COMBINATORIAL CHARACTERIZATIONS OF POLARIZATIONS OF POWERS OF THE GRADED MAXIMAL IDEAL

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This dissertation is dedicated to the study of combinatorial characterizations of polarizations of powers of the graded maximal ideal in a polynomial ring, and applications of these characterizations to questions in algebra, geometry, and combinatorics. We first characterize polarizations of powers of the graded maximal ideal in terms of their graphs of linear syzygies, and apply this characterization to study their Alexander duals and the question of when the Stanley–Reisner ideals of polarizations are shellable. We then give a novel characterization of polarizations of the same class of ideals in terms of hook tableaux. Finally, we show that any triangulation of a product of simplices gives rise to a polarization of a power of a graded maximal ideal.

BIOGRAPHICAL SKETCH

Ayah Almousa was born in Fridley, Minnesota in March 1993, and spent the majority of her childhood in Milwaukee, Wisconsin. She received her Bachelor of Science degrees in Physics and Mathematics from the University of Wisconsin-Madison in May 2015, and began a PhD program at Cornell University the same year. She spent over half of the year 2019 at University of Bergen in Norway collaborating with Gunnar Fløystad.

To all my friends and family who convinced me that I could do this.

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CHAPTER 1

INTRODUCTION

Intuitively, a *polarization* of a monomial ideal I is a *squarefree* monomial ideal $\widetilde{I}$ that has many of the same homological properties of $\tilde{I}$. The first incarnation of polarizations appeared in Hartshorne's 1966 thesis [23], in which he used polarizations to construct so-called "distractions" and which he used to prove connectedness of the Hilbert scheme. Since then, polarizations have been ubiquitous in the study of monomial ideals due to the ease with which they allow one to pass from an arbitrary monomial ideal to a squarefree one. In this way, polarizations give access to combinatorial and topological tools that only exist for squarefree monomial ideals, such as Stanley–Reisner theory.

Historically in the literature, there has been one particular construction of a polarization which is most commonly used; we refer to this ideal as the *standard polarization*. However, it eventually became clear that this was not the only way to polarize an ideal. Several authors have considered and used polarizations other than the standard polarization over the years, such as the *shifting operator* introduced by Aramova, Herzog, and Hibi [3] and later generalized by Murai [36]; or the polarization introduced by Nagel and Reiner in [38] in their construction of the "complex of boxes", which gives rise to a minimal, linear, cellular resolution of all strongly stable ideals. Despite this, there has been no systematic study of the variety of polarizations and properties which characterize them.

In this dissertation, we undertake this task for the case of powers of the graded maximal ideal in a polynomial ring. We present three distinct combinatorial characterizations of polarizations in this case and apply them to study questions in algebra and combinatorics. Informally, the key idea uniting all of these characterizations is to think of a "potential" polarizations as coming from a set of isotone maps from the generators of the ideal to the Boolean poset, and to ask what properties these maps must satisfy in order to give a polarization of a power of the maximal ideal. The first characterization is in terms of subgraphs of the graph of linear syzygies between the generators of the "potential" polarization; we frequently refer to this condition as the "spanning tree condition". In later chapters, we show that this "spanning tree condition" can be stated in other equivalent ways. In Part II, we show that the spanning tree condition is equivalent to showing that a certain set of hook tableaux which arise from the isotone maps span a certain Schur module. In Part III, we show that the spanning tree condition is also equivalent to the "hexagon axiom" introduced by Galashin, Nenashev, and Postnikov in [20] in their axiomatic characterization of triangulations of root polytopes.

This dissertation is organized as follows. Part I is dedicated to studying polarization in terms of the linear syzygy edges between their generantors. In Chapter 2, we give a complete characterization of polarizations of powers of the maximal ideal in terms of their graphs of linear syzygies. In Chapter 3, we utilize this characterization to construct the Alexander dual of any polarization of a power of the graded maximal ideal. In Chapter 4, we apply the constructions of previous chapters to show that the Stanley–Reisner complexes of polarizations of $(x, y, z)^d$ are always simplicial balls. Chapter 5 is dedicated to studying necessary and sufficient conditions on potential graphs of linear syzygies to determine when a set of valid isotone maps (as defined in Chapter 2) exists. In particular, we give an algorithm for computing a set of isotone maps (and therefore, potentially, a polarization) from a graph satisfying certain properties. Chapters 1-4 are based on joint work with Gunnar Fløystad and Henning Lohne.

In Part II, we study the L-complex of Buchsbaum and Eisenbud. In Chapter 6, we show that the L-complex is cellular, and is in fact supported on a CW-complex obtained from the so-called *hypersimplicial complex* (introduced by Batzies and Welker in [5]) via discrete Morse theory. In Chapter 7, we rephrase the "spanning tree condition" from Part I in terms of a condition on the hook tableaux corresponding to the linear syzygy edges. Chapter 7 is based on joint work with Keller VandeBogert.

Finally, in Part III, we relate triangulations of a product of simplices $\Delta^{n-1} \times \Delta^{d-1}$ to polarizations of powers of the graded maximal ideal. In particular, we show that every so-called *trianguloid* (introduced by Galashin, Nenashev, and Postnikov in [20]) of a product of simplices gives rise to a polarization of $(x_1, \ldots, x_n)^d$.

Part I

Polarizations and Linear Syzygy Edges

CHAPTER 2

POLARIZATIONS OF POWERS OF GRADED MAXIMAL IDEALS

This chapter is based on joint work with Gunnar Fløystad and Henning Lohne.

2.1 Polarizations in the literature

We begin by presenting the "classical" construction of polarizations of monomial ideals, which we call the "standard polarization". Afterwards, we introduce other types of polarizations which have appeared in the literature to date and discuss their applications.

It is common for authors to define a polarization of a monomial ideal using the following construction, which outputs what we will call the *standard polarization* of a monomial ideal. The standard polarization has been used extensively in the literature for various purposes in algebra and combinatorics; see, for instance, work of Faridi [13], Fröberg [19], Herzog [24], Schwartau [47], and Rota and Stein [44].

Construction 2.1.1 (Standard Polarization)**.** Let I be a monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. Let d_i be the largest power of the variable x_i which divides a minimal generator of I. Let $\check{X}_i = \{x_{i1}, x_{i2}, \ldots, x_{id_i}\}$ for each $i \in [n]$, and construct the new polynomial ring $\widetilde{S} = k[\check{X}_1, \ldots, \check{X}_n]$ with the union of all of these variables.

Take each generator of I of the form

$$x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

and make the following monomial

$$(x_{11}x_{12}\dots x_{1a_1})\cdot(x_{21}x_{22}\dots x_{2a_2})\dots(x_{n1}\dots x_{na_n})$$

a minimal generator of $\widetilde{I} \subset \widetilde{S}$.

Call $\tilde{I}$ the *standard polarization* of I. To recover the quotient ring S/I from $\widetilde{S}/\widetilde{I}$, simply cut down by the regular sequence of variable differences

$$\sigma = \bigcup_{i=1}^{n} \{x_{i1} - x_{i2}, x_{i1} - x_{i3}, \dots, x_{i1} - x_{id_i}\}.$$

Although the standard polarization is what is historically meant in the literature by a "polarization" of a monomial ideal, in recent years it has become evident that there are other ways to polarize an ideal. The following construction was introduced by Nagel and Reiner in [38] for strongly stable ideals. It was further studied by Yanagawa in [52].

Construction 2.1.2 (Box Polarization)**.** Let I be a monomial ideal in the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. Let d_i be the largest power of the variable x_i which divides a minimal generator of I. Let $\check{X}_i = \{x_{i1}, x_{i2}, \dots, x_{id_i}\}$ for each $i \in [n]$, and construct the new polynomial ring $\widetilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ with the union of all of these variables.

Take each generator of I of the form

$$x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$$

and make the following monomial

$$(x_{11}x_{12}\dots x_{1a_1})\cdot(x_{2a_1+1}x_{22}\dots x_{2a_1+a_2})\dots(x_{n,a_1+\cdots+a_{n-1}+1}\dots x_{n,a_1+\cdots+a_n})$$

a minimal generator of $\widetilde{I} \subset \widetilde{S}$. Call $\tilde{I}$ the *box polarization* of I.

Another instance of a non-standard polarization which has previously appeared in the literature is the *shifting operation* introduced by Aramova, Herzog, and Hibi in [3] for generic initial ideals. Murai further generalized this operation in [36] for the purposes of proving a conjecture of Kalai on generic initial ideals of Stanley–Reisner ideals of squeezed spheres. Yanagawa carried out further study of this operation in [52]. We present here Yanagawa's formulation of Murai's operator.

Construction 2.1.3 (Shifting Operator)**.** Let $T = k[x_1, \ldots, x_N]$ be a polynomial ring over a field k with $N \gg 0$. Let $\mathbf{m}$ be a monomial in the polynomial ring $S = k[x_1, \ldots, x_n]$, and write $\mathbf{m}$ in the form

$$\mathbf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \le \alpha_1 \le \cdots \le \alpha_e \le n.$$

Define $(-)^{\sigma(a)}$ to be the operation sending $\mathbf{m}$ to

$$\mathbf{m}^{\sigma(a)} := \prod_{i=1}^{e} x_{\alpha_i + a_{i-1}} \in T.$$

For a monomial ideal $I \subset S$ with minimal generating set $G(I)$, set

$$I^{\sigma(a)} := \left(\mathbf{m}^{\sigma(a)} \mid \mathbf{m} \in G(I)\right) \subset T.$$

If $a_{i+1} > a_i$ for all i, then $\mathbf{m}^{\sigma(a)}$ is a squarefree monomial ideal. In particular, if $a_i = i$ for all i, then $(-)^{\sigma(a)}$ coincides with the *squarefree operation* $(-)^{\sigma}$, which plays an important role in the construction of the symmetic shifting of a simplicial complex; see [3] for this formulation, or see [28] for Kalai's original formulation of a shifted complex.

2.2 Separations and polarizations

In this section we recall the basic notions of separation of a monomial ideals and separated models, as introduced by Fløystad, Greve, and Herzog in [16]. We also define a polarization of a monomial ideal as a separation which is a squarefree monomial ideal.

Notation 2.2.1. If R is a set, let $k[x_R]$ be the polynomial ring in the variables x_r where $r \in R$. If $S \to R$ is a map of sets, it induces a k-algebra homomorphism $k[x_S] \to k[x_R]$ by mapping x_s to x_r if $s \mapsto r$.

Definition 2.2.2 (Separation, Separated Model)**.** Let $R' \xrightarrow{p} R$ be a surjection of finite sets such that $|R'| = |R| + 1$. Let r_1 and r_2 be the two distinct elements of R' which map to a single element r in R. Let I be a monomial ideal in the polynomial ring $k[x_R]$ and J a monomial ideal in $k[x_{R'}]$. Then J is a *simple separation* of I if the following holds:

i. The monomial ideal I is the image of J by the map $k[x_{R'}] \to k[x_R]$.

ii. Both the variables x_{r_1} and x_{r_2} occur in some minimal generators of J (usually in distinct generators).

iii. The variable difference $x_{r_1} - x_{r_2}$ is a non-zero divisor in the quotient ring $k[x_{R'}]/J$.

More generally, if $R' \xrightarrow{p} R$ is a surjection of finite sets and $I \subseteq k[x_R]$ and $J \subseteq k[x_{R'}]$ are monomial ideals such that J is obtained by a succession of simple separations of I, then J is a *separation* of I. J a *separated model* (of I) if there are no possible nontrivial separations of J.

Observation 2.2.3. The minimal generators of the separation J and the ideal I are in one-to-one correspondence. More generally, the graded Betti numbers of J and I coincide, since we get from $k[x_{R'}]/J$ to $k[x_R]/I$ by dividing out by a regular sequence of linear forms.

In [2], Altmann, Bigdeli, Herzog, and Lu show that simple separations may be considered as deformations of the ideal I.

Any monomial ideal may be separated to its standard polarization. So clearly any separated model is a squarefree monomial ideal. The standard polarization may, however, be further separable, so it may not be a separated model.

Example 2.2.4. Consider the ideal $I = (x^2y^2, x^2z^2, y^2z^2)$ in the polynomial ring $k[x, y, z]$. The "standard polarization" of I from Construction 2.1.1 is

$$\tilde{I} = (x_1x_2y_1y_2, x_1x_2z_1z_2, y_1y_2z_1z_2).$$

This may be further separated to

$$J = (x_1x_2y_1y_2, x_1'x_2'z_1z_2, y_1y_2z_1z_2).$$

With the notion of separations in hand, we may now introduce a more general definition of polarizations, which was likely first noted by Yanagawa in [52].

Definition 2.2.5 (Polarization)**.** Let $I \subseteq k[x_R]$ be a monomial ideal and $R' \to R$ be a surjection of finite sets. An ideal $J \subseteq k[x_{R'}]$ is a *polarization* of I if J is squarefree and a separation of I.

This general notion of polarization is likely first defined in [52]. By the example above it is not true that any polarization is a separated model. However, as we will see in the next section, it turns out these notions are equivalent for Artinian monomial ideals.

We conclude this section with a general lemma which will be useful in the proof of the main theorem of this chapter.

Lemma 2.2.6. *Let I be a monomial ideal in $k[x_0, x_1, \ldots, x_n]$ such that each generator of I is squarefree in the x_0-variable. Then if $(x_0 - x_1) \cdot f$ is in I, then for every monomial m in f we have that $x_0 m$ and $x_1 m$ are in I.*

Proof. Let $f = x_0^a f_a + x_0^{a-1} f_{a-1} + \cdots + f_0$ where each f_p have no x_0-terms. Then if $(x_0 - x_1)f$ is in I, the only terms with x_0^{a+1} are the terms in $x_0^{a+1} f_a$, and so these are in I since we are in a $\mathbb{Z}^n$-graded setting. But since I is squarefree in x_0, we have $x_0 f_a$ in I and so $x_0^a f_a$ in I. In this way we may "pull out" one variable at a time to find that all terms $x_0^p f_p$ are in I for $p \geq 1$.

Then in $(x_0 - x_1) f_0$, the terms with x_0 are those in $x_0 f_0$. Hence $x_0 f_0$ is in I and so $x_0 f$ is in I. Again since I is multigraded, each monomial term $x_0 m$ is in I. We also get $x_1 f \in I$ and then each $x_1 m \in I$. □

2.3 Polarizations of Artinian monomial ideals

In this section, we restrict our study of polarizations to the case of Artinian monomial ideals in a polynomial ring $S = k[x_1, \ldots, x_n]$. In this case, for every index i, some power $x_i^{d_i}$ is a minimal generator of I.

Notation 2.3.1. Let I be an Artinian monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. For each $i \in [n]$, let d_i be the power of x_i giving a minimal generator of I. Let $\check{X}_i = \{x_{i1}, x_{i2}, \ldots, x_{id'_i}\}$ for each $i \in [n]$ be a set of new d'_i variables (where $d'_i \geq d_i$), and construct a new polynomial ring $\widetilde{S} = k[\check{X}_1, \ldots, \check{X}_n]$ with the union of all of these variables. Denote by π the homomorphism

$$\begin{aligned} \pi : \tilde{S} &\to S \\ x_{ij} &\mapsto x_i. \end{aligned}$$

Let $\widetilde{I} \subset \widetilde{S}$ be a polarization of I. Then each monomial generator $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ of I corresponds to the squarefree monomial

$$m(\mathbf{a}) = m_1(\mathbf{a}) \cdot m_2(\mathbf{a}) \cdots m_n(\mathbf{a})$$

in $\widetilde{I}$, where $m_i(\mathbf{a})$ is a squarefree monomial of degree a_i in the variables in the set $\check{X}_i$.

To get from $\widetilde{S}/\widetilde{I}$ to the quotient ring S/I, divide out by a regular sequence consisting of variable differences $x_{ip} - x_{iq}$. For each i, choose $(d_i - 1)$ linearly independent such variable differences. Any such sequence of variable differences in any order will do.

To obtain an intermediate separation of I, choose surjections $p_i : \check{X}_i \to \check{X}'_i$. This gives a map of polynomial rings

$$\widetilde{S} \to k[\check{X}'_1, \ldots, \check{X}'_n].$$

The image of the polarization $\widetilde{I}$ is an ideal I' in $k[\check{X}'_1, \ldots, \check{X}'_n]$ such that I' is a separation of I. Now, to get from $\widetilde{S}/\widetilde{I}$ to $k[\check{X}'_1, \ldots, \check{X}'_n]/I'$, divide out by a regular sequence of variable differences $x_{ia} - x_{ib}$ where for each i, x_{ia} and x_{ib} are in the same fiber $p_i^{-1}(x')$ of p_i, where there are $|p_i^{-1}(x')| - 1$ linearly independent such variable differences for each fiber.

