}$$

Further, since $\boldsymbol{\Lambda} = \boldsymbol{\eta}_a \boldsymbol{\Lambda}^a$ for the *direction-independent* vector $\boldsymbol{\Lambda}^a$ we have that, $\boldsymbol{\Lambda}$ is reflection-odd

$$\boldsymbol{\Lambda}(\vec{\eta}) = -\boldsymbol{\Lambda}(-\vec{\eta})\,. \tag{3.4.32}$$

For a reflection-even $\boldsymbol{E}_{ab}$, from Eqs. (3.4.11) and (3.4.29), it follows that using a logarithmic translation we can demand that the potential $\boldsymbol{E}$ is also reflection-even so that

$$\boldsymbol{E}(\vec{\eta}) = \boldsymbol{E}(-\vec{\eta})\,. \tag{3.4.33}$$

Having fixed the logarithmic translations in this way, $\boldsymbol{E}_{ab} = 0$ then implies that $\boldsymbol{E} = 0$. In particular, for Minkowski spacetime we have

$$\boldsymbol{E} = 0\,, \quad \boldsymbol{B}_{ab} = 0\,, \quad \boldsymbol{\beta}_{ab} = 0 \quad \text{(on Minkowski spacetime)}\,. \tag{3.4.34}$$

Note that when $\boldsymbol{E}_{ab} = 0$, $\boldsymbol{\beta}_{ab}$ is conformally-invariant (see Eq. (3.4.23)) and the conditions Eq. (3.4.34) do not depend on the conformal factor chosen for Minkowski spacetime. These conditions will ensure that our all our charges will vanish on Minkowski spacetime. Thus, from here on we will assume that the logarithmic translations have been fixed as above that is, we work the choice of $C^{>1}$ differential structure at i^0 where the parity condition Eq. (3.4.33) is satisfied.

3.5 | Metric perturbations and symplectic current at i^0

Now consider a one-parameter family of asymptotically flat physical metrics $\hat{g}_{ab}(\lambda)$ where $\hat{g}_{ab} = \hat{g}_{ab}(\lambda = 0)$ is some chosen background spacetime. Define the physical metric perturbation $\hat{\gamma}_{ab}$ around the background $\hat{g}_{ab}$ by

$$\hat{\gamma}_{ab} = \delta \hat{g}_{ab} := \frac{d}{d\lambda} \hat{g}_{ab}(\lambda)\Big|_{\lambda=0}\,. \tag{3.5.1}$$

We will use "δ" to denote perturbations of other quantities defined in a similar way.

As discussed above, the conformal factor Ω can be chosen universally, i.e., independently of the choice of the physical metric. Then, the unphysical metric perturbation satisfies

$$\delta g_{ab} = \gamma_{ab} = \Omega^2 \hat{\gamma}_{ab} \,, \tag{3.5.2}$$

and we also have

$$\delta \eta_a = \delta \nabla_a \Omega^{1/2} = 0 \,, \quad \delta \eta^a = \delta(g^{ab}\eta_b) = -\gamma^{ab}\eta_b \,. \tag{3.5.3}$$

Now we investigate the conditions on the unphysical perturbation γ_{ab} which preserve asymptotic flatness and the universal structure at i^0 described in Sec. 3.4.1. First, recall that since the unphysical metric g_{ab} is $C^{>0}$ and universal at i^0, it follows that the unphysical metric perturbation γ_{ab} is $C^{>0}$ and $\gamma_{ab}|_{i^0} = 0$. Therefore

$$\boldsymbol{\gamma}_{ab}(\vec{\eta}) := \lim_{\to i^0} \Omega^{-1/2} \gamma_{ab} \text{ is } C^{>-1} \,, \tag{3.5.4}$$

From Eqs. (3.5.3) and (3.5.4) we also see that $\delta \boldsymbol{\eta}^a = 0$. Thus, the metric perturbation also preserves the spatial directions $\vec{\eta}$ at i^0, the space of directions $\mathscr{H}$ and the metric $\boldsymbol{h}_{ab}$ on it.

Now consider the universal structure given by the equivalence classes of $C^{>1}$ curves through i^0 as described in Sec. 3.4.1. Consider the equivalence class of a fixed curve Γ_v with tangent v^a. For this equivalence class to be preserved, the perturbation of Eq. (3.4.27) must vanish. Evaluating this condition using Eqs. (3.5.3) and (3.5.4), we obtain the condition

$$\boldsymbol{h}_a{}^b \boldsymbol{\eta}^c \boldsymbol{\gamma}_{bc}(\vec{\eta}) = 0 \,. \tag{3.5.5}$$

In summary, Eqs. (3.5.4) and (3.5.5) are the asymptotic conditions on the unphysical metric perturbations which preserve the asymptotic flatness and the universal structure at i^0.

The metric perturbation $\boldsymbol{\gamma}_{ab}$ can be directly related to the perturbations of the gravitational potentials $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ defined in Eq. (3.4.9). Perturbing Eq. (3.4.1) to evaluate $\Omega^{1/2} \delta S_{ab}$ and taking the limit to i^0 using Eqs. (3.5.3) and (3.5.4), we get

$$\delta \boldsymbol{S}_{ab} = \lim_{\to i^0} \Omega^{1/2} \delta S_{ab} = 4 \boldsymbol{\partial}_{(a} \boldsymbol{\gamma}_{b)c} \boldsymbol{\eta}^c + 4 \boldsymbol{\eta}_{(a} \boldsymbol{\gamma}_{b)c} \boldsymbol{\eta}^c + 2 \boldsymbol{\gamma}_{ab} - 4 \boldsymbol{\gamma}_{cd} \boldsymbol{\eta}^c \boldsymbol{\eta}^d \boldsymbol{g}_{ab} \,. \tag{3.5.6}$$

Using the definition of the gravitational potentials Eq. (3.4.9) and Eq. (3.5.5), we obtain

$$\delta \boldsymbol{E} = 2 \boldsymbol{\gamma}_{ab} \boldsymbol{\eta}^a \boldsymbol{\eta}^b \,, \tag{3.5.7a}$$

$$\delta \boldsymbol{K}_{ab} = -2 \boldsymbol{h}_a{}^c \boldsymbol{h}_b{}^d \boldsymbol{\gamma}_{cd} - \boldsymbol{h}_{ab} \delta \boldsymbol{E} \,. \tag{3.5.7b}$$

Using Eqs. (3.5.5) and (3.5.7), we can reconstruct the metric perturbation $\boldsymbol{\gamma}_{ab}(\vec{\eta})$ in terms of the perturbed gravitational potentials on $\mathscr{H}$ as

$$\boldsymbol{\gamma}_{ab}(\vec{\eta}) = \tfrac{1}{2} \left[\delta \boldsymbol{E} (\boldsymbol{\eta}_a \boldsymbol{\eta}_b - \boldsymbol{h}_{ab}) - \delta \boldsymbol{K}_{ab} \right] . \tag{3.5.8}$$

The linearized Einstein equations for $\boldsymbol{\gamma}_{ab}$ in the form Eq. (3.5.8) are then equivalent to the linearizations of Eqs. (3.4.12) and (3.4.13).

Next, we consider the behaviour of the symplectic current of vacuum general relativity near i^0. Recall that the symplectic current is given by

$$\omega_{abc} = -\tfrac{1}{16\pi} \hat{\varepsilon}_{abcd} \hat{w}^d \quad \text{with} \quad \hat{w}^a = \hat{P}^{abcdef} \hat{\gamma}_{2bc} \hat{\nabla}_d \hat{\gamma}_{1ef} - [1 \leftrightarrow 2] \,, \tag{3.5.9}$$

where "[1 ↔ 2]" denotes the preceding expression with the 1 and 2, labeling the perturbations, interchanged and the tensor $\hat{P}^{abcdef}$ is given by

$$\hat{P}^{abcdef} = \hat{g}^{ae} \hat{g}^{fb} \hat{g}^{cd} - \tfrac{1}{2} \hat{g}^{ad} \hat{g}^{be} \hat{g}^{fc} - \tfrac{1}{2} \hat{g}^{ab} \hat{g}^{cd} \hat{g}^{ef} - \tfrac{1}{2} \hat{g}^{bc} \hat{g}^{ae} \hat{g}^{fd} + \tfrac{1}{2} \hat{g}^{bc} \hat{g}^{ad} \hat{g}^{ef} \,. \tag{3.5.10}$$

To analyze the behaviour of the symplectic current in the limit to i^0, we first express it in terms of quantities in the unphysical spacetime using

$$\varepsilon_{abcd} = \Omega^4 \hat{\varepsilon}_{abcd} \,, \quad P^{abcdef} = \Omega^{-6} \hat{P}^{abcdef} \,, \quad \gamma_{ab} = \Omega^2 \hat{\gamma}_{ab} \,, \tag{3.5.11}$$

where P^{abcdef} is defined through the unphysical metric by the same expression as Eq. (3.5.10). Using these, and converting the physical derivative operator $\hat{\nabla}$ to the unphysical one ∇ as

$$\hat{\nabla}_d \hat{\gamma}_{1ef} = \nabla_d \hat{\gamma}_{1ef} + \Omega^{-1}[\hat{\nabla}_d \Omega \hat{\gamma}_{1ef} + \hat{\nabla}_e \Omega \hat{\gamma}_{1df} - g_{ed} \hat{\nabla}^a \Omega \hat{\gamma}_{1af} + (e \leftrightarrow f)] \,, \tag{3.5.12}$$

we obtain

$$\begin{aligned} \omega_{abc} &= -\tfrac{1}{16\pi} \varepsilon_{abcd} w^d \,, \\ \text{with} \quad w^a &= \Omega^{-2} P^{abcdef} \gamma_{2bc} \nabla_d \gamma_{1ef} + \Omega^{-3} \gamma_1^{ab} \nabla_b \Omega \gamma_{2c}{}^c - [1 \leftrightarrow 2] \,. \end{aligned} \tag{3.5.13}$$

Converting to quantities which are direction-dependent at i^0 and using Eq. (3.5.4), we see that $\Omega^{3/2} \omega_{abc}$ is $C^{>-1}$. The pullback $\underleftarrow{\boldsymbol{\omega}}$ to $\mathscr{H}$ of $\lim_{\to i^0} \Omega^{3/2} \omega_{abc}$ is given by

$$\underleftarrow{\boldsymbol{\omega}} = -\frac{1}{16\pi} \boldsymbol{\varepsilon}_3 \, \boldsymbol{\eta}^a \left(2\boldsymbol{\eta}^b \boldsymbol{\gamma}_{2ab} \boldsymbol{\gamma}_1 - \tfrac{1}{2} \boldsymbol{\gamma}_{1ab} \boldsymbol{\partial}^b \boldsymbol{\gamma}_2 + \boldsymbol{\gamma}_1^{bc} \boldsymbol{\partial}_c \boldsymbol{\gamma}_{2ab} - \tfrac{1}{2} \boldsymbol{\gamma}_1 \boldsymbol{\partial}^b \boldsymbol{\gamma}_{2ab} \right) - [1 \leftrightarrow 2] \,. \tag{3.5.14}$$

This expression can be considerably simplified by rewriting it in terms of the perturbed gravitational potentials $\delta \boldsymbol{E}$ and $\delta \boldsymbol{K}_{ab}$ using Eq. (3.5.8). An easy but long computation gives

$$\underleftarrow{\boldsymbol{\omega}} = \frac{1}{64\pi} \boldsymbol{\varepsilon}_3 \left(\delta_1 \boldsymbol{K} \delta_2 \boldsymbol{E} - \delta_2 \boldsymbol{K} \delta_1 \boldsymbol{E} \right) , \tag{3.5.15}$$

where, as before, $\boldsymbol{K} := \boldsymbol{h}^{ab} \boldsymbol{K}_{ab}$.

3.6 | Asymptotic symmetries at i^0: The $\mathfrak{spi}$ algebra

In this section we analyze the asymptotic symmetries at i^0. We show that the diffeomorphisms of the physical spacetime which preserve the asymptotic flatness of the spacetime in spatial limits (defined by Def. 3.2.1) generate an infinite-dimensional algebra denoted $\mathfrak{spi}$. This asymptotic symmetry algebra was obtained in [16, 17] by analyzing the infinitesimal diffeomorphisms which preserve the universal structure at i^0. Here, we provide an alternative derivation by considering the physical perturbations generated by infinitesimal diffeomorphisms and demanding that the corresponding unphysical perturbations satisfy the asymptotic conditions Eqs. (3.5.4) and (3.5.5).

Consider an infinitesimal diffeomorphism generated by a vector field $\hat{\xi}^a$ in the physical spacetime, and let $\xi^a = \hat{\xi}^a$ be the corresponding vector field in the unphysical spacetime. For ξ^a to be a representative of an asymptotic symmetry at i^0, the infinitesimal diffeomorphism generated by ξ^a must preserve the universal structure at i^0. Firstly, the infinitesimal diffeomorphism must keep the the point i^0 fixed and preserve the $C^{>1}$ differential structure at i^0. Thus, ξ^a must be $C^{>0}$ at i^0 and $\xi^a|_{i^0} = 0$. This implies that $\Omega^{-1/2}\xi^a$ is $C^{>-1}$ at i^0. We denote

$$\boldsymbol{X}^a(\vec{\eta}) := \lim_{\to i^0} \Omega^{-1/2}\xi^a\,. \tag{3.6.1}$$

Now consider the physical metric perturbation $\hat{\gamma}^{(\xi)}_{ab} = \delta_\xi \hat{g}_{ab} := \pounds_\xi \hat{g}_{ab}$ corresponding to an infinitesimal diffeomorphism generated by ξ^a. The corresponding unphysical metric perturbation is given by

$$\gamma^{(\xi)}_{ab} = \Omega^2 \pounds_\xi \hat{g}_{ab} = \pounds_\xi g_{ab} - 4\Omega^{-1/2}\xi^c \eta_c g_{ab}\,. \tag{3.6.2}$$

Since $\gamma^{(\xi)}_{ab}$ must satisfy the asymptotic conditions at i^0 in Eqs. (3.5.4) and (3.5.5), we have that $\gamma^{(\xi)}_{ab}$ is $C^{>0}$ at i^0 and $\gamma^{(\xi)}_{ab}|_{i^0} = 0$. To see the implications of these conditions, we evaluate the condition $\gamma^{(\xi)}_{ab}|_{i^0} = 0$ using Eqs. (3.6.1) and (3.6.2) which gives

$$\boldsymbol{\eta}_a \boldsymbol{X}^a(\vec{\eta}) = 0\,, \quad \boldsymbol{D}_{(a}\boldsymbol{X}_{b)} = 0\,, \tag{3.6.3}$$

that is, the vector field $\boldsymbol{X}^a$ is tangent to $\mathscr{H}$ and is a Killing vector field on it. Thus, $\boldsymbol{X}^a$ is an element of the Lorentz algebra $\mathfrak{so}(1,3)$. Some useful properties of these Killing vectors and their relationship to infinitesimal Lorentz transformations in the tangent space Ti^0 are collected in Appendix A.5.1.

Further, since both $\gamma^{(\xi)}_{ab}$ and $\pounds_\xi g_{ab}$ are $C^{>0}$, we must have that $\Omega^{-1/2}\xi^a\eta_a$ is also $C^{>0}$. Since $\Omega^{-1/2}\xi^a\eta_a|_{i^0} = 0$ (which follows from Eqs. (3.6.1) and (3.6.3)) we have that $\Omega^{-1}\xi^a\eta_a$ is $C^{>-1}$ at i^0 so we define

$$\boldsymbol{f}(\vec{\eta}) := \lim_{\to i^0} \Omega^{-1}\xi^a\eta_a\,. \tag{3.6.4}$$

The function $\boldsymbol{f}$ on $\mathscr{H}$ then parametrizes the *supertranslations*. A vector field that generates a supertranslation can be obtained as follows. Consider ξ^a such that the corresponding $\boldsymbol{X}^a$ (Eq. (3.6.1)) vanishes and $\boldsymbol{\chi}^a := \lim_{\to i^0} \Omega^{-1}\xi^a$ is $C^{>-1}$ so that $\boldsymbol{f} = \boldsymbol{\chi}^a \boldsymbol{\eta}_a$. Now consider the metric perturbation Eq. (3.6.2) corresponding to such a vector field. From Eq. (3.5.5) we must have

$$\boldsymbol{h}_a{}^b \boldsymbol{\eta}^c \boldsymbol{\gamma}^{(\xi)}_{bc} = 0\,, \tag{3.6.5}$$

where, as before, $\boldsymbol{\gamma}^{(\xi)}_{ab} = \lim_{\to i^0} \Omega^{-1/2}\gamma^{(\xi)}_{ab}$. Evaluating this condition using Eq. (3.6.2) and $\boldsymbol{\chi}^a = \lim_{\to i^0} \Omega^{-1}\xi^a$, we get

$$\boldsymbol{h}_{ab}\boldsymbol{\chi}^b = -\boldsymbol{D}_a \boldsymbol{f}\,. \tag{3.6.6}$$

Thus a pure supertranslation $\boldsymbol{f}$ is represented by a vector field ξ^a such that

$$\lim_{\to i^0} \Omega^{-1}\xi^a = \boldsymbol{f}\boldsymbol{\eta}^a - \boldsymbol{D}^a \boldsymbol{f}\,. \tag{3.6.7}$$

In summary, the asymptotic symmetries at i^0 are parameterized by a pair $(\boldsymbol{f}, \boldsymbol{X}^a)$ where $\boldsymbol{f}$ is a smooth function and $\boldsymbol{X}^a \in \mathfrak{so}(1,3)$ is a smooth Killing vector field on $\mathscr{H}$.

The Lie algebra structure of these symmetries can be obtained as follows. Let ξ_1^a and ξ_2^a be the vector fields representing the asymptotic Spi-symmetries $(\boldsymbol{f}_1, \boldsymbol{X}_1^a)$ and $(\boldsymbol{f}_2, \boldsymbol{X}_2^a)$ respectively. Then the Lie bracket $[\xi_1, \xi_2]^a = \xi_1^b \nabla_b \xi_2^a - \xi_2^b \nabla_b \xi_1^a$ of the representatives induces a Lie bracket on the Spi-symmetries. Using Eqs. (3.6.1), (3.6.3) and (3.6.4) the induced Lie bracket on the Spi-symmetries can be computed to be

$$\begin{aligned} (\boldsymbol{f}, \boldsymbol{X}^a) &= [(\boldsymbol{f}_1, \boldsymbol{X}_1^a), (\boldsymbol{f}_2, \boldsymbol{X}_2^a)]\,, \\ \text{with} \quad \boldsymbol{f} &= \boldsymbol{X}_1^b \boldsymbol{D}_b \boldsymbol{f}_2 - \boldsymbol{X}_2^b \boldsymbol{D}_b \boldsymbol{f}_1\,, \\ \boldsymbol{X}^a &= \boldsymbol{X}_1^b \boldsymbol{D}_b \boldsymbol{X}_2^a - \boldsymbol{X}_2^b \boldsymbol{D}_b \boldsymbol{X}_1^a\,. \end{aligned} \tag{3.6.8}$$

Thus the Spi symmetries form a Lie algebra $\mathfrak{spi}$ with the above Lie bracket structure. Note that if $\boldsymbol{X}_1^a = \boldsymbol{X}_2^a = 0$ then $\boldsymbol{f} = \boldsymbol{X}^a = 0$ — the supertranslations form an infinite-dimensional abelian subalgebra $\mathfrak{s}$. Further if $\boldsymbol{X}_1^a = 0$ and $\boldsymbol{X}_2^a \neq 0$ then $\boldsymbol{X}^a = 0$ and so we see that the supertranslations

$\mathfrak{s}$ are a Lie ideal in $\mathfrak{spi}$. The quotient algebra $\mathfrak{spi}/\mathfrak{s}$ is then isomorphic to the algebra of Killing fields on $\mathscr{H}$ i.e. the Lorentz algebra $\mathfrak{so}(1,3)$. Thus the Spi symmetry algebra has the structure of a semi-direct sum

$$\mathfrak{spi} \cong \mathfrak{so}(1,3) \ltimes \mathfrak{s}\,. \tag{3.6.9}$$

The $\mathfrak{spi}$ algebra also has a preferred 4-dimensional subalgebra $\mathfrak{t}$ of *translations*. These are obtained as the supertranslations $\boldsymbol{f}$ satisfying the additional condition

$$\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f} = 0\,. \tag{3.6.10}$$

The space of solutions to the above condition is indeed 4-dimensional — this can be seen from the argument in Remark 3.6.1 below, or by solving the equation in a suitable coordinate system on $\mathscr{H}$; see Eqs. D.204 and D.205 of [92] or Eq. C.12 of [10]. Moreover, from Eq. (3.6.8), it can be verified that the Lie bracket of a translation with any other element of $\mathfrak{spi}$ is again a translation, that is, the translations $\mathfrak{t}$ are a 4-dimensional Lie ideal of $\mathfrak{spi}$.

Remark 3.6.1 (Translation vectors at i^0). Let $\boldsymbol{v}^a$ be a direction-independent vector at i^0, and $\boldsymbol{v}^a = \boldsymbol{f}\boldsymbol{\eta}^a + \boldsymbol{f}^a$ where $\boldsymbol{\eta}_a\boldsymbol{f}^a = 0$. Then, since $\boldsymbol{v}^a$ is direction-independent, we have

$$0 = \boldsymbol{\partial}_a\boldsymbol{v}_b = \boldsymbol{D}_a\boldsymbol{f}_b + \boldsymbol{h}_{ab}\boldsymbol{f} + \boldsymbol{\eta}_b(\boldsymbol{D}_a\boldsymbol{f} - \boldsymbol{f}_a)\,, \tag{3.6.11}$$

which then implies $\boldsymbol{f}_a = \boldsymbol{D}_a\boldsymbol{f}$ and that $\boldsymbol{f}$ satisfies Eq. (3.6.10). Thus, any vector $\boldsymbol{v}^a \in Ti^0$ gives rise to a Spi-translation in $\mathfrak{t}$. Conversely, given any translation $\boldsymbol{f} \in \mathfrak{t}$, the vector at i^0 defined by (note the sign difference in the hyperboloidal component relative to Eq. (3.6.7))

$$\boldsymbol{v}^a := \boldsymbol{f}\boldsymbol{\eta}^a + \boldsymbol{D}^a\boldsymbol{f}\,, \tag{3.6.12}$$

is direction-independent i.e., $\boldsymbol{v}^a \in Ti^0$. Thus, the Spi-translations $\mathfrak{t}$ can be represented by vectors in Ti^0.

Remark 3.6.2 (Conformal transformation of Spi symmetries). Let $(\boldsymbol{f}, \boldsymbol{X}^a)$ be a Spi symmetry defined by a vector field ξ^a as above, i.e.,

$$\boldsymbol{X}^a := \lim_{\to i^0} \Omega^{-1/2} \xi^a \,, \quad \boldsymbol{f} := \lim_{\to i^0} \Omega^{-1} \xi^a \eta_a \,. \tag{3.6.13}$$

For a fixed ξ^a, consider the change in the conformal factor $\Omega \mapsto \omega\Omega$. Then, from Remark 3.2.1, we have the transformations

$$\boldsymbol{X}^a \mapsto \boldsymbol{X}^a \,, \quad \boldsymbol{f} \mapsto \boldsymbol{f} + \tfrac{1}{2} \pounds_{\boldsymbol{X}} \boldsymbol{\alpha} \,. \tag{3.6.14}$$

Note that a pure supertranslation $(\boldsymbol{f}, \boldsymbol{X}^a = 0)$ is conformally-invariant, while a "pure Lorentz" symmetry $(\boldsymbol{f} = 0, \boldsymbol{X}^a)$ is not invariant but shifts by a supertranslation given by $\frac{1}{2} \pounds_{\boldsymbol{X}} \boldsymbol{\alpha}$. This further reflects the semi-direct structure of the $\mathfrak{spi}$ algebra given in Eq. (3.6.9).

* * *

To find the charge corresponding to the Spi-symmetries, we need to evaluate the symplectic current given in Eq. (3.5.15) when the perturbation denoted by δ_2 is generated by a Spi-symmetry. So we now calculate the perturbations $\delta_{(\boldsymbol{f},\boldsymbol{X})} \boldsymbol{E}$ and $\delta_{(\boldsymbol{f},\boldsymbol{X})} \boldsymbol{K}$ in the gravitational potentials corresponding to the metric perturbation given in Eq. (3.6.2).

The potentials $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ are defined in terms of (a rescaled) limit of S_{ab} by Eq. (3.4.9). Consider then the change in S_{ab} under the perturbation Eq. (3.6.2). The second term on the right-hand-side of Eq. (3.6.2) is a linearized conformal transformation (see Remark 3.2.1) with $\boldsymbol{\alpha} = -2\boldsymbol{f}$. Thus, the change in $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ induced by this linearized conformal transformation is given by (see Remark 3.4.1)

$$\delta_{\boldsymbol{f}} \boldsymbol{E} = 0 \,, \quad \delta_{\boldsymbol{f}} \boldsymbol{K}_{ab} = 4(\boldsymbol{D}_a \boldsymbol{D}_b \boldsymbol{f} + \boldsymbol{f} \boldsymbol{h}_{ab}) \,. \tag{3.6.15}$$

The first term on the right hand side of Eq. (3.6.2) is a linearized diffeomorphism and, since S_{ab} is a local and covariant functional of g_{ab}, the corresponding perturbation in S_{ab} is $\pounds_\xi S_{ab}$. Explicitly computing the Lie derivative using Eqs. (3.6.1) and (3.6.3) gives

$$\delta_{\boldsymbol{X}} \boldsymbol{S}_{ab} = \lim_{\to i^0} \Omega^{1/2} \pounds_\xi S_{ab} = \boldsymbol{X}^c \boldsymbol{\partial}_c \boldsymbol{S}_{ab} + 2\boldsymbol{S}_{c(a} \boldsymbol{\eta}_{b)} \boldsymbol{X}^c + 2\boldsymbol{S}_{c(a} \boldsymbol{\partial}_{b)} \boldsymbol{X}^c \,. \tag{3.6.16}$$

Then, from the definition of the gravitational potentials Eq. (3.4.9) we have

$$\delta_{\boldsymbol{X}} \boldsymbol{E} = \pounds_{\boldsymbol{X}} \boldsymbol{E} \,, \quad \delta_{\boldsymbol{X}} \boldsymbol{K}_{ab} = \pounds_{\boldsymbol{X}} \boldsymbol{K}_{ab} \,. \tag{3.6.17}$$

As a result, under a general Spi symmetry parametrized by $(\boldsymbol{f}, \boldsymbol{X}^a)$ we have

$$\delta_{(\boldsymbol{f},\boldsymbol{X})} \boldsymbol{E} = \pounds_{\boldsymbol{X}} \boldsymbol{E} \,, \quad \delta_{(\boldsymbol{f},\boldsymbol{X})} \boldsymbol{K}_{ab} = \pounds_{\boldsymbol{X}} \boldsymbol{K}_{ab} + 4(\boldsymbol{D}_a \boldsymbol{D}_b \boldsymbol{f} + \boldsymbol{h}_{ab} \boldsymbol{f}) \,. \tag{3.6.18}$$

Note that our parity condition Eq. (3.4.33) does not place any further restrictions on these symmetries.

Remark 3.6.3 (Special choices of conformal factor)*.* The freedom in the conformal factor can be used to impose further restrictions on the potential $\boldsymbol{K}_{ab}$. We note the following two conditions that have been used in prior work.

(1) From Eq. (3.4.22) we see that $\boldsymbol{K} := \boldsymbol{h}^{ab} \boldsymbol{K}_{ab}$ transforms as

$$\boldsymbol{K} \mapsto \boldsymbol{K} - 2(\boldsymbol{D}^2 \boldsymbol{\alpha} + 3\boldsymbol{\alpha}) \,. \tag{3.6.19}$$

Now given a choice of conformal factor so that $\boldsymbol{K} \neq 0$, we can always solve a linear hyperbolic equation for $\boldsymbol{\alpha}$ on $\mathscr{H}$ and choose a new conformal factor (as in Remark 3.2.1) so that in the new conformal completion $\boldsymbol{K} = 0$. This is the choice made in [88, 91, 92]. With this restriction on $\boldsymbol{K}$ we see from Eq. (3.6.18) that the allowed supertranslations are reduced to functions $\boldsymbol{f}$ which satisfy

$$\boldsymbol{D}^2 \boldsymbol{f} + 3\boldsymbol{f} = 0 \,. \tag{3.6.20}$$

(2) Consider the restricted class of spacetimes where $\boldsymbol{B}_{ab} = 0$. Then, the tensor $\boldsymbol{K}_{ab}$ can be written in terms of a scalar potential $\boldsymbol{k}$ as in Eq. (3.4.16). Comparing Eq. (3.4.16) with Eq. (3.4.22), we see that we can choose $\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{k}$. Then, we can choose a new conformal factor (as in Remark 3.2.1) so that in the new conformal completion $\boldsymbol{K}_{ab} = 0$. This is the choice made in [16, 17]. With this restriction we see from Eq. (3.6.18) that the allowed supertranslations are reduced to the translation algebra (Eq. (3.6.10)) and the full asymptotic symmetry algebra

reduces to the Poincaré algebra.

We emphasize that to keep our expressions as general as possible, we will *not* impose any such conditions on the conformal factor in our analysis and will work with the full $\mathfrak{spi}$ algebra. However, we will argue that our results reduce to those of [16, 92] when the corresponding restrictions are imposed.

3.7 | Spi-charges

In this section, we compute the charges associated with the Spi-symmetries. Following our strategy, we consider the symplectic current $\underleftarrow{\boldsymbol{\omega}}$ where one of the perturbations, δ_2, is a perturbation generated by an asymptotic Spi-symmetry represented by $(\boldsymbol{f}, \boldsymbol{X}^a)$. Using Eqs. (3.5.15) and (3.6.18) we have

$$\underleftarrow{\boldsymbol{\omega}}(\delta g, \delta_{(\boldsymbol{f},\boldsymbol{X})} g) = \frac{1}{64\pi} \boldsymbol{\varepsilon}_3 \left[\delta \boldsymbol{K} \pounds_{\boldsymbol{X}} \boldsymbol{E} - \delta \boldsymbol{E} \pounds_{\boldsymbol{X}} \boldsymbol{K} - 4 \delta \boldsymbol{E} (\boldsymbol{D}^2 \boldsymbol{f} + 3 \boldsymbol{f}) \right] . \tag{3.7.1}$$

We show next that, under suitable conditions, the above expression can be written as a total derivative on $\mathscr{H}$ that is,

$$\underleftarrow{\boldsymbol{\omega}}(\delta g, \delta_{(\boldsymbol{f},\boldsymbol{X})} g) = -\boldsymbol{\varepsilon}_3 \; \boldsymbol{D}^a \boldsymbol{Q}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X})) \, , \tag{3.7.2}$$

where $\boldsymbol{Q}_a$ is a local and covariant functional of its arguments on $\mathscr{H}$.

It will be convenient to do this separately for supertranslations and Lorentz symmetries. In Sec. 3.7.1, we will find that for supertranslations the functional $\boldsymbol{Q}_a$ is integrable, and defines the *supermomentum* charges on cross-sections S of $\mathscr{H}$. Then we will show in Sec. 3.7.2 that for Lorentz symmetries $\boldsymbol{Q}_a$, is not integrable in general. In this case, we will adopt the prescription of Wald and Zoupas with suitable modifications to define an integrable charge for Lorentz symmetries. Finally, as noted in Remark 3.6.2, a “pure Lorentz” symmetry is not conformally-invariant but rather shifts by a supertranslation under the action of a conformal transformation. We will show in Sec. 3.7.3 that the Lorentz charge similarly shifts by a supertranslation charge under a conformal transformation, in accord with the semi-direct structure of the $\mathfrak{spi}$ algebra (Eq. (3.6.9)).

1 Charges for supertranslations: Spi-supermomentum

To define the charge for the supertranslations, consider Eq. (3.7.1) for a pure supertranslation $(\boldsymbol{f}, \boldsymbol{X}^a = 0)$

$$\begin{aligned} \underleftarrow{\boldsymbol{\omega}}(\delta g, \delta_{\boldsymbol{f}} g) &= -\frac{1}{16\pi}\boldsymbol{\varepsilon}_3 \; \delta \boldsymbol{E}(\boldsymbol{D}^2 \boldsymbol{f} + 3\boldsymbol{f})\,, \\ &= -\frac{1}{16\pi}\boldsymbol{\varepsilon}_3 \boldsymbol{D}^a \delta(\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{f} - \boldsymbol{f}\boldsymbol{D}_a\boldsymbol{E})\,, \end{aligned} \tag{3.7.3}$$

where the second line uses Eq. (3.4.12). In this case, the symplectic current can be written in the form Eq. (3.7.2) where the $\boldsymbol{Q}_a$ is manifestly integrable. Thus, we define the Spi *supermomentum* charge at a cross-section S of $\mathscr{H}$ by

$$\mathcal{Q}[\boldsymbol{f}; S] = \frac{1}{16\pi}\int_S \boldsymbol{\varepsilon}_2 \; \boldsymbol{u}^a(\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{f} - \boldsymbol{f}\boldsymbol{D}_a\boldsymbol{E})\,. \tag{3.7.4}$$

Here we have chosen the charge to vanish on Minkowski spacetime where $\boldsymbol{E} = 0$ (see Eq. (3.4.34)). The corresponding flux is given by (using Eq. (3.4.12))

$$\mathcal{F}[\boldsymbol{f}; \Delta\mathscr{H}] := \mathcal{Q}[\boldsymbol{f}; S_2] - \mathcal{Q}[\boldsymbol{f}; S_1] = -\frac{1}{16\pi}\int_{\Delta\mathscr{H}} \boldsymbol{\varepsilon}_3 \; \boldsymbol{E}(\boldsymbol{D}^2\boldsymbol{f} + 3\boldsymbol{f})\,. \tag{3.7.5}$$

When $\boldsymbol{f} \in \mathfrak{t}$ is a Spi-translation the charge Eq. (3.7.4) can be written in an alternative form as follows. Using Eqs. (3.4.11) and (3.4.12) we have the identity

$$\begin{aligned} -\boldsymbol{f}\boldsymbol{D}_a\boldsymbol{E} + \boldsymbol{E}\boldsymbol{D}_a\boldsymbol{f} &= 2\boldsymbol{E}_{ab}\boldsymbol{D}^b\boldsymbol{f} + \boldsymbol{D}^b\left(\boldsymbol{D}_{[a}\boldsymbol{E}\boldsymbol{D}_{b]}\boldsymbol{f}\right) \\ &\quad - \tfrac{1}{2}\left[\boldsymbol{D}_a\boldsymbol{E}(\boldsymbol{D}^2\boldsymbol{f} + 3\boldsymbol{f}) - \boldsymbol{D}^b\boldsymbol{E}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f})\right]\,. \end{aligned} \tag{3.7.6}$$

The second term on the right-hand-side corresponds to an exact 2-form and vanishes upon integrating on S while the last line vanishes for translations due to Eq. (3.6.10). Hence, the charge for any translation $\boldsymbol{f} \in \mathfrak{t}$ can be written as

$$\mathcal{Q}[\boldsymbol{f}; S] = \frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2 \; \boldsymbol{u}^a\boldsymbol{E}_{ab}\boldsymbol{D}^b\boldsymbol{f}\,, \tag{3.7.7}$$

which reproduces the charge for translations given in [16]. Using Eq. (3.6.10), the flux of translations vanishes across any region $\Delta\mathscr{H}$ and thus the translation charge is independent of the choice of cross-section S. Using the isomorphism between Spi-translations $\boldsymbol{f}$ and vectors $\boldsymbol{v}^a$ in Ti^0 (see Remark 3.6.1), the translation charge in Eq. (3.7.7) defines a 4-momentum vector $\boldsymbol{P}^a$ at i^0 such that

$$\boldsymbol{P}^a \boldsymbol{v}_a = \mathcal{Q}[\boldsymbol{f}; S] . \tag{3.7.8}$$

Note that this relation is well-defined at i^0 since the translation charge is independent of the cross-section S. The vector $\boldsymbol{P}^a$ is precisely the ADM 4-momentum at i^0 [101] and also coincides with the limit to i^0 of the Bondi 4-momentum on null infinity [11] (the corresponding result for all the supertranslation charges was proven in [10]).

The charge expression in Eq. (3.7.4) agrees with the results of Compère and Dehouck [92]. Note that when the conformal factor is chosen so that $\boldsymbol{K} = 0$, the supertranslation algebra is reduced to the subalgebra satisfying Eq. (3.6.20) and the flux corresponding to such supertranslations vanishes across any region $\Delta\mathscr{H}$. As was shown in [10], to relate the supertranslation symmetries and charges at spatial infinity to the ones on null infinity, it is sufficient that the *total* flux of these charges vanishes on *all* of $\mathscr{H}$, and the flux need not vanish across some local region $\Delta\mathscr{H}$. Thus the restriction on the conformal factor imposing $\boldsymbol{K} = 0$ is not necessary.

2 | Lorentz charges with $\boldsymbol{B}_{ab} = 0$

Next we will obtain a charge formula for the Lorentz symmetries. As emphasized in [16, 17], to obtain such a charge formula one needs to consider the "subleading" piece of the magnetic part of the Weyl tensor. Thus, in the following we will make the additional assumption that $\boldsymbol{B}_{ab} = 0$ and that the "subleading" magnetic part $\boldsymbol{\beta}_{ab}$ defined in Eq. (3.4.14) exists. However, in Appendix A.8 we will show how the restriction that $\boldsymbol{B}_{ab}$ vanishes can be lifted and obtain a charge formula in that case.

For a "pure Lorentz" symmetry ($\boldsymbol{f} = 0, \boldsymbol{X}^a$), we have from Eq. (3.7.1)

$$\underleftarrow{\boldsymbol{\omega}}(\delta g, \delta_{\boldsymbol{X}} g) = \frac{1}{64\pi} \boldsymbol{\varepsilon}_3 (\pounds_{\boldsymbol{X}} \boldsymbol{E} \delta \boldsymbol{K} - \pounds_{\boldsymbol{X}} \boldsymbol{K} \delta \boldsymbol{E}) . \tag{3.7.9}$$

We now want to write this as a total derivative of the form Eq. (3.7.2). To do so consider the following tensor

$$\boldsymbol{W}_{ab} := \boldsymbol{\beta}_{ab} + \tfrac{1}{8}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\boldsymbol{E}\boldsymbol{K}^d{}_{b)} - \tfrac{1}{16}\boldsymbol{\varepsilon}_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E}\,. \tag{3.7.10}$$

Using Eqs. (3.4.13a), (3.4.15) and (3.4.20), we obtain

$$\boldsymbol{D}^a\boldsymbol{W}_{ab} = 0\,, \quad \boldsymbol{h}^{ab}\boldsymbol{W}_{ab} = 0\,. \tag{3.7.11}$$

Note that $\boldsymbol{W}_{ab}$ is not a symmetric tensor. Further using Eqs. (3.7.10) and (A.5.3) we have

$$\boldsymbol{D}^a[\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b] = \tfrac{1}{8}\boldsymbol{X}^a\boldsymbol{D}_a\boldsymbol{E}\boldsymbol{K}\,, \tag{3.7.12}$$

where ${}^{\star}\boldsymbol{X}^a := \tfrac{1}{2}\boldsymbol{\varepsilon}^{abc}\boldsymbol{D}_b\boldsymbol{X}_c$ is the "dual" Killing vector field to $\boldsymbol{X}^a$ (see Eq. (A.5.4)). Therefore, Eq. (3.7.9) can be written as

$$\underleftarrow{\boldsymbol{\omega}}(\delta g, \delta_{\boldsymbol{X}} g) = \frac{1}{8\pi}\boldsymbol{\varepsilon}_3\boldsymbol{D}^a\left[\delta\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b - \tfrac{1}{8}\delta\boldsymbol{E}\boldsymbol{K}\boldsymbol{X}_a\right]\,, \tag{3.7.13}$$

which is again of the form Eq. (3.7.2). However the functional $\boldsymbol{Q}_a$ in this case is not integrable, in general. To see this consider

$$\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\boldsymbol{Q}_a[\delta g; \boldsymbol{X}] = -\frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\left[\delta\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b - \tfrac{1}{8}\delta\boldsymbol{E}\boldsymbol{K}\boldsymbol{X}_a\right]\,, \tag{3.7.14}$$

and compute an antisymmetrized second variation to get

$$\begin{aligned}\int_S \boldsymbol{\varepsilon}_2\boldsymbol{u}^a\left(\delta_1\boldsymbol{Q}_a[\delta_2 g; \boldsymbol{X}] - \delta_2\boldsymbol{Q}_a[\delta_1 g; \boldsymbol{X}]\right) &= \tfrac{1}{64\pi}\int_S \boldsymbol{\varepsilon}_2\boldsymbol{u}^a\boldsymbol{X}_a\left(\delta_1\boldsymbol{K}\delta_2\boldsymbol{E} - \delta_2\boldsymbol{K}\delta_1\boldsymbol{E}\right)\\ &= -\int_S \boldsymbol{X}\cdot\underleftarrow{\boldsymbol{\omega}}(\delta_1 g, \delta_2 g)\,.\end{aligned} \tag{3.7.15}$$

If Eq. (3.7.14) were integrable then the above antisymmetrized second variation would vanish for all perturbations and all cross-sections S. However, since we allow arbitrary perturbations of both $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$, the expression on the right hand side vanishes if and only if the Lorentz vector field happens to be tangent to the cross-section S. However a general Lorentz vector field is not tangent to cross-sections of $\mathscr{H}$. Thus, the expression Eq. (3.7.14) is not integrable and cannot be used to define the charge of Lorentz symmetries.

To remedy this, we use the Wald-Zoupas prescription, as in Sec. 2.5. Let $\boldsymbol{\Theta}(g;\delta g)$ be a 3-form on $\mathscr{H}$ which is a symplectic potential for the pullback of the symplectic current (Eq. (3.5.15)) to $\mathscr{H}$, that is,

$$\underleftarrow{\boldsymbol{\omega}}(g;\delta_1 g,\delta_2 g) = \delta_1\boldsymbol{\Theta}(g;\delta_2 g) - \delta_2\boldsymbol{\Theta}(g;\delta_1 g)\,, \tag{3.7.16}$$

for all backgrounds and all perturbations. We also require that the choice of $\boldsymbol{\Theta}$ satisfy the following conditions

(1) $\boldsymbol{\Theta}$ is locally and covariantly constructed out of the dynamical fields $(\boldsymbol{E}, \boldsymbol{K}_{ab})$, their perturbations, and finitely many of their derivatives, along with the "universal background structure" $\boldsymbol{h}_{ab}$ present on $\mathscr{H}$.

(2) $\boldsymbol{\Theta}$ is independent of any arbitrary choices made in specifying the background structure, in particular, $\boldsymbol{\Theta}$ is conformally-invariant.

(3) $\boldsymbol{\Theta}(g;\delta g) = 0$ for Minkowski spacetime for *all* perturbations δg.

Following the Wald-Zoupas prescription, we define the charge $\mathcal{Q}[\boldsymbol{X}^a; S]$ associated with a Lorentz symmetry through

$$\delta\mathcal{Q}[\boldsymbol{X}^a; S] := \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{Q}_a(\delta g; \boldsymbol{X}^a) + \int_S \boldsymbol{X}\cdot\boldsymbol{\Theta}(\delta g)\,. \tag{3.7.17}$$

From Eqs. (3.7.15) and (3.7.16), it follows that this defining relation is integrable and thus defines a charge $\mathcal{Q}[\boldsymbol{X}^a; S]$ once we pick a reference solution where the charge vanishes.

For the 3-form $\boldsymbol{\Theta}$ we choose

$$\boldsymbol{\Theta}(g;\delta g) := -\frac{1}{64\pi}\boldsymbol{\varepsilon}_3 \boldsymbol{E}\delta\boldsymbol{K}\,. \tag{3.7.18}$$

It can be verified that this choice satisfies all the criteria listed below Eq. (3.7.16). In particular $\boldsymbol{\Theta}$ is conformally-invariant, and since for Minkowski spacetime $\boldsymbol{E} = 0$ (Eq. (3.4.34)), $\boldsymbol{\Theta} = 0$ on Minkowski spacetime for *all* perturbations. Note that this choice for $\boldsymbol{\Theta}$ is not unique; however, we will argue in

Appendix A.7 that the ambiguity in the choice of $\mathbf{\Theta}$ does not affect our final charge expression.

With the choice Eq. (3.7.18) and Eqs. (3.7.14) and (3.7.17), we have

$$\delta \mathcal{Q}[\boldsymbol{X}^a; S] = -\frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a \delta[\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b - \tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a]\,, \tag{3.7.19}$$

We define the unperturbed charge by picking the reference solution to be Minkowski spacetime which satisfies $\boldsymbol{E} = 0$ and $\boldsymbol{\beta}_{ab} = 0$ (Eq. (3.4.34)). Thus, we have the charge

$$\mathcal{Q}[\boldsymbol{X}^a; S] = -\frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a [\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b - \tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a]\,, \tag{3.7.20}$$

The corresponding flux of the Lorentz charges is given by

$$\mathcal{F}[\boldsymbol{X}^a, \Delta\mathscr{H}] = -\frac{1}{64\pi}\int_{\Delta\mathscr{H}} \boldsymbol{\varepsilon}_3\ \boldsymbol{E}\pounds_{\boldsymbol{X}}\boldsymbol{K}\,. \tag{3.7.21}$$

Note that the flux is essentially given by $\mathcal{F}[\boldsymbol{X}^a, \Delta\mathscr{H}] = \int_{\Delta\mathscr{H}} \mathbf{\Theta}(g; \delta_{\boldsymbol{X}} g)$ which is analogous to 2.5.27.

When the conformal factor is chosen so that $\boldsymbol{K}_{ab} = 0$ then the Lorentz charge reduces to

$$\mathcal{Q}[\boldsymbol{X}^a; S] = -\frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a \boldsymbol{\beta}_{ab}{}^{\star}\boldsymbol{X}^b\,, \tag{3.7.22}$$

which is the expression given in [16]. Note that when the conformal factor is chosen such that $\boldsymbol{K} = 0$, the expression Eq. (3.7.14) is manifestly integrable and our "correction term" $\mathbf{\Theta}$ (Eq. (3.7.18)) vanishes. In both these cases, the flux of the Lorentz charges vanishes across any region $\Delta\mathscr{H}$, i.e., the Lorentz charges are identically conserved. Further, since the vector fields $\boldsymbol{X}^a$ correspond precisely to infinitesimal Lorentz transformations $\mathbf{\Lambda}_{ab}$ in Ti^0 (see Eq. (A.5.6)), the charge in this case defines an "angular momentum" tensor $\boldsymbol{J}^{ab}$ at i^0 through

$$\boldsymbol{J}^{ab}\mathbf{\Lambda}_{ab} = \mathcal{Q}[\boldsymbol{X}^a; S]\,, \tag{3.7.23}$$

where the right hand side is independent of the cross-section since the charge is conserved.

3 | Transformation of charges under conformal changes

We now consider the transformation of the charges and fluxes for a Spi symmetry under changes of the choice of conformal factor as discussed in Remark 3.2.1.

Consider a pure supertranslation symmetry $(\boldsymbol{f}, \boldsymbol{X}^a = 0)$. As shown in Remark 3.6.2, a pure supertranslation is conformally-invariant. Further from Remark 3.4.1 the potential $\boldsymbol{E}$ is also conformally-invariant. Thus, the charge and flux of supertranslations in Eqs. (3.7.4) and (3.7.5) are also conformally-invariant.

However a "pure Lorentz" symmetry $(\boldsymbol{f} = 0, \boldsymbol{X}^a)$ is not conformally-invariant (see Remark 3.6.2), and hence we expect that the charge and flux of a Lorentz symmetry must transform nontrivially under changes of the conformal factor. Consider first the flux of Lorentz charges given by Eq. (3.7.21). Using the transformation of $\boldsymbol{K}_{ab}$ (Eq. (3.4.22)), we see that this flux expression transforms as

$$\mathcal{F}[\boldsymbol{X}^a; \Delta\mathscr{H}] \mapsto \mathcal{F}[\boldsymbol{X}^a; \Delta\mathscr{H}] + \frac{1}{32\pi} \int_{\Delta\mathscr{H}} \boldsymbol{\varepsilon}_3 \boldsymbol{E} (\boldsymbol{D}^2 \pounds_{\boldsymbol{X}} \boldsymbol{\alpha} + 3 \pounds_{\boldsymbol{X}} \boldsymbol{\alpha}) \,. \tag{3.7.24}$$

Comparing the second term on the right-hand-side with Eq. (3.7.5), we see that it is precisely the flux of a supertranslation given by $(-\tfrac{1}{2} \pounds_{\boldsymbol{X}} \boldsymbol{\alpha})$. Thus, under a change of conformal factor the Lorentz flux shifts by the flux of a supertranslation

$$\mathcal{F}[\boldsymbol{X}^a; \Delta\mathscr{H}] \mapsto \mathcal{F}[\boldsymbol{X}^a; \Delta\mathscr{H}] + \mathcal{F}[-\tfrac{1}{2} \pounds_{\boldsymbol{X}} \boldsymbol{\alpha}; \Delta\mathscr{H}] \,. \tag{3.7.25}$$

One can similarly verify that the Lorentz charge Eq. (3.7.20) also shifts by the charge of a supertranslation. The explicit computation is a bit tedious and is presented in Appendix A.6. However, we can derive the transformation of the Lorentz charge by a more general argument which we present below. This argument also holds in the more general case when $\boldsymbol{B}_{ab} \neq 0$ considered in Appendix A.8 below.

From the transformation of the flux Eq. (3.7.25), we can deduce that the Lorentz charge expression

Eq. (3.7.20) must transform as

$$\mathcal{Q}[\boldsymbol{X}^a; S] \mapsto \mathcal{Q}[\boldsymbol{X}^a; S] + \mathcal{Q}[-\tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}; S] + \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\mu}_a[\boldsymbol{\alpha}]\,, \tag{3.7.26}$$

where the second term on the right-hand-side is the charge of a supertranslation $(-\tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha})$ and the third term is a possible additional term determined by a covector $\boldsymbol{\mu}_a$ which depends linearly on $\boldsymbol{\alpha}$ and is divergence-free, $\boldsymbol{D}^a\boldsymbol{\mu}_a[\boldsymbol{\alpha}] = 0$ for *all* $\boldsymbol{\alpha}$. Since $\boldsymbol{\alpha}$ is a free function on $\mathscr{H}$ we can apply Theorem 1 with $\boldsymbol{\alpha}$ as the "dynamical field". Thus, from Eq. (A.5.10) we conclude that the final integral above vanishes and that the Lorentz charge shifts by the charge of a supertranslation $(-\tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha})$.

$$\mathcal{Q}[\boldsymbol{X}^a; S] \mapsto \mathcal{Q}[\boldsymbol{X}^a; S] + \mathcal{Q}[-\tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}; S]\,. \tag{3.7.27}$$

If we restrict to the choice of conformal factor where $\boldsymbol{K}_{ab} = 0$, so that the asymptotic symmetries are reduced to the Poincaré algebra and $\boldsymbol{\alpha}$ is a Spi-translation satisfying Eq. (3.6.10) then Eq. (3.7.27) reproduces the transformation law given in Eq. 29 of [16] and Eq. 6.8 of [17].

Consider the charge of any Spi-symmetry represented by $(\boldsymbol{f}, \boldsymbol{X}^a)$. Then, under a conformal transformation, the same Spi-symmetry is represented by $(\boldsymbol{f} + \tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}, \boldsymbol{X}^a)$ (see Remark 3.6.2). The total charge of the Spi-symmetry transforms as

$$\begin{aligned} \mathcal{Q}[\boldsymbol{f}; S] + \mathcal{Q}[\boldsymbol{X}^a; S] \mapsto & \mathcal{Q}[\boldsymbol{f} + \tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}; S] + \mathcal{Q}[\boldsymbol{X}^a; S] + \mathcal{Q}[-\tfrac{1}{2}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}; S]\,, \\ & = \mathcal{Q}[\boldsymbol{f}; S] + \mathcal{Q}[\boldsymbol{X}^a; S]\,, \end{aligned} \tag{3.7.28}$$

that is, the charge of any Spi-symmetry is independent of the choice of conformal factor — the change in the function $\boldsymbol{f}$ representing the symmetry is exactly compensated by the change in the Lorentz charge given in Eq. (3.7.27).

3.8 | Future directions

An avenue for future investigation based on the analysis presented here would be to quantize the asymptotic fields on $\mathscr{H}$ in the spirit of the asymptotic quantization program on null infinity [19] (see also [102]). This could lead to the possibility of relating the asymptotic "in-states" on past null infinity to the "out-states" on future null infinity, similar to the matching conditions in the classical theory, and provide further insight into the structure of quantum scattering.

We also note that the asymptotic fields at spatial infinity in both Maxwell theory and general relativity are described by smooth tensor fields living on a unit-hyperboloid $\mathscr{H}$. As is well-known, $\mathscr{H}$ is precisely 3-dimensional de Sitter spacetime. It would be interesting to see if insights from the de Sitter/CFT correspondence [103] can be applied to develop a holographic understanding of electromagnetism and general relativity in asymptotically-flat spacetimes at spatial infinity, perhaps similar to [104].

Chapter 4

Extensions of the asymptotic symmetry algebra of general relativity

(Adapted with permission from [6])

Chapter summary

In this chapter, we study a proposal for an extension of the Bondi-Metzner-Sachs algebra to include arbitrary infinitesimal diffeomorphisms on a 2-sphere, first made in [44]. To realize the elements of this extended algebra as asymptotic symmetries, we work with an extended class of spacetimes in which the unphysical metric at null infinity is not universal. We show that the symplectic current evaluated on these extended symmetries is divergent in the limit to null infinity. We also show that this divergence cannot be removed by a local and covariant redefinition of the symplectic current. This implies that at the very least, the usual framework of the covariant phase space formalism must be augmented to make sense of this extension. Using techniques inspired by holographic renormalization in Anti de Sitter spacetimes, this question was recently revisited in [54] although we will not include that analysis here. As a subsidiary result, we also show that the aforementioned extended algebra does not have a preferred subalgebra of translations and therefore does not admit a universal definition of Bondi 4-momentum.

4.1 | Context

As discussed chapter 1, there have been various recent proposals for extensions of the BMS algebra. It was first proposed by Barnich and Troessaert [105, 106] (see also [65]) that the BMS algebra should be extended to include the entire infinite-dimensional Virasoro algebra, which consists of all *local* conformal Killing fields on a 2-sphere[1]. The vector fields in the Virasoro algebra which are not Lorentz vector fields are necessarily singular at isolated points on a 2-sphere. An alternative proposal by Campiglia and Laddha [44, 45] was to extend the Lorentz transformations by including all smooth infinitesimal diffeomorphisms on a 2-sphere. The conservation law (i.e. Ward identity) for the charges at null infinity corresponding to such extensions is then claimed to be equivalent to the subleading soft theorem of [107]. In just the last year, various other extensions of the BMS algebra have also been proposed; see e.g, [46, 108]. We will not concern ourselves with these recent proposals in this chapter although some discussion on them may be found in [54]. Instead, we will instead restrict ourselves to analyzing the extension to smooth 2-sphere diffeomorphisms [44, 45] mentioned above.

The main quantity of interest in our analysis is the symplectic current which is derived from the Lagrangian of general relativity and was discussed at length in Sec. 2.5. As detailed there, if suitable asymptotic conditions are satisfied then the symplectic current has a finite limit to null infinity. Then, the integral of the symplectic current over null infinity gives a symplectic form on the phase space of general relativity. If one of the perturbations, say $\delta_2 \hat{g}_{ab}$, is taken to be the perturbation generated by some asymptotic symmetry, then the symplectic form leads to an expression for the generator of that symmetry on phase space. Note that the crucial thing here is that the symplectic current needs to have a finite limit to null infinity, otherwise the generator would not be defined.

It follows from the analysis of chapter 2 that for asymptotically-flat spacetimes, the symmetries in the usual BMS algebra have well-defined generators in the sense described above. Here, we

[1] These symmetries are often called *superrotations* [105, 106], although more recently that term has come to be used for the smooth infinitesimal diffeomorphisms on the 2-sphere [23]. Another terminology for the smooth diffeomorphisms is *super-Lorentz* transformations, with the odd parity ones being called superrotations and the even parity ones being called superboosts [37].

are interested in whether generators corresponding to the extension of the BMS algebra by all diffeomorphisms of a 2-sphere correspond to well defined symmetries. We provide evidence to the contrary. In particular, we show that the symplectic current of general relativity diverges in the limit to null infinity, in general, when one of the perturbations is generated by an extended BMS symmetry (which is not a BMS symmetry). This divergence was also previously encountered in the computations of Compère, Fiorucci and Ruzziconi [37].

A potential loophole in this argument is that one can exploit an ambiguity in the symplectic current to render it finite in the limit to null infinity (see Ref. [109] for a general discussion of such renormalization in a different context). The ambiguity is of the form $\boldsymbol{\omega} \to \boldsymbol{\omega} + d[\delta_1 \boldsymbol{Y}(\hat{g}; \delta_2 \hat{g}) - (1 \leftrightarrow 2)]$, for some two-form $\boldsymbol{Y}$ which constructed out of the dynamical fields and their variations [53]. Recently, Compère, Fiorucci and Ruzziconi have shown that one can indeed obtain a finite symplectic current using this method, and they find expressions for charges corresponding to all the symmetries of the extended algebra, including the general 2-sphere diffeomorphisms [37].

However, as noted by the authors themselves, their prescription relies on a particular choice of coordinates with the result that the two form $\boldsymbol{Y}$ and the final, finite symplectic current are not local, covariant function of the dynamical fields. Thus, it is not clear that the expressions obtained in Ref. [37] for charges are unique. For instance, if one repeated the construction using Newman-Unti coordinates instead of Bondi coordinates, it is not clear if equivalent results would be obtained. We will in fact show that one *cannot* eliminate the divergences in the symplectic current by exploiting the ambiguity in a local and covariant manner. We refer the reader to [54] for treatments of this divergence when covariance requirements are relaxed.

This result suggests that the general 2-sphere diffeomorphisms do not give rise to well defined charges and fluxes. However, a possible loophole is that requiring that all the quantities in the construction be local and covariant [53] is too strong a restriction, and instead one should only impose this requirement on physically measurable quantities. For example it might be possible that the presymplectic form (obtained by integrating the presymplectic current over a Cauchy surface) may be independent of the arbitrary choice of coordinate system used in Ref. [37], despite the fact that the presymplectic current $\boldsymbol{\omega}$ does depend on this choice. It would be interesting to investigate this possibility further but we do not do so here.

The remainder of this chapter is organized as follows. In Sec. 4.2, we consider the extended phase space proposed in [44] which leads to an extension of the BMS algebra to include arbitrary infinitesimal diffeomorphisms of a 2-sphere. In Sec. 4.3, we show that the symplectic current evaluated on these extended symmetries diverges in the limit to null infinity. We also show that any local and covariant ambiguities in the symplectic current cannot get rid of this divergent behavior. We consider other issues associated with this extension of the BMS algebra in Sec. 4.4. In Appendix A.9, we show that the extension of the BMS algebra by all diffeomorphisms of a 2-sphere does not contain a preferred translation subalgebra and discuss the implications of this result.

Note that in this chapter, we make a slight change of notation compared to chapter 2 and denote $\gamma_{ab} := \delta g_{ab}$ (c.f. Eq. (2.3.8)).

4.2 | An extended field configuration space and extended algebra

As discussed at length in chapter 2, a priori, the conformal completion of a spacetime depends on the physical spacetime $(\hat{M}, \hat{g}_{ab})$ under consideration. However, if (M, g_{ab}, Ω) and (M', g'_{ab}, Ω') are the unphysical spacetimes corresponding to *any* two asymptotically-flat physical spacetimes, then M' can be identified with M using a diffeomorphism such that $\mathscr{I}'$ maps to $\mathscr{I}$[2], $\Omega' = \Omega$ in a neighborhood of $\mathscr{I}$ and $g'_{ab}|_{\mathscr{I}} = g_{ab}|_{\mathscr{I}}$. As a result of this identification, we can work on a single manifold M with boundary $\mathscr{I}$ and treat Ω and $g_{ab}|_{\mathscr{I}}$ as *universal* within the entire class of asymptotically-flat spacetimes, in the sense that they can be chosen to be independent of the choice of the physical spacetime. Specifically, we can fix a metric $g_{0\,ab}$ on $\mathscr{I}$ and a conformal factor Ω_0 in a neighborhood $\mathcal{N}$ of $\mathscr{I}$, and define the field configuration space

$$\Gamma_0 = \left\{(M, g_{ab}, \Omega) |\; g_{ab\,|\mathscr{I}} = g_{0\,ab\,|\mathscr{I}}, \quad \Omega = \Omega_0 \text{ on } \mathcal{N}, \quad \nabla_a \nabla_b \Omega \mathrel{\hat{=}} 0\right\}. \tag{4.2.1}$$

[2] We will not need to distinguish between past and future null infinity in this chapter and the discussion can be applied to either of them. For this reason, we will just use "null infinity" (denoted by $\mathscr{I}$) without specifying "past" or "future."

The asymptotic symmetries at $\mathscr{I}$ of the field configuration space Γ_0 are the infinitesimal diffeomorphisms generated by vector fields ξ^a in M which extend smoothly to $\mathscr{I}$, and whose pullbacks preserve the asymptotic flatness conditions and map Γ_0 into itself (modded out by the trivial diffeomorphisms whose asymptotic charges vanish which, in vacuum general relativity, correspond to vector fields that vanish on $\mathscr{I}$ [53]). It was shown in chapter 2 that these correspond to the BMS algebra. Recall that metric perturbations within this field configuration space satisfy

$$\gamma_{ab} = \Omega \tau_{ab} \ , \quad \gamma_{ab} n^b = \Omega^2 \tau_a \ , \tag{4.2.2}$$

for some tensor fields τ_{ab} and τ_a that extend smoothly to $\mathscr{I}$.

In the following subsection we show that by weakening the universal structure near $\mathscr{I}$ — equivalently, by extending the class of allowed metrics — one obtains a bigger asymptotic symmetry algebra at null infinity which includes all the smooth diffeomorphisms of a 2-sphere. This is the algebra proposed by Campiglia and Laddha [44, 45].

1 Extended field configuration space

It is clear from the preceding discussion that to obtain any extension of the BMS algebra, one must enlarge the class of metrics under consideration. One option might be to suitably weaken the definition of asymptotic flatness. An alternative approach, which we follow here, is to enlarge the definition (4.2.1) of the field configuration space by relaxing the requirement that the unphysical metric evaluated on $\mathscr{I}$ be universal. The motivation for this enlargement is questionable, since the new metrics that are being added are related to metrics already included in the space Γ_0 by diffeomorphisms and by the conformal transformations (2.2.8). This issue is discussed further in Sec. 4.4.0.2 below. Nevertheless, we shall proceed and consider the extended class of metrics proposed by Campiglia and Laddha [45].

In the definition of the extended field configuration space, we will continue to require that the unphysical metric g_{ab} be smooth at $\mathscr{I}$. We will also continue to choose the conformal factor Ω so that the Bondi condition (2.2.6) holds. Now, if we are given an unphysical spacetime (M, g_{ab}, Ω), we can define tensors n^a and $\underleftarrow{\varepsilon_{abc}}$ intrinsic to $\mathscr{I}$ by

$$n^a = g^{ab}\nabla_a\Omega|_{\mathscr{I}}, \tag{4.2.3a}$$

$$\varepsilon_{abcd}|_{\mathscr{I}} = 4\varepsilon_{[abc}n_{d]}|_{\mathscr{I}}, \tag{4.2.3b}$$

and by defining $\underleftarrow{\varepsilon_{abc}}$ to be the pullback of ε_{abc} to $\mathscr{I}$.[3] It follows from the Bondi condition (2.2.6) that

$$\pounds_{\hat{n}}\underleftarrow{\varepsilon_{abc}} \mathrel{\hat{=}} 0. \tag{4.2.4}$$

We now fix a choice of tensors n_0^a, $\underleftarrow{\varepsilon_{0\,abc}}$ obtained in this way, fix a choice of conformal factor Ω_0 on a neighborhood $\mathcal{N}$ of $\mathscr{I}$, and define the extended field configuration space Γ_{ext} to be [compare Eq. (4.2.1)]

$$\Gamma_{\text{ext}} = \Big\{(M, g_{ab}, \Omega)|\; n^a = n_0^a, \quad \underleftarrow{\varepsilon_{abc}} = \underleftarrow{\varepsilon_{0\,abc}}, \quad \Omega = \Omega_0 \text{ on } \mathcal{N}, \quad \nabla_a\nabla_b\Omega|_{\mathscr{I}} = 0\Big\}\,. \tag{4.2.5}$$

This is the definition proposed by Campiglia and Laddha [45], in which the 3-volume form $\bar{\varepsilon}_{abc}$ and the normal $\hat{n}^a$ at $\mathscr{I}$ are universal. Note that Γ_0 is a proper subset of Γ_{ext}, if we choose the fields $\hat{n}_0^a$ and $\bar{\varepsilon}_{0\,abc}$ to be those associated with $g_{0\,ab}|_{\mathscr{I}}$.

2 Extended algebra

We now derive the form of the symmetry algebra for the field configuration space (4.2.5). Since the fields $\underleftarrow{\varepsilon_{abc}}$ and n^a are universal their perturbations must vanish and so

$$\begin{aligned} \delta\underleftarrow{\varepsilon_{abc}} \mathrel{\hat{=}} 0 \implies g^{ab}\gamma_{ab} \mathrel{\hat{=}} 0 \implies g^{ab}\gamma_{ab} = \Omega\sigma\,, \\ \delta n^a \mathrel{\hat{=}} 0 \implies \gamma_{ab}n^b = \Omega\chi_a\,, \end{aligned} \tag{4.2.6}$$

[3] Note that ε_{abc} is ambiguous up to $\varepsilon_{abc} \mapsto \varepsilon_{abc} + \alpha_{[ab}n_{c]}$ but this does not affect the pullback $\underleftarrow{\varepsilon_{abc}}$.

for some fields σ and χ_a which extend smoothly to $\mathscr{I}$. Perturbing the Bondi condition (2.2.6) we have

$$\delta(\nabla_a n_b) \hat{=} 0 \implies n^c \nabla_c \gamma_{ab} \hat{=} 2n_{(a}\chi_{b)}, \tag{4.2.7}$$

and taking the trace and using Eq. (4.2.6) gives

$$n^a \chi_a \hat{=} 0. \tag{4.2.8}$$

Thus, the unphysical metric perturbations γ_{ab} in the extended class satisfy

$$g^{ab}\gamma_{ab} = \Omega\sigma\,, \quad \gamma_{ab}n^b = \Omega\chi_a\,, \quad n^a\chi_a \hat{=} 0\,, \quad n^c\nabla_c\gamma_{ab} \hat{=} 2n_{(a}\chi_{b)}\,. \tag{4.2.9}$$

To find the asymptotic symmetries of this extended class of spacetimes, let $\delta g^{(\xi)}_{ab}$ be the unphysical metric perturbation generated by a diffeomorphism along ξ^a. Imposing the requirements (4.2.9), we find that ξ^a still satisfies $\pounds_\xi n^a \hat{=} -\alpha_{(\xi)} n^a$ and $\pounds_n \alpha_{(\xi)} \hat{=} 0$. However, since the unphysical metric is no longer universal at $\mathscr{I}$, $\delta g^{(\xi)}_{ab}|_{\mathscr{I}}$ is no longer required to vanish and so the condition

$$\pounds_\xi q_{ab} \hat{=} 2\alpha_{(\xi)} q_{ab}\,, \tag{4.2.10}$$

no longer holds.[4] Using $g^{ab}\gamma^{(\xi)}_{ab} \hat{=} 0$ we have instead

$$\pounds_\xi \underleftarrow{\varepsilon_{abc}} \hat{=} 3\alpha_{(\xi)} \underleftarrow{\varepsilon_{abc}}, \tag{4.2.11}$$

where $\underleftarrow{\varepsilon_{abc}}$ is the 3-volume element (4.2.3b) on $\mathscr{I}$.

Modding out by the trivial diffeomorphisms as before[5], we thus find that this extension of the BMS algebra $\mathfrak{b}_{\text{gen}}$ [44] of Γ_{ext} is generated by vector fields ξ^a on $\mathscr{I}$ which are tangent to $\mathscr{I}$ and

[4] The last condition in Eq. (4.2.9) does not impose additional restrictions on ξ^a at $\mathscr{I}$.

[5] The trivial vector fields are those for which the integral of symplectic current evaluated on $\delta\hat{g}_{ab}$ and $\pounds_\xi \hat{g}_{ab}$ vanishes [53]. However, this quantity cannot be evaluated since the symplectic current diverges on $\mathscr{I}$, as we show in the next section. So we simply assume here that the trivial vector fields ξ^a are those which vanish on $\mathscr{I}$, as for the standard definition of field configuration space. The symmetry algebra could thus change given a finite renormalized symplectic current; see Sec. 4.4.0.2 below for further discussion of this point.

which satisfy

$$\pounds_\xi n^a \mathrel{\hat{=}} -\alpha_{(\xi)} n^a, \tag{4.2.12a}$$

$$\pounds_\xi \underleftarrow{\varepsilon_{abc}} \mathrel{\hat{=}} 3\alpha_{(\xi)} \underleftarrow{\varepsilon_{abc}}, \tag{4.2.12b}$$

where the function $\alpha_{(\xi)}$ is smooth and satisfies $\pounds_n \alpha_{(\xi)} \mathrel{\hat{=}} 0$. A small point on terminology is in order. In recent years, the term *generalized* BMS algebra has become commonplace for the algebra of the aforementioned symmetries, as opposed to *extended* BMS algebra which is used for the extension of the BMS algebra which includes the full Virasoro algebra. We will therefore use *generalized* BMS algebra to refer to this algebra from here on. The structure of the generalized BMS algebra can be analyzed as before. There is an infinite-dimensional abelian Lie ideal $\mathfrak{s}$ of supertranslations as before. However as a direct consequence of dropping Eq. (4.2.10) in favor of Eq. (4.2.12b), the factor algebra $\mathfrak{b}_{\text{gen}}/\mathfrak{s}$ is now isomorphic to the Lie algebra $\mathfrak{diff}(\mathbb{S}^2)$ of *all* smooth infinitesimal diffeomorphisms of $\mathbb{S}^2$. Hence we have $\mathfrak{b}_{\text{gen}} = \mathfrak{diff}(\mathbb{S}^2) \ltimes \mathfrak{s}$.

Thus, by weakening the universal structure near $\mathscr{I}$ — equivalently, extending the class of allowed perturbations — one obtains a bigger asymptotic symmetry algebra at null infinity which includes all the smooth diffeomorphisms of a 2-sphere.

4.3 | The symplectic current of general relativity at null infinity

In this section we evaluate the symplectic current of general relativity for the extended class of perturbations detailed in Sec. 4.2. We will show that any choice of symplectic current, which is local and covariant, necessarily diverges in the limit to $\mathscr{I}$.

1 | The symplectic current for general perturbations

We now investigate whether the symplectic current has a limit to $\mathscr{I}$ when ξ^a is a generator of the generalized BMS algebra. To analyze the behavior of the symplectic current at $\mathscr{I}$, it is convenient to express the symplectic current in terms of unphysical quantities which extend smoothly to $\mathscr{I}$ and are non-vanishing there in general. Thus we define

$$\varepsilon_{abcd} = \Omega^4 \hat{\varepsilon}_{abcd}\,, \quad P^{abcdef} = \Omega^{-6} \hat{P}^{abcdef}\,, \quad \gamma_{ab} = \Omega^2 \hat{\gamma}_{ab}\,. \tag{4.3.1}$$

The relation between the physical derivative operator $\hat{\nabla}$ and unphysical derivative operator ∇ acting on any covector v_a is given by

$$\begin{aligned} \hat{\nabla}_a v_b &= \nabla_a v_b + C^c{}_{ab} v_c \\ \text{with} \quad C^c{}_{ab} &= 2\Omega^{-1} \delta^c_{(a} n_{b)} - \Omega^{-1} n^c g_{ab} \end{aligned} \tag{4.3.2}$$

Inserting Eq. (4.3.1) into Eq. (2.5.7), using Eq. (4.3.2), one obtains:

$$\begin{aligned} &\boldsymbol{\omega} \equiv \omega_{abc} = \tfrac{1}{16\pi} \varepsilon_{dabc} w^d\,, \\ &\text{with} \quad w^a = \Omega^{-2} P^{abcdef} \gamma_{2bc} \nabla_d \gamma_{1ef} + \Omega^{-3} \gamma_1^{ab} n_b \gamma_{2c}{}^c - [1 \leftrightarrow 2]\,. \end{aligned} \tag{4.3.3}$$

Note that if both perturbations γ_{1ab} and γ_{2ab} are tangent to the standard field configuration space (4.2.1) and so satisfy the usual conditions (4.2.2), we have

$$w^a = P^{abcdef} \tau_{2bc} \nabla_d \tau_{1ef} + \tau_1{}^{ab} \tau_{2b} + \tau_1{}^a \tau_{2c}{}^c - [1 \leftrightarrow 2] + O(\Omega)\,, \tag{4.3.4}$$

and in this case the symplectic current is finite at $\mathscr{I}$. In particular, when one of the perturbations is generated by an infinitesimal diffeomorphism corresponding to the BMS algebra, one can define the corresponding charges and fluxes following the procedure in [53].

2 | Divergence of the symplectic current on the extended phase space

Now consider the case where one of the perturbations, say γ_{1ab}, satisfies the usual set of conditions 4.2.2, while the other one, γ_{2ab}, lies in Γ_{ext} and so satisfies only the weaker set of conditions (4.2.9). If in this case the symplectic current has a finite limit to $\mathscr{I}$ then one can hope to define charges and fluxes for the generalized BMS algebra taking $\gamma_{2ab} = \gamma^{(\xi)}_{ab}$ where $\xi^a|_{\mathscr{I}}$ is an element of $\mathfrak{b}_{\text{gen}}$. However, as we now show, the symplectic current (4.3.3) in this case necessarily diverges in the limit to $\mathscr{I}$, unless γ_{2ab} also lies in the standard field configuration space Γ_0 [i.e. satisfies the standard conditions (4.2.2)], or the perturbation γ_{1ab} contains no gravitational radiation, i.e., has vanishing perturbed News tensor.

If the symplectic current $\boldsymbol{\omega}$ has a limit to $\mathscr{I}$ then its pullback to $\mathscr{I}$ will be proportional to $\bar{\boldsymbol{\varepsilon}}_3\ n_a w^a$ where $\bar{\boldsymbol{\varepsilon}}_3 \equiv \bar{\varepsilon}_{abc}$ is the 3-volume element on $\mathscr{I}$. It will suffice for our purposes to show that $n_a w^a$ does not have a limit to $\mathscr{I}$. Using Eq. (4.2.2) for γ_{1ab} and Eqs. (4.2.6) for γ_{2ab} we have

$$\begin{aligned} n_a w^a = \Omega^{-1} \left[n_a P^{abcdef} \gamma_{2bc} \nabla_d \tau_{1ef} - n_a P^{abcdef} \tau_{1bc} \nabla_d \gamma_{2ef} \right] \\ + \chi_{2a} \tau_1{}^a + \tfrac{1}{2} \sigma_2 \tau_1{}^a n_a - \tfrac{3}{2} \Omega^{-1} n^a \chi_{2a} \tau_{1b}{}^b - \tfrac{1}{2} \Omega^{-2} n_a n^a \gamma_2{}^{bc} \tau_{1bc} + \tfrac{1}{2} \Omega^{-1} n_a n^a \sigma_2 \tau_{1b}{}^b\,. \end{aligned} \tag{4.3.5}$$

Due to Eqs. (2.2.7) and (4.2.9), the second line in Eq. (4.3.5) has a finite limit to $\mathscr{I}$. Now if $n_a w^a$ has a finite limit to $\mathscr{I}$ then we must have $\lim_{\to \mathscr{I}} \Omega n_a w^a = 0$. From Eq. (4.3.5) we have

$$\Omega n_a w^a = n_a P^{abcdef} \gamma_{2bc} \nabla_d \tau_{1ef} - n_a P^{abcdef} \tau_{1bc} \nabla_d \gamma_{2ef} + O(\Omega), \tag{4.3.6}$$

where $O(\Omega)$ indicates terms which vanish in the limit to $\mathscr{I}$. Evaluating the first term of Eq. (4.3.6) at $\mathscr{I}$, using Eq. (4.2.9), we have

$$\begin{aligned} n_a P^{abcdef} \gamma_{2bc} \nabla_d \tau_{1ef} &\mathrel{\widehat{=}} n^a \gamma_2{}^{bc} \nabla_c \tau_{1ab} - \tfrac{1}{2} n^a \gamma_2{}^{bc} \nabla_a \tau_{1bc} \\ &= -\tfrac{1}{2} \gamma_2{}^{bc} n^a \nabla_a \tau_{1bc}. \end{aligned} \tag{4.3.7}$$

Similarly for the second term in (Eq. (4.3.6)) we have (from Eqs. (4.2.2) and (4.2.9))

$$n_a P^{abcdef} \tau_{1bc} \nabla_d \gamma_{2ef} \mathrel{\widehat{=}} -\tfrac{1}{2} \tau_1{}^{bc} n^a \nabla_a \gamma_{2bc} = 0. \tag{4.3.8}$$

Thus Eq. (4.3.6) can be written as

$$\Omega n_a w^a = -\tfrac{1}{2} \gamma_2{}^{ab} n^c \nabla_c \tau_{1ab} + O(\Omega). \tag{4.3.9}$$

We now simplify the above expression further using the vacuum Einstein equation. Perturbing

Eq. (2.2.1), for the perturbation γ_{1ab} we have (see also Eq. (67) of [53])

$$\delta_1 S_{ab} \mathrel{\hat{=}} 2n_a\tau_{1b} + 2n_b\tau_{1a} - n^c\nabla_c\tau_{1ab} - n^c\tau_{1c}g_{ab}. \tag{4.3.10}$$

Using Eq. (4.3.10) in Eq. (4.3.9) to eliminate the derivative of τ_{1ab}, and Eqs. (4.2.2), (4.2.9) we get

$$\Omega n_a w^a = \tfrac{1}{2}\gamma_2{}^{ab}\delta_1 S_{ab} + O(\Omega). \tag{4.3.11}$$

Further, since $\gamma_2{}^{ab}$ is tangent to $\mathscr{I}$ (which follows from Eq. (4.2.6)) we can replace S_{ab} by its pullback to $\mathscr{I}$ $\underleftarrow{S_{ab}}$ to get

$$\Omega n_a w^a = \tfrac{1}{2}\gamma_2{}^{ab}\delta_1 \underleftarrow{S_{ab}} + O(\Omega) \tag{4.3.12}$$

Recall that for an asymptotically flat spacetime the News tensor on $\mathscr{I}$ is defined by

$$N_{ab} = \underleftarrow{S_{ab}} - \rho_{ab}, \tag{4.3.13}$$

where ρ_{ab} is the unique symmetric tensor field on $\mathscr{I}$ constructed from the (usual) universal structure at $\mathscr{I}$ in Theorem 5 of [62]. Thus, for the perturbation $\gamma_{1\,ab}$ we have $\delta_1\rho_{ab} = 0$ and we can replace $\delta_1\underleftarrow{S_{ab}}$ in Eq. (4.3.12) with $\delta_1 N_{ab}$ to get

$$\Omega n_a w^a = \tfrac{1}{2}\gamma_2{}^{ab}\delta_1 N_{ab} + O(\Omega) \tag{4.3.14}$$

For general perturbations γ_{1ab} the perturbed News $\delta_1 N_{ab}$ does not vanish on $\mathscr{I}$, indicating the presence of (linearized) gravitational radiation, although it is subject to the constraints

$$g^{ab}\delta_1 N_{ab} = 0, \quad n^a\delta_1 N_{ab} = 0. \tag{4.3.15}$$

If the quantity (4.3.14) vanishes for all perturbations $\gamma_{1\,ab}$, then $\gamma_{2\,ab}$ must be of the form $\alpha g_{ab} + n_{(a}v_{b)} + O(\Omega)$, but it then follows from Eqs. (4.2.9) that $\gamma_{2\,ab} = O(\Omega)$.

We therefore conclude that

(1) The symplectic current has a finite limit to $\mathscr{I}$ for *all* perturbations γ_{1ab} that are tangent to the standard phase space Γ_0, if and only if $\gamma_{2ab}|_{\mathscr{I}} = 0$, that is, γ_{2ab} also is tangent to Γ_0. In particular when $\gamma_{2ab} = \gamma^{(\xi)}_{ab}$ is a perturbation generated by an infinitesimal diffeomorphism ξ^a, then $\gamma^{(\xi)}_{ab}|_{\mathscr{I}} = 0$ and thus $\xi^a|_{\mathscr{I}}$ is an element of the usual BMS algebra $\mathfrak{b}$.

(2) The symplectic current has a finite limit to $\mathscr{I}$ for any $\gamma_{2ab} = \gamma^{(\xi)}_{ab}$ generated by an infinitesimal diffeomorphism ξ^a in $\mathfrak{b}_{\text{gen}}$ which is not in $\mathfrak{b}$, if and only if γ_{1ab} has vanishing perturbed News, that is, γ_{1ab} is non-radiating at $\mathscr{I}$.

We emphasize that we have shown that (except in the cases discussed above) the limit to $\mathscr{I}$ of the symplectic current $\boldsymbol{\omega}$ as a 3-form does not exist. That is, the symplectic current diverges as we approach *any* point of $\mathscr{I}$ along *any* curve in the unphysical spacetime independently of any choice of coordinates.

We now compare our result with the procedure used by Campiglia and Laddha [45], who obtained finite charges associated with generators of the generalized BMS algebra. Their procedure can be described as follows. In the physical spacetime pick some Bondi coordinate system (r, u, x^A) near $\mathscr{I}$. Consider the surfaces Σ_t given by $t = u + r =$ constant and integrate the symplectic current $\boldsymbol{\omega}$ on Σ_t with the perturbation $\delta_2 \hat{g}_{ab} = \pounds_\xi \hat{g}_{ab}$ generated by some diffeomorphism in $\mathfrak{b}_{\text{gen}}$ and $\delta_1 \hat{g}_{ab}$ lying in Γ_0. This integral can be rewritten as an integral over a 2-sphere of $u =$ constant on Σ_t. Then as $u \to -\infty$ this integral diverges linearly in u if the vector field ξ^a is an element of $\mathfrak{b}_{\text{gen}}$ which is not in the usual BMS algebra $\mathfrak{b}$. To get a finite symplectic form for all symmetries in $\mathfrak{b}_{\text{gen}}$, [44] then imposes the boundary condition $C_{AB} \sim 1/u^{1+\epsilon}$ along *every* Σ_t where C_{AB} is a subleading piece of the physical metric on the 2-spheres in Bondi coordinates (see Eq. (2.6.5)). The symplectic form on $\mathscr{I}$ is then defined as the $t \to \infty$ limit of this symplectic form on the surfaces Σ_t.

We note that, in contrast to our approach, the procedure used by [45] is not covariant. In particular their boundary condition $C_{AB} \sim 1/u^{1+\epsilon}$ along *every* Σ_t is not invariant under supertranslations (this was noted also by [45]). Thus, if this condition holds in one choice of Bondi coordinate system, it fails to hold in another Bondi coordinate system related to the first by a supertranslation. Similarly, this condition fails to hold if one instead integrates the symplectic current on some different family of surfaces which are supertranslated relative to their choice of Σ_t. We also note that for the "soft charge" on $\mathscr{I}$ defined in [44,45] to be finite one needs to impose $C_{AB} \sim 1/|u|^{1+\epsilon}$ along $\mathscr{I}$ as $u \to \pm\infty$ (the "soft charge" vanishes for elements of the Lorentz algebra $\mathfrak{so}(1,3)$ upon integration over the 2-spheres and this restriction is not required). As is well-known [78], this implies that the memory effect in such spacetimes must vanish which is a severe restriction on the class of spacetimes.

3 Ambiguities in the symplectic current

As shown in the previous section, the symplectic current of a perturbation in the standard phase space Γ_0 with any element of the generalized BMS algebra (which is not in the usual BMS algebra) is not finite at $\mathscr{I}$. However, the symplectic current of general relativity is not uniquely determined by its Lagrangian, and it was claimed in [37] that the symplectic current can be made finite at $\mathscr{I}$ by a suitable choice of such an ambiguity. Since the computations of [37] are tied to a Bondi coordinate system, it is not apparent if their choice of the ambiguity is local and covariant. In this section we show that any ambiguity in the symplectic current, which is local and covariant, cannot be used to make the symplectic current finite at $\mathscr{I}$, in general.

The symplectic potential $\boldsymbol{\theta}$ defined by Eq. (2.5.2) is ambiguous up to the addition of a local and covariant 3-form $\boldsymbol{X}(\hat{g};\delta\hat{g})$ which is linear in $\delta\hat{g}$ and closed, i.e. $d\boldsymbol{X} = 0$. It can be shown quite generally [110] that such a closed form must be exact i.e. $\boldsymbol{X}(\hat{g};\delta\hat{g}) = d\boldsymbol{Y}(\hat{g};\delta\hat{g})$ for some 2-form $\boldsymbol{Y}$ which is local and covariant and linear in $\delta\hat{g}$. Thus, the ambiguity in the symplectic potential is

$$\boldsymbol{\theta}(\hat{g};\delta\hat{g}) \mapsto \boldsymbol{\theta}(\hat{g};\delta\hat{g}) + d\boldsymbol{Y}(\hat{g};\delta\hat{g}) \tag{4.3.16}$$

From Eq. (2.5.4), the corresponding ambiguity in the symplectic current is given by[6]

$$\boldsymbol{\omega}(\hat{g};\delta_1\hat{g},\delta_2\hat{g}) \mapsto \boldsymbol{\omega}(\hat{g};\delta_1\hat{g},\delta_2\hat{g}) + d\left[\boldsymbol{Z}(\hat{g};\delta_1\hat{g},\delta_2\hat{g}) - \boldsymbol{Z}(\hat{g};\delta_2\hat{g},\delta_1\hat{g})\right] \tag{4.3.17}$$

[6] Note that the equations of motion are unaffected by the change $\boldsymbol{L} \mapsto \boldsymbol{L} + d\boldsymbol{K}$ in the Lagrangian. This does not affect the symplectic current since $\delta_1\delta_2\boldsymbol{K} - \delta_2\delta_1\boldsymbol{K} = 0$.

where, for later convenience, we have defined the 2-form

$$\boldsymbol{Z}(\hat{g};\delta_1\hat{g},\delta_2\hat{g}) = \delta_1\boldsymbol{Y}(\hat{g};\delta_2\hat{g})\,, \tag{4.3.18}$$

Note that it follows from Eq. (4.3.18) that any such $\boldsymbol{Z}$ must satisfy the condition

$$\delta_3\boldsymbol{Z}(\hat{g};\delta_1\hat{g},\delta_2\hat{g}) - \delta_1\boldsymbol{Z}(\hat{g};\delta_3\hat{g},\delta_2\hat{g}) = 0 \tag{4.3.19}$$

for *arbitrary* perturbations $\delta_3\hat{g}$ of the metric, even those that do not lie in Γ_0.

We first define the notion of a scaling dimension for tensors following [111]. A tensor $L^{a\cdots}{}_{b\ldots}$ with u upper and l lower indices constructed out of the unphysical metric g_{ab} and the conformal factor Ω is said to have a scaling dimension s, if under a scaling of the conformal factor $\Omega \mapsto \lambda\Omega$ and the metric $g_{ab} \mapsto \lambda^2 g_{ab}$ by a constant λ, we have $L^{a\cdots}{}_{b\ldots} \mapsto \lambda^{s-u+l}L^{a\ldots}{}_{b\ldots}$. Note that the scaling dimension is independent of the tensor index positions and is additive under tensor products. One sees from this that the relevant scaling dimensions are:

$$\Omega : 1\,, \quad g_{ab} : 0\,, \quad \varepsilon_{abcd} : 0\,, \quad \nabla : -1\,, \quad n_a : 0\,. \tag{4.3.20}$$

Since we are interested in the behavior of $\boldsymbol{Z}$ near $\mathscr{I}$, it is useful to write everything in terms of unphysical quantities which are smooth at $\mathscr{I}$. Since $\boldsymbol{Z}(\delta_1\hat{g},\delta_2\hat{g})$ is linear in both physical metric perturbations, in terms of the unphysical perturbations $\gamma_{1ab}, \gamma_{2ab}$ we must have

$$\boldsymbol{Z}(\gamma_1,\gamma_2) = \sum_{p,q} \boldsymbol{W}^{abcde_1\ldots e_p f_1\ldots f_q}(\nabla_{e_1}\cdots\nabla_{e_p}\gamma_{1ab})(\nabla_{f_1}\cdots\nabla_{f_q}\gamma_{2cd}) \tag{4.3.21}$$

where p and q (each ranging from 0 to some finite value) count the number of derivatives of γ_{1ab} and γ_{2ab}, respectively. Here $\boldsymbol{W}^{abcde_1\ldots e_p f_1\ldots f_q}$ are some local and covariant tensor-valued 2-forms which are local functionals of the unphysical metric, the unphysical Riemann tensor and its derivatives and the conformal factor. The scaling dimension of γ_{1ab} and γ_{2ab} is 0, and since the scaling dimension of the symplectic potential $\boldsymbol{\theta}$ is -3 it follows from Eq. (4.3.16) that the scaling dimension of $\boldsymbol{Z}$ is -2. Therefore, the scaling dimension of $\boldsymbol{W}^{abcde_1\ldots e_p f_1\ldots f_q}$ is $-2+p+q$.

Now we analyze the possible forms of $\boldsymbol{W}^{abcde_1\dots e_p f_1\dots f_q}$ that can appear in Eq. (4.3.21). Note that our goal is to find a $\boldsymbol{Z}$ that can get rid of the divergence in the symplectic current in the limit to $\mathscr{I}$. From Eq. (4.3.14) we see that this diverging term depends analytically on the background unphysical metric. Thus, in any candidate expression for $\boldsymbol{Z}$ of the form (4.3.21), we can assume that $\boldsymbol{W}^{abcde_1\dots e_p f_1\dots f_q}$ is an analytic functional of its arguments.[7] Using the Einstein equation (2.2.1), we can eliminate the covariant derivatives of n_a in favor of S_{ab} and its derivatives. Similarly, the unphysical Riemann tensor and its derivatives can be rewritten in terms of S_{ab} and the Weyl tensor C_{abcd} and their derivatives using

$$R_{abcd} = C_{abcd} + g_{a[c}S_{d]b} - g_{b[c}S_{d]a} \tag{4.3.22}$$

Thus a typical term in $\boldsymbol{W}^{abcde_1\dots e_p f_1\dots f_q}$ can be schematically written in the form[8]

$$\Omega^v \prod_{i=1}^{r} (\nabla)^{s_i} S_{ab} \prod_{j=1}^{u} (\nabla)^{t_j} (C_{abcd}) \times (\text{terms with 0 scaling dimension})\,, \tag{4.3.23}$$

where we have suppressed contractions with the metric g_{ab} for simplicity of notation. In the expression above, $r, u \geq 0$ count the number of factors involving S_{ab} and C_{abcd} respectively and $s_i, t_j \geq 0$ count the number of derivatives occurring in each such term. Note that v is allowed to be negative. Comparing the scaling dimensions of Eq. (4.3.23) and $\boldsymbol{W}^{abcde_1\dots e_p f_1\dots f_q}$ gives

$$-2 + p + q = v - \sum_i^r s_i - 2r - \sum_j^u t_j - 2u\,, \tag{4.3.24}$$

where we have used the fact that the scaling dimension of both S_{ab} and C_{abcd} is -2. From the expressions above, we see that $v \geq -2$. Let's consider the "most singular" term where $v = -2$, and thus $p = q = r = u = 0$; this term does not contain any S_{ab} or C_{abcd} and has no derivatives of the

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[7] We emphasize that $\boldsymbol{W}^{abcde_1\dots e_p f_1\dots f_q}$ being analytic in its functional dependence is unrelated to the analyticity of the unphysical metric on the spacetime manifold. We do not impose any analyticity conditions on the spacetimes under consideration.

[8] Recall that by the peeling theorem for an asymptotically-flat spacetime, C_{abcd} vanishes and $\Omega^{-1}C_{abcd}$ has a finite limit at $\mathscr{I}$. Thus, in Eq. (4.3.23) we can use $\Omega^{-1}C_{abcd}$ instead; this only changes the last term in Eq. (4.3.24) to $-3u$ and does not affect the rest of the argument. We use the Weyl tensor C_{abcd} since we allow the background spacetime to satisfy some "extended" notion of asymptotic flatness for which the peeling theorem might not hold.

perturbations $\gamma_{1ab}, \gamma_{2ab}$. Then, Eq. (4.3.21) simplifies to the form

$$\boldsymbol{Z}(\gamma_1, \gamma_2) = \boldsymbol{W}^{abcd}\gamma_{1ab}\gamma_{2cd} + O(\Omega^{-1})\,, \tag{4.3.25}$$

where $\boldsymbol{W}^{abcd} = \Omega^{-2} \times$ (terms with 0 scaling dimension). Recall that when γ_{1ab} is a perturbation in Γ_0 we have $\gamma_{1ab} = \Omega\tau_{1ab}$ where τ_{1ab} is in general smooth and non-vanishing at $\mathscr{I}$. In this case, the "most singular" term we have considered in Eq. (4.3.25) diverges as Ω^{-1} near $\mathscr{I}$. This is precisely the term one would need to cancel the diverging part of the symplectic current in Eq. (4.3.14).

Let us now figure out what the 2-form $\boldsymbol{W}^{abcd}$ can be. Notice that Ω can only appear with a power -2 in the expression for $\boldsymbol{W}^{abcd}$, in particular, any terms with 0 scaling dimension that we need cannot be constructed by multiplying some powers of Ω with something with a negative scaling dimension. Since $\boldsymbol{W}^{abcd}$ must be local and covariant, the only quantities available are $g_{ab}, \varepsilon_{abcd}$ and n_a — note that any derivatives of these will have negative scaling dimension. Using Eqs. (2.2.7) and (4.2.9), one sees that there are just two possible terms which appear at order Ω^{-2}, in terms of which we can write Eq. (4.3.25) as

$$\boldsymbol{Z}(\gamma_1, \gamma_2) \equiv Z_{ab}(\gamma_1, \gamma_2) = \Omega^{-2}(A\varepsilon_{ab}{}^{cd} + B\delta^c_{[a}\delta^d_{b]})g^{ef}\gamma_{1ce}\gamma_{2df} + O(\Omega^{-1}) \tag{4.3.26}$$

where A, B are some constants. Since we have only computed the $\boldsymbol{Z}$ up to terms of $O(\Omega^{-1})$, our consistency condition Eq. (4.3.19), must also hold to this order. However, it is easy to verify that Eq. (4.3.26) fails to satisfy this condition since $\delta_3 g_{ab}|_{\mathscr{I}} \neq 0$ for an arbitrary perturbation in the extended class of perturbations. That is, there does not exist an ambiguity $\boldsymbol{Y}$ in the symplectic potential such that Eq. (4.3.26) is of the form Eq. (4.3.18). Thus we conclude that *any* choice of the symplectic current for general relativity, which is local and covariant, must diverge in the limit to $\mathscr{I}$, in general, when at least one of the perturbations is taken to be in the extended class of allowed perturbations.

4.4 | Other Issues

We now consider some other arguments for and against the extension of the BMS algebra. We focus on two specific issues: the desirability of having a definition of Bondi 4-momentum and the freedom in choosing a field configuration space.

1 Existence of Bondi four-momentum

As discussed in chapter 2, the standard BMS algebra $\mathfrak{b}$ contains a preferred four dimensional subalgebra of translations, associated with the existence of Bondi 4-momentum. By contrast, the generalized BMS algebra $\mathfrak{b}_{\text{gen}}$ does not, as we show explicitly in Appendix A.9. Therefore, there is no universal definition of Bondi 4-momentum in any context where $\mathfrak{b}_{\text{gen}}$ is the asymptotic symmetry algebra. This lack of a definition of Bondi 4-momentum would seem to be a difficulty for any physical interpretation of the generalized BMS algebra.

However, the notion of Bondi 4-momentum would still apply in the context of the symmetry algebra $\mathfrak{b}_{\text{gen}}$, but in a solution-dependent manner. Specifically, given a solution (M, g_{ab}, Ω), one can define the field configuration space (4.2.1) associated with that solution, and from it obtain an associated translation subgroup of $\mathfrak{b}_{\text{gen}}$ and corresponding 4-momentum charge. The 4-momenta associated with two different solutions need not be comparable, as in general they would lie in different spaces. This status of 4-momentum in the generalized algebra would be analogous to the status of angular momentum in the standard BMS context. There, stationary solutions determine preferred Poincaré subalgebras of the BMS algebra, with associated linear and angular momentum charges, but the angular momentum charges associated with two different stationary solutions need not be comparable as they live in different spaces.

2 Choice of field configuration space

In this chapter, we considered an enlargement of the field configuration space Γ_0 to a larger space Γ_{ext} which contains additional unphysical metrics (M, g_{ab}, Ω) that are related to metrics already in Γ_0 by diffeomorphisms and conformal transformations. This raises the question of what criterion can one use to define field configuration spaces in general? How much gauge (here diffeomorphism and conformal) freedom can or should be fixed?

A key consideration is that the phase space of the theory is constructed from the field configuration space Γ_0 or Γ_{ext} by modding out by degeneracies of the presymplectic form [49, 53, 112]. The construction of the symmetry algebra also mods out by these degeneracies (see footnote 5 above). The degeneracy directions correspond to gauge transformations (diffeomorphism or conformal) which vanish sufficiently rapidly near the boundary. Therefore, in defining the initial field configuration space, it should not matter how much gauge freedom is fixed since any residual gauge freedom will be removed in the construction of the final phase space and symmetry algebra. However, one must be careful that one fixes only "true gauge" degrees of freedom, that is, degeneracy directions of the presymplectic form.

The question then is whether the standard configuration space Γ_0 of Eq. (4.2.1) has already fixed some degrees of freedom which are physical and not gauge (i.e. do not correspond to degeneracy directions of the presymplectic form). Unfortunately, it is not straightforward to answer this question, since as we have shown, for the relevant metric perturbations the presymplectic current is either divergent on $\mathscr{I}$, or if one uses the renormalized presymplectic current of Ref. [37], the presymplectic form may not be covariant. If one can indeed find a covariant presymplectic form, one can then check whether Γ_0 fixes any physical degrees of freedom. If so, this would be an argument in favor of extensions of the BMS algebra.

4.5 | Outlook

As mentioned at the start of this chapter, a possible loophole in the conclusions presented in this chapter may be that imposing locality and covariance, which are both central to the covariant phase space framework, at all stages of the computation is too strong a restriction. To derive a consistent (augmented) framework for studying *all* recently proposed extensions of the BMS algebra and calculate the corresponding charges was the goal of a recent paper [54] to which we refer the reader for further discussion on this topic.

Chapter 5

Conservation of asymptotic charges from past to future null infinity: Lorentz charges in general relativity

(Adapted with permission from [7])

Chapter summary

In this chapter, we show that the asymptotic charges associated with Lorentz symmetries on past and future null infinity match in the limit to spatial infinity in a class of spacetimes that are asymptotically-flat in the sense of Ashtekar and Hansen. Combined with the results of [10], this shows that *all* BMS charges on past and future null infinity match in the limit to spatial infinity in this class of spacetimes, proving a relationship that was conjectured by Strominger. Assuming additional suitable conditions are satisfied at timelike infinities, this proves that the flux of all BMS charges is conserved in any classical gravitational scattering process in these spacetimes.

5.1 Context

As remarked in chapter 1, in general relativity in four dimensions, the asymptotic symmetry groups at past and future null infinity in asymptotically-flat spacetimes are the (a priori independent) BMS groups (often denoted by BMS^- and BMS^+ respectively). It is natural to ask how these groups are related in the limit to spatial infinity along past and future null infinity. As remarked in chapter 3, it has in fact been conjectured by Strominger [22] that the generators of these groups match (up to antipodal reflection) in the limit to spatial infinity and, moreover, that the associated charges on cross-sections of past and future null infinity are equal in this limit. If this matching of symmetries and charges can be proven, it would imply the existence of a *global* "diagonal" asymptotic symmetry group for classical general relativity and the existence of an infinite number of conservation laws in classical gravitational scattering. The content of these conservation laws would be that for each generator of BMS^- and its corresponding[1] generator in BMS^+, the difference of the associated charges evaluated on cross-sections of past and future null infinity would equal the difference of the incoming flux at past null infinity and the outgoing flux at future null infinity in the region between the two cross-sections (see Eq. (6.3.24)). Moreover, if appropriate conditions are obeyed at timelike infinities such that the BMS charges all go to zero in the limit to timelike infinities (see Remark 4.6 of [10]), for each such pair of identified generators of BMS^- and BMS^+, the total incoming flux through past null infinity would equal the total outgoing flux through future null infinity. Strominger has further conjectured that the diagonal asymptotic symmetry group obtained from the aforementioned matching of symmetries is the symmetry group of the scattering matrix in quantum gravity.

Recall that group structure of the BMS group is that it is the semi-direct product of an infinite dimensional group of supertranslations with the Lorentz group. The existence of matching conditions on asymptotic symmetries and charges has by now been proven for the supertranslation subgroups of the BMS^+ and BMS^- groups and for the associated supermomentum charges [10, 88].[2] For translations (which, recall, form a subgroup of the group of supertranslations) these results reduce to the older result of [11] which showed that the Bondi 4-momentum on past and future null infinity is equal to the ADM 4-momentum in the limit to spatial infinity. However, for Lorentz symmetries and their associated charges, no such matching conditions have as yet been proven, except for the case of

[1] By "corresponding" here we mean the generator of BMS^+ which this generator of BMS^- matches up to antipodal reflection in the limit to spatial infinity where this limit is dictated by the equations that govern BMS symmetries on $\mathscr{I}$ (Eq. (5.3.1)).

[2] See also [9, 87] for proofs of similar matching conditions for Maxwell theory in asymptotically-flat spacetimes.

angular momentum in stationary spacetimes [113]. The goal of this chapter is to supplement the result of [10] by proving the matching of Lorentz symmetries and their associated charges in general (and in particular non-stationary) asymptotically-flat spacetimes. Altogether, this will complete the proof of matching of the full BMS$^-$ and BMS$^+$ groups and *all* associated charges in the limit to spatial infinity. Consequently, this will also complete the proof of Strominger's conjecture referred to above.

The main tool we will use in our analysis is the covariant formulation of asymptotic-flatness due to Ashtekar and Hansen [16] which was discussed in detail in chapter 3. Since we will be interested in considering limits of quantities defined on null infinity to spatial infinity, we will then conformally-complete this hyperboloid into a cylinder, $\mathscr{C}$, (as discussed in [9, 10]) whose boundaries are diffeomorphic to the space of (rescaled) null directions at spatial infinity. We will then fix the conformal freedom in a neighborhood of i^0 and show how this leads to the antipodal matching of Lorentz symmetries at past and future null infinity in the limit to spatial infinity. We will then show that assuming a certain continuity condition on the Weyl tensor (Eq. (5.5.9)) on the future (past) boundary of $\mathscr{C}$,[3] the Lorentz charges on limiting cross-sections of future (past) null infinity match the Lorentz charges at spatial infinity. As a consequence of Einstein's equations on $\mathscr{H}$, the Lorentz charges (in our conformal frame) are conserved on $\mathscr{H}$ which implies that their values on the past and future boundaries of $\mathscr{C}$ are equal. It then follows that the Lorentz charges at future null infinity match those at past null infinity in the limit to spatial infinity. This result along with the proof of matching of supertranslation symmetries and the associated supermomentum charges in [10] completes the proof of matching of all BMS symmetries and charges.

The rest of this chapter is organized as follows. In Sec. 5.2, we review the basic setup needed to study the matching of asymptotic symmetries and charges, borrowing heavily from the description given in [10]. In Sec. 5.3, we give a brief review of the asymptotic symmetry groups at null and spatial infinity and study various properties of the associated generators that we will need later in our analysis. We then discuss how fixing the conformal freedom near spatial infinity allows us to isolate Lorentz subgroups of the asymptotic symmetry groups at null and spatial infinity and how

[3] This continuity condition is shown to hold in the Kerr-Newman family of spacetimes in Appendix A.13; we discuss its status in more general spacetimes in Sec. 5.6.

the antipodal matching of Lorentz symmetries on past and future null infinity (in the limit to spatial infinity) comes about in Sec. 5.4. In Sec. 5.5, we show that the Lorentz charges on null infinity match those at spatial infinity and that this leads to the matching of the Lorentz charges on past and future null infinity. We conclude in Sec. 5.6 with a discussion of the assumptions used in our analysis and some open questions. We collect some results that are needed for the calculations in the body of the chapter in the appendices.

5.2 | Relating past and future null infinity: the construction

In this section, we review various elements of the construction that we will use to relate past and future null infinity in asymptotically-flat spacetimes and address "the matching problem," that is, the question of how asymptotic symmetries and charges defined on cross-sections of past and future null infinity are related in the limit to spatial infinity. This construction was developed in [10] (see also [9]) and some accompanying results were derived in [5] (discussed in chapter 3). Here, we will simply borrow results from these sources without attempting to derive or prove them. Some results from chapter 2 and chapter 3 will be repeated here for completeness and coherence.

Asymptotic structure at null and spatial infinity: We will work in a class of spacetimes which are asymptotically-flat at null and spatial infinity. This notion of asymptotic flatness is defined using an Ashtekar-Hansen structure [16, 17] and this structure was described in detail in chapter 3.

To define the charge for asymptotic Lorentz symmetries at spatial infinity, which will be part of our analysis in this chapter, we will also need access to a "subleading" piece of the magnetic part of Weyl tensor for which one has to restrict to a class of spacetimes where $\boldsymbol{B}_{ab} = 0$ (where $\boldsymbol{B}_{ab}$) was defined in Eq. (3.4.5). While it is possible to define Lorentz charges in cases where $\boldsymbol{B}_{ab} \neq 0$ (see [92], A.8), that formula is significantly more complicated and we will not analyze that case here. We recall that the condition $\boldsymbol{B}_{ab} = 0$ is satisfied (at least) in any asymptotically flat spacetime which is *either* stationary *or* axisymmetric [97] and the recent discussion in [98] suggests that it may hold more generally. Having set $\boldsymbol{B}_{ab} = 0$, we then require that

$$\boldsymbol{\beta}_{ab} := \lim_{\to i^0} {*C}_{acbd}\eta^c\eta^d\,, \tag{5.2.1}$$

exists as a $C^{>-1}$ tensor field at i^0. This defines for us the aforementioned subleading magnetic field. It follows from Eq. (5.2.1) that $\boldsymbol{\beta}_{ab}$ is tangent to $\mathscr{H}$, symmetric and traceless with the respect to $\boldsymbol{h}_{ab}$. In what follows, we will also need the equations of motion for $\boldsymbol{\beta}_{ab}$. Our main calculations will be performed in a conformal frame where $\boldsymbol{K}_{ab} = 0$ (see Sec. 5.4) and in this frame, these equations are given by [16, 114]

$$\boldsymbol{D}^b\boldsymbol{\beta}_{ab} = 0\,. \tag{5.2.2a}$$

$$\boldsymbol{D}^2\boldsymbol{\beta}_{ab} - 2\boldsymbol{\beta}_{ab} = -\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{E}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{E}\,, \tag{5.2.2b}$$

where $\boldsymbol{E}_{ab}$ and $\boldsymbol{E}$ were defined in Eqs. (3.4.5) and (3.4.11).

Remark 5.2.1 (Conformal transformations of the asymptotic fields). Recall that Def. 3.2.1 implies that the allowed conformal freedom $\Omega \mapsto \omega\Omega$ is such that $\omega > 0$ is a positive function which is smooth on $M - i^0$, $C^{>0}$ at i^0 and satisfies $\omega|_{i^0} = 1$. This implies that along spatial directions that limit to i^0, we can write

$$\omega = 1 + \Omega^{1/2}\alpha\,, \tag{5.2.3}$$

where α is $C^{>-1}$ at i^0. We denote $\boldsymbol{\alpha} = \lim_{\to i^0} \alpha$. Recall that we showed in chapter 3 that $\boldsymbol{E}_{ab}$, $\boldsymbol{B}_{ab}$ and $\boldsymbol{E}$ are conformally invariant while

$$\boldsymbol{K}_{ab} \mapsto \boldsymbol{K}_{ab} - 2(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\alpha} + \boldsymbol{h}_{ab}\boldsymbol{\alpha})\,. \tag{5.2.4}$$

We now introduce some quantities at null infinity that we will need in our analysis. We denote

$$\Phi := \tfrac{1}{4}\nabla_a n^a|_{\mathscr{I}}\,, \quad \Phi|_{i^0} = 2\,, \tag{5.2.5}$$

where the second equality follows from condition (4.c). Under conformal transformations (Remark 5.2.1),

$$\Phi \to \omega^{-1}(\Phi + \pounds_n \ln \omega)\,. \tag{5.2.6}$$

Since S_{ab} is smooth at $\mathscr{I}$ by the conditions in Def. 3.2.1, Eq. (3.4.1) implies

$$\lim_{\to \mathscr{I}} \Omega^{-1} n^a n_a = 2\Phi\,, \quad \nabla_a n_b \mathrel{\hat{=}} \Phi g_{ab}\,, \tag{5.2.7}$$

that is, the vector field n^a is a null geodesic generator of $\mathscr{I}^{\pm} \cong \mathbb{R} \times \mathbb{S}^2$ which is future pointing on $\mathscr{I}^+$ and past pointing on $\mathscr{I}^-$. Further, we denote the pullback of g_{ab} to $\mathscr{I}$ by q_{ab}. This defines a degenerate metric on $\mathscr{I}$ with $q_{ab}n^b = 0$. It is convenient to introduce a foliation of $\mathscr{I}$ by a family of cross-sections diffeomorphic to $\mathbb{S}^2$. The pullback of q_{ab} to any cross-section S defines a Riemannian metric on S.[4] Then, for any choice of foliation, there is a unique *auxiliary normal* vector field l^a at $\mathscr{I}$ such that

$$l^a l_a \mathrel{\hat{=}} 0\,, \quad l^a n_a \mathrel{\hat{=}} -1\,, \quad q_{ab} l^b = 0\,. \tag{5.2.8}$$

We further have

$$q_{ab} \mathrel{\hat{=}} g_{ab} + 2n_{(a} l_{b)}\,, \quad \varepsilon_{abc} \mathrel{\hat{=}} l^d \varepsilon_{dabc}\,, \quad \varepsilon_{ab} \mathrel{\hat{=}} n^c \varepsilon_{cab} \tag{5.2.9}$$

where ε_{abc} defines a volume element on $\mathscr{I}$ and ε_{ab} is the area element on any cross-section of the foliation. Evaluating the pullback of $\pounds_n g_{ab}$ and using Eq. (5.2.7), we have on $\mathscr{I}$

$$\pounds_n q_{ab} \mathrel{\hat{=}} 2\Phi q_{ab}\,, \tag{5.2.10}$$

that is, Φ measures the expansion of the chosen cross-sections of $\mathscr{I}$ along the null generator n^a while their shear and twist vanish identically. We also define

$$\tau_a := q_a{}^c n^b \nabla_b l_c\,, \tag{5.2.11}$$

which satisfies $n^b \nabla_b l_a \mathrel{\hat{=}} \tau_a - \Phi l_a$. We see that τ_a represents the change in the direction of l_a along the null generators of n^a. The shear of the auxiliary normal l^a on the cross-sections S of the foliation

[4] Recall that in chapter 2, we defined a "pushforward," Q_{ab}, of this metric to the unphysical spacetime. For the purposes of this chapter, the distinction between q_{ab} and Q_{ab} will not be important and so we will stick to using q_{ab} for simplicity.

is defined by

$$\sigma_{ab} := \mathrm{STF}\, \nabla_a l_b\,, \tag{5.2.12}$$

where STF denotes the operation of taking the symmetric trace-free projection of a tensor onto a cross-section. The twist $\varepsilon^{ab}\nabla_a l_b$ vanishes since l_a is normal to the cross-sections while the expansion of l^a is given by

$$\vartheta(l^a) := q^{ab}\nabla_a l_b\,. \tag{5.2.13}$$

For any smooth v_a satisfying $n^a v_a \mathrel{\widehat{=}} l^a v_a \mathrel{\widehat{=}} 0$, we define the derivative $\mathscr{D}_a$ on the cross-sections by

$$\mathscr{D}_a v_b := q_a{}^c q_b{}^d \nabla_c v_d\,. \tag{5.2.14}$$

It is easily verified that $\mathscr{D}_a \varepsilon_{bc} \mathrel{\widehat{=}} 0$ and $\mathscr{D}_a q_{bc} \mathrel{\widehat{=}} 0$.

In this chapter, we will work in a class of spacetimes where the peeling theorem holds. It follows then that $C_{abcd} = 0$ at $\mathscr{I}$, and thus $\Omega^{-1}C_{abcd}$ admits a limit to $\mathscr{I}$ (see, e.g, Theorem 11 of [62]). Recall from chapter 2 that in any choice of foliation of $\mathscr{I}$, we can define the fields

$$\mathcal{R}_{ab} := (\Omega^{-1}C_{cdef})q_a{}^c n^d q_b{}^e n^f\,, \qquad \mathcal{S}_a := (\Omega^{-1}C_{cdef})l^c n^d q_a{}^e n^f \tag{5.2.15a}$$

$$\mathcal{P} := (\Omega^{-1}C_{cdef})l^c n^d l^e n^f\,, \qquad \mathcal{P}^* := \tfrac{1}{2}(\Omega^{-1}C_{cdef})l^c n^d \varepsilon^{ef} \tag{5.2.15b}$$

$$\mathcal{J}_a := (\Omega^{-1}C_{cdef})n^c l^d q_a{}^e l^f\,, \qquad \mathcal{I}_{ab} := (\Omega^{-1}C_{cdef})q_a{}^c l^d q_b{}^e l^f \tag{5.2.15c}$$

These tensors are all orthogonal to n^a and l^a in all indices and therefore can be taken to be tensor fields on the cross-sections of the chosen foliation of $\mathscr{I}$. For the fields defined in Eq. (5.2.15), Eq. (2.2.3a) implies the following evolution equations along $\mathscr{I}$

$$(\pounds_n + 2\Phi)\mathcal{S}_a = (\mathscr{D}^b + \tau^b)\mathcal{R}_{ab}\,, \tag{5.2.16a}$$

$$(\pounds_n + 3\Phi)\mathcal{P} = (\mathscr{D}^a + 2\tau^a)\mathcal{S}_a - \sigma^{ab}\mathcal{R}_{ab}\,, \tag{5.2.16b}$$

$$(\pounds_n + 3\Phi)\mathcal{P}^* = -\varepsilon^{ab}(\mathscr{D}_a + 2\tau_a)\mathcal{S}_b + \varepsilon_b{}^c\sigma^{ab}\mathcal{R}_{ac}\,, \tag{5.2.16c}$$

$$(\pounds_n + 2\Phi)\mathcal{J}_a = \tfrac{1}{2}(\mathscr{D}_b + 3\tau_b)(q_a{}^b\mathcal{P} - \varepsilon_a{}^b\mathcal{P}^*) - 2\sigma_a{}^b\mathcal{S}_b\,, \tag{5.2.16d}$$

$$(\pounds_n + \Phi)\mathcal{I}_{ab} = (q_a{}^c q_b{}^d - \tfrac{1}{2}q_{ab}q^{cd})(\mathscr{D}_c + 4\tau_c)\mathcal{J}_d - \tfrac{3}{2}\sigma_{ac}(q_b{}^c\mathcal{P} - \varepsilon_b{}^c\mathcal{P}^*)\,. \tag{5.2.16e}$$

Finally, recall that the News tensor is defined by

$$N_{ab} := 2(\pounds_n - \Phi)\sigma_{ab}\,, \tag{5.2.17}$$

which satisfies $N_{ab}n^b \hat{=} 0$, $N_{ab}q^{ab} \hat{=} 0$ and is conformally-invariant on $\mathscr{I}$. The News tensor is related to S_{ab} by Eq. (A.1.6)

$$N_{ab} \hat{=} \mathrm{STF}\left[S_{ab} - 2\Phi\sigma_{ab} + 2(\mathscr{D}_a\tau_b + \tau_a\tau_b)\right]\,, \tag{5.2.18}$$

and to the Weyl tensor on $\mathscr{I}$ by (from Eq. (2.2.37))

$$\mathcal{R}_{ab} \hat{=} \tfrac{1}{2}\pounds_n N_{ab} \tag{5.2.19a}$$

$$\mathcal{S}_a \hat{=} \tfrac{1}{2}\mathscr{D}^b N_{ab}\,. \tag{5.2.19b}$$

The space $\mathscr{C}$ of null and spatial directions at i^0: As detailed in [10], to study limits of quantities defined at null infinity to spatial infinity, one needs to rescale n^a to obtain a set of "good" (non-vanishing) null directions at i^0 and, in addition, conformally complete $\mathscr{H}$ into a cylinder, denoted by $\mathscr{C}$. The boundaries of $\mathscr{C}$, denoted by $\mathscr{N}^\pm$, correspond to the space of (rescaled) null directions at i^0 (see Fig. 2 of [10] for an illustration) that are antipodally mapped onto each other by the reflection map (Eq. (3.2.7)). To carry out this rescaling of directions, in a neighborhood of i^0 in M (from hereon in, we use M to denote such a neighborhood unless otherwise specified), one defines

$$N^a := \frac{1}{2}\Sigma n^a = \frac{1}{2}\Sigma\nabla^a\Omega\,, \tag{5.2.20}$$

where Σ, called the rescaling function, satisfies the properties listed below.

Definition 5.2.1 (Rescaling function Σ)**.** We take Σ to be a function in M such that

(1) $\Sigma^{-1} > 0$ is smooth on $M - i^0$

(2) Σ^{-1} is $C^{>0}$ at i^0 in both null and spatial directions,

(3) $\Sigma^{-1}|_{i^0} = 0$, $\lim_{\to i^0} \nabla_a\Sigma^{-1} \neq 0$ and

(4) $\Sigma\pounds_n\Sigma^{-1} = 2$ at i^0 and on $\mathscr{I}$

Note that Σ is not uniquely defined and the freedom in picking a Σ is detailed in Remark 2.3 of [10]. Note also that N^a is $C^{>-1}$ at i^0 and $\boldsymbol{N}^a = \lim_{\to i^0} N^a \neq 0$ along both null and spatial directions.

Since $\Sigma^{-1}|_{i^0} = 0$ and Σ^{-1} is $C^{>0}$, there exists a function $\boldsymbol{\Sigma}(\vec{\eta})$, which is $C^{>-1}$ along spatial directions, such that

$$\boldsymbol{\Sigma}^{-1}(\vec{\eta}) = \lim_{\to i^0}(\Omega^{1/2}\Sigma)^{-1}\,. \tag{5.2.21}$$

We also define a rescaled auxiliary normal L^a in M by

$$L^a := -\nabla^a\Sigma^{-1} + \frac{1}{2}N^a\nabla^b\Sigma^{-1}\nabla_b\Sigma^{-1} - \frac{1}{2}\Omega\Sigma\overline{L}^b\nabla^a\nabla_b\Sigma^{-1}\,, \tag{5.2.22}$$

where

$$\overline{L}^a := -\nabla^a\Sigma^{-1} + \frac{1}{2}N^a\nabla_b\Sigma^{-1}\nabla_b\Sigma^{-1}\,. \tag{5.2.23}$$

Our L^a, we note, is different from the expression for L^a (here denoted as $\overline{L}^a$) in [10] where the last term was absent.[5] [6] Note also that L^a is $C^{>-1}$ at i^0 and $\lim_{\to i^0} L^a \neq 0$ in both null and spatial

[5] Note that the expression for $\overline{\boldsymbol{L}}^a$ in Eq. (2.33) of [10] was erroneously written. The corrected version of that expression is $\overline{\boldsymbol{L}}^a = -\boldsymbol{h}^a{}_b\boldsymbol{D}^b\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}^a(\frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{h}^{bc}\boldsymbol{D}_b\boldsymbol{\Sigma}^{-1}\boldsymbol{D}_c\boldsymbol{\Sigma}^{-1} - \frac{1}{2}\boldsymbol{\Sigma}^{-1})$. However, since this correction does not effect $\boldsymbol{h}^a{}_b\overline{\boldsymbol{L}}^b$, it does not effect any of the conclusions in [10].

[6] As discussed in Appendix A.12, the proof of matching of supertranslation charges in [10] goes through unchanged with the choice of L^a used in this chapter.

directions. Further, using Eq. (5.2.20) and condition (4), we have

$$N^a L_a \hat{=} -1\,, \quad L^a L_a \hat{=} 0\,. \tag{5.2.24}$$

The pullback of L_a to $\mathscr{I}$ equals the pullback of $-\nabla_a \Sigma^{-1}$ and therefore L^a defines a rescaled auxiliary normal to a foliation of $\mathscr{I}$ by a family of cross-sections S_Σ with Σ^{-1} = constant. It follows from Def. 5.2.1 and condition (6) that the limiting cross-section S_Σ as $\Sigma^{-1} \to 0$, is diffeomorphic to $\mathscr{N}^\pm$. The auxiliary normal to this foliation, l^a, satisfying Eq. (5.2.8), is obtained by

$$l^a := \tfrac{1}{2}\Sigma L^a\,, \tag{5.2.25}$$

which we also take to define our choice of extension of l^a into M. In the foliation of S_Σ cross-sections, we have (using Eqs. (5.2.20) and (5.2.24))

$$N^a|_{\mathscr{I}} \equiv \partial_{\Sigma^{-1}}\,, \tag{5.2.26a}$$

$$n^a|_{\mathscr{I}} \equiv 2\Sigma^{-1}\partial_{\Sigma^{-1}}\,. \tag{5.2.26b}$$

We turn now to the conformal completion of $\mathscr{H}$. Let $\boldsymbol{\Sigma}$ be the function induced on $\mathscr{H}$ by $\boldsymbol{\Sigma}(\vec{\eta})$ (defined in Eq. (5.2.21)). Let $(\tilde{\mathscr{H}}, \tilde{\boldsymbol{h}}_{ab})$ be a conformal-completion of $(\mathscr{H}, \boldsymbol{h}_{ab})$ with the metric $\tilde{\boldsymbol{h}}_{ab} := \boldsymbol{\Sigma}^2 \boldsymbol{h}_{ab}$. Then there exists a diffeomorphism from $\tilde{\mathscr{H}}$ onto $\mathscr{C}$ (see Eq. B.16 of [9]) such that $\mathscr{H}$ is mapped onto $\mathscr{C} \setminus \mathscr{N}^\pm$ and $\boldsymbol{\Sigma}$, as a function on $\mathscr{C} \setminus \mathscr{N}^\pm$, extends smoothly to the boundaries $\mathscr{N}^\pm$ where

$$\boldsymbol{\Sigma}|_{\mathscr{N}^\pm} = 0\,. \tag{5.2.27}$$

Note that we will implicitly use this diffeomorphism to treat fields defined on $\mathscr{H}$ as fields on $\mathscr{C}$ throughout this chapter. Note also that the rescaled metric

$$\tilde{q}_{ab} := \Sigma^2 q_{ab}\,, \tag{5.2.28}$$

on S_Σ is such that as $\Sigma^{-1} \to 0$, $\lim_{\to i^0} \tilde{q}_{ab}(\vec{N})$ exists along null directions $\vec{N}$ and defines a direction-dependent Riemannian metric $\tilde{\boldsymbol{q}}_{ab}$ on the space of null directions $\mathscr{N}^{\pm}$. Moreover, this metric coincides with the metric induced on $\mathscr{N}^{\pm}$ by $\tilde{\boldsymbol{h}}_{ab}$ on $\mathscr{C}$, that is,

$$\tilde{\boldsymbol{q}}_{ab} = \lim_{\to \mathscr{N}^{\pm}} (\tilde{\boldsymbol{h}}_{ab} + \boldsymbol{D}_a \boldsymbol{\Sigma} \boldsymbol{D}_b \boldsymbol{\Sigma})\,. \tag{5.2.29}$$

Similarly, we have the rescaled area element

$$\tilde{\varepsilon}_{ab} := \Sigma^2 \varepsilon_{ab}\,, \tag{5.2.30}$$

on the foliation S_Σ. This induces an area element $\tilde{\boldsymbol{\varepsilon}}_{ab}$ on $\mathscr{N}^{\pm}$ such that

$$\tilde{\boldsymbol{\varepsilon}}_{ab} = \lim_{\to \mathscr{N}^{\pm}} \boldsymbol{U}^c \tilde{\boldsymbol{\varepsilon}}_{cab} = \lim_{\to \mathscr{N}^{\pm}} \pm \boldsymbol{\Sigma}^2 \boldsymbol{\varepsilon}_{ab}\,, \tag{5.2.31}$$

where $\boldsymbol{U}^a := \boldsymbol{h}^a{}_b \boldsymbol{L}^b$ and $\tilde{\boldsymbol{\varepsilon}}_{abc} := \boldsymbol{\Sigma}^3 \boldsymbol{\varepsilon}_{abc}$ is the volume element on $\mathscr{C}$ defined by the metric $\tilde{\boldsymbol{h}}_{ab} = \boldsymbol{\Sigma}^2 \boldsymbol{h}_{ab}$. Note also that

$$\lim_{\to \mathscr{N}\pm} \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a = \pm \lim_{\to \mathscr{N}\pm} \boldsymbol{\Sigma}^{-2} \boldsymbol{u}^a \neq 0\,. \tag{5.2.32}$$

Null-regular spacetimes at i^0: In [10], it was shown that the spacetimes in which the supertranslations charges on $\mathscr{I}^-$ and $\mathscr{I}^+$ match in the limit to i^0 are spacetimes with an Ashtekar-Hansen structure (Def. 3.2.1) where

(1) the rescaled quantity

$$\Sigma^{-3} \Omega^{-1} C_{abcd} l^a n^b l^c n^d \text{ is } C^{>-1} \text{ in } \textit{both} \text{ null and spatial directions at } i^0 \tag{5.2.33}$$

(2) in the limit to i^0 along each null generator of $\mathscr{I}$

$$N_{ab} = O(\Sigma^{-(1+\epsilon)})\,, \quad \mathcal{R}_{ab} = O(\Sigma^{-(1+\epsilon)}) \text{ as } \Sigma^{-1} \to 0 \text{ along } \mathscr{I} \tag{5.2.34}$$

for the vector field l^a (defined by Eqs. (5.2.22) and (5.2.25); see footnote 6 as well). Such spacetimes are called *null-regular* at i^0. It was also shown that these spacetimes satisfy the property that $\boldsymbol{E}_{ab}$

is even under the reflection isometry given in Eq. (3.2.7). Further, as discussed in chapter 3, one can use logarithmic translations to set $\boldsymbol{E}$ to be reflection-even. This then removes the ambiguity in the Ashtekar-Hansen structure. Throughout this chapter, we will always work in these null-regular spacetimes. Note that we do not require that the News tensor vanish in any open region of $\mathscr{I}$ and therefore do *not* impose that our spacetimes be stationary.