Lemma 2.3.2. *Let $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ be minimal generators of a monomial ideal I and $m(\mathbf{a})$ and $m(\mathbf{b})$ the corresponding generators in a polarization of I. Fix an index i. If $a_i \leq b_i$ and $a_j \geq b_j$ for every $j \neq i$, then the i'th part $m_i(\mathbf{a})$ divides $m_i(\mathbf{b})$.*

Proof. Proceed by induction on $d = b_i - a_i$. If $d = 0$ then clearly $\mathbf{b} = \mathbf{a}$ and there is nothing to prove. We may also assume that $a_i \geq 1$, since otherwise there is nothing to prove. Suppose, seeking contradiciton, that $m_i(\mathbf{a})$ does not divide $m_i(\mathbf{b})$. Then we may factor $m_i(\mathbf{b})$ as $t_i(\mathbf{b}) \cdot n_i(\mathbf{b})$ where $t_i(\mathbf{b})$ has degree $d+1$ and has no common variable with $m_i(\mathbf{a})$. (We are of course using here that $m_i(\mathbf{b})$ is squarefree.) For simplicity, re-index variables so that $t_i(\mathbf{b}) = x_{i1}x_{i2}\cdots x_{i,d+1}$. We now in $k[\check{X}_1, \ldots, \check{X}_n]/\widetilde{I}$ divide out by all the variable differences involving $\check{X}_j$-variables where $j \neq i$, and by all variable differences $x_{ir} - x_{i,r+1}$ for $r = d+2, \ldots, d_i - 1$. Thus we are collapsing all the $\check{X}_j$-variables into the single variable x_j and the variables $x_{i,d+2}, \ldots, x_{i,n_i}$ into a single variable x_i. We get a quotient ring

$$k[x_{i1}, \ldots, x_{i,d+1}, x_1, \ldots, x_n]/I', \tag{2.1}$$

where I' is a separation of I. Note that $m(\mathbf{a})$ collapses to $x^{\mathbf{a}}$ in I'.

Consider now the variable difference $x_{i,d+1} - x_i$ in the polynomial ring above. We see that

$$\begin{aligned} &(x_{i,d+1} - x_i)x_{i1}\cdots x_{id} \cdot x_i^{a_i - 1} \prod_{j \neq i} x_j^{a_j} \\ &= x_{i1}\cdots x_{i,d+1} \cdot x_i^{a_i-1} \prod_{j\neq i} x_j^{a_j} - x_{i1}\cdots x_{i,d} \cdot x_i^{a_i} \prod_{j \neq i} x_j^{a_j} \end{aligned} \tag{2.2}$$

vanishes in the quotient ring (2.1): the first term is divisible by the image of $m(\mathbf{b})$ in I' (note that $b_i = (d+1) + (a_i - 1)$), and the second term is divisible by the image of $m(\mathbf{a})$. Since $x_{i,d+1} - x_i$ is not a zero divisor (it belongs to a regular sequence), we get from (2.2) that

$$\mathbf{n} = x_{i1} \cdots x_{id} \cdot x_i^{a_i - 1} \prod_{j \neq i} x_j^{a_j} \tag{2.3}$$

is in I'. Now if $d = 1$, this monomial has $\mathbb{Z}^n$-degree $\mathbf{a}$. But the monomial $x^{\mathbf{a}}$ is in I', with the same degree. Since these are the $\mathbb{Z}^n$-degree of a generator of I, there can only be a single monomial in I' with this $\mathbb{Z}^n$-degree. We get a contradiction. Now suppose $d \geq 2$. Then $\mathbf{n}$ is divisible by a generator $m'(\mathbf{c})$ in I' which can *not* be $x^{\mathbf{a}}$. We will have each $c_j \leq a_j$ for $j \neq i$, and so $c_i > a_i$. Furthermore, we have $b_i > c_i$ since $\mathbf{n}$ in (2.3) has i-degree $d + a_i - 1 = b_i - 1$. By induction on d, considering the polarized ideal J, the i'th part $m_i(\mathbf{a})$ here divides the i'th part $m_i(\mathbf{c})$. But then going to I' then $x_i^{a_i}$ divides the image of $m_i'(\mathbf{c})$, and so $x_i^{a_i}$ would divide $\mathbf{n}$ of (2.3), a contradiction. $\square$

Remark 2.3.3. If $m(\mathbf{a})$ is a minimal generator of $\widetilde{I}$, then Lemma 2.3.2 implies that $m_i(\mathbf{a})$ will divide $m_i(0, \ldots, n_i, \ldots, 0)$ which of course is just $m(0, \ldots, n_i, \ldots, 0)$. Thus, if the polarization of $x_i^{d_i}$ is $x_{i1}x_{i2} \cdots x_{id_i}$, then every x_i-variable occurring in the minimal generators of J are among these variables, and so we may take $\check{X}_i = \{x_{i1}, \ldots, x_{id_i}\}$.

As we saw in Example 2.2.4, the following result is quite particular to Artinian monomial ideals.

Corollary 2.3.4. *Every polarization of an Artinian monomial ideal I is a separated model for I.*

Proof. If the polarization J was not a separated model, then let J' be a further simple separation. Since I in $k[x_1, \ldots, x_n]$ is an Artinian monomial ideal, every variable x_i of course occurs in a minimal generator of I, in fact $x_i^{d_i}$ is a minimal generator. Then if J' is in $k[\check{X}_1', \ldots, \check{X}_n']$ then every variable in this polynomial ring must also occur in a generator of J', by the definition of a separation. By the above Lemma 2.3.2 and Remark 2.3.3, if $x_i^{d_i}$ polarizes to $x_{i1} \cdots x_{id_i}$ then $\check{X}_i' = \{x_{i1}, \ldots, x_{id_i}\}$. But J is obtained from J' by dividing out by a variable difference $x_{ia} - x_{ib}$. Then the image of $x_{i1} \cdots x_{id_i}$ in J would not be squarefree, a contradiction. □

2.4 Isotone maps and linear syzygy edges

The material in this section is a summary of the combinatorial characterization of polarizations of powers of the graded maximal ideal $\mathfrak{m}$ in a polynomial ring given by the author, Fløystad, and Lohne in [1]. The idea is to put a set of partial orders $\geq_i$ on the lattice points of a dilated simplex (which are in bijection with the generators of a power of $\mathfrak{m}$) and view a "potential polarization" as coming from a set of isotone maps from the lattice points of the dilated simplex to the Boolean lattice. One may visualize these potential polarizations as a graph of linear syzygies among the generators of the "potentially polarized" ideal. Theorem 2.5.1 gives a complete characterization of which of these isotone maps give "honest" polarizations in terms of a combinatorial condition on the corresponding graph of linear syzygies.

Notation 2.4.1. Fix integers n and d, and let $S = k[x_1, \ldots, x_n]$ be a polynomial over a field k. Let $\check{X}_i = \{x_{i1}, \ldots, x_{id}\}$ be a set of variables, and let $\tilde{S} = k[\check{X}_1, \ldots, \check{X}_n]$ be a polynomial ring in the union of all these variables. Denote by $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S.

Denote by $\Delta^{\mathbb{Z}}(n, d) = \Delta(n, d) \cap \mathbb{Z}^n$ the set of lattice points of the dilated simplex $d \cdot \Delta^{n-1}$, i.e., the set of tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers with $\sum_i^n a_i = d$. Consider the polytopal CW-complex with the underlying space $d \cdot \Delta_{n-1}$, with

CW-complex structure induced by intersection with the cubical CW-complex structure on $\mathbb{R}^n$ given by the integer lattice $\mathbb{Z}^n$. Denote by $\mathcal{T}(n,d)$ the one-skeleton of this cell complex.

Observation 2.4.2. The elements of $\Delta^{\mathbb{Z}}(n,d)$ are exactly the exponent vectors of the minimal generating set of the ideal $\mathfrak{m}^d$.

Notation 2.4.3. Let $e_i \in \mathbb{N}^n$ be the ith unit vector in $\mathbb{N}^n$. For a given $\mathbf{a}$, denote by $\mathrm{Supp}(\mathbf{a})$ the *support* of $\mathbf{a}$, that is, the set of all i such that $a_i > 0$. If B is a subset of $[n]$, denote by $\mathbb{1}_B$ the n-tuple $\sum_{i\in B} e_i$. For example, if $B = [n]$, then $\mathbb{1}_B = (1,\dots,1)$.

In the following definitions, we introduce some key subgraphs of $\mathcal{T}(n,d)$ which will be critical for characterizing polarizations of $\mathfrak{m}^d$ combinatorially.

Definition 2.4.4 (Complete down-graph)**.** Given $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$ and $i,j \in \mathrm{Supp}(\mathbf{c})$, there is an edge between $\mathbf{c}-e_i$ and $\mathbf{c}-e_j$ in $\mathcal{T}(n,d)$ denoted $(\mathbf{c};i,j)$. Every edge in $\mathcal{T}(n,d)$ can be realized as an edge $(\mathbf{c};i,j)$ for unique $\mathbf{c}, i$, and j. An n-tuple $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$ induces a subgraph of $\mathcal{T}(n,d)$ called the *complete down-graph* $D(\mathbf{c})$ on the points $\mathbf{c}-e_i$ for $i \in \mathrm{Supp}(\mathbf{c})$. If $R \subseteq [n]$, denote by $D_R(\mathbf{c})$ the complete graph with edges $(\mathbf{c};r,s)$ for $r,s \in R$.

Definition 2.4.5 (Complete up-graph)**.** Any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d-1)$ also determines a subgraph of $\mathcal{T}(n,d)$: the *complete up-graph* $U(\mathbf{a})$ consisting of points $\mathbf{a}+e_i$ for $i = 1,\dots,n$ with edges $(\mathbf{a}+e_i+e_j;i,j)$ for $i \neq j$.

Remark 2.4.6. The complete down-graph $D(\mathbf{c})$ induces a simplex of full dimension $d-1$ if and only if $c_i \geq 1$ for all i, i.e., $\mathbf{c}$ has full support. For each $\mathbf{a}$ in $\Delta^{\mathbb{Z}}(n,d-1)$, the induced simplex of the up-graph $U(\mathbf{a})$ always has full dimension $d-1$.

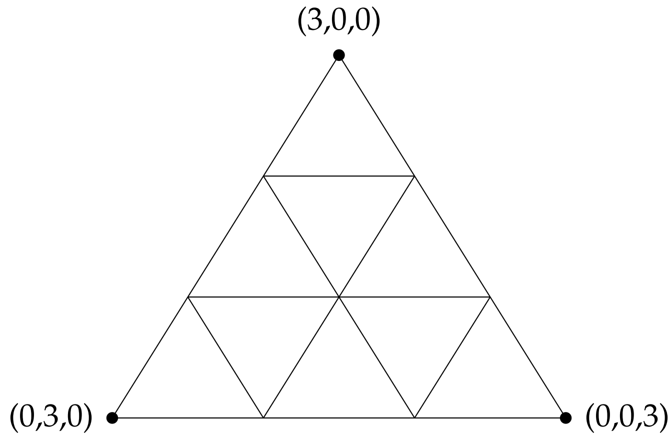


Figure 2.1: The graph $\mathcal{T}(3,3)$.

Example 2.4.7. The graph $\mathcal{T}(3,3)$ pictured in Figure 2.1 has three "complete down-triangles" with full support corresponding to the vectors $(2,1,1), (1,2,1)$, and $(1,1,2)$ in $\Delta^{\mathbb{Z}}(n, d+1)$. It also has six "complete up-triangles".

We now introduce a set of partial orders $\geq_i$ for each $i \in [n]$.

Definition 2.4.8 (The Partial Order $\geq_i$)**.** Adopt notation and hypotheses of Notation 2.4.1. Fix an index $1 \leq i \leq n$. Define $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$ to be the poset with ground set $\Delta^{\mathbb{Z}}(n,d)$ and partial order $\geq_i$ such that $\mathbf{b} \geq_i \mathbf{a}$ if $b_i \geq a_i$ and $b_j \leq a_j$ for $j \neq i$.

Observation 2.4.9. The partial order $\geq_i$ as in Definition 2.4.8 is graded, where $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d)$ has rank a_i.

The maps in the following construction will be play an important role in the combinatorial characterization of $\mathfrak{m}^d$.

Construction 2.4.10 (Isotone Maps)**.** Adopt notation and hypotheses of Notation 2.4.1. Let $\mathcal{B}_d$ be the Boolean poset on $[d]$ and $\{X_i\}_{1 \leq i \leq n}$ be a set of rank-preserving isotone maps

$$X_i : (\Delta^{\mathbb{Z}}(n,d), \leq_i) \to \mathcal{B}_d.$$

For any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d)$, let $m_i(\mathbf{a}) = \prod_{j \in X_i(\mathbf{a})} x_{ij}$ and $m(\mathbf{a}) = \prod_{i=1}^n m_i(\mathbf{a})$. Let J be the ideal in $k[\check{X}_1, \ldots, \check{X}_n]$ generated by the $m(\mathbf{a})$.

Definition 2.4.11 (Linear Syzygy Edge)**.** Let $(\mathbf{c}; i, j)$ be an edge of $\mathcal{T}(n,d)$, where $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$. Then $(\mathbf{c}; i, j)$ is a *linear syzygy edge* (or *LS-edge*) if there is a monomial $\mathbf{m}$ of degree $d-1$ such that

$$m(\mathbf{c} - e_i) = x_{jr} \cdot \mathbf{m} \quad \text{and} \quad m(\mathbf{c} - e_j) = x_{is} \cdot \mathbf{m},$$

for suitable variables $x_{jr} \in \check{X}_j$ and $x_{is} \in \check{X}_i$. This edge gives a linear syzygy between the monomials $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$. Equivalently, in terms of the isotone maps,

$$X_p(\mathbf{c} - e_i) = X_p(\mathbf{c} - e_j)$$

for every $p \neq i, j$. Observe that both $m_i(\mathbf{c} - e_i)$ and $m_j(\mathbf{c} - e_j)$ are common factors of $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$.

Sometimes, one may wish to consider whether two elements of $\Delta^{\mathbb{Z}}(n,d)$ would share a linear syzygy edge with respect to a *subset* of $[n]$.

Definition 2.4.12 (R-Linear Syzygy Edge)**.** Let $R \subseteq [n]$ and $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ with R contained in the support of $\mathbf{c}$. Let $r, s \in R$. Define $(\mathbf{c}; r, s)$ to be an *R-linear syzygy edge* if

$$X_p(\mathbf{c} - e_r) = X_p(\mathbf{c} - e_s) \text{ for } p \in R \setminus \{r, s\}.$$

By the isotonicity of the X_p, for $p = r, s$,

$$X_r(\mathbf{c} - e_r) \subseteq X_r(\mathbf{c} - e_s), \quad X_s(\mathbf{c} - e_s) \subseteq X_s(\mathbf{c} - e_r).$$

Let $D_R(\mathbf{c})$ be the complete graph with edges $(\mathbf{c}; r, s)$ for $r, s \in R$.

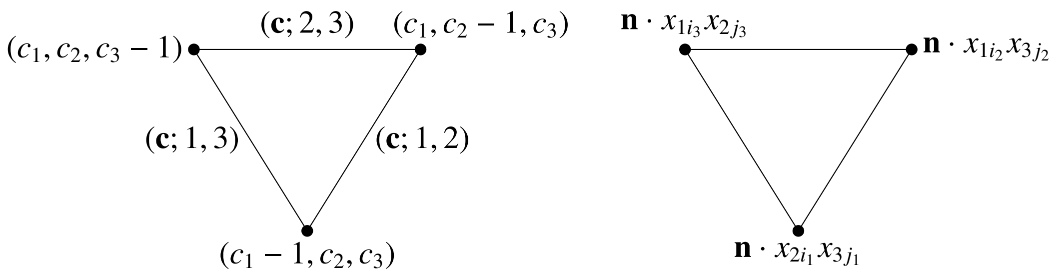


Figure 2.2: A down-triangle and its labeled monomials

The following lemma tells us that the monomials assigned to vertices of a down-triangle by a set of isotone maps must have a common factor which is easy to describe.

Lemma 2.4.13. *Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d)$ have support $C \subseteq \{1, 2, \ldots, n\}$. The monomials assigned to the vertices in the down-graph $D(\mathbf{c})$ by the maps X_i have a common factor of degree $\mathbf{c} - \mathbb{1}_C$. This common factor is $\prod_{i \in C} m_i(\mathbf{c} - e_i)$.*

Proof. Fix an element $j \in C$. For the order $\geq_k$ we have $\mathbf{c} - e_j \geq_k \mathbf{c} - e_k$ for every $k \in C$. Hence $X_k(\mathbf{c} - e_k)$ is contained in $X_k(\mathbf{c} - e_j)$ for every $k \in C$. Thus $m(\mathbf{c} - e_j)$ has $m_k(\mathbf{c} - e_k)$ as a factor for each $k \in C$. □

Example 2.4.14. Let $m = 3$ and $\mathbf{c} = (c_1, c_2, c_3)$ be in $\Delta_3^+(n + 1)$. On the left in Figure 2.2 is the down triangle $D(\mathbf{c})$. Let

$$\mathbf{n} = m_1(\mathbf{c} - e_1) \cdot m_2(\mathbf{c} - e_2) \cdot m_3(\mathbf{c} - e_3).$$

Then the monomials associated to the vertices of this down-triangle are shown to the right in Figure 2.2.