Choice of conformal frame: One can use the conformal freedom $\Omega \to \omega\Omega$ discussed in Remark 5.2.1 and the corresponding change in Φ, given in Eq. (5.2.6), to go to a conformal frame where $\Phi = 2$ not just at i^0 but in a neighborhood of i^0. The appropriate ω can be picked by solving the following ordinary differential equation $\pounds_n \ln \omega = 2\omega - \Phi$ (for some initial Φ) in a neighborhood of i^0. This was done, e.g, in [11, 115]. In what follows, we will also work in a conformal frame where this is true and so all our subsequent calculations will be performed assuming that we work on a portion of $\mathscr{I}$ that is in a neighborhood of i^0 where $\Phi = 2$. Since the asymptotic charges at both null and spatial infinity are conformally invariant (see, e.g, [4, 5]), making this choice to prove matching of asymptotic charges entails no loss of generality. Note also that after having picked this conformal frame, one still has the residual conformal freedom given by $\Omega \to \omega\Omega$ where $\pounds_n \omega \mathrel{\hat{=}} 0$. We will restrict this freedom further in Sec. 5.4.

Choice of rescaling function: Note that since asymptotic charges at both null and spatial infinity are independent of the choice of rescaling function, we can use any choice of rescaling function to study the matching of asymptotic charges. A particularly convenient choice is one where the metric on cross-sections of $\mathscr{I}$ (in a neighborhood of i^0) is $q_{ab} \mathrel{\hat{=}} \Sigma^{-2} s_{ab}$ where s_{ab} is the unit round sphere metric. This choice can always be made when $\Phi = 2$ (see e.g, Appendix. B of [10]) and in the rest of our analysis we will work with this choice.

Choice of foliation: As in [10], in the rest of this chapter, we will work in a context where $\mathscr{I}$ is foliated by $\Sigma^{-1} =$ constant cross-sections, S_Σ . This implies that

$$\mathscr{D}_a \Sigma^{-1} \mathrel{\hat{=}} 0\,, \tag{5.2.35}$$

on any cross-section of the foliation. It was shown in [10] that this choice can be made in any conformal frame and that using Eq. (5.2.35), this implies that $\tau_a \mathrel{\hat{=}} 0$ (see Eq. 2.31 and footnote 5 of [10]). Therefore, $\tau_a \mathrel{\hat{=}} 0$ in the rest of our analysis.

Limits of integrals to $\mathscr{I}$ and $\mathscr{H}$: In this chapter, we will need to consider limits of certain integrated quantities to cross-sections of $\mathscr{I}$ and $\mathscr{H}$ (see Sec. 5.5). In these cases, the limits to cross-sections of $\mathscr{I}$ will be taken along a sequence of null hypersurfaces, that exists in a neigborhood of i^0 in the unphysical spacetime. Each of these null surfaces is foliated by constant Ω spheres $\mathcal{S}'$, that limit to cross-sections S_Σ of $\mathscr{I}$. These null surfaces are taken to be generated by an affine, null vector field, denoted by K^a (defined in Eq. (5.5.14)). This vector field is such that $\lim_{\to i^0} \Omega^{1/2} K^a$ is direction-dependent. Moreover, we require that the null normal K_a be such that $\lim_{\to i^0} K_a$ is direction-dependent. Note that this difference in scaling between the null generator and null normal uses the fact that on a null surface, they can be scaled arbitrarily with respect to each other. The limit to cross-sections of $\mathscr{H}$ is taken along spacelike hypersurfaces that go to spatial infinity. These surfaces are foliated by spheres, $\mathcal{S}'$, that limit to cross-sections of $\mathscr{H}$.

This completes our review of the construction needed for the calculations in this chapter.

5.3 Asymptotic symmetries at spatial infinity

1 Behaviour of the BMS symmetries at i^0

In this section, we derive the behavior of BMS symmetries in the limit to $\mathscr{N}^\pm$ along $\mathscr{I}^\pm$. We start by giving a brief review of BMS symmetries (see, chapter 2 and [4] for a more detailed discussion).

BMS symmetries at null infinity are defined by diffeomorphisms that preserve the universal structure at $\mathscr{I}$ (that is, the structure common to all physical spacetimes that satisfy Def. 3.2.1). This universal structure is given by the equivalence class $[n^a, q_{ab}]$ with $(n^a, q_{ab}) \sim (\omega^{-1} n^a, \omega^2 q_{ab})$, where ω is a positive function which is smooth on $M - i^0$, $C^{>0}$ at i^0 and satisfies $\omega|_{i^0} = 1$. The diffeomorphisms on $\mathscr{I}$ which preserve this universal structure are generated by vector fields ξ^a, in the physical spacetime, which extend smoothly to $\mathscr{I}$, are tangent to $\mathscr{I}$ and satisfy

$$\pounds_\xi n^a \hat{=} -\alpha_{(\xi)} n^a \,, \quad \pounds_\xi q_{ab} \hat{=} 2\alpha_{(\xi)} q_{ab} \,, \tag{5.3.1}$$

for some function $\alpha_{(\xi)}$ which depends on ξ^a, is smooth on $\mathscr{I}$, $C^{>0}$ in spatial directions at i^0 and satisfies $\alpha_{(\xi)}|_{i^0} = 0$ (which follows from the fact that $\omega|_{i^0} = 1$).

Since ξ^a is tangent to $\mathscr{I}$, we can write

$$\xi^a \hat{=} \beta n^a + q^a{}_b X^b \,, \tag{5.3.2}$$

where $\beta := f - l_a X^a$. Eq. (5.3.1) then gives[7]

$$(\pounds_n - 4)\, q^b{}_a X_b \hat{=} 0 \,, \tag{5.3.3a}$$

$$(q_a{}^c q_b{}^d - \tfrac{1}{2} q_{ab} q^{cd}) \mathscr{D}_{(c}\, q^e{}_{d)} X_e \hat{=} 0 \,, \tag{5.3.3b}$$

$$\alpha_{(\xi)} \hat{=} \pounds_n \beta \hat{=} 2\beta + \tfrac{1}{2}\mathscr{D}_a(q^a{}_b X^a) \,, \implies (\pounds_n - 2)\beta \hat{=} \tfrac{1}{2}\mathscr{D}_a(q^a{}_b X^a) \,. \tag{5.3.3c}$$

BMS symmetries on $\mathscr{I}$ are parametrized by (f, X^a) that satisfy these conditions. It can be shown (see, e.g [4]) that symmetries of the form $(f, X^a = 0)$ form an infinite dimensional subalgebra, $\mathfrak{s}$, which is a Lie ideal of the BMS algebra, $\mathfrak{b}$. These symmetries are called BMS supertranslations and they satisfy $(\pounds_n - 2) f \hat{=} 0$ (which follows from Eq. (5.3.3c) with $X^a = 0$). Further, one can see from Eq. (5.3.3b) that $q^a{}_b X^b$ satisfies the conformal Killing equation on cross-sections of $\mathscr{I}$ and since these cross-sections are diffeomorphic to $\mathbb{S}^2$, it follows that $q^a{}_b X^b$ are elements of the Lorentz algebra, $\mathfrak{so}(1,3)$. One can show that the Lie bracket of a BMS supertranslation and a Lorentz symmetry is a BMS supertranslation and therefore, the Lorentz algebra forms a quotient subalgebra of $\mathfrak{b}$. The

[7] Recall that we have specialized here to $\Phi = 2$ and $\tau_a \hat{=} 0$. The corresponding expressions in arbitrary conformal frames and arbitrary foliations of $\mathscr{I}$ may be found in Appendix. A of [4].

structure of $\mathfrak{b}$ is therefore that of a semidirect sum,

$$\mathfrak{b} \cong \mathfrak{so}(1,3) \ltimes \mathfrak{s} \,. \tag{5.3.4}$$

Finally, there is 4-dimensional Lie ideal, $\mathfrak{t}$, of $\mathfrak{s}$ which corresponds to BMS translations. These are BMS supertranslations which satisfy

$$\text{STF}\, \mathscr{D}_a \mathscr{D}_b f \mathrel{\hat{=}} 0 \,. \tag{5.3.5}$$

Next, we study the behavior of ξ^a in the limit to $\mathscr{N}^\pm$ along $\mathscr{I}$ by solving Eqs. (5.3.3a)–(5.3.3c) in a coordinate system adapted to $\mathscr{I}$ that is well defined in a neighborhood of i^0 (constructed in Appendix. B of [9]). In these coordinates, $q_{ab} \mathrel{\hat{=}} \Sigma^{-2} s_{ab}$ where s_{ab} is the unit round sphere metric, given in stereographic coordinates by (Eq. (A.10.2a))

$$s_{AB} \equiv 2P^{-2} dz d\bar{z} \,, \tag{5.3.6}$$

where $P := \frac{1+z\bar{z}}{\sqrt{2}}$. Written in these coordinates,

$$q^a{}_b X^b \mathrel{\hat{=}} P X \partial_{\bar{z}} + P \overline{X} \partial_z \,. \tag{5.3.7}$$

Here $X \mathrel{\mathring{=}} s = -1$ which is determined by the fact that $q^a{}_b X^b$ is invariant under spin transformations (see Eq. (A.10.5) for the definition of spin transformations). Using the definition of $\bar{\eth}$ in Eq. (A.10.6), Eq. (5.3.3b), written in stereographic coordinates, is the same as

$$\bar{\eth} X \mathrel{\hat{=}} 0 \,, \tag{5.3.8}$$

which, in terms of spherical harmonics on the unit-sphere, implies that X is $\ell = 1$. Note also that,

$$\mathscr{D}_a (q^a{}_b X^b) \mathrel{\hat{=}} (\eth\, X + \bar{\eth}\, \overline{X}) \,. \tag{5.3.9}$$

With $n^a \mathrel{\hat{=}} 2\Sigma^{-1}\partial_{\Sigma^{-1}}$ (Eq. (5.2.26b)), Eqs. (5.3.3a) and (5.3.3c) lead to

$$2\Sigma^{-1}\partial_{\Sigma^{-1}}(\Sigma^{-2}X) - 4(\Sigma^{-2}X) \mathrel{\hat{=}} 0\,, \quad 2\Sigma^{-1}\partial_{\Sigma^{-1}}\beta - 2\beta \mathrel{\hat{=}} \tfrac{1}{2}\mathscr{D}_a(q^a{}_bX^b)\,. \tag{5.3.10}$$

The first equation above implies that X is constant in Σ^{-1} and therefore has a well defined limit as $\Sigma^{-1} \to 0$. It then follows from Eq. (5.3.9) that $\mathscr{D}_a(q^a{}_bX^b)$ is also constant in Σ^{-1}. Further, the solution to the second equation above gives

$$\beta \mathrel{\hat{=}} \Sigma^{-1}\beta_0 - \tfrac{1}{4}\mathscr{D}_a(q^a{}_bX^b)\,, \tag{5.3.11}$$

where β_0 is a constant in Σ^{-1}. Using $\alpha_{(\xi)} \mathrel{\hat{=}} \pounds_n\beta$, we then obtain

$$\alpha_{(\xi)} \mathrel{\hat{=}} 2\Sigma^{-1}\beta_0\,. \tag{5.3.12}$$

Recall that a BMS supertranslation is given by $\xi^a|_{X^a=0}$ evaluated on $\mathscr{I}$ and we see that in this case, $\Sigma\,\beta(X^a = 0) = \Sigma f = \beta_0$ has a non-vanishing limit to $\mathscr{N}^\pm$. As already shown above, $q^a{}_bX^b$ has a limit as $\Sigma^{-1} \to 0$ and we see therefore that a BMS symmetry on $\mathscr{N}^\pm$ is given by $(\Sigma f, q^a{}_bX^b)$ where $q^a{}_bX^b$ is a conformal Killing vector field tangent to $\mathscr{N}^\pm$.

2 | Spi symmetries on the space of null directions $\mathscr{N}^\pm$

In this section, we will study some properties of the asymptotic symmetries at spatial infinity, called *spi* symmetries, that we will need in our analysis.

Spi symmetries correspond to diffeomorphisms that preserve the universal structure at spatial infinity. The consequences of this were discussed chapter 3 where it was shown that this implies that the generators of these diffeomorphisms, ξ^a, are such that $\lim_{\to i^0}\Omega^{-1/2}\xi^a$ is direction-dependent and ξ^a satisfies $\pounds_\xi g_{ab} = 4\Omega^{-1/2}\xi^c\eta_c g_{ab}$ on $\mathscr{H}$. Spi symmetries are parametrized on $\mathscr{H}$ by $(\boldsymbol{f}, \boldsymbol{X}^a)$ where

$$\boldsymbol{X}^a := \lim_{\to i^0}\Omega^{-1/2}\xi^a\,, \quad \boldsymbol{f} := \lim_{\to i^0}\Omega^{-1}\xi^a\eta_a\,, \tag{5.3.13}$$

where $\boldsymbol{f}$ is a smooth function on $\mathscr{H}$ which parameterizes spi supertranslations. The action of a spi supertranslation on the asymptotic fields on $\mathscr{H}$ is that of a linearized conformal transformation with $\boldsymbol{\alpha} = -2\boldsymbol{f}$ (where $\boldsymbol{\alpha}$ was defined in Remark 5.2.1). Further, $\boldsymbol{X}^a$ satisfies $\boldsymbol{\eta}_a \boldsymbol{X}^a = \boldsymbol{D}_{(a}\boldsymbol{X}_{b)} = 0$ on $\mathscr{H}$ which implies that $\boldsymbol{X}^a$ is an element of the Lorentz algebra, $\mathfrak{so}(1,3)$. These symmetries comprise the $\mathfrak{spi}$ algebra which is very similar in its structure to the BMS algebra in that, $\mathfrak{spi} \cong \mathfrak{so}(1,3) \ltimes \mathfrak{s}$; that is, it is given by the semi-direct sum of spi-supertranslations, $\mathfrak{s}$, (which form an infinite dimensional Lie ideal of $\mathfrak{spi}$) with the Lorentz algebra which like in BMS algebra forms a quotient subalgebra of $\mathfrak{spi}$. Finally, the $\mathfrak{spi}$ algebra also has a 4-dimensional subalgebra $\mathfrak{t}$ of spi translations which forms a Lie ideal of $\mathfrak{t}$. These are given by the spi-supertranslations f which satisfy the additional condition

$$\boldsymbol{D}_a \boldsymbol{D}_b \boldsymbol{f} + \boldsymbol{h}_{ab} \boldsymbol{f} = 0\,. \tag{5.3.14}$$

The limiting behavior of spi supertranslations to $\mathscr{N}^{\pm}$ was studied in [10] where it was shown that for spi-supertranslations that match onto BMS-supertranslations at null infinity in the limit to spatial infinity, $\boldsymbol{F} := \boldsymbol{\Sigma f}$ has a limit to $\mathscr{N}^{\pm}$. Since only these spi supertranslations are relevant for the matching problem we are studying here, we restrict our attention only to them. In this chapter, we will also need some properties of $\boldsymbol{X}^a$, including its limiting behavior to $\mathscr{N}^{\pm}$. We turn to deriving that next.

Lorentz symmetries on $\mathscr{H}$

To study the limiting behaviour of $\boldsymbol{X}^a$, we explicitly solve $\boldsymbol{D}_{(a}\boldsymbol{X}_{b)} = 0$ and analyze the behavior of the solutions to this equation in the limit to $\mathscr{N}^{\pm}$. We use coordinates $(\alpha, z, \bar{z})$ on $\mathscr{H}$ (see Appendix. B of [10] for details), in which the metric on $\mathscr{H}$, $\boldsymbol{h}_{ab}$, has the following form

$$\boldsymbol{h}_{ab} \equiv -\frac{1}{(1-\alpha^2)^2} d\alpha^2 + \frac{1}{1-\alpha^2} s_{AB} d\theta^A d\theta^B\,. \tag{5.3.15}$$

Here $\theta^A = (z, \bar{z})$ and $s_{AB} = 2P^{-2} dz d\bar{z}$. Note also that here α is related to τ used in Eq. (3.2.6) by $\alpha := \tanh\tau$, $-1 \leq \alpha \leq 1$ and $\alpha \to \pm 1$ corresponds to the limits to $\mathscr{N}^{\pm}$.

The vector field $\boldsymbol{X}^a$ can be written as

$$\boldsymbol{X}^a = Z\partial_\alpha + (PX)\partial_{\overline{z}} + (P\overline{X})\partial_z \,, \tag{5.3.16}$$

where

$$Z \mathrel{\hat{=}} s = 0 \,, \quad X \mathrel{\hat{=}} s = -1 \,. \tag{5.3.17}$$

As before, the spin weights are determined by the fact that $\boldsymbol{X}^a$ is invariant under spin transformations. Note also that Z is a real function while X is complex on $\mathscr{H}$. In these coordinates, the components of the equation $\boldsymbol{D}_{(a}\boldsymbol{X}_{b)} = 0$ are given by

$$0 = (1-\alpha^2)\partial_\alpha Z + 2\alpha Z \,, \tag{5.3.18a}$$

$$0 = (1-\alpha^2)\partial_\alpha X - \overline{\eth} Z \,, \tag{5.3.18b}$$

$$0 = \overline{\eth} X \,, \tag{5.3.18c}$$

$$0 = (1-\alpha^2)(\eth\, X + \overline{\eth}\, \overline{X}) + 2\alpha Z \,, \tag{5.3.18d}$$

where the operators $\eth$ and $\overline{\eth}$ are defined in Eq. (A.10.6). Note that Eq. (5.3.18c) is the same equation that we encountered for X in Sec. 5.3.1 which, in terms of spherical harmonics on the unit-sphere means that X is $\ell = 1$ (or, stated in a conformally invariant way, that $\boldsymbol{X}^A$ is a conformal Killing vector field). Using this, Eq. (5.3.18b) implies that Z is also $\ell = 1$. As a result, the solution to Eq. (5.3.18a) is given by

$$Z = (1-\alpha^2) \sum_{m=-1}^{m=1} K_m Y^{s=0}_{\ell=1,m} \,, \tag{5.3.19}$$

where K_m labels three complex constants. Using Eq. (A.10.10a), the fact that Z is real relates these constants through

$$K_0 \in \mathbb{R} \,, \quad K_{-1} = -\overline{K_1} \,, \tag{5.3.20}$$

leaving us with three independent real constants. Similarly, using

$$X = \sum_{m=-1}^{m=1} X_m(\alpha) Y^{s=-1}_{\ell=1,m} \,, \tag{5.3.21}$$

we see that Eq. (5.3.18b) becomes

$$0 = \partial_\alpha X_m(\alpha) - K_m\,, \tag{5.3.22}$$

and so we have

$$X_m(\alpha) = R_m + \alpha K_m\,. \tag{5.3.23}$$

Moreover, Eq. (5.3.18d) implies that

$$R_m = (-)^{m+1}\overline{R_{-m}}\,. \tag{5.3.24}$$

As a result, R_m also labels three real constants (with $m = -1, 0, 1$). We then see that K_m and R_m each represent three real numbers. They parametrize boosts and rotations respectively which can be seen by noting the fact that the divergence of $\boldsymbol{X}^A$ on a cross-section of $\mathscr{H}$, which is given by $\eth X + \bar{\eth}\,\overline{X}$, is zero when $K_m = 0\,\forall m$ and only non-zero when $\exists\, m : K_m \neq 0$.

The expression for the Lorentz charge on $\mathscr{H}$ depends on ${}^\star\boldsymbol{X}^a$ (see Eq. (5.5.5)), which is defined by ${}^\star\boldsymbol{X}^a := \frac{1}{2}\boldsymbol{\varepsilon}^{abc}\boldsymbol{D}_b\boldsymbol{X}_c$. The properties of ${}^\star\boldsymbol{X}^a$ are discussed in Appendix A.5. In particular, it is shown that it satisfies

$$\boldsymbol{D}_{(a}{}^\star\boldsymbol{X}_{b)} = 0\,, \tag{5.3.25}$$

and therefore, ${}^\star\boldsymbol{X}^a$ also represents a Lorentz symmetry on $\mathscr{H}$.

To study the behavior of ${}^\star\boldsymbol{X}^a$ in the limit to $\mathscr{N}^\pm$, we start with

$${}^\star\boldsymbol{X}^a = ({}^\star Z)\partial_\alpha + P({}^\star X)\partial_{\bar{z}} + P(\overline{{}^\star X})\partial_z\,. \tag{5.3.26}$$

Evaluating this explicitly and rewriting the resulting expression using $\eth$ and $\bar{\eth}$, we obtain[8]

$$\begin{aligned} {}^\star Z &= -\frac{i}{2}(1-\alpha^2)(\eth X - \bar{\eth}\,\overline{X})\,, \\ {}^\star X &= \frac{i}{2}\left[\bar{\eth} Z + (1-\alpha^2)\partial_\alpha X + 2\alpha X\right] \\ &= i\left(\bar{\eth} Z + \alpha X\right)\,, \end{aligned} \tag{5.3.27}$$

[8] In our conventions, the volume form on $\mathscr{H}$ in $(\alpha, z, \bar{z})$ coordinates is given by $\boldsymbol{\varepsilon}_{abc} \equiv \frac{2i}{(1+z\bar{z})^2(1-\alpha^2)^2}\,d\alpha\wedge dz\wedge d\bar{z}$.

where the last equality uses Eq. (5.3.18b). Then, using the expressions for Z and X obtained above as well as Eqs. (5.3.20) and (5.3.24), we obtain

$$ {}^\star Z = (1-\alpha^2)\sum_{m=-1}^{m=1} iR_m Y^{s=0}_{\ell=1,m}\,, \quad {}^\star X = \sum_{m=-1}^{m=1} i(K_m + \alpha R_m) Y^{s=-1}_{\ell=1,m}\,. \tag{5.3.28} $$

We therefore see that this "dual" transformation, $\boldsymbol{X}^a \to {}^\star\boldsymbol{X}^a$, effectively interchanges K_m and R_m i.e. boosts and rotations. Note also that in the limit $\alpha \to \pm 1$, ${}^\star Z \to 0$ and one can check by explicitly evaluating the Hodge dual in $(z, \overline{z})$ coordinates that

$$ {}^\star\boldsymbol{X}^a\big|_{\mathscr{N}^\pm} = -{}^*\boldsymbol{X}^a\big|_{\mathscr{N}^\pm}\,, \tag{5.3.29} $$

where ${}^*\boldsymbol{X}^a := \tilde{\boldsymbol{\varepsilon}}_b{}^a \boldsymbol{X}^b$.

We now study the transformation of $\boldsymbol{X}^a$ under the reflection map $(\alpha, \theta^A) \to (-\alpha, -\theta^A)$. Note that K_m and R_m are reflection-even since they are constants. From the transformation under the parity operation of the spin weighted spherical harmonics (Eq. (A.10.10b)), we can conclude that

$$ Z(-\alpha, -\theta^A) = -Z(\alpha, \theta^A)\,, \tag{5.3.30} $$

as well as

$$ \begin{aligned} \overline{X}(-\alpha,-\theta^A) &= \sum_{m=-1}^{m=1}\left[\overline{R_m} - \alpha\overline{K_m}\right]\overline{Y^{s=-1}_{\ell=1,m}}(-\theta^A) \\ &= \sum_{m=-1}^{m=1}\left[(-)^{m+1}R_{-m} - \alpha(-)^m K_{-m}\right](-)^{m-1} Y^{s=1}_{\ell=1,-m}(-\theta^A) \\ &= e^{2i\phi}\sum_{m=-1}^{m=1}\left[R_m + \alpha K_m\right] Y^{s=-1}_{\ell=1,m}(\theta^A) = e^{2i\phi} X(\alpha,\theta^A)\,, \end{aligned} \tag{5.3.31} $$

where $e^{2i\phi} = z/\overline{z}$. Similarly, $X(-\alpha, -\theta^A) = e^{-2i\phi}\overline{X}(\alpha, \theta^A)$. Using the fact that under antipodal map ($\theta^a \to -\theta^a : z \to -1/\overline{z}$)

$$
\begin{aligned}
&P\partial_z \to e^{-2i\phi} P\partial_{\overline{z}}\,, \\
&P\partial_{\overline{z}} \to e^{2i\phi} P\partial_z\,,
\end{aligned}
\tag{5.3.32}
$$

we find that $\boldsymbol{X}^a \to \boldsymbol{X}^a$ under the reflection map and $\boldsymbol{X}^a$ is hence even under this map. It is straightforward to show that in the same way, ${}^\star\boldsymbol{X}^a$ is also even under the reflection map.

Recall from Eq. (5.3.19) that $Z|_{\mathscr{N}^\pm} = 0$ and therefore $\boldsymbol{X}^a$ becomes tangent to $\mathscr{N}^\pm$ in the limit. Thus, on $\mathscr{N}^\pm$ a spi symmetry is given by $(\boldsymbol{F}^\pm, \boldsymbol{X}^a)$ where $\boldsymbol{F} = \boldsymbol{\Sigma f}$ and $\boldsymbol{X}^a$ is a conformal Killing vector field tangent to $\mathscr{N}^\pm$.

5.4 | Fixing the supertranslation freedom at i^0

Our goal now is to show the matching of Lorentz charges at past and future null infinity in the limit to spatial infinity. We will show that this matching follows from requiring the continuity, at $\mathscr{N}^\pm$, of a quantity constructed from the Weyl tensor and a vector field in spacetime that limits to a BMS symmetry on $\mathscr{I}$ with its BMS supertranslation part being zero and a spi symmetry on $\mathscr{H}$ with its spi supertranslation part being zero. This will define for us our notion of a "pure" Lorentz symmetry. Recall from the discussion of the asymptotic symmetry algebras in Sec. 5.3.1 and Sec. 5.3.2 that the Lorentz algebra forms quotient subalgebras of $\mathfrak{bms}$ and $\mathfrak{spi}$ and therefore Lorentz symmetries are only defined as equivalence classes of symmetries that are related by supertranslations. Therefore, the notion of a "pure" Lorentz symmetry only makes sense when the supertranslation freedom is fixed. We will show below that restricting the conformal freedom in spatial directions near spatial infinity restricts the allowed spi supertranslations. This will then be used to restrict the allowed BMS supertranslations by requiring the continuity of a quantity constructed from S_{ab} in the limit to spatial infinity along both null and spatial directions.

The Ashtekar-Hansen gauge: Recall from Sec. 5.2 that we picked a conformal frame in a neighborhood of i^0 where $\Phi = 2$. This choice fixes the dependence of ω on the null generators of $\mathscr{I}$ near i^0. However, it does not restrict $\lim_{\to i^0} \alpha = \boldsymbol{\alpha}$ (defined below Eq. (5.2.3)). One can use this freedom to do a conformal transformation such that $\boldsymbol{K}_{ab} \to \boldsymbol{K}_{ab} - 2(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\alpha} + \boldsymbol{h}_{ab}\boldsymbol{\alpha}) = 0$, as discussed in detail in Remark 3.6.3. This is what we refer to as the "Ashtekar-Hansen gauge" since this choice was first made in [16]. Note that this does not exhaust the freedom in the choice of $\boldsymbol{\alpha}$. In particular, it leaves "un-fixed" the spi supertranslations that satisfy $\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f} = 0$ which precisely correspond to spi translations (see Eq. (5.3.14)). We will return to these at the end of this section.

We now show that the condition $\boldsymbol{K}_{ab} = 0$ can be used to restrict the supertranslation freedom at null infinity. Consider the quantity $\Sigma^{-1}S_{ab}m^am^b$ in a neighborhood of i^0 in the (unphysical) spacetime. We take its limits to $\mathscr{N}^\pm$ along both null and spatial directions and require that this quantity be continuous on $\mathscr{N}^\pm$. We take these limits along the null and spacelike hypersurfaces described in Sec. 5.2. Here, m^a along with its complex conjugate $\overline{m}^a$, forms an orthonormal basis on cross-sections of the surfaces described in Sec. 5.2 such that $m_am^a = \overline{m}_a\overline{m}^a = 0$, $m_a\overline{m}^a = 1$ and q'_{ab}, the metric on cross-sections, S', of the surfaces described in Sec. 5.2, is such that $q'_{ab} = 2m_{(a}\overline{m}_{b)}$. Note also that this metric limits to the intrinsic metric on cross-sections of $\mathscr{I}$ and $\mathscr{H}$. On $\mathscr{I}$, we have

$$\Sigma^{-1}S_{ab}m^am^b = \Sigma S_{ab}\tilde{m}^a\tilde{m}^b\,, \tag{5.4.1}$$

where $\tilde{m}^a$ satisfies $\tilde{m}^a = \Sigma^{-1}m^a$ and is such that $\tilde{q}_{ab} = 2\tilde{m}_{(a}\overline{\tilde{m}}_{b)}$. Recall that $\tilde{q}_{ab}$ is the rescaled metric that limits, along $\mathscr{I}$ (Eq. (5.2.28)), to a direction-dependent metric $\tilde{\boldsymbol{q}}_{ab}$ on $\mathscr{N}^\pm$. Next, note that in the limit to i^0 along spatial directions, we have

$$\lim_{\to i^0} \Sigma^{-1}S_{ab}m^am^b = \boldsymbol{\Sigma}^{-1}\boldsymbol{S}_{ab}\boldsymbol{m}^a\boldsymbol{m}^b = \boldsymbol{\Sigma}\boldsymbol{K}_{ab}\tilde{\boldsymbol{m}}^a\tilde{\boldsymbol{m}}^b = \boldsymbol{\Sigma}\,\mathrm{STF}\,\boldsymbol{K}_{ab}\,. \tag{5.4.2}$$

Here, the direction-dependent limit of m^a to i^0 has been denoted by $\boldsymbol{m}^a$. Further, $\tilde{\boldsymbol{m}}^a$ is such that $\tilde{\boldsymbol{m}}^a = \boldsymbol{\Sigma}^{-1}\boldsymbol{m}^a$ and $\tilde{\boldsymbol{q}}_{ab} = 2\tilde{\boldsymbol{m}}_{(a}\overline{\tilde{\boldsymbol{m}}}_{b)}$. Note also that the third equality in Eq. (5.4.2) follows from STF $\boldsymbol{K}_{ab} =$ STF $\boldsymbol{S}_{ab}$ (which follows from Eq. (3.4.9)). We then see that when $\boldsymbol{K}_{ab} = 0$, assuming continuity of $\Sigma^{-1}S_{ab}m^a m^b$ at $\mathscr{N}^\pm$ implies that in the limit to $\mathscr{N}^\pm$ along $\mathscr{I}$, we have

$$\lim_{\to\mathscr{N}^\pm} \Sigma S_{ab}\tilde{m}^a\tilde{m}^b = 0 \implies \lim_{\to\mathscr{N}^\pm} \mathrm{STF}(\Sigma S_{ab}) = 0\,, \tag{5.4.3}$$

which, using

$$N_{ab} \mathrel{\hat{=}} \mathrm{STF}\, S_{ab} - 2\Phi\sigma_{ab}\,, \tag{5.4.4}$$

(which follows from Eq. (5.2.18) with $\tau_a \mathrel{\hat{=}} 0$) and the fact that $N_{ab} = O(\Sigma^{-(1+\epsilon)})$ as $\Sigma^{-1} \to 0$ along $\mathscr{I}$ (see Sec. 5.2), implies

$$\lim_{\to\mathscr{N}^\pm} \Sigma\,\sigma_{ab} = 0\,. \tag{5.4.5}$$

Requiring this fall-off on σ_{ab} reduces the supertranslation freedom at null infinity to translations since general supertranslations (that are not translations) do not preserve this fall-off.

To recap, the fall-offs along $\mathscr{I}$ that we will assume to hold are (for some small $\epsilon > 0$)

$$\begin{aligned} &N_{ab} = O\left(\Sigma^{-(1+\epsilon)}\right), \quad \mathcal{R}_{ab} = O\left(\Sigma^{-(1+\epsilon)}\right) \\ &\sigma_{ab} = O(\Sigma^{-(1+\epsilon)})\,, \text{ as } \Sigma^{-1} \to 0 \text{ along } \mathscr{I}\,. \end{aligned} \tag{5.4.6}$$

where the first two were part of the definition of null-regular spacetimes (Sec. 5.2) and the fall-off of σ_{ab} follows from Eq. (5.4.5). These fall-offs imply certain conditions on the behavior of $\mathcal{P}^*$ in the limit $\Sigma^{-1} \to 0$ along $\mathscr{I}$ which we turn to deriving next.

Fall-off of $\mathcal{P}^*$: In our foliation of $\mathscr{I}$, recall that $\mathcal{P}^*$ (defined in Eq. (5.2.15)) can be related to the shear and News tensors through

$$\mathcal{P}^* \mathrel{\hat{=}} \varepsilon^{ab}\left[\mathscr{D}_a\mathscr{D}_c\sigma_b{}^c - \tfrac{1}{2}N_{ac}\sigma_b{}^c\right]\,. \tag{5.4.7}$$

Let us consider the behavior of this quantity as $\Sigma^{-1} \to 0$. Using Eq. (5.4.6), we see that

$$\varepsilon^{ab}\mathscr{D}_a\mathscr{D}_c\sigma_b{}^c = O(\Sigma^{-(-3+\epsilon)})\,, \tag{5.4.8a}$$

$$\varepsilon^{ab} N_{ac} \sigma_b{}^c = O(\Sigma^{-(-2+2\epsilon)}) . \tag{5.4.8b}$$

Note also that since σ_{ab} is a symmetric and trace-free tensor, in terms of spherical harmonics on the unit-sphere, it is supported only on $\ell \geq 2$ tensor harmonics. Since $X^b q^a{}_b$ comprises purely $\ell = 1$ vector spherical harmonics (as shown in Sec. 5.3.1), as a consequence of the orthogonality of spherical harmonics, the first term in $\mathcal{P}^*$ drops out when integrated against $X^b q^a{}_b$ on a unit-sphere. Using this, we obtain that $\int_{S_\Sigma} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b \, \mathscr{D}_a \mathcal{P}^* = -\frac{1}{2} \int_{S_\Sigma} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b \, \varepsilon^{de} \mathscr{D}_a (N_{dc} \sigma_e{}^c)$, where $\tilde{\varepsilon}_2$ is the unit area element and where we have implicitly used the fact that S_Σ are $\Sigma =$ constant cross-sections and therefore factors of Σ^{-1} can be pulled outside the integral over S_Σ. This goes to 0 as $\Sigma^{-1} \to 0$ because of Eq. (5.4.8b) and the fact that $X^b q^a{}_b$ has a finite limit as $\Sigma^{-1} \to 0$, as shown in Sec. 5.3.1. Hence, we have

$$\lim_{\Sigma^{-1} \to 0} \int_{S_\Sigma} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b \, \mathscr{D}_a \mathcal{P}^* = 0 . \tag{5.4.9}$$

The same argument also shows that

$$\lim_{\Sigma^{-1} \to 0} \int_{S_\Sigma} \tilde{\varepsilon}_2 \Sigma^{-2} X^b \varepsilon^a{}_b \, \mathscr{D}_a \mathcal{P}^* = 0 . \tag{5.4.10}$$

Although we have now fixed the supertranslation freedom at both null and spatial infinity, to unambiguously define a notion of "pure" Lorentz symmetry, we still have to contend with the translation freedom. To fix that, we proceed as follows. We consider a vector field, ξ^a, that limits to a BMS symmetry at $\mathscr{I}$ and a spi symmetry on $\mathscr{H}$. We require that $\lim_{\to i^0} \Sigma \alpha_{(\xi)} = \lim_{\to i^0} \Omega^{-1} \Sigma \, \xi^a \, \nabla_a \Omega$ vanishes along both null and spatial directions. In the latter limit, this implies $\lim_{\to i^0} \Omega^{-1} \boldsymbol{\Sigma} \, \xi^a \eta_a = 0$ which means $\boldsymbol{\Sigma f} = 0 \Rightarrow \boldsymbol{f} = 0$ on $\mathscr{H}$. Therefore, this condition sets the translations that were left unfixed in going to the Ashtekar-Hansen gauge to zero and $\lim_{\to i^0} \Omega^{-1/2} \xi^a$, subject to these conditions, gives us our notion of a "pure" Lorentz symmetry on $\mathscr{H}$. To see what $\lim_{\to i^0} \Sigma \alpha_{(\xi)} = 0$ taken along $\mathscr{I}$ implies, recall from Eq. (5.3.12) that $\Sigma \alpha_{(\xi)} = \Sigma \pounds_n \beta = 2\beta_0$. Since β_0 is independent of Σ^{-1}, requiring $\lim_{\to i^0} \Sigma \alpha_{(\xi)} = 0$ along $\mathscr{I}$ implies that it vanishes everywhere (in the neighborhood of i^0 on $\mathscr{I}$ where we have set $\Phi = 2$). This gives us our notion of a "pure" Lorentz symmetry on $\mathscr{I}$. Note that since we have only specified the asymptotic behavior of "pure" Lorentz symmetries, we are free to extend them into the spacetime in any way. Denoting "pure" Lorentz symmetries on $\mathscr{I}$ by X^a, we pick this extension to be one which satisfies $\Sigma^{-2}K^a\nabla_a X^b \hat{=} 0$ where K^a is defined by Eq. (5.5.14) and is the (affine) generator of null surfaces along which we will consider limits to $\mathscr{I}$ (discussed in Sec. 5.2). This will turn out to simplify some of our later calculations.

Having clarified what we mean by "pure" Lorentz, we will refer to these simply as Lorentz symmetries henceforth and to the associated charges as Lorentz charges.

Remark 5.4.1 (Matching of Lorentz symmetries)*.* It follows from the analysis in Sec. 5.3.1 and 5.3.2 that the Lorentz symmetries as defined above correspond to conformal killing vector fields on $\mathscr{N}^\pm$ in the limits along $\mathscr{I}^\pm$ as well as $\mathscr{C}$. There is therefore an isomorphism between Lorentz symmetries at null and spatial infinity in this limit. Moreover, since, as shown in Sec. 5.3.2, $\boldsymbol{X}^a$ is even under the reflection map on $\mathscr{H}$ which maps $\mathscr{N}^-$ to $\mathscr{N}^+$, we see that in the limit to spatial infinity, Lorentz symmetries at past and future null infinity match each other up to antipodal reflection.

5.5 Matching the Lorentz charges

In this section, we consider limits of the Lorentz charges along $\mathscr{I}^\pm$ and $\mathscr{C}$ to $\mathscr{N}^\pm$. A priori, the limits to $\mathscr{N}^\pm$ along $\mathscr{I}^\pm$ and $\mathscr{C}$ are completely independent. However, we will show that there is a tensorial quantity in spacetime that reduces to the limiting expressions for these charges in each of these limits. Therefore, if we assume that the aforementioned quantity is continuous at $\mathscr{N}^\pm$, the Lorentz charges at null and spatial infinity match in the limit to $\mathscr{N}^\pm$. We will then discuss how this leads to the matching of Lorentz charges between past and future null infinity. Comments on the validity of our continuity assumption are deferred to the next section.

Consider first the charges at null infinity. From the results of chapter 2, it follows that in our chosen foliation, the expression for the charge associated with a BMS symmetry, ξ^a, on a finite cross-section, S_Σ, of $\mathscr{I}^+$ is given by (see 5.2.15)

$$\mathcal{Q}[\xi^a; S_\Sigma] \hat{=} -\frac{1}{8\pi}\int_{S_\Sigma} \boldsymbol{\varepsilon}_2 \left[\beta(\mathcal{P} + \tfrac{1}{2}\sigma^{ab}N_{ab}) + q^a{}_b X^b \mathcal{J}_a + q^a{}_b X^b \sigma_{ab}\mathscr{D}_c\sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab}\mathscr{D}_c(q^c{}_d X^d)\right], \tag{5.5.1}$$

We now consider its behavior in the limit $\Sigma^{-1} \to 0$. First, solving the evolution equations for $\mathcal{P}$ and $\mathcal{J}_a$ given in Eq. (5.2.16) using $\tau_a \mathrel{\hat{=}} 0$, $n^a \mathrel{\hat{=}} 2\Sigma^{-1}\partial_{\Sigma^{-1}}$, $\Phi = 2$, the fall-offs given in Eq. (5.4.6) as well as Eqs. (5.2.19b), (5.2.30), (5.4.7) and (5.4.10), we find that

$$\lim_{\Sigma^{-1}\to 0} \int_{S_\Sigma} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b (\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a \mathcal{P}) \text{ is finite}\,, \tag{5.5.2}$$

where $\tilde{\varepsilon}_2$ denotes the unit area element and we have dropped terms that integrate to zero because of the orthogonality of spherical harmonics. We then take the limit $\Sigma^{-1} \to 0$ of Eq. (A.1.9) using Eq. (5.4.6). For a Lorentz symmetry (that is, where $\beta_0 = 0$), we see, using Eqs. (5.3.11) and (5.5.2), that in the limit to $\mathscr{N}^+$, the charge becomes

$$\mathcal{Q}[X^a; \mathscr{N}^+] \mathrel{\hat{=}} -\frac{1}{8\pi} \int_{\mathscr{N}^+} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b (\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a \mathcal{P})\,. \tag{5.5.3}$$

It can be shown by similarly taking limits to $\mathscr{N}^-$ along $\mathscr{I}^-$ that the charge associated with a Lorentz symmetry on $\mathscr{N}^-$ is given by

$$\mathcal{Q}[X^a; \mathscr{N}^-] \mathrel{\hat{=}} -\frac{1}{8\pi} \int_{\mathscr{N}^-} \tilde{\varepsilon}_2 \Sigma^{-2} X^b q^a{}_b (\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a \mathcal{P})\,. \tag{5.5.4}$$

We now turn to the charges at spatial infinity. The charge associated with a Lorentz symmetry, $\boldsymbol{X}^a$, in Ashtekar-Hansen gauge on a cross-section $\mathcal{S}$ of $\mathscr{H}$, is given by [16]

$$\mathcal{Q}[\boldsymbol{X}^a, \mathcal{S}] = -\frac{1}{8\pi} \int_{\mathcal{S}} \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b\,, \tag{5.5.5}$$

where recall that $\boldsymbol{u}^a$ is the future-directed timelike normal on $\mathcal{S}$ and ${}^*\boldsymbol{X}^b$ was defined in Eq. (5.3.26). It follows from Eq. (5.2.2a) and Eq. (5.3.25) that this charge is conserved on $\mathscr{H}$. This implies that $\boldsymbol{\beta}_{ab}$ has a definite behavior under the reflection map defined in Eq. (3.2.7), which can be deduced as follows. Consider two cross-sections $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathscr{H}$ such that $\Upsilon \circ \mathcal{S}_1 = \mathcal{S}_2$ where $\Upsilon\circ$ denotes the action of the reflection map. Then, charge conservation implies that

$$
\begin{aligned}
0 = \int_{\mathcal{S}_2} \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b - \int_{\mathcal{S}_1} \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b &= \int_{\mathcal{S}_2} \left[\boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b - \Upsilon \circ (\boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b)\right], \\
&= \int_{\mathcal{S}_2} \left[\boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b + \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \Upsilon \circ (\boldsymbol{\beta}_{ab} {}^\star\boldsymbol{X}^b)\right], \qquad (5.5.6)
\end{aligned}
$$

where in the last equality, we have used the fact that $\Upsilon \circ (\boldsymbol{\varepsilon}_2 \boldsymbol{u}^a) = -\boldsymbol{\varepsilon}_2 \boldsymbol{u}^a$. This follows because $\Upsilon \circ \boldsymbol{\varepsilon}_2 = -\boldsymbol{\varepsilon}_2$ and because $\boldsymbol{u}^a$ is future-directed on both $\mathcal{S}_1$ and $\mathcal{S}_2$ and therefore does not get acted on by the reflection map. Moreover, since ${}^\star\boldsymbol{X}^a$ is even under the reflection map on $\mathscr{H}$ (as shown in Sec. 5.3.2), it follows that the charge only receives contribution from reflection-odd solutions of $\boldsymbol{\beta}_{ab}$. Using this as motivation, in the rest of this chapter we will restrict our attention to only reflection-odd solutions for $\boldsymbol{\beta}_{ab}$ (derived in Appendix A.11).[9] Since we have specialized to spacetimes where $\boldsymbol{E}_{ab}$ is reflection-even as remarked in Sec. 5.2, it follows from Eq. (5.2.2b) and the fact that the reflection map preserves the volume form on $\mathscr{H}$ that the reflection-odd solutions for $\boldsymbol{\beta}_{ab}$ satisfy the equation

$$
(\boldsymbol{D}^2 - 2)\boldsymbol{\beta}_{ab} = 0\,. \qquad (5.5.7)
$$

In fact, one can show by solving for the reflection-even solutions to this equation in the same way as for the reflection-odd solutions in Appendix A.11 that the $\ell = 1$ reflection-even solution diverges in the limits to $\mathscr{N}^\pm$ and therefore for these solutions the Lorentz charge diverges in these limits as well. While we have not shown it explicitly, we expect this property to remain true even for the reflection-even solutions to Eq. (5.2.2b). The fact that these reflection-even solutions are clearly pathological serves as further motivation for discarding them.

As shown in Appendix A.11 and Sec. 5.3.2, reflection-odd solutions for $\boldsymbol{\beta}_{ab}$ and ${}^\star\boldsymbol{X}^a$ have a finite limit to $\mathscr{N}^\pm$. Using this and Eqs. (5.2.31) and (5.2.32), we see that the limit of the charge in Eq. (5.5.5) to $\mathscr{N}^\pm$ is non-vanishing and is given by

$$
\mathcal{Q}[\boldsymbol{X}^a, \mathscr{N}^\pm] = -\frac{1}{8\pi} \int_{\mathscr{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a \boldsymbol{\beta}_{ab} {}^*\boldsymbol{X}^b = \frac{1}{8\pi} \int_{\mathscr{N}^\pm} \tilde{\boldsymbol{\varepsilon}}_2 \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^a \boldsymbol{\beta}_{ab} \tilde{\boldsymbol{\varepsilon}}_c{}^b \boldsymbol{X}^c \qquad (5.5.8)
$$

9 As we show in Appendix A.13, this condition on $\boldsymbol{\beta}_{ab}$ is satisfied in the Kerr-Newman family of spacetimes; for Kerr spacetimes this was also shown in Appendix. C of [16].

where the last equality uses Eq. (5.3.29).

Let us now consider the following quantity in a neighborhood of i^0 in the unphysical spacetime,

$$-\frac{1}{8\pi}\int_{\mathcal{S}'} \tilde{\varepsilon}_2\, \Omega^{-1/2}\Sigma^{-1} \, *\, C_{acbd}\, U^a(\nabla^c\Omega^{1/2})(\nabla^d\Omega^{1/2})(\Omega^{-1/2}X^b)\,, \tag{5.5.9}$$

where, as before, $\tilde{\varepsilon}_2$ is the unit area element, L^a was defined in Eq. (5.2.22) and U^a is a $C^{>-1}$ vector field at i^0, is defined as

$$U^a = L^a - N^a(-\Omega^{-1}\Sigma^{-2} + \frac{1}{2}\nabla_b\Sigma^{-1}\nabla^b\Sigma^{-1})\,, \tag{5.5.10}$$

whose limit to $\mathscr{H}$ satisfies $\boldsymbol{U}^a = \boldsymbol{h}^a{}_b\boldsymbol{L}^b$. Further, X^a is such that $\Omega^{-1/2}X^a$ limits to a Lorentz symmetry on $\mathscr{H}$ and X^a limits to a Lorentz symmetry on $\mathscr{I}$. We now explore how this quantity behaves in the limit to $\mathscr{N}^\pm$ along $\mathscr{C}$ and along $\mathscr{I}^\pm$.

1 Limit to $\mathscr{N}^\pm$ along $\mathscr{C}$

Consider first the limit of Eq. (5.5.9) to cross-sections of $\mathscr{H}$ along the sequence of spacelike hypersurfaces described in Sec. 5.2. Using $\lim_{\to i^0}\Omega^{-1/2}X^a = \boldsymbol{X}^a$ in addition to Eq. (3.2.1) and Eq. (5.2.21), we see that in the limit to $\mathscr{H}$, Eq. (5.5.9) becomes

$$-\frac{1}{8\pi}\int_{\mathcal{S}} \tilde{\boldsymbol{\varepsilon}}_2\boldsymbol{\Sigma}^{-1} * \boldsymbol{C}_{acbd}\boldsymbol{\eta}^c\boldsymbol{\eta}^d\boldsymbol{U}^a\boldsymbol{X}^b\,, \tag{5.5.11}$$

where $\mathcal{S}$ is a cross-section of $\mathscr{H}$. Using Eq. (5.2.1), we obtain

$$\lim_{\to\mathscr{N}^\pm}\frac{-1}{8\pi}\int_{\mathcal{S}} \tilde{\boldsymbol{\varepsilon}}_2\, \boldsymbol{\Sigma}^{-1} * \boldsymbol{C}_{acbd}\boldsymbol{\eta}^c\boldsymbol{\eta}^d\boldsymbol{U}^a\boldsymbol{X}^b = -\frac{1}{8\pi}\int_{\mathscr{N}^\pm}\tilde{\boldsymbol{\varepsilon}}_2\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a\boldsymbol{\beta}_{ab}\boldsymbol{X}^b\,, \tag{5.5.12}$$

This, from Eq. (5.5.8), corresponds to the Lorentz charge, on $\mathscr{N}^\pm$, for any Lorentz symmetry given by $\tilde{\boldsymbol{\varepsilon}}_b{}^a\boldsymbol{X}^b$.

2 Limit to $\mathscr{N}^{\pm}$ along $\mathscr{I}^{\pm}$

Consider now the limit of Eq. (5.5.9) to $\mathscr{N}^{+}$ along $\mathscr{I}^{+}$.[10] Throughout this calculation, we will often implicitly use Eq. (5.2.15), the symmetries of the Weyl tensor and the fact that under the replacement $C_{abcd} \to *C_{abcd}$, the expressions for the Weyl tensor components in Eq. (5.2.15) change as follows

$$\mathcal{P} \to \mathcal{P}^{*}\,, \quad \mathcal{S}_a \to \mathcal{S}_b\, \varepsilon^b{}_a\,, \quad \mathcal{J}_a \to \mathcal{J}_b\, \varepsilon_a{}^b\,, \quad \mathcal{R}_{ab} \to \mathcal{R}_a{}^c\, \varepsilon_{cb}\,. \tag{5.5.13}$$

As described in Sec. 5.2, we will take the limit to cross-sections S_Σ of $\mathscr{I}^{+}$ along a sequence of null hypersurfaces, that exist in a neighborhood of i^0, that intersect these cross-sections. We will then take the limit $\Sigma^{-1} \to 0$ along $\mathscr{I}^{+}$ which will define for us the limit of our expression to $\mathscr{N}^{+}$. We define

$$K^a := l^a - \Omega \alpha^a\,, \tag{5.5.14}$$

where K^a is the affine null generator of the aforementioned null surfaces. Here, $\alpha^a \mathrel{\widehat{=}} -l^b \nabla_b l^a$. As indicated, this expression only fixes α^a at $\mathscr{I}^{+}$ and its expression away from $\mathscr{I}^{+}$ is chosen to ensure that $K^a \nabla_a K^b = K^a K_a = 0$ all along the null surfaces. Converting the integrand in Eq. (5.5.9) into quantities defined on $\mathscr{I}^{+}$ using Eqs. (5.2.20), (5.2.25) and (5.5.10) (and relabeling the indices for later convenience), we get

$$\Omega^{-1/2}\Sigma^{-1}\; * C_{acbd}\, U^a (\nabla^c \Omega^{1/2})(\nabla^d \Omega^{1/2})(\Omega^{-1/2} X^b) = \frac{\Omega^{-2}\Sigma^{-2}}{2} * C_{bcde}\, n^d\, n^b\, l^c\, X^e\,. \tag{5.5.15}$$

Using $\alpha^a \mathrel{\widehat{=}} -l^b \nabla_b l^a$, it follows that

$$K^a\, n_a = -1 + O(\Omega^2)\,. \tag{5.5.16}$$

[10] The reader who wishes to skip the details of this somewhat involved calculation may jump directly to Eq. (5.5.38) where the final expression is given.

We can then write

$$\lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Omega^{-2}\,\Sigma^{-2}\, *C_{bcde}\, n^d\, n^b l^c\, X^e = \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Omega\, K^a \nabla_a(\Sigma^{-2}\,\Omega^{-2} * C_{bcde}\, n^d\, n^b\, l^c\, X^e)$$
$$- \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, K^a \nabla_a(\Omega^{-1}\,\Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c\, X^e)\,. \tag{5.5.17}$$

Consider the first term on the right hand side of Eq. (5.5.17). This can be written as

$$\begin{aligned}
&\lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Omega\, K^a \nabla_a(\Sigma^{-2}\,\Omega^{-2} * C_{bcde}\, n^d\, n^b\, l^c\, X^e)\\
&= \lim_{S'\to S_\Sigma} \frac{\Omega}{2}\int_{S'} \tilde{\varepsilon}_2\, K^a \nabla_a(\Sigma^{-2}\,\Omega^{-2} * C_{bcde}\, n^d\, n^b\, l^c\, X^e)\\
&= \lim_{\Omega\to 0} \Omega \frac{d}{d\Omega}\Big(\int_{S'} \frac{\tilde{\varepsilon}_2}{2} \Sigma^{-2}\Omega^{-1} \mathcal{S}_a\, \varepsilon^a{}_b\, X^b\Big) - \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Sigma^{-2}\,\vartheta(K^a)\, \mathcal{S}_c\, \varepsilon^c{}_b X^b\,,
\end{aligned} \tag{5.5.18}$$

where in the first equality, we used the fact that S' denote $\Omega =$ constant cross-sections to move Ω outside the integral and in the second and third equalities, we used Eq. (5.5.13) and Eq. 2.23 of [116], which, translated into our notation, states that[11]

$$\frac{d}{d\Omega}\int_{S'} \tilde{\epsilon}_2\, B = \int_{S'} \tilde{\epsilon}_2\, \left[(K^a\nabla_a B + \vartheta(K^a)B)(-n_b K^b)^{-1}\right], \tag{5.5.19}$$

for some scalar B that has a finite integral over cross-sections S' as $S' \to S_\Sigma$. Additionally, we have used Eq. (5.5.16), along with the fact that the limit $S' \to S_\Sigma$ coincides with $\Omega \to 0$. Note that the expression in the round brackets in the first term in Eq. (5.5.18) is finite on $\mathscr{I}^+$ for the following reason. Since $\mathcal{S}_a \,\hat{=}\, \frac{1}{2}\mathscr{D}^b N_{ab}$ (Eq. (5.2.19b)), using Eq. (5.2.35), up to a total derivative term that would drop out upon integrating over S_Σ, $\tilde{\varepsilon}_2 \Sigma^{-2} \mathcal{S}_a\, \varepsilon^a{}_b\, X^b = -\frac{\tilde{\varepsilon}_2\,\Sigma^{-2}}{2} N_{ab}\, \mathscr{D}^a \varepsilon^b{}_c X^c$. Since $\varepsilon^a{}_b X^b$ is a conformal killing vector on S_Σ (as shown Sec. 5.3.1), this vanishes upon contraction with N_{ab} since N_{ab} is a symmetric traceless tensor. As a result, $\int_{S'} \tilde{\varepsilon}_2 \Sigma^{-2} \mathcal{S}_a\, \varepsilon^a{}_b\, X^b$ is $O(\Omega)$ and hence

$$\lim_{\Omega\to 0} \Omega \frac{d}{d\Omega}\Big(\int_{S'} \frac{\tilde{\varepsilon}_2}{2} \Sigma^{-2}\Omega^{-1} \mathcal{S}_a\, \varepsilon^a{}_b\, X^b\Big) = 0\,. \tag{5.5.20}$$

[11] Note that the definition of expansion used in this chapter (Eq. (5.2.13)) is twice the definition in Eq. 2.25 of [116].

As a result, we have

$$\lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Omega^{-2}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b l^c\, X^e = \lim_{S'\to S_\Sigma} \frac{-1}{2}\int_{S'} \tilde{\varepsilon}_2\, \Sigma^{-2}\, \vartheta(K^a)\, \mathcal{S}_c\, \varepsilon^c{}_b X^b$$
$$- \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, K^a \nabla_a(\Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c\, X^e)\,. \tag{5.5.21}$$

Using $\Sigma^{-2} K^a \nabla_a X^b \mathrel{\hat{=}} 0$ (recall the discussion at the end of Sec. 5.4), the second term on the right hand side above becomes

$$\lim_{S'\to S_\Sigma} \frac{-1}{2}\int_{S'} \tilde{\varepsilon}_2\, X^e\, K^a \nabla_a(\Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c)\,. \tag{5.5.22}$$

Since X^e is tangent to $\mathscr{I}^+$, we write it as $X^e = -n^e l_b X^b + q^e{}_b X^b + O(\Omega)$, where

$$q^e{}_b = \delta^e{}_b + n^e l_b + n_b l^e\,. \tag{5.5.23}$$

Then Eq. (5.5.22) becomes

$$\lim_{S'\to S_\Sigma} \frac{-1}{2}\int_{S'} \tilde{\varepsilon}_2\, (-n^e l_f X^f + q^e{}_f X^f)\, K^a \nabla_a(\Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c)$$
$$= \lim_{S'\to S_\Sigma} \frac{-1}{2}\int_{S'} \tilde{\varepsilon}_2\; \Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c\, [l_f X^f K^a \nabla_a n^e - X^f\, K^a \nabla_a q^e{}_f]$$
$$- \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, X^f\, K^a \nabla_a(\Omega^{-1}\, \Sigma^{-2} *C_{bcde}\, n^d\, n^b\, l^c\, q^e{}_f)\,. \tag{5.5.24}$$

Using $\nabla_a n_b \mathrel{\hat{=}} 2 g_{ab}$ (from Eq. (A.1.3), adapted to $\Phi = 2$) and Eq. (5.5.23), the right hand side can be expanded out and written as

$$\lim_{S'\to S_\Sigma} \int_{S'} \tilde{\varepsilon}_2\; \Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c\, l^e\, K_f X^f - \lim_{S'\to S_\Sigma} \frac{1}{2}\int_{S'} \tilde{\varepsilon}_2\, X^f\, K^a \nabla_a(\Omega^{-1}\, \Sigma^{-2}\, *C_{bcde}\, n^d\, n^b\, l^c\, q^e{}_f)\,, \tag{5.5.25}$$

where in simplifying this expression, we used $X^a n_a \mathrel{\hat{=}} 0$. Note that since $K^a \mathrel{\hat{=}} l^a$ and $l_a X^a \mathrel{\hat{=}} \frac{1}{4}\mathscr{D}_a(q^a{}_b X^b)$ (from Eqs. (5.3.2) and (5.3.11)), the first term above becomes

$$\frac{1}{4}\int_{S_\Sigma} \tilde{\varepsilon}_2\; \Sigma^{-2}\, \mathcal{P}^* \mathscr{D}_a(q^a{}_b X^b) = -\frac{1}{4}\int_{S_\Sigma} \tilde{\varepsilon}_2\; \Sigma^{-2}\, q^a{}_b X^b\, \mathscr{D}_a \mathcal{P}^*\,, \tag{5.5.26}$$

where, because the integrand of this term is finite on $\mathscr{I}^+$, we took the limit and evaluated it directly on S_Σ and in the last step we did an integration by parts. Note that as we take $\Sigma^{-1} \to 0$ on $\mathscr{I}^+$, this term will go to zero from Eq. (5.4.9). As a result, we can discard this term. Turn now to the second term in Eq. (5.5.25)

$$\lim_{S' \to S_\Sigma} \frac{-1}{2} \int_{S'} \tilde{\varepsilon}_2 \, X^f \, K^a \nabla_a (\Omega^{-1} \, \Sigma^{-2} * C_{bcde} \, n^d \, n^b \, l^c \, q^e{}_f) \\ = \lim_{S' \to S_\Sigma} \frac{-1}{2} \int_{S'} \tilde{\varepsilon}_2 \, X^f \, K^a \nabla_a (\Omega^{-1} \, \Sigma^{-2} * C_{bcde} \, l^d n^c n^e q^b{}_f) \,, \tag{5.5.27}$$

where we have simply relabeled the indices for later convenience. To evaluate this term, consider the Bianchi identity for the Hodge dual of the Weyl tensor.

$$\nabla_{[a}(\Omega^{-1} * C_{bc]de}) = 0 \,, \tag{5.5.28}$$

which can be rewritten as

$$\nabla_a(\Omega^{-1} \, \Sigma^{-2} \, * C_{bcde}) + \nabla_c(\Omega^{-1} \, \Sigma^{-2} \, * C_{abde}) + \nabla_b(\Omega^{-1} \, \Sigma^{-2} \, * C_{cade}) - \Omega^{-1} * C_{bcde} \nabla_a \Sigma^{-2} \\ - \Omega^{-1} * C_{cade} \nabla_b \Sigma^{-2} - \Omega^{-1} * C_{abde} \nabla_c \Sigma^{-2} = 0 \,. \tag{5.5.29}$$

Contracting this equation with $K^a \, l^d \, n^c \, n^e \, q^b{}_f$, we obtain

$$\frac{1}{2} K^a \nabla_a (\Omega^{-1} \Sigma^{-2} * C_{bcde} \, l^d n^c n^e q^b{}_f) = \frac{1}{2} \Big[K^a \Omega^{-1} \, \Sigma^{-2} \, * C_{bcde} \nabla_a (l^d n^c n^e q^b{}_f) \\ - n^c \nabla_c (\Omega^{-1} \Sigma^{-2} * C_{abde} \, K^a l^d n^e q^b{}_f) - q^b{}_f \nabla_b (\Omega^{-1} \Sigma^{-2} * C_{cade} K^a l^d n^c n^e) \\ + \Omega^{-1} \Sigma^{-2} * C_{abde} n^c \nabla_c (K^a l^d n^e q^b{}_f) + q^b{}_f \, \Omega^{-1} \, \Sigma^{-2} \, * C_{cade} \nabla_b (K^a l^d n^c n^e) \\ + \Omega^{-1} * C_{bcde} K^a l^d n^c n^e q^b{}_f \nabla_a \, \Sigma^{-2} + \Omega^{-1} * C_{cade} K^a l^d n^c n^e q^b{}_f \nabla_b \Sigma^{-2} + \\ \Omega^{-1} * C_{abde} K^a l^d n^e q^b{}_f n^c \nabla_c \Sigma^{-2} \Big] . \tag{5.5.30}$$

Each of the terms on the right hand side of this expression is individually finite on $\mathscr{I}^+$ and therefore we can evaluate this expression directly on $\mathscr{I}^+$. Using $n^c \nabla_c \, \varepsilon_a{}^b \mathrel{\widehat{=}} 0$ (which holds since $\tau_a \mathrel{\widehat{=}} 0$), Eq. (5.2.15), Eq. (5.5.13), $K^a \hat{=} l^a$ (from Eq. (5.5.14)) and $\Sigma \pounds_n \Sigma^{-1} \hat{=} 2$ (condition (4)), the two terms on the second line in this expression combine with the second term on the fourth line to give

$$-\frac{\varepsilon_b{}^a}{2} n^c \nabla_c (\Sigma^{-2}\,\mathcal{J}_a) + \frac{\Sigma^{-2}}{2} \mathscr{D}_b \mathcal{P}^* = \frac{\varepsilon_b{}^a}{2}(2\Sigma^{-2}\mathcal{J}_a - \Sigma^{-2}\pounds_n \mathcal{J}_a - 4\Sigma^{-2}\mathcal{J}_a) + \frac{\Sigma^{-2}}{2}\,\mathscr{D}_b\mathcal{P}^*\,. \tag{5.5.31}$$

This can be simplified using the evolution equation for $\mathcal{J}_a$ (Eq. (5.2.16d) with $\tau_a \hat{=} 0$) and we obtain

$$\begin{aligned}
&-\frac{\varepsilon_b{}^a}{2} n^c \nabla_c (\Sigma^{-2}\,\mathcal{J}_a) + \frac{\Sigma^{-2}}{2} \mathscr{D}_b \mathcal{P}^* \\
&= \frac{\varepsilon_b{}^a}{2}(2\Sigma^{-2}\mathcal{J}_a - 4\Sigma^{-2}\mathcal{J}_a) + \frac{\Sigma^{-2}}{2}\,\mathscr{D}_b\mathcal{P}^* - \frac{\Sigma^{-2}\varepsilon_b{}^a}{4}(\mathscr{D}_a\mathcal{P} - \varepsilon_a{}^c\mathscr{D}_c\mathcal{P}^*) + \Sigma^{-2}\,\varepsilon_b{}^c\sigma_c{}^a\mathcal{S}_a \\
&+ 2\Sigma^{-2}\,\varepsilon_b{}^a\mathcal{J}_a\,.
\end{aligned} \tag{5.5.32}$$

We now evaluate the remaining terms in Eq. (5.5.30). In what follows, we drop terms proportional to $\Sigma^{-2}\mathscr{D}_a\mathcal{P}^*$ everywhere. This is because they contribute terms proportional to $\int_{S_\Sigma} \tilde{\varepsilon}_2\,\Sigma^{-2}\,X^a q_a{}^b \mathscr{D}_b\mathcal{P}^*$ to Eq. (5.5.27) and therefore, from Eq. (5.4.9), drop out in the limit $\Sigma^{-1} \to 0$ which we take in the end. We use $\cdots$ to indicate that these terms have been suppressed in our expressions. We obtain, altogether, that

$$\begin{aligned}
&\frac{1}{2}K^a\nabla_a(\Omega^{-1}\Sigma^{-2} * C_{bcde} l^d n^c n^e q^b{}_f) \hat{=} -\Sigma^{-2}\varepsilon_f{}^a\mathcal{J}_a - \frac{\Sigma^{-2}}{4}\varepsilon_f{}^a\mathscr{D}_a\mathcal{P} + \Sigma^{-2}\,\varepsilon_f{}^a\sigma_a{}^b\mathcal{S}_b \\
&- \frac{\Omega^{-1}}{2}\,\Sigma^{-2} * C_{bcde}\, q^b{}_f\, n^c\, n^e\, \alpha^d + \Sigma^{-2}\mathcal{S}_a\,\varepsilon_b{}^a\,\sigma^b_f - \frac{\Sigma^{-2}}{2}\,\vartheta(l^a)\varepsilon^b{}_f\,\mathcal{S}_b \\
&+ \frac{\Omega^{-1}}{2} * C_{bcde} l^d n^c n^e q^b{}_f K^a \nabla_a \Sigma^{-2} + \cdots\,.
\end{aligned} \tag{5.5.33}$$

Note that since ε_{ab} is antisymmetric and σ_{ab} is symmetric and trace-free, $\varepsilon_{ba}\sigma^a{}_c$ is a symmetric tensor (see, e.g, Appendix. D of [4] for a proof). Using this, the third term in the first line above cancels with the second term in the second line. Next, using $K^a\nabla_a\Sigma^{-2} \hat{=} l^a\nabla_a\Sigma^{-2} \hat{=} -\frac{1}{2}\nabla_a\Sigma^{-1}\nabla^a\Sigma^{-1}$ (where the last equality follows from Eqs. (5.2.20), (5.2.22) and (5.2.25) and condition (4)), along with Eqs. (5.2.15) and (5.5.13) in the last term, we get

$$\frac{1}{2}K^a\nabla_a(\Omega^{-1}\Sigma^{-2} * C_{bcde}l^dn^cn^eq^b{}_f) \mathrel{\hat{=}} -\Sigma^{-2}\,\varepsilon_f{}^a\,\mathcal{J}_a - \frac{\Sigma^{-2}}{4}\varepsilon_f{}^a\,\mathscr{D}_a\mathcal{P}$$
$$-\frac{\Omega^{-1}}{2}\,\Sigma^{-2} * C_{bcde}\,q^b{}_f\,n^c\,n^e\,\alpha^d - \frac{\Sigma^{-2}}{2}\,\vartheta(l^a)\,\varepsilon^b{}_f\,\mathcal{S}_b - \frac{\varepsilon^b{}_f}{4}\,S_b\nabla_a\Sigma^{-1}\nabla^a\Sigma^{-1} + \cdots\,. \tag{5.5.34}$$

Let us now consider the term $-\frac{\Omega^{-1}}{2}\,\Sigma^{-2} * C_{bcde}\,q^b{}_f\,n^c\,n^e\,\alpha^d$ in Eq. (5.5.34). Using $\alpha^a \mathrel{\hat{=}} -l^b\nabla_b l^a$ as well as Eqs. (5.2.15) and (5.5.13), this term can be written as

$$-\frac{\Sigma^{-2}\,\varepsilon^c{}_f}{2}(n_b\,\mathcal{S}_c l^a\nabla_a l^b - \mathcal{R}_{bc}\,l^a\nabla_a l^b)\,. \tag{5.5.35}$$

Then using Eqs. (A.12.5) and (A.12.14) to simplify this, we obtain

$$\frac{1}{2}K^a\nabla_a(\Omega^{-1}\Sigma^{-2} * C_{bcde}l^dn^cn^eq^b{}_f) \mathrel{\hat{=}} -\Sigma^{-2}\varepsilon_f{}^a\mathcal{J}_a - \frac{\Sigma^{-2}}{4}\varepsilon_f{}^a\,\mathscr{D}_a\mathcal{P} - \frac{\Sigma^{-2}}{2}\vartheta(l^a)\,\varepsilon^b{}_f\,\mathcal{S}_b + \cdots\,. \tag{5.5.36}$$

Putting all of this together, we have

$$\lim_{S'\to S_\Sigma}\frac{1}{2}\int_{S'}\tilde{\varepsilon}_2\,\Omega^{-2}\,\Sigma^{-2}\; * C_{bcde}\,n^d\,n^b l^c\,X^e = \lim_{S'\to S_\Sigma}\int_{S'}\frac{-\tilde{\varepsilon}_2}{2}\,\Sigma^{-2}\,\vartheta(K^a)\,\mathcal{S}_c\,\varepsilon^c{}_b X^b$$
$$-\lim_{S'\to S_\Sigma}\frac{1}{2}\int_{S'}\tilde{\varepsilon}_2\,K^a\nabla_a(\Omega^{-1}\,\Sigma^{-2}\; * C_{bcde}\,n^d\,n^b\,l^c\,X^e) = \int_{S_\Sigma}\tilde{\varepsilon}_2\,[\Sigma^{-2}X^b\varepsilon_b{}^a(\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a\mathcal{P}) + \cdots]\,, \tag{5.5.37}$$

where we used $\vartheta(K^a) \mathrel{\hat{=}} \vartheta(l^a)$ to simplify the final expression. As shown in Eq. (5.5.2), the right hand side of Eq. (5.5.37) is finite in the limit $\Sigma^{-1} \to 0$. As a result, the limit of Eq. (5.5.9) to $\mathscr{N}^+$ along $\mathscr{I}^+$ gives

$$-\frac{1}{8\pi}\int_{\mathscr{N}^+}\tilde{\varepsilon}_2[\Sigma^{-2}X^b\,\varepsilon_b{}^a(\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a\mathcal{P})]\,, \tag{5.5.38}$$

which, from Eq. (5.5.3), corresponds to the limit of the charge associated with any Lorentz symmetry given by $\tilde{\varepsilon}_b{}^a X^b$ (using the fact that $\tilde{\varepsilon}_b{}^a = \varepsilon_b{}^a$). In the same way, the limit of Eq. (5.5.9) to $\mathscr{N}^-$ along $\mathscr{I}^-$ also yields

$$-\frac{1}{8\pi}\int_{\mathscr{N}^-}\tilde{\varepsilon}_2[\Sigma^{-2}X^b\,\varepsilon_b{}^a(\mathcal{J}_a + \frac{1}{4}\mathscr{D}_a\mathcal{P})]\,. \tag{5.5.39}$$

Therefore, assuming continuity of Eq. (5.5.9) at $\mathscr{N}^{\pm}$, Lorentz charges, in the limit to $\mathscr{N}^{\pm}$ along $\mathscr{I}^{\pm}$, match the Lorentz charges in the limit to $\mathscr{N}^{\pm}$ along $\mathscr{C}$. Since, as discussed earlier, the Lorentz charges (in Ashtekar-Hansen gauge) on $\mathscr{H}$ are conserved and therefore their values on the $\mathscr{N}^{\pm}$ are the same, it follows that the Lorentz charges on $\mathscr{I}^{+}$ and $\mathscr{I}^{-}$ match in the limit to spatial infinity.

Using the proof of matching of supertranslation symmetries and the associated supermomentum charges in [10], we then conclude that *all* BMS symmetries on past and future null infinity match antipodally in the limit to spatial infinity and their charges become equal in this limit. This immediately implies the following infinitely many conservation laws (one for each pair of "matched" generators ξ^{a+} and ξ^{a-})

$$\mathcal{Q}\left[\xi^{a+}; S_{\Sigma}^{+}\right] - \mathcal{Q}\left[\xi^{a-}; S_{\Sigma}^{-}\right] = \mathcal{F}^{+}\left[\xi^{a+}; \Delta\mathscr{I}^{+}\right] + \mathcal{F}^{-}\left[\xi^{a-}; \Delta\mathscr{I}^{-}\right] , \tag{5.5.40}$$

where ξ^{a+} denotes a BMS symmetry on $\mathscr{I}^{+}$, ξ^{a-} denotes the BMS symmetry on $\mathscr{I}^{-}$ that this matches onto in the limit to spatial infinity, $\mathcal{F}^{+}$ ($\mathcal{F}^{-}$) denotes the incoming flux[12] of charge associated with a BMS symmetry on $\mathscr{I}^{+}$ ($\mathscr{I}^{-}$), $\Delta\mathscr{I}^{+}$ ($\Delta\mathscr{I}^{-}$) denotes a portion of $\mathscr{I}^{+}$ ($\mathscr{I}^{-}$) between spatial infinity and a cross-section, S_{Σ}^{+} (S_{Σ}^{-}), of future (past) null infinity. As discussed in [10], if suitable fall-off conditions are satisfied on the future and past boundaries of future and past null infinity (i.e in the limit to timelike infinities) such that the BMS charges go to zero in those limits then we obtain the global conservation law

$$\mathcal{F}^{+}\left[\xi^{a+}; \Delta\mathscr{I}^{+}\right] + \mathcal{F}^{-}\left[\xi^{a-}; \Delta\mathscr{I}^{-}\right] = 0\,, \tag{5.5.41}$$

that is, the total incoming flux equals the total outgoing flux for all BMS symmetries and therefore that the flux is conserved in any classical gravitational scattering process from $\mathscr{I}^{-}$ to $\mathscr{I}^{+}$.

[12] In the orientation conventions picked in this chapter, as in [9, 10], the fluxes at both $\mathscr{I}^{\pm}$ are incoming.

5.6 | Discussion and future directions

We showed the antipodal matching of Lorentz symmetries and the equality of the associated charges on past and future null infinity in the limit to spatial infinity in a class of spacetimes that are asymptotically-flat at null and spatial infinity in the sense of Asthekar and Hansen. Combined with the result of [10] where the matching of supertranslation symmetries and supermomentum charges was similarly shown, this proves the matching of *all* BMS symmetries and charges in these spacetimes. While we did not require that our spacetimes be stationary, we did make the following assumptions about our class of spacetimes: (1) we assumed that $\boldsymbol{B}_{ab} = 0$, which, as discussed earlier, is known to be true (at least) in asymptotically-flat spacetimes that are *either* stationary *or* axisymmetric; (2) that $\boldsymbol{\beta}_{ab}$ is odd under the reflection map on $\mathscr{H}$ in our class of spacetimes; (3) that the peeling theorem holds and therefore $\Omega^{-1}C_{abcd}$ admits a limit to $\mathscr{I}$; (4) that the spacetimes are null-regular at i^0 in the sense of Eqs. (5.2.33) and (5.2.34); (5) that the trace-free projection of (rescaled) S_{ab} is continuous at $\mathscr{N}^\pm$ (in the precise sense discussed in Sec. 5.4); and (6) that the Weyl tensor obeys the condition that Eq. (5.5.9) is continuous at $\mathscr{N}^\pm$. As we show in Appendix A.13, our assumptions all hold in the Kerr-Newman family of spacetimes.[13] However, they do not hold, for example, in polyhomogenous spacetimes [117] where the peeling theorem is not satisfied. Although we have not investigated this in detail, we also expect our assumptions to hold in (at least a subset of) the class of spacetimes considered by Christodoulou-Klainerman (CK) [118]. Determining exactly how big a class of asymptotically-flat spacetimes admits our assumptions is very much an open question. Indeed it would be interesting to also rigorously analyze if they hold, for example, in the spacetimes considered by Bieri [119, 120]. In spacetimes where our assumptions obviously fail, for example polyhomogenous spacetimes, it would be interesting to understand how our procedure may be extended to study the matching of asymptotic symmetries and charges.

It would also be interesting to investigate matching conditions between symmetries and charges

[13] What makes it easy to check our assumptions, in particular condition (6) above, in these spacetimes is that an explicit conformal completion that includes i^0 and $\mathscr{I}^\pm$ is known for these spacetimes. Given such explicit conformal completions for more general spacetimes, it should be straightforward to check if our assumptions are satisfied in those spacetimes.

defined on black hole horizons [55] and those associated with null infinity to see if flux conservation laws on Hawking radiation can be similarly obtained. This would require analyzing symmetries and charges in the limit to timelike infinities using the framework developed in [121]. We leave a detailed study of this to future work.

Chapter 6

The classical dynamics of gauge theories in the deep infrared

(Adapted with permission from [8])

Chapter summary

Recent activity aimed at studying asymptotic symmetries and charges has led to the result that gauge and gravitational theories in asymptotically flat spacetimes possess infinitely many charges associated, respectively, with large gauge transformations and diffeomorphisms that are nontrivial at infinity. Assuming certain regularity conditions near spatial infinity and appropriate fall-offs near timelike infinity (see, e.g, [7, 9, 10]), the results of [9, 10] along with the discussion in chapter 5, [7] imply that these charges are conserved in any classical scattering process. The question then is how these charges *constrain* classical scattering. In the absence of matter or stress-energy, these "new" charges are functionals purely of the zero-frequency components of the radiative fields and are *soft* in that sense. It has been claimed in the literature that the *hard* (non-zero energy) and soft degrees of freedom in a theory evolve in a manner such that they are decoupled from each other and, therefore, that these charges do not impose any non-trivial constraints on the scattering of hard degrees of freedom. We provide evidence to the contrary. In particular, we explicitly show that the hard and soft degrees of freedom in a theory with non-linear equations of motion couple under time evolution. We do this by performing a perturbative classical computation of the scattering map in a $U(1)$ gauge theory coupled to a massless charged scalar field in four-dimensional Minkowski spacetime. While it is true that the evolution of the soft and hard sectors is mutually decoupled at low orders in the coupling constant, we show that it is in fact coupled at higher orders (in particular, we calculate a contribution at quartic order in the coupling constant that demonstrates this). We show that this coupling cannot be removed by perturbative field redefinitions and use this to conclude that the conservation of these infinitely many charges yields nontrivial constraints on the scattering of hard degrees of freedom. Since the essential ingredient in our analysis was the fact the equations of motion of the theory were non-linear, we expect this result to generalize to general relativity and for these infinitely many conserved charges to play a role in constraining gravitational scattering processes and, in particular, black hole evaporation.

6.1 | Context

As discussed in the introduction, the study of asymptotic symmetries and charges in recent years has led to the demonstration of the existence of infinitely many new charges in gauge theories and gravity. In particular, it has been argued that black hole possess an infinite number of soft "hair" that may constrain black hole evaporation. While this proposal has led to a lot of activity in recent years aimed at understanding how large gauge transformations add new hair to black holes which can then be used to enumerate black hole microstates [122–125], an important criticism of it has also emerged. This criticism, which is interesting on its own with regards to the dynamics of the soft sector in gauge theories, has been aimed at questioning how much interplay there is between the soft and hard (non-zero energy) sectors of a gauge theory in any scattering process. Starting with the works of Bousso, Mirbabayi and Porratti [126, 127], several papers [128–131] have argued that the newly discovered charges constrain a sector of the theory that is decoupled from all the "hard" degrees of freedom which include the Hawking radiation emitted from a black hole. As a result, it has been claimed that the conservation of these soft charges does not have any utility as constraints that correlate the initial and final hard states in a scattering problem.

Without delving into the deeper question of whether these charges imply more hair for black holes that help solve the black hole information paradox, we restrict ourselves in this chapter to addressing the question of whether the soft sector in a non-linear theory decouples from the hard sector in a scattering process. To study this problem, we study electromagnetism coupled to a massless charged scalar field in four dimensional Minkowski spacetime. This is an interesting toy model for studying soft-hard (de-)coupling. This is because the equations of motion of this theory are non-linear and can therefore allow for the existence of soft-hard coupling as opposed to vacuum electromagnetism where the equations of motion are linear and the soft and hard sectors are trivially decoupled (a result that is well known and one that we will recover in our analysis). At the same time however, this problem is much easier than studying the scattering problem in full, non-linear general relativity. By studying this toy model, we expect to be able to understand the effect of non-linearities and extrapolate the conclusion to the case of general relativity.

Before proceeding further, let us describe our set up in more detail and summarize our conclusions. We show how starting from the presymplectic form of electromagnetism coupled to a massless charged scalar field and systematically factoring out its degeneracy directions, the phase space, at (both past and future) null infinity[1], can be parameterized by two quantities. The first of these is the coefficient of the $1/r$ term in the expansion of the scalar field near null infinity,[2] denoted by $\chi_{+}(u, \theta^A)$ on $\mathscr{I}^+$ and $\chi_{-}(v, \theta^A)$ on $\mathscr{I}^-$. The second quantity corresponds to the angular components of the $O(r^0)$ piece of vector potential which we denote by $\mathcal{A}_{+A}(u, \theta^A)$ on $\mathscr{I}^+$ and $\mathcal{A}_{-A}(v, \theta^A)$ on $\mathscr{I}^-$. We split [3] $\mathcal{A}_{-A}(v, \theta^A)$ [and analogously $\mathcal{A}_{+A}(u, \theta^A)$] into a piece that goes to zero on the two ends of $\mathscr{I}^-$ plus two pieces that encode its values on the two ends of $\mathscr{I}^-$[see Eq. (6.2.46c)]. These two pieces (the average and difference of its limiting values) parameterize the "soft" degrees of freedom [see Eq. (6.2.48)] of the theory. This choice of terminology is inspired by the fact that the difference of the two limiting values of $\mathcal{A}_{-A}(v, \theta^A)$ is the zero frequency mode of the radiative data which is given by $\mathcal{F}_{-vA}$ while the average of the aforementioned limiting values is the mode conjugate to it (see Sec. 6.2.5). Note also that the average and difference of these asymptotic values are

[1] In the rest of this chapter, we will use null infinity to refer to both past and future null infinity unless explicitly stated otherwise.

[2] When referring to the fields discussed in this chapter, we will switch back and forth between the terms "degree of freedom," "field," "phase space coordinate," and "variable' . They are all intended to mean the same thing.

[3] The arbitrariness in this choice of splitting is discussed in Sec. 6.2.5.

the coefficients of the singular pieces at zero-frequency of the Fourier transform of $\mathcal{A}_{-A}(v,\theta^A)$. To see this, recall that the Fourier transform of a function, $f(v)$, that has non-zero limiting values as $v \to \pm\infty$ is given by $\tilde{f}(\omega) = \text{p.v.}\, \frac{i}{\omega}\tilde{h}(\omega) + \delta(\omega)\, avg[f]$, where "tilde" denotes Fourier transform, p.v. denotes the Cauchy principal value, $\text{avg}[f] := \lim_{v\to\infty} \frac{f(v)+f(-v)}{2}$ and $h(v) = \partial_v f(v)$. Variables which vanish in these asymptotic limits have no such singular pieces and can therefore be thought of as "hard" degrees of freedom. The piece of $\mathcal{A}_{-A}$ (and similarly $\mathcal{A}_{+A}$) that goes to zero on the two ends of $\mathscr{I}^-$ therefore counts as a "hard" degree of freedom. We will also take $\chi(v)$ to be compact support in v and therefore that too will be a "hard" degree of freedom in our analysis. Our goal is to study how the soft and hard degrees of freedom, as defined here, mix under evolution governed by equations of motion from $\mathscr{I}^-$ to $\mathscr{I}^+$. To set up the classical scattering problem in this theory, we consider an ansatz [see Eq. (6.3.13)] such that the equations of motion of the theory are source-free wave equations at leading order in the perturbative parameter that we call α. The equations of motion at order $O(\alpha^2)$ and higher, however, are non-linear and have non-zero source terms. To address the question of how soft and hard degrees of freedom mix under evolution, we first solve these equations explicitly to $O(\alpha^2)$. We find that at $O(\alpha)$ they evolve completely independently of each other. At $O(\alpha^2)$, we find that $\chi(u,\theta^A)|_{\mathscr{I}^+}$, computed in Lorenz gauge, picks up a term that represents an interaction between the soft and hard variables. We show, however, that there exists a redefinition of fields (thought of as phase space coordinates) under which this "mixing" of soft and hard degrees of freedom can be redefined away. We argue, however, that is just a feature of the scattering map at low orders in α. To show that this notion of "factorization" breaks down at higher orders, we then specialize our initial data to the case where only $\chi_{-}(v,\theta^A)$ is non-zero (and is a function of compact support in v) while $\mathcal{A}_{-A}$ is exactly zero [see Eq. (6.5.19)]. This corresponds to a scenario where the initial data is "purely hard." For this initial data, we re-compute the expression for $\chi(u,\theta^A)|_{\mathscr{I}^+}$. Using conservation of charge [see Eq. (6.3.24)], we then deduce that at $O(\alpha^4)$, $\Delta\mathcal{A}^{(4)}_{+A} := \mathcal{A}^{(4)}_A(u=+\infty,\theta^A) - \mathcal{A}^{(4)}_A(u=-\infty,\theta^A)$ is non-zero. This is a demonstration of coupling between the soft and hard degrees of freedom because it corresponds to a situation where a soft variable in the phase space at $\mathscr{I}^+$, namely $\Delta\mathcal{A}_{+A}$ is a function of initial data that is purely hard. We go on to then show that no perturbative redefinition of fields gets rid of this coupling of soft and hard variables.

The rest of this chapter is organized as follows. In Sec. 6.2, we analyze the asymptotic data and the phase space of this theory at past and future null infinity. We evaluate the presymplectic form and systematically mod out by degeneracies to obtain the non-degenerate symplectic form which gives Dirac brackets on the phase space of the theory. In Sec. 6.3 and Sec. 6.4, we compute the scattering map of this theory from past to future null infinity in Lorenz gauge. We compute this map to first and second orders in α. The details of these computations are given in Appendix A.14 and Appendix A.17 while the results, specialized to the case of $l = 1$ spherical harmonic for the vector potential and $l = 0, 1$ spherical harmonics for the scalar field, are cited in this chapter. The calculation of the scattering map at cubic and quartic orders in α and the failure of field redefinitions to remove the soft-hard coupling at quartic order is discussed in Sec. 6.5. Some calculations and results that supplement the discussion in this chapter are included in the appendices.

NB: In this chapter, we will use D_a to denote the gauge covariant derivative, $D_a := \nabla_a - iA_a$, h_{AB} to denote the unit two-sphere metric and D_A to denote the covariant derivative with respect to h_{AB}, which is a slight departure in notation from previous chapters.

6.2 | Asymptotic data and configuration phase space of electromagnetism coupled to a charged scalar field.

In this section we review the phase space and symplectic form of electromagnetism coupled to a charged scalar field in Minkowski spacetime, expressed in terms of data at past null infinity and at future null infinity. We closely follows the recent treatment of Strominger [23], specialized to a charged scalar source, but with one or two key modifications which we point out below. A treatment of the construction in a different language has been given by Ashtekar [132].

To construct this phase space, we would like to fix gauge degrees of freedom in order to obtain good coordinates on phase space and a non-degenerate symplectic form that can be inverted to obtain Dirac brackets. Gauge transformations can be divided into two categories, those which correspond to degeneracy directions of the presymplectic form (“true” gauge transformations) and those which do not. The latter category includes the so-called “large gauge transformations” that have non-trivial behavior at infinity and that correspond to the infinity of conserved charges in gauge theories discovered in the past few years [133–135]. Our strategy here will be to fix all of the true gauge degrees of freedom, but avoid fixing any of the gauge degrees of freedom that correspond to nondegenerate directions of the presymplectic form, since otherwise we would obscure the interesting new conservation laws.

1 Foundations

The action of the theory is

$$S = -\frac{1}{4e^2}\int d^4x\sqrt{-g}F_{ab}F^{ab} - \int d^4x\sqrt{-g}\,(D^a\Phi)^* D_a\Phi, \tag{6.2.1}$$

where A_a is the vector potential, Φ is a complex scalar field, e is the electric charge, $D_a = \nabla_a - iA_a$, and $F_{ab} = \nabla_a A_b - \nabla_b A_a$. The equations of motion are

$$\Box A_a - \nabla_b\nabla_a A^b = e^2 j_a = -ie^2(\Phi\nabla_a\Phi^* - \Phi^*\nabla_a\Phi) + 2e^2 A_a\Phi^*\Phi, \tag{6.2.2a}$$

$$\Box\Phi = 2iA^a\nabla_a\Phi + A^a A_a\Phi + i\Phi\nabla_a A^a. \tag{6.2.2b}$$

The theory is invariant under the local gauge transformations

$$A_a \to A_a + \nabla_a\varepsilon, \qquad \Phi \to e^{i\varepsilon}\Phi, \tag{6.2.3}$$

which may or may not be true gauge transformations.

We now consider asymptotic conditions near future null infinity $\mathscr{I}^+$. We use retarded coordinates $(u, r, \theta^1, \theta^2) = (u, r, \theta^A)$ in terms of which the metric is

$$ds^2 = -du^2 - 2dudr + r^2 h_{AB}d\theta^A d\theta^B, \tag{6.2.4}$$

where h_{AB} is the unit metric on the two-sphere. We assume the following asymptotic behavior of the components of the Maxwell tensor in the limit to $\mathscr{I}^+$, that is, $r \to \infty$ at fixed u:

$$\begin{aligned}
F_{ur} &= \frac{1}{r^2}\mathcal{F}_{+ur} + \frac{1}{r^3}\hat{\mathcal{F}}_{+ur} + O\left(\frac{1}{r^4}\right), && (6.2.5a)\\
F_{uA} &= \mathcal{F}_{+uA} + \frac{1}{r}\hat{\mathcal{F}}_{+uA} + O\left(\frac{1}{r^2}\right), && (6.2.5b)\\
F_{rA} &= \frac{1}{r^2}\mathcal{F}_{+rA} + \frac{1}{r^3}\hat{\mathcal{F}}_{+rA} + O\left(\frac{1}{r^4}\right), && (6.2.5c)\\
F_{AB} &= \mathcal{F}_{+AB} + \frac{1}{r}\hat{\mathcal{F}}_{+AB} + O\left(\frac{1}{r^2}\right). && (6.2.5d)
\end{aligned}$$

We are using a notational convention where caligraphic quantities are used to represent the pieces of fields that appear at leading order in an expansion in $1/r$, the $+$ subscripts denote quantities on $\mathscr{I}^+$, and hatted caligraphic quantities are subleading. The scalings of the leading terms can be deduced from the physical arguments given by Strominger [23], or from demanding smoothness of the solution on the conformal completion of the spacetime [132].

We assume the following asymptotic behavior of the vector potential and scalar field as $r \to \infty$ at fixed u, slightly more general than that of [23]:

$$\begin{aligned}
A_A &= \mathcal{A}_{+A} + \frac{1}{r}\hat{\mathcal{A}}_{+A} + O\left(\frac{1}{r^2}\right), && (6.2.6a)\\
A_u &= \mathcal{A}_{+u} + \frac{1}{r}\hat{\mathcal{A}}_{+u} + \frac{1}{r^2}\hat{\hat{\mathcal{A}}}_{+u} + O\left(\frac{1}{r^3}\right), && (6.2.6b)\\
A_r &= \frac{1}{r^2}\mathcal{A}_{+r} + \frac{1}{r^3}\hat{\mathcal{A}}_{+r} + O\left(\frac{1}{r^4}\right), && (6.2.6c)\\
\Phi &= \frac{1}{r}\chi_+ + \frac{1}{r^2}\hat{\chi}_+ + O\left(\frac{1}{r^3}\right). && (6.2.6d)
\end{aligned}$$

The expansion coefficients in the expansions (6.2.5) and (6.2.6) are then related by

$$\begin{aligned}
\mathcal{F}_{+ur} &= \partial_u\mathcal{A}_{+r} + \hat{\mathcal{A}}_{+u}, \quad \hat{\mathcal{F}}_{+ur} = \partial_u\hat{\mathcal{A}}_{+r} + 2\hat{\hat{\mathcal{A}}}_{+u}, && (6.2.7a)\\
\mathcal{F}_{+uA} &= \partial_u\mathcal{A}_{+A} - D_A\mathcal{A}_{+u}, \quad \hat{\mathcal{F}}_{+uA} = \partial_u\hat{\mathcal{A}}_{+A} - D_A\hat{\mathcal{A}}_{+u}, && (6.2.7b)\\
\mathcal{F}_{+rA} &= -D_A\mathcal{A}_{+r} - \hat{\mathcal{A}}_{+A}, && (6.2.7c)\\
\mathcal{F}_{+AB} &= D_A\mathcal{A}_{+B} - D_B\mathcal{A}_{+A}, && (6.2.7d)
\end{aligned}$$

where D_A is a covariant derivative with respect to the two-sphere metric h_{AB}.

The components of the current (6.2.2a) are

$$j_u = \mathcal{J}_{+u}/r^2 + O(r^{-3}), \quad j_r = \mathcal{J}_{+r}/r^4 + O(r^{-5}), \quad j_{+A} = \mathcal{J}_A/r^2 + O(r^{-3}), \tag{6.2.8}$$

where

$$\mathcal{J}_{+u} = -i(\chi_+\partial_u\chi_+^* - \chi_+^*\partial_u\chi_+) + 2\mathcal{A}_{+u}\chi_+^*\chi_+, \tag{6.2.9a}$$

$$\mathcal{J}_{+r} = i(\chi_+\hat{\chi}_+^* - \chi_+^*\hat{\chi}_+) + 2\mathcal{A}_{+r}\chi_+^*\chi_+, \tag{6.2.9b}$$

$$\mathcal{J}_{+A} = -i(\chi_+\partial_A\chi_+^* - \chi_+^*\partial_A\chi_+) + 2\mathcal{A}_{+A}\chi_+^*\chi_+. \tag{6.2.9c}$$

The leading order pieces of Maxwell's equations are [23]

$$\partial_u\mathcal{F}_{+ur} + D^A\mathcal{F}_{+Au} = e^2\mathcal{J}_{+u}, \tag{6.2.10a}$$

$$\hat{\mathcal{F}}_{+ur} + D^A\mathcal{F}_{+Ar} = e^2\mathcal{J}_{+r}, \tag{6.2.10b}$$

$$-\partial_u\mathcal{F}_{+rA} + \hat{\mathcal{F}}_{+uA} + D^C\mathcal{F}_{+CA} = e^2\mathcal{J}_{+A}, \tag{6.2.10c}$$

where $D^A = h^{AB}D_B$.

The expansion coefficients of the various fields transform under gauge transformations as follows. Under the gauge transformation (6.2.3) with

$$\varepsilon = \varepsilon_+ + \frac{1}{r}\hat{\varepsilon}_+ + \frac{1}{r^2}\hat{\hat{\varepsilon}}_+ + O\left(\frac{1}{r^3}\right), \tag{6.2.11}$$

we have

$$\mathcal{A}_{+u} \to \mathcal{A}_{+u} + \partial_u\varepsilon_+, \quad \hat{\mathcal{A}}_{+u} \to \hat{\mathcal{A}}_{+u} + \partial_u\hat{\varepsilon}_+, \quad \hat{\hat{\mathcal{A}}}_{+u} \to \hat{\hat{\mathcal{A}}}_{+u} + \partial_u\hat{\hat{\varepsilon}}_+, \tag{6.2.12a}$$

$$\mathcal{A}_{+A} \to \mathcal{A}_{+A} + D_A\varepsilon_+, \quad \hat{\mathcal{A}}_{+A} \to \hat{\mathcal{A}}_{+A} + D_A\hat{\varepsilon}_+, \tag{6.2.12b}$$

$$\mathcal{A}_{+r} \to \mathcal{A}_{+r} - \hat{\varepsilon}_+, \quad \hat{\mathcal{A}}_{+r} \to \hat{\mathcal{A}}_{+r} - 2\hat{\hat{\varepsilon}}_+, \tag{6.2.12c}$$

$$\chi_+ \to e^{i\varepsilon_+}\chi_+, \quad \hat{\chi}_+ \to e^{i\varepsilon_+}(\hat{\chi}_+ + i\hat{\varepsilon}_+\chi_+). \tag{6.2.12d}$$

A similar analysis can be carried out for the limiting behavior of the fields near past null infinity $\mathscr{I}^-$. We use advanced coordinates v, r, θ^A given by $v = u + 2r$. The expansion of the Maxwell tensor at $\mathscr{I}^-$, as $r \to \infty$ at fixed v, is similar to the expansion (6.2.5) and is given by

$$F_{vr} = \frac{1}{r^2}\mathcal{F}_{\text{-}vr} + \frac{1}{r^3}\hat{\mathcal{F}}_{\text{-}vr} + O\left(\frac{1}{r^4}\right), \tag{6.2.13a}$$

$$F_{vA} = \mathcal{F}_{\text{-}vA} + \frac{1}{r}\hat{\mathcal{F}}_{\text{-}vA} + O\left(\frac{1}{r^2}\right), \tag{6.2.13b}$$

$$F_{rA} = \frac{1}{r^2}\mathcal{F}_{\text{-}rA} + \frac{1}{r^3}\hat{\mathcal{F}}_{\text{-}rA} + O\left(\frac{1}{r^4}\right), \tag{6.2.13c}$$

$$F_{AB} = \mathcal{F}_{\text{-}AB} + \frac{1}{r}\hat{\mathcal{F}}_{\text{-}AB} + O\left(\frac{1}{r^2}\right). \tag{6.2.13d}$$

Here the subscripts - denote expansion coefficients of an expansion near $\mathscr{I}^-$. The corresponding expansions of the vector potential and scalar field are

$$A_A = \mathcal{A}_{\text{-}A} + \frac{1}{r}\hat{\mathcal{A}}_{\text{-}A} + O\left(\frac{1}{r^2}\right), \tag{6.2.14a}$$

$$A_v = \mathcal{A}_{\text{-}v} + \frac{1}{r}\hat{\mathcal{A}}_{\text{-}v} + \frac{1}{r^2}\hat{\hat{\mathcal{A}}}_{\text{-}v} + O\left(\frac{1}{r^3}\right), \tag{6.2.14b}$$

$$A_r = \frac{1}{r^2}\mathcal{A}_{\text{-}r} + \frac{1}{r^3}\hat{\mathcal{A}}_{\text{-}r} + O\left(\frac{1}{r^4}\right), \tag{6.2.14c}$$

$$\Phi = \frac{1}{r}\chi_{\text{-}} + \frac{1}{r^2}\hat{\chi}_{\text{-}} + O\left(\frac{1}{r^3}\right), \tag{6.2.14d}$$

and the gauge transformation parameter can be expanded as

$$\varepsilon = \varepsilon_{\text{-}} + \frac{1}{r}\hat{\varepsilon}_{\text{-}} + \frac{1}{r^2}\hat{\hat{\varepsilon}}_{\text{-}} + O\left(\frac{1}{r^3}\right). \tag{6.2.15}$$

2 Space of solutions of the field equations

We will restrict attention to field configurations on $\mathscr{I}^-$ which satisfy three conditions:

(1) The limits

$$\lim_{v\to\pm\infty} \mathcal{A}_{\text{-}A}, \qquad \lim_{v\to\pm\infty} \mathcal{A}_{\text{-}r}, \qquad \lim_{v\to\pm\infty} \mathcal{A}_{\text{-}v}, \tag{6.2.16}$$

exist, as functions on the two-sphere.

(2) The field $\mathcal{A}_{-A}$ satisfies the following fall off conditions near timelike infinity and spatial infinity

$$\partial_v \mathcal{A}_{-A} \sim \frac{1}{|v|^{1+\epsilon}}, \qquad v \to \pm\infty, \tag{6.2.17}$$

for some $\epsilon > 0$. This condition is sufficient to ensure the convergence of the symplectic form (6.2.43) below.

(3) The initial data χ_- for the scalar field falls off like[4]

$$\chi_- \sim 1/|v|, \qquad v \to \pm\infty. \tag{6.2.18}$$

We will assume that these conditions are preserved by the scattering process, so that solutions which obey these conditions at $\mathscr{I}^-$ also satisfy analogous conditions at $\mathscr{I}^+$.

We introduce the following notations for the limiting values of fields on $\mathscr{I}^-$ at past timelike infinity i^- and at spatial infinity i^0. For any function $f_- = f_-(v, \theta^A)$ defined on $\mathscr{I}^-$ we define

$$f_{\substack{-\\-}}(\theta^A) = \lim_{v\to-\infty} f_-(v,\theta^A), \qquad f_{\substack{-\\+}}(\theta^A) = \lim_{v\to\infty} f_-(v,\theta^A). \tag{6.2.19}$$

Similarly for functions f_+ defined on $\mathscr{I}^+$ we denote the limiting functions at i^+ and at i^0 by

$$f_{\substack{+\\+}}(\theta^A) = \lim_{u\to\infty} f_+(u,\theta^A), \qquad f_{\substack{+\\-}}(\theta^A) = \lim_{u\to-\infty} f_+(u,\theta^A). \tag{6.2.20}$$

We now assume that for solutions to the field equations we have the following behavior near past timelike infinity [23]:

$$\mathcal{F}_{\substack{-\\-}vr}(\theta^A) = 0, \tag{6.2.21a}$$
$$\mathcal{F}_{\substack{-\\-}rA}(\theta^A) = 0, \tag{6.2.21b}$$
$$\mathcal{F}_{\substack{-\\-}AB}(\theta^A) = 0, \tag{6.2.21c}$$
$$\hat{\chi}_{\substack{-\\-}}(\theta^A) = 0. \tag{6.2.21d}$$

[4] Our assumed fall off for the scalar field is stronger than that for the gauge field. We are excluding for simplicity any nontrivial soft behavior in the scalar sector.

These conditions can be derived if the initial data for χ_{-} is of compact support on $\mathscr{I}^-$. In this case there exists a neighborhood of i^- in which Φ vanishes, and so in that neighborhood A_a satisfies the homogeneous Maxwell equations with no sources. Solutions of these equations satisfy the conditions (6.2.21a) – (6.2.21c) (see Appendix A.14). Similar arguments can be given for the condition (6.2.21d). We will assume that the conditions (6.2.21) continue to be valid under the weaker assumption (6.2.18) on the behavior of χ_{-}; this should follow from continuity of the dependence of the solutions of the equations of motion on the initial data on $\mathscr{I}^-$. Conditions analogous to (6.2.21) at future timelike infinity i^+, namely

$$\mathcal{F}_{\substack{+\\+}ur}(\theta^A) = 0, \tag{6.2.22a}$$

$$\mathcal{F}_{\substack{+\\+}rA}(\theta^A) = 0, \tag{6.2.22b}$$

$$\mathcal{F}_{\substack{+\\+}AB}(\theta^A) = 0, \tag{6.2.22c}$$

$$\hat{\chi}_{\substack{+\\+}}(\theta^A) = 0, \tag{6.2.22d}$$

should similarly follow from the assumptions on the final data on $\mathscr{I}^+$ that we have discussed.

Another key property of the solutions is the validity of the matching conditions

$$\mathcal{F}_{\substack{+\\-}ur}(\theta^A) = \mathcal{P}_*\mathcal{F}_{\substack{-\\+}vr}(\theta^A), \tag{6.2.23a}$$

$$\mathcal{F}_{\pm AB}(\theta^A) = -\mathcal{P}_*\mathcal{F}_{\substack{-\\+}AB}(\theta^A). \tag{6.2.23b}$$

Here $\mathcal{P} : S^2 \to S^2$ is the antipodal inversion mapping given by $(\theta, \varphi) \to (\pi - \theta, \pi + \varphi)$, and $\mathcal{P}_*$ is the pullback $(\mathcal{P}_* f)(\theta, \varphi) = f(\pi - \theta, \pi + \varphi)$ for functions f. These identities are related to the existence of charges related to large gauge transformations and were discovered by Strominger [23, 136]. They were proven rigorously in Minkowski spacetime by Campiglia and Eyheralde [137], and generalized to all asymptotically flat spacetimes by Prabhu [9].

We next claim that solutions are determined up to gauge by specifying on $\mathscr{I}^-$ the initial data

$$\mathcal{A}_{-A},\ \mathcal{A}_{-v},\ \mathcal{A}_{-r}, \chi_{-}. \tag{6.2.24}$$

Subleading fields can be obtained from these fields from the analog on $\mathscr{I}^-$ of the asymptotic expansion (6.2.10) of Maxwell equations. For example, the subleading field $\hat{\mathcal{A}}_{-v}$ can be obtained from the leading fields (6.2.24) and from $\mathcal{F}_{-vr}$, from the analog of Eq. (6.2.7a) on $\mathscr{I}^-$. In turn, the field $\mathcal{F}_{-vr}$ can be obtained from its evolution equation, the analog on $\mathscr{I}^-$ of Eq. (6.2.10a), together with the initial condition (6.2.21a) at $v = -\infty$. Similarly $\hat{\mathcal{A}}_{-A}$ is obtained from $\mathcal{F}_{-rA}$ from the analogs of Eqs. (6.2.7c), the evolution equation (6.2.10c) and the initial condition (6.2.21b). Similar arguments apply to the subleading scalar field $\hat{\chi}_-$ using an expansion of the scalar field equation (6.2.2b).

Therefore we can take the four fields (6.2.24) on $\mathscr{I}^-$ to parameterize the phase space (up to gauge transformations which we discuss below). Similarly the fields

$$\mathcal{A}_{+A},\ \mathcal{A}_{+u},\ \mathcal{A}_{+r}, \chi_+ \tag{6.2.25}$$

on $\mathscr{I}^+$ also uniquely determine the solution, up to gauge transformations.

3 Presymplectic form

The presymplectic form obtained from the action (6.2.1) depends on a pair of linearized perturbations $(\delta_1 A_a, \delta_1\Phi)$ and $(\delta_2 A_a, \delta_2\Phi)$ about a background solution A_a, Φ [53]. It is given by the integral over any complete Cauchy surface Σ,

$$\Omega_\Sigma(A_a, \Phi; \delta_1 A_a, \delta_1\Phi, \delta_2 A_a, \delta_2\Phi) = \int_\Sigma \omega_{abc} \tag{6.2.26}$$

of the 3-form

$$\omega_{abc} = \frac{1}{e^2}\epsilon_{abcd}\delta_1 F^{df}\delta_2 A_f + \epsilon_{abcd}(D^d\delta_1\Phi^*\delta_2\Phi + D^d\delta_1\Phi\delta_2\Phi^*) - (1 \leftrightarrow 2). \tag{6.2.27}$$

Here the orientation of Σ is that determined by $t^a \epsilon_{abcd}$ where t^a is any future-pointing timelike vector field. We define[5]

$$\Omega_{\mathscr{I}^-} = -\int_{\mathscr{I}^-} \omega_{abc}, \qquad \Omega_{\mathscr{I}^+} = \int_{\mathscr{I}^+} \omega_{abc}, \tag{6.2.28}$$

the limiting integrals over past and future null infinity. The limits to $\mathscr{I}^+$ and $\mathscr{I}^-$ of ω_{abc} exist by virtue of our assumed expansions (6.2.5), (6.2.6), (6.2.13) and (6.2.14). We will discuss in Sec. 6.2.5 below the conditions necessary for the integrals (6.2.28) to converge and be finite.

Before considering the presymplectic forms (6.2.28) for general perturbations, it will be useful to first specialize to the case where the second perturbation is a pure gauge transformation,

$$\delta_1 A_a = \delta A_a, \qquad \delta_1 \Phi = \delta \Phi, \qquad \delta_2 A_a = \delta_\varepsilon A_a = \nabla_a \varepsilon, \qquad \delta_2 \Phi = \delta_\varepsilon \Phi = i\varepsilon\Phi, \tag{6.2.29}$$

In this case ω_{abc} is always exact [53], and here we have $\omega_{abc} = 3\nabla_{[a} Q_{bc]}$, where

$$Q_{ab} = \frac{1}{2e^2} \varepsilon \, \epsilon_{abcd} \, \delta F^{cd}. \tag{6.2.30}$$

We assume expansions of the form (6.2.11) for the gauge transformation function ε near $\mathscr{I}^+$ and $\mathscr{I}^-$, and we further assume that at each angle θ^A the leading order coefficients ε_+, ε_- asymptote to constants as $|u|$ or $|v|$ go to infinity, to be consistent the assumed asymptotic behavior of the fields at $i^\pm$ and i^0 discussed above Eq. (6.2.20). Following the notational conventions (6.2.19) and (6.2.20) these limiting values will be denoted $\varepsilon_{\substack{+\\+}}$, $\varepsilon_{\substack{+\\-}}$, $\varepsilon_{\substack{-\\+}}$ and $\varepsilon_{\substack{-\\-}}$. Converting the integrals over $\mathscr{I}^+$ and $\mathscr{I}^-$ to integrals over their boundaries using Eq. (6.2.30) we obtain

$$\begin{aligned}
\Omega_{\mathscr{I}^+} &= \frac{1}{e^2} \int d^2\Omega \left[-\delta \mathcal{F}_{\substack{+\\+} ru} \varepsilon_{\substack{+\\+}} + \delta \mathcal{F}_{\substack{+\\-} ru} \varepsilon_{\substack{+\\-}} \right], &\text{(6.2.31a)}\\
\Omega_{\mathscr{I}^-} &= \frac{1}{e^2} \int d^2\Omega \left[-\delta \mathcal{F}_{\substack{-\\-} rv} \varepsilon_{\substack{-\\-}} + \delta \mathcal{F}_{\substack{-\\+} rv} \varepsilon_{\substack{-\\+}} \right]. &\text{(6.2.31b)}
\end{aligned}$$

We now simplify using the asymptotic conditions (6.2.21a) and (6.2.22a), and the matching condition

[5] We insert a minus sign into the definition (6.2.28) of $\Omega_{\mathscr{I}^-}$ in order that Eq. (6.2.33) below with a plus sign follows from $0 = \int_M d\omega = \int_{\mathscr{I}^-} \omega + \int_{\mathscr{I}^+} \omega$, modulo the issues discussed in the rest of this subsection.

(6.2.23a). This gives

$$\Omega_{\mathscr{I}^+} - \Omega_{\mathscr{I}^-} = \frac{1}{e^2} \int d^2\Omega \delta \mathcal{F}_{\substack{+\\-}ru} \left[\varepsilon_{\substack{+\\-}} - \mathcal{P}_* \varepsilon_{\substack{-\\+}} \right]. \tag{6.2.32}$$

Now we would like to specialize our definition of the field configuration space to make

$$\Omega_{\mathscr{I}^+} = \Omega_{\mathscr{I}^-} \tag{6.2.33}$$

for general on-shell perturbations, that is, to make the presymplectic forms on past null infinity and future null infinity coincide. In other words, the scattering map from data at past null infinity to data at future null infinity should be a symplectomorphism (the classical version of unitarity in the quantum theory). From Eq. (6.2.32) we see that the condition (6.2.33) cannot be preserved under general transformations of the form $A_a \to A_a + \nabla_a \varepsilon$. In the next section we will discuss a specialization of the definition of the field configuration space suggested by Strominger [135] which makes the quantity (6.2.32) vanish, and thus removes the obstruction to achieving (6.2.33) for general perturbations.

4 | Gauge specialization and scattering map

We now fix some of the degrees of freedom that correspond to degeneracy directions of the presymplectic form (6.2.26), and so correspond to true gauge degrees of freedom. First, we use the free function ε_{+} to set $\mathcal{A}_{+u}$ to zero, using Eq. (6.2.12a). In doing so we specialize to $\varepsilon_{\substack{+\\-}} = \varepsilon_{+}(u = -\infty) = 0$, in order to correspond to a degeneracy direction of (6.2.31a), from Eq. (6.2.22a). Next, we use the free function $\hat{\varepsilon}_{+}$ to set $\mathcal{A}_{+r}$ to zero, from Eq. (6.2.12c). We perform similar specializations at $\mathscr{I}^-$. Summarizing, we have fixed the gauge so that

$$\mathcal{A}_{+u} = \mathcal{A}_{+r} = \mathcal{A}_{-v} = \mathcal{A}_{-r} = 0. \tag{6.2.34}$$

The remaining gauge freedom that acts on the data on $\mathscr{I}^+$ and $\mathscr{I}^-$ consists of functions $\varepsilon_{+} = \varepsilon_{+}(\theta^A)$ and $\varepsilon_{-} = \varepsilon_{-}(\theta^A)$ that are functions of angle only. In particular we have $\varepsilon_{\substack{+\\-}} = \varepsilon_{\substack{+\\+}} = \varepsilon_{+}$ and $\varepsilon_{\substack{-\\+}} = \varepsilon_{\substack{-\\-}} = \varepsilon_{-}$.

We define the even and odd linear combinations of these gauge transformations via

$$\begin{aligned}\varepsilon_{\rm e} &= \frac{1}{2}(\varepsilon_{+} + \mathcal{P}_{*}\varepsilon_{-}), &\quad (6.2.35a)\\ \varepsilon_{\rm o} &= \frac{1}{2}(\varepsilon_{+} - \mathcal{P}_{*}\varepsilon_{-}). &\quad (6.2.35b)\end{aligned}$$

We next decompose the fields $\mathcal{A}_{+A}$ and $\mathcal{A}_{-A}$ into electric and magnetic parity pieces on the two sphere, as

$$\begin{aligned}\mathcal{A}_{+A} &= D_A\Psi^{\rm e}_{+} + \varepsilon_{AB}h^{BC}D_C\Psi^{\rm m}_{+}, &\quad (6.2.36a)\\ \mathcal{A}_{-A} &= D_A\Psi^{\rm e}_{-} + \varepsilon_{AB}h^{BC}D_C\Psi^{\rm m}_{-}, &\quad (6.2.36b)\end{aligned}$$

which determines the potentials $\Psi^{\rm e}_{+}$ etc. up to their $l = 0$ parts which we take to vanish. Under the gauge transformation given by Eqs. (6.2.11) and (6.2.15) we have

$$\begin{aligned}\Psi^{\rm e}_{-} &\to \Psi^{\rm e}_{-} + \varepsilon_{-}, \quad \Psi^{\rm e}_{+} \to \Psi^{\rm e}_{+} + \varepsilon_{+}, &\quad (6.2.37a)\\ \Psi^{\rm m}_{-} &\to \Psi^{\rm m}_{-}, \quad \Psi^{\rm m}_{+} \to \Psi^{\rm m}_{+}, &\quad (6.2.37b)\end{aligned}$$

assuming that ε_{-} and ε_{+} have no $l = 0$ pieces.

Finally, following Strominger [135], we specialize our definition of the field configuration space by imposing the condition[6]

$$\Psi^{\rm e}_{\pm} = \mathcal{P}_{*}\Psi^{\rm e}_{\mp}, \quad (6.2.39)$$

which can be achieved by using the odd gauge transformation (6.2.35b), from Eqs. (6.2.37a) and (6.2.36). The condition (6.2.39) then eliminates the odd gauge transformation freedom, and consequently the quantity (6.2.32) vanishes as desired, removing the obstruction to the scattering symplectomorphism property (6.2.33) discussed in the last section. Although not included in this thesis, in an appendix of [8], this result is generalized and it is shown that the symplectic forms on

[6] Note that the corresponding relation for the magnetic potentials $\Psi^{\rm m}$,

$$\Psi^{\rm m}_{\pm} = \mathcal{P}_{*}\Psi^{\rm m}_{\mp}, \quad (6.2.38)$$

is a consequence of the matching condition (6.2.23b) together with Eqs. (6.2.7d) and (6.2.36).

$\mathscr{I}^-$ and $\mathscr{I}^+$ coincide in general, and not just for field variations of the form (6.2.29). It is also shown there that the condition (6.2.39) follows from imposing a Lorenz-like gauge condition on the fields to leading order in an expansion around spatial infinity.

The condition (6.2.39) is called a "matching condition" in Refs. [127, 135]. However its status is very different from that of the matching conditions (6.2.23), being a specialization of the definition of the configuration space rather than a property of generic physical solutions. Note also that it is not accurate to call it a gauge specialization, even though it is enforced by making use of the transformations (6.2.35b), since true gauge transformations are defined in terms of degeneracy directions of the presymplectic form, and the specialization (6.2.39) is needed before a unique presymplectic form can be defined.

The condition (6.2.39) together with the gauge fixing (6.2.34) determines a unique gauge for initial data on $\mathscr{I}^-$ and final data on $\mathscr{I}^+$ which we will call the *preferred asymptotic gauge*. From Eq. (6.2.24), the space Γ_- of initial data in this gauge consists of the pairs $(\mathcal{A}_{-A}, \chi_{-})$ on $\mathscr{I}^-$. Similarly the space Γ_+ of final data consists of the pairs $(\mathcal{A}_{+A}, \chi_{+})$ on $\mathscr{I}^+$.

We do not fix the even transformation freedom (6.2.35a), as it corresponds to a non-degenerate direction of the presymplectic form, so it is a physical symmetry transformation rather than a gauge freedom. The corresponding charges are the new conserved charges of [23]. Specifically, the variation in the charge is given by [cf. Eq. (8) of Ref. [53]] $\delta Q_\varepsilon = \Omega(A_a, \Phi; \delta A_a, \delta\Phi, \delta_\varepsilon A_a, \delta_\varepsilon \Phi)$, and using Eqs. (6.2.31) and (6.2.35a) and integrating in phase space gives [23]

$$Q_\varepsilon = \frac{1}{e^2}\int d^2\Omega \mathcal{F}_{\pm ru}\varepsilon_{\pm} = \frac{1}{e^2}\int d^2\Omega \mathcal{F}_{\mp rv}\varepsilon_{\mp}. \tag{6.2.40}$$

There are also magnetic conserved charges analogous to the electric charges (6.2.40) [136], which we review in Appendix A.16. For simplicity however, we restrict attention to the sector of the theory where all the magnetic charges (A.16.4) vanish, which by Eqs. (6.2.7d), (6.2.36), (A.16.5) and (A.16.6) is equivalent to the requirement that

$$\Psi^{\mathrm{m}}_{\pm} = \Psi^{\mathrm{m}}_{\mp} = 0. \tag{6.2.41}$$

To summarize, the field configuration space of the theory is given by the set of fields that obey the asymptotic conditions (6.2.6) and (6.2.14), the matching condition (6.2.39), the vanishing magnetic charges condition (6.2.41), and for which the initial data on $\mathscr{I}^-$ obeys the conditions (6.2.16) – (6.2.18) and the final data satisfies analogous conditions on $\mathscr{I}^+$. The phase space Γ of the theory is the on-shell subspace of the field configuration space, modded out by degeneracy directions of the presymplectic form [112], which correspond to gauge transformations that act trivially on the boundaries $\mathscr{I}^-$ and $\mathscr{I}^+$. This phase space is in one-to-one correspondence with the space Γ_- of initial data $(\mathcal{A}_{\text{-}A}, \chi_{\text{-}})$ on $\mathscr{I}^-$ in the preferred asymptotic gauge. It is similarly in one-to-one correspondence with the space Γ_+ of final data $(\mathcal{A}_{\text{+}A}, \chi_{\text{+}})$ on $\mathscr{I}^+$ in the preferred asymptotic gauge. In the remainder of the chapter we will be concerned with properties of the scattering map

$$\mathcal{S} : \Gamma_- \to \Gamma_+ \tag{6.2.42}$$

which is a symplectomorphism. While our phase space, Γ, is well defined, it will be useful to consider a larger phase space, Γ_{extended} where the matching condition Eq. (6.2.39) is not satisfied. As we show in Appendix A.14, Lorenz gauge solutions lie in this extended phase space. We will refer to the evolution map from $\Gamma_{-_{\text{extended}}}$ and $\Gamma_{+_{\text{extended}}}$ as a "scattering map" in the sense that it maps a phase space on $\mathscr{I}^-$ to a phase space on $\mathscr{I}^+$. However, as shown in Eq. (6.2.32), when the matching condition Eq. (6.2.39) is not satisfied, the symplectic forms on $\mathscr{I}^-$ and $\mathscr{I}^+$ do not coincide and so, strictly speaking, the scattering map is only a map between Γ_- and Γ_+, as depicted in Eq. (6.2.42).

5 Soft and hard variables and Poisson brackets

The presymplectic form (6.2.26) when evaluated on the space Γ_- of initial data $(\mathcal{A}_{\text{-}A}, \chi_{\text{-}})$ is now non-degenerate, so from now on we will call it the symplectic form. It can be written as

$$\begin{aligned}\Omega_{\mathscr{I}^-}(A_a, \Phi; \delta_1 A_a, \delta_1\Phi, \delta_2 A_a, \delta_2\Phi) &= \int dv \int d^2\Omega \left[\frac{1}{e^2} h^{AB} \partial_v \delta_1 \mathcal{A}_{\text{-}A} \delta_2 \mathcal{A}_{\text{-}B} - (1 \leftrightarrow 2)\right] \\ &+ \int dv \int d^2\Omega \left[\partial_v \delta_1 \chi^*_{\text{-}} \delta_2 \chi_{\text{-}} + \partial_v \delta_1 \chi_{\text{-}} \delta_2 \chi^*_{\text{-}} - (1 \leftrightarrow 2)\right].\end{aligned} \tag{6.2.43}$$

In terms of the potentials $\Psi^{\rm e}$ and $\Psi^{\rm m}$ defined in Eq. (6.2.36) the symplectic form is

$$\Omega_{\mathscr{I}^-} = -\frac{1}{e^2}\int dv \int d^2\Omega \left[\partial_v \Psi^{\rm e}_{1\,-} D^2 \Psi^{\rm e}_{2\,-} + \partial_v \Psi^{\rm m}_{1\,-} D^2 \Psi^{\rm m}_{2\,-} - (1 \leftrightarrow 2)\right]$$
$$+ \int dv \int d^2\Omega \left[\partial_v \delta_1 \chi^*_- \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi^*_- - (1 \leftrightarrow 2)\right]. \qquad (6.2.44)$$

Here for simplicity we have written $\Psi^{\rm e}_1$ instead of $\delta_1 \Psi^{\rm e}$ etc.