The following lemma turns out to be a useful tool for induction, and for applications in later sections.

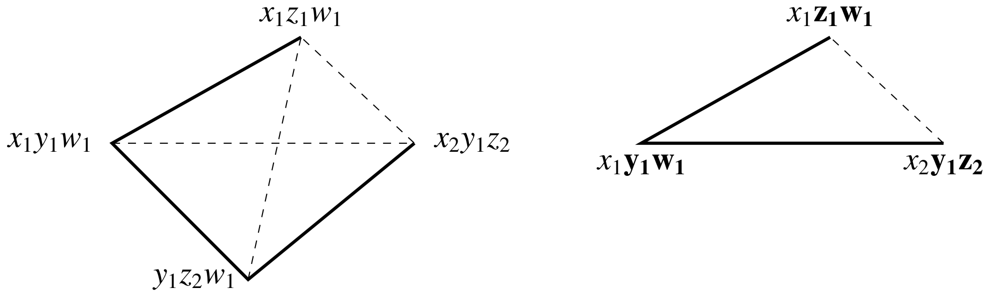


Figure 2.3: R-linear syzygy edges where $R = \{2, 3, 4\}$.

Lemma 2.4.15. *Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$. If the set of linear syzygy edges in* $\mathrm{LS}(\mathbf{c})$ *contains a spanning tree for $D(\mathbf{c})$, then for each $R \subseteq \mathrm{supp}(\mathbf{c})$, the set of R-linear syzygy edges contains a spanning tree for $D_R(\mathbf{c})$.*

Example 2.4.16. Consider the case of four variables and $\mathbf{c} = (1, 1, 1, 1)$. Write x, y, z, w for x_1, x_2, x_3, x_4, respectively. On the left of Figure 2.3 is the down-graph $D(\mathbf{c})$ with the three thick edges the linear syzygy edges.

Let $R = \{2, 3, 4\}$. On the right is the down-graph $D_R(\mathbf{c})$ where the two thick edges are the R-linear syzygy edges and the relevant variables marked in bold.

Proof of Lemma 2.4.15. Let Q be the complement of R in $\mathrm{Supp}\,(\mathbf{c})$. Let r and s be two elements in R. There is a path from $\mathbf{c} - e_r$ to $\mathbf{c} - e_s$ in $D(\mathbf{c})$ consisting of linear syzygy edges. It may be broken up into smaller paths: From $\mathbf{c} - e_r = \mathbf{c} - e_{r_0}$ to $\mathbf{c} - e_{r_1}$, from $\mathbf{c} - e_{r_1}$ to $\mathbf{c} - e_{r_2}$, ..., from $\mathbf{c} - e_{r_{p-1}}$ to $\mathbf{c} - e_{r_p} = \mathbf{c} - e_s$ where on the path from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ the only vertices $\mathbf{c} - e_q$ with $q \in R$ are the end vertices $q = r_{i-1}$ and $q = r_i$ while the in-between vertices $\mathbf{c} - e_q$ all have $q \in Q$. We claim that each edge from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ is an R-linear syzygy edge. This will prove the lemma.

Let the path from $\mathbf{c} - r_{i-1}$ to $\mathbf{c} - r_i$ be

$$\mathbf{c} - e_{r_{i-1}} = \mathbf{c} - e_{q_0}, \mathbf{c} - e_{q_1}, \ldots, \mathbf{c} - e_{q_t} = \mathbf{c} - e_{r_i}$$

where $q_1, \ldots, q_{t-1}$ are all in Q. We must show that

$$X_p(\mathbf{c} - e_{r_{i-1}}) = X_p(\mathbf{c} - e_{r_i}) \text{ for } p \in R \setminus \{r_{i-1}, r_i\}. \tag{2.4}$$

But since the edges on the path are linear syzygy edges we have

$$X_p(\mathbf{c} - e_{q_{j-1}}) = X_p(\mathbf{c} - e_{q_j}) \text{ for } p \in \operatorname{supp}(\mathbf{c}) \setminus \{q_{j-1}, q_j\}.$$

Since $q_1, \ldots, q_{t-1}$ are not in R we get (2.4) □

2.5 Proof of the main theorem

We conclude this chapter by presenting and proving a complete combinatorial characterization of all polarizations of $\mathfrak{m}^d$ in terms of their graphs of linear syzygies.

Theorem 2.5.1. *Adopt notation 2.4.1. A set of isotone maps $X_1, \ldots, X_n$ as in Construction 5.0.11 determines a polarization of the ideal $(x_1, \ldots, x_n)^d$ if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, the linear syzygy edges $\mathrm{LS}(\mathbf{c})$ contain a spanning tree for the down-graph $D(\mathbf{c})$.*

Example 2.5.2. Figure 2.4 depicts the graph of linear syzygies for a polarization of $(x, y, z)^3$. Notice that at most one edge is removed from each down-triangle, so it satisfies the spanning tree condition of Theorem 2.5.1.

Before stating the technical propositions and lemmas required to prove Theorem 2.5.1, we establish some notation we will be used without comment for the remainder of this section.

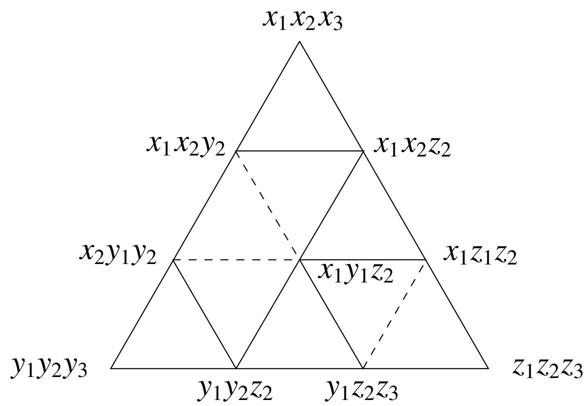


Figure 2.4: An example of a polarization of $(x, y, z)^3$.

Notation 2.5.3. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be two n-tuples in $\Delta^{\mathbf{z}}(n, d)$. Let $[n] = A \cup B$ be the disjoint set partition such that $a_i \geq b_i$ for $i \in A$ and $a_i < b_i$ for $i \in B$. Define

$$d(\mathbf{a}, \mathbf{b}) := \sum_{i \in B} (b_i - a_i) = \sum_{i \in A} (a_i - b_i) \tag{2.5}$$

to be a measure of the distance between $\mathbf{a}$ and $\mathbf{b}$. Note that the distance may be measured using only the index set B which in turn depends on the ordered set $(\mathbf{a}, \mathbf{b})$. It should thus really be written $B(\mathbf{a}, \mathbf{b})$. When we measure the distance between two vertices, the first will normally be denoted by a variation on $\mathbf{a}$ and the second a variation on $\mathbf{b}$.

We have the partial order $\mathbf{a} \leq \mathbf{b}$ if each $a_i \leq b_i$. The least upper bound for $\mathbf{a}$ and $\mathbf{b}$ in this partial order is

$$\mathbf{a} \vee \mathbf{b} = (\max\{a_1, b_1\}, \dots, \max\{a_n, b_n\}).$$

The main ingredient of the proof of Theorem 2.5.1 is the following technical Proposition.

Proposition 2.5.4. *Let* $\mathbf{a}, \mathbf{b} \in \Delta^{\mathbb{Z}}(n, d)$. *Suppose that for every* $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d + 1)$*, the linear syzygy edges* $\mathrm{LS}(\mathbf{c})$ *contains a spanning tree for the down-graph* $D(\mathbf{c})$*. Then there* *is a path*

$$\mathbf{a} = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_N = \mathbf{b}$$

in $\Delta^{\mathbb{Z}}(n, d)$ *such that:*

1. *Every* $\mathbf{b}_i \leq \mathbf{a} \vee \mathbf{b}$*,*
2. *Every* $m(\mathbf{b}_i)$ *divides the least common multiple lcm*$(m(\mathbf{a}), m(\mathbf{b}))$*,*
3. *The edge from* $\mathbf{b}_{i-1}$ *to* $\mathbf{b}_i$ *is a linear syzygy edge for each* i*.*

We call such a path an LS-path from $\mathbf{a}$ *to* $\mathbf{b}$*.*

We begin by proving Proposition 2.5.4 when the distance between $\mathbf{a}$ and $\mathbf{b}$ is one.

Lemma 2.5.5. *When the distance between* $\mathbf{a}$ *and* $\mathbf{b}$ *is one, there is an LS-path from* $\mathbf{a}$ *to* $\mathbf{b}$.

Proof. In this case there is a unique $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ such that $\mathbf{a} = \mathbf{c} - e_i$ and $\mathbf{b} = \mathbf{c} - e_j$, and then $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$. Let T in the linear syzygy edges $\mathrm{LS}(\mathbf{c})$ be a spanning tree for $D(\mathbf{c})$. Then there is a unique path from $\mathbf{a}$ to $\mathbf{b}$ in T. We show that for any $m(\mathbf{u})$ on this path, $m(\mathbf{u})$ divides $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. It is enough to show for any $k \in \mathrm{Supp}(\mathbf{c})$, any x_k-variable in $m(\mathbf{u})$ is contained in either the x_k-variables of $m(\mathbf{a})$ or the x_k-variables of $m(\mathbf{b})$. There are two cases to consider.

Case 1. If the path from $\mathbf{a}$ to $\mathbf{b}$ does not contain $\mathbf{c} - e_k$, then since all edges on the path are linear syzygy edges, $X_k(\mathbf{b}_{i-1}) = X_k(\mathbf{b}_i)$ for every i.

Case 2. It the path from $\mathbf{a}$ to $\mathbf{b}$ contains $\mathbf{c} - e_k$, say this is $\mathbf{b}_t$, then:

- $X_k(\mathbf{b}_{i-1}) = X_k(\mathbf{b}_i)$ for $i < t$ and these are all equal to $X_k(\mathbf{a})$,

- $X_k(\mathbf{b}_i) = X_k(\mathbf{b}_{i+1})$ for $i > t$ and these are all equal to $X_k(\mathbf{b})$,
- $X_k(\mathbf{b}_t) = X_k(\mathbf{c} - e_k)$ is contained in both $X_k(\mathbf{a})$ and $X_k(\mathbf{b})$ by the isotonicity of X_k.

$\square$

Proof of Proposition 2.5.4. This proof has been divided up into three parts. In Part A, we define the setting, establish notation, and split our desired path into three pieces. In Part B, we prove several facts which give more information about one of these pieces of the path. Finally, in Part C, we connect these three paths and show this new path has all of the desired properties.

Part A. In this part we define the setting. Take distinct $\mathbf{a}$ and $\mathbf{b}$ in $\Delta^{\mathbb{Z}}(n, d)$. Assume the distance $d(\mathbf{a}, \mathbf{b}) \geq 2$ since the case of distance 1 is done above. Let

$$B = B(\mathbf{a}, \mathbf{b}) = \{i \,|\, b_i > a_i\}, \quad A_{>} = \{i \,|\, a_i > b_i\}, \quad A_{=} = \{i \,|\, a_i = b_i\},$$

and $A = A_{>} \cup A_{=}$. We want to consider $\mathbf{b}'$ which are in some sense "close" to $\mathbf{b}$. Let $P(\mathbf{b})$ consist of all $\mathbf{b}' \in \Delta^{\mathbb{Z}}(n, d)$ such that

(1.)
  - For i in B, $b_i = b'_i$
  - For i in A, $b'_i \leq a_i$.

(2.) There is some LS-path from $\mathbf{b}'$ to $\mathbf{b}$ where the vertices $\mathbf{u}$ on the path satisfy
  - $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ (which, since we are assuming an LS-path, follows from 1 above),
  - $m(\mathbf{u})$ divides $\operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}))$.

Now let the subset A_1 of A consist of all coordinate indices i in A such that there is some $\mathbf{b}'$ in $P(\mathbf{b})$ with strict inequality $b'_i < a_i$. Let A_0 be the complement $A \setminus A_1$. It is the intersection of all the $A_=$ associated to $\mathbf{b}'$ in $P(\mathbf{b})$. In particular note that:

(i) $A_1 \supseteq A_>$ (since $\mathbf{b} \in P(\mathbf{b})$), so A_1 is not empty and $A_0 \subseteq A_=$.

(ii) $b'_i = a_i = b_i$ for $i \in A_0$,

(iii) $d(\mathbf{a}, \mathbf{b}') = d(\mathbf{a}, \mathbf{b})$ for $\mathbf{b}' \in P(\mathbf{b})$

where (iii) follows because the B-sets $B(\mathbf{a}, \mathbf{b}') = B(\mathbf{a}, \mathbf{b})$ and the distance may be measured by this; see equation (2.5) in Notation 2.5.3.

Choose $\beta \in B$ and let $R = A_1 \cup \{\beta\}$. Consider the down-graph $D_R(\mathbf{a} + e_\beta)$. With β fixed, there is by Lemma 2.4.15 an R-linear syzygy edge $(\mathbf{a} + e_\beta; \beta, \alpha)$ for some α in A_1. This is an edge from $\mathbf{a}$ to $\mathbf{a} + e_\beta - e_\alpha$. Since $\alpha \in A_1$ there is a $\mathbf{b}' \in P(\mathbf{b})$ with $b'_\alpha < a_\alpha$. Then the B-sets (see Notation 2.5.3) $B(\mathbf{a} + e_\beta - e_\alpha, \mathbf{b}') \subseteq B(\mathbf{a}, \mathbf{b}')$, with equality unless $a_\beta + 1 = b_\beta$ in which case the the former set comes from removing β from the latter. In any case one has

$$d(\mathbf{a} + e_\beta - e_\alpha, \mathbf{b}') = d(\mathbf{a}, \mathbf{b}') - 1 \; (= d(\mathbf{a}, \mathbf{b}) - 1).$$

By induction on distance there is an LS-path

$$\mathbf{a} + e_\beta - e_\alpha = \mathbf{b}^0, \mathbf{b}^1, \cdots, \mathbf{b}^N = \mathbf{b}'. \tag{2.6}$$

So we have that

- each $m(\mathbf{b}^j)$ divides $\operatorname{lcm}(m(\mathbf{a} + e_\beta - e_\alpha), m(\mathbf{b}'))$, and
- each $\mathbf{b}^j \le (\mathbf{a} + e_\beta - e_\alpha) \vee \mathbf{b}' \le \mathbf{a} \vee \mathbf{b}$.

There may be elements on this LS-path such that $m(\mathbf{b}^j)$ does not divide $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. But $m(\mathbf{b}^j)$ will divide if we get sufficiently close to $\mathbf{b}'$ as we show in Fact 2.5.7 below. If $\mathbf{a} + e_\beta - e_\alpha$ and $\mathbf{b}'$ have equal i'th coordinates for every $i \in B$, the distance $d(\mathbf{a}, \mathbf{b})$ would be 1, but we are assuming the distance is ≥ 2. So $\mathbf{a} + e_\beta - e_\alpha$ and $\mathbf{b}'$ do not have equal i'th coordinate for every $i \in B$. Let $\mathbf{b}^p$ be the last element on the path (2.6) for which $b_k^p \neq b_k'$ for some $k \in A_0 \cup B$.

Part B. In this part, in Facts 2.5.6, 2.5.7, and 2.5.8, we investigate in detail the path from $\mathbf{b}^p$ to $\mathbf{b}' = \mathbf{b}^N$.

Fact 2.5.6. *There is a unique $k \in A_0 \cup B$ such that $b_k^p \neq b_k'$. For every $i \in A_0 \cup B$ and $j = p, \ldots, N$ we have $b_i^j = b_i' = b_i$, save for $j = p$ and $i = k$ when $b_k^p = b_k^{p+1} - 1$ which is $b_k' - 1 = b_k - 1$.*

Proof. Clearly, by the definition of $\mathbf{b}^p$, we have that the b_i^j are all equal to b_i' for $i \in A_0 \cup B$ and $j = p+1, \ldots, N$. If $i \in B$ then $b_i' = b_i$ since $\mathbf{b}' \in P(\mathbf{b})$. If $i \in A_0$ then $b_i' = b_i$ by the equation (ii) in Part A of this proof. Furthermore we must have $b_k^p = b_k^{p+1} \pm 1$ which is $b_k' \pm 1$. Note that

$$b_i^p \leq \max\{a_i, b_i\} = b_i, \quad i \in A_0 \cup B. \tag{2.7}$$

Since the edge from $\mathbf{b}^p$ to $\mathbf{b}^{p+1}$ is an LS-edge, there are exactly two coordinates k, ℓ where $\mathbf{b}^p$ and $\mathbf{b}^{p+1}$ are distinct. Since $b_k^{p+1} = b_k$ we must by (2.7) have $b_k^p = b_k^{p+1} - 1$. Then we will have $b_\ell^p = b_\ell^{p+1} + 1$. If $\ell \in A_0 \cup B$ then $b_\ell^{p+1} = b_\ell' = b_\ell$ which together with the inequality (2.7) gives a contradiction. Thus we have a unique k in $A_0 \cup B$. Whence when $i \in A_0 \cup B$ and $i \neq k$ we have $b_i^p = b_i^{p+1} = b_i' = b_i$. □

Fact 2.5.7. *For all $\mathbf{b}^j$ with $j = p, \ldots, N$ we have*

i) $\mathbf{b}^j \leq \mathbf{a} \vee \mathbf{b}$, *ii) $m(\mathbf{b}^j)$ divides $lcm(m(\mathbf{a}), m(\mathbf{b}))$.*

Proof. Part i) is already noted when we defined the path from $\mathbf{a} + e_\beta - e_\alpha$ to $\mathbf{b}'$. Now we do part ii). We know that $m_t(\mathbf{b}^j)$ divides $\mathrm{lcm}(m_t(\mathbf{a} + e_\beta - e_\alpha), m_t(\mathbf{b}'))$ for every t.

(a.) Let $t \in A_1$. There is an $R = A_1 \cup \{\beta\}$-linear syzygy between $m(\mathbf{a})$ and $m(\mathbf{a} + e_\beta - e_\alpha)$ and so $m_t(\mathbf{a}) = m_t(\mathbf{a} + e_\beta - e_\alpha)$ for $t \in A_1 \setminus \{\alpha\}$. For $t = \alpha$ then $m_\alpha(\mathbf{a} + e_\beta - e_\alpha)$ divides $m_\alpha(\mathbf{a})$ by X_α being isotone. From this and the defining requirements on $\mathbf{b}'$, it follows that $m_t(\mathbf{b}^j)$ divides $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ for $t \in A_1$.

(b.) Let now $t \in A_0 \cup B$. The edges on the path between $\mathbf{b}^p$ and $\mathbf{b}'$ are LS-edges. It follows then from Fact 2.5.6 that for each $t \in (A_0 \cup B) \setminus \{k\}$ and $j = p, \ldots, N$ that $m_t(\mathbf{b}^j) = m_t(\mathbf{b}')$. When $t = k$, since $b^p_k = b^{p+1}_k - 1$ the part $m_k(\mathbf{b}^p)$ divides $m_k(\mathbf{b}^{p+1})$ and we will further have all $m_k(\mathbf{b}^j)$ equal for $j = p + 1, \ldots, N$, since these $\mathbf{b}^j$ are related by LS-edges, and have the same k'th coordinate. The upshot is that also $m_k(\mathbf{b}^j)$ divides $m_k(\mathbf{b}')$. Thus $m_t(\mathbf{b}^j)$ divides $m_t(\mathbf{b}')$ for every $t \in A_0 \cup B$. Since $\mathbf{b}' \in P(\mathbf{b})$, the $m_t(\mathbf{b}')$ divide $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ and we are done. □

The following is the main technical detail that makes the proof work. It ensures that we can use induction on distance in Part C. To achieve this we need A_0 as small as possible, and therefore introduce the neighbourhood $P(\mathbf{b})$ of $\mathbf{b}$. (But this had to be balanced against A_1, the complement of A_0 in A, not being too big in order to construct the LS-path in (2.6) by induction.)

Fact 2.5.8. *The coordinate $k \in B$.*

Proof. By Fact 2.5.7 $m(\mathbf{b}^j)$ divides $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ and $\mathbf{b}^j \leq \mathbf{a} \vee \mathbf{b}$ for $j = p, \ldots, N$. If $k \in A_0$ then by Fact 2.5.6 $b^p_i = b'_i = b_i$ for each $i \in B$. So $\mathbf{b}^p$ fulfills the requirement to be in $P(\mathbf{b})$. But then by the equation (ii) in Part A, one has that $b^p_k = a_k = b_k$ contradicting Fact 2.5.6. $\square$

Part C. In this last part we splice three paths together to make an LS-path from $\mathbf{a}$ to $\mathbf{b}$. Consider the distance between $\mathbf{a}$ and $\mathbf{b}^p$. For $i \in A$ we have $b^p_j \leq a_j$ since $\mathbf{b}^p \leq \mathbf{a} \vee \mathbf{b}$. For $i \in B \setminus \{k\}$ we have $b^p_i = b'_i = b_i > a_i$ and when $i = k$ then $a_k < b_k$ and $b^p_k = b_k - 1$. Thus, by looking at the terms with coordinates in B, the distance $d(\mathbf{a}, \mathbf{b}^p) = d(\mathbf{a}, \mathbf{b}) - 1$. By induction there is an LS-path from $\mathbf{a}$ to $\mathbf{b}^p$. We now have three LS-paths which we splice:

- The LS-path from $\mathbf{a}$ to $\mathbf{b}^p$. All $m(\mathbf{u})$ on this path have i) $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}^p \leq \mathbf{a} \vee \mathbf{b}$ and ii) $m(\mathbf{u})$ divides $\operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}^p))$ which again divides $\operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ by Fact 2.5.7.
- The LS-path from $\mathbf{b}^p$ to $\mathbf{b}'$. We refer to Fact 2.5.7 concerning the terms here.
- The LS-path from $\mathbf{b}'$ to $\mathbf{b}$ with properties required by the definition of $P(\mathbf{b})$.

Splicing these three LS-paths together we get a path of linear syzygy edges from $m(\mathbf{a})$ to $m(\mathbf{b})$ where all $m(\mathbf{u})$ on this path have i) $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ and ii) $m(\mathbf{u})$ divides $\operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. Thus we have an LS-path from $\mathbf{a}$ to $\mathbf{b}$. $\square$

We show first Part a., that if the isotone maps $\{X_i\}$ give a polarization, then for each $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ the linear syzygy edges $\mathrm{LS}(\mathbf{c})$ of the down-graph $D(\mathbf{c})$ contain a spanning tree for this down-graph.

Proof of Theorem 2.5.1, ($\Longrightarrow$). We assume that the isotone maps $\{X_i\}$ give an ideal J which is a polarization. We shall prove that every down-graph $D(\mathbf{c})$ contains a spanning tree of linear syzygy edges. For simplicity we shall assume $\operatorname{Supp}(\mathbf{c})$

has full support $[n] = \{1, 2, \ldots, n\}$. The arguments work just as well in the general case. Since by Lemma 2.4.13

$$\mathbf{m} = \prod_{i=1}^{n} m_i(\mathbf{c} - e_i)$$

of degree $\mathbf{c} - \mathbf{1}$ is a divisor of $m(\mathbf{c} - e_v)$ for any $\mathbf{c} - e_v$ in $D(\mathbf{c})$, we may write $m(\mathbf{c} - e_v) = \mathbf{m} \cdot n(\mathbf{c} - e_v)$ where $n(\mathbf{c} - e_v)$ has degree $\mathbf{1} - e_v$. For two distinct vertices $\mathbf{c} - e_v$ and $\mathbf{c} - e_w$ in $D(\mathbf{c})$ we define the distance $d(m(\mathbf{c} - e_v), m(\mathbf{c} - e_w))$ to be the number of $k \in [m]$ such that either:

- The (unique) x_k-variables of $n(\mathbf{c} - e_v)$ and of $n(\mathbf{c} - e_w)$ are distinct,
- $k = v$ (then $n(\mathbf{c} - e_v)$ has no x_v-variable),
- $k = w$ (then $n(\mathbf{c} - e_w)$ has no x_w-variable),

Note that if the distance between $m(\mathbf{c} - e_v)$ and $m(\mathbf{c} - e_w)$ is 2, then the set of k's is $\{v, w\}$ and there is a linear syzygy between these monomials. Suppose now the vertices of $D(\mathbf{c})$ can be divided into two distinct subsets V_1 and V_2 such that there is no linear syzygy edge between a vertex in V_1 and a vertex in V_2.

Let $\mathbf{c} - e_v$ in V_1 and $\mathbf{c} - e_w$ in V_2 be such that the distance d between $m(\mathbf{c} - e_v)$ and $m(\mathbf{c} - e_w)$ is minimal. We must have $d \geq 3$ and the number of vertices $m \geq 3$. For simplicity we may assume $v = 1$ and $w = 2$ and that we may write

$$n(\mathbf{c} - e_2) = x_{1i_1} x_{3i_3} \cdots x_{ni_n}, \quad n(\mathbf{c} - e_1) = x_{2j_2} x_{3j_3} \cdots x_{nj_n},$$

where $x_{pi_p} \neq x_{pj_p}$ for $p = 3, \ldots, d$ and $x_{pi_p} = x_{pj_p}$ for $p > d$ where $d \geq 3$.

Consider the graded ring $k[\check{X}_1, \ldots, \check{X}_n]/J$ and divide out by the regular sequence $x_{pi_p} - x_{pj_p}$ for $p = 4, \ldots, d$. This is a regular sequence, since we assume we have a polarization. We get a quotient algebra $k[\check{X}'_1, \dots, \check{X}'_n]/J'$ and denote by x_p the class $\overline{x_{pi_p}} = \overline{x_{pj_p}}$ for $p \geq 4$. In J' we have generators

$$\overline{m}(\mathbf{c} - e_2) = \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c} - e_2), \quad \overline{n}(\mathbf{c} - e_2) = x_{1i_1} x_{3i_3} x_4 \cdots x_n$$

$$\overline{m}(\mathbf{c} - e_1) = \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c} - e_1), \quad \overline{n}(\mathbf{c} - e_1) = x_{2j_2} x_{3j_3} x_4 \cdots x_n$$

Now $x_{3i_3} - x_{3j_3}$ is a non-zero divisor of $k[\check{X}'_1, \cdots, \check{X}'_n]/J'$. Consider

$$(x_{3i_3} - x_{3j_3}) x_{1i_1} x_{2j_2} x_4 \cdots x_n \cdot \overline{\mathbf{m}}.$$

It is zero in this quotient ring, and so

$$\overline{\mathbf{m}'} = x_{1i_1} x_{2j_2} x_4 \cdots x_n \cdot \overline{\mathbf{m}}$$

is zero in this quotient ring and so must be a generator of J' of degree $\mathbf{c} - e_3$. But then the generator of this degree in the polarization J must be

$$\mathbf{m}' = x_{1i_1} x_{2j_2} x_{4k_4} \cdots x_{nk_n} \cdot \mathbf{m}$$

where each k_p is either i_p or j_p. Hence all $k_p = i_p = j_p$ for $p > d$. But then we see that the distance between $\mathbf{m}'$ and $m(\mathbf{c} - e_2)$ is $\leq d - 1$ and similarly the distance between $\mathbf{m}'$ and $m(\mathbf{c} - e_1)$ is $\leq d - 1$. Whether $\mathbf{m}'$ is now in V_1 or in V_2 we see that this contradicts d being the minimal distance. □

Proof of Theorem 2.5.1, ($\Longleftarrow$). We shall now prove that if each down-graph $D(\mathbf{c})$ contains a spanning tree of linear syzygy edges, then J will be a polarization. Order the variables in each $\check{X}_i$ in a sequence $x_{i1}, x_{i2}, \dots, x_{in}$. Let $\check{X}'_i$ consist of $x_{i1}, \dots, x_{ip_i}, x_i$ so we have a surjection $\check{X}_i \to \check{X}'_i$ for each i sending x_{ij} to itself for $j \leq p_i$, and to x_i for $j > p_i$. Denote the image of J in $k[\check{X}'_1, \dots, \check{X}'_n]$ by J' and the image of $m(\mathbf{a})$ by $m'(\mathbf{a})$. The quotient ring $k[\check{X}'_1, \dots, \check{X}'_n]/J'$ is obtained from $k[\check{X}_1, \dots \check{X}_n]/J$ by dividing out by variable differences $x_{ij} - x_{i,j+1}$ for $i = 1, \dots n$ and

$j > p_i$. Assume that this is a regular sequence. We show that if we now divide out by $x_{i,p_i} - x_{i,p_i+1}$ this is a non-zero divisor of $k[\check{X}'_1, \ldots, \check{X}'_n]/J'$. By continuing we get eventually that $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^d$ is a regular quotient of $k[\check{X}_1, \ldots, \check{X}_n]/J$ and so J is a polarization of $(x_1, \ldots, x_n)^d$.

Write $x'_i = x_{i,p_i}$. Suppose $(x'_i - x_i) \cdot \mathbf{f} = 0$ in $k[\check{X}'_1, \ldots, \check{X}'_n]/J'$ where $\mathbf{f}$ is a polynomial. By Lemma 2.2.6 (with $x'_i = x_{i,p_i}$ taking the place of x_0), we must show that if $\mathbf{m}$ is a monomial such that $x'_i \cdot \mathbf{m} = 0$ and $x_i \cdot \mathbf{m} = 0$, then $\mathbf{m} = 0$ in the quotient ring. So some generator $m'(\mathbf{a})$ of J' divides $x'_i\mathbf{m}$ and some generator $m'(\mathbf{b})$ divides $x_i\mathbf{m}$.

By Proposition 2.5.4 there is an LS-path from $m(\mathbf{a})$ to $m(\mathbf{b})$ consisting of linear syzygy edges and such that each $\mathbf{u}$ on this path has $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ and $m(\mathbf{u})$ on this path divides $\mathrm{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. The image $m'(\mathbf{u})$ then divides $x'_i x_i \mathbf{m}$. We will show by induction on the length of the path that some monomial $m'(\mathbf{u})$ on this path divides $\mathbf{m}$, and so $\mathbf{m}$ is zero in the quotient ring $k[\check{X}'_1, \ldots, \check{X}'_n]/J'$.

If the path has length one, there is a linear syzygy edge between $m(\mathbf{a})$ and $m(\mathbf{b})$. Write

$$x'_i \cdot \mathbf{m} = m'(\mathbf{a}) \cdot n^0(\mathbf{a}), \quad x_i \cdot \mathbf{m} = m'(\mathbf{b}) \cdot n^0(\mathbf{b}).$$

Write also $\mathbf{m} = (x'_i)^p (x_i)^q \cdot \mathbf{n}$ where $\mathbf{n}$ does not contain x'_i or x_i. If none of $m'(\mathbf{a})$ or $m'(\mathbf{b})$ divides $\mathbf{m}$, then

$$m'(\mathbf{a}) = (x'_i)^{p+1}(x_i)^{q'} \cdot n^1(\mathbf{a}), \quad m'(\mathbf{b}) = (x'_i)^{p'}(x_i)^{q+1} \cdot n^1(\mathbf{b}),$$

where $p' \leq p$ and $q' \leq q$ (and $n^1(\mathbf{a})$ and $n^1(\mathbf{b})$ do not contain x'_i or x_i). But since the edge from $\mathbf{a}$ to $\mathbf{b}$ is a linear syzygy edge, we must have $p' = p$, $q' = q$. But a linear syzygy edge involves variables of distinct x_i-type, which is not so here. Thus one of $m'(\mathbf{a})$ or $m'(\mathbf{b})$ must divide $\mathbf{m}$.

Suppose now the path has length ≥ 2.

Case $a_i \geq b_i$. Let $\mathbf{a}$ to $\mathbf{a}'$ be the first edge along the path. Then the coordinate $a'_i \leq a_i$.

If i) the coordinates a_i and a'_i are equal, the x_i-type variables of $m(\mathbf{a})$ and $m(\mathbf{a}')$ are the same, since this is a linear syzygy edge. Since $m'(\mathbf{a})$ divides $x'_i\mathbf{m}$ we get that $m'_i(\mathbf{a}')$ divides $x'_i\mathbf{m}$. If ii) $a'_i < a_i$ then when going from $m(\mathbf{a})$ to $m(\mathbf{a}')$ some x_i-type variable drops out from $m_i(\mathbf{a})$ by isotonicity of X_i and so also $m'_i(\mathbf{a}')$ divides $x'_i\mathbf{m}$.

Since the path from $\mathbf{a}$ to $\mathbf{b}$ is an LS-path, $m'(\mathbf{a}')$ divides $x'_i x_i \mathbf{m}$. For $j \neq i$, then $m'_j(\mathbf{a}')$ must divide $\mathbf{m}$ since $m'_j(\mathbf{a}')$ contains no x_i-type variable. The upshot is that $m'(\mathbf{a}')$ divides $x'_i\mathbf{m}$. Considering the LS-path from $\mathbf{a}'$ to $\mathbf{b}$, by induction on path length, some $m'(\mathbf{u})$ along this path divides $\mathbf{m}$.

Case $a_i \leq b_i$. Let $\mathbf{b}$ to $\mathbf{b}'$ be the first edge along the path going from $\mathbf{b}$ to $\mathbf{a}$. Then the coordinate $b'_i \leq b_i$. Now the same argument as in the case above works in this case. □

CHAPTER 3

ALEXANDER DUALS OF POLARIZATIONS OF POWERS OF THE MAXIMAL IDEAL

This chapter is based on joint work with Gunnar Fløystad and Henning Lohne.

In this chapter, we will describe the Alexander dual ideal of any polarization of the ideal $(x_1, \ldots, x_n)^d$. The description is a direct construction involving the isotone maps $\{X_i\}_{i\in[n]}$ from the previous chapter.