We next make a change of coordinates on Γ_-, from the set $\left[\Psi^{\rm e}_-(v,\theta^A), \Psi^{\rm m}_-(v,\theta^A), \chi_-(v,\theta^A)\right]$ to a new set

$$\left[\tilde{\Psi}^{\rm e}_-(v,\theta^A), \Delta\Psi^{\rm e}_-(\theta^A), \overline{\Psi}^{\rm e}_-(\theta^A), \Psi^{\rm m}_-(v,\theta^A), \chi_-(v,\theta^A)\right] \qquad (6.2.45)$$

defined as follows. We pick a smooth function $g(v)$ which increases monotonically from $g(-\infty) = -1/2$ to $g(\infty) = 1/2$. We define

$$\Delta\Psi^{\rm e}_- = \Psi^{\rm e}_{\substack{- \\ +}} - \Psi^{\rm e}_{\substack{- \\ -}}, \qquad (6.2.46a)$$

$$\overline{\Psi}^{\rm e}_- = \frac{1}{2}\left(\Psi^{\rm e}_{\substack{- \\ +}} + \Psi^{\rm e}_{\substack{- \\ -}}\right), \qquad (6.2.46b)$$

$$\tilde{\Psi}^{\rm e}_-(v,\theta^A) = \Psi^{\rm e}_-(v,\theta^A) - \overline{\Psi}^{\rm e}_-(\theta^A) - g(v)\Delta\Psi^{\rm e}_-(\theta^A). \qquad (6.2.46c)$$

Note that it follows from these definitions that

$$\tilde{\Psi}^{\rm e}_{\substack{- \\ +}} = \tilde{\Psi}^{\rm e}_{\substack{- \\ -}} = 0. \qquad (6.2.47)$$

We similarly define the phase space variables

$$\left[\tilde{\Psi}^{\rm e}_+(u,\theta^A), \Delta\Psi^{\rm e}_+(\theta^A), \overline{\Psi}^{\rm e}_+(\theta^A), \Psi^{\rm m}_+(u,\theta^A), \chi_+(v,\theta^A)\right] \qquad (6.2.48)$$

at future null infinity $\mathscr{I}^+$, using the same function g. Following the terminology of Ref. [23] we will call the quantities χ_-, $\tilde{\Psi}^{\rm e}_-$, and $\Psi^{\rm m}_-$ which depend on v "hard" variables, and the quantities $\overline{\Psi}^{\rm e}_-$, $\Delta\Psi^{\rm e}_-$ which do not depend on v "soft" variables. As discussed at the start of this chapter, these are related to the coefficients of the singular pieces in the fourier transform of $\mathcal{A}_A$.

It may seem strange that we have to introduce an arbitrary function $g(v)$ in order to to separate out the soft variables from the hard variables. However, it is necessary to make such a choice in order to get a complete separation. Of course, the choice of $g(v)$ is arbitrary and no observable quantities will depend on this choice. Any two phase space coordinate systems corresponding to two different choices of $g(v)$ are related by a symplectomorphism [see Eqs. (6.2.51) below, which are independent of g].

We note that a transformation of phase space variables similar to our transformation (6.2.46) was used in Refs. [23, 127], except that those authors did not include the third term on the right hand side of Eq. (6.2.46c). As a consequence, their variables are not independent, obeying the constraint (in our notation) of $\Delta\Psi^{\rm e}_{-} = \tilde{\Psi}^{\rm e}_{-}(u=\infty) - \tilde{\Psi}^{\rm e}_{-}(u=-\infty)$. This lack of independence of phase space coordinates is the key reason why the results on the coupling of hard and soft degrees of freedom in [127] differ from the ones presented here. Rewriting the symplectic form (6.2.44) in terms of the new variables (6.2.45) gives

$$\begin{aligned} \Omega_{\mathscr{I}^-} \quad = \quad & -\frac{1}{e^2}\int dv \int d^2\Omega \left[\partial_v \tilde{\Psi}^{\rm e}_{1\,-} D^2 \tilde{\Psi}^{\rm e}_{2\,-} + \partial_v \Psi^{\rm m}_{1\,-} D^2 \Psi^{\rm m}_{2\,-} - (1 \leftrightarrow 2)\right] \\ & -\frac{1}{e^2}\int d^2\Omega \left\{ D^2 \Delta\Psi^{\rm e}_{1\,-} \left[\overline{\Psi}^{\rm e}_{2\,-} + 2\int dv g' \tilde{\Psi}^{\rm e}_{2-}\right] - (1 \leftrightarrow 2)\right\} \\ & + \int dv \int d^2\Omega \left[\partial_v \delta_1 \chi^*_- \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi^*_- - (1 \leftrightarrow 2)\right]. \end{aligned} \tag{6.2.49}$$

We see that the soft and hard phase space variables (6.2.45) are not symplectically orthogonal, that is, there are nonvanishing Poisson brackets between the hard and soft variables. This can be remedied by defining the new soft variable

$$\hat{\Psi}^{\rm e}_{-}(\theta^A) = \overline{\Psi}^{\rm e}_{-}(\theta^A) + 2\int dv g'(v) \tilde{\Psi}^{\rm e}_{-}(v, \theta^A). \tag{6.2.50}$$

Then the soft variables $\Delta\Psi^{\rm e}_{-}(\theta^A)$, $\hat{\Psi}^{\rm e}_{-}(\theta^A)$ and the hard variables $\tilde{\Psi}^{\rm e}_{-}(v,\theta^A), \Psi^{\rm m}_{-}(v,\theta^A), \chi_{-}(v,\theta^A)$ are symplectically orthogonal, from Eq. (6.2.49). The corresponding nonzero Poisson brackets can be obtained from Eqs. (6.2.49) and (6.2.50) and are [7]

$$\left\{\tilde{\Psi}^{\mathrm{e}}_{-}(v,\theta)\ ,\ D^2\tilde{\Psi}^{\mathrm{e}}_{-}(v',\theta')\right\} = \frac{e^2}{2}\left[\Theta(v-v')-\frac{1}{2}\right]\left[\delta^{(2)}(\theta,\theta')-\frac{1}{4\pi}\right], \tag{6.2.51a}$$

$$\left\{\Psi^{\mathrm{m}}_{-}(v,\theta)\ ,\ D^2\Psi^{\mathrm{m}}_{-}(v',\theta')\right\} = \frac{e^2}{2}\left[\Theta(v-v')-\frac{1}{2}\right]\left[\delta^{(2)}(\theta,\theta')-\frac{1}{4\pi}\right], \tag{6.2.51b}$$

$$\left\{\hat{\Psi}^{\mathrm{e}}_{-}(\theta)\ ,\ D^2\Delta\Psi^{\mathrm{e}}_{-}(\theta')\right\} = -e^2\left[\delta^{(2)}(\theta,\theta')-\frac{1}{4\pi}\right], \tag{6.2.51c}$$

$$\left\{\chi_{-}(v,\theta)\ ,\chi^*_{-}(v',\theta')\right\} = -\frac{1}{2}\left[\Theta(v-v')-\frac{1}{2}\right]\delta^{(2)}(\theta,\theta'). \tag{6.2.51d}$$

Here $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$, and $\delta^{(2)}$ is the covariant delta function on the unit sphere. The formulae (6.2.51a) – (6.2.51c) contain factors of D^2 inside the Poisson brackets, but they determine the corresponding formulae without factors of D^2 since all of the functions of θ (except χ_{-}) have no $l=0$ components.

Note that it follows from Eqs. (6.2.46) and (6.2.51) that the field $\Psi^{\mathrm{e}}_{-}(v,\theta)$ satisfies the same Poisson bracket relation (6.2.51a) as the field $\tilde{\Psi}^{\mathrm{e}}_{-}(v,\theta)$. However, one cannot replace Eqs. (6.2.51a) and (6.2.51c) with the single equation (6.2.51a) with $\tilde{\Psi}^{\mathrm{e}}_{-}(v,\theta)$ replaced by $\Psi^{\mathrm{e}}_{-}(v,\theta)$, because limits $u\to\pm\infty$ of the Poisson brackets do not coincide with Poisson brackets of limits [23]. We remark that the soft degrees of freedom $(\hat{\Psi}^{\mathrm{e}}_{-},\Delta\Psi^{\mathrm{e}}_{-})$ can alternatively be described in the language of edge modes [138–141], as was conjectured in Ref. [23]. This equivalence is outlined in an appendix of [8].

Turn now to the situation at future null infinity $\mathscr{I}^+$. An analogous analysis yields versions of the formulae (6.2.49) – (6.2.51) with v replaced by u everywhere, and with the subscripts $+$ replaced by subscripts $-$.

Finally, we note that the main differences between the construction of the phase space given here and previous treatments [19, 23, 132] are as follows:

- Ashtekar [19, 132] constructs the phase space by imposing $\Psi^{\mathrm{e}}_{+\atop -}=\Psi^{\mathrm{e}}_{-\atop +}=0$ instead of the matching condition (6.2.39) suggested by Strominger. This eliminates one of the physical degrees of freedom and also the symmetry that underlies the conservation laws (6.2.40). Including the

[7] These Poisson brackets agree with those of Ref. [23] but not those of Ref. [19]. We believe that the right hand side of Eq. (C.5) of Ref. [19] should be proportional to the derivative of a delta function if instead of a step function.

extra degree of freedom will modify the character of the quantum theory constructed in [19]. Note also that as pointed out in [135] (and as can be deduced from Eq. 6.2.51c), the existence of two conjugate soft degrees of freedom is necessary for the action of the charge to generate the symmetry on phase space (see, e.g, Eq. 4.10 of [135]).

- In fixing the gauge we restrict to degeneracy directions of the presymplectic form, and demonstrate that the gauge conditions (6.2.34) used are degeneracy directions[8].

- In separating out hard and soft variables, we use a coordinate system on phase space in which all the variables are independent, unlike the set of variables in Refs. [23, 127]. This change does not affect the computation of Poisson brackets, but will be important in the decoupling discussion below.

6.3 | Foundations for computation of classical scattering map

Having completed our analysis of the symmetries, charges and asymptotics of the theory, and the definition of the phase space, we now turn to an exploration of the dynamics of the theory in the deep infrared. Our goal is to determine the extent to which the conserved charges (6.2.40) constrain in a nontrivial way the dynamics of the theory. To do this, we compute the scattering map (6.2.42) in perturbation theory. This section will derive the general formalism and compute the linear order scattering, and the following sections will address higher order scattering.

1 | Asymptotic field expansions in a more general class of gauges

It will be convenient to use Lorenz gauge for our explicit computations, since the equations of motion reduce to simple wave equations in this gauge. However, the form (6.2.6) of the asymptotic expansions that we have assumed are insufficiently general for this purpose, as Lorenz gauges generically requires logarithmic terms when sources are present [142, 143]. Indeed, starting from the expansions (6.2.6)

[8] An exception is the gauge condition (6.2.39) that we have adopted which is not a degeneracy direction of the presymplectic forms (6.2.28). However $\Omega_{\mathscr{I}^+}$ and $\Omega_{\mathscr{I}^-}$ do not coincide until after this condition is imposed, so one can argue that (6.2.28) is not the correct presymplectic form until after the condition is imposed.

we obtain

$$\begin{aligned}
\nabla_a A^a &= -\frac{2}{r}\mathcal{A}_{+u} + \frac{1}{r^2}\left[-\partial_u \mathcal{A}_{+r} - \hat{\mathcal{A}}_{+u} + D^A \mathcal{A}_{+A}\right] + O\left(\frac{1}{r^3}\right) \\
&= -\frac{2}{r}\mathcal{A}_{+u} + \frac{1}{r^2}\left[e^2 \int_u^\infty du' \mathcal{J}_{+u}(u') + D^A \mathcal{A}_{\ddagger A} - \int_u^\infty du' D^2 \mathcal{A}_{+u}\right] + O\left(\frac{1}{r^3}\right), \quad (6.3.1)
\end{aligned}$$

where we have used the asymptotic Maxwell's equations (6.2.10) and the boundary conditions (6.2.22) to rewrite the coefficient of the $1/r^2$ term. Thus our assumed expansions are incompatible with Lorenz gauges, since when the first term in Eq. (6.3.1) vanishes the second term is generically nonzero and cannot be made to vanish using the gauge transformations (6.2.12).

We therefore generalize the form of the expansion (6.2.6) to encompass Lorenz gauges, following Refs. [142, 143]. In the limit to $\mathscr{I}^+$, we assume

$$A_A = \mathcal{A}_{+A} - \frac{\ln r}{r} D_A \tilde{\mathcal{A}}_{+r} + \frac{1}{r}\hat{\mathcal{A}}_{+A} - \frac{\ln r}{2r^2} D_A \tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^2}\hat{\hat{\mathcal{A}}}_{+A} + O\left(\frac{\ln r}{r^3}\right), \quad (6.3.2a)$$

$$A_u = \mathcal{A}_{+u} - \frac{\ln r}{r}\partial_u \tilde{\mathcal{A}}_{+r} + \frac{1}{r}\hat{\mathcal{A}}_{+u} - \frac{\ln r}{2r^2}\partial_u \tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^2}\hat{\hat{\mathcal{A}}}_{+u} + O\left(\frac{\ln r}{r^3}\right), \quad (6.3.2b)$$

$$A_r = \frac{Q}{r} + \frac{\ln r}{r^2}\tilde{\mathcal{A}}_{+r} + \frac{1}{r^2}\mathcal{A}_{+r} + \frac{\ln r}{r^3}\tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^3}\hat{\mathcal{A}}_{+r} + O\left(\frac{\ln r}{r^4}\right), \quad (6.3.2c)$$

$$\Phi = e^{iQ\ln r}\left[\frac{1}{r}\chi_+ - i\frac{\ln r}{r^2}\tilde{\mathcal{A}}_{+r}\,\chi_+ + \frac{1}{r^2}\hat{\chi}_+ + O\left(\frac{1}{r^3}\right)\right]. \quad (6.3.2d)$$

Here the notational conventions are as follows. Caligraphic font quantities are coefficients in the double expansion in $1/r$ and $\ln r/r$, functions of u and θ^A. The quantities $\mathcal{A}_{+A}$, $\mathcal{A}_{+u}$ and $\mathcal{A}_{+r}$ without any tildes or carets are the leading order fields discussed in Sec. 6.2.1 above. Quantities with one or more carets like $\hat{\mathcal{A}}_{+r}$ are coefficients of subleading terms in the $1/r$ expansion, while quantities with one or more tildes like $\tilde{\mathcal{A}}_{+r}$ are coefficients of log terms, new in this section. Finally Eqs. (6.3.2c) and (6.3.2d) depend on a quantity Q which is a constant (the total ingoing or outgoing charge multiplied by 4π).

The specific relations between the coefficients of the log terms in Eqs. (6.3.2) are chosen to ensure that the expansions (6.2.5) of the Maxwell tensor and (6.2.8) of the current are still valid in this context. The formulae (6.2.7) for the expansion coefficients are replaced by

$$\begin{aligned}
\mathcal{F}_{+ur} &= \partial_u \mathcal{A}_{+r} + \partial_u \tilde{\mathcal{A}}_{+r} + \hat{\mathcal{A}}_{+u}, \quad \hat{\mathcal{F}}_{+ur} = \partial_u \hat{\mathcal{A}}_{+r} + 2\hat{\hat{\mathcal{A}}}_{+u} + \frac{1}{2}\partial_u \tilde{\tilde{\mathcal{A}}}_{+r}, && \text{(6.3.3a)}\\
\mathcal{F}_{+uA} &= \partial_u \mathcal{A}_{+A} - D_A \mathcal{A}_{+u}, \quad \hat{\mathcal{F}}_{+uA} = \partial_u \hat{\mathcal{A}}_{+A} - D_A \hat{\mathcal{A}}_{+u}, \quad \hat{\hat{\mathcal{F}}}_{+uA} = \partial_u \hat{\hat{\mathcal{A}}}_{+A} - D_A \hat{\hat{\mathcal{A}}}_{+u}, && \text{(6.3.3b)}\\
\mathcal{F}_{+rA} &= -D_A \mathcal{A}_{+r} - \hat{\mathcal{A}}_{+A} - D_A \tilde{\mathcal{A}}_{+r}, \quad \hat{\mathcal{F}}_{+rA} = -D_A \hat{\mathcal{A}}_{+r} - 2\hat{\hat{\mathcal{A}}}_{+A} - \frac{1}{2} D_A \tilde{\tilde{\mathcal{A}}}_{+r}, && \text{(6.3.3c)}\\
\mathcal{F}_{+AB} &= D_A \mathcal{A}_{+B} - D_B \mathcal{A}_{+A}, \quad \hat{\mathcal{F}}_{+AB} = D_A \hat{\mathcal{A}}_{+B} - D_B \hat{\mathcal{A}}_{+A}. && \text{(6.3.3d)}
\end{aligned}$$

The gauge transformation expansion (6.2.11) is replaced by the more general version

$$\varepsilon = \delta Q \ln r + \varepsilon_+ + \frac{\ln r}{r}\tilde{\varepsilon}_+ + \frac{1}{r}\hat{\varepsilon}_+ + \frac{\ln r}{r^2}\tilde{\tilde{\varepsilon}}_+ + \frac{1}{r^2}\hat{\hat{\varepsilon}}_+ + O\left(\frac{1}{r^3}\right), \qquad \text{(6.3.4)}$$

under which we have

$$\begin{aligned}
Q &\to Q + \delta Q, && \text{(6.3.5a)}\\
\mathcal{A}_{+u} &\to \mathcal{A}_{+u} + \partial_u \varepsilon_+, \quad \hat{\mathcal{A}}_{+u} \to \hat{\mathcal{A}}_{+u} + \partial_u \hat{\varepsilon}_+, \quad \hat{\hat{\mathcal{A}}}_{+u} \to \hat{\hat{\mathcal{A}}}_{+u} + \partial_u \hat{\hat{\varepsilon}}_+, && \text{(6.3.5b)}\\
\mathcal{A}_{+A} &\to \mathcal{A}_{+A} + D_A \varepsilon_+, \quad \hat{\mathcal{A}}_{+A} \to \hat{\mathcal{A}}_{+A} + D_A \hat{\varepsilon}_+, \quad \hat{\hat{\mathcal{A}}}_{+A} \to \hat{\hat{\mathcal{A}}}_{+A} + D_A \hat{\hat{\varepsilon}}_+, && \text{(6.3.5c)}\\
\mathcal{A}_{+r} &\to \mathcal{A}_{+r} + \tilde{\varepsilon}_+ - \hat{\varepsilon}_+, \quad \hat{\mathcal{A}}_{+r} \to \hat{\mathcal{A}}_{+r} + \tilde{\tilde{\varepsilon}}_+ - 2\hat{\hat{\varepsilon}}_+, && \text{(6.3.5d)}\\
\tilde{\mathcal{A}}_{+r} &\to \tilde{\mathcal{A}}_{+r} - \tilde{\varepsilon}_+, \quad \tilde{\tilde{\mathcal{A}}}_{+r} \to \tilde{\tilde{\mathcal{A}}}_{+r} - 2\tilde{\tilde{\varepsilon}}_+, && \text{(6.3.5e)}\\
\chi_+ &\to e^{i\varepsilon_+}\chi_+, \quad \hat{\chi}_+ \to e^{i\varepsilon_+}(\hat{\chi}_+ + i\hat{\varepsilon}_+\chi_+). && \text{(6.3.5f)}
\end{aligned}$$

The Lorenz gauge condition can be written using the expansion (6.3.2) as

$$\begin{aligned}
0 &= \nabla_a A^a = \frac{1}{r}\left[-2\mathcal{A}_{+u}\right] + \frac{1}{r^2}\left[Q - \mathcal{A}_{+r,u} - \hat{\mathcal{A}}_{+u} + D^A \mathcal{A}_{+A} + \tilde{\mathcal{A}}_{+r,u}\right]\\
&\quad + \frac{\ln r}{r^3}\left[-\tilde{\tilde{\mathcal{A}}}_{+r,u} - D^2 \tilde{\mathcal{A}}_{+r}\right] + \frac{1}{r^3}\left[\tilde{\mathcal{A}}_{+r} - \hat{\mathcal{A}}_{+r,u} - \hat{\mathcal{A}}_{+u} + D^A \hat{\mathcal{A}}_{+A} + \frac{1}{2}\tilde{\tilde{\mathcal{A}}}_{+r,u}\right]\\
&\quad + O\left(\frac{\ln r}{r^4}\right), && \text{(6.3.6)}
\end{aligned}$$

where $D^2 = h^{AB} D_A D_B$. Setting the coefficients of this expansion to zero gives four conditions for Lorenz gauge, and starting from a general gauge of the form (6.3.2) one can check that it is possible to use the transformation freedom (6.3.5) to satisfy these conditions.

2 Transformation to preferred asymptotic gauge

We now discuss the transformation from Lorenz gauge to the preferred asymptotic gauge discussed in Sec. 6.2 above, which is defined by the expansion (6.2.6) and the conditions (6.2.34) and (6.2.39). We start from an expansion of the form (6.3.2), specialized to Lorenz gauge (which implies $\mathcal{A}_{+u} = 0$):

$$\begin{aligned}
\underline{A}_A &= \underline{\mathcal{A}}_{+A} - \frac{\ln r}{r} D_A \underline{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r}\underline{\hat{\mathcal{A}}}_{+A} - \frac{\ln r}{2r^2} D_A \underline{\tilde{\tilde{\mathcal{A}}}}_{+r} + \frac{1}{r^2}\underline{\hat{\hat{\mathcal{A}}}}_{+A} + O\left(\frac{\ln r}{r^3}\right), && (6.3.7a)\\
\underline{A}_u &= \frac{\ln r}{r}\partial_u \underline{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r}\underline{\hat{\mathcal{A}}}_{+u} - \frac{\ln r}{2r^2}\partial_u \underline{\tilde{\tilde{\mathcal{A}}}}_{+r} + \frac{1}{r^2}\underline{\hat{\hat{\mathcal{A}}}}_{+u} + O\left(\frac{\ln r}{r^3}\right), && (6.3.7b)\\
\underline{A}_r &= \frac{Q}{r} + \frac{\ln r}{r^2}\underline{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^2}\underline{\mathcal{A}}_{+r} + \frac{\ln r}{r^3}\underline{\tilde{\tilde{\mathcal{A}}}}_{+r} + \frac{1}{r^3}\underline{\hat{\mathcal{A}}}_{+r} + O\left(\frac{\ln r}{r^4}\right), && (6.3.7c)\\
\underline{\Phi} &= e^{iQ\ln r}\left[\frac{1}{r}\underline{\chi}_{+} - i\frac{\ln r}{r^2}\underline{\tilde{\mathcal{A}}}_{+r}\underline{\chi}_{+} + \frac{1}{r^2}\underline{\hat{\chi}}_{+} + O\left(\frac{\ln r}{r^3}\right)\right]. && (6.3.7d)
\end{aligned}$$

Here and throughout underlined quantities refer to quantities in Lorenz gauge. We now make a gauge transformation of the form (6.3.4) with the expansion coefficients chosen to be

$$\delta Q = -Q, \quad \varepsilon_{+} = 0, \quad \tilde{\varepsilon}_{+} = \underline{\tilde{\mathcal{A}}}_{+r}, \quad \hat{\varepsilon}_{+} = \underline{\mathcal{A}}_{+r} + \underline{\tilde{\mathcal{A}}}_{+r}, \quad \tilde{\tilde{\varepsilon}}_{+} = \frac{1}{2}\underline{\tilde{\tilde{\mathcal{A}}}}_{+r}, \tag{6.3.8}$$

which enforces the required conditions (6.2.6) and the first two equations of (6.2.34) by Eqs. (6.3.5). We make a similar gauge transformation near $\mathscr{I}^-$ to enforce (6.2.14) and the last two equations of (6.2.34).

We have not yet enforced the condition (6.2.39) of the preferred asymptotic gauge. To do so we use an odd transformation of the form (6.2.35b). The gauge transformation function is determined by the condition (6.2.39) of preferred asymptotic gauge, together with the condition (A.14.22) of asymptotic Lorenz gauge, which is valid for interacting solutions as well as free solutions as discussed in Appendix A.15. The resulting transformation between Lorenz gauge fields and preferred asymptotic gauge fields is

$$\begin{aligned} \Psi^{\mathrm{e}}_{+} &= \underline{\Psi}^{\mathrm{e}}_{+} + \varepsilon_{+}, \qquad \Psi^{\mathrm{e}}_{-} = \underline{\Psi}^{\mathrm{e}}_{-} - \mathcal{P}_{*}\varepsilon_{+}, && (6.3.9a)\\ \Psi^{\mathrm{m}}_{+} &= \underline{\Psi}^{\mathrm{m}}_{+}, \qquad \Psi^{\mathrm{m}}_{-} = \underline{\Psi}^{\mathrm{m}}_{-}, && (6.3.9b)\\ \chi_{+} &= e^{i\varepsilon_{+}}\underline{\chi}_{+}, \qquad \chi_{-} = e^{-i\mathcal{P}_{*}\varepsilon_{+}}\underline{\chi}_{-}. && (6.3.9c) \end{aligned}$$

with

$$\varepsilon_{+} = \frac{1}{2}\left(\mathcal{P}_{*}\underline{\Psi}^{\mathrm{e}}_{-} - \underline{\Psi}^{\mathrm{e}}_{+}\right). \tag{6.3.10}$$

The inverse transformation is given by the same formulae (6.3.9) but with ε_{+} now expressed in terms of the preferred asymptotic gauge fields:

$$\varepsilon_{+} = \frac{1}{2}\left(\Psi^{\mathrm{e}}_{+} - \mathcal{P}_{*}\Psi^{\mathrm{e}}_{-}\right). \tag{6.3.11}$$

3 Perturbative framework

The scattering map (6.2.42) can be written schematically as

$$\begin{aligned} \chi_{+}(u,\theta^{A}) &= \chi_{+}[u,\theta^{A};\chi_{-},\mathcal{A}_{-A}], && (6.3.12a)\\ \mathcal{A}_{+A}(u,\theta^{A}) &= \chi_{+}[u,\theta^{A};\chi_{-},\mathcal{A}_{-A}], && (6.3.12b) \end{aligned}$$

where the functional dependence on the initial data is indicated by the square brackets. We will compute this map perturbatively by considering general Lorenz gauge solutions, and by transforming from Lorenz gauge to preferred asymptotic gauge using Eqs. (6.3.9).

We make the following ansatz for the scalar field and vector potential

$$\begin{aligned} \Phi &= \alpha\Phi^{(1)} + \alpha^{2}\Phi^{(2)} + \alpha^{3}\Phi^{(3)} + O(\alpha^{4}), && (6.3.13a)\\ A^{a} &= \alpha A^{(1)\,a} + \alpha^{2}A^{(2)\,a} + \alpha^{3}A^{(3)\,a} + O(\alpha^{4}), && (6.3.13b) \end{aligned}$$

where α is the perturbative expansion parameter.[9] There is a corresponding expansion for the initial data $\mathcal{A}_{-A}$, χ_- on $\mathscr{I}^-$ in the preferred asymptotic gauge given by Eqs. (6.2.14), (6.2.34) and (6.2.39):

$$\chi_- = \alpha\chi_-^{(1)} + \alpha^2\chi_-^{(2)} + \alpha^3\chi_-^{(3)} + O(\alpha^4), \tag{6.3.14a}$$

$$\mathcal{A}_{-A} = \alpha\mathcal{A}_{-A}^{(1)} + \alpha^2\mathcal{A}_{-A}^{(2)} + \alpha^3\mathcal{A}_{-A}^{(3)} + O(\alpha^4). \tag{6.3.14b}$$

We will take the second order initial data $\chi_-^{(2)}$ and $\mathcal{A}_{-A}^{(2)}$ and higher order initial data to vanish. This is done for convenience and incurs no loss in generality, since terms linear, quadratic, cubic etc. in the first order fields $\chi_-^{(1)}$ and $\mathcal{A}_{-A}^{(1)}$ give complete information about the scattering map. The corresponding expansion of the final data at $\mathscr{I}^+$ in preferred asymptotic gauge is

$$\chi_+ = \alpha\chi_+^{(1)} + \alpha^2\chi_+^{(2)} + \alpha^3\chi_+^{(3)} + O(\alpha^4), \tag{6.3.15a}$$

$$\mathcal{A}_{+A} = \alpha\mathcal{A}_{+A}^{(1)} + \alpha^2\mathcal{A}_{+A}^{(2)} + \alpha^3\mathcal{A}_{+A}^{(3)} + O(\alpha^4). \tag{6.3.15b}$$

We will denote the Lorenz-gauge fields with underlines, and write them as $\underline{\Phi}$, $\underline{A}_a$ etc. The general equations of motion (6.2.2) reduce in this gauge to

$$\Box\underline{\Phi} = 2i\underline{A}_a\nabla^a\underline{\Phi} + \underline{A}^a\underline{A}_a\underline{\Phi}\,, \tag{6.3.16a}$$

$$\Box\underline{A}^a = -ie^2(\underline{\Phi}\nabla^a\underline{\Phi}^* - \underline{\Phi}^*\nabla^a\underline{\Phi}) + 2e^2\underline{A}^a\underline{\Phi}^*\underline{\Phi}\,. \tag{6.3.16b}$$

Now using the expansions (6.3.13) yields the leading order equations of motion

$$\Box\underline{\Phi}^{(1)} = 0\,, \tag{6.3.17a}$$

$$\Box\underline{A}^{(1)\,a} = 0\,, \tag{6.3.17b}$$

[9] This expansion is equivalent to expanding in powers of the charge e at fixed Φ and fixed A_a/e.

the subleading order equations

$$\Box\underline{\Phi}^{(2)} = 2i\underline{A}^{(1)\,a}\nabla_a\underline{\Phi}^{(1)}\,, \tag{6.3.18a}$$

$$\Box\underline{A}^{(2)\,a} = -ie^2\underline{\Phi}^{(1)}\nabla^a\underline{\Phi}^{(1)*} + ie^2\underline{\Phi}^{(1)*}\nabla^a\underline{\Phi}^{(1)}, \tag{6.3.18b}$$

and the subsubleading equations

$$\Box\underline{\Phi}^{(3)} = 2i\underline{A}^{(1)\,a}\nabla_a\underline{\Phi}^{(2)} + 2i\underline{A}^{(2)\,a}\nabla_a\underline{\Phi}^{(1)} + \underline{A}^{(1)\,a}\underline{A}^{(1)}_a\underline{\Phi}^{(1)}\,, \tag{6.3.19a}$$

$$\begin{aligned}\Box\underline{A}^{(3)\,a} = {} & -ie^2\underline{\Phi}^{(1)}\nabla^a\underline{\Phi}^{(2)*} + ie^2\underline{\Phi}^{(1)*}\nabla^a\underline{\Phi}^{(2)} - ie^2\underline{\Phi}^{(2)}\nabla^a\underline{\Phi}^{(1)*} + ie^2\underline{\Phi}^{(2)*}\nabla^a\underline{\Phi}^{(1)} \\ & +2e^2\underline{A}^{(1)\,a}\underline{\Phi}^{(1)*}\underline{\Phi}^{(1)}\,.\end{aligned} \tag{6.3.19b}$$

4 First order solutions and scattering map

Appendix A.14 reviews the general solutions of the leading order Lorenz gauge equations of motion (6.3.17) which have nontrivial soft charges. These solutions satisfy our assumed expansions (6.2.6) and (6.2.14) near $\mathscr{I}^+$ and $\mathscr{I}^-$, and our asymptotic gauge conditions (6.2.34). They do not satisfy the matching condition (6.2.39), a reflection of the fact that the Lorenz and preferred asymptotic gauges do not coincide in general.

From these solutions one can evaluate the free field scattering map $\mathcal{S}:\Gamma_-\to\Gamma_+$. One might expect this map to be trivial for free solutions, and to reduce essentially to the identity map (up to antipodal identification). However, the presence of soft degrees of freedom makes the situation slightly more complicated, and in particular the identity map would not be consistent with the matching condition (6.2.39) at spatial infinity. The scattering map when written in terms of the potentials $\Psi^{\rm e}$ and $\Psi^{\rm m}$, and denoting Lorenz gauge fields with underlines, is [cf. Eq. (A.14.24)]

$$\underline{\Psi}^{\rm e}_{+}(u,\theta^A) = \mathcal{P}_*\left[\underline{\Psi}^{\rm e}_{-}(u,\theta^A) - \underline{\Psi}^{\rm e}_{+}(\theta^A) + \underline{\Psi}^{\rm e}_{-}(\theta^A)\right], \tag{6.3.20a}$$

$$\underline{\Psi}^{\rm m}_{+}(u,\theta^A) = -\mathcal{P}_*\underline{\Psi}^{\rm m}_{-}(u,\theta^A), \tag{6.3.20b}$$

$$\underline{\chi}_{+}(u,\theta^A) = -\mathcal{P}_*\,\underline{\chi}_{-}(u,\theta^A). \tag{6.3.20c}$$

We can rewrite this scattering map in terms of the preferred asymptotic gauge fields using the gauge transformation (6.3.9) applied to both the initial and final fields. The result for the scalar field is

$$\chi_{+} = -\exp\left[2i\mathcal{P}_*(\Psi^{\rm e}_{\overset{+}{-}} - \Psi^{\rm e}_{\overset{-}{-}})\right]\mathcal{P}_*\chi_{-}. \tag{6.3.21}$$

However the phase factor here, while present in free field evolution, is a non-linear effect that should be discarded in a perturbative expansion. We will show below that non-linear interactions have the effect of removing this phase factor, cf. Eqs. (6.4.5). Discarding this phase factor the full free field scattering map is, from Eqs. (6.3.9) and (6.3.20),

$$\begin{aligned}
\Psi^{\rm e}_{+} &= \mathcal{P}_*\left[\Psi^{\rm e}_{-} - \Psi^{\rm e}_{\overset{-}{-}} + \Psi^{\rm e}_{\overset{-}{+}}\right], && (6.3.22a)\\
\Psi^{\rm m}_{+} &= -\mathcal{P}_*\,\Psi^{\rm m}_{-}, && (6.3.22b)\\
\chi_{+} &= -\mathcal{P}_*\chi_{-}. && (6.3.22c)
\end{aligned}$$

Note that some of the signs in Eq. (6.3.22a) differ from those in Eq. (6.3.20a). The scattering map (6.3.22) manifestly satisfies the matching condition (6.2.39). It also preserves the symplectic form (6.2.44), as it should, which provides a nontrivial consistency check of some of the coefficients of the u-independent terms.

We can rewrite the free field scattering map (6.3.22) in terms of the soft variables $(\Delta\Psi^{\rm e}, \overline{\Psi}^{\rm e})$ and hard variables $(\tilde{\Psi}^{\rm e}, \Psi^{\rm m}, \chi)$ defined in Sec. 6.2.5 above. The result is

$$\begin{aligned}
\mathcal{P}_*\Delta\Psi^{\rm e}_{+} &= \Delta\Psi^{\rm e}_{-}, && (6.3.23a)\\
\mathcal{P}_*\overline{\Psi}^{\rm e}_{+} &= \overline{\Psi}^{\rm e}_{-} + \Delta\Psi^{\rm e}_{-}, && (6.3.23b)\\
\mathcal{P}_*\chi_{+} &= -\chi_{-}, && (6.3.23c)\\
\mathcal{P}_*\tilde{\Psi}^{\rm e}_{+} &= \tilde{\Psi}^{\rm e}_{-}, && (6.3.23d)\\
\mathcal{P}_*\Psi^{\rm m}_{+} &= -\Psi^{\rm m}_{-}. && (6.3.23e)
\end{aligned}$$

Note that the hard and soft sectors are decoupled to this order, with the soft sector evolving via Eqs. (6.3.23a) – (6.3.23b) and the hard sector via Eqs. (6.3.23c) – (6.3.23e).

5 | General parameterization of the scattering map

The scattering map $\mathcal{S} : \Gamma_- \to \Gamma_+$ must satisfy a number of constraints. In this section we derive the most general form of the map that obeys all the constraints, as a foundation for later analysis. The various constraints are:

- The scattering map must satisfy the conservation law (6.2.40) for the soft charges Q_ε. From Eqs. (6.2.10a), (6.2.21a), (6.2.22a), (6.2.7b) (6.2.34), (6.2.36) and (6.2.46a) this conservation law can be written as [23]

$$Q_{+}(\theta) + \frac{1}{e^2} D^2 \Delta \Psi^{\mathrm{e}}_{+}(\theta) = \mathcal{P}_* \left[Q_{-}(\theta) + \frac{1}{e^2} D^2 \Delta \Psi^{\mathrm{e}}_{-}(\theta) \right], \tag{6.3.24}$$

  where

$$Q_{-} = \int dv \mathcal{J}_{-v}, \qquad Q_{+} = \int du \mathcal{J}_{+u} \tag{6.3.25}$$

  are the total ingoing and outgoing charges per unit angle. Since we have one such conservation law for every angle, there are actually an infinite number of them. Note that since we have specialized to the sector where the magnetic charges (A.16.4) vanish, we have no corresponding constraint from the magnetic charges.

- It must be compatible with the transformations of initial and final data associated with the residual even transformations ε_{e} discussed in Sec. 6.2.4 above, which act on the physical phase space since they are not degeneracy directions of the presymplectic form. Specifically, under the transformation of initial data

$$\chi_{-} \to e^{i\varepsilon} \chi_{-}, \quad \mathcal{A}_{-A} \to \mathcal{A}_{-A} + D_A \varepsilon, \tag{6.3.26}$$

  where $\varepsilon = \varepsilon(\theta)$, the final data must transform as

$$\chi_{+} \to e^{i\mathcal{P}_* \varepsilon} \chi_{+}, \quad \mathcal{A}_{+A} \to \mathcal{A}_{+A} + D_A \mathcal{P}_* \varepsilon. \tag{6.3.27}$$

- It must obey the gauge specialization condition (6.2.39) that was imposed in order to correlate the gauge freedom on $\mathscr{I}^-$ and $\mathscr{I}^+$ in such a way as to allow the scattering map to be a symplectomorphism, as discussed in Sec. 6.2.4.

- It must transform appropriately under Poincaré symmetries. While this is an important constraint we will not make it explicit in this section.

Taken together, these requirements strongly constrain the scattering map.

For the analysis in this section it will be convenient to use as the basic variables particular combinations of the preferred asymptotic gauge fields, namely $\Psi^{\rm m}$, χ, $\overline{\Psi}^e$ and [cf. Eqs. (6.2.46) above]

$$\breve{\Psi}^{\rm e} \equiv \Psi^{\rm e} - \overline{\Psi}^e = \tilde{\Psi}^{\rm e} + g\Delta\Psi^{\rm e}. \tag{6.3.28}$$

The general scattering map (6.3.12) can be written in terms of these variables as

$$\begin{aligned}
\mathcal{P}_*\breve{\Psi}^{\rm e}_{+} &= \breve{\Psi}^{\rm e}_{-} + \mathcal{H}^{\rm e}\left[u, \breve{\Psi}^{\rm e}_{-}, \overline{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, \chi_{-}\right], & (6.3.29a)\\
\mathcal{P}_*\overline{\Psi}^{\rm e}_{+} &= \overline{\Psi}^{\rm e}_{-} + \Delta\Psi^{\rm e}_{-} + \mathcal{G}\left[\breve{\Psi}^{\rm e}_{-}, \overline{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, \chi_{-}\right], & (6.3.29b)\\
\mathcal{P}_*\Psi^{\rm m}_{+} &= -\Psi^{\rm m}_{-} + \mathcal{H}^{\rm m}\left[u, \breve{\Psi}^{\rm e}_{-}, \overline{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, \chi_{-}\right], & (6.3.29c)\\
\mathcal{P}_*\chi_{+} &= -\chi_{-} + \mathcal{K}\left[u, \breve{\Psi}^{\rm e}_{-}, \overline{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, \chi_{-}\right], & (6.3.29d)
\end{aligned}$$

in terms of some functionals $\mathcal{H}^{\rm e}$, $\mathcal{H}^{\rm m}$, $\mathcal{G}$ and $\mathcal{K}$, where the functional dependence on the initial data is indicated by the square brackets. Here for convenience we have separated out the terms that arise in the free evolution (6.3.23), so that the functionals parameterize the non-linear interactions. The functionals do depend on the angles θ^A but we have suppressed this dependence for simplicity.

We start by imposing the transformation property (6.3.26) and (6.3.27). The functionals $\mathcal{H}^{\rm e}$, $\mathcal{H}^{\rm m}$ and $\mathcal{G}$ need to be invariant under the transformation, while $\mathcal{K}$ needs to transform by a phase. Choosing $\varepsilon = -\overline{\Psi}^{\rm e}_{-}$, the invariance implies for example that

$$\mathcal{H}^{\rm e}\left[u, \breve{\Psi}^{\rm e}_{-}, \overline{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, \chi_{-}\right] = \mathcal{H}^{\rm e}\left[u, \breve{\Psi}^{\rm e}_{-}, 0, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right]. \tag{6.3.30}$$

Hence by redefining the functionals the scattering map can be written in the general form

$$\begin{aligned}
\mathcal{P}_*\breve{\Psi}^{\rm e}_{+} &= \breve{\Psi}^{\rm e}_{-} + \mathcal{H}^{\rm e}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.31a)\\
\mathcal{P}_*\overline{\Psi}^{\rm e}_{+} &= \overline{\Psi}^{\rm e}_{-} + \Delta\Psi^{\rm e}_{-} + \mathcal{G}\left[\breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.31b)\\
\mathcal{P}_*\Psi^{\rm m}_{+} &= -\Psi^{\rm m}_{-} + \mathcal{H}^{\rm m}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.31c)\\
\mathcal{P}_*\chi_{+} &= -\chi_{-} + \exp\left[i\overline{\Psi}^{\rm e}_{-}\right]\mathcal{K}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right]. && (6.3.31d)
\end{aligned}$$

Next we impose the matching condition (6.2.39). From the definition (6.3.28) of $\breve{\Psi}^{\rm e}$ we have that $\breve{\Psi}^{\rm e}_{+}(u=\infty) = -\breve{\Psi}^{\rm e}_{+}(u=-\infty)$, and $\mathcal{H}^{\rm e}$ must also have this property by Eq. (6.3.31a). Defining the functional

$$\mathcal{H}^{\rm e}_{\infty} = \lim_{u\to\infty}\mathcal{H}^{\rm e}(u) = -\lim_{u\to-\infty}\mathcal{H}^{\rm e}(u), \tag{6.3.32}$$

we find from Eqs. (6.2.39) and (6.3.31) that $\mathcal{G} = \mathcal{H}^{\rm e}_{\infty}$. Therefore the scattering map can be written as

$$\begin{aligned}
\mathcal{P}_*\breve{\Psi}^{\rm e}_{+} &= \breve{\Psi}^{\rm e}_{-} + \mathcal{H}^{\rm e}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.33a)\\
\mathcal{P}_*\overline{\Psi}^{\rm e}_{+} &= \overline{\Psi}^{\rm e}_{-} + \Delta\Psi^{\rm e}_{-} + \mathcal{H}^{\rm e}_{\infty}\left[\breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.33b)\\
\mathcal{P}_*\Psi^{\rm m}_{+} &= -\Psi^{\rm m}_{-} + \mathcal{H}^{\rm m}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right], && (6.3.33c)\\
\mathcal{P}_*\chi_{+} &= -\chi_{-} + \exp\left[i\overline{\Psi}^{\rm e}_{-}\right]\mathcal{K}\left[u, \breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right]. && (6.3.33d)
\end{aligned}$$

Finally we impose the conservation laws (6.3.24). Taking the limits $u \to \pm\infty$ of Eq. (6.3.33a) and using the definitions (6.2.46a), (6.3.28) and (6.3.32) yields

$$\mathcal{P}_*\Delta\Psi^{\rm e}_{+} = \Delta\Psi^{\rm e}_{-} + 2\mathcal{H}^{\rm e}_{\infty}\left[\breve{\Psi}^{\rm e}_{-}, \Psi^{\rm m}_{-}, e^{-i\overline{\Psi}^{\rm e}_{-}}\chi_{-}\right]. \tag{6.3.34}$$

It follows that $\mathcal{H}^{\rm e}_{\infty}$ is essentially the change in the electromagnetic memory $\Delta\Psi^{\rm e}$ [144, 145] between $\mathscr{I}^-$ and $\mathscr{I}^+$. Now combining this with the conservation law (6.3.24) gives

$$\mathcal{H}^{\rm e}_{\infty} = -\frac{1}{2}e^2 D^{-2}\Delta Q(\theta), \tag{6.3.35}$$

where the total change in the charge per unit angle is $\Delta Q = \mathcal{P}_* Q_+(\theta) - Q_-(\theta)$, which can be computed from the functional $\mathcal{K}$ from Eqs. (6.2.9a), (6.3.24), (6.3.25) and (6.3.33d). This tells us that the functionals are not all independent, as $\mathcal{H}^{\rm e}_\infty$ can be computed from $\mathcal{K}$.

To summarize, we have derived a general parameterization of the scattering map that is consistent with all of the constraints listed above, given by Eq. (6.3.33), assuming that the various functionals transform appropriately under Poincaré transformations.

We can rewrite the scattering map (6.3.33) in terms of asymptotic Lorenz gauge fields using the gauge transformation (6.3.9). The arguments of the functionals in Eq. (6.3.33) are invariant under the transformation, and so we can simply insert underlines on all of these arguments to indicate asymptotic Lorenz gauge fields. The gauge transformation function (6.3.11) evaluates to $\varepsilon_+ = \mathcal{P}_* \Delta \Psi^{\rm e}_- + \mathcal{P}_* \mathcal{H}^{\rm e}_\infty$, and the final result is

$$\begin{aligned}
\mathcal{P}_* \underline{\breve{\Psi}}^{\rm e}_+ &= \underline{\breve{\Psi}}^{\rm e}_- + \mathcal{H}^{\rm e}\left[u, \underline{\breve{\Psi}}^{\rm e}_-, \underline{\Psi}^{\rm m}_-, e^{-i\underline{\overline{\Psi}}^{\rm e}_-}\underline{\chi}_-\right], && (6.3.36a)\\
\mathcal{P}_* \underline{\overline{\Psi}}^{\rm e}_+ &= \underline{\overline{\Psi}}^{\rm e}_- - \Delta \underline{\Psi}^{\rm e}_- - \mathcal{H}^{\rm e}_\infty\left[\underline{\breve{\Psi}}^{\rm e}_-, \underline{\Psi}^{\rm m}_-, e^{-i\underline{\overline{\Psi}}^{\rm e}_-}\underline{\chi}_-\right], && (6.3.36b)\\
\mathcal{P}_* \underline{\Psi}^{\rm m}_+ &= -\underline{\Psi}^{\rm m}_- + \mathcal{H}^{\rm m}\left[u, \underline{\breve{\Psi}}^{\rm e}_-, \underline{\Psi}^{\rm m}_-, e^{-i\underline{\overline{\Psi}}^{\rm e}_-}\underline{\chi}_-\right], && (6.3.36c)\\
\mathcal{P}_* \underline{\chi}_+ &= \exp\left\{-2i\mathcal{H}^{\rm e}_\infty\left[\underline{\breve{\Psi}}^{\rm e}_-, \underline{\Psi}^{\rm m}_-, e^{-i\underline{\overline{\Psi}}^{\rm e}_-}\underline{\chi}_-\right] - 2i\Delta \underline{\Psi}^{\rm e}_-\right\}\left\{-\underline{\chi}_- + \exp\left[i\underline{\overline{\Psi}}^{\rm e}_-\right]\mathcal{K}\left[u, \underline{\breve{\Psi}}^{\rm e}_-, \underline{\Psi}^{\rm m}_-, e^{-i\underline{\overline{\Psi}}^{\rm e}_-}\underline{\chi}_-\right]\right\}.
\end{aligned}$$

This has the same form as the original scattering map (6.3.33) except for sign flips in two of the terms in Eq. (6.3.36b) and the overall phase factor in the scalar field (6.3.36d).

6.4 Second order dynamics

1 Computation of scattering map to second order

We now turn to the computation of the scattering map at second order, which involves solving the second order equations of motion (6.3.18). We specialize to the case of a scalar field that has support only on $l = 0, 1$ spherical harmonics and a vector potential that has support only on $l = 1$ harmonics. The details of the explicit computation for $\underline{\chi}^{(2)}_+(u, \theta^A)$ are relegated to Appendix A.17. The results

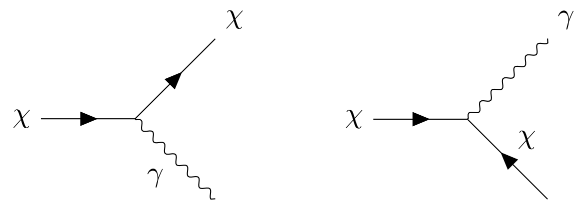

Figure 6.1: We denote the scalar with χ and the photon with γ. The graph on the left depicts an incoming scalar and a photon going into a scalar while the one the right shows two incoming scalars going into a photon.

for the final data for the second order fields are

$$\underline{\chi}_{+}^{(2)}(u, \theta^A) \;=\; -4i\gamma^i n_i \mathcal{P}_* \underline{\chi}_{-}^{(1)}(u)\,, \tag{6.4.1a}$$

$$\underline{\mathcal{A}}_{+\,A}^{(2)}(u, \theta^A) \;=\; 0\,, \tag{6.4.1b}$$

where γ_i was defined in Eq. (A.14.13). Note from Eq. (A.14.13) that γ^i parameterizes the change in the incoming vector potential $\underline{\mathcal{A}}_{-\,A}^{(1)}$ between $v = \infty$ at i^0 and $v = -\infty$ at i^-. It is hence the soft part of the $l = 1$ piece of the conserved charge, Q_ε, of Eq. (6.2.40). The non-linear term in (6.4.1a) therefore represents an interaction between the hard and soft degrees of freedom. Note that γ^i above encodes the soft behavior of the vector potential and $\underline{\chi}_{+}^{(2)}(u, \theta^A)$ vanishes when $\gamma_i = 0$. This result is consistent with the fact that in massless scalar QED, in Lorenz gauge, the three point diagram for an incoming scalar and photon going into an outgoing scalar (shown in the first graph on the left in Fig. 6.1) vanishes by energy-momentum conservation unless the photon has exactly zero energy. The second graph in Fig. 6.1 also vanishes by energy-momentum conservation in the case when the incoming scalar fields have strictly non-zero energy and momentum (which is true in our case since the scalar field initial data has no soft behavior in our setup). This argument can be used to deduce Eq. 6.4.1b. [10]

It is also interesting to note that the aforementioned map in Lorenz gauge is not continuous, in the following sense. Consider a sequence ${}^{(n)}\mathcal{A}_{-\,A}$ of incoming configurations, each of which has no

[10] Here, we are appealing to the fact that tree level diagrams capture classical scattering processes. A detailed calculation confirming this may be found in [8].

soft part (${}^{(n)}\gamma_i = 0$), which converges pointwise to $\mathcal{A}_{\bar{-}\,A}$:

$$\lim_{n\to\infty} {}^{(n)}\mathcal{A}_{\bar{-}\,A}(v,\theta^A) = \mathcal{A}_{\bar{-}\,A}(v,\theta^A). \tag{6.4.2}$$

For each element in the sequence, the result (6.4.1a) vanishes, and so the $n \to \infty$ limit of the scattered fields also has this property. However the result (6.4.1a) does not vanish for the scattering of the $n \to \infty$ pointwise limit (6.4.2) of the initial data. Thus, the scattering of soft degrees of freedom cannot be obtained by simply taking a naive limit of hard scattering. Equivalently, the scattering map is not determined by its restriction to initial data that consists of smooth wavepacket states.

We note the solution for the scalar field to second order given by Eqs. (A.14.20a) and (6.4.1a) can be rewritten so that the non-linear interaction appears as an overall multiplicative phase factor:

$$\underline{\chi}_{+}(u,\theta^A) = -\exp\left[-2i\mathcal{P}_*\underline{\Psi}^{\mathrm{e}}_{\bar{+}}(\theta^A)\right]\mathcal{P}_*\underline{\chi}_{-}(u,\theta^A) + O(\alpha^3)\,, \tag{6.4.3}$$

where we have used Eqs. (6.2.36b), (A.14.13), (A.14.15) and (A.14.18b) [see also Eq. (A.14.10)]. This fact will be important below. The remaining piece of the scattering map to second order, given by Eqs. (A.14.20b) and (6.4.1b), can be rewritten in terms of the potentials (6.2.36) as

$$\underline{\Psi}^{\mathrm{e}}_{+}(u,\theta^A) \;=\; \mathcal{P}_*\left[\underline{\Psi}^{\mathrm{e}}_{-}(u,\theta^A) - \underline{\Psi}^{\mathrm{e}}_{\bar{+}}(\theta^A)\right] + O(\alpha^3)\,, \tag{6.4.4a}$$

$$\underline{\Psi}^{\mathrm{m}}_{+}(u,\theta^A) \;=\; -\mathcal{P}_*\,\underline{\Psi}^{\mathrm{m}}_{-}(u,\theta^A) + O(\alpha^3)\,, \tag{6.4.4b}$$

where we have used $\mathcal{P}_*\varepsilon_{AB} = -\varepsilon_{AB}$.

We now transform from Lorenz gauge to the preferred asymptotic gauge which is defined by the validity of the expansions (6.2.6) and (6.2.14) and the conditions (6.2.34) and (6.2.39). The details of this gauge transformation were worked out in Sec. 6.3.1 above[11], and are specified in Eqs. (6.3.9). They are expressed in terms of the potentials Ψ^{e} and Ψ^{m} in Eqs. (6.3.9a) and (6.3.9b). Combining this gauge transformation with the second order scattering map given in Eqs. (6.4.3) and (6.4.4)

[11] Note that since the Lorenz gauge solutions satisfy the matching condition (6.2.39) with a sign flip, this gauge transformation is odd in the terminology of Sec. 6.2.4 above.

gives the scattering map in our new gauge:

$$\begin{aligned}
\chi_{+}(u,\theta^A) &= -\mathcal{P}_*\chi_{-}(u,\theta^A) + O(\alpha^3)\,, && \text{(6.4.5a)}\\
\Psi^{\mathrm{e}}_{+}(u,\theta^A) &= \mathcal{P}_*\left[\Psi^{\mathrm{e}}_{-}(u,\theta^A) - \Psi^{\mathrm{e}}_{\substack{-\\-}}(\theta^A)\right] + O(\alpha^3)\,, && \text{(6.4.5b)}\\
\Psi^{\mathrm{m}}_{+}(u,\theta^A) &= -\mathcal{P}_*\,\Psi^{\mathrm{m}}_{-}(u,\theta^A) + O(\alpha^3)\,. && \text{(6.4.5c)}
\end{aligned}$$

Note that in this computation, there is a cancellation between the phase factor in the scattering map (6.4.3), that arises from the non-linearity in the equations of motion, and the phase factors in the gauge transformations (6.3.9). This cancellation leads to the final, simple form (6.4.5).

The final form (6.4.5) of the scattering map is valid in a specific gauge which satisfies our conditions (6.2.34) and (6.2.39). However, because it was computed starting from Lorenz gauge, it obeys the additional restrictions A.14.6, A.14.9 (see also the discussion in Appendix A.15). We would like to compute the form of the scattering map in a more general class of gauges in which the the restrictions A.14.6, A.14.9 are not imposed. To achieve this, we perform a general even gauge transformation [of the form (6.2.37a) with $\varepsilon_{-} = \mathcal{P}_*\varepsilon_{+}$]. Writing the gauge transformation parameter in terms of $\Psi^{\mathrm{e}}_{\substack{-\\+}}$, the scattering map in this more general class of gauges is

$$\begin{aligned}
\chi_{+} &= -\mathcal{P}_*\chi_{-} + O(\alpha^3)\,, && \text{(6.4.6a)}\\
\Psi^{\mathrm{e}}_{+} &= \mathcal{P}_*\left[\Psi^{\mathrm{e}}_{-} - \Psi^{\mathrm{e}}_{\substack{-\\-}} + \Psi^{\mathrm{e}}_{\substack{-\\+}}\right] + O(\alpha^3)\,, && \text{(6.4.6b)}\\
\Psi^{\mathrm{m}}_{+} &= -\mathcal{P}_*\left[\Psi^{\mathrm{m}}_{-} - \Psi^{\mathrm{m}}_{\substack{-\\+}}\right] + O(\alpha^3)\,. && \text{(6.4.6c)}
\end{aligned}$$

Note that this scattering map preserves the symplectic form (6.2.44) to $O(\alpha^2)$.

2 | Decoupled hard and soft sectors at second order

Finally, we rewrite the scattering map (6.4.6) in terms of the phase space coordinates introduced in Sec. 6.2.5 above, which gives a separation into soft and hard degrees of freedom which are symplectically orthogonal. Combining Eqs. (6.2.46), (6.2.50), (6.2.41) and (6.4.6) we find

$$\chi_{+}(u,\theta^A) = -\mathcal{P}_*\chi_{-}(u,\theta^A) + O(\alpha^3)\,, \tag{6.4.7a}$$

$$\tilde{\Psi}^{\mathrm{e}}_{+}(u,\theta^A) = \mathcal{P}_*\tilde{\Psi}^{\mathrm{e}}_{-}(u,\theta^A) + O(\alpha^3)\,, \tag{6.4.7b}$$

$$\Psi^{\mathrm{m}}_{+}(u,\theta^A) = -\mathcal{P}_*\Psi^{\mathrm{m}}_{-}(u,\theta^A) + O(\alpha^3)\,, \tag{6.4.7c}$$

$$\Delta\Psi^{\mathrm{e}}_{+}(\theta^A) = \mathcal{P}_*\Delta\Psi^{\mathrm{e}}_{-}(\theta^A) + O(\alpha^3)\,, \tag{6.4.7d}$$

$$\hat{\Psi}^{\mathrm{e}}_{+}(\theta^A) = \mathcal{P}_*\left[\hat{\Psi}^{\mathrm{e}}_{-}(\theta^A) + \Delta\Psi^{e}_{-}(\theta^A)\right] + O(\alpha^3)\,. \tag{6.4.7e}$$

The second order scattering map (6.4.7) factors into two scattering maps, one in the hard sector given by Eqs. (6.4.7a) – (6.4.7c), and one in the soft sector given by Eqs. (6.4.7d) and (6.4.7e). This factorization is exactly of the kind discussed[12] in Ref. [127] (although there it was claimed to hold to all orders). In the next section, we will show that this factorization property breaks down at higher orders in perturbation theory.

6.5 | Third and fourth order dynamics

In the previous section we showed that the classical scattering map factorizes into decoupled hard and soft sectors at quadratic order. We now proceed to third and fourth orders in perturbation theory. We will show in this section that no factorization into decoupled sectors in this approximation is possible, in three steps. First, in Sec. 6.5.1, we will derive which cubic and quartic terms in the scattering map are invariant under linear or perturbative field redefinitions of the hard and soft sectors. We will call such terms *invariant couplings*, as they cannot be removed by field redefinitions. Second, in Sec. 6.5.2, we will show that the total change in electromagnetic memory of a scattering process is such an invariant coupling. Finally in Sec. 6.5.3 we will show by means of an explicit calculation that the change in electromagnetic memory is nonvanishing in general at quartic order.

[12] The scattering map found in Ref. [127] reduces to the identity in the soft sector (up to a factor of the pullback $\mathcal{P}_*$), in disagreement with our Eqs. (6.4.7d) and (6.4.7e), because of their use of phase space coordinates that are not independent, cf. the discussion in Sec. 6.2.5 above.

1 Definition of invariant couplings

The general form (6.3.33) of the scattering map can be written schematically as

$$\overline{y}^a = L^a{}_b y^b + M^a{}_{bcd} y^b y^c y^d + N^a{}_{bcde} y^b y^c y^d y^e + O(y^5), \tag{6.5.1}$$

where y^a are abstract phase space coordinates and the map is parameterized in terms of some phase space tensors $L^a{}_b$, $M^a{}_{bcd}$ and $N^a{}_{bcde}$. Here we are using a notation where the indices $a, b, \ldots$ run over the various fields

$$\left(\tilde{\Psi}^{\rm e}, \Psi^{\rm m}, \chi, \Delta\Psi^e, \overline{\Psi}^e\right), \tag{6.5.2}$$

and also encode the dependence of these fields on the coordinates v, θ^A, so that contractions over these indices encompass integrals over these variables. The unbarred coordinates y^a refer to the initial data on $\mathscr{I}^-$, while the barred coordinates $\overline{y}^a$ refer to the final data on $\mathscr{I}^+$. Finally the schematic scattering map (6.5.1) encodes the fact that there are no quadratic terms when using preferred asymptotic gauge, cf. Eq. (6.4.7) above, so the leading non-linearities arise at cubic order or higher order.

The phase space coordinates can be decomposed into "hard" and "soft" components,

$$y^a = (h^A, s^\Gamma), \tag{6.5.3}$$

where the hard variables h^A refer to the first three fields in (6.5.2), which depend on v and θ^A, while the soft variables s^Γ refer to the last two fields in (6.5.2), which depend only on θ^A. As discussed in Sec. 6.4.2, the two sectors are uncoupled in the linear order scattering map (6.3.23), so the tensor $L^a{}_b$ is block diagonal with vanishing off-diagonal blocks:

$$L^\Gamma{}_A = 0, \quad L^A{}_\Gamma = 0. \tag{6.5.4}$$

The diagonal block $L^A{}_B$ in the hard sector is given by Eqs. (6.3.23c) – (6.3.23e) , while the diagonal block $L^\Gamma{}_\Sigma$ in the soft sector is given by Eqs. (6.3.23a) and (6.3.23b).