In the Section 3.1, we will introduce *rainbow monomial ideals* and show that the class of rainbow monomial ideals with linear resolution is exactly Alexander dual to the class of polarizations of Artinian monomial ideals. In Section 3.2, we restrict our attention to polarizations of powers of the graded maximal ideal and use the constructions in Chapter 2 to give a construction of the Alexander dual. In Section 3.3, we rephrase the main theorem of this chapter in terms of simplified versions χ_i of the X_i, which are isotone maps $\chi_i : \Delta^{\mathbb{Z}}(n, d) \to \{0 < 1\}$, and present some key lemmas and definitions. The proofs of these key lemmas are in Section 3.4.

3.1 Alexander duals of polarizations of Artinian monomial ideals

First, we recall the definition of the Alexander dual of an ideal.

Definition 3.1.1 (Alexander Dual)**.** Let I be a squarefree monomial ideal in a polynomial ring S. The *Alexander dual* ideal $I^\vee$ of I is the monomial ideal in S

whose monomials are precisely those that have nontrivial common divisor with every monomial in I, or equivalently, every generator of I.

Definition 3.1.2 (Rainbow Monomial)**.** Consider each set of variables $\check{X}_i$, $i = 1, \ldots, n$ as a *color* class of monomials. A monomial $x_{1i_1}x_{2i_2}\cdots x_{ni_n}$ with one variable of each color is a *rainbow* monomial.

Proposition 3.1.3. *The class of ideals generated by rainbow monomials and with m-linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in m variables:*

a. Let J be a polarization of an Artinian monomial ideal I in $k[x_1, \ldots, x_n]$. The Alexander dual ideal of J is generated by rainbow monomials and has m-linear resolution.

b. If an ideal J' is generated by rainbow monomials and has m-linear resolution (and every variable in the ambient ring occurs in some generator of the ideal), then its Alexander dual J is a polarization of an Artinian monomial ideal in m variables.

Proof. a. Since I is Cohen-Macaulay of codimension m, the same is true for J. Then the Alexander dual of J is generated in degree m and has m-linear resolution [11]. But if $\mathbf{m}$ is a generator for this Alexander dual, it has a common variable with $x_{i1}x_{i2}\cdots x_{in_i}$ (the polarization of $x_i^{n_i}$) for every $i = 1, \ldots, m$. Hence $\mathbf{m}$ must have a variable of each of the m colors.

b. By [11], the Alexander dual J of J' is Cohen-Macaulay of codimension m. For each color class $\check{X}_i = \{x_{i1}, \ldots, x_{in_i}\}$, the ideal J will contain the monomial which is the product of all these variables.

If we for every color class i divide the quotient ring $k[\check{X}_1, \ldots, \check{X}_n]/J$ by all the variable differences $x_{i,j} - x_{i,j-1}$ for $j = 2, \ldots, d_i$, we get a quotient ring $k[x_1, \ldots, x_n]/I$

where I is Artinian since it contains $x_i^{d_i}$ for each i. Hence like J the ideal I is Cohen-Macaulay of codimension n. Then the sequence we divided out by must have been a regular sequence, and so J is a polarization of I. $\square$

Remark 3.1.4. Considering the $\check{X}_i$ as color classes, both the Artinian ideal I, the polarization J and its Alexander dual are generated by colored monomials. Such ideals and the associated simplicial complexes have been considered in various settings, like balanced simplicial complexes by Stanley [49], relating to the colorful topological Helly theorem by Kalai and Meshulam, [29], and resolutions of such ideals by the second author [14].

In the following sections, we describe the Alexander dual of any polarization J of a maximal ideal power. But while the description of the generators of J is rather direct, it is highly redundant, and it is not obvious from the description that there are actually always $\binom{n+m-1}{m}$ generators of the Alexander dual.

3.2 Statement and examples

In this section, we present our construction of the Alexander dual of a polarization of $\mathfrak{m}^d$ and some examples.

Construction 3.2.1. Let $J \subset k[\check{X}_1, \ldots, \check{X}_n]$ be a polarization of the ideal $(x_1, \ldots, x_n)^d$ in $k[x_1, \ldots, x_n]$. For any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n, d-1)$ we have the up-graph $U(\mathbf{a})$, with vertices $\mathbf{a} + e_j$ for $j = 1, \ldots, n$. At the vertex $\mathbf{a} + e_j$ we have the x_j-type variables $X_j(\mathbf{a} + e_j)$. We take the product of all these variable sets:

$$\mathbb{M}(\mathbf{a}) = \prod_{j=1}^{n} X_j(\mathbf{a} + e_j).$$

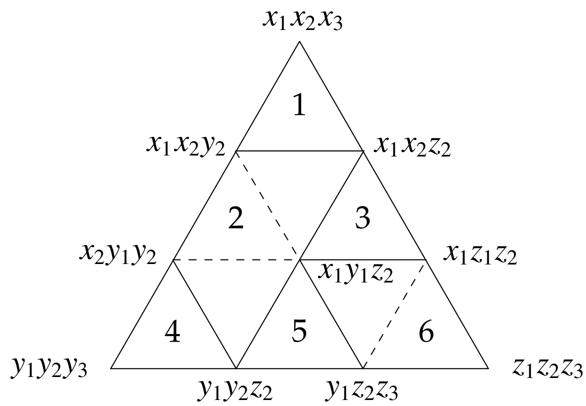


Figure 3.1: The up-triangles of a polarization of $(x, y, z)^3$

It consists of monomials $x_{1i_1}x_{2i_2}\cdots x_{ni_n}$ where x_{ji_j} is in $X_j(\mathbf{a}+e_j)$. Let I be the ideal generated by the monomials in the union of all the $\mathbb{M}(\mathbf{a})$ for $\mathbf{a} \in \Delta^{\mathbb{Z}}(n, d-1)$.

Theorem 3.2.2. *The ideal I in Construction 3.2.1 is the Alexander dual $J^\vee$ of J.*

Example 3.2.3. Consider again the polarization J from Example 2.5.2. Its graph of linear syzygies is again given in Figure 3.1, this time labeling each of the up-triangles in the graph with integers 1-6.

The up-triangle i corresponds to an element $a_i \in \Delta_3(2)$, where $a_1 = (2,0,0), a_2 = (1,1,0), a_3 = (1,0,1), a_4 = (0,2,0), a_5 = (0,1,1)$, and $a_6 = (0,0,2)$. The sets of monomials $\mathbb{M}(\mathbf{a}_i)$ are:

$$\mathbb{M}(\mathbf{a}_1) = \{\mathbf{x_1y_2z_2}, \mathbf{x_2y_2z_2}, \mathbf{x_3y_2z_2}\}$$

$$\mathbb{M}(\mathbf{a}_2) = \{\mathbf{x_1y_1z_2}, x_1y_2z_2, \mathbf{x_2y_1z_2}, x_2y_2z_2\}$$

$$\mathbb{M}(\mathbf{a}_3) = \{\mathbf{x_1y_1z_1}, x_1y_1z_2, \mathbf{x_2y_1z_1}, x_2y_1z_2\}$$

$$\mathbb{M}(\mathbf{a}_4) = \{x_2y_1z_2, x_2y_2z_2, \mathbf{x_2y_3z_2}\}$$

$$\mathbb{M}(\mathbf{a}_5) = \{x_1y_1z_2, \mathbf{x_1y_1z_3}, x_1y_2z_2, \mathbf{x_1y_2z_3}\}$$

$$\mathbb{M}(\mathbf{a}_6) = \{x_1y_1z_1, x_1y_1z_2, x_1y_1z_3\}$$

The boldface monomials are the ten distinct monomials we find from this process, which in fact generate the Alexander dual I of J.

We shall go through several steps in proving the above theorem. It turns out that we will be able to abstract the situation so our arguments will only involve a collection of isotone maps

$$\chi_i : \Delta^{\mathbb{Z}}(n, d) \to \{0 < 1\}, \quad i = 1, \ldots, n \tag{3.1}$$

where $\Delta^{\mathbb{Z}}(n, d)$ has the partial order $\geq_i$, and such that $\chi_i(\mathbf{b}) = 0 \iff b_i = 0$.

Remark 3.2.4. When $n = 3$, the maps χ_i are in bijection with ways of stacking coins in the plane, where the bottom row consists of $d + 1$ consecutive coins. In particular, this means there are C_{d+1} many such maps, where C_d is the dth Catalan number; see [50, Exercise 6.19 hhh]).

First we establish some notation which will be used for the rest of the chapter.

Notation 3.2.5. For a monomial $\mathbf{m} \in k[\check{X}_1, \ldots, \check{X}_n]$ and index $i \in [n]$, define maps

$$\chi_{i,\mathbf{m}} : \Delta^{\mathbb{Z}}(n,d) \to \{0 < 1\}$$
$$\mathbf{b} \mapsto \begin{cases} 0, & \text{no variable of } X_i(\mathbf{b}) \text{ is in } \mathbf{m}. \\ 1, & \text{some variable of } X_i(\mathbf{b}) \text{ is in } \mathbf{m}. \end{cases}$$

We will frequently drop the $\mathbf{m}$ from the notation and simply denote these maps by χ_i when the monomial $\mathbf{m}$ is understood.

Observation 3.2.6. The following properties of the maps $\chi_{i,\mathbf{m}}$ follow directly from the properties of the maps X_i.

(i) The map $\chi_{i,\mathbf{m}}$ is an isotone.

(ii) If $(\mathbf{c}; j, k)$ is a linear syzygy edge for the isotone maps $\{X_i\}_{i\in[n]}$, then $\chi_{i,\mathbf{m}}(\mathbf{c} - e_j) = \chi_{i,\mathbf{m}}(\mathbf{c} - e_k)$ for every $i \neq j, k$.

(iii) An edge $(\mathbf{c}; i, j)$ of $\Delta^{\mathbb{Z}}(n,d)$ is a linear syzygy edge for the collection $\{X_i\}_{i\in[n]}$ if $\chi_p(\mathbf{c} - e_i) = \chi_p(\mathbf{c} - e_j)$ for every $p \neq i, j$.

Furthermore, we have the following observations about monomials $\mathbf{m}$ in the Alexander dual of J.

(a) If a monomial $\mathbf{m} \in k[\check{X}_1, \ldots, \check{X}_n]$ is in the Alexander dual of J, then it has a common variable with every $m(\mathbf{b})$; equivalently, for every $\mathbf{b} \in \Delta^{\mathbb{Z}}(n,d)$, $\mathbf{m}$ has a common variable with some $m_i(\mathbf{b})$ for $i = 1, \ldots, n$. This holds if and only if for every such $\mathbf{b}$, there is some i with $\chi_{i,\mathbf{m}}(\mathbf{b}) = 1$.

(b) The monomial $\mathbf{m}$ is in I if and only if for some $\mathbf{a} \in \Delta^{\mathbb{Z}}(n, d-1)$, it has a common variable with every $X_j(\mathbf{a} + e_j)$ for $j = 1, \ldots, n$. Thus for such an $\mathbf{a}$ we have $\chi_{j,\mathbf{m}}(\mathbf{a} + e_j) = 1$ for every j.

The goal now is to abstract the negations of statements (a) and (b) in the observation above in order to prove Theorem 3.2.2.

Definition 3.2.7 (Full Zero Point, Zero Corner)**.** A multidegree $\mathbf{b} \in \Delta^{\mathbb{Z}}(n,d)$ is a *full zero point* for the collection $\{\chi_i\}_{i\in[n]}$ if $\chi_i(\mathbf{b}) = 0$ for every $i = 1, \ldots, n$. An up-simplex $U(\mathbf{a})$ of $\Delta^{\mathbb{Z}}(n,d)$ *has a zero corner* if $\chi_i(\mathbf{a} + e_i) = 0$ for some i.

We prove the following.

Theorem 3.2.8. *Let $\{\chi_i\}_{i\in[n]}$ be a set of isotone maps as in 3.1 such that for every down-graph of $\Delta^{\mathbb{Z}}(n,d)$, the linear syzygy edges for $\{\chi_i\}$ contains a spanning tree. Then $\{\chi_i\}$ has a full zero point in $\Delta^{\mathbb{Z}}(n,d)$ if and only if every up-graph of $\Delta^{\mathbb{Z}}(n,d)$ has a zero corner.*

As a consequence we get Theorem 3.2.2.

Proof of Theorem 3.2.2. If $\mathbf{m} \notin I$, then every up-graph in $\Delta^{\mathbb{Z}}(n,d)$ has a zero corner. If $\mathbf{m}$ is not in the Alexander dual of J, then it has a full zero point for the $\chi_{i,\mathbf{m}}$'s. Hence by Theorem 3.2.8 I will be the Alexander dual of J. □

3.3 Definitions and key lemmas

In order to facilitate our arguments we need to have a more flexible framework to work in. For $\mathbf{g} \in \mathbb{N}^n$ with $|\mathbf{g}| \leq d$ and $S \subseteq [n]$, let $\Delta_S(d,\mathbf{g})$ be the induced subgraph of $\mathcal{T}(n,d)$ (see Notation 2.4.1) whose vertices are the degrees $\mathbf{b} \geq \mathbf{g}$ such that $\operatorname{supp}(\mathbf{b} - \mathbf{g}) \subseteq S$. This means:

$$i)\ |\mathbf{b}| = d, \quad ii)\ b_i = g_i \text{ for } j \in [n] \setminus S, \quad iii)\ b_i \geq g_i \text{ for } i \in S.$$

We omit n in $\Delta_S(d, \mathbf{g})$ since n is fixed througout. We sometimes write $\Delta_S(\mathbf{g})$ for $\Delta_S(d, \mathbf{g})$, but on a few occasions we will want $(d-1)$ or $(d+1)$ instead of d as an argument, and use the full notation.

The convex hull of $\Delta_S(\mathbf{g})$ in $\mathbb{R}^S$ is a simplex of dimension $|S|-1$ if $|\mathbf{g}| < n$. The *size* of this simplex is $n - |\mathbf{g}|$. For $\mathbf{g} = \mathbf{0}$, the zero degree, we have $\Delta_{[n]}(\mathbf{0}) = \Delta^{\mathbb{Z}}(n, d)$ as defined earlier. Note that $\Delta_{S'}(\mathbf{g}')$ is a non-empty subset of $\Delta_S(\mathbf{g})$ if and only if the following three properties hold:

$$i)\ S' \subseteq S, \quad ii)\ \mathbf{g}' \geq \mathbf{g}, \quad iii)\ d_i' = g_i \text{ for } i \in [n] \setminus S.$$

Example 3.3.1. Let $S = \{2, 3, 4\}$ and $\mathbf{g} = (1, 0, 0, 0)$. Then $\Delta_S(\mathbf{g})$ is the induced subgraph of $\mathcal{T}(4, 3)$ depicted in Figure 3.2; its convex hull is a simplex of dimension 2, and its size is 2. If $S' = \{2, 4\}$, then $\Delta_{S'}(\mathbf{g})$ is the subgraph of $\Delta_S(\mathbf{g})$ depicted by the thick line in Figure 3.2. Its convex hull is a simplex of dimension 1, and its size is also 2.

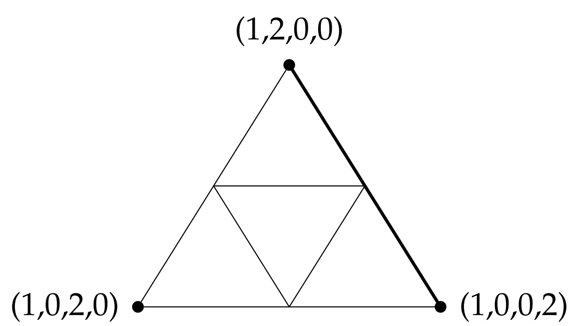


Figure 3.2: A subgraph of $\mathcal{T}(4, 3)$.

Definition 3.3.2. For $\mathbf{a} \in \Delta_S(d-1, \mathbf{g})$ we get an induced subgraph $U_S(\mathbf{a}; \mathbf{g})$ of $\Delta_S(\mathbf{g})$ with vertices $\{\mathbf{a} + e_i \mid i \in S\}$. This is a complete graph on $|S|$ vertices whose convex hull is of dimension $|S|-1$. The graph $U_S(\mathbf{a}; \mathbf{g})$ is an *up-graph*. The $i \in S$ are the *corners* of the up-graph.

For $\mathbf{c} \in \Delta_S(n+1, \mathbf{g})$ we get an induced subgraph $D_S(\mathbf{c}; \mathbf{g})$ of $\Delta_S(\mathbf{g})$ with vertices $\{\mathbf{c} - e_i \mid i \in \mathrm{supp}(\mathbf{c} - \mathbf{g})\}$. This is a complete graph on $|\mathrm{supp}(\mathbf{c} - \mathbf{g})|$ vertices whose convex hull is a simplex of dimension $|\mathrm{supp}(\mathbf{c} - \mathbf{g})| - 1$. The graph $D_S(\mathbf{c}; \mathbf{g})$ is a *down-graph*.

Suppose that for each $j \in S$ we have isotone maps, where we have given $\Delta_S(\mathbf{g} + e_j)$ the $\geq_j$-ordering:

$$\chi_j : \Delta_S(\mathbf{g} + e_j) \to \{0 < 1\}.$$

Note that for $\mathbf{b} \in \Delta_S(\mathbf{g})$, the $\chi_j(\mathbf{b})$ are defined precisely for $j \in \mathrm{supp}(\mathbf{b} - \mathbf{g})$.

Definition 3.3.3. A vertex $\mathbf{b} \in \Delta_S(\mathbf{g})$ is a *full zero point* for $\{\chi_i\}_{i \in S}$ if $\chi_i(\mathbf{b}) = 0$ for every $i \in \mathrm{supp}(\mathbf{b} - \mathbf{g})$. An up-graph $U_S(\mathbf{a}; \mathbf{g})$ *has a zero corner* for $\{\chi_i\}_{i \in S}$ if $\chi_j(\mathbf{a} + e_j) = 0$ for some $j \in S$.

An edge $(\mathbf{c}; r, s)$ of $\Delta_S(\mathbf{g})$ (note that then $\{r, s\} \subseteq S$) is a *linear syzygy edge* for the $\{\chi_j\}_{j \in S}$ if

$$\chi_j(\mathbf{c} - e_r)) = \chi_j(\mathbf{c} - e_s), \quad \text{for } j \in \mathrm{supp}(\mathbf{c} - \mathbf{g}) \setminus \{r, s\}.$$

Lemma 3.3.4. *Suppose we have isotone maps $\{\chi_j\}_{j \in S}$ for $\Delta_S(\mathbf{g})$ such that the linear syzygy edges in every down-graph of $\Delta_S(\mathbf{g})$ contains a spanning tree.*

Let $\Delta_R(\mathbf{g}')$ be a non-empty subgraph of $\Delta_S(\mathbf{g})$. For $j \in R$ let $\overline{\chi}_j$ be the restriction of χ_j to isotone maps associated to $\Delta_R(\mathbf{g}')$. Then each down-graph of $\Delta_R(\mathbf{g}')$ contains a spanning tree of linear syzygy edges for the $\{\overline{\chi}_j\}_{j \in R}$.

Proof. The proof of this statement is essentially the same as for Lemma 2.4.15. □

3.4 Proofs of technical lemmas

We prove this direction of Theorem 3.2.8.

Lemma 3.4.1. *Let $S \subseteq [n]$ have cardinality ≥ 2. Suppose every down-graph of $\Delta_S(\mathbf{g})$ contains a spanning tree of linear syzygies for the isotone maps $\{\chi_j\}_{j \in S}$. Let $\mathbf{b} \in \Delta_S(\mathbf{g})$ and $p \in S$. Suppose $\chi_i(\mathbf{b}) = 0$ for every $i \in \mathrm{supp}(\mathbf{b} - \mathbf{g}) \setminus \{p\}$. Then every up-graph $U_S(\mathbf{a}; \mathbf{g})$ in $\Delta_S(\mathbf{g})$ with $a_p \geq b_p$ has a zero corner for the $\{\chi_i\}_{i \in S}$.*

Corollary 3.4.2. *Suppose the maps $\{\chi_i\}_{i \in S}$ have a full zero point in $\Delta_S(\mathbf{g})$. Then every up-graph in $\Delta_S(\mathbf{g})$ has a zero corner.*

Proof of Lemma 3.4.1 and Corollary 3.4.2. We prove these in tandem by induction on the cardinality $|S|$. We prove Corollary 3.4.2 by assuming Lemma 3.4.1. Then we prove Lemma 3.4.1 by assuming Corollary 3.4.2 has been proven for all $\Delta_{S'}(\mathbf{g})$ when the cardinality $|S'| < |S|$.

Assume we have shown Lemma 3.4.1. Let $\mathbf{b}$ be a full zero point for $\Delta_S(\mathbf{g})$ and consider an up-graph $U_S(\mathbf{a}; \mathbf{g})$ in $\Delta_S(\mathbf{g})$. Since $|\mathbf{b}| = d$ and $|\mathbf{a}| = d - 1$, then at least for one p we have $a_p \geq b_p$. Then Lemma 3.4.1 implies that $U_S(\mathbf{a}; \mathbf{g})$ is an up-graph with a zero corner, proving Corollary 3.4.2.

We now show Lemma 3.4.1. For simplicity we assume $p = 1$. First we do the case $|S| = 2$, say $S = \{1, 2\}$. Then $\chi_2(\mathbf{b}) = 0$. By isotonicity of χ_2 we have $\chi_2(\mathbf{b} + \lambda e_1 - \lambda e_2) = 0$ for $\lambda \geq 0$. But letting $\lambda = a_1 - b_1$, the point $\mathbf{b} + \lambda e_1 - \lambda e_2 = \mathbf{a} + e_2$ is a zero corner for $U_S(\mathbf{a}; \mathbf{g})$.

Assume now $|S| \geq 3$ and Corollary 3.4.2 holds for S' with $2 \leq |S'| < |S|$. We argue by induction on the difference $a_1 - b_1$.

Case 1: $a_1 = b_1$. Let $S' = S \setminus \{1\}$ and $\mathbf{g}' = \mathbf{g} + (b_1 - g_1)e_1$. Then $\mathbf{b} \in \Delta_{S'}(\mathbf{g}')$ and $\mathbf{b}$ is a full zero point for the $\{\overline{\chi}_i\}_{i\in S'}$. By Corollary 3.4.2 every up-graph $U_{S'}(\mathbf{a};\mathbf{g}')$ in $\Delta_{S'}(\mathbf{g}')$ has a zero corner for $\{\overline{\chi}_i\}_{i\in S'}$. Since $\mathbf{a} \in \Delta_{S'}(d-1,\mathbf{g}')$ iff $\mathbf{a} \in \Delta_S(d-1,\mathbf{g})$ with $a_1 = d_1' = b_1$, up-graphs $U_{S'}(\mathbf{a};\mathbf{g}')$ in $\Delta_{S'}(\mathbf{g}')$ correspond to up-graphs $U_S(\mathbf{a};\mathbf{g})$ with $a_1 = b_1$. Then every up-graph $U_S(\mathbf{a};\mathbf{g})$ with $a_1 = b_1$ has a zero corner for $\{\chi_i\}_{i\in S}$.

Case 2: $a_1 > b_1$. Consider the down-graph $D_S(\mathbf{b}+e_1)$. By assumption on the maps χ_i, at least one edge $(\mathbf{b}+e_1;1,r)$ is a linear syzygy edge, so $\chi_j(\mathbf{b}+e_1-e_r) = \chi_j(\mathbf{b}) = 0$ for every $j \in \operatorname{supp}(\mathbf{b}-\mathbf{g}) \setminus \{1,r\}$. But $\mathbf{b}+e_1-e_r \leq_r \mathbf{b}$, so $\chi_r(\mathbf{b}+e_1-e_r) \leq \chi_p(\mathbf{b}) = 0$. Therefore $\chi_j(\mathbf{b}+e_1-e_r) = 0$ for every $j \in \operatorname{supp}(\mathbf{b}-\mathbf{g}) \setminus \{1\}$. Since the difference in first coordinates of $\mathbf{a}$ and $\mathbf{b}+e_1-e_r$ is one less than $a_1 - b_1$, by induction the up-graph $U_S(\mathbf{a};\mathbf{g})$ has a zero corner for $\{\chi_j\}_{j\in S}$. □

We now prove the other direction of Theorem 3.2.8. We need the following specific lemma. It says that you can "pull a point" with specific properties in a given direction and into a simplex of a smaller size.

Lemma 3.4.3. *Let $\mathbf{b} \in \Delta_S(\mathbf{g})$ with $|\mathbf{g}| \leq d-1$, $1 \in S$, and $b_1 \leq g_1 + 1$. Suppose*

$$\chi_i(\mathbf{b}) = 0 \textit{ for } i \in \operatorname{supp}(\mathbf{b}-\mathbf{g}) \setminus \{1\}, \quad \chi_1(\mathbf{b}) = 1 \textit{ if } 1 \in \operatorname{supp}(\mathbf{b}-\mathbf{g}).$$

Let $p \in S \setminus \{1\}$ and $\mathbf{g}' = \mathbf{g} + e_p$. Then there is some $\mathbf{b}' \in \Delta_S(\mathbf{g}')$ with $b_1' \leq d_1' + 1 (= g_1 + 1)$ such that

$$\chi_i(\mathbf{b}') = 0 \textit{ for } i \in \operatorname{supp}(\mathbf{b}'-\mathbf{g}') \setminus \{1\}, \quad \chi_1(\mathbf{b}') = 1 \textit{ if } 1 \in \operatorname{supp}(\mathbf{b}'-\mathbf{g}'). \tag{3.2}$$

Proof. If $b_p \geq d_p + 1$ we simply let $\mathbf{b}' = \mathbf{b}$. So suppose $b_p = d_p$. Looking at the down-graph $D_S(\mathbf{b}+e_p;\mathbf{g})$ there is a linear syzygy edge $(\mathbf{b}+e_p;p,q)$ for some $q \in S \setminus \{p\}$. It goes from $\mathbf{b}$ to $\mathbf{b}' = \mathbf{b}+e_p-e_q$. Let us show that this $\mathbf{b}'$ has the desired properties.

We have $\mathbf{b}' - \mathbf{g}' = \mathbf{b} - \mathbf{g} - e_q$ and so $\mathrm{supp}(\mathbf{b}' - \mathbf{g}')$ is $\mathrm{supp}(\mathbf{b} - \mathbf{g})$ with q possibly removed. We see that $p \notin \mathrm{supp}(\mathbf{b}' - \mathbf{g}')$, and so $\chi_i(\mathbf{b}') = \chi_i(\mathbf{b})$ for $i \in \mathrm{supp}(\mathbf{b}' - \mathbf{g}') \setminus \{q\}$.

Case 1: $q = 1$**.** In this case $b'_1 = g_1$ so $1 \notin \mathrm{supp}(\mathbf{b}' - \mathbf{g}')$ and (3.2) holds.

Case 2: $q \neq 1$**.** Note $\mathbf{b}' \leq_q \mathbf{b}$. So if q is contained in $\mathrm{supp}(\mathbf{b}' - \mathbf{g}')$ then $\chi_q(\mathbf{b}') \leq \chi_q(\mathbf{b}) = 0$. Furthermore $\chi_1(\mathbf{b}') = \chi_1(\mathbf{b}) = 1$ if $1 \in \mathrm{supp}(\mathbf{b}' - \mathbf{g}')$. So again (3.2) holds. □

Proposition 3.4.4. *Let $|\mathbf{g}| \leq d - 1$ and $|S| \geq 2$. Suppose every up-graph in $\Delta_S(\mathbf{g})$ has a zero corner for the $\{\chi_i\}_{i \in S}$. Then there is an element of $\Delta_S(\mathbf{g})$ which is a full zero point.*

Proof. We prove by induction on the size $(n - |\mathbf{g}|)$ and cardinality $|S|$ that the above holds. If $|\mathbf{g}| = d - 1$, then $\Delta_S(\mathbf{g})$ equals the up-graph $U_S(\mathbf{g}; \mathbf{g})$. If we have a zero at corner $p \in S$, so $\chi_p(\mathbf{g} + e_p) = 0$, then $\mathbf{b} = \mathbf{g} + e_p$ is a full zero point of $\Delta_S(\mathbf{g})$ since $\mathrm{supp}(\mathbf{b} - \mathbf{g}) = \{p\}$.

Now pick an element of S, say $1 \in S$. By induction on size, $\Delta_S(\mathbf{g} + e_1)$ has a full zero point $\mathbf{b}$. If $b_1 > g_1 + 1$ then $\mathrm{supp}(\mathbf{b} - \mathbf{g}) = \mathrm{supp}(\mathbf{b} - (\mathbf{g} + e_1))$ and $\mathbf{b}$ is a full zero point for $\Delta_S(\mathbf{g})$. So suppose $b_1 = g_1 + 1$. If $\chi_1(\mathbf{b}) = 0$, it is a full zero point in $\Delta_S(\mathbf{g})$. Otherwise we have:

$$\chi_i(\mathbf{b}) = 0 \text{ for } i \in \mathrm{supp}(\mathbf{b} - \mathbf{g}) \setminus \{1\}, \quad \chi_1(\mathbf{b}) = 1.$$

First consider when S has cardinality two, say $S = \{1, 2\}$, let $\mathbf{a} = \mathbf{b} - e_1$ and consider the up-graph $U_S(\mathbf{a}; \mathbf{g})$. Since $\chi_1(\mathbf{a} + e_1) = 1$, we must in the other corner of this up-graph have $\chi_2(\mathbf{a} + e_2) = 0$. Since 1 is not in the support of $(\mathbf{a} + e_2) - \mathbf{g}$, then $\mathbf{b} = \mathbf{a} + e_2$ is a full zero point.

So let S have cardinality ≥ 3 and put $S' = S \setminus \{1\}$. We show now that each up-graph in $\Delta_{S'}(\mathbf{g})$ has a zero corner. By induction we then have a full zero $\mathbf{b^0}$ in $\Delta_{S'}(\mathbf{g})$. Since 1 is not in the support of $\mathbf{b^0} - \mathbf{g}$ this $\mathbf{b^0}$ is also a full zero point in $\Delta_S(\mathbf{g})$ and we are done.

So let $U_{S'}(\mathbf{a};\mathbf{g})$ be an up-graph in $\Delta_{S'}(\mathbf{g})$. We also have the up-graph $U_S(\mathbf{a};\mathbf{g})$ in $\Delta_S(\mathbf{g})$. If there is some $p \geq 2$ such that $a_p > b_p$ we apply Lemma 3.4.3 and "pull" the point $\mathbf{b}$ to a point $\mathbf{b}'$ in a smaller sized $\Delta_S(\mathbf{g} + e_p)$, which still contains $U_S(\mathbf{a};\mathbf{g})$. In this way we continue until we have a $\mathbf{b}$ with either i) $a_p \leq b_p$ for every $p \geq 2$, or ii) the size of $\Delta_S(\mathbf{g})$ has become 1. But with this size we have $\mathbf{a} = \mathbf{g}$ and so $a_p \leq b_p$ for every $p \geq 2$ in any case. We also have $a_1 = g_1 \leq b_1 \leq g_1 + 1$ and recall that $|\mathbf{b}| = d$ and $|\mathbf{a}| = d - 1$.

Case $b_1 = a_1$**.** Then $\mathbf{b} = \mathbf{a} + e_p$ for some $p \geq 2$ and $p \in \mathrm{supp}(\mathbf{b} - \mathbf{g})$. Then

$$\chi_p(\mathbf{a} + e_p) = \chi_p(\mathbf{b}) = 0$$

and so $U_{S'}(\mathbf{a};\mathbf{g})$ has a zero corner.

Case $b_1 = a_1 + 1$**.** Then $\mathbf{b} = \mathbf{a} + e_1$ and $1 \in \mathrm{supp}(\mathbf{b} - \mathbf{g})$ so

$$\chi_1(\mathbf{a} + e_1) = \chi_1(\mathbf{b}) = 1.$$

Since $U_S(\mathbf{a};\mathbf{g})$ has a zero corner we must have $\chi_p(\mathbf{a} + e_p) = 0$ for some $p \in S'$ and so $U_{S'}(\mathbf{a};\mathbf{g})$ has a zero corner. $\square$

CHAPTER 4

POLARIZATIONS DEFINE SHELLABLE SIMPLICIAL COMPLEXES

This chapter is joint work with Gunnar Fløystad and Henning Lohne.

In this chapter we show that the Alexander dual of any polarization of the power $(x, y, z)^n$ is a monomial ideal with linear quotients. This is equivalent to the polarization defining a shellable simplicial complex via the Stanley-Reisner correspondence, [25, Prop.8.2.5]. By a result of Björner [7, Thm.11.4], this immediately implies that these polarizations define simplicial balls.

4.1 Balls and spheres

The notion of a shelling first appeared in work of Schläfli in the late 1800s, who used (without proof) the assumption that the boundary of a convex polytope is shellable in order to compute its Euler characteristic (see [46]). This assumption was not actually proven until 1970 by Bruggesser and Mani [8], and has since been utilized in the proofs of celebrated theorems such as McMullen's proof of the Upper Bound Theorem [32]. Shellability has been and continue to be a critical tool in algebra, geometry, and combinatorics in large part due to its strong topological implications.

We begin by recalling the definition of a shellable simplicial complex.

Definition 4.1.1 (Shelling order)**.** An ordering $F_1, \ldots, F_t$ of the facets of a simplicial complex Δ is a *shelling order* if, for each j with $1 < j \leq t$, the intersection

$$\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j$$

is a nonempty union of facets of ∂F_j, the boundary of F_j. If there exists a shelling order of Δ, then Δ is called *shellable*.

We now recall a famous theorem of Danaraj and Klee, which describes a case where shellability has strong topological consequences beyond just describing the homotopy type.

Theorem 4.1.2. *Let Δ be a pure shellable d-dimensional simplicial complex in which every codimension 1 face is contained in at most 2 facets. Then Δ is homeomorphic to a d-sphere or a d-ball. Moreover, Δ is homeomorphic to a d-sphere if and only if every codimension 1 face is contained in exactly two facets.*

It turns out that the faces of the Stanley–Reisner complex of any polarization of an Artinian monomial ideal satisfy the condition appearing in Theorem 4.1.2.