We will see later in this section that the hard and soft sectors are coupled via the higher order terms in Eq. (6.5.1) which mix the two sectors together. Our goal here is to determine when these couplings can be removed by perturbative field redefinitions of the variables that defined our hard and soft sectors. We consider field redefinitions of the form

$$y^a = z^a + \Upsilon^a{}_{bcd} z^b z^c z^d + \Xi^a{}_{bcde} z^b z^c z^d z^e + O(z^5), \tag{6.5.5}$$

which defines new phase space coordinates z^a, together with an identical transformation for the barred variables. Here we have assumed that the transformation is the identity to linear order, as linear transformations are considered separately below.[13]

Using the transformation (6.5.5) and its inverse, we can write the scattering map (6.5.1) in terms of the new phase space variables z^a. The result is

$$\bar{z}^a = \hat{L}^a{}_b z^b + \hat{M}^a{}_{bcd} z^b z^c z^d + \hat{N}^a{}_{bcde} z^b z^c z^d z^e + O(z^5), \tag{6.5.6}$$

where the transformed tensors are

$$\begin{aligned}
\hat{L}^a{}_b &= L^a{}_b, && (6.5.7\text{a})\\
\hat{M}^a{}_{bcd} &= M^a{}_{bcd} + L^a{}_e \Upsilon^e{}_{bcd} - \Upsilon^a{}_{efg} L^e{}_b L^f{}_c L^g{}_d, && (6.5.7\text{b})\\
\hat{N}^a{}_{bcde} &= N^a{}_{bcde} + L^a{}_f \Xi^f{}_{bcde} - \Xi^a{}_{fghi} L^f{}_b L^g{}_c L^h{}_d L^i{}_e. && (6.5.7\text{c})
\end{aligned}$$

We denote by $\mathcal{V}$ the linear space of tensors $(M^a{}_{bcd}, N^a{}_{bcde})$. Consider now linear maps $\ell : \mathcal{V} \to \mathbf{R}$, elements of the dual space $\mathcal{V}^*$. For example $\ell(M, N)$ could be a particular component of the tensor N. We define $\mathcal{W}_{\text{mixed}}$ to be the subspace of $\mathcal{V}^*$ that is spanned by components of M and N with both hard and soft indices, excluding the purely soft components $(M^\Sigma{}_{\Gamma\Delta\Upsilon}, N^\Sigma{}_{\Gamma\Delta\Upsilon\Lambda})$ and purely hard components $(M^A{}_{BCD}, N^A{}_{BCDE})$. We will call maps ℓ in $\mathcal{W}_{\text{mixed}}$ *couplings*, since they couple the hard and soft sectors together in the dynamics, and we will compute one such coupling in the

[13] We have also excluded any quadratic terms in the transformation (6.5.5) since these would generically generate quadratic terms in the scattering map. These terms can in general be important for our argument. However, they do not effect any of our conclusions. Their effect is nonetheless addressed in [8].

scattering map explicitly in Sec. 6.5.3 below. We define a linear map $\mathfrak{A} : \mathcal{V} \to \mathcal{V}$ that takes

$$\mathfrak{A} : (\Upsilon^a{}_{bcd}, \Xi^a{}_{bcde}) \to \left(L^a{}_e \Upsilon^e{}_{bcd} - \Upsilon^a{}_{efg} L^e{}_b L^f{}_c L^g{}_d \; , \; L^a{}_f \Xi^f{}_{bcde} - \Xi^a{}_{fghi} L^f{}_b L^g{}_c L^h{}_d L^i{}_e \right). \qquad (6.5.8)$$

We define the subspace $\mathcal{W}_{\rm invariant}$ of $\mathcal{V}^*$ to be the set of maps ℓ for which $\ell \circ \mathfrak{A} = 0$, which is the kernel of the transpose of $\mathfrak{A}$. The space $\mathcal{W}_{\rm invariant}$ depends on the linear order scattering map $L^a{}_b$, for example if $L^a{}_b = \delta^a{}_b$ then $\mathcal{W}_{\rm invariant}$ is the entire space $\mathcal{V}^*$. The key property of this space is that nonzero couplings in $\mathcal{W}_{\rm invariant} \cap \mathcal{W}_{\rm mixed}$ cannot be set to zero using the field redefinitions (6.5.5), from Eqs. (6.5.7) and (6.5.8).

Turn now to linear field redefinitions of the form

$$y^a = \Omega^a{}_b z^b. \qquad (6.5.9)$$

The transformed scattering map is again of the form (6.5.6) with the transformed tensors being

$$\hat{L}^a{}_b = \left(\Omega^{-1}\right)^a{}_c L^c{}_d \Omega^d{}_b, \qquad (6.5.10a)$$

$$\hat{M}^a{}_{bcd} = \left(\Omega^{-1}\right)^a{}_e M^e{}_{fgh} \Omega^f{}_b \Omega^g{}_c \Omega^h{}_d, \qquad (6.5.10b)$$

$$\hat{N}^a{}_{bcde} = \left(\Omega^{-1}\right)^a{}_f N^f{}_{ghij} \Omega^g{}_b \Omega^h{}_c \Omega^i{}_d \Omega^j{}_e. \qquad (6.5.10c)$$

We restrict attention to transformations $\Omega^a{}_b$ which preserve the decoupling (6.5.4) of the two sectors at linear order:

$$\hat{L}^\Gamma{}_A = 0, \quad \hat{L}^A{}_\Gamma = 0. \qquad (6.5.11)$$

Without loss of generality for analyzing decoupling we can assume that

$$\Omega^\Gamma{}_\Sigma = \delta^\Gamma{}_\Sigma, \quad \Omega^A{}_B = \delta^A{}_B. \qquad (6.5.12)$$

The conditions (6.5.11) are then equivalent to, from Eqs. (6.5.4) and (6.5.10a),

$$\Omega^A{}_\Sigma L^\Sigma{}_\Gamma = L^A{}_B \Omega^B{}_\Gamma, \qquad \Omega^\Gamma{}_B L^B{}_A = L^\Gamma{}_\Sigma \Omega^\Sigma{}_A. \qquad (6.5.13)$$

For each such map $\Omega^a{}_b$, we define the map $\mathfrak{A}_\Omega : \mathcal{V} \to \mathcal{V}$ by

$$\mathfrak{A}_\Omega : (M^a{}_{bcd}, N^a{}_{bcde}) \to \left(M^a{}_{bcd} - \Omega^{-1}{}^a{}_e M^e{}_{fgh}\Omega^f{}_b\Omega^g{}_c\Omega^h{}_d \;,\; N^a{}_{bcde} - \Omega^{-1}{}^a{}_f N^f{}_{ghij}\Omega^g{}_b\Omega^h{}_c\Omega^i{}_d\Omega^j{}_e\right). \tag{6.5.14}$$

We define the subspace $\mathcal{W}'_{\text{invariant}}$ of $\mathcal{V}^*$ to be the set of maps ℓ for which $\ell \circ \mathfrak{A}_\Omega = 0$ for all $\Omega^a{}_b$ satisfying the conditions (6.5.12) and (6.5.13). Equivalently, $\mathcal{W}'_{\text{invariant}}$ is the intersection of the kernels of the transposes of the maps $\mathfrak{A}_\Omega$. From Eqs. (6.5.10) and (6.5.14) we see that nonzero couplings ℓ in $\mathcal{W}'_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$ cannot be set to zero using the field redefintions (6.5.9).

We will refer to the couplings in $\mathcal{W}_{\text{invariant}} \cap \mathcal{W}'_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$ as *invariant couplings*. These couplings are not altered by either type of transformation, linear or non-linear. Which specific cubic and quartic couplings are invariant depends on the details of the free field scattering map $L^a{}_b$.

2 Change in electromagnetic memory is an invariant coupling

In Sec. 6.3.5 above, we noted that the functional $\mathcal{H}^{\text{e}}_\infty$ in the general parameterization (6.3.33) of the scattering map corresponds to the change in electromagnetic memory. We now show that this quantity is an invariant coupling.

We consider in particular the memory produced when there is an incoming scalar field but no vector potential, $\mathcal{H}^{\text{e}}_\infty[0, 0, \chi_-]$. We show in Sec. 6.5.3 below that this quantity is of order $O(\alpha^4)$, so it contributes to the tensor $N^a{}_{bcde}$. Since the scalar field is assumed to have no soft part [cf. Eq. (6.2.18) above] the corresponding component is

$$\ell(M, N) = N^\Lambda{}_{ABCD} \tag{6.5.15}$$

where the Λ index corresponds to the field $\Delta\Psi^{\text{e}}$, and the indices A, B, C and D correspond to the field χ, from Eqs. (6.3.33b), (6.3.34) and (6.5.3).

We now consider the action of the map (6.5.8) on the coupling (6.5.15). From Eqs. (6.5.4), (6.5.8) and (6.5.15) we have that $\ell \circ \mathfrak{A} = 0$ if

$$L^{\Lambda}{}_{\Sigma}\Xi^{\Sigma}{}_{ABCD} = \Xi^{\Lambda}{}_{EFGH}L^{E}{}_{A}L^{F}{}_{B}L^{G}{}_{C}L^{H}{}_{D}, \tag{6.5.16}$$

where the indices Λ, A, B, C and D have the values discussed after Eq. (6.5.15), and $\Xi^{a}{}_{bcde}$ is the transformation tensor defined in Eq. (6.5.5). By using the linear order scattering map (6.3.23), we see that the effect of the mappings $L^{E}{}_{A}$ on the right hand side is to replace χ_{-} with $-\mathcal{P}_{*}\chi_{-}$. Similarly the effect of the mapping $L^{\Lambda}{}_{\Sigma}$ on the left hand side is to act with the pullback $\mathcal{P}_{*}$. Therefore the condition (6.5.16) can be written as

$$\mathcal{P}_{*}\, F[\chi_{-}] = F[-\mathcal{P}_{*}\chi_{-}], \tag{6.5.17}$$

where we have defined the quartic functional $F[\chi_{-}]$ to be $\Xi^{\Lambda}{}_{ABCD}y^{A}y^{B}y^{C}y^{D}$ with the same values of Λ and A, B, C, D. On the right hand side, we can pull the pullback operator $\mathcal{P}_{*}$ through the covariant functional F, assuming that the phase space coordinate transformation (6.5.5) does not violate covariance on the two-sphere. Also the minus sign in the argument of the right hand side will cancel out, since the term is a quartic function of this argument. Therefore the condition (6.5.17) is satisfied, and so the coupling (6.5.15) is invariant under the perturbative redefinitions (6.5.5) and is an element of $\mathcal{W}_{\rm invariant} \cap \mathcal{W}_{\rm mixed}$.

We now turn to invariance under the linear phase space transformations (6.5.9). These transformations are strongly constrained by the conditions (6.5.12) and (6.5.13) and by the explicit form (6.3.23) of the free field scattering map. The most general linear map consistent with these conditions is[14]

$$\Delta\Psi^{\rm e} \;\rightarrow\; \Delta\Psi^{\rm e}, \tag{6.5.18a}$$

$$\overline{\Psi}^{\rm e} \;\rightarrow\; \overline{\Psi}^{\rm e} + \sigma[\tilde{\Psi}^{\rm e}], \tag{6.5.18b}$$

$$\chi \;\rightarrow\; \chi, \tag{6.5.18c}$$

$$\tilde{\Psi}^{\rm e} \;\rightarrow\; \tilde{\Psi}^{\rm e} + \beta[\Delta\Psi^{\rm e}], \tag{6.5.18d}$$

$$\Psi^{\rm m} \;\rightarrow\; \Psi^{\rm m}, \tag{6.5.18e}$$

where β and σ are linear functionals of their arguments. The electromagnetic memory produced by an incoming scalar field with no incoming vector potential is invariant under these transformations. This follows from the fact that it is the change in the quantity $\Delta\Psi^{\rm e}$, which is invariant by Eq. (6.5.18a), and it is a functional only of χ, which is invariant by (6.5.18c).

3 Change in electromagnetic memory is nonzero

To show that the hard and soft variables in this theory are coupled at higher orders in the perturbative expansion in α, in this section, we specialize our initial data further and take it to be given by

$$\underline{\mathcal{A}}_{A_-}(v,\theta^A) = 0\,, \tag{6.5.19a}$$

$$\underline{\chi}_-(v,\theta^A) = \frac{(1-i)\sqrt{1+i}}{4\sqrt{\pi}}(1+\cos\theta)\exp[-iv - (\frac{1-i}{8})v^2]\,. \tag{6.5.19b}$$

The initial data for the scalar field, $\underline{\chi}_-(v,\theta^A)$, is chosen so that the incoming charge $\int dv \mathcal{J}_{-v} = -i\int dv(\underline{\chi}_-\partial_v\underline{\chi}_-^* - \underline{\chi}_-^*\partial_v\underline{\chi}_-)$ is non-zero. Moreover, it is a mixture of $l=0$ and $l=1$ spherical harmonics on $\mathbb{S}^2 \subset \mathscr{I}^-$ and as such this initial data is a special case of the data considered in 6.4. Note that our initial data is purely hard since the only non-zero field on $\mathscr{I}^-$, $\underline{\chi}_-(v,\theta^A)$ has compact support in v. One can also see from the equations of motion [Eqs. (6.3.16a), (6.3.16b)] that for this

[14] These transformations encompass transformations caused by changes in the choice of function $g(v)$ in the definition (6.2.46c), and also the change of variables (6.2.50) used to diagonalize the Poisson brackets.

choice of initial data, $\underline{\chi}_{+}(u,\theta^A)$ will be non-zero at orders $O(\alpha)$, $O(\alpha^3)$ and higher while $\underline{\mathcal{A}}_{+A}(u,\theta^A)$ will be non-zero at $O(\alpha^4)$ and higher (we have already argued in the previous section that the $O(\alpha^2)$ piece of $\underline{\mathcal{A}}^{(2)}_{+A}$ is zero).

Let us consider evaluating the difference

$$\int du\, \mathcal{J}_{+u}(u,\theta^A) - \int dv\, \mathcal{J}_{-v}(v,\theta^A)\,. \tag{6.5.20}$$

Since $\underline{\Phi}^{(1)}$ is the solution to $\Box\underline{\Phi}^{(1)} = 0$, it follows that Eq. (A.14.20a) still holds. Using Eq. (6.2.9a) and the fall-offs given in Eqs. (6.3.7b) and (6.3.7d), we obtain

$$\int du\, \mathcal{J}_{+u}(u,\theta^A) - \int dv\, \mathcal{J}_{-v}(v,\theta^A) = \int du\, \mathcal{J}^{(4)}_{+u}(u,\theta^A) = i\int du\, [\underline{\chi}_{+}^{(3)*}\partial_u\underline{\chi}_{+}^{(1)} + \underline{\chi}_{+}^{(1)*}\partial_u\underline{\chi}_{+}^{(3)} - c.c]\,, \tag{6.5.21}$$

where we suppress the arguments of these fields everywhere to keep the notation simple. We also omit putting underlines under $\mathcal{J}_{+u}$ and $\mathcal{J}_{-v}$ since these are gauge invariant quantities. Using the fact that $\underline{\chi}_{+}^{(1)}(u,\theta^A) \to 0$ as $u \to \pm\infty$ and that $\underline{\chi}_{+}^{(3)}(u,\theta^A)$ is finite as $u \to \pm\infty$, this can be rewritten as

$$\int du\, \mathcal{J}^{(4)}_{+u}\,(u,\theta^A) = 2i\int du\, [\underline{\chi}_{+}^{(3)*}\partial_u\underline{\chi}_{+}^{(1)} - c.c]\,. \tag{6.5.22}$$

Writing this in terms of the Fourier transforms of the fields as functions of ω [15], we get

$$\int du\, \mathcal{J}^{(4)}_{+u}\,(u,\theta^A) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, \omega\, [\tilde{\underline{\chi}}_{+}^{(3)*}\tilde{\underline{\chi}}_{+}^{(1)} + \tilde{\underline{\chi}}_{+}^{(3)}\tilde{\underline{\chi}}_{+}^{(1)*}]\,. \tag{6.5.23}$$

To evaluate this, we would need to calculate $\underline{\chi}_{+}^{(3)}(u,\theta^A)$ for which the relevant pieces of the equations of motion are

$$\Box\underline{\Phi}^{(3)} = 2i\underline{A}^{(2)a}\nabla_a\underline{\Phi}^{(1)}\,, \tag{6.5.24a}$$

$$\Box\underline{A}^{a(2)} = -ie^2(\underline{\Phi}^{(1)}\nabla^a\underline{\Phi}^{(1)*} - c.c)\,. \tag{6.5.24b}$$

Let us first solve for $A^{a(2)}$. Using a Fourier mode expansion for $\underline{\Phi}^{(1)}(x)$ (see Eq. (A.17.17) for

[15] We take these to be defined by $\underline{\chi}_{+}^{(3)}(u,\theta^A) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega u}\tilde{\underline{\chi}}_{+}^{(3)}(\omega,\theta^A)$ and a similar equation for $\underline{\chi}_{+}^{(1)}(u,\theta^A)$.

more details), Eq. (6.5.24b) is given by

$$\square \underline{A}^{a(2)} = -e^2 \int d\Omega_p \int d\Omega_k \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp \, \frac{p^2 k^2}{2\sqrt{|p||k|}} \left[a_{\overline{k}} p^a a^*_{\overline{p}} e^{-i(k-p)t + i\vec{x}\cdot(\vec{k}-\vec{p})} + c.c \right], \tag{6.5.25}$$

where p^a denotes the 4-vector which in components is $p^a := \{p\,,\vec{p}\}$[16]. Using the retarded propagator in Fourier space,

$$\tilde{G}(k) = \frac{-1}{\vec{k}^2 - (k + i\epsilon)^2}\,, \tag{6.5.26}$$

we see that the solution is given by

$$\underline{A}^{a(2)}(x) = e^2 \int d\Omega_p \int d\Omega_k \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp \frac{p^2 k^2}{2\sqrt{|p||k|}} p^a \left[\frac{a_{\overline{k}} a^*_{\overline{p}} e^{-i(k-p)t + i\vec{x}\cdot(\vec{k}-\vec{p})}}{(\vec{k}-\vec{p})^2 - (k - p + i\epsilon)^2} + \frac{a^*_{\overline{k}} a_{\overline{p}} e^{+i(k-p)t - i\vec{x}\cdot(\vec{k}-\vec{p})}}{(\vec{k}-\vec{p})^2 - (k - p - i\epsilon)^2} \right]. \tag{6.5.27}$$

Note that here, we are using x to denote functional dependence on the 4 spacetime coordinates. We can now use this to solve for $\underline{\Phi}^{(3)}(x)$ using Eq. (6.5.24a). A useful way of extracting the $1/r$ piece of this solution near future null infinity can be derived using results given in [146] which we review in Appendix A.17.2. Using the equation above and the formula derived in Eq. (A.17.27), the Fourier transform in u of the $1/r$ piece of $\underline{\Phi}^{(3)}(x)$ on $\mathscr{I}^+$, labeled $\tilde{\underline{\chi}}_+^{(3)}(\omega, \theta^A)$, is given by

$$\begin{aligned} -2i\pi\, \tilde{\underline{\chi}}_+^{(3)}(\omega, \theta^A) &= -\int d^4y \, (\underline{A}^{a(2)} \nabla_a \underline{\Phi}^{(1)}) e^{i\omega y^0 - i\omega \vec{y}\cdot\hat{x}} \\ &= -\, e^2 \int d^4y \int d\Omega_p \int d\Omega_k \int d\Omega_q \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{p^2 k^2 q^2}{2^{3/2}\sqrt{|p||k||q|}} (iq^a p_a) \\ &\quad \left[\frac{a_{\overline{k}} a^*_{\overline{p}} a_{\overline{q}} e^{-i(k-p+q-\omega)y^0 + i\vec{y}\cdot(\vec{k}-\vec{p}+\vec{q}-\omega\hat{x})}}{(\vec{k}-\vec{p})^2 - (k-p+i\epsilon)^2} + \frac{a^*_{\overline{k}} a_{\overline{p}} a_{\overline{q}} e^{-i(p-k-\omega+q)y^0 + i\vec{y}\cdot(\vec{p}-\vec{k}+\vec{q}-\omega\hat{x})}}{(\vec{k}-\vec{p})^2 - (k-p-i\epsilon)^2} \right], \end{aligned} \tag{6.5.28}$$

where $\hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We split up the integral $\int_{-\infty}^{\infty} dq \to \int_{-\infty}^{0} dq + \int_0^{\infty} dq$ and then

[16] In what follows, we will use this notation for q^a and k^a as well, that is, $q^a := \{q\,,\vec{q}\}$ and $k^a := \{k\,,\vec{k}\}$.

do the $\int d^4y$ and $\int d^3q$ integrals from which we obtain [17]

$$\frac{i}{8\pi^3 e^2}\,\underset{+}{\tilde{\underline{\chi}}}^{(3)}(\omega,\theta^A) = \int d\Omega_p \int d\Omega_k \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dp \Big[\frac{p^2k^2}{2^{3/2}\sqrt{|p||k||\tilde{q}|}}(i\tilde{q}^a p_a)\frac{a_{\overline{k}}a^*_{\overline{p}}a_{\overline{\tilde{q}}}\,\delta(k-p-\omega+|\vec{k}-\vec{p}-\omega\hat{x}|)}{(\vec{k}-\vec{p})^2-(k-p+i\epsilon)^2}$$
$$+\frac{p^2k^2}{2^{3/2}\sqrt{|p||k||\tilde{q}|}}(-i\tilde{q}^a p_a)\frac{a_{\overline{k}}a^*_{\overline{p}}a_{-\overline{\tilde{q}}}\,\delta(k-p-\omega-|\vec{k}-\vec{p}-\omega\hat{x}|)}{(\vec{k}-\vec{p})^2-(k-p+i\epsilon)^2}$$
$$+\frac{p^2k^2}{2^{3/2}\sqrt{|p||k||\tilde{q}|}}(i\tilde{q}^a p_a)\frac{a^*_{\overline{k}}a_{\overline{p}}a_{\overline{\tilde{q}}}\,\delta(-k+p-\omega+|\vec{k}-\vec{p}+\omega\hat{x}|)}{(\vec{k}-\vec{p})^2-(k-p-i\epsilon)^2}$$
$$+\frac{p^2k^2}{2^{3/2}\sqrt{|p||k||\tilde{q}|}}(-i\tilde{q}^a p_a)\frac{a^*_{\overline{k}}a_{\overline{p}}a_{-\overline{\tilde{q}}}\,\delta(-k+p-\omega-|\vec{k}-\vec{p}+\omega\hat{x}|)}{(\vec{k}-\vec{p})^2-(k-p-i\epsilon)^2}\Big], \tag{6.5.29}$$

where we use $\tilde{q}^a$ and $\tilde{q}$ to denote values of the four-momentum q^a and its time component, q, due to the delta-function integrations in each term. We then use this in Eq. (6.5.23). One can show that the resulting expression is finite as $\epsilon \to 0$ and so we take $\epsilon \to 0$ at this stage. Then, doing the integration over ω and denoting

$$f_{\overline{k}} = \sqrt{2}i\pi\,\frac{k}{\sqrt{|k|}}a_{\overline{k}}\,, \tag{6.5.30}$$

(see Eq. (A.17.18) and the discussion below it) we obtain

$$\int du\,\mathcal{J}^{(4)}_{+u}(u,\theta^A) \sim e^2\int d\Omega_p\int d\Omega_k\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dp\Big[\frac{k^3\,p^2\,(\hat{k}\cdot\hat{x}-1)(\hat{p}\cdot\hat{k}-1)}{|-k+p+\vec{k}\cdot\hat{x}-\vec{p}\cdot\hat{x}|^3}\cdot\frac{|k-p-\omega_1|}{k-p-\omega_1}\cdot$$
$$\Big(f_{\overline{k}}f^*_{\overline{p}}f_{-\underline{q}_{(1)}}f^*_{\overline{\omega}_1}+f^*_{\overline{k}}f_{\overline{p}}f_{\underline{q}_{(1)}}f^*_{-\overline{\omega}_1}\Big)-c.c\Big], \tag{6.5.31}$$

where $\sim$ is used to suppress constant factors that will be not be important for our calculation,

$$\underline{q}_{(1)} := k-p-\omega_1\ ,\ \omega_1 := \frac{kp(-\hat{p}\cdot\hat{k}+1)}{-k+p+\vec{k}\cdot\hat{x}-\vec{p}\cdot\hat{x}}\,, \tag{6.5.32}$$

and $\hat{p}$ and $\hat{k}$ are unit vectors in the direction of $\vec{p}$ and $\vec{k}$. For our choice of initial data,

$$f_{\overline{k}} = (1+\hat{k}\cdot\hat{z})\exp[-(i+1)(k-1)^2]\,, \tag{6.5.33}$$

[17] For the one-dimensional Dirac delta function, we use $\int dt e^{i\omega t} = 2\pi\delta(\omega)$. For a 4-vector $k^a = \{k,\vec{k}\}$, we have: $\int d^4y\, e^{-iky^0+i\vec{k}\cdot\vec{y}} = (2\pi)^4\delta(k)\delta^{(3)}(\vec{k})$.

and similarly

$$
\begin{aligned}
f_{\overline{p}} &= (1 + \hat{p} \cdot \hat{z}) \exp[-(i+1)(p-1)^2]\,, \\
f_{\overline{\underline{q}}_{(1)}} &= (1 + \frac{\vec{k} - \vec{p} - \omega_1 \hat{x}}{k - p - \omega_1} \cdot \hat{z}) \exp[-(i+1)(\underline{q}_{(1)} - 1)^2]\,, \\
f_{\overline{\omega}_1} &= (1 + \hat{x} \cdot \hat{z}) \exp[-(i+1)(\omega_1 - 1)^2]\,.
\end{aligned} \tag{6.5.34}
$$

where $\hat{z}$ is the unit vector pointing towards the north pole on $\mathbb{S}^2$. Using this, we evaluate Eq. (6.5.31) numerically. We find that the result is non-zero which implies that [18]

$$
\int du\, \mathcal{J}_{+u}(u, \theta^A) - \int dv\, \mathcal{J}_{-v}(v, \theta^A) = \int du\, \mathcal{J}^{(4)}_{+u}(u, \theta^A) \neq 0\,. \tag{6.5.35}
$$

Using the charge conservation law given in Eq. (6.3.24), this implies that

$$
D^2 \Delta \Psi^{(e)}_{+}(\theta^A) - D^2 \Delta \Psi^{(e)}_{-}(\theta^A) \neq 0\,. \tag{6.5.36}
$$

Since $\Delta\Psi^{(e)}_{-}(\theta^A)$ was zero for our initial data, this shows that purely hard initial data on $\mathscr{I}^-$ can evolve to final data on $\mathscr{I}^+$ where the soft variable $\Delta\Psi^{(e)}_{+}(\theta)$ is non-vanishing. This is a demonstration of how the evolution of soft and hard degrees of freedom is coupled under scattering. Since this is a non-zero change in the memory observable that results from an incoming scalar field with no incoming vector potential, the results of the previous sections imply that this coupling cannot be removed using the perturbative redefinition of fields discussed earlier, thereby showing that the soft and hard degrees of freedom are coupled in an invariant manner. Given the similarity between non-linear gauge theories and gravity, we expect such an effect to exist in general relativity as well and for future experiments like LISA [147], where we expect to detect gravitational memory effects (see, e.g, [148]), to confirm the existence of this coupling.

[18] Numerically evaluating the integral on the right hand side of Eq. (6.5.31) at $\hat{x} = \hat{z}$ or $\theta = 0$, we get that it equals $(-9.54 \pm 0.29)e^2$.

6.6 | Future directions

We expect the coupling of soft and hard sectors to be a generic feature of scattering in any non-linear theory, in particular general relativity. As a result, we expect the gravitational analogues of the infinite number of conservation laws discussed in this chapter (which would be the conservation laws for the BMS charges discussed in chapter 5) to yield an infinity of non-trivial constraints on the scattering of hard degrees of freedom, a prime example of which would be Hawking radiation emitted from an evaporating black hole. This is in line with the suggestions made in [38, 149]. It would be interesting to investigate this further and explicitly study the role played by the BMS charges in constraining Hawking radiation in a black hole evaporation process. It would also be interesting to attempt to write an explicit expression for an entangled pure state that describes the final state of an evaporated black hole (the existence of which is suggested by our calculations in this chapter and by the arguments of Strominger [150]) such that tracing over the soft degrees of freedom leads to a thermal density matrix for Hawking radiation that is consistent with the results of [41]. If such a state is shown to exist, this result would imply that black hole evaporation is, in its entirety, a unitary process and that the apparent "loss of information" is a consequence of having traced over the soft degrees of freedom in the final state. On an independent note, it would also be interesting to study the quantization of the gauge theory studied in this chapter, while carefully accounting for the soft degrees of freedom.

Chapter 7

Appendix

(Adapted with permission from [4–8])

The appendices collected in this chapter contain several results that supplement the discussion in the main body of this thesis.

A.1 Non Bondi frames at null infinity

In chapter 2, we worked in a conformal frame where

$$\Phi := \tfrac{1}{4}\nabla_a n^a \,\hat{=}\, 0\,, \tag{A.1.1}$$

which, as a result of the Einstein equation (Eq. (2.2.1)), implies the conditions Eqs. (2.2.6) and (2.2.7). This choice, however, was made purely for convenience and is not essential to any of the results presented there. In this appendix, we state some of our main results in general conformal frames where $\Phi \neq 0$ and therefore the Bondi condition does not hold. In this context, one is allowed more general conformal transformations of the form

$$\Omega \mapsto \omega\Omega\,, \quad \text{where} \quad \pounds_n \omega \,\hat{\neq}\, 0\,. \tag{A.1.2}$$

Using the fact that S_{ab} is smooth at $\mathscr{I}$, Eq. (2.2.1) implies that in general

$$\nabla_a n_b \mathrel{\widehat{=}} \Phi g_{ab} \,, \quad \lim_{\to \mathscr{I}} \Omega^{-1} n_a n^a = 2\Phi \,, \tag{A.1.3}$$

which generalize Eqs. (2.2.6) and (2.2.7). In these conformal frames, the pullback of the unphysical metric to $\mathscr{I}$, q_{ab}, satisfies

$$\pounds_n q_{ab} \mathrel{\widehat{=}} 2\Phi q_{ab} \,. \tag{A.1.4}$$

It is important to note that Φ is universal in the sense of Sec. 3.4.1, that is, it is independent of the physical spacetime under consideration and can, without any loss of generality, be picked to be the same for the conformal completion of any asymptotically-flat physical spacetime. Hence, $\delta\Phi \mathrel{\widehat{=}} 0$ on phase space.

In these general conformal frames, evolution equations for components of the Weyl tensor on $\mathscr{I}$ given in Eq. (5.2.16) also get generalized and may be found in Eq. 3.3 of [10]. Moreover, the definition of the News tensor is generalized to

$$N_{ab} := 2 Q_a{}^c Q_b{}^d (\pounds_n - \Phi) \sigma_{cd} \,, \tag{A.1.5}$$

where σ_{ab} is (still) defined by Eq. (2.2.16) and has conformal weight 1. In addition, Eq. (2.2.36) is generalized to

$$N_{ab} \mathrel{\widehat{=}} \text{STF} \left[S_{ab} - 2\Phi\sigma_{ab} + 2 \left(\mathscr{D}_a + \tau_a \right) \tau_b \right] \,. \tag{A.1.6}$$

Note that these expressions for the News tensor are defined in any conformal frame and are conformally invariant under all conformal transformations, including those of the form Eq. (A.1.2), whereas, for example, the expression due to Geroch, discussed in Eq. (2.2.39), is only defined in frames where the Bondi condition holds.

In general conformal frames where $\Phi \neq 0$, a BMS symmetry is given by $\xi^a \mathrel{\widehat{=}} \beta n^a + X^a$ where

$X^a l_a \,\hat{=}\, 0$. Here, $\beta\,, X^a$ satisfy the relations (which are derived using the same method as for Eq. (2.4.21))

$$(\pounds_n - \Phi)\beta \,\hat{=}\, \tfrac{1}{2}(\mathscr{D}_a - 2\tau_a)X^a\,, \quad \pounds_n X^a \,\hat{=}\, n^a X_b \tau^b\,, \quad \text{STF}\, \mathscr{D}_a X_b \,\hat{=}\, 0\,, \quad \alpha_{(\xi)} \,\hat{=}\, \Phi\beta + \tfrac{1}{2}\mathscr{D}_a X^a\,. \tag{A.1.7}$$

It is worth emphasizing that this is *not* a different symmetry algebra than the one considered in the body of the paper. The observation that, for example, in these conformal frames a pure supertranslation, $\xi^a \,\hat{=}\, f n^a$, is no longer constrained to satisfy $\pounds_n f \,\hat{=}\, 0$ and that f can have a non-trivial functional dependence on the normal direction along $\mathscr{I}$, should not be taken to mean that symmetry algebra has become bigger. The point is simply that supertranslations are associated with conformally-weighted functions which have different-looking functional forms in different conformal frames.

With these considerations all of our computations can be carried out in a similar manner; one only has to keep track of the chosen Φ through the calculations. For any BMS symmetry with β and X^a subject to Eq. (A.1.7) and N_{ab} given by Eq. (A.1.5), the expressions for the WZ flux (Eq. (2.5.57)) and charge (Eq. (2.5.44)), remain unchanged in these conformal frames and are given by

$$\mathcal{F}[\xi; \Delta\mathscr{I}] \,\hat{=}\, -\frac{1}{16\pi}\int_{\Delta\mathscr{I}} \varepsilon_3 N^{ab}[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta + \pounds_X \sigma_{ab} - \tfrac{1}{2}(\mathscr{D}_c X^c)\sigma_{ab}]\,, \tag{A.1.8}$$

and

$$\mathcal{Q}[\xi; S] \,\hat{=}\, -\frac{1}{8\pi}\int_S \varepsilon_2 \left[\beta(\mathcal{P} + \tfrac{1}{2}\sigma^{ab}N_{ab}) + X^a \mathcal{J}_a + X^a \sigma_{ab}(\mathscr{D}_c - \tau_c)\sigma^{bc} - \tfrac{1}{4}\sigma_{ab}\sigma^{ab}(\mathscr{D}_c - 2\tau_c)X^c\right]\,, \tag{A.1.9}$$

respectively. These expressions are invariant under all conformal transformations including those of the form Eq. (A.1.2).

A.2 | Comparison to other BMS charge formulae

In this appendix we compare the Wald-Zoupas prescription for asymptotic charges, discussed in chapter 2, to other prescriptions for the BMS charge/flux. In particular, we consider the Hamiltonian formulation of Ashtekar and Streubel, the Noether/Komar charge formula (and their linkage versions), and the charges defined by Penrose using twistors at null infinity.

1 | Ashtekar-Streubel flux and charge

A formula for the flux corresponding to BMS symmetries was given by Ashtekar and Streubel in [151]. Rewriting the expression given in Eq. 4.14 of [151] in our conventions, we obtain

$$\mathcal{F}^{(\mathrm{AS})}[\xi;\Delta\mathscr{I}] \mathrel{\hat=} -\frac{1}{16\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\varepsilon}_3 N^{ab}\left(\pounds_\xi D_a - D_a\pounds_\xi\right) l_b\,, \tag{A.2.1}$$

where D_a is the derivative operator induced on $\mathscr{I}$ by the unphysical derivative operator ∇_a as defined on pp. 46 of [62]. The derivative operator D_a satisfies

$$D_a n^b \mathrel{\hat=} 0\,,\quad D_a q_{bc} \mathrel{\hat=} 0\,,\quad D_a \underleftarrow{v_b} \mathrel{\hat=} (\delta_a{}^c + n_a l^c)(\delta_b{}^d + n_d l^b)\nabla_c v_d\,, \tag{A.2.2}$$

where v_a is any covector field in in the unphysical spacetime and $\underleftarrow{v_a}$is its pullback to $\mathscr{I}$.

To compare the flux given in Eq. (A.2.1) to the WZ flux, we need to compute the quantity $\mathrm{STF}[(\pounds_\xi D_a - D_a\pounds_\xi)\, l_b]$ for a BMS symmetry ξ^a. First, note that $(\pounds_\xi D_a - D_a\pounds_\xi)\, l_b$ can be expressed in terms of the Riemann tensor associated with D_a which can, in turn, be written in terms of S_{ab} using Eq. 3.5 of [151]. This gives

$$\left(\pounds_\xi D_a - D_a\pounds_\xi\right) l_b \mathrel{\hat=} \xi^c\left(q_{b[c}S_{a]}{}^d + S_{b[c}\delta_{a]}{}^d\right) l_d - l_c D_a D_b \xi^c\,. \tag{A.2.3}$$

For a BMS symmetry $\xi^a \mathrel{\hat{=}} \beta n^a + X^a$, we get

$$\mathrm{STF}[(\pounds_\xi D_a - D_a \pounds_\xi)\, l_b] \mathrel{\hat{=}} \mathrm{STF}\left[\tfrac{1}{2} X_b S_a{}^c l_c + \tfrac{1}{2} S_{ab}\beta + D_a D_b \beta - l_c D_a D_b X^c\right] . \tag{A.2.4}$$

Next, we use the following relation

$$\tfrac{1}{2} Q_b{}^c S_{ac} l^a \mathrel{\hat{=}} \mathscr{D}^a \sigma_{ab} - \tfrac{1}{2}\mathscr{D}_b \vartheta \,, \tag{A.2.5}$$

where $\vartheta \mathrel{\hat{=}} Q^{ab}\nabla_a l_b$ is the expansion of l^a. Therefore, the first term in Eq. (A.2.4) gives

$$\mathrm{STF}[\tfrac{1}{2} X_b S_a{}^c l_c] \mathrel{\hat{=}} \mathrm{STF}\left[X_b \mathscr{D}^c \sigma_{ac} - \tfrac{1}{2} X_b \mathscr{D}_a \vartheta\right] . \tag{A.2.6}$$

Using **??**, the second term in Eq. (A.2.4) is

$$\mathrm{STF}[\tfrac{1}{2} S_{ab}\beta] \mathrel{\hat{=}} \beta\, \mathrm{STF}[\tfrac{1}{2} N_{ab} - (\mathscr{D}_a \tau_b + \tau_a \tau_b)] \,. \tag{A.2.7}$$

The third term in Eq. (A.2.4) can be expressed as

$$\mathrm{STF}\, D_a D_b \beta \mathrel{\hat{=}} \mathrm{STF}(\mathscr{D}_a \mathscr{D}_b \beta - \pounds_n \beta \nabla_a l_b) = \mathrm{STF}\left[\mathscr{D}_a \mathscr{D}_b \beta - \tfrac{1}{2}(\mathscr{D}_c - 2\tau_c) X^c \sigma_{ab}\right] , \tag{A.2.8}$$

where we have used $\pounds_n \beta \mathrel{\hat{=}} \frac{1}{2}(\mathscr{D}_a - 2\tau_a) X^a$ (Eq. (2.4.21a)). For the fourth term in Eq. (A.2.4), using $l_a X^a \mathrel{\hat{=}} 0$ we have $-l_c D_a D_b X^c = D_a (X^c D_b l_c) + D_b X^c D_a l_c$. We substitute $D_a l_b \mathrel{\hat{=}} \sigma_{ab} + \frac{1}{2} Q_{ab}\vartheta - l_a \tau_b$ and take the STF of the resulting expression. Using $\mathrm{STF}\, \mathscr{D}_a X_b \mathrel{\hat{=}} 0$ (Eq. (2.4.21c)), we get

$$\mathrm{STF}[-l_c D_a D_b X^c] \mathrel{\hat{=}} \mathrm{STF}[\mathscr{D}_a (X^c \sigma_{bc}) + \tfrac{1}{2} X_b \mathscr{D}_a \vartheta - X^c \tau_c \sigma_{ab} + \sigma_a{}^c \mathscr{D}_b X_c] \,. \tag{A.2.9}$$

Putting together Eqs. (A.2.6)–(A.2.9) we get

$$\begin{aligned} \mathrm{STF}[(\pounds_\xi D_a - D_a \pounds_\xi)\, l_b] \mathrel{\hat{=}} \mathrm{STF}\Big[&\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a + \tau_a)(\mathscr{D}_b - \tau_b)\beta \\ &+ X_b \mathscr{D}^c \sigma_{ac} - \tfrac{1}{2}\mathscr{D}_c X^c \sigma_{ab} + \mathscr{D}_a (X^c \sigma_{bc}) + \sigma_a{}^c \mathscr{D}_b X_c\Big] . \end{aligned} \tag{A.2.10}$$

Finally, we simplify the second line above using Prop. A.3.1. As a result, we find

$$\mathcal{F}^{(\mathrm{AS})}[\xi;\Delta\mathscr{I}] \mathrel{\hat{=}} -\frac{1}{16\pi}\int_{\Delta\mathscr{I}} \varepsilon_3 \; N^{ab}\left[\tfrac{1}{2}\beta N_{ab} + (\mathscr{D}_a+\tau_a)(\mathscr{D}_b-\tau_b)\beta + \pounds_X\sigma_{ab} - \tfrac{1}{2}\sigma_{ab}\mathscr{D}_c X^c\right] . \tag{A.2.11}$$

This is the same as the WZ flux formula given in Eq. (2.5.57). A charge expression whose flux is given by the Ashtekar-Streubel formula, for the special case where $\tau_a = \vartheta = 0$, was also considered in [152]. Our charge expression, when adapted to $\tau_a = \vartheta = 0$, differs from the one in [152] which we believe to be a result of sign errors in some of the terms given in [152].

2 Komar formulae and linkage charges

The Komar formulae [153] are formulae (see Eq. C.12 of [4]) for the conserved charges associated with the Killing vector fields of a spacetime. For these spacetimes, the Komar charges match the WZ charges and they both reproduce, for example, the standard expressions for mass and angular momentum in the Kerr spacetime. Note however that there is no first principles derivation for the Komar formulae and some factors in them are essentially "put in by hand" while the WZ charges have a rigorous derivation procedure, as we saw in chapter 2, and, unlike the Komar formulae, are not restricted to the Killing symmetries of a spacetime. The linkage charges [72, 73, 116] (see Eqs. C.16 and C.18 of [4]) are attempts to generalize the Komar formulae to all BMS symmetries. For BMS translations, these coincide with the WZ charge; for BMS Lorentz symmetries on a cross-section, they are equal to *half* the WZ charge while for general supertranslations, they differ from the WZ charge completely. In fact, for supertranslations which are not translations, these charges are non-zero even in Minkowski spacetime when the cross-section is not shear-free. It follows that the flux of the linkage charges between generic, non shear-free cross-sections does not vanish in Minkowski spacetime. In the interest of brevity, we skip all the calculations that prove the assertions made here but they may be found in full detail in Appendix. C of [4]. Suffice it to say that from this discussion, we see that the Komar formulae are restricted in their application while linkage charges violate item (5) on our list of criteria for a physically reasonable notion of asymptotic charge that was given at the start of chapter 2. The WZ charges are therefore more useful and better physically motivated and in this sense "more correct" than the Komar/linkage charges.

3 | Twistor charge

Another expression for the charge associated with a BMS symmetry was given by Penrose [154] motivated by twistor theory (see also [155–157]). Penrose's formula only holds on a fixed cross section of $\mathscr{I}$ where it is possible to ioslate a Poincaré subalgebra of the BMS algebra (recall the discussion in Sec. 2.4). In this special case, this formula can be shown to match the WZ charge (the reader is once again referred to [4] for the details). However, the fact that Penrose's formula only works in the aforementioned restricted context means that its utility compared to the WZ charge is limited.

A.3 | Symmetric and tracefree tensors in two dimensions

This appendix collects some useful identities for symmetric, tracefree tensors in two dimensions. In the body of chapter 2 these are applied to tensors on a cross-section S of $\mathscr{I}$. However, they hold for any 2-dimensional orientable manifold S with a Riemannian metric Q_{ab}. Therefore, for simplicity, we drop the hats on all equalities.

For a tensor field A_{ab}, let STF A_{ab} be the symmetric tracefree part as defined in Eq. (2.2.13). Then, we can decompose A_{ab} into an antisymmetric part, a pure trace part, and a symmetric, tracefree part:

$$A_{ab} = A_{[ab]} + \tfrac{1}{2}A^c{}_c Q_{ab} + \mathrm{STF}\, A_{ab}. \tag{A.3.1}$$

We then have the following proposition whose detailed proof may be found in Appendix. D of [4].

Proposition A.3.1. *Let A_{ab} and B_{ab} be symmetric and tracefree tensors on S; then the following identities hold:*

$$\mathrm{STF}\,(A_{ac}B^c{}_b) = 0\,, \tag{A.3.2}$$

$$\mathscr{D}_{[a}A_{b]c} = Q_{c[a}\mathscr{D}^d A_{b]d}\,, \tag{A.3.3}$$

where $\mathscr{D}$ is the covariant derivative compatible with Q_{ab}, and

$$\text{STF}\left(\pounds_X A_{ab} - \tfrac{1}{2}A_{ab}\mathscr{D}_c X^c\right) = \text{STF}\left[X_a \mathscr{D}^c A_{bc} + \mathscr{D}_a(X^c A_{bc}) + A_a{}^c \mathscr{D}_{[b}X_{c]}\right], \tag{A.3.4}$$

for any vector field X^a on S.

Further, we also have the following.

Proposition A.3.2. *Let A_{ab} be a symmetric, trace-free tensor on S that satisfies $\mathscr{D}_a A^{ab} = 0$. Then A_{ab} vanishes.*

Unlike the first proposition above, which holds for any two-dimensional manifold S, this is a consequence of the fact that any cross-section S of $\mathscr{I}$ has the topology of $\mathbb{S}^2$. The proof is based on the one given in Lemma 5 of [62] (for similar discussion, see Sec. A.4 of [152] or Appendix. C of [101]) and may be found in Appendix. D of [4].

A.4 | Coordinates, universal structure and asymptotic expansions near i^0

In this appendix, to supplement the discussion in chapter 3, we construct a suitable asymptotic coordinate system near spatial infinity. Using these coordinates, we explicitly demonstrate the universal structure near i^0 described in Sec. 3.4.1. We also describe the asymptotic expansion of the unphysical and physical metrics in these coordinates, thus making contact with the expansions used in previous works [90–92].

Consider the unphysical spacetime (M, g_{ab}) obtained from some physical spacetime satisfying Def. 3.2.1. The unphysical metric $\boldsymbol{g}_{ab}$ at i^0 induces a metric which is isometric to the Minkowski metric in the tangent space Ti^0. Thus we can introduce asymptotically Cartesian coordinates (t, x, y, z) so that i^0 is at the origin of this coordinate system and

$$\boldsymbol{g}_{ab} \equiv -dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{A.4.1}$$

Note that $x^i = (t, x, y, z)$ define a C^1 coordinate system at i^0. To define a $C^{>1}$ differential structure, we allow any other coordinate chart $x'^i(x)$ such that

$$\frac{\partial^2 x'^i(x)}{\partial x^j \partial x^k} \text{ and } \frac{\partial^2 x^i(x')}{\partial x'^j \partial x'^k} \text{ are } C^{>-1} \text{ at } i^0 \,. \tag{A.4.2}$$

A collection of all coordinate charts related by Eq. (A.4.2) defines a choice of $C^{>1}$-structure on M at i^0; see [95] and Appendix A of [9] for details.

It is more convenient to use coordinates which are adapted to the space of unit spacelike directions $\mathscr{H}$. Thus, we define (ρ, τ) by

$$\rho^2 := -t^2 + x^2 + y^2 + z^2 \,, \quad \tanh \tau := \frac{t}{\sqrt{x^2 + y^2 + z^2}} \,. \tag{A.4.3}$$

In these coordinates the metric in Ti^0 takes the form

$$\boldsymbol{g}_{ab} \equiv d\rho^2 + \rho^2 \left(-d\tau^2 + \cosh^2 \tau s_{AB} d\theta^A d\theta^B \right) \,, \tag{A.4.4}$$

where s_{AB} is the unit metric on $\mathbb{S}^2$ in some coordinates θ^A which can be the usual (θ, ϕ) coordinates. Note that the coordinates (ρ, τ, θ^A) are *not* $C^{>1}$ coordinates — the bases $(d\rho, \rho d\tau, \rho d\theta^A)$ are not continuous but are direction-dependent at i^0.

The unit spatial directions $\vec{\eta}$ then correspond to the unit vectors ∂_ρ in Ti^0 which are parameterized by (τ, θ^A). The space of directions $\mathscr{H}$ is then the surface $\rho = 1$ in Ti^0 with the induced metric

$$\boldsymbol{h}_{ab} \equiv -d\tau^2 + \cosh^2 \tau s_{AB} d\theta^A d\theta^B \,. \tag{A.4.5}$$

The reflection of the directions $\vec{\eta} \mapsto -\vec{\eta}$ then induces the reflection isometry

$$(\tau, \theta^A) \mapsto (-\tau, -\theta^A) \,, \tag{A.4.6}$$

on $\mathscr{H}$, where $\theta^A \mapsto -\theta^A$ is the antipodal reflection on $\mathbb{S}^2$.

So far, we have only considered the structure at i^0. Now, we extend the metric away from i^0.

Since the unphysical metric g_{ab} is $C^{>0}$ and limits to $\boldsymbol{g}_{ab}$ at i^0 (where $\rho = 0$), it can be verified that g_{ab} admits an expansion in ρ of the form

$$\begin{aligned} g_{ab} \equiv & \left[1 + \sigma\rho + o(\rho)\right]^2 d\rho^2 + 2\left[\rho A_a + o\left(\rho\right)\right] d\rho(\rho dy^a) \\ & + \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} + o(\rho)\right](\rho dy^a)(\rho dy^b)\,, \end{aligned} \tag{A.4.7}$$

where $y^a = (\tau, \theta^A)$ are coordinates on the unit hyperboloid, and $h^{(0)}_{ab} \equiv \boldsymbol{h}_{ab}$ is the unit hyperboloid metric. The expansion coefficients σ, A_a and $h^{(1)}_{ab}$ can be thought of as tensor fields on $\mathscr{H}$. The $o(\rho)$ denotes terms which falloff faster than ρ in the limit to i^0, that is, $\lim\limits_{\rho\to 0} \rho^{-1} o(\rho) = 0$.

For the conformal factor, one can choose

$$\Omega = \rho^2\,, \tag{A.4.8}$$

which can be verified to satisfy all the conditions in Def. 3.2.1, that is, in the limit $\rho \to 0$, $\Omega = 0$, $\nabla_a\Omega = 0$ and $\nabla_a\nabla_b\Omega = 2\boldsymbol{g}_{ab}$. Before considering the physical metric let us analyze the universal structure at i^0.

From the discussion above, it is clear that the metric $\boldsymbol{g}_{ab}$ and the space of directions $\mathscr{H}$ is universal, that is, independent of which unphysical metric is chosen. What is the structure corresponding to the equivalence classes of $C^{>1}$ curves described in Sec. 3.4.1? Consider the $C^{>1}$ curves Γ_v through i^0 with tangents $v^a \equiv \partial_\rho$ in these coordinates. Further, with the choice of conformal factor in Eq. (A.4.8) we have

$$\eta^a = \nabla^a\Omega^{1/2} \equiv (1 - 2\rho\sigma)\frac{\partial}{\partial\rho} + \rho h^{(0)ab} A_b \frac{\partial}{\rho\partial y^a} + o(\rho)\,. \tag{A.4.9}$$

From Eq. (3.4.27) we see that the curves Γ_v (with tangent $v^a \equiv \partial_\rho$) will be equivalent to the curves Γ_η (with tangent η^a) for all spacetimes if we can always choose A_a to vanish. This can be accomplished using the freedom in the choice of the hyperboloid coordinates y^a at "next order" in ρ. Consider the coordinate transformation[1]

$$\rho \mapsto \rho\,, \quad y^a \mapsto y^a + \rho h^{(0)ab} A_b\,. \tag{A.4.10}$$

By rewriting this in terms of the Cartesian coordinates $x^i = (t, x, y, z)$, it can be verified that the transformation Eq. (A.4.10) is a $C^{>1}$ coordinate transformation (Eq. (A.4.2)). It can be also be verified that using this transformation the $d\rho dy^a$ term in the metric, i.e. A_a, vanishes in the new coordinates. Thus, the curves Γ_v and Γ_η can always be chosen to be equivalent. Further, this choice can always be made in any choice of the physical spacetime. Thus, the equivalence classes of $C^{>1}$ curves through i^0 is also universal.

Having made this choice the unphysical metric takes the form

$$g_{ab} \equiv \left[1 + \sigma\rho + o(\rho)\right]^2 d\rho^2 + \rho\, o(\rho) d\rho dy^a + \rho^2 \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} + o(\rho)\right] dy^a dy^b\,. \tag{A.4.11}$$

To get the form of the physical metric $\hat{g}_{ab} = \Omega^{-2} g_{ab}$, we use Eq. (A.4.8) and define the Beig-Schmidt coordinate $\rho_{(\mathrm{BS})} := 1/\rho$ to obtain

$$\begin{aligned}\hat{g}_{ab} \equiv & \left[1 + \frac{\sigma}{\rho_{(\mathrm{BS})}} + o(1/\rho_{(\mathrm{BS})})\right]^2 d\rho^2_{(\mathrm{BS})} + \rho_{(\mathrm{BS})} o(1/\rho_{(\mathrm{BS})}) d\rho_{(\mathrm{BS})} dy^a \\ & + \rho^2_{(\mathrm{BS})} \left[h^{(0)}_{ab} + \frac{h^{(1)}_{ab}}{\rho_{(\mathrm{BS})}} + o(1/\rho_{(\mathrm{BS})})\right] dy^a dy^b\,,\end{aligned} \tag{A.4.12}$$

This is the form of the physical metric assumed by Beig and Schmidt [90].

The asymptotic potentials Eq. (3.4.9) are related to the metric coefficients in the above expansion by

$$\boldsymbol{E} \equiv 4\sigma\,, \quad \boldsymbol{K}_{ab} \equiv -2(h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab})\,. \tag{A.4.13}$$

From these the asymptotic Weyl tensors can be computed using Eq. (3.4.11). Note that the parity condition Eq. (3.4.33) imposed on $\boldsymbol{E}$ to eliminate the logarithmic translation ambiguity then corresponds to requiring

$$\sigma(\tau, \theta^A) = \sigma(-\tau, -\theta^A)\,. \tag{A.4.14}$$

[1] This is essentially the unphysical spacetime version of the coordinate transformations consider in Lemma 2.2 of [90].

From Eq. (A.4.13) it straightforward to see that our expression for supertranslation charges Eq. (3.7.4) matches the expression obtained by Compère and Dehouck in Eq. 4.88 of [92].

For the "subleading" magnetic Weyl tensor $\boldsymbol{\beta}_{ab}$ (defined by Eq. (3.4.14) when $\boldsymbol{B}_{ab} = 0$) to exist, we need additional regularity conditions on the metric expansion Eq. (A.4.7). Thus, to define $\boldsymbol{\beta}_{ab}$ we assume the "next order" expansion

$$\begin{aligned} g_{ab} \equiv & \left[1 + \sigma\rho + o(\rho)\right]^2 d\rho^2 + \rho\, o(\rho) d\rho dy^a \\ & + \rho^2 \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} + \rho^2 h^{(2)}_{ab} + o(\rho^2)\right] dy^a dy^b \,, \end{aligned} \tag{A.4.15}$$

where $h^{(2)}_{ab}$ is a smooth tensor on $\mathscr{H}$. Computing $\boldsymbol{\beta}_{ab}$ using this expansion and 3.4.14 as well as $\boldsymbol{B}_{ab} = 0$, we obtain

$$\boldsymbol{\beta}_{ab} = \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c h^{(2)d}_{b)} - \tfrac{1}{8}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c \boldsymbol{E}\boldsymbol{K}^d{}_{b)} - \tfrac{1}{16}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^d{}_e \,. \tag{A.4.16}$$

When the conformal factor is chosen so that $\boldsymbol{K}_{ab} = 0$, the above expression simplifies considerably. In this case, our Lorentz charge matches the one found by Compère, Dehouck and Virmani [91]. We discuss the case when $\boldsymbol{B}_{ab} \neq 0$ in Appendix A.8.

A.5 | Some useful relations on $\mathscr{H}$

In this appendix we collect some relations on the unit-hyperboloid $\mathscr{H}$ which are useful for the analysis in chapter 3.

The Riemann tensor of $\mathscr{H}$ is given by

$$\boldsymbol{\mathcal{R}}_{abcd} = \boldsymbol{h}_{ac}\boldsymbol{h}_{bd} - \boldsymbol{h}_{ad}\boldsymbol{h}_{bc} \,. \tag{A.5.1}$$

Using this, it is easy to derive simple expressions for the action of derivatives on tensor fields on $\mathscr{H}$ (see also Appendix A of [90]).

1 Killing vector fields

Let $\boldsymbol{X}^a$ be a Killing vector field on $\mathscr{H}$, so that $\boldsymbol{D}_{(a}\boldsymbol{X}_{b)} = 0$. For any Killing vector field, using Eq. C.3.6 of [61] and Eq. (A.5.1), we have

$$\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{X}_c = \boldsymbol{\mathcal{R}}_{cbad}\boldsymbol{X}^d = \boldsymbol{h}_{ac}\boldsymbol{X}_b - \boldsymbol{h}_{ab}\boldsymbol{X}_c\,. \tag{A.5.2}$$

Contracting the indices a and b we get

$$\boldsymbol{D}^2\boldsymbol{X}_a + 2\boldsymbol{X}_a = 0\,. \tag{A.5.3}$$

Define the "dual" vector field ${}^\star\boldsymbol{X}^a$ on $\mathscr{H}$ for any Killing vector field $\boldsymbol{X}^a$ by

$${}^\star\boldsymbol{X}^a := \tfrac{1}{2}\boldsymbol{\varepsilon}^{abc}\boldsymbol{D}_b\boldsymbol{X}_c\,. \tag{A.5.4}$$

Then, using Eq. (A.5.2), we have

$$\boldsymbol{D}_a{}^\star\boldsymbol{X}_b = \boldsymbol{\varepsilon}_{abc}\boldsymbol{X}^c\,, \quad \boldsymbol{X}^a = -\tfrac{1}{2}\boldsymbol{\varepsilon}^{abc}\boldsymbol{D}_b{}^\star\boldsymbol{X}_c = -{}^\star({}^\star\boldsymbol{X})^a\,, \quad \boldsymbol{D}_a\boldsymbol{X}_b = -\boldsymbol{\varepsilon}_{abc}{}^\star\boldsymbol{X}^c\,. \tag{A.5.5}$$

In particular $\boldsymbol{D}_{(a}{}^\star\boldsymbol{X}_{b)} = 0$ and so ${}^\star\boldsymbol{X}^a$ is also a Killing vector field on $\mathscr{H}$. In a suitable choice of coordinates on $\mathscr{H}$, this relation maps Lorentz rotations and Lorentz boosts into each other; see Appendix B of [91].

The relationship between the Killing vector fields on $\mathscr{H}$ and Lorentz transformations in the tangent space Ti^0 is as follows. Let $\boldsymbol{\Lambda}_{ab}$ be a *direction-independent* antisymmetric tensor at i^0 corresponding to an infinitesimal Lorentz transformation in Ti^0. Then the *direction-dependent* vector field defined by[2]

$$\boldsymbol{X}^a(\vec{\eta}) := \boldsymbol{\Lambda}^{ab}\boldsymbol{\eta}_b\,, \tag{A.5.6}$$

is tangent to $\mathscr{H}$. Further, since $\boldsymbol{\Lambda}_{ab}$ is direction-independent, $\boldsymbol{\partial}_c\boldsymbol{\Lambda}_{ab} = 0$. Projecting the indices of

[2] The relation Eq. (A.5.6) is the "dual" of the relation used below Eq. 27 of [16].

$\boldsymbol{\partial}_c \boldsymbol{\Lambda}_{ab} = 0$ tangent and normal to $\mathscr{H}$ in all possible ways, it follows that $\boldsymbol{X}^a$ is a Killing vector field on $\mathscr{H}$ and

$$\boldsymbol{\Lambda}_{ab} = -\boldsymbol{D}_a \boldsymbol{X}_b - 2\boldsymbol{\eta}_{[a} \boldsymbol{X}_{b]} = \boldsymbol{\varepsilon}_{abc} {}^\star\boldsymbol{X}^c + \boldsymbol{\eta}_{[a} \boldsymbol{\varepsilon}_{b]cd} \boldsymbol{D}^{c\star} \boldsymbol{X}^d \,, \tag{A.5.7}$$

where the last equality uses Eq. (A.5.5). Similarly, it can be shown that if $\boldsymbol{X}^a$ is the Killing vector field on $\mathscr{H}$ corresponding to $\boldsymbol{\Lambda}_{ab}$ through Eq. (A.5.6), then $(-{}^\star\boldsymbol{X}^a)$ is the Killing vector field on $\mathscr{H}$ corresponding to the "dual" Lorentz transformation $*\boldsymbol{\Lambda}_{ab} := \frac{1}{2}\boldsymbol{\varepsilon}_{ab}{}^{cd}\boldsymbol{\Lambda}_{cd}$.