Lemma 4.1.3. *Let $\Delta(J)$ be the simplicial complex associated to the polarization J of an Artinian monomial ideal I. Then every codimension one face of $\Delta(J)$ is contained in one or two facets. If I is not a complete intersection, then at least once there is a codimension one face contained in exactly one facet.*

Proof. Let Δ_i be the simplex on $\{(i, j) \mid j = 1, \ldots, d_i\}$. The squarefree monomial $x_{i1}x_{i2} \cdots x_{id_i}$ in $k[\check{X}_i]$ defines the sphere which is the boundary $\partial\Delta_i$ of this simplex. The natural polarization in $k[\check{X}_1, \ldots, \check{X}_n]$ of the complete intersection $(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n})$ then defines the sphere of codimension n which is the join $S = \mathop{*}\limits_{i=1}^{n} \partial\Delta_i$. Every codimension one face is here on precisely two facets.

The simplicial complex $\Delta(J)$ is a Cohen-Macaulay subcomplex of S with the same dimension as S. If $\Delta(J)$ is not all of S, let F be a facet of $\Delta(J)$ and G a facet of S not in $\Delta(J)$. Since S is strongly connected, there is a path of facets

$$F = F_0, F_1, \ldots, F_r = G$$

such that $F_i \cap F_{i+1}$ has codimension one for each i, [25, Prop.9.1.12]. Let p be maximal such that F_p is in $\Delta(J)$. Then $F_p \cap F_{p+1}$ is only on the facet F_p in $\Delta(J)$. $\square$

This leads to the following natural question:

Question 4.1.4. *Do polarizations of Artinian monomial ideals have shellable simplicial complexes?*

The standard polarization of an Artinian monomial ideal was shown to be shellable independently by A. Soleyman Jahan in [48] and Faridi [13]. In [37], Murai uses this fact to conclude that standard polarizations give simplicial balls. More generally, Ali, Fløystad, and Nematbakhsh [10] show that so-called *letterplace ideals* define simplicial balls by showing that these simplicial complexes are shellable. Letterplace ideals are introduced in [16] and are polarizations of Artinian monomial ideals. The article [15] discusses such Artinian monomial ideals more in depth.

In the next section, we will show in the case of *three* variables that the Alexander dual of any polarization J has linear quotients, which is equivalent to J being a shellable simplicial complex. Thus, in this case the simplicial complex $\Delta(J)$ is shellable and hence a simplicial ball.

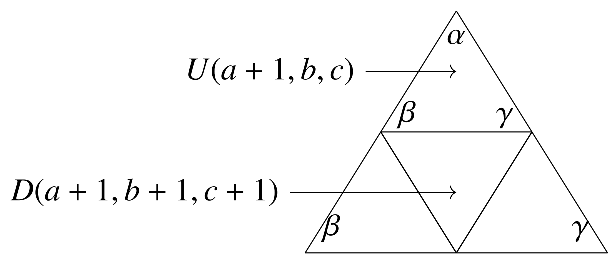


Figure 4.1: A sub-graph of $(x, y, z)^d$

4.2 Polarizations are shellable

Notation 4.2.1. An element (a, b, c) in $\Delta_3(d-1)$ corresponds to an up-triangle in $\mathcal{T}(3, d)$. If $x_\alpha \in X(a+1, b, c)$ we say that x_α (or just α) is an x-variable belonging to the up-triangle $U(a, b, c)$. Similarly if $y_\beta \in Y(a, b+1, c)$ and $z_\gamma \in Z(a, b, c+1)$. We also say the monomial $x_\alpha y_\beta z_\gamma$ (or just $\alpha\beta\gamma$) belongs to (a, b, c).

Lemma 4.2.2. *Suppose $\alpha\beta\gamma$ belongs to the up-triangle $U(a+1, b, c)$ in $\Delta_3(d-1)$, see Figure 4.1.*

a. Then either the up-triangle $U(a, b+1, c)$ or the up-triangle $U(a, b, c+1)$ has a monomial $\alpha'\beta\gamma$ belonging to them.

b. If α either belongs to the up-triangle $U(a, b+1, c)$ or to $U(a, b, c+1)$, then $\alpha\beta\gamma$ will belong to one of the up-triangles $U(a, b, c+1)$ or in $U(a, b+1, c)$.

Proof. Consider the up-triangles in Figure 4.1. In the middle we have a down-triangle $D(a+1, b+1, c+1) \in \mathcal{T}(3, d)$. Note that since Y is isotone, β will be in both the up-triangles $U(a+1, b, c)$ and $U(a, b+1, c)$ and since Z isotone γ in both $U(a+1, b, c)$ and $U(a, b, c+1)$.

a. If the edge $((a+1, b+1, c+1); 1, 2)$ is a linear syzygy edge, then also γ belongs to $U(a, b+1, c)$, and if $((a+1, b+1, c+1); 1, 3)$ is a linear syzygy edge then β belongs to $U(a, b, c+1)$. Since at least one of them is a linear syzygy edge we are done.

b. If $((a, b+1, c+1); 2, 3)$ is a linear syzygy edge, α is either in none or in both the two lower up-triangles. It then follows by part a that $\alpha\beta\gamma$ belongs to one of these up-triangles.

If $((a, b+1, c+1); 2, 3)$ is not a linear syzygy edge, the two other edges are linear syzygy edges. By the argument in part a, both the lower up-triangles contains β and γ and so at least on of them contains $\alpha\beta\gamma$. $\square$

Let $\check{X}$ be a set of x-variables (with various indices) and $\check{Y}$ and $\check{Z}$ be sets of y- and z-variables.

Lemma 4.2.3. *Let I be an ideal generated by a subset of monomials in the product set $\check{X} \cdot \check{Y} \cdot \check{Z}$. Let $x_\alpha y_\beta z_\gamma$ be in $\check{X} \cdot \check{Y} \cdot \check{Z}$ but not in I. Then $I : x_\alpha y_\beta z_\gamma$ is generated by variables if and only if for every $x_{\alpha'} y_{\beta'} z_{\gamma'} \in I$ one of the variables $x_{\alpha'}, y_{\beta'}$ or $z_{\gamma'}$ is in the colon ideal.*

Proof. Note that by the construction of I and definition of $x_\alpha y_\beta z_\gamma$, none of the variables x_α, y_β, or z_γ can be in $I : x_\alpha y_\beta z_\gamma$.

It is straightforward to verify that the first assertion implies the second. Assume the second assertion holds. Then, if say $y_{\beta'} z_{\gamma'}$ is in the colon ideal, then $x_\alpha y_\beta y_{\beta'} z_\gamma z_{\gamma'}$ is in I. So at least some $x_\alpha y_{\tilde{\beta}} z_{\tilde{\gamma}}$ is in I, where $\tilde{\beta} = \beta'$ or $\tilde{\gamma} = \gamma'$. But by assumption then either $y_{\beta'}$ or $z_{\gamma'}$ is in the colon ideal. This implies the colon ideal is generated by variables. $\square$

We now consider the monomials $x_\alpha y_\beta z_\gamma$ belonging to the up-triangles $U(a, b, c) \in \Delta_3(d-1)$ and shall provide a total order on these monomials. First consider the partial order on *triples* where $(a, b, c) \geq (a', b', c')$ if $a \geq a'$ and take any linear extension on this to get a total order $\geq$ on triples.

Now for each up-triangle $U(a, b, c)$ we shall make a total order on the (degree 3) *monomials* belonging to it. For each $X(a+1, b, c)$ choose any total order of the x-variables. To order the variables in $Y(a, b+1, c)$ we have an ascending chain

$$Y(a, 1, d-a-1) \subseteq Y(a, 2, d-a-2) \subseteq \cdots \subseteq Y(a, d-a, 0). \tag{4.1}$$

We order the variables such that each new variable popping up in the chain is less than the foregoing variables. Similarly for the variables in $Z(a, b, c+1)$ we have a chain

$$Z(a, d-a-1, 1) \subseteq Z(a, d-a-2, 2) \subseteq \cdots \subseteq Z(a, 0, d-a),$$

and we order the variables such that each new variables popping up in the chain is less than the foregoing variables. The monomials belonging to $U(a, b, c)$ correspond to

$$X(a+1, b, c) \times Y(a, b+1, c) \times Z(a, b, c+1).$$

We get the partial product order on this and take a linear extension of this partial order.

We now order the monomials associated to the up-triangles in $\mathcal{T}(3, d)$ as follows. If $\alpha'\beta'\gamma'$ occurs first in (a', b', c') and $\alpha\beta\gamma$ occurs first in (a, b, c), then

$$\alpha'\beta'\gamma' > \alpha\beta\gamma \tag{4.2}$$

if $(a', b', c') > (a, b, c)$, or if $(a', b', c') = (a, b, c)$ and the order of (4.2) is given by the order on the monomials belonging to the up-triangle $U(a, b, c)$.

Proposition 4.2.4. *The ideal generated by all the variables belonging to the up-triangles of $\mathcal{T}(3, d)$, has linear quotients given by the ordering of the monomials above.*

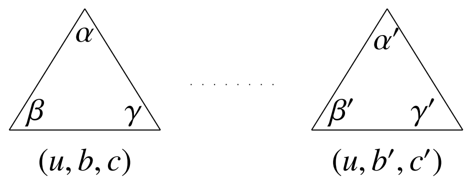


Figure 4.2: The situation in Part 1 of the proof of Proposition 4.2.4.

Proof. Let $\alpha\beta\gamma$ occur for the first time in the up-triangle (u, b, c) and let I be the ideal generated by all the larger monomials. We shall show that $I : x_\alpha y_\beta z_\gamma$ is generated by variables and use Lemma 4.2.3.

1. Let $\alpha'\beta'\gamma'$ be in (u, b', c') where $(u, b', c') > (u, b, c)$. Suppose $c' \geq c$, see Figure 4.2. and so γ belongs to (u, b', c'), since the map $Z : \Delta_3(n) \to B(n)$ is isotone.

a. If $\beta \geq \beta'$ then β will be in $U(u, b'', c'')$ where $b'' \leq b'$ so $c'' \geq c'$. Then β will also be in $U(u, b', c')$ and since γ is in $U(u, b', c')$ we will have $\alpha'\beta\gamma$ belonging to $U(u, b', c')$. If $\alpha = \alpha'$ then $\alpha\beta\gamma$ would occur in $U(u, b', c')$ contradicting that $\alpha\beta\gamma$ first occurs in $U(u, b, c)$. So $\alpha \neq \alpha'$ and this gives $x_{\alpha'}$ in the colon ideal.

b. Assume now that $\beta < \beta'$. Note that since $b' \leq b$ we have β' belonging to $U(u, b, c)$. Then $\alpha\beta'\gamma$ is already in I by the ordering on the monomials belonging to $U(u, b, c)$, and hence β' is in the colon ideal.

2. A symmetric argument works when $\alpha'\beta'\gamma'$ is in $U(u, b', c')$ and $b' \geq b$.

3. Assume now that $\alpha'\beta'\gamma'$ belongs to the up-triangle $U(u + 1, b', c')$ where the sum of these coordinates is $d - 1$. Either $b' \geq b$ or $c' \geq c$. Suppose the latter, see Figure 4.3. Then β' belongs to the up-triangle $U(u, b, c)$ due to Y being isotone.

a. If $\beta' > \beta$ in the order given by (4.1), then $\alpha\beta'\gamma > \alpha\beta\gamma$ and so the former belongs to I and β' is in the colon ideal $I : x_\alpha y_\beta z_\gamma$.

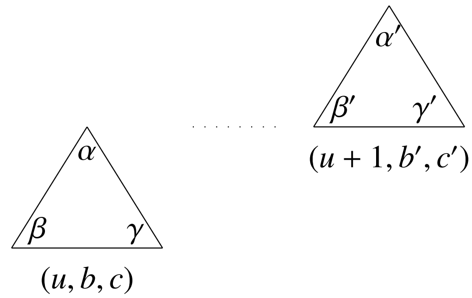


Figure 4.3: The situation in Part 3 of the proof of Proposition 4.2.4.

b. If $\beta' = \beta$ note that $y_\beta = y_{\beta'}$ is in $Y(u, b-1, c+1)$ since $b > b'$. By Lemma 4.2.2 (applied in the y-direction, not x-direction) either $\alpha\beta\gamma$ is in $U(u+1, b, c)$ or in $U(u, b-1, c+1)$. The latter must be the case since $\alpha\beta\gamma$ first occurs in $U(u, b, c)$.

c. If $\beta' < \beta$, then since β' belongs to $U(u, b, c)$, β must belong to $U(u, b-1, c+1)$ and by Lemma 4.2.2 $\alpha\beta\gamma$ will belong to $U(u, b-1, c+1)$.

In case b. and c. we may continue like this and push $\alpha\beta\gamma$ stepwise to the right, until we get to $(u, b-r, c+r)$ where $c+r = c'+1$, and $(u, b-r, c+r) = (u, b', c'+1)$, so $\alpha\beta\gamma$ is in both $U(u, b'+1, c')$ and $U(u, b', c'+1)$. Note that by X being isotone α belongs to $(u+1, b', c')$. We show that one of α', β', or γ' is in the colon ideal.

i. If $\beta = \beta'$ and $\gamma = \gamma'$ then $\alpha'\beta\gamma$ belongs to $U(u+1, b', c')$. Since $\alpha\beta\gamma$ occurs first in $U(u, b, c)$, we cannot have $\alpha = \alpha'$ and so $x_{\alpha'}$ is in the colon ideal.

ii. If $\beta \neq \beta'$ and $\gamma = \gamma'$ then $\alpha\beta'\gamma$ belongs to $U(u+1, b', c')$ and so $y_{\beta'}$ is in the colon ideal.

iii. If $\beta = \beta'$ and $\gamma \neq \gamma'$ then $\alpha\beta\gamma'$ belongs to $U(u+1, b', c')$ and so $z_{\gamma'}$ is in the colon ideal.

iv. Suppose that $\beta \neq \beta'$ and $\gamma \neq \gamma'$. If the edge $((u+1, b'+1, c'+1); 1, 2)$ is a linear syzygy edge then γ is in $Z(u+1, b', c'+1)$ and so $\alpha\beta'\gamma$ belongs to $U(u+1, b', c')$. If $((u+1, b'+1, c'+1); 1, 3)$ is a linear syzygy edge then $\beta \in Y(u+1, b'+1, c')$ and so $\alpha\beta\gamma'$ is belongs to $U(u+1, b', c')$.

4. Suppose then that $\alpha'\beta'\gamma'$ is in $U(u+r, b', c')$ where $r \geq 2$.

a. If $b' \leq b$ and $c' \leq c$ (then at least one inequality is strict) then $\alpha\beta'\gamma'$ is in $U(u+r, b', c')$ since the map X is isotone, and α also belongs to either $U(u+r-1, b'+1, c')$ or $U(u+r-1, b', c'+1)$. Hence by Lemma 4.2.2 $\alpha\beta'\gamma'$ is in one of these up-triangles. We may continue until either $u+r-1 = u+1$, treated in Case 3., or until $b' > b$ or $c' > c$. Assume $c' > c$.

b. We then assume $\alpha'\beta'\gamma'$ is in $(u+r, b', c')$ where $r \geq 2$ and $c' > c$. Note that by Y being isotone and $b' < b$, β' will belong to $U(u, b, c)$ and to $U(u, b-1, c+1)$.

b1. If $\beta' > \beta$, then $\alpha\beta'\gamma > \alpha\beta\gamma$ and so $y_{\beta'}$ is in the colon ideal.

b2. If $\beta = \beta'$ then β belongs to $U(u+r, b', c')$. By Y being isotone, β belongs to $U(u, b-1, c+1)$.

b3. If $\beta' < \beta$, then β is in the up-triangle $U(u, b-1, c+1)$. In both cases b2 and b3, by Lemma 4.2.2 $\alpha\beta\gamma$ is either in up-triangle $U(u+1, b-1, c)$, not possible, or in $U(u, b-1, c+1)$.

In this way we may continue going rightwards until we get to $(u, b-t, c+t)$ with $c+t = c'$. Then $(u, b'+r, c')$ contains $\alpha\beta\gamma$ and so α is in $(u+r, b', c')$ and $(u+r-1, b'+1, c')$. Then $\alpha\beta'\gamma'$ is in $(u+r, b', c')$ and since $\alpha\beta\gamma$ occurs first in $U(a, b, c)$ this is not equal to $\alpha\beta'\gamma'$. By Lemma 4.2.2 we may push it down to level $u+r-1$. In this way we can continue until we get $\alpha\beta'\gamma'$ on level $u+1$ which is treated in Case 3. $\square$

CHAPTER 5

WHEN DO ISOTONE MAPS EXIST?

All of the combinatorial characterizations of polarizations of $\mathfrak{m}^d$ given in previous chapters rely on the existence of the rank-preserving isotone maps X_i in Construction 2.4.10. However, it is not clear when a graph of linear syzygies gives rise to such a set of isotone maps. The goal of this chapter is to give necessary and sufficient conditions for a graph of linear syzygies of $\mathfrak{m}^d$ to induce isotone maps as in Construction 2.4.10. These conditions, labelled (G1)-(G4), are presented in Theorem 5.0.4. The results in this section are a critical stepping stone in the direction of being able to generate all polarizations of $(x_1, \ldots, x_n)^d$ for arbitrary n and d.