2 Integral identity for symmetric tensors on $\mathscr{H}$

Let $\boldsymbol{T}_{ab}$ be any symmetric tensor on $\mathscr{H}$. Then $\boldsymbol{T}_{ab}$, its curl and divergence are related by the identity

$$-2\boldsymbol{T}_{ab} {}^\star\boldsymbol{X}^b + 2\boldsymbol{\varepsilon}_{cd(a} \boldsymbol{D}^c \boldsymbol{T}^d{}_{b)} \boldsymbol{X}^b - \boldsymbol{D}_c \boldsymbol{T}^{cb} \boldsymbol{D}_a {}^\star\boldsymbol{X}_b = \boldsymbol{D}^b \left(\boldsymbol{\varepsilon}_{abc} \boldsymbol{T}^c{}_d \boldsymbol{X}^d + 2\boldsymbol{T}^c{}_{[a} \boldsymbol{D}_{b]} {}^\star\boldsymbol{X}_c \right) \,, \tag{A.5.8}$$

where $\boldsymbol{X}^a$ is any Killing vector on $\mathscr{H}$ and ${}^\star\boldsymbol{X}^a$ is the corresponding "dual" Killing vector (Eq. (A.5.4)). This identity can be verified by expanding out the right hand side and using Eqs. (A.5.1), (A.5.4) and (A.5.5). Note that the right hand side of Eq. (A.5.8) corresponds to an exact 2-form on $\mathscr{H}$, and thus vanishes when integrated over any cross-section S of $\mathscr{H}$. This gives us the following useful integral identity on any cross-section S

$$\int_S \boldsymbol{\varepsilon}_2 \; \boldsymbol{u}^a \boldsymbol{T}_{ab} {}^\star\boldsymbol{X}^b = \int_S \boldsymbol{\varepsilon}_2 \; \boldsymbol{u}^a \left[\boldsymbol{\varepsilon}_{cd(a} \boldsymbol{D}^c \boldsymbol{T}^d{}_{b)} \boldsymbol{X}^b - \tfrac{1}{2} \boldsymbol{D}_c \boldsymbol{T}^{cb} \boldsymbol{D}_a {}^\star\boldsymbol{X}_b \right] \,. \tag{A.5.9}$$

3 Closed and exact forms

For some results in chapter 3, we needed to argue that certain 2-forms on $\mathscr{H}$ which are closed are also exact, so that their integral on cross-sections of $\mathscr{H}$ vanishes. In general, not all closed 2-forms on $\mathscr{H}$ are exact since the topology of $\mathscr{H}$ is $\mathbb{S}^2 \times \mathbb{R}$ and the second de Rahm cohomology group is nontrivial. However, when the closed 2-forms considered are local and covariant functionals of suitable fields (as described below) then they can be shown to be exact by a general theorem of Wald [110].

In the theorem stated below, the differential forms $\mu[\phi, \psi]$ under consideration will be functionals of two types of fields. The "dynamical fields", denoted by ϕ, are arbitrary cross-sections of some vector bundle, and we require that $d\mu = 0$ for every cross-section ϕ. The form μ also can depend on some "background fields", denoted by ψ. The "background fields" ψ need not have a linear structure and, unlike the dynamical fields, are allowed to satisfy (possibly nonlinear) differential equations. Now we can state the theorem from [110].

Theorem 1 ([110]). *Let $\mu[\phi, \psi]$ be a p-form on a d-dimensional manifold M with $p < d$, which is a local and covariant functional of a collection of two sets of fields (ϕ, ψ) (as described above) and finitely many of their derivatives on M. Then, if for* any *"background fields" ψ*

(1) $d\mu[\phi, \psi] = 0$ for all *cross-sections of the vector bundle of "dynamical fields" ϕ and*

(2) $\mu[\phi, \psi] = 0$ for the zero cross-section $\phi = 0$

then there exists a $(p-1)$-form $\nu[\phi, \psi]$ which is a local and covariant functional of (ϕ, ψ) and finitely many of their derivatives such that $\mu[\phi, \psi] = d\nu[\phi, \psi]$. That is the closed p-form μ is also exact.

It is worth emphasizing that for this theorem to be applicable, the "dynamical fields" need to have a linear structure as the cross-sections of some vector bundle and further, the p-form μ must be closed for all possible cross-sections of this vector bundle, i.e., one must be able to freely specify the "dynamical fields" and all of their derivatives at any point of M. In contrast, the "background fields" ψ, need not have a linear structure and are allowed to satisfy differential equations. In fact, the set of "background fields" can even be empty. The proof in [110] also provides a constructive procedure for finding the $(p-1)$-form ν although we will not need to use this construction.

For our applications of this theorem, we will be concerned with closed 2-forms on $\mathscr{H}$. When such a 2-form satisfies the conditions required for Theorem 1, using the volume element $\boldsymbol{\varepsilon}_{abc}$ on $\mathscr{H}$, we will write this 2-form in terms of a covector $\boldsymbol{\mu}_a$ such that $\boldsymbol{D}^a \boldsymbol{\mu}_a = 0$. Then, we will conclude that this 2-form is exact and thus

$$\boldsymbol{D}^a \boldsymbol{\mu}_a[\phi, \psi] = 0 \implies \int_S \boldsymbol{\varepsilon}_2 \, \boldsymbol{u}^a \boldsymbol{\mu}_a[\phi, \psi] = 0 \,, \tag{A.5.10}$$

for any cross-section S of $\mathscr{H}$ with $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{u}^a$ being the area element and normal to S. The choice of the "dynamical fields" ϕ depends on the particular case in consideration. Since the fields $\boldsymbol{E}$, $\boldsymbol{K}_{ab}$ and $\boldsymbol{\beta}_{ab}$ satisfy differential equations of motion (Eqs. (3.4.12), (3.4.13) and (3.4.20)) they cannot be used as the "dynamical fields". Similarly, the Lorentz vector fields $\boldsymbol{X}^a$ form a 6-dimensional vector space and cannot be arbitrary sections of some vector bundle. Therefore, these can also not be used as "dynamical fields" in applications of Theorem 1. Thus, these fields, along with the metric and volume form on $\mathscr{H}$, will always be in the collection of "background fields" ψ. In contrast, the supertranslation symmetries $\boldsymbol{f}$, the freedom in the conformal factor $\boldsymbol{\alpha}$ (Remark 3.2.1) and the scalar potential $\boldsymbol{k}$ for $\boldsymbol{K}_{ab}$ (when $\boldsymbol{B}_{ab} = 0$) are free functions on $\mathscr{H}$ and will be used as "dynamical fields" in our applications of this theorem.

A.6 | Conformal transformation of the Spi Lorentz charge

In Sec. 3.7.3 we argued that under conformal transformations the Lorentz charge at spatial infinity shifts by the charge of a spi supertranslation (Eq. (3.7.27)). In this appendix we describe the explicit computation of this transformation.

Using Eqs. (3.4.22) and (3.4.23), and the fact that $\boldsymbol{E}$ is conformally-invariant, we have the following transformation for the tensor $\boldsymbol{W}_{ab}$ defined in Eq. (3.7.10) under changes of the conformal factor

$$\boldsymbol{W}_{ab} \mapsto \boldsymbol{W}_{ab} + \tfrac{1}{4}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\left[\boldsymbol{D}_{b)}\boldsymbol{E}\boldsymbol{D}^d\boldsymbol{\alpha} + \boldsymbol{D}^d\boldsymbol{E}\boldsymbol{D}_{b)}\boldsymbol{\alpha}\right] + \tfrac{1}{8}\boldsymbol{\varepsilon}_{abc}\boldsymbol{D}^c\boldsymbol{E}(\boldsymbol{D}^2\boldsymbol{\alpha} + 3\boldsymbol{\alpha})\,. \tag{A.6.1}$$

Thus, we have (note that the Lorentz vector does not transform under changes of the conformal factor; see Remark 3.6.2)

$$\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b \mapsto \boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b + \tfrac{1}{4}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}{}^{\star}\boldsymbol{X}^b + \tfrac{1}{8}(\boldsymbol{D}^2\boldsymbol{\alpha} + 3\boldsymbol{\alpha})\boldsymbol{D}^b\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{X}_b\,, \tag{A.6.2}$$

where we have defined the shorthand $\boldsymbol{T}_{ab} := \boldsymbol{D}_a\boldsymbol{E}\boldsymbol{D}_b\boldsymbol{\alpha} + \boldsymbol{D}_b\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{\alpha}$ and used the last identity in

Eq. (A.5.5). Now using the identity Eq. (A.5.9) (with $\boldsymbol{X}^a$ replaced by ${}^\star\boldsymbol{X}^a$), we have

$$\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}{}^\star\boldsymbol{X}^b = -\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\left[\tfrac{1}{2}\boldsymbol{D}_c\boldsymbol{T}^{cb}\boldsymbol{D}_a\boldsymbol{X}_b + \boldsymbol{T}_{ab}\boldsymbol{X}^b\right]. \tag{A.6.3}$$

A straightforward but tedious computation using the definition of $\boldsymbol{T}_{ab}$, Eqs. (3.4.12), (A.5.1) and (A.5.3) gives

$$\begin{aligned}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}{}^\star\boldsymbol{X}^b = \int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\big[&-\tfrac{1}{2}(\boldsymbol{D}^2\boldsymbol{\alpha}+3\boldsymbol{\alpha})(2\boldsymbol{E}\boldsymbol{X}_a + \boldsymbol{D}^b\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{X}_b)\\ &+(\boldsymbol{E}\boldsymbol{D}_a\pounds_{\boldsymbol{X}}\boldsymbol{\alpha} - \boldsymbol{D}_a\boldsymbol{E}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha})\big],\end{aligned} \tag{A.6.4}$$

where we have dropped terms that integrate to zero on S. Using the equation above in Eq. (A.6.2), we get

$$\begin{aligned}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b \mapsto \int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b + \tfrac{1}{4}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\big[&(\boldsymbol{E}\boldsymbol{D}_a\pounds_{\boldsymbol{X}}\boldsymbol{\alpha} - \boldsymbol{D}_a\boldsymbol{E}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha})\\ &-(\boldsymbol{D}^2\boldsymbol{\alpha}+3\boldsymbol{\alpha})\boldsymbol{E}\boldsymbol{X}_a\big].\end{aligned} \tag{A.6.5}$$

Further, from Eq. (3.4.22), we also have

$$-\tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a \mapsto -\tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a + \tfrac{1}{4}(\boldsymbol{D}^2\boldsymbol{\alpha}+3\boldsymbol{\alpha})\boldsymbol{E}\boldsymbol{X}_a. \tag{A.6.6}$$

Thus,

$$\begin{aligned}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\left[\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b - \tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a\right] \mapsto &\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\left[\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b - \tfrac{1}{8}\boldsymbol{K}\boldsymbol{E}\boldsymbol{X}_a\right]\\ &+\tfrac{1}{4}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\left(\boldsymbol{E}\boldsymbol{D}_a\pounds_{\boldsymbol{X}}\boldsymbol{\alpha} - \boldsymbol{D}_a\boldsymbol{E}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}\right).\end{aligned} \tag{A.6.7}$$

The Lorentz charge Eq. (3.7.20) then transforms as

$$\mathcal{Q}[\boldsymbol{X}^a;S] \mapsto \mathcal{Q}[\boldsymbol{X}^a;S] - \frac{1}{16\pi}\int_S \boldsymbol{\varepsilon}_2\ \boldsymbol{u}^a\tfrac{1}{2}\left(\boldsymbol{E}\boldsymbol{D}_a\pounds_{\boldsymbol{X}}\boldsymbol{\alpha} - \boldsymbol{D}_a\boldsymbol{E}\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}\right). \tag{A.6.8}$$

Comparing this with Eq. (3.7.4), we recognize the last integral above as the charge of the supertranslation given by $\boldsymbol{f} - {}^1\!/_2\pounds_{\boldsymbol{X}}\boldsymbol{\alpha}$. Thus, the Lorentz charge shifts by the charge of a supertranslation under changes of the conformal factor, as argued in Sec. 3.7.3.

A.7 Ambiguities in the Spi-charges

In this section, we analyze the ambiguities in our procedure for defining the Spi charges, discussed in chapter 3. We show our Spi charges are unambiguously defined by the choice of the symplectic current for general relativity made in Eqs. (3.5.14) and (3.5.15).

Recall that our charges on a cross-section S of $\mathscr{H}$ are defined by

$$\delta \mathcal{Q}[(\boldsymbol{f}, \boldsymbol{X}^a); S] := \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{Q}_a(\delta g; (\boldsymbol{f}, \boldsymbol{X}^a)) + \int_S \boldsymbol{X} \cdot \boldsymbol{\Theta}(\delta g)\,, \tag{A.7.1}$$

with $\mathcal{Q} = 0$ on Minkowski spacetime, which acts as our reference solution. The covector $\boldsymbol{Q}_a$ is a local and covariant functional of its arguments, is linear in the metric perturbations and the asymptotic symmetry and satisfies Eq. (3.7.2). The 3-form $\boldsymbol{\Theta}$ is a symplectic potential for $\underleftarrow{\boldsymbol{\omega}}$ and satisfies Eq. (3.7.16).

Given a fixed choice of the symplectic current, it follows from Eqs. (3.7.2) and (3.7.16) that the ambiguities in the choice of $\boldsymbol{Q}_a$ and the $\boldsymbol{\Theta}$ are of the form

$$\begin{aligned} \boldsymbol{Q}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X})) &\mapsto \boldsymbol{Q}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X})) + \boldsymbol{\mu}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X}))\,, \\ \boldsymbol{\Theta}(g, \delta g) &\mapsto \boldsymbol{\Theta}(g, \delta g) + \boldsymbol{\varepsilon}_3 \delta \boldsymbol{\Xi}(g)\,, \end{aligned} \tag{A.7.2}$$

where the covector $\boldsymbol{\mu}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X}))$ is a local and covariant functional of its arguments, is linear in the metric perturbations and the asymptotic symmetry, and further satisfies

$$\boldsymbol{D}^a \boldsymbol{\mu}_a(g; \delta g; (\boldsymbol{f}, \boldsymbol{X})) = 0\,, \tag{A.7.3}$$

for all background spacetimes and perturbations (satisfying the background and linearized equations of motion respectively) and all asymptotic symmetries. Moreover, $\boldsymbol{\Xi}$ is any local and covariant functional of the background spacetime metric.

Under these ambiguities, the definition of $\delta\mathcal{Q}$ (Eq. (A.7.1)) changes by

$$\delta\mathcal{Q}[(\boldsymbol{f},\boldsymbol{X}^a);S] \mapsto \delta\mathcal{Q}[(\boldsymbol{f},\boldsymbol{X}^a);S] + \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\mu}_a(g;\delta g;(\boldsymbol{f},\boldsymbol{X})) - \delta\int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{X}_a \boldsymbol{\Xi}(g)\,. \tag{A.7.4}$$

Since the integrated charge $\mathcal{Q}$ is fixed by the requirement that it vanish on Minkowski spacetime (where $\boldsymbol{E} = \boldsymbol{\beta}_{ab} = 0$), we only need to analyze the ambiguities in $\delta\mathcal{Q}$.

We now argue that the last two integrals above must vanish under the following assumptions

(1) $\boldsymbol{\mu}_a$ and $\boldsymbol{\Xi}$ are local and covariant functionals of their arguments as mentioned above with $\boldsymbol{\mu}_a$ satisfying Eq. (A.7.3).

(2) The Lorentz charge $\mathcal{Q}[(\boldsymbol{f}=0,\boldsymbol{X}^a);S]$ must match the Ashtekar-Hansen expression when the conformal factor is chosen such that $\boldsymbol{K}_{ab}=0$.

(3) The total charge $\mathcal{Q}[(\boldsymbol{f},\boldsymbol{X}^a);S]$ of a Spi symmetry (including both the supertranslation and Lorentz pieces) is conformally-invariant.

Consider first the $\boldsymbol{\mu}_a$-ambiguity and the case of a pure supertranslation $(\boldsymbol{f},\boldsymbol{X}^a=0)$. Since the ambiguity $\boldsymbol{\mu}_a$ is linear in $\boldsymbol{f}$, we have $\boldsymbol{\mu}_a(g;\delta g;\boldsymbol{f}=0)=0$. Further since $\boldsymbol{\mu}_a$ is divergence-free (Eq. (A.7.3)), we can use the implication Eq. (A.5.10) of Theorem 1 with $\boldsymbol{f}$ as the"dynamical field" to conclude that the second integral on the right-hand-side of Eq. (A.7.4) vanishes on any cross-section S for a supertranslation.

Next, consider the $\boldsymbol{\mu}_a$-ambiguity with a Lorentz transformation $(\boldsymbol{f}=0,\boldsymbol{X}^a)$. As already described, we cannot use $\boldsymbol{X}^a$ as the "dynamical" fields in our applications of Theorem 1. Instead, we proceed in a different way. Consider the scalar potential $\boldsymbol{k}$ for the tensor $\boldsymbol{K}_{ab}$ (Eq. (3.4.16)). Since $\boldsymbol{k}$ is a completely free function on $\mathscr{H}$, it is allowed to be an arbitrary cross-section of a vector bundle on $\mathscr{H}$. Further, whenever $\boldsymbol{k}=0$ we have $\boldsymbol{K}_{ab}=0$ and by our assumption the Lorentz charge must the one found by Ashtekar and Hansen. Thus, the ambiguity $\boldsymbol{\mu}_a=0$ whenever $\boldsymbol{k}=0$ for all background spacetimes and all Lorentz vector fields $\boldsymbol{X}^a$. Now using $\boldsymbol{k}$ as the "dynamical field", from Theorem 1 in the form Eq. (A.5.10), we conclude again that the second integral on the right-hand-side of Eq. (A.7.4) vanishes on any cross-section S for a Lorentz symmetry. Thus, the

$\boldsymbol{\mu}_a$-ambiguity does not affect $\delta\mathcal{Q}$.

Finally, consider the $\boldsymbol{\Xi}$-ambiguity in the choice of $\boldsymbol{\Theta}$. In Sec. 3.7.3 we showed that the total charge $\mathcal{Q}$ for a Spi-symmetry $(\boldsymbol{f}, \boldsymbol{X}^a)$ is invariant under conformal transformations with our choice of $\boldsymbol{\Theta}$ (Eq. (3.7.18)) which implies that the charge of a "pure Lorentz" symmetry must shift by a charge of a supertranslation under changes of the conformal factor (see Eq. (3.7.27)). It follows that for the redefined Lorentz charge to transform correctly, the integral contributed by $\boldsymbol{\Xi}$ in Eq. (A.7.4) must be conformally-invariant. Further, for the redefined Lorentz charge to match the one found by Ashtekar and Hansen the integral contributed by $\boldsymbol{\Xi}$ in Eq. (A.7.4) must vanish whenever $\boldsymbol{K}_{ab} = 0$. Since $\boldsymbol{K}_{ab}$ can be chosen to vanish by a choice of conformal factor (see Remark 3.6.3) this implies the $\boldsymbol{\Xi}$-ambiguity does not affect $\delta\mathcal{Q}$.

In summary, our charges are unambiguously determined by the pullback of the symplectic current Eq. (3.5.15).

Here we remark that the symplectic current 3-form itself is *not* uniquely determined by the Lagrangian of the theory but is ambiguous up to

$$\omega(g; \delta_1 g, \delta_2 g) \mapsto \omega(g; \delta_1 g, \delta_2 g) + d\left[\delta_1 \nu(g; \delta_2 g) - \delta_2 \nu(g; \delta_1 g)\right] , \tag{A.7.5}$$

where $\nu(g; \delta g)$ is a local and covariant 2-form and is linear in the perturbation δg. We have not analyzed the effect of this ambiguity on our charges.

A.8 | Lorentz charges with $\boldsymbol{B}_{ab} \neq 0$

In Sec. 3.7.2, to define the Lorentz charges at i^0, we imposed the condition $\boldsymbol{B}_{ab} = 0$ to access the "subleading" magnetic part $\boldsymbol{\beta}_{ab}$ of the asymptotic Weyl tensor (see Eq. (3.4.14)). In this section, we show how we can define a "subleading" magnetic Weyl tensor piece and the Lorentz charges even when $\boldsymbol{B}_{ab} \neq 0$.

If $\boldsymbol{B}_{ab}$ does not vanish,then the "subleading" piece as defined by Eq. (3.4.14) does not exist in the limit. However, consider the derivative of the magnetic part of the Weyl tensor along η^a:

$$\lim_{\to i^0} \Omega^{1\!/\!2} \eta^e \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) = \boldsymbol{\eta}^e \boldsymbol{\partial}_e \boldsymbol{B}_{ab} = 0 \,. \tag{A.8.1}$$

Since the limit of the above quantity vanishes, we can now demand that its "next order" part exist, that is,

$$\boldsymbol{H}_{ab}(\vec{\eta}) := \lim_{\to i^0} \eta^e \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) \quad \text{is } C^{>-1} \,. \tag{A.8.2}$$

The tensor field $\boldsymbol{H}_{ab}(\vec{\eta})$ is not tangential to $\mathscr{H}$. We can compute

$$\begin{aligned} \boldsymbol{H}_{ab}(\vec{\eta}) \boldsymbol{\eta}^b &= \lim_{\to i^0} \eta^b \eta^e \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) = - \lim_{\to i^0} \eta^e \nabla_e \eta^b (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) \\ &= \tfrac{1}{4} \boldsymbol{B}_{ab} \boldsymbol{D}^b \boldsymbol{E} \,, \end{aligned} \tag{A.8.3}$$

where in the first line we have used the fact that $*C_{abcd}$ is antisymmetric in the last two indices and to get the second line, we replaced the derivative of η^a using the Einstein equation Eq. (3.4.1), and used Eqs. (3.4.5) and (3.4.10). Note that $\boldsymbol{H}_{ab}(\vec{\eta}) \boldsymbol{\eta}^a \boldsymbol{\eta}^b = 0$ and thus the only remaining part of $\boldsymbol{H}_{ab}$ is its projection onto $\mathscr{H}$ on both indices. We use this projection to define the "subleading" magnetic part of the Weyl tensor, that is, instead of Eq. (3.4.14), we use

$$\boldsymbol{\beta}_{ab} := \boldsymbol{h}_a{}^c \boldsymbol{h}_b{}^d \boldsymbol{H}_{cd}(\vec{\eta}) \,. \tag{A.8.4}$$

As before, $\boldsymbol{\beta}_{ab}$ is a symmetric and traceless tensor field on $\mathscr{H}$. Note that when $\boldsymbol{B}_{ab} = 0$, this new definition is completely equivalent to the previous one in Eq. (3.4.14) (see also [16]).

The generalization of the equation of motion Eq. (3.4.20) is rather tedious to obtain. We want to compute

$$\begin{aligned} \boldsymbol{\partial}^b \boldsymbol{H}_{ab} &= \lim_{\to i^0} \Omega^{1\!/\!2} \nabla^b \left[\eta^e \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) \right] \\ &= \lim_{\to i^0} \left[(\nabla^b \eta^e) \Omega^{1\!/\!2} \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) + \Omega^{1\!/\!2} \eta^e \nabla^b \nabla_e (\Omega^{1\!/\!2} * C_{acbd} \eta^c \eta^d) \right] \,. \end{aligned} \tag{A.8.5}$$

In the first term we substitute the derivative of η^a using Eq. (3.4.1) and then evaluate the limit of the expression using Eqs. (3.4.6), (3.4.9), (3.4.10) and (A.8.3). For the second term on the right-hand-side, we first commute the derivatives and introduce terms involving the Riemann tensor of the unphysical spacetime. The term with the derivatives ∇^b and ∇_e interchanged vanishes in the limit while the Riemann tensor terms can be computed by decomposing the Riemann tensor in terms of the Weyl tensor using Eq. (4.3.22). Then we can evaluate the limit using Eqs. (3.4.6), (3.4.9) and (3.4.10). The final limit gives the equation

$$\boldsymbol{\partial}^b \boldsymbol{H}_{ab} = -\tfrac{1}{4}\boldsymbol{\partial}_c \boldsymbol{B}_{ab}\boldsymbol{K}^{bc} - \tfrac{1}{4}\boldsymbol{\partial}^b \boldsymbol{B}_{ab}\boldsymbol{E} + \tfrac{5}{4}\boldsymbol{B}_{ab}\boldsymbol{D}^b\boldsymbol{E} + \tfrac{1}{4}\boldsymbol{\varepsilon}_{cda}\boldsymbol{E}^c{}_b\boldsymbol{K}^{db} - \tfrac{1}{4}\boldsymbol{\eta}_a\boldsymbol{B}_{bc}\boldsymbol{K}^{bc} - \boldsymbol{\eta}_a\boldsymbol{B}_{bc}\boldsymbol{E}^{bc} \,. \tag{A.8.6}$$

Using Eq. (A.8.3) and the equation of motion Eq. (3.4.7), it can be verified that the contraction of the above equation with $\boldsymbol{\eta}^a$ is trivial. Projecting the index a onto $\mathscr{H}$, we then get the equation of motion for $\boldsymbol{\beta}_{ab}$ as

$$\boldsymbol{D}^b \boldsymbol{\beta}_{ab} = \tfrac{1}{4}\boldsymbol{\varepsilon}_{cda}\boldsymbol{E}^c{}_b\boldsymbol{K}^{bd} + \tfrac{5}{4}\boldsymbol{B}_{ab}\boldsymbol{D}^b\boldsymbol{E} - \tfrac{1}{4}\boldsymbol{D}_a\boldsymbol{B}_{bc}\boldsymbol{K}^{bc}\,, \tag{A.8.7}$$

which reduces to Eq. (3.4.20) when $\boldsymbol{B}_{ab} = 0$.

To define the Lorentz charge, we now construct the generalization of the tensor $\boldsymbol{W}_{ab}$ (Eq. (3.7.10)). Note that the only essential properties of $\boldsymbol{W}_{ab}$ used to obtain Eq. (3.7.13) are that $\boldsymbol{W}_{[ab]} = -\frac{1}{16}\boldsymbol{\varepsilon}_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E}$ and that $\boldsymbol{D}^a\boldsymbol{W}_{ab} = 0$ using the equation of motion for $\boldsymbol{\beta}_{ab}$. We will further require that $\boldsymbol{W}_{ab}$ is also traceless.

To find such a $\boldsymbol{W}_{ab}$, first note that the last term in Eq. (A.8.7) can be written as the divergence of a symmetric tensor using Eqs. (3.4.8), (3.4.11) and (3.4.13)

$$-\tfrac{1}{4}\boldsymbol{D}_b\boldsymbol{B}_{ac}\boldsymbol{K}^{ac} = -\tfrac{1}{16}\boldsymbol{D}^a\left[-2\boldsymbol{B}_{ab}\boldsymbol{K} + 2\boldsymbol{h}_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{K}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{K} - \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^d{}_e\right]. \tag{A.8.8}$$

Note that although the tensor in the square brackets is *not* traceless, we can add to it the following symmetric tensor

$$-\tfrac{5}{8}\left[2\boldsymbol{B}_{c(a}\boldsymbol{K}^c_{b)} - \boldsymbol{h}_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \boldsymbol{B}_{ab}\boldsymbol{K}\right], \tag{A.8.9}$$

which has vanishing divergence and thus does not affect the left-hand-side. With this we define

$$\begin{aligned}\boldsymbol{W}_{ab} := \boldsymbol{\beta}_{ab} &+ \tfrac{1}{8}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}^c\boldsymbol{E}\boldsymbol{K}^d{}_{b)} - \tfrac{1}{16}\boldsymbol{\varepsilon}_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E} \\ &- \tfrac{3}{2}\boldsymbol{B}_{ab}\boldsymbol{E} + \tfrac{5}{4}\boldsymbol{B}_{c(a}\boldsymbol{K}^c_{b)} - \tfrac{1}{2}\boldsymbol{h}_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \tfrac{3}{4}\boldsymbol{B}_{ab}\boldsymbol{K} \\ &- \tfrac{1}{16}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^d{}_e - \tfrac{1}{16}\boldsymbol{\varepsilon}_{cd(a}\boldsymbol{K}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{K}\,,\end{aligned} \tag{A.8.10}$$

which satisfies

$$\boldsymbol{W}_{[ab]} = -\tfrac{1}{16}\boldsymbol{\varepsilon}_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E}\,, \quad \boldsymbol{D}^a\boldsymbol{W}_{ab} = 0\,, \quad \boldsymbol{h}^{ab}\boldsymbol{W}_{ab} = 0\,. \tag{A.8.11}$$

Then the Lorentz charge formula takes the same form as in Eq. (3.7.20) with $\boldsymbol{W}_{ab}$ now defined as in Eq. (A.8.10). The flux of this charge is still given by Eq. (3.7.21).

Note that when $\boldsymbol{B}_{ab} = 0$, the second line in Eq. (A.8.10) vanishes but the terms in the third line are in general nonvanishing. Let us denote these terms by a symmetric tensor $\boldsymbol{T}_{ab}$. It follows from Eq. (A.8.8) that $\boldsymbol{T}_{ab}$ is divergence-free when $\boldsymbol{B}_{ab} = 0$. Thus $\boldsymbol{D}^a(\boldsymbol{T}_{ab}{}^\star\boldsymbol{X}^b) = 0$ and $\boldsymbol{T}_{ab}{}^\star\boldsymbol{X}^b = 0$ when the scalar potential $\boldsymbol{k}$ for $\boldsymbol{K}_{ab}$ (Eq. (3.4.16)) vanishes. Using the scalar potential $\boldsymbol{k}$ as the "dynamical field" in Theorem 1, it follows from Eq. (A.5.10) that these terms do not contribute to the Lorentz charge expression. Thus, when $\boldsymbol{B}_{ab} = 0$, the Lorentz charge defined using Eq. (A.8.10) coincides with the one defined previously in Sec. 3.7.2.

One can show that under conformal transformations,

$$\boldsymbol{\beta}_{ab} \mapsto \boldsymbol{\beta}_{ab} - \boldsymbol{\varepsilon}_{cd(a}\boldsymbol{E}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{\alpha} - \tfrac{3}{2}\boldsymbol{B}_{ab}\boldsymbol{\alpha} + \tfrac{1}{2}\boldsymbol{D}_c\boldsymbol{B}_{ab}\boldsymbol{D}^c\boldsymbol{\alpha}\,, \tag{A.8.12}$$

and that Eq. (A.8.7) is invariant. The explicit computation of the transformation of the Lorentz charge presented in Appendix A.6 now becomes much more complicated. However, the general argument presented in Sec. 3.7.3 still holds. In summary, even without the assumption $\boldsymbol{B}_{ab} = 0$, we have a satisfactory definition of Lorentz charges at spatial infinity.

The Lorentz charges for $\boldsymbol{B}_{ab} \neq 0$ case were also derived by Compère and Dehouck [92] (with $\boldsymbol{K} = 0$) using an asymptotic expansion in Beig-Schmidt coordinates which in the unphysical spacetime coordinates used in Appendix A.4 reads

$$\begin{aligned} g_{ab} \equiv & [1+\sigma\rho+o(\rho)]^2\, d\rho^2 + \rho\, o(\rho) d\rho dy^a \\ & + \rho^2 \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} - \rho^2 \ln\rho\ i_{ab} + \rho^2 h^{(2)}_{ab} + o(\rho^2) \right] dy^a dy^b\,. \end{aligned} \tag{A.8.13}$$

For $\boldsymbol{\beta}_{ab}$ as defined by Eq. (A.8.4), to exist we set the logarithmic term $i_{ab} = 0$. With this condition the $\boldsymbol{\beta}_{ab}$ is related to the curl of the metric coefficient $h^{(2)}_{ab}$ with additional terms whose form is rather complicated (as compared to Eq. (A.4.16) when $\boldsymbol{B}_{ab} = 0$). Note that when $\boldsymbol{K} = 0$, our $\boldsymbol{W}_{ab}$ is a symmetric, divergence-free and traceless tensor and thus we expect that our charge expression in this case matches with the one derived in [92] in terms of $h^{(2)}_{ab}$ although we have not checked this explicitly.

When the logarithmic term i_{ab} does not vanish, our definition Eq. (A.8.4) cannot be used for the "subleading" magnetic part of the Weyl tensor. We have not explored this case in detail but we expect the following strategy to be useful. We can assume that

$$\Omega^{1/2} * C_{acbd}\eta^c\eta^d = B_{ab} + \Omega^{1/2} \ln\Omega^{1/2} b_{ab} + \Omega^{1/2}\beta_{ab} + o(\Omega^{1/2})\,, \tag{A.8.14}$$

where each of the tensors B_{ab}, b_{ab} and β_{ab} are symmetric and orthogonal to η^a and admit a $C^{>-1}$ limit to i^0. Using such an expansion in the Hodge dual of Eq. (3.4.3a) we can derive the equations of motion for the limits of B_{ab}, b_{ab} and β_{ab}. Since the expression for the symplectic current Eq. (3.7.9) is unchanged, we can use these equations of motion to define an analogue of the tensor $\boldsymbol{W}_{ab}$ and the Lorentz charges. From the point of view of matching these charges to those on null infinity, we expect that the spacetimes with such a logarithmic behaviour at spatial infinity would be related to polyhomogenous spacetimes at null infinity which were defined in [158].

A.9 | There is no preferred translation subalgebra in the generalized BMS algebra

In this appendix we show that the generalized BMS algebra (discussed in chapter 4) does not contain any preferred subalgebra (i.e. a Lie ideal) of translations. Since the asymptotic symmetry algebra is common to all spacetimes under consideration, its Lie bracket is independent of the choice of background spacetime. Thus we can compute the Lie bracket on *any* choice of background spacetime, and in particular we can take the background physical spacetime to be Minkowski. Let us choose a conformal completion for Minkowski so that the induced metric q_{ab} on $\mathscr{I}$ is that of a unit round metric on $\mathbb{S}^2$ and let $\mathscr{D}$ denote the covariant derivative of the unit round metric . Let u be an affine parameter along the null geodesics of n^a such that that $n^a \nabla_a u \mathrel{\hat{=}} 1$.

From Eq. (4.2.12), any element ξ^a of the algebra $\mathfrak{b}_{\text{gen}}$ can be written as

$$\xi^a \mathrel{\hat{=}} X^a + \tfrac{1}{2}(u - u_0)\mathscr{D}_b X^b n^a + f' n^a \tag{A.9.1}$$

where f' is any function on $\mathbb{S}^2$ (representing a supertranslation), X^a is a vector field on $\mathbb{S}^2$ while the function $\alpha_{(\xi)} = \frac{1}{2}\mathscr{D}_a X^a$. The Lie bracket of a supertranslation $fn^a \in \mathfrak{s}$ and ξ^a is then

$$[fn, \xi]^a = \beta n^a \quad \text{where } \beta \mathrel{\hat{=}} -X^a \mathscr{D}_a f + \tfrac{1}{2}\mathscr{D}_a X^a f \tag{A.9.2}$$

It is straightforward to check that $\pounds_n \beta \mathrel{\hat{=}} 0$ and so βn^a is a supertranslation in $\mathfrak{s}$.

If translations are a Lie ideal in $\mathfrak{b}_{\text{gen}}$ then βn^a would also be a translation whenever fn^a is translation. To investigate this we proceed as follows. Since fn^a is a translation, f is a $\ell = 0, 1$ spherical harmonic on $\mathbb{S}^2$ — it is well-known that the limit of translations in Minkowski spacetime to $\mathscr{I}$ are precisely such vector fields. Let X^a be a ℓ'-vector harmonic so that

$$X^a \mathrel{\hat{=}} \mathscr{D}^a F + \varepsilon^{ab}\mathscr{D}_b G \tag{A.9.3}$$

for some functions F and G which are ℓ'-spherical harmonics. In the case that ξ^a is an element of the BMS algebra $\mathfrak{b}$ so that X^a is an element of the Lorentz algebra $\mathfrak{so}(1,3)$, both F and G are spherical harmonics with $\ell' = 1$. The function F corresponds to Lorentz boosts while G corresponds to Lorentz rotations. When ξ^a is an element of the generalized BMS algebra $\mathfrak{b}_{\text{gen}}$, $\ell' \geq 1$ and then $\ell' > 1$ modes of F and G can be thought of as "extended" boosts and rotations.

Using the decomposition Eq. (A.9.3) in Eq. (A.9.2) we have

$$\beta = -\mathscr{D}^a F \mathscr{D}_a f + \tfrac{1}{2}\mathscr{D}^2 F f + \varepsilon^{ab}\mathscr{D}_a G \mathscr{D}_b f \tag{A.9.4}$$

Now we wish to find the spherical harmonic mode L of β when f is a translation i.e. $\ell = 0, 1$-harmonic mode while the harmonic mode of F and G can be $\ell' \geq 1$. It is useful to consider the following different cases.

Case 1: f is time translation, $\ell = 0$ Then,

$$\beta = \tfrac{1}{2}\mathscr{D}^2 F f = -\tfrac{1}{2}\ell'(\ell'+1) F f \tag{A.9.5}$$

so

$$\mathscr{D}^2\beta = -\ell'(\ell'+1)\beta = -L(L+1)\beta \tag{A.9.6}$$

Thus, $\beta = 0$ if $F = 0$ else β is a $L = \ell'$ mode. When $X^a \in \mathfrak{so}(1,3)$, $\ell' = 1$ and this can be interpreted as the fact that a time translation is invariant under Lorentz rotations given by G but changes by a spatial translation under Lorentz boosts given by F.

Case 2: f is spatial translation i.e. $\ell = 1$, $F = 0$ and $G \neq 0$ Then we have,

$$\beta = \varepsilon^{ab}\mathscr{D}_a G \mathscr{D}_b f \tag{A.9.7}$$

and

$$\begin{aligned} \mathscr{D}^2\beta &= \left[-\ell'(\ell'+1) - \ell(\ell+1) + 2\right]\beta + 2\varepsilon^{ab}\mathscr{D}_c\mathscr{D}_a G\mathscr{D}^c\mathscr{D}_b f \\ &= -\ell'(\ell'+1)\beta = -L(L+1)\beta \end{aligned} \tag{A.9.8}$$

where in the last line we use $\ell = 1$ and that $\mathscr{D}_a\mathscr{D}_b f = -q_{ab} f$ for such functions. Thus, β is a $L = \ell'$ mode. Thus, when $X^a \in \mathfrak{so}(1,3)$, $\ell' = 1$, a spatial translation changes by another spatial translation under Lorentz rotations given by G.

Case 3: f is spatial translation i.e $\ell = 1$, $F \neq 0$ and $G = 0$

$$\beta = -\mathscr{D}_a F\mathscr{D}_a f + \tfrac{1}{2}\mathscr{D}^2 F f \tag{A.9.9}$$

To find the L-mode of β, we multiply the above equation with the (complex conjugate) spherical harmonic $\overline{Y}_{L,M}$ and integrate over $\mathbb{S}^2$ to get (we have left the area element of the unit-metric on $\mathbb{S}^2$ implicit for notational convenience)

$$\int \beta\overline{Y}_{L,M} = -\int \mathscr{D}^a F\mathscr{D}_a f\overline{Y}_{L,M} - \tfrac{1}{2}\ell'(\ell'+1)\int Ff\overline{Y}_{L,M} \tag{A.9.10}$$

The first term on the right-hand-side can be rewritten using repeated integration-by-parts as

$$\begin{aligned} -\int \mathscr{D}^a F\mathscr{D}_a f\overline{Y}_{L,M} &= \int F\mathscr{D}^2 f\overline{Y}_{L,M} + \int F\mathscr{D}_a f\mathscr{D}^a\overline{Y}_{L,M} \\ &= \int F\mathscr{D}^2 f\overline{Y}_{L,M} - \int \mathscr{D}_a F f\mathscr{D}^a\overline{Y}_{L,M} - \int Ff\mathscr{D}^2\overline{Y}_{L,M} \\ &= \int F\mathscr{D}^2 f\overline{Y}_{L,M} + \int \mathscr{D}^2 F f\overline{Y}_{L,M} - \int Ff\mathscr{D}^2\overline{Y}_{L,M} + \int \mathscr{D}_a F\mathscr{D}^a f\overline{Y}_{L,M} \\ -\int \mathscr{D}^a F\mathscr{D}_a f\overline{Y}_{L,M} &= \tfrac{1}{2}\int F\mathscr{D}^2 f\overline{Y}_{L,M} + \tfrac{1}{2}\int \mathscr{D}^2 F f\overline{Y}_{L,M} - \tfrac{1}{2}\int Ff\mathscr{D}^2\overline{Y}_{L,M} \\ &= \tfrac{1}{2}\left[-\ell(\ell+1) - \ell'(\ell'+1) + L(L+1)\right]\int Ff\overline{Y}_{L,M} \end{aligned} \tag{A.9.11}$$

Thus, we have

$$\int \beta\overline{Y}_{L,M} = \left[-\tfrac{1}{2}\ell(\ell+1) - \ell'(\ell'+1) + \tfrac{1}{2}L(L+1)\right]\int Ff\overline{Y}_{L,M} \tag{A.9.12}$$

Expanding the functions F and f in terms of the corresponding spherical harmonics $Y_{\ell',m'}$ and $Y_{\ell,m}$ respectively, we can write the final integral in terms of the $3j$-symbols (see Sec. 34 of [159]) (or in terms of the *Clebsch-Gordon* coefficients, Sec. 3.7 of [160]) as

$$\int Y_{\ell',m'} Y_{\ell,m} \overline{Y}_{L,M} = (-1)^M \sqrt{\frac{(2\ell+1)(2\ell'+1)(2L+1)}{4\pi}} \begin{pmatrix} \ell, & \ell', & L \\ 0, & 0, & 0 \end{pmatrix} \begin{pmatrix} \ell, & \ell', & L \\ m, & m', & -M \end{pmatrix} \tag{A.9.13}$$

Since f is a spatial translation with $\ell = 1$, we have

$$\int \beta \overline{Y}_{L,M} \propto \left[-1 - \ell'(\ell'+1) + \tfrac{1}{2} L(L+1)\right] \begin{pmatrix} 1, & \ell', & L \\ 0, & 0, & 0 \end{pmatrix} \begin{pmatrix} 1, & \ell', & L \\ m, & m', & -M \end{pmatrix} \tag{A.9.14}$$

where we have ignored non-zero constant factors. The right-hand-side is non-vanishing if and only if (see Sec. 34 of [159])

$$\begin{aligned} -1 - \ell'(\ell'+1) + \tfrac{1}{2} L(L+1) &\neq 0 \\ 1 + \ell' + L \quad &\text{is even} \\ \ell' - 1 \leq L \leq \ell' + 1\,, \quad M &= m + m' \end{aligned} \tag{A.9.15}$$

These conditions on L can be satisfied if and only if (we do not need the conditions on M for our argument)

$$L = \begin{cases} 0 & \text{for } \ell' = 1 \\ \ell' - 1 \quad \text{or} \quad \ell' + 1 & \text{for } \ell' \geq 2 \end{cases} \tag{A.9.16}$$

Note that for the $\ell' = 1$ case, the value $L = \ell' + 1 = 2$ is ruled out by the first condition in Eq. (A.9.15). Thus, when $X^a \in \mathfrak{so}(1,3)$, $\ell' = 1$, a spatial translation changes by a time translation under Lorentz boosts given by F.

For the usual BMS algebra $\mathfrak{b}$ with $X^a \in \mathfrak{so}(1,3)$ and $\ell' = 1$, we see that in each case β is a spherical harmonic with $L = 0, 1$ that is βn^a is a translation. Thus the translation subalgebra is preserved under the Lie bracket of $\mathfrak{b}$ i.e. there is a preferred 4-dimensional Lie ideal of translations in $\mathfrak{b}$. For the generalized BMS algebra $\mathfrak{b}_{\text{gen}}$ with $X^a \in \mathfrak{diff}(\mathbb{S}^2)$ and $\ell' \geq 2$, the translations fn^a, in general, change by βn^a where β contains a spherical harmonic as high as $L = \ell' + 1$. Thus, translations are not preserved by the Lie bracket of $\mathfrak{b}_{\text{gen}}$ and are not a preferred subalgebra (Lie ideal) of $\mathfrak{b}_{\text{gen}}$. The above argument can be generalized to show that there is, in fact, no finite-dimensional Lie ideal of generalized BMS algebra.

The absence of a preferred translation algebra poses a problem for the prescription used by [45] to define a symplectic form on $\mathscr{I}$. As discussed above, the boundary condition imposed by [45] near spatial infinity to obtain a finite symplectic form for $\mathfrak{b}_{\text{gen}}$ is not invariant under general supertranslations, but is invariant under translations in a specific choice of Bondi coordinates. However, as we have shown, there is no preferred notion of pure translations in $\mathfrak{b}_{\text{gen}}$. Thus, the translation invariance of the boundary condition in [45] is also unclear.

A.10 | Spin-weighted spherical harmonics

In this appendix, we include some results on stereographic coordinates, spin transformations and spin-weighted spherical harmonics that are used in chapter 5.

Stereographic coordinates: Stereographic coordinates $\theta^A = (z, \bar{z})$ are related to the usual spherical polar coordinates (θ, ϕ) by

$$z = e^{i\phi} \cot {}^{\theta}\!/_{2} \,, \quad \bar{z} = e^{-i\phi} \cot {}^{\theta}\!/_{2} \,. \tag{A.10.1}$$

The unit-sphere metric and the area-element in these coordinates are given by

$$s_{AB} \equiv 2P^{-2} dz d\bar{z} \,, \tag{A.10.2a}$$

$$\varepsilon_{AB} \equiv iP^{-2} dz \wedge d\bar{z} = -\sin\theta d\theta \wedge d\phi \,, \tag{A.10.2b}$$

where

$$P := \frac{1 + z\bar{z}}{\sqrt{2}} \,. \tag{A.10.3}$$

The antipodal map is implemented by the transformation

$$\theta^A \mapsto -\theta^A : z \mapsto -1/\bar{z} \implies P^{-1}dz \mapsto e^{2i\phi}P^{-1}d\bar{z}\,, \tag{A.10.4}$$

where $e^{2i\phi} = z/\bar{z}$.

Spin transformations: A function η is said to have spin-weight s if

$$P^{-1}dz \mapsto e^{i\lambda}P^{-1}dz \implies \eta \mapsto e^{is\lambda}\eta\,, \tag{A.10.5}$$

where λ is any smooth function on $\mathbb{S}^2$. We denote this as $\eta \mathrel{\hat{=}} s$. On spin-weighted functions, we defined the operators ð and $\bar{\eth}$ by

$$\eth\,\eta := P^{1-s}\frac{\partial}{\partial\bar{z}}(P^s\eta)\,, \quad \bar{\eth}\eta := P^{1+s}\frac{\partial}{\partial z}(P^{-s}\eta)\,. \tag{A.10.6}$$

Note that if η has spin-weight s then $\eth\,\eta$ has spin-weight $s+1$ and $\bar{\eth}\eta$ has spin weight $s-1$. Further, we have the relations

$$(\eth\,\bar{\eth} - \bar{\eth}\,\eth)\eta = -s\eta\,, \quad \mathscr{D}^2\eta = (\eth\,\bar{\eth} + \bar{\eth}\,\eth)\eta\,, \tag{A.10.7}$$

where $\mathscr{D}^2$ is the laplacian corresponding to the unit round metric on $\mathbb{S}^2$.

Spin-weighted spherical harmonics: The *spin-weighted spherical harmonics* $Y^s_{\ell,m}(\theta^A)$ are eigenfunctions of the laplacian with spin-weight s. They satisfy [161,162]

$$\mathscr{D}^2 Y^s_{\ell,m} = -\left[\ell(\ell+1) - s^2\right] Y^s_{\ell,m}\,. \tag{A.10.8}$$

$$\eth\,Y^s_{\ell,m} = -\sqrt{\frac{(\ell-s)(\ell+s+1)}{2}}\,Y^{s+1}_{\ell,m}\,, \quad \bar{\eth}Y^s_{\ell,m} = \sqrt{\frac{(\ell+s)(\ell-s+1)}{2}}\,Y^{s-1}_{\ell,m}\,. \tag{A.10.9}$$

Note that the harmonic $Y^s_{\ell,m}$ is non-vanishing only for $\ell \geq |s|$ and $\ell \geq |m|$. An explicit expression for $Y^s_{\ell,m}$ as functions of the coordinates $(z,\bar{z})$ is given in Eq. 3.9.20 of [162].

Further, under complex conjugation and antipodal reflection we have

$$\overline{Y^s_{\ell,m}}(\theta^A) = (-)^{m+s} Y^{-s}_{\ell,-m}(\theta^A)\,, \tag{A.10.10a}$$

$$Y^s_{\ell,m}(-\theta^A) = (-)^{\ell+s} e^{2is\phi} Y^{-s}_{\ell,m}(\theta^A)\,. \tag{A.10.10b}$$

A.11 | Solutions for reflection-odd β_{ab}

As discussed in chapter 5, in the matching analysis of asymptotic charges, we restrict ourselves to solutions for $\boldsymbol{\beta}_{ab}$ that are odd under the reflection map defined in Eq. (3.2.7). In this appendix, we solve the equations of motion that govern these solutions, that is, Eqs. (5.2.2a) and (5.5.7), and study the limits of the solutions to $\mathscr{N}^\pm$. As in Sec. 5.3.2, we make use of $(\alpha, z, \bar{z})$ coordinates in which the metric is given by Eq. (5.3.15) and the limits to $\mathscr{N}^\pm$ correspond to $\alpha \to \pm 1$. Using the fact that $\boldsymbol{\beta}_{ab}$ is traceless, its components can be written as

$$\boldsymbol{\beta}_{\alpha\alpha} = H\,, \quad \boldsymbol{\beta}_{\alpha z} = P^{-1} J\,, \quad \boldsymbol{\beta}_{zz} = P^{-2} G\,, \quad \boldsymbol{\beta}_{z\bar{z}} = \tfrac{1}{2} P^{-2}(1-\alpha^2) H\,, \tag{A.11.1}$$

where

$$H \mathrel{\hat{=}} s = 0\,, \quad J \mathrel{\hat{=}} s = -1\,, \quad G \mathrel{\hat{=}} s = -2\,, \tag{A.11.2}$$

Note that the condition that $\boldsymbol{\beta}_{ab}$ be reflection-odd gives us the following conditions

$$H(-\alpha, -\theta^A) = -H(\alpha, \theta^A)\,, \quad J(-\alpha, -\theta^A) = e^{-2i\phi}\overline{J}(\alpha, \theta^A)\,, \quad G(-\alpha, -\theta^A) = -e^{-4i\phi}\overline{G}(\alpha, \theta^A)\,. \tag{A.11.3}$$

In terms of these components, $\boldsymbol{D}^b \boldsymbol{\beta}_{ab} = 0$ corresponds to the following equations

$$0 = (1-\alpha^2)\partial_\alpha H - \alpha H - \eth\, J - \bar{\eth}\, \overline{J}\,, \tag{A.11.4a}$$

$$0 = (1-\alpha^2)\partial_\alpha J - \tfrac{1}{2}(1-\alpha^2)\bar{\eth} H - \eth\, G\,, \tag{A.11.4b}$$

while $(\boldsymbol{D}^2 - 2)\boldsymbol{\beta}_{ab} = 0$ corresponds to

$$0 = (1-\alpha^2)\partial_\alpha^2 H - 4\alpha\partial_\alpha H - (\mathscr{D}^2 + 2)H\,, \tag{A.11.5a}$$

$$0 = (1-\alpha^2)\partial_\alpha^2 J - 4\alpha\partial_\alpha J - (\mathscr{D}^2 + 1)J + 2\alpha\bar{\eth}H\,, \tag{A.11.5b}$$

$$0 = (1-\alpha^2)\partial_\alpha^2 G - 4\alpha\partial_\alpha G - \mathscr{D}^2 G + 4\alpha\bar{\eth}J\,, \tag{A.11.5c}$$

where $\mathscr{D}^2$ denotes the laplacian with respect to the unit-sphere metric. Note also that to simplify the first and second equations above, we used Eq. (A.11.4a) and Eq. (A.11.4b) respectively. Expanded in terms of spin-weighted spherical harmonics, we have

$$H = \sum_{\ell,m} H_{\ell,m}(\alpha) Y_{\ell,m}^{s=0}(\theta^A)\,, \quad J = \sum_{\ell\geq 1,m} J_{\ell,m}(\alpha) Y_{\ell,m}^{s=-1}(\theta^A)\,, \quad G = \sum_{\ell\geq 2,m} G_{\ell,m}(\alpha) Y_{\ell,m}^{s=-2}(\theta^A)\,. \tag{A.11.6}$$

The conditions for the solutions to be reflection-odd in Eq. (A.11.3) then imply the following relations

$$H_{\ell,m}(-\alpha) = (-1)^{\ell+1} H_{\ell,m}(\alpha)\,,\; J_{\ell,m}(-\alpha) = (-1)^{\ell+m}\overline{J_{\ell,-m}}(\alpha)\,,\; G_{\ell,m}(-\alpha) = (-1)^{1+\ell+m}\overline{G_{\ell,-m}}(\alpha)\,. \tag{A.11.7}$$

Note also that since H is real, it follows from Eq. (A.10.10a) that

$$H_{\ell,m} = (-1)^m \overline{H_{\ell,-m}}\,. \tag{A.11.8}$$

Then, Eq. (A.11.4) becomes

$$0 = (1-\alpha^2)\frac{d}{d\alpha}H_{\ell,m} - \alpha H_{\ell,m} + \sqrt{\tfrac{\ell(\ell+1)}{2}}\left(J_{\ell,m} + (-1)^m \overline{J_{\ell,-m}}\right). \tag{A.11.9a}$$

$$0 = (1-\alpha^2)\frac{d}{d\alpha}J_{\ell,m} - \tfrac{1}{2}(1-\alpha^2)\sqrt{\tfrac{\ell(\ell+1)}{2}}H_{\ell,m} + \sqrt{\tfrac{(\ell-1)(\ell+2)}{2}}G_{\ell,m}\,. \tag{A.11.9b}$$

Moreover, using Eq. (A.10.8) and Eq. (A.11.8), Eq. (A.11.5) becomes

$$0 = (1-\alpha^2)\frac{d^2}{d\alpha^2}H_{\ell,m} - 4\alpha\frac{d}{d\alpha}H_{\ell,m} + [\ell(\ell+1)-2]\,H_{\ell,m}\,, \tag{A.11.10a}$$

$$0 = (1-\alpha^2)\frac{d^2}{d\alpha^2}J_{\ell,m} - 4\alpha\frac{d}{d\alpha}J_{\ell,m} + [\ell(\ell+1)-2]\,J_{\ell,m} + 2\alpha\sqrt{\tfrac{\ell(\ell+1)}{2}}H_{\ell,m}\,, \tag{A.11.10b}$$

$$0 = (1-\alpha^2)\frac{d^2}{d\alpha^2}\overline{J_{\ell,m}} - 4\alpha\frac{d}{d\alpha}\overline{J_{\ell,m}} + [\ell(\ell+1)-2]\,\overline{J_{\ell,m}} + 2(-1)^m\alpha\sqrt{\tfrac{\ell(\ell+1)}{2}}H_{\ell,-m}\,, \tag{A.11.10c}$$

$$0 = (1-\alpha^2)\frac{d^2}{d\alpha^2}G_{\ell,m} - 4\alpha\frac{d}{d\alpha}G_{\ell,m} + [\ell(\ell+1)-4]\,G_{\ell,m} + 4\alpha\sqrt{\tfrac{(\ell-1)(\ell+2)}{2}}J_{\ell,m}\,, \tag{A.11.10d}$$

where in each of these equations, we have suppressed the dependence of $H_{\ell,m}$, $J_{\ell,m}$ and $G_{\ell,m}$ on α. Note also that Eq. (A.11.10c) was obtained by taking the complex conjugate of Eq. (A.11.10b) and using Eq. (A.11.8).

We now proceed to solving these equations, starting first by deriving explicit solutions for the $\ell = 0$ and $\ell = 1$ cases. Note that $G = 0$ in both these cases. For $\ell = 0$, $J = 0$ and we have the solution

$$H_{\ell=0,m=0}(\alpha) = (1-\alpha^2)^{-1}(H_0 + H_1\alpha)\,, \tag{A.11.11}$$

which satisfies Eq. (A.11.9a) only for $H_0 = H_1 = 0$. Therefore all $\ell = 0$ solutions are zero.

For $\ell = 1$, we have the solutions

$$\begin{aligned} H_{\ell=1,m}(\alpha) &= H^{(0)}_m + H^{(1)}_m(\alpha(1-\alpha^2)^{-1} + \tanh^{-1}\alpha)\,, \\ J_{\ell=1,m}(\alpha) &= J^{(0)}_m + J^{(1)}_m\alpha\,. \end{aligned} \tag{A.11.12}$$

Putting these in Eq. (A.11.9) gives us the conditions

$$H^{(1)}_m = 0\,, \quad J^{(0)}_m + (-)^m\overline{J^{(0)}_{-m}} = 0\,, \quad H^{(0)}_m = J^{(1)}_m + (-)^m\overline{J^{(1)}_{-m}}\,, \quad H^{(0)}_m = 2J^{(1)}_m\,, \tag{A.11.13}$$

and the last two equations together imply

$$J^{(1)}_m - (-)^m\overline{J^{(1)}_{-m}} = 0\,. \tag{A.11.14}$$

With these constraints, the free functions correspond to 6 degrees of freedom

$$J^{(0)}_{m=1} \in \mathbb{C}\,, \quad iJ^{(0)}_{m=0} \in \mathbb{R}\,, \quad J^{(1)}_{m=1} \in \mathbb{C}\,, \quad J^{(1)}_{m=0} \in \mathbb{R}\,. \tag{A.11.15}$$

We will see below that these encode the 6 charges associated with Lorentz boosts and rotations. Note that we have shown that the $\ell = 0, 1$ solutions are finite in the limits $\alpha \to \pm 1$. We now proceed to showing that this finiteness property holds for all ℓ. Note that the solutions of Eq. (A.11.10a) for $\ell \geq 1$ are spanned by

$$(1-\alpha^2)^{-1/2} P^{-1}_{\ell}(\alpha)\,, \quad (1-\alpha^2)^{-1/2} Q^{-1}_{\ell}(\alpha) \tag{A.11.16}$$

where $P^{-1}_{\ell}(\alpha)$ and $Q^{-1}_{\ell}(\alpha)$ are the Legendre functions, which satisfy

$$P^{-1}_{\ell}(-\alpha) = (-1)^{\ell-1} P^{-1}_{\ell}(\alpha)\ , \quad Q^{-1}_{\ell}(-\alpha) = (-1)^{\ell} Q^{-1}_{\ell}(\alpha)\,. \tag{A.11.17}$$

We see that reflection-odd condition on $H_{\ell,m}$ in Eq. (A.11.7) picks out the solutions spanned by $(1-\alpha^2)^{-1/2} P^{-1}_{\ell}(\alpha)$ for $H_{\ell,m}$. Hence, we have that

$$H_{\ell>1,m}(\alpha) = c_m (1-\alpha^2)^{-1/2} P^{-1}_{\ell}(\alpha)\,, \tag{A.11.18}$$

for some constants c_m. It follows from the following recursion relation

$$(1-\alpha^2)^{-1/2} P^{-1}_{\ell}(\alpha) = \frac{-1}{2^{\ell}\ell!(1-\alpha^2)} \left(\frac{d}{d\alpha}\right)^{\ell-1} (\alpha^2-1)^{\ell} \ \text{ for } \ell > 1\,, \tag{A.11.19}$$

that these solutions are finite as $\alpha \to \pm 1$ for all $\ell > 1$. Note also that Eqs. (A.11.10b) and (A.11.10c) can be combined to give

$$\begin{aligned} 0 = (1-\alpha^2)\frac{d^2}{d\alpha^2}(J_{\ell,m} - (-1)^m \overline{J_{\ell,-m}}) - 4\alpha \frac{d}{d\alpha}(J_{\ell,m} - (-1)^m \overline{J_{\ell,-m}}) \\ + \left[\ell(\ell+1) - 2\right](J_{\ell,m} - (-1)^m\, \overline{J_{\ell,-m}}) \end{aligned} \tag{A.11.20}$$

This shows that $J_{\ell,m}(\alpha) - (-1)^m \overline{J_{\ell,-m}}(\alpha)$ satisfies the same equation as $H_{\ell,m}$ and therefore the solutions to this equation are also spanned by linear combinations of

$$(1-\alpha^2)^{-1/2}P_\ell^{-1}(\alpha)\,, \quad (1-\alpha^2)^{-1/2}Q_\ell^{-1}(\alpha)\,. \tag{A.11.21}$$

Using the reflection-odd condition on $J_{\ell,m}$ given in Eq. (A.11.7), we see that $J_{\ell,m}(\alpha) - (-1)^m\overline{J_{\ell,-m}}(\alpha) = J_{\ell,m}(\alpha) - (-1)^\ell J_{\ell,m}(-\alpha)$ which is odd (even) under $\alpha \to -\alpha$ for even (odd) ℓ. From Eq. (A.11.17), we see that this property only holds for the the solutions spanned by $(1-\alpha^2)^{-1/2}P_\ell^{-1}(\alpha)$ which leads us to discard the solutions spanned by $(1-\alpha^2)^{-1/2}Q_\ell^{-1}(\alpha)$. Hence, we have

$$J_{\ell>1,m}(\alpha) - (-1)^m\overline{J_{\ell>1,-m}}(\alpha) = d_m(1-\alpha^2)^{-1/2}P_\ell^{-1}(\alpha)\,, \tag{A.11.22}$$

for some constants d_m. Using Eq. (A.11.19) again, we conclude that the solutions for $J_{\ell,m} - (-1)^m\overline{J_{\ell,-m}}$ are finite as $\alpha \to \pm 1$ for all ℓ. Next, note that we can rewrite Eq. (A.11.9a) as

$$J_{\ell,m} + (-1)^m\overline{J_{\ell,-m}} = \sqrt{\frac{2}{\ell(\ell+1)}}\left[(\alpha^2-1)\frac{d}{d\alpha}H_{\ell,m} + \alpha H_{\ell,m}\right] \quad (\text{for } \ell \geq 1)\,. \tag{A.11.23}$$

Using Eq. (A.11.19), we see that $(\alpha^2-1)\frac{d}{d\alpha}H_{\ell,m}$ is finite as $\alpha \to \pm 1$ which, using Eq. (A.11.23) and the finiteness of $H_{\ell,m}$ in these limits, implies that $J_{\ell,m} + (-1)^m\overline{J_{\ell,-m}}$ is also finite as $\alpha \to \pm 1$ for all $\ell \geq 1$. Using the finiteness of $J_{\ell,m} - (-1)^m\overline{J_{\ell,-m}}$ for all ℓ in the limit $\alpha \to \pm 1$, we then conclude that $J_{\ell,m}(\alpha)$ is finite as $\alpha \to \pm 1$ for all $\ell \geq 1$.

Finally, we note from Eq. (A.11.9b) that

$$G_{\ell,m} = \sqrt{\frac{2}{(\ell-1)(\ell+2)}}\left[(\alpha^2-1)\frac{d}{d\alpha}J_{\ell,m} - \tfrac{1}{2}(\alpha^2-1)\sqrt{\tfrac{\ell(\ell+1)}{2}}H_{\ell,m}\right] \quad (\text{for } \ell \geq 2)\,. \tag{A.11.24}$$

Using Eqs. (A.11.22) and (A.11.23) to obtain the functional behavior of $J_{\ell,m}(\alpha)$ and using Eq. (A.11.19), one can also show that $(\alpha^2-1)\frac{d}{d\alpha}J_{\ell,m}$ is finite as $\alpha \to \pm 1$. This, using the finiteness of $H_{\ell,m}$ in these limits, shows that $G_{\ell,m}$ is finite as $\alpha \to \pm 1$ for all $\ell \geq 2$. This completes our proof of the fact that the reflection-odd solutions for $\boldsymbol{\beta}_{ab}$ are finite in the $\alpha \to \pm 1$ limits for all ℓ.

One can show that in the coordinates used in this section, $\lim\limits_{\to \mathscr{N}^\pm} \boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a|_{\mathscr{N}^\pm} \equiv \pm\partial_\alpha$. We therefore see from Eq. (5.5.8) that the (integrand of) Lorentz charge at $\mathscr{N}^\pm$ is proportional to $\boldsymbol{\beta}_{\alpha A}\boldsymbol{X}^A$ where

$A = (z, \bar{z})$. Since X^A (as shown in Sec. 5.3.2) is an $\ell = 1$ vector field, it follows that the charge receives contribution only from $J_{\ell=1,m}$. The six degrees of freedom of $J_{\ell=1,m}$ (Eq. (A.11.15)) therefore encode the six Lorentz charges associated with boosts and rotations.

A.12 | Affinity of l^a

For the calculation in Sec. 5.5, we need the expression for the affinity of l^a. Recall that we are working in a conformal frame where $\Phi = 2$ and so from Eq. (A.1.3) we have

$$\nabla_a n_b \mathrel{\hat{=}} 2 g_{ab} \,. \tag{A.12.1}$$

Using Eq. (5.2.25), we have

$$l^a \nabla_a l_b = \frac{1}{4} \Sigma L^a (\Sigma \nabla_a L_b + L_b \nabla_a \Sigma) \,. \tag{A.12.2}$$

Then, using Eqs. (5.2.20) and (5.2.22) and Eq. (A.12.1), we get

$$\begin{aligned} \nabla_a L_b \mathrel{\hat{=}} & -\nabla_a \nabla_b \Sigma^{-1} + \frac{1}{2} \Sigma n_b \nabla_a \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} + \frac{1}{4} \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} (n_b \nabla_a \Sigma + 2 \Sigma g_{ab}) \\ & - \frac{1}{2} \nabla_a (\Omega \Sigma \overline{L}^c \nabla_b \nabla_c \Sigma^{-1}) \,, \end{aligned} \tag{A.12.3}$$

which gives

$$\begin{aligned} l^a \nabla_a l_b \mathrel{\hat{=}} & \frac{1}{4} \Sigma L_b L^a \nabla_a \Sigma + \frac{1}{4} \Sigma^2 L^a [-\nabla_a \nabla_b \Sigma^{-1} + \frac{1}{2} \Sigma n_b \nabla_a \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} \\ & + \frac{1}{4} \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} (n_b \nabla_a \Sigma + 2 \Sigma g_{ab})] - \frac{1}{8} \Sigma^2 L^a \nabla_a (\Omega \Sigma \overline{L}^c \nabla_b \nabla_c \Sigma^{-1}) \,. \end{aligned} \tag{A.12.4}$$

Let us use this to compute, $q^b{}_c\, l^a \nabla_a l_b$ on $\mathscr{I}$. Using $q^a{}_b\, n_a \mathrel{\hat{=}} q^a{}_b\, L_a \mathrel{\hat{=}} 0$ and $L^a \nabla_a \Omega \mathrel{\hat{=}} -2\Sigma^{-1}$ (where the last equation follows from Eqs. (5.2.20) and (5.2.24)), we have

$$q^b_c\, l^a \nabla_a l_b \mathrel{\hat{=}} -\frac{1}{4} \Sigma^2 q^b{}_c L^a \nabla_a \nabla_b \Sigma^{-1} + \frac{1}{4} \Sigma^2 q^b{}_c \overline{L}^a \nabla_a \nabla_b \Sigma^{-1} \mathrel{\hat{=}} 0 \,, \tag{A.12.5}$$

where the last equality follows because $L^a \hat{=} \overline{L}^a$ (see Eq. (5.2.23)). Next, we evaluate $n^b l^a \nabla_a l^b$ on $\mathscr{I}$. Using Eqs. (5.2.20), (5.2.24) and (A.12.2), we have

$$\begin{aligned} n^b l^a \nabla_a l_b &= \frac{1}{4}\Sigma^2 L^a n^b \nabla_a L_b + \frac{1}{4}\Sigma L^a n^b L_b \nabla_a \Sigma \\ &\hat{=} \frac{1}{2}\Sigma L^a N^b \nabla_a L_b - \frac{1}{2} L^a \nabla_a \Sigma \,. \end{aligned} \tag{A.12.6}$$

Note that using Eq. (5.2.22)

$$\begin{aligned} L^a \nabla_a \Sigma &= -\nabla^a \Sigma^{-1} \nabla_a \Sigma + \frac{1}{2} N^a \nabla_a \Sigma \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} - \frac{1}{2}\Omega\Sigma\overline{L}^b \nabla^a \nabla_b \Sigma^{-1} \nabla_a \Sigma \\ &\hat{=} \frac{1}{2}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} \,, \end{aligned} \tag{A.12.7}$$

where the second equality uses $N^a \nabla_a \Sigma^{-1} \hat{=} 1$ (which follows from condition (4) and Eq. (5.2.20)) and the fact that $\Omega \hat{=} 0$. Moreover,

$$\begin{aligned} L^a N^b \nabla_a L_b &= -L^a L_b \nabla_a N^b + L^a \nabla_a (N^b L_b) \hat{=} -L^a L_b (-\Sigma N^b \nabla_a \Sigma^{-1} + \Sigma\, \delta^b{}_a) + L^a \nabla_a (N^b L_b) \\ &\hat{=} -L^a \Sigma \nabla_a \Sigma^{-1} + L^a \nabla_a (-N^b \nabla_b \Sigma^{-1} + \frac{1}{2} N^b N_b \nabla_c \Sigma^{-1} \nabla^c \Sigma^{-1} - \frac{1}{2}\Omega\Sigma\overline{L}^b N^c \nabla_c \nabla_b \Sigma^{-1}) \,, \end{aligned} \tag{A.12.8}$$

where the first line uses Eq. (5.2.20) and Eq. (A.12.1) while the second line uses Eqs. (5.2.22) and (5.2.24). Further, using $N^a N_a = \Sigma^2 \Omega + O(\Omega^2)$ (which follows from Eq. (5.2.20) and the fact that $\lim_{\to \mathscr{I}} \Omega^{-1} n^a n_a = 2\Phi = 4$), and $L^a \nabla_a \Omega \hat{=} -2\Sigma^{-1}$, this becomes

$$\begin{aligned} L^a N^b \nabla_a L_b &\hat{=} -L^a \Sigma \nabla_a \Sigma^{-1} + L^a \nabla_a (-N^b \nabla_b \Sigma^{-1} + \frac{1}{2}\Sigma^2 \Omega \nabla_b \Sigma^{-1} \nabla^b \Sigma^{-1}) + \overline{L}^a N^b \nabla_b \nabla_a \Sigma^{-1} \\ &\hat{=} -L^a \Sigma \nabla_a \Sigma^{-1} - L^a \nabla_a (N^b \nabla_b \Sigma^{-1}) - \Sigma \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} + \overline{L}^a N^b \nabla_b \nabla_a \Sigma^{-1} \,, \end{aligned} \tag{A.12.9}$$

Putting all of this together, we have

$$\begin{aligned}
n^b l^a \nabla_a l_b &\mathrel{\widehat{=}} \frac{1}{2}\Sigma L^a N^b \nabla_a L_b - \frac{1}{2} L^a \nabla_a \Sigma \\
&\mathrel{\widehat{=}} -\frac{1}{4}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} + \frac{1}{2}(-L^a \Sigma^2 \nabla_a \Sigma^{-1} - \Sigma L^a \nabla_a (N^b \nabla_b \Sigma^{-1}) - \Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1}) \\
&+ \frac{1}{2}\Sigma \overline{L}^a N^b \nabla_b \nabla_a \Sigma^{-1} \,.
\end{aligned} \tag{A.12.10}$$

Note that using Eq. (5.2.22) and $N^a \nabla_a \Sigma^{-1} \mathrel{\widehat{=}} 1$, we have

$$L^a \nabla_a \Sigma^{-1} \mathrel{\widehat{=}} -\nabla^a \Sigma^{-1} \nabla_a \Sigma^{-1} + \frac{1}{2}\nabla^a \Sigma^{-1} \nabla_a \Sigma^{-1} = -\frac{1}{2}\nabla^a \Sigma^{-1} \nabla_a \Sigma^{-1} \,, \tag{A.12.11}$$

and therefore

$$-\frac{1}{4}\Sigma^2 \nabla^a \Sigma^{-1} \nabla_a \Sigma^{-1} - \frac{1}{2}\Sigma^2 L^a \nabla_a \Sigma^{-1} \mathrel{\widehat{=}} 0 \,, \tag{A.12.12}$$

Hence, we get

$$\begin{aligned}
n^b l^a \nabla_a l_b &\mathrel{\widehat{=}} -\frac{1}{2}(\Sigma L^a \nabla_a (N^b \nabla_b \Sigma^{-1}) + \Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} - \Sigma \overline{L}^b N^a \nabla_a \nabla_b \Sigma^{-1}) \\
&\mathrel{\widehat{=}} -\frac{1}{2}(\Sigma L^a \nabla_b \Sigma^{-1} \nabla_a N^b + \Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1}) \\
&\mathrel{\widehat{=}} -\frac{1}{2}(\Sigma L^a \nabla_b \Sigma^{-1} (\frac{1}{2} n^b \nabla_a \Sigma + \Sigma \delta_a{}^b) + \Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1}) \\
&\mathrel{\widehat{=}} -\frac{1}{2} L^a \nabla_a \Sigma - \frac{1}{2}\Sigma^2 L^a \nabla_a \Sigma^{-1} - \frac{1}{2}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} \\
&\mathrel{\widehat{=}} -\frac{1}{4}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} + \frac{1}{4}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} - \frac{1}{2}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} \,,
\end{aligned} \tag{A.12.13}$$

where in going to the second line, we used the fact that $L^a \mathrel{\widehat{=}} \overline{L}^a$ and in the subsequent steps we used condition (4) and Eqs. (5.2.20), (A.12.1) and (A.12.11). We therefore see that

$$n^b l^a \nabla_a l_b \mathrel{\widehat{=}} -\frac{1}{2}\Sigma^2 \nabla_a \Sigma^{-1} \nabla^a \Sigma^{-1} \,. \tag{A.12.14}$$

Relation to the extension for the auxiliary normal used in [10]

In the matching analysis carried out for supertranslations and supermomenta in [10], a different extension of the vector fields l^a and L^a into the spacetime was used. Denoting the expressions used in that paper and the ones in chapter 5 (see Eq. (5.2.22)) by $\bar{l}^a$ and l^a respectively, we note that

$$l^a = \bar{l}^a - \frac{1}{2}\Omega\Sigma\bar{l}^b\nabla^a\nabla_b\Sigma^{-1}\,. \tag{A.12.15}$$

and correspondingly

$$L^a = \overline{L}^a - \frac{1}{2}\Omega\Sigma\overline{L}^b\nabla^a\nabla_b\Sigma^{-1}\,. \tag{A.12.16}$$

We see that on $\mathscr{I}$, the expressions agree which implies that all of the results pertaining to null infinity in [10] are valid for the l^a and L^a defined here well. We also note that in the limit to $\mathscr{H}$, the second term in Eq. (A.12.16) can be written as

$$\lim_{\to i^0} -\frac{1}{2}\Omega\Sigma\overline{L}^b\nabla^a\nabla_b\Sigma^{-1} = -\frac{1}{2}(-\boldsymbol{D}^a\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}\boldsymbol{D}^b\boldsymbol{\Sigma}^{-1}\boldsymbol{D}^a\boldsymbol{D}_b\boldsymbol{\Sigma}^{-1})\,, \tag{A.12.17}$$

where we have used $\overline{\boldsymbol{L}}^a = -\boldsymbol{h}^a{}_b\boldsymbol{D}^b\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}^a(\frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{h}^{bc}\boldsymbol{D}_b\boldsymbol{\Sigma}^{-1}\boldsymbol{D}_c\boldsymbol{\Sigma}^{-1} - \frac{1}{2}\boldsymbol{\Sigma}^{-1})$ (from Eq. (5.2.23)), as well as Eqs. (3.2.2)–(3.2.4) and (5.2.21). This term can be explicitly evaluated in suitable coordinates on $\mathscr{H}$ and shown to be exactly zero for $\boldsymbol{\Sigma} = \sqrt{1-\alpha^2}$ which is a particular choice for $\boldsymbol{\Sigma}$ (see Appendix B of [10]). For an arbitrary $\boldsymbol{\Sigma}$, given by $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\sqrt{1-\alpha^2}$ (see Remark 2.3 and the discussion below Eq. 2.40 of [10] for a description of the freedom in choosing $\boldsymbol{\Sigma}$) where $\boldsymbol{\sigma}$ is an arbitrary smooth function on $\mathscr{H}$, one can similarly show that this term vanishes fast enough such that $\lim_{\to\mathscr{N}^\pm}\boldsymbol{\Sigma}^{-1}\boldsymbol{h}^a{}_b\boldsymbol{L}^b = \lim_{\to\mathscr{N}^\pm}\boldsymbol{\Sigma}^{-1}\boldsymbol{h}^a{}_b\overline{\boldsymbol{L}}^b$ which implies that $\lim_{\to\mathscr{N}^\pm}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^a = \lim_{\to\mathscr{N}^\pm}\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{U}}^b$ where $\overline{\boldsymbol{U}}^a := \boldsymbol{h}^a{}_b\overline{\boldsymbol{L}}^b$. This is sufficient to show that the matching analysis performed in [10] holds for the "modified" l^a and L^a used in chapter 5 as well.

A.13 | Matching of Lorentz charges in Kerr-Newman spacetimes

The Kerr-Newman metric is the most general stationary and axisymmetric black hole solution to the vacuum Einstein-Maxwell equations. It is well known that these spacetimes satisfy the peeling property along with the condition that $\boldsymbol{B}_{ab} = 0$. Moreover, since these spacetimes are stationary, the matching of Lorentz charges in these spacetimes was already shown in [113]. Nonetheless, in this appendix, for completeness, we explicitly verify that all the assumptions we made for proving the matching of Lorentz charges are satisfied in this class of spacetimes. Showing that our assumptions hold at least in this class of spacetimes will be an important consistency check of our results.

We work with the conformal completion of these spacetimes given by Herberthson [163] in which the unphysical metric g_{ab} is $C^{>0}$ in both null and spatial directions at i^0. Our calculation has some overlap with the one in Appendix. C of [16] where a different conformal completion was used to study properties of the Kerr spacetime. The line element corresponding to the physical Kerr-Newman metric is given in Boyer-Lindquist coordinates (see, e.g, [164]; note that we multiply the metric in Eq. 3.2 of [164] with an overall minus sign since their metric signature is $(+,-,-,-)$ while we use $(-,+,+,+)$)

$$d\hat{s}^2 = -\frac{\Delta}{R^2}\left(a\sin^2\theta \mathrm{d}\varphi - \mathrm{d}t\right)^2 + \frac{\sin^2\theta}{R^2}\left[\left(r^2+a^2\right)\mathrm{d}\varphi - a\ \mathrm{d}t\right]^2 + \frac{R^2}{\Delta}\mathrm{d}r^2 + R^2\ \mathrm{d}\theta^2\,, \tag{A.13.1}$$

where

$$\Delta := r^2 + a^2 + 2Mr + Q^2\,, \quad R^2 := r^2 + a^2\cos^2\theta\,. \tag{A.13.2}$$

Here, M denotes the mass of the black hole, Q denotes its electric charge while a is related to the angular momentum, J, by $a := \frac{J}{M}$. Following [163], we define coordinates (v, w) that are well-behaved in a neighborhood of spatial infinity and null infinity, implicitly through

$$t = \frac{w-v}{2vw} + 2M\log(\frac{w}{v})\,, \quad r = \frac{v+w}{2vw}\,. \tag{A.13.3}$$

To obtain the unphysical line element, we use the conformal factor

$$\Omega = vw\,, \tag{A.13.4}$$

whereby $g_{ab} = \Omega^2 \hat{g}_{ab}$. Note that in these coordinates, $\mathscr{I}^+$ corresponds to the surface $v = 0$, i^0 corresponds to the point $w = v = 0$ while $\mathscr{I}^-$ corresponds to the surface $w = 0$. Further, we take the rescaling function, Σ, to be given by

$$\Sigma = (w+v)^{-1} \exp(-2M(w+v))\,. \tag{A.13.5}$$

We take $\mathscr{I}^+$ to be foliated by Σ^{-1} = constant cross-sections for this choice of Σ. In terms of this Σ,

$$n^a|_{\mathscr{I}^+} \equiv \frac{2w}{1+2Mw}\partial_w \mathrel{\widehat{=}} 2\Sigma^{-1}\partial_{\Sigma^{-1}}\,, \tag{A.13.6}$$

which is consistent with Eq. (5.2.26b). Moreover, using Eq. (5.2.5) to compute Φ, we get

$$\Phi = \frac{2}{1+2Mw}\,, \tag{A.13.7}$$

and therefore, $\lim\limits_{w\to 0} \Phi = 2$ which is consistent with Eq. (5.2.5). Further, l^a can be computed from Eqs. (5.2.22) and (5.2.25) and shown to satisfy $l^a n_a \mathrel{\widehat{=}} -1$. Next, we write the unphysical metric in the coordinates defined in Eq. (A.13.3). To access the non-trivial components of the Weyl tensor near null infinity, we first expand the resulting expression around $v = 0$ up to $O(v^3)$ and compute the corresponding expression for the Weyl tensor (using the RGTC package in MATHEMATICA [3]). We see that $\Omega^{-1}C_{abcd}$ has a limit to null infinity and therefore, the peeling property is satisfied in these spacetimes. Further, using the resulting expression, along with the expression for U^a computed from Eq. (5.5.10) for our choice of Ω and Σ, and Eq. (5.2.30), we then compute the integrand of Eq. (5.5.9) and see that it has a finite limit to $\mathscr{I}^+$. We can therefore dispense with the limiting procedure involving the null surfaces described in Sec. 5.5, evaluate this integral on $\mathscr{I}^+$ and then take the limit to $\mathscr{N}^+$ which corresponds to taking $w \to 0$. We find that

$$\lim_{w\to 0}\lim_{v\to 0}\,[-\frac{1}{8\pi}\int_{S'}\tilde{\varepsilon}_2\,\Omega^{-1/2}\Sigma^{-1}\,{*C}_{acbd}\,U^a(\nabla^c\Omega^{1/2})(\nabla^d\Omega^{1/2})(\Omega^{-1/2}X^b)] = -\frac{3}{8\pi}\int_{\mathscr{N}^+}d\theta d\phi\, aM\sin^2\theta X^\theta\,, \tag{A.13.8}$$

where X^θ denotes the θ-component of the Lorentz symmetry on $\mathscr{N}^+$. Since the Lorentz symmetries in the limit along $\mathscr{I}^+$ to $\mathscr{N}^+$ match the corresponding limit of the Lorentz symmetries at spatial infinity (see Remark 5.4.1), we can evaluate this expression using the explicit expressions for the Lorentz symmetries at spatial infinity, derived in Appendix. B of [91]. The only non-trivial contribution comes from the limit of the boost vector field, given in Eq. (B.118) of [91]. Translated into our notation, it is $\boldsymbol{X}^a_{(boost)} \equiv \cos\theta(1-\alpha^2)\partial_\alpha - \alpha\sin\theta\partial_\theta$. Note that as remarked below Eq. (5.5.38), the right hand side of Eq. (A.13.8) actually corresponds to the charge associated with $\tilde{\varepsilon}_\theta{}^\phi X^\theta$ and therefore this charge corresponds to a rotation in the ϕ - direction. We get that the integral on the right hand side of Eq. (A.13.8) evaluates to

$$\frac{3}{8\pi}\int_{\mathscr{N}^+}d\theta d\phi\, aM\sin^3\theta = aM = J\,, \tag{A.13.9}$$

as expected. To study limits to i^0 along spatial directions, we define coordinates (ρ,α) by

$$\alpha := \frac{w-v}{w+v}\,,\quad \rho := \sqrt{vw}\,, \tag{A.13.10}$$

where $\rho \to 0$ corresponds to the limit to i^0 and $\rho = 1$ corresponds to the unit hyperboloid of spatial directions, $\mathscr{H}$, in Ti^0. We then write the metric in these coordinates and expand it around $\rho = 0$ to $O(\rho^4)$ to access the non-zero Weyl tensor components. We find that $\boldsymbol{B}_{ab} = 0$. Moreover, calculating $\boldsymbol{\beta}_{ab}$ using Eq. (5.2.1) yields

$$\begin{aligned}\boldsymbol{\beta}_{ab} \equiv\, & 6aM\cos\theta\,(\rho d\alpha)^2 - 6aM\alpha\sin\theta(\rho d\alpha)(\rho d\theta) - 3aM(\alpha^2-1)\cos\theta(\rho d\theta)^2 \\ & - 3aM(\alpha^2-1)\cos\theta\sin^2\theta(\rho d\phi)^2\,,\end{aligned} \tag{A.13.11}$$

where we have written the final expression in a $(\rho d\alpha, \rho d\theta, \rho d\phi)$ basis which is a $C^{>-1}$ basis at i^0. Note that this expression is consistent with $\boldsymbol{\beta}_{ab}$ being odd under the reflection map on $\mathscr{H}$ under which $(\alpha, \theta, \phi) \to (-\alpha, \pi - \theta, \phi \pm \pi)$ (cf. Eq. (3.2.7)). Next, computing the limit of Eq. (5.5.9) to $\mathscr{H}$ using our expressions for Ω, Σ and U^a in these coordinates, and then taking the limit to $\mathscr{N}^+$ (that is, $\alpha \to 1$), we get

$$\begin{aligned}\lim_{\alpha\to 1}\lim_{\rho\to 0}\big[-\frac{1}{8\pi}\int_{\mathcal{S}'}\tilde{\varepsilon}_2\,\Omega^{-1/2}\Sigma^{-1}\ *C_{acbd}\,U^a(\nabla^c\Omega^{1/2})(\nabla^d\Omega^{1/2})(\Omega^{-1/2}X^b)\big] \\ = -\frac{3}{8\pi}\int_{\mathscr{N}^+} d\theta d\phi\, aM\sin^2\theta X^\theta = J\,,\end{aligned} \tag{A.13.12}$$

where we have used X^θ to denote the limit of $\boldsymbol{X}^\theta$ to $\mathscr{N}^+$ and in the last equality, we have its explicit expression discussed above. Note also that in taking this limit, we have used the fact that $\boldsymbol{X}^\alpha \to 0$ as $\alpha \to 1$ (see Eqs. (5.3.16) and (5.3.19)). We therefore see that our continuity condition on Eq. (5.5.9) is satisfied at $\mathscr{N}^+$ in these spacetimes. In the same way, one can show that it is also satisfied at $\mathscr{N}^-$. Further, one can check by explicit computation that the condition Eq. (5.2.33) is also satisfied. Moreover, since these spacetimes are stationary, $N_{ab} = \mathcal{R}_{ab} = 0$ and so both conditions Eq. (5.2.34) are trivially satisfied. One can similarly check that the trace-free projection of ΣS_{ab} is continuous (and in fact 0) at $\mathscr{N}^\pm$. We therefore conclude that all our assumptions (summarized in Sec. 5.6) are satisfied in these spacetimes.

A.14 | Free Lorenz gauge solutions with nontrivial soft charges

To supplement the discussion in chapter 6, in this appendix we review the solutions of the free, noninteracting theory which satisfy everywhere the Lorenz gauge condition. We show that these solutions satisfy the gauge-invariant asymptotic properties at i^+ and i^- given in Eqs. (6.2.21) and (6.2.22). We then generalize the solutions to a larger class which satisfy Lorenz gauge only asymptotically, and deduce the general form of the Lorenz gauge scattering map for free solutions.

1 Global Lorenz gauge

In inertial coordinates (t, x^i) the homogeneous global Lorenz gauge solutions are

$$\underline{A}_0 = 0, \tag{A.14.1a}$$

$$\underline{A}_i = \sum_{l\geq 0}\left\{\partial_{ijL}\left[\frac{D_{jL}(t-r)}{r} - \frac{D_{jL}(t+r)}{r}\right] - \partial_L\left[\frac{D''_{iL}(t-r)}{r} - \frac{D''_{iL}(t+r)}{r}\right]\right\} + \sum_{l\geq 0}\epsilon_{ipq}\partial_{pL}\left[\frac{C_{qL}(t-r)}{r} - \frac{C_{qL}(t+r)}{r}\right]. \tag{A.14.1b}$$

The notation here follows Ref. [165] and is as follows. The symbol L is the multi-index $L = (i_1, i_2, \ldots i_l)$, and D_{iL}, C_{iL} are Cartesian tensors which are symmetric and trace-free on all of their indices. The symbol ∂_L means $\partial_{i_1}\ldots\partial_{i_l}$. Here and throughout underlines mean that the corresponding quantities are in Lorenz gauge. Note that, for a given solution, the tensors D_{iL} and C_{iL} are not unique, since the solution (A.14.1) is invariant under transformations of the form

$$D_{iL}(x) \to D_{iL}(x) + \delta D_{iL}(x), \quad C_{iL}(x) \to C_{iL}(x) + \delta C_{iL}(x), \tag{A.14.2}$$

for $l \geq 0$, where δD_{iL} and δC_{iL} are a polynomials in x of degree $2l+2$.

We will restrict attention to solutions for which the asymptotic behavior of the symmetric tracefree tensors D_{iL} and C_{iL} as $x \to \infty$ is given by

$$D_{iL}(x) = \tilde{D}_{+iL}(x) + P_{+iL}(x), \quad C_{iL}(x) = \tilde{C}_{+iL}(x) + Q_{+iL}(x), \tag{A.14.3}$$

where $\tilde{D}_{+iL}, \tilde{C}_{+iL}$ go to 0 as $x \to \infty$, and P_{+iL} and Q_{+iL} are polynomials in x of degree $l+2$ and $l+1$ respectively. Similarly as $x \to -\infty$ we require that

$$D_{iL}(x) = \tilde{D}_{-iL}(x) + P_{-iL}(x), \quad C_{iL}(x) = \tilde{C}_{-iL}(x) + Q_{-iL}(x), \tag{A.14.4}$$

where $\tilde{D}_{-iL}, \tilde{C}_{-iL} \to 0$ as $x \to -\infty$, and P_{-iL} and Q_{-iL} are again polynomials in x of degree $l+2$ and $l+1$ respectively. Because of the invariance property (A.14.2), the solution (A.14.1) depends only on the differences $\Delta P_{iL} = P_{+iL} - P_{-iL}$ and $\Delta Q_{iL} = Q_{+iL} - Q_{-iL}$ between these polynomials and not on $P_{\pm iL}$ or $Q_{\pm iL}$ individually. The assumptions (A.14.3) and (A.14.4) are compatible with the large r field expansions (6.2.6) and (6.2.14) that we have assumed[3], and also yield solutions with finite energy.

The coefficients of the large r expansions of the fields near $\mathscr{I}^+$ are given by $\underline{\mathcal{A}}_{+u} = \underline{\mathcal{A}}_{+r} = 0$, and

$$\underline{\mathcal{A}}_{+A}(u, \theta^A) = \sum_l (-1)^{l+1} n^L e^i_A \left[\tilde{D}^{(l+2)}_{+iL}(u) + \epsilon_{ipq} n^p \tilde{C}^{(l+1)}_{+qL}(u) \right], \tag{A.14.5}$$

where $n^i = x^i/r$, $e^i_A = D_A n^i$, $n^L = n^{i_1} \dots n^{i_l}$, and the superscripts $(l+2)$ and $(l+1)$ indicate the number of derivatives taken. This expression satisfies the condition

$$\underline{\mathcal{A}}_{+A} \to 0 \tag{A.14.6}$$

as $u \to \infty$ at i^+, which yields the condition (6.2.22c), from Eq. (6.2.7d). In the other limit $u \to -\infty$ at i^0 we have, from Eqs. (A.14.3) and (A.14.4),

$$\underline{\mathcal{A}}_{+A}(u, \theta^A) \to \underline{\mathcal{A}}_{\substack{+\\-}A}(\theta^A) = \sum_l (-1)^l n^L e^i_A \left[\Delta P^{(l+2)}_{iL} + \epsilon_{ipq} n^p \Delta Q^{(l+1)}_{qL} \right]. \tag{A.14.7}$$

Here the right hand side is independent of u since ΔP_{iL} is a polynomial of order $l+2$ and ΔQ_{qL} is a polynomial of order $l+1$.

Similar results apply for the limiting behavior of the solutions near $\mathscr{I}^-$. The expansion coefficients are $\underline{\mathcal{A}}_{-v} = \underline{\mathcal{A}}_{-r} = 0$, and

$$\underline{\mathcal{A}}_{-A}(v, \theta^A) = \sum_l n^L e^i_A \left[\tilde{D}^{(l+2)}_{-iL}(v) - \epsilon_{ipq} n^p \tilde{C}^{(l+1)}_{-qL}(v) \right]. \tag{A.14.8}$$

This satisfies the condition as $v \to -\infty$ at i^-

$$\underline{\mathcal{A}}_{-A} \to 0, \tag{A.14.9}$$

[3] This would no longer be true if the polynomials $P_{\pm iL}$ and $Q_{\pm iL}$ were of higher degree than $l+2$ and $l+1$.

yielding the condition (6.2.21c), while as $v \to \infty$ at i^0 we have, from Eqs. (A.14.3) and (A.14.4)

$$\underline{\mathcal{A}}_{-A}(v,\theta^A) \to \underline{\mathcal{A}}_{\substack{-\\+}A}(\theta^A) = \sum_l n^L e^i_A \left[\Delta P^{(l+2)}_{iL} - \epsilon_{ipq} n^p \Delta Q^{(l+1)}_{qL}\right]. \tag{A.14.10}$$

One can check that these Lorenz gauge solutions (A.14.1) satisfy the matching conditions (6.2.23a) and (6.2.23b) [or equivalently (6.2.38)] discussed in chapter 6. Note, however, that they do not satisfy the matching condition (6.2.39), instead this relation is satisfied with a sign flip, since from Eqs. (A.14.7) and (A.14.10) we have

$$\underline{\mathcal{A}}_{\substack{+\\-}A} = -\mathcal{P}_* \underline{\mathcal{A}}_{\substack{-\\+}A}. \tag{A.14.11}$$

This is discussed further in Sec. 6.3.2 above.

By expanding the solution (A.14.1) to subleading order in $1/r$ near $\mathscr{I}^+$ and $\mathscr{I}^-$ and reading off the subleading coefficients $\hat{\mathcal{A}}_{+u}$, $\hat{\mathcal{A}}_{+A}$, $\hat{\mathcal{A}}_{-v}$ and $\hat{\mathcal{A}}_{-A}$, one can verify the limiting behavior at i^- and i^+ given by Eqs. (6.2.21a), (6.2.21b), (6.2.22a) and (6.2.22b). Similar analyses for free massless scalar field solutions establishes (6.2.21d) and (6.2.22d).

Let us now specialize these solutions to the $l = 0, 1$ sectors for the scalar field, and to the $l = 1$ sector for the vector potential which is the case we consider in the body of chapter 6. In Lorentzian coordinates (t, x^i) these solutions are

$$\underline{\Phi}^{(1)} = -\frac{\psi_0(t-r)}{r} + \frac{\psi_0(t+r)}{r} + \partial_i\left[\frac{\psi_i(t+r)}{r} - \frac{\psi_i(t-r)}{r}\right], \tag{A.14.12a}$$

$$\underline{A}^{(1)}_t = 0\,, \tag{A.14.12b}$$

$$\underline{A}^{(1)}_i = \partial_{ij}\left[\frac{D_j(t-r) - D_j(t+r)}{r}\right] - \left[\frac{D''_i(t-r) - D''_i(t+r)}{r}\right] + \epsilon_{ipq}\partial_p\left[\frac{C_q(t-r) - C_q(t+r)}{r}\right], \tag{A.14.12c}$$

for some functions ψ_0, D_i and C_i, where primes denote derivatives with respect to the argument and $\partial_{ij} \equiv \partial_i\partial_j$. The boundary conditions on these functions at large values of their arguments are given in Eqs. (A.14.3) and (A.14.4). By exploiting the redefinition freedoms (A.14.2) we can take these boundary conditions to be

$$D_i(x) \rightarrow 0, \quad x \rightarrow -\infty, \tag{A.14.13a}$$

$$D_i(x) - \alpha_i - \beta_i x - \gamma_i x^2 \rightarrow 0, \quad x \rightarrow +\infty, \tag{A.14.13b}$$

and

$$C_i(x) \rightarrow 0, \quad x \rightarrow -\infty, \tag{A.14.14a}$$

$$C_i(x) - \lambda_i - \kappa_i x \rightarrow 0, \quad x \rightarrow +\infty, \tag{A.14.14b}$$

where α_i, β_i, γ_i, λ_i and κ_i are constants. Following the specialization (6.2.41) we will take

$$\kappa_i = 0. \tag{A.14.15}$$

Finally, we impose on ψ_0 and ψ_i the conditions

$$\psi_0(x) = O\left(1/|x|\right), \quad x \rightarrow \pm\infty, \tag{A.14.16}$$

$$\psi_i(x) = O\left(1/|x|\right), \quad x \rightarrow \pm\infty, \tag{A.14.17}$$

(cf. the discussion in Sec. 6.2.2 above).

Next, by writing the solution (A.14.12) in advanced polar coordinates $(v, r, \theta^A) = (v, r, \theta^1, \theta^2)$ where $v = t + r$, and by performing the expansion (6.2.14), we can read off the leading order initial data defined by Eqs. (6.3.14) [see also Eq. (A.14.8)] :

$$\underline{\chi}_{-}^{(1)}(v, \theta^A) = \psi_0(v) + n^i \psi_i'(v), \tag{A.14.18a}$$

$$\underline{\mathcal{A}}_{-A}^{(1)}(v, \theta^A) = e^i_A \left[D_i''(v) - \epsilon_{ipq} n_p C_q'(v)\right]. \tag{A.14.18b}$$

The corresponding final data is [see Eq. (A.14.5)]

$$\begin{aligned} \underline{\chi}^{(1)}_{+}(u,\theta^A) &= -\psi_0(u) + n^i\psi_i'(u), && \text{(A.14.19a)} \\ \underline{\mathcal{A}}^{(1)}_{+A}(u,\theta^A) &= -e^i_A\left[D_i''(u) - 2\gamma_i + \epsilon_{ipq}n_pC_q'(u)\right]. && \text{(A.14.19b)} \end{aligned}$$

Note that from Eqs. (A.14.13) and (A.14.14), the initial and final data Eqs. (A.14.18) and (A.14.19) satisfy our gauge condition (6.2.34) but not (6.2.39). Instead, (6.2.39) is satisfied with a sign flip. We remedy this with a gauge transformation in the body of chapter 6 below Eq. (6.4.4). By comparing Eqs. (A.14.19) and (A.14.18), we can read off the leading order scattering map in Lorenz gauge (also derived more generally in Appendix A.14). It is given by

$$\begin{aligned} \underline{\chi}^{(1)}_{+}(u,\theta^A) &= -\mathcal{P}_*\,\underline{\chi}^{(1)}_{-}(u,\theta^A)\,, && \text{(A.14.20a)} \\ \underline{\mathcal{A}}^{(1)}_{+A}(u,\theta^A) &= \mathcal{P}_*\left[\underline{\mathcal{A}}^{(1)}_{-A}(u,\theta^A) - \underline{\mathcal{A}}^{(1)}_{\mp A}(\theta^A)\right]. && \text{(A.14.20b)} \end{aligned}$$

It is easy to see that this map preserves the symplectic form (6.2.43), as it should, using Eqs. (A.14.6) and (A.14.9).

2 Asymptotic Lorenz gauge

The solutions (A.14.1) are the most general free solutions that obey the Lorenz gauge condition everywhere in spacetime. However one can obtain a more general class of solutions in *asymptotic Lorenz gauge*, in which one imposes the Lorenz gauge condition only at $r \geq R$ for some R. Specifically for $r \geq R$ one can transform the solutions according to

$$\underline{A}_a \to \underline{A}_a + \nabla_a\varepsilon, \qquad \underline{\Phi} \to e^{i\varepsilon}\underline{\Phi}, \qquad \varepsilon = \sum_{l\geq 1} D_L\partial_L\left(\frac{u^l+v^l}{r}\right), \tag{A.14.21}$$

where D_L for $l \geq 1$ are some constant traceless symmetric tensors. For $r < R$ one can use any smooth extension of ε. This transformation preserves the asymptotic gauge conditions (6.2.34), and the initial and final data transform as $\underline{\mathcal{A}}_{+A} \to \underline{\mathcal{A}}_{+A} + D_A\varepsilon_+$, $\underline{\mathcal{A}}_{-A} \to \underline{\mathcal{A}}_{-A} + D_A\varepsilon_-$, where $\varepsilon_+(\theta^A)$ is a freely specifiable function with only $l \geq 1$ components and $\varepsilon_- = \mathcal{P}_*\varepsilon_+$. The transformation is of the even form (6.2.35), and thus is not gauge but instead is a mapping from solutions to physically distinct solutions, as discussed in Sec. 6.2.4.

The general asymptotic Lorenz gauge solutions will no longer satisfy the conditions (A.14.6) and (A.14.9) at past and future timelike infinity. However there is a combination of these conditions which is unaffected by the transformation (A.14.21), and which is still valid for the general solutions, namely

$$\underline{\Psi}^{\rm e}_{\substack{+\\+}} = \mathcal{P}_* \underline{\Psi}^{\rm e}_{\substack{-\\-}}. \tag{A.14.22}$$

Scattering map for free solutions in asymptotic Lorenz gauge

Recall that the scattering map that relates the initial data on $\mathscr{I}^-$ to the final data on $\mathscr{I}^+$ for the free global Lorenz gauge solutions is

$$\begin{aligned} \underline{\chi}_{+}(u,\theta^A) &= -\mathcal{P}_* \underline{\chi}_{-}(u,\theta^A), & \text{(A.14.23a)} \\ \underline{\mathcal{A}}_{+A}(u,\theta^A) &= \mathcal{P}_* \left[\underline{\mathcal{A}}_{-A}(u,\theta^A) - \underline{\mathcal{A}}_{\substack{-\\+}A}(\theta^A)\right]. & \text{(A.14.23b)} \end{aligned}$$

This scattering map can be generalized to the class of asymptotic Lorenz gauge solutions generated by the transformation (A.14.21) by expressing the gauge transformation function ε_{+} in terms of the new initial data. Writing the result in terms of the potentials (6.2.36), making use of the vanishing magnetic charges condition (6.2.41) and using $\mathcal{P}_*\varepsilon_{AB} = -\varepsilon_{AB}$ gives

$$\begin{aligned} \underline{\Psi}^{\rm e}_{+}(u,\theta^A) &= \mathcal{P}_* \left[\underline{\Psi}^{\rm e}_{-}(u,\theta^A) - \underline{\Psi}^{\rm e}_{\substack{-\\+}}(\theta^A) + \underline{\Psi}^{\rm e}_{\substack{-\\-}}(\theta^A)\right], & \text{(A.14.24a)} \\ \underline{\Psi}^{\rm m}_{+}(u,\theta^A) &= -\mathcal{P}_* \underline{\Psi}^{\rm m}_{-}(u,\theta^A), & \text{(A.14.24b)} \\ \underline{\chi}_{+}(u,\theta^A) &= -\mathcal{P}_* \underline{\chi}_{-}(u,\theta^A). & \text{(A.14.24c)} \end{aligned}$$

A.15 | Properties of interacting Lorenz gauge solutions

In this appendix we deduce some properties of nonlinear Lorenz gauge solutions that are applicable to the analysis presented in chapter 6. Consider first solutions in global Lorenz gauge. We claim that the conditions (A.14.6) and (A.14.9) at future and past timelike infinity are still satisfied by these solutions. To see this, consider the version of the theory on the Einstein static universe obtained by making a conformal transformation. In this version spacetime is compact and all the fields are bounded. Consider now a small neighborhood $\mathcal{V}$ of future timelike infinity i^+. At each order in perturbation theory, one can obtain the fields inside $\mathcal{V}$ by specifying the initial conditions on the Cauchy surface obtained by taking the intersection of $\partial\mathcal{V}$ with the image of Minkowski spacetime on the Einstein static universe cylinder. The fields inside $\mathcal{V}$ (and in particular the limit to i^+ of the fields on $\mathscr{I}^+$) are given as a sum of a homogeneous solution determined by the initial data, and an inhomogeneous solution determined by the sources inside $\mathcal{V}$ with zero initial data. However, we know from Appendix A.14 that the homogeneous solutions must satisfy the vanishing condition (A.14.6) at i^+, since they are free Lorenz gauge solutions. The inhomogeneous solution can give a nonvanishing contribution to the limit, however this can be made arbitrarily small by taking the size of the neighborhood to zero, since the sources are bounded. We conclude that the conditions (A.14.6) and (A.14.9) are satisfied.

Just as for the free solutions, more general nonlinear solutions in asymptotic Lorenz gauge obtained from the transformation (A.14.21) will no longer satisfy the conditions (A.14.6) and (A.14.9), but will satisfy the condition (A.14.22).

A.16 | Field configuration space in the magnetic sector

In this appendix we discuss a difference between the space of solutions for the electric potential Ψ^{e}_{-} and the magnetic potential Ψ^{m}_{-}, to supplement the discussion contained in chapter 6. For the class of Lorenz gauge free solutions discussed in Appendix A.14 these potentials vanish at i^-, from Eq.

(A.14.9):

$$\underset{-}{\Psi}^{\mathrm{e}} = 0, \qquad \underset{-}{\Psi}^{\mathrm{m}} = 0, \tag{A.16.1}$$

and this generalizes to the solutions of the interacting theory. However, by using the transformation freedom (6.2.35a) we can make $\underset{-}{\Psi}^{\mathrm{e}}$ be nonzero, from Eq. (6.2.37a). Thus in the full space of solutions with our preferred asymptotic gauge conditions (6.2.34) and (6.2.39) [which do not include Lorenz gauge] we have that both $\underset{-}{\Psi}^{\mathrm{e}}$ and $\underset{+}{\Psi}^{\mathrm{e}}$ are nonzero in general.

For the magnetic variables the story is somewhat different. Starting from the class of solutions which satisfy (A.16.1), one can attempt to obtain a larger class of solutions by analogy with the procedure for the electric variables, by making a transformation of the initial data on $\mathscr{I}^-$ of the form

$$\underset{-}{\Psi}^{\mathrm{e}} \to \underset{-}{\Psi}^{\mathrm{e}}, \qquad \underset{-}{\Psi}^{\mathrm{m}} \to \underset{-}{\Psi}^{\mathrm{m}} + \tilde{\varepsilon}_{-}, \tag{A.16.2}$$

where $\tilde{\varepsilon}_{-}$ is a function on $\mathscr{I}^-$ with no $l = 0$ component that is independent of v. Although the transformation (A.16.2) is not a gauge transformation, it is a kind of magnetic analog of the gauge transformation given by Eqs. (6.2.37) [136]. By solving the equations of motion one can determine the effects of this transformation of the initial data on the entire solution, and in particular the data on $\mathscr{I}^+$ transforms as [cf. Eq. (6.2.35a) above]

$$\underset{+}{\Psi}^{\mathrm{e}} \to \underset{+}{\Psi}^{\mathrm{e}}, \qquad \underset{+}{\Psi}^{\mathrm{m}} \to \underset{+}{\Psi}^{\mathrm{m}} + \tilde{\varepsilon}_{+}. \tag{A.16.3}$$

where $\tilde{\varepsilon}_{+} = \mathcal{P}_{*}\tilde{\varepsilon}_{-}$. The magnetic charges

$$\tilde{Q}_{\tilde{\varepsilon}} = \frac{1}{e^2}\int d^2\Omega\, \tilde{\varepsilon}_{+} \underset{+}{\mathcal{F}}_{AB} = \frac{1}{e^2}\int d^2\Omega\, \tilde{\varepsilon}_{-} \underset{+}{\mathcal{F}}_{AB} \tag{A.16.4}$$

associated with these transformations [cf. Eq. (6.2.23b) above] can be derived in exact parallel with the derivation for the electric case discussed in Sec. 6.2.4 [136]. However, the solutions generated by the transformations (A.16.2) are generically singular in the interior of the spacetime. One example of such a solution is a static magnetic dipole at the origin $r = 0$. In our discussion, we restricted attention to initial data on $\mathscr{I}_{-}$ which evolves into smooth solutions in the interior of the spacetime, which requires that

$$\Psi^{\mathrm{m}}_{\frac{-}{-}} = 0, \tag{A.16.5}$$

and disallows the transformations (A.16.2). A similar analysis at $\mathscr{I}^+$ yields the condition

$$\Psi^{\mathrm{m}}_{\frac{+}{+}} = 0. \tag{A.16.6}$$

The other limits $\Psi^{\mathrm{m}}_{\frac{+}{-}}$ and $\Psi^{\mathrm{m}}_{\frac{-}{+}}$ of the magnetic potentials at spatial infinity are generally nonzero (see Appendix A.14). However, in our analysis, we restricted attention to the sector of the theory in which they vanish, as discussed around Eq. (6.2.41).

A.17 | Computation of Lorenz gauge scattering map

In this appendix, we solve the second order Lorenz gauge equations of motion for the scalar field (6.3.18a), with source terms obtained from the leading order solutions (A.14.12). We use retarded coordinates (u, x^i) or (u, r, n^i) where $\mathrm{r} = |\vec{x}|$, $u = t - r$ and $\mathbf{n} = \vec{x}/r$ is the unit radial vector. We also include some supplementary results that are needed for the computation of the cubic order scalar field solution in Sec. 6.5.2.

1 | Scalar field solution

The wave equation (6.3.18a) for the second order piece of the scalar field is of the form $\Box\underline{\Phi}^{(2)} = S$ for some source S, whose general solution in retarded coordinates is

$$\underline{\Phi}^{(2)}(u, r, \mathbf{n}) = -\frac{1}{4\pi}\int d^3\mathbf{y}\frac{S(u + r - \rho - R, \mathbf{y})}{R}\,, \tag{A.17.1}$$

where $R = |\mathbf{x} - \mathbf{y}|$ and $\rho = |\mathbf{y}|$. Our source has $l = 0, 1, 2$ modes. Let us analyze them separately, starting with the $l = 1$ mode. This can is be written as $S = S_i(u, r)n^i$, yielding

$$\underline{\Phi}^{(2)}_{(l=1)}(u, r, \mathbf{n}) = -\frac{n^i}{2}\int_0^\infty d\rho\int_{-1}^1 d\mu\frac{\rho^2\mu}{R}S_i(u + r - \rho - R, \rho)\,, \tag{A.17.2}$$

where $\mu = \mathbf{n} \cdot \mathbf{n}'$ and $\mathbf{n}' = \mathbf{y}/\rho$. We next make a change of variables of integration from ρ, μ to Δu, Δv defined by

$$\Delta u = -r + \rho + R\,, \qquad \Delta v = r + \rho - R\,, \tag{A.17.3}$$

where $R = \sqrt{r^2 + \rho^2 - 2r\rho\mu}$. This gives

$$\underline{\Phi}^{(2)}_{(l=1)}(u, r, \mathbf{n}) = \frac{n^i}{8r} \int_0^\infty d\Delta u \int_0^{2r} d\Delta v \left[\Delta u - \Delta v - \frac{\Delta u \Delta v}{r}\right] S_i(u - \Delta u, \Delta u/2 + \Delta v/2). \tag{A.17.4}$$

We now discuss taking the limit $r \to \infty$ to future null infinity. The leading order piece of the solution (A.17.4), proportional to $1/r$, is given by replacing the upper bound on the Δv integral with ∞, and dropping the third term in the large square brackets. The approximation is delicate however since it depends on the behavior of the function S_i near $\mathscr{I}^+$, and in general terms proportional to $\log r$ can arise. In our case we will find that the integral defining the coefficient of the leading order $1/r$ term converges, indicating that the approximation is valid for the leading order term.

We next insert into Eq. (A.17.4) the expression for S_i obtained from Eqs. (6.3.18a) and (A.14.12) and take the large r limit. This yields

$$\underline{\Phi}^{(2)}_{(l=1)}(u, r, \mathbf{n}) = -4i \frac{n^i}{r} \int_0^\infty d\Delta u \int_0^\infty d\Delta v\, \mathcal{I}_i(u, \Delta u, \Delta v) + O\left(\frac{\ln r}{r^2}\right), \tag{A.17.5}$$

where the integrand is

$$\begin{aligned}\mathcal{I}_i(u, \Delta u, \Delta v) = \frac{\Delta v - \Delta u}{\Delta u + \Delta v} (\partial_{\Delta u} + \partial_{\Delta v}) \left\{ \frac{1}{\Delta u + \Delta v} \left[D_i(u - \Delta u) - D_i(u + \Delta v)\right] \right\} \\ \times (\partial_{\Delta u} + \partial_{\Delta v}) \left\{ \frac{1}{\Delta u + \Delta v} \left[\psi_0(u - \Delta u) - \psi_0(u + \Delta v)\right] \right\}.\end{aligned} \tag{A.17.6}$$

To explicitly evaluate the integral (A.17.5), we expand out the integrand (A.17.6) and evaluate the derivatives. Now although the total integrand is smooth at $\Delta u = \Delta v = 0$, the individual terms diverge there. To evaluate the integrals of the individual terms we introduce a lower cutoff ν in the Δu and Δv integrals and use integrations by parts. When we add the integrals of the individual terms, the divergent terms proportional to ν^{-1} and ν^{-2} cancel. Taking the cutoff to zero, we find, after a long but straightforward calculation,

$$\underline{\Phi}^{(2)}_{(l=1)}(u,r,\mathbf{n}) = -4i\frac{n^i}{r}\psi_0(u)\gamma_i + O\left(\frac{\ln r}{r^2}\right) , \tag{A.17.7}$$

where γ_i is defined in Eq. (A.14.13b).

Let us now consider the contribution to the source term Eq. (6.3.18a) from the ψ_i terms in Eq. (A.14.12a). [We drop the C_q terms appearing from Eq. (A.14.12c) since these terms do not contribute to the final answer because of Eq. (A.14.14).] This is given by

$$\begin{aligned} S_{(l=0,\,l=2)}(u,v,r) = {} & \frac{4in^in^j}{r}\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\,\partial_r^2\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big] \\ & - \frac{2i}{r^2}(\delta^{ij}-n^in^j)\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\partial_r\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big] \\ & + \frac{2i}{r^2}(\delta^{ij}-n^in^j)\big[D_i''(u)-D_i''(v)\big]\partial_r\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big] , \end{aligned} \tag{A.17.8}$$

where the subscript denotes the fact that this term is comprised of $l=0$ and $l=2$ modes. Breaking up n^in^j into a symmetric-traceless and a pure trace part, we can extract each of these modes. The $l=0$ mode is given by

$$\begin{aligned} S_{(l=0)}(u,v,r) = {} & \frac{4i}{3r}\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\,\partial_r^2\big[\frac{\psi^i(u)-\psi^i(v)}{r}\big] - \frac{4i}{3r^2}\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\partial_r\big[\frac{\psi^i(u)-\psi^i(v)}{r}\big] \\ & + \frac{4i}{3r^2}\big[D_i''(u)-D_i''(v)\big]\partial_r\big[\frac{\psi^i(u)-\psi^i(v)}{r}\big] \end{aligned} \tag{A.17.9}$$

This yields

$$\underline{\Phi}^{(2)}_{(l=0)}(u,r) = -\frac{1}{2r}\int_0^\infty d\Delta u\int_0^\infty d\Delta v\,\mathcal{J}(u,\Delta u,\Delta v) + O\left(\frac{\ln r}{r^2}\right), \tag{A.17.10}$$

where the integrand is

$$\mathcal{J}(u,\Delta u,\Delta v) = \frac{\Delta v-\Delta u}{4}S_{(l=0)}(u-\Delta u, u+\Delta v, \frac{\Delta u+\Delta v}{2}) \tag{A.17.11}$$

The $l = 2$ mode is given by

$$\begin{aligned} S_{(l=2)}(u,v,r) = &\frac{4i(n^i n^j - \frac{1}{3}\delta^{ij})}{r}\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\,\partial_r^2\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big] \\ &+ \frac{2i}{r^2}(n^i n^j - \frac{1}{3}\delta^{ij})\partial_r\big[\frac{D_i(u)-D_i(v)}{r}\big]\partial_r\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big] \\ &- \frac{2i}{r^2}(n^i n^j - \frac{1}{3}\delta^{ij})\big[D_i''(u)-D_i''(v)\big]\partial_r\big[\frac{\psi_j(u)-\psi_j(v)}{r}\big]\,, \end{aligned} \tag{A.17.12}$$

where we have suppressed the $\mathbf{n}$ dependence in the argument of $S_{(l=2)}$ for ease of notation. This yields

$$\underline{\Phi}^{(2)}_{(l=2)}(u,r,\mathbf{n}) = -\frac{1}{2r}(n_i n_j - \frac{1}{3}\delta_{ij})\int_0^\infty d\Delta u\int_0^\infty d\Delta v\,\mathcal{K}^{ij}(u,\Delta u,\Delta v) + O\left(\frac{\ln r}{r^2}\right), \tag{A.17.13}$$

where the integrand is

$$\mathcal{K}^{ij}(u,\Delta u,\Delta v) = \frac{\Delta u^2 - 4\Delta u\Delta v + \Delta v^2}{6(\Delta u + \Delta v)}S_{(l=2)}(u-\Delta u, u+\Delta v, \frac{\Delta u + \Delta v}{2}) \tag{A.17.14}$$

The calculation for Eq.(A.17.10) proceeds exactly like the one described earlier to obtain Eq. (A.17.7). Eq. (A.17.13), however, is hard to evaluate analytically and so we resort to numerical methods to solve the integral. We do this by putting in various functions that satisfy our fall-off conditions (A.14.13), (A.14.16). Our final answer for this part of the solution is

$$\begin{aligned} \underline{\Phi}^{(2)}_{(l=0)}(u,r,\mathbf{n}) + \underline{\Phi}^{(2)}_{(l=2)}(u,r,\mathbf{n}) &= \frac{1}{r}\left[\frac{4i}{3}\gamma^j\psi_j'(u) + 4i(n^i n^j - \frac{1}{3}\delta^{ij})\gamma_i\psi_j'(u)\right] + O\left(\frac{\ln r}{r^2}\right)\,, \\ &= 4in^i n^j\gamma_i\psi_j'(u) + O(\frac{\ln r}{r^2}) \end{aligned} \tag{A.17.15}$$

Hence, using Eq. (A.14.18), we obtain the following expression for the final data of the second order scalar field

$$\underline{\chi}_{+}^{(2)}(u,\theta^A) = -4i\gamma_i n^i\mathcal{P}_*\underline{\chi}_{-}^{(1)}(u,\theta^A). \tag{A.17.16}$$

2 | Details of the cubic order scalar field calculation

To calculate $\underset{+}{\chi}^{(3)}(u,\theta^A)$ which we need to evaluate $\int du\, \mathcal{J}_{+u}(u,\theta^A) - \int dv\, \mathcal{J}_{-v}(v,\theta^A)$ in Sec. 6.5.3, we will make use of Fourier mode expansions. Below, we write out the mode expansion for a massless complex scalar field that satisfies the free wave equation and use this opportunity to lay out our conventions. We take a specific scalar field profile that has support on only $l=0$ and $l=1$ modes. As such, we write

$$\begin{aligned}\underline{\Phi}^{(1)}(x) &= \int \frac{d^3k}{\sqrt{2k}}\, g(\hat{k}) \left[c_k e^{-ikt+i\vec{k}\cdot\vec{x}} + b_k^{\star} e^{ikt-i\vec{k}\cdot\vec{x}}\right] \\ &= \int d\Omega_k\, g(\hat{k}) \left[\int_0^\infty dk \frac{k^2}{\sqrt{2k}} c_k e^{-ikt+i\vec{k}\cdot\vec{x}} + \int_{-\infty}^0 dk \frac{k^2}{\sqrt{2|k|}}(-ib^*_{-k}) e^{-ikt+i\vec{k}\cdot\vec{x}}\right] \\ &= \int d\Omega_k \int_{-\infty}^\infty dk\, g(\hat{k}) \frac{k^2}{\sqrt{2|k|}} a_k e^{-ikt+i\vec{k}\cdot\vec{x}} = \int d\Omega_k \int_{-\infty}^\infty dk\, \frac{k^2}{\sqrt{2|k|}} a_{\overline{k}} e^{-ikt+i\vec{k}\cdot\vec{x}}\,, \end{aligned} \tag{A.17.17}$$

where $g(\hat{k})$ encodes the dependence of the mode functions on $\hat{k}$ and we have taken that to be the same for both mode functions. In the second term in the second step, we took $k \to -k$ and consequently $\vec{k} = k\hat{k} \to -k\hat{k} = -\vec{k}$. We have defined a new complex valued function a_k such that for $k > 0$, $a_k = c_k$ and for $k < 0$, $a_k = -ib^*_{-k}$. In the last step, we have also defined $a_{\overline{k}} := g(\hat{k})a_k$. To relate these mode functions to our chosen initial data profile for the scalar field on $\mathscr{I}^-$, we convert this to v, r coordinates, take the large r limit at constant v, and evaluate the momentum integrals using a saddle point approximation. This gives us the relation

$$f_{\overline{k}} = \sqrt{2} i\pi \frac{k}{\sqrt{|k|}} a_{\overline{k}}\,, \tag{A.17.18}$$

where $f_{\overline{k}}$ is the momentum space profile of our initial scalar field on $\mathscr{I}^-$. In our calculation, we pick this profile to be

$$f_{\overline{k}} = (1 + \hat{k}\cdot\hat{z}) \exp[-(i+1)(k-1)^2]\,, \tag{A.17.19}$$

where $\hat{z}$ is the unit vector pointing towards the north pole on $\mathbb{S}^2$. In position space, this corresponds to [4]

$$\underline{\chi}_{-}(v,\theta^A) = \frac{(1-i)\sqrt{1+i}}{4\sqrt{\pi}}(1+\cos\theta)\exp[-iv - (\frac{1-i}{8})v^2]\,. \tag{A.17.20}$$

Our goal is to find $\underline{\chi}_{+}^{(3)}(u,\theta^A)$, that is, the leading $1/r$ piece in a $1/r$ expansion of $\underline{\Phi}^{(3)}(x)$ near $\mathscr{I}^+$ where $\underline{\Phi}^{(3)}(x)$ solves Eq. (6.5.24a). A useful way of doing this, for a generic field, $F(x)$ satisfying

$$\Box F(x) = -j(x)\,, \tag{A.17.21}$$

in Minkowski spacetime was given in [146] where they were interested in extracting the $1/r$ piece at large r and fixed t. We now adapt their derivation to our limit of interest, namely large r at fixed u which corresponds to the limit to $\mathscr{I}^+$.

The retarded solution to Eq. (A.17.21) is given by

$$F(x) = -\int d^4y\, G_r(x,y)j(y)\,, \tag{A.17.22}$$

where $G_r(x,y)$ is the retarded Green's function

$$G_r(x,y) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \int d^3\vec{\ell}\, e^{-i\omega(t-y^0)+i\vec{\ell}\cdot(\vec{x}-\vec{y})}\frac{1}{(\omega+i\epsilon)^2-\vec{\ell}^2}\,, \tag{A.17.23}$$

where we have made explicit the fact that we denote ℓ^0 by ω and $x^a = (t,\vec{x})$ while $y^a = (y^0,\vec{y})$. This gives us

$$F(x) = -\frac{1}{(2\pi)^4}\int d^4y \int_{-\infty}^{\infty} d\omega \int d^3\vec{\ell}\, e^{-i\omega t+i\omega y^0+i\vec{\ell}\cdot(\vec{x}-\vec{y})}\frac{j(y)}{(\omega+i\epsilon)^2-\vec{\ell}^2}\,. \tag{A.17.24}$$

We now write $t = u + r$,where $|\vec{x}| = r$, and repeat the calculation given in Eqs. (2.7-2.9) of [146]. Defining $\vec{\ell}_{\|}$ and $\vec{\ell}_{\perp}$ to be the components of $\vec{\ell}$ parallel and perpendicular to $(\vec{x}-\vec{y})$ respectively, we

[4] See the discussion below Eq. (6.5.19) for an explanation of this choice.

obtain

$$F(u,\vec{x}) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \int d^4y \int d^2\ell_\perp d\ell_\parallel \, e^{i\omega y^0 - i\omega(u+r) + i\ell_\parallel |\vec{x}-\vec{y}|} \frac{j(y)}{(\omega+i\epsilon)^2 - \ell_\parallel^2 - \vec{\ell}_\perp^{\,2}} \, . \quad \text{(A.17.25)}$$

Next, we do the integral over $\vec{\ell}_\parallel$ using contour integration where the we close the contour for $\ell_{||}$ integration in the upper-half plane. We then do the integration over $\vec{\ell}_\perp$, which at large $|\vec{x}-\vec{y}|$ can be done using the saddle point approximation where the saddle point lies at $\vec{\ell}_\perp = 0$. Doing both these integrals and then taking $\epsilon \to 0$, we obtain

$$F(u,\vec{x}) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\omega \int d^4y \, e^{i\omega y^0 - i\omega(u+r) + i\omega|\vec{x}-\vec{y}|} \frac{j(y)}{|\vec{x}-\vec{y}|} \, . \quad \text{(A.17.26)}$$

Assuming $|\vec{x}| \gg |\vec{y}|$, using $|\vec{x}-\vec{y}| = r - y\,\hat{x}\cdot\hat{y} + O(1/r)$, we obtain

$$F(u,\vec{x}) \approx \frac{1}{8\pi^2 r}\int_{-\infty}^{\infty} d\omega \int d^4y \, j(y) \, e^{-i\omega u + i\omega y^0 - i\omega \hat{x}\cdot\vec{y}} \, . \quad \text{(A.17.27)}$$

Note that the assumption $|\vec{x}| \gg |\vec{y}|$ is justified (at least) when the source term, $j(y)$ has an exponential fall-off at large $|\vec{y}|$ at fixed y^0 which is true in our application of this formula because of our choice of initial data [Eq. (A.17.20)].

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