To show that our conditions are sufficient, we give an explicit algorithm (see Construction 5.0.11) to determine a set of isotone maps from a graph of linear syzygies, and we prove in Proposition 5.0.14 that the maps output by the algorithm are indeed isotone and satisfy the condition $(*)$ in Setup 5.0.1. This algorithm extends work of Lohne in [30], where he gives such an algorithm for determining polarizations of $(x, y, z)^d$ from potential graphs of linear syzygies. Note, however, that the three variable case is quite straightforward: any subgraph of $\mathcal{T}(n, d)$ containing all of the "boundary" edges gives rise to a set of isotone maps. As we will see in Example 5.0.2, this is not the case for $n > 3$, even if the graph of linear syzygies corresponds to a set of linear syzygies in a valid free resolution of $\mathfrak{m}^d$.

As it turns out, the main difficulty of this section is the correct formulation of the conditions (G1)-(G4) of Theorem 5.0.4; once these conditions are written down, it is a straightforward yet tedious verification that the desired algorithm is well defined.

We begin by establishing some notation for the rest of the chapter.

Notation 5.0.1. Fix integers n and d, and let $S = k[x_1, \dots, x_n]$ be a polynomial over a field k. Let $\check{X}_i = \{x_{i1}, \dots, x_{id}\}$ be a set of variables, and let $\tilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ be a polynomial ring in the union of all these variables. Denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of S.

Denote by $\Delta^{\mathbb{Z}}(n, d) = d\Delta^{n-1} \cap \mathbb{Z}^n$ the set of lattice points of the dilated $(n-1)$-simplex $d\Delta^{n-1}$, i.e., the set of tuples $\mathbf{a} = (a_1, \dots, a_n)$ of non-negative integers with $\sum_i^n a_i = d$. Denote by $\mathcal{T}(n, d)$ the one-skeleton of the hypersimplicial complex $\mathcal{H}_n^d$ from Definition 6.3.2.

Let G be a subgraph of $\mathcal{T}(n, d)$. Let $\mathcal{P}_i = (\Delta^{\mathbb{Z}}(n, d), \geq_i)$ be the poset with ground set $\Delta^{\mathbb{Z}}(n, d)$ and partial order $\geq_i$ from Definition 2.4.8, and let $\mathcal{B}_d$ be the Boolean poset on $[d]$. Let $\{X_i\}_{i \in [n]}$ be a set of isotone maps

$$X_i : \mathcal{P}_i \to \mathcal{B}_d$$

such that rank k elements of $\mathcal{P}_i$ map to rank k elements of $\mathcal{B}_d$ and the following property holds:

($*$) $(\mathbf{c}; i, j)$ is an edge in G if and only if $X_p(\mathbf{c} - e_i) = X_p(\mathbf{c} - e_j)$ for all $p \neq i, j$.

Let P be a poset and let $\mathbf{a}$ and $\mathbf{b}$ be elements in the ground set of P. The notation $\mathbf{a} \gtrdot \mathbf{b}$ means that $\mathbf{a}$ covers $\mathbf{b}$ in P. In this case, we will frequently say that $\mathbf{a}$ is a *parent* of $\mathbf{b}$ and that $\mathbf{b}$ is a *child* of $\mathbf{a}$.

For any $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, let $\mathrm{LS}(\mathbf{c})$ be the set of linear syzygy edges in the complete down-graph $D(\mathbf{c})$. For any $\mathbf{b} \in \Delta^{\mathbb{Z}}(n, d-1)$, denote by $\mathrm{LS}_i(\mathbf{b})$ the set of linear syzygy edges in the induced subgraph of $U(\mathbf{b})$ on the vertex set $\{\mathbf{b} + e_j \mid j \neq i\}$.

As the following example shows, not all subgraphs $G \subset \mathcal{T}(n,d)$ give rise to a well-defined set of isotone maps $\{X_i\}_{i\in[n]}$ satisfying $(*)$, even if the graph corresponds to the graph of syzygies among the generators of $\mathfrak{m}^d$ appearing in a minimal free resolution of $\mathfrak{m}^d$.

Example 5.0.2. Let $I = (x_1, x_2, x_3, x_4)^2$ in $S = k[x_1, x_2, x_3, x_4]$ where k is any field. Let $\mathbb{F}$ be the L-complex (see Chapter 6) of Definition 6.2.3 resolving I, where $\psi : F \to S$ maps the basis element f_i to the variable x_i. The basis elements of $L^1_2(F)$ correspond to elements of the form $(f_{a_1} \wedge f_{a_2}) \otimes (f_{b_1})$ such that $a_1 < a_2$ and $a_1 \leq b_1$, as described in Proposition 6.2.6. Let $G_{\mathbb{F}}$ correspond to graph of linear syzygies in the presenting step of $\mathbb{F}$. Then there are no linear syzygy edges between any of the vertices corresponding to the vertices $(1,1,0,0)$, $(1,0,1,0)$ or $(1,0,0,1)$. Now, suppose there exists an isotone map $X_1 : \mathcal{P}_1 \to \mathcal{B}_2$ satisfying $(*)$. Then $X_1(1,1,0,0) \neq X_1(1,0,1,0) \neq X_1(1,0,0,1)$, but all must be equal to rank 1 elements of $\mathcal{B}_2$. However, $\mathcal{B}_2$ has only 2 elements of rank 1, so this is not possible. Therefore, there is no valid set of isotone maps X_i satisfying $(*)$ induced by $G_{\mathbb{F}}$.

The example above gives key insights into what properties are necessary for a graph of linear syzygies to induce isotone maps. Before we give these conditions, we establish some notation which will be used throughout the remainder of this chapter.

Notation 5.0.3. For any $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, let $\mathrm{LS}(\mathbf{c})$ be the set of linear syzygy edges in the complete down-graph $D(\mathbf{c})$. For any $\mathbf{b} \in \Delta^{\mathbb{Z}}(n, d-1)$, denote by $\mathrm{LS}_i(\mathbf{b})$ the set of linear syzygy edges in the induced subgraph of $U(\mathbf{b})$ on the vertex set $\{\mathbf{b} + \epsilon_j \mid j \neq i\}$.

In the following theorem, we give a complete list of necessary and sufficient conditions for a graph of linear syzygies to give rise to a valid set of isotone maps satisfying ($*$) from Setup 5.0.1. Showing necessity is a straightforward verification. Sufficiency follows from the explicit construction of these isotone maps in Construction 5.0.11.

Theorem 5.0.4. *Adopt notation and hypotheses of Setup 5.0.1. If X_i is any map with source $\mathcal{P}_i$ satisfying ($*$), then X_i is well-defined if and only if G satisfies the following properties:*

(G1) For any $\mathbf{b} \in \mathcal{P}_i$ of rank k, there are at most k connected components of $\mathrm{LS}_i(\mathbf{b} - e_i)$ (see Notation 5.0.3), and each is a complete graph.

(G2) Any element of $\mathcal{P}_i$ has at most two parents which are not adjacent in G who share a parent.

(G3) Suppose that $\mathbf{a}$ and $\mathbf{b}$ both cover elements $\mathbf{f}$ and $\mathbf{g}$ such that $\mathbf{f}$ and $\mathbf{g}$ are in two distinct connected components of $\mathrm{LS}_i(\mathbf{c} - \epsilon_i)$ for some $\mathbf{c}$. Then $\mathbf{a}$ and $\mathbf{b}$ must be adjacent. Similarly, if $\mathbf{a}$ and $\mathbf{b}$ are covered by elements $\mathbf{f}$ and $\mathbf{g}$ such that $\mathbf{f}$ and $\mathbf{g}$ are in two different connected components of $\mathrm{LS}_i(\mathbf{c} - \epsilon_i)$ for some $\mathbf{c}$, then $\mathbf{a}$ and $\mathbf{b}$ are adjacent.

(G4) Suppose $\mathbf{f}$ and $\mathbf{g}$ are adjacent in G. Let $\mathbf{f}$ be a parent of elements $\mathbf{a}$ and $\mathbf{d}$ in $\mathcal{P}_i$ and let $\mathbf{g}$ be a parent of elements $\mathbf{b}$ and $\mathbf{d}$. If $\mathbf{d}$ and $\mathbf{a}$ are adjacent in G and $\mathbf{a}$ and $\mathbf{b}$ are adjacent in G, then $\mathbf{d}$ must also be adjacent to $\mathbf{b}$ in G. See Figure 5.1 for a .

In Figure 5.1, we give a visual depiction of the conditon (G4). Thick edges correspond to edges in the graph G and thin edges correspond to covering relations in the poset $\mathcal{P}_i$.

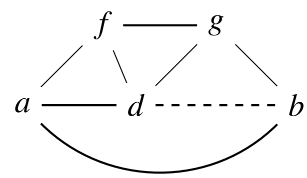


Figure 5.1: A depiction of condition (G4).

Proof. The existence of X_i given conditions (G1)-(G4) follows from Construction 5.0.11. Now suppose that G does not satisfy any of conditions (G1)-(G4) in Theorem 5.0.4.

(G1) Let $\mathbf{a}$ be an element of rank k in $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$. It has at most $n-1$ children. In $\mathcal{B}_d$, any element of rank k has $\binom{k}{k-1} = k$ children. If $\mathrm{LS}_i(\mathbf{a} - e_i)$ has more than k components, then each component must map to a distinct descendant of a rank k element of $\mathcal{B}_d$, which is not possible. Moreover, if any of these connected components are not a complete graph, then there are two vertices $\mathbf{a}$ and $\mathbf{b}$ which are not adjacent but map to the same element of $\mathcal{B}_d$, contradicting the assumption that X_i satisfies $(*)$.

(G2) Suppose $\mathbf{a}$ has three non-adjacent parents $\mathbf{b}_1, \mathbf{b}_1, \mathbf{b}_3 \in \mathcal{P}_i$ of rank k. If $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$ all share a parent, then $|X_i(\mathbf{b}_1) \cap X_i(\mathbf{b}_2) \cap X_i(\mathbf{b}_3)| = k-2$. But $X_i(\mathbf{a}) = X_i(\mathbf{b}_1) \cap X_i(\mathbf{b}_2) \cap X_i(\mathbf{b}_3)$, so it cannot map to a rank k element of $\mathcal{B}_d$, a contradiction.

(G3) Suppose $\mathbf{a}$ and $\mathbf{b}$ are rank $k > 0$ elements of $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$ and $\mathbf{a}$ is not adjacent to $\mathbf{b}$. Then $X_i(\mathbf{a}) \cup X_i(\mathbf{b})$ is a rank $k+1$ element of $\mathcal{B}_d$. In particular, there is exactly one possible image under the map X_i for a parent of both $\mathbf{a}$ and $\mathbf{b}$. Similarly, $X_i(\mathbf{a}) \cap X_i(\mathbf{b})$ is a rank $k-1$ element of $\mathcal{B}_d$, which must correspond to the image of any child of $\mathbf{a}$ and $\mathbf{b}$ under X_i.

(G4) Suppose $\mathbf{a}$ is adjacent to $\mathbf{d}$ and $\mathbf{b}$. If X_i satisfies condition $(*)$, then $X_i(\mathbf{a}) = X_i(\mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are adjacent in G.

□

The following lemma tells us that the "boundary edges" of the dilated simplex must be contained in the graph of linear syzygies in order for it to give rise to a set of isotone maps.

Lemma 5.0.5. *Adopt notation and hypotheses of Setup 5.0.1, and assume G satisfies conditions (G1)-(G4) in Theorem 5.0.4. Then for all $0 \leq k \leq d$, the edge $(ke_i + (d-k+1)e_j\,; i, j)$ is in G.*

Proof. This follows from property (G1) in Theorem 5.0.4. □

It turns out that this property is both necessary and sufficient in the case where $n = 3$ for isotone maps X_i satisfying $(*)$ to exist.

Observation 5.0.6. It is straightforward to verify that if $n = 3$, any graph of linear syzygies such that the only edges removed from $\mathrm{sk}_1(\mathcal{H}_3^d)$ are those contained in maximal down-triangles satisfies conditions (G1)-(G4) above.

We now partition $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$ into a set of chains we call $C^{\mathbf{p}}$ such that for all $\mathbf{a} \in C^{\mathbf{p}}$, one has that $a_k = p_k$ for all $k \neq i, j$ (where $j = 2$ if $i = 1$, and $j = 1$ for all other i). In particular, every element of $\Delta^{\mathbb{Z}}(n,d)$ is contained in a unique $C^{\mathbf{p}}$. All of these chains can be extended in a natural way to maximal chains which we call $\overline{C^p}$. We also define a total ordering $\prec$ on the chains $C^{\mathbf{p}}$ and $\overline{C}^{\mathbf{p}}$ which will later be used to construct the desired X_i maps in Construction 5.0.11.

Notation 5.0.7. If C is any chain in $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$, denote by C_k the unique rank k element of C, if it exists. For simplicity, if $i = 1$, set $j = 2$; for $i \neq 1$, set $j = 1$. Denote by $\mathbb{0} = (0, \ldots, 0) \in \mathbb{N}^n$. For any $\mathbf{p} \in \mathbb{N}^n$, let $|\mathbf{p}| = p_1 + p_2 + \cdots + p_n$.

Definition 5.0.8 (Chains $C^{\mathbf{p}}$). Let $\mathbf{p} \in \mathbb{N}^n$ with $i, j \notin \mathrm{Supp}(\mathbf{p})$ and $|\mathbf{p}| = r \leq d$. Define $C^{\mathbf{p}}$ to be the $(d-r)$-chain such that $C^{\mathbf{p}}_{k-r} = C^{\mathbb{0}}_k - re_i + \mathbf{p}$. In particular, every $\mathbf{a} \in C^{\mathbf{p}}$ satisfies that $a_k = p_k$ for all $k \neq i, j$.

For any $C^{\mathbf{p}}$ with $|\mathbf{p}| = r$, where $1 \leq r \leq d$, set $m := \min\{\ell \mid \ell \in \mathrm{Supp}(\mathbf{p})\}$. Let $\bar{C}^{\mathbf{p}}$ denote the maximal chain from extending $C^{\mathbf{p}}$ such that $\bar{C}^{\mathbf{p}}_k = \bar{C}^{\mathbf{p}-e_m}$ for all $k > d - r$.

If $\mathbf{p}, \mathbf{q} \in \mathbb{N}^n$ with $i, j \notin \mathrm{Supp}(\mathbf{p} + \mathbf{q})$ and $|\mathbf{p}|, |\mathbf{q}| \leq d$, define the total order $\prec$ on the chains $C^{\mathbf{p}}$ where $C^{\mathbf{p}} \prec C^{\mathbf{q}}$ if $|\mathbf{p}| < |\mathbf{q}|$ or $|\mathbf{p}| = |\mathbf{q}|$ and $p_\ell < q_\ell$ for the first ℓ where they differ. Similarly, define $\overline{C}^{\mathbf{p}} \prec \overline{C}^{\mathbf{q}}$ if $C^{\mathbf{p}} \prec C^{\mathbf{q}}$.

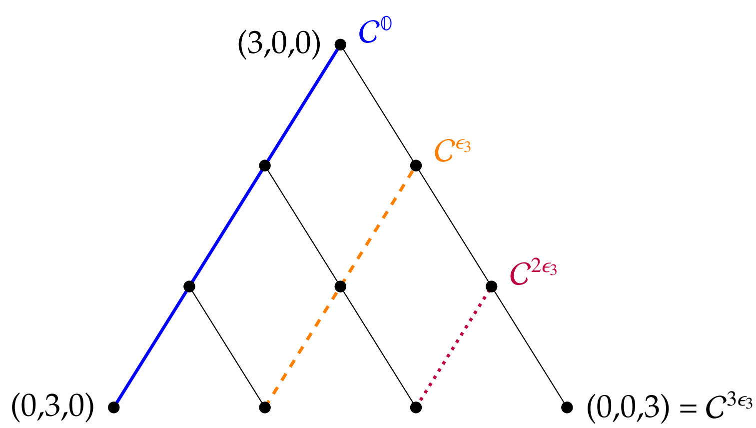


Figure 5.2: Chains of Definition 5.0.8 for $(\Delta^{\mathbb{Z}}(3,3), \geq_1)$.

Example 5.0.9. Figures 5.2 and 5.3 give labeled chains for $\Delta^{\mathbb{Z}}(3,3)$ and $\Delta^{\mathbb{Z}}(4,3)$, respectively. Observe that if $\mathbf{a} = de_k$ for some $k \neq i, j$, then $\mathbf{a}$ is contained in the unique one-element chain C^{de_k}. In Figure 5.3, the chains are ordered as follows:

$$C^{\mathbb{0}} \prec C^{e_3} \prec C^{e_4} \prec C^{2e_3} \prec C^{e_3+e_4} \prec C^{2e_4} \prec C^{3e_3} \prec C^{3e_4}.$$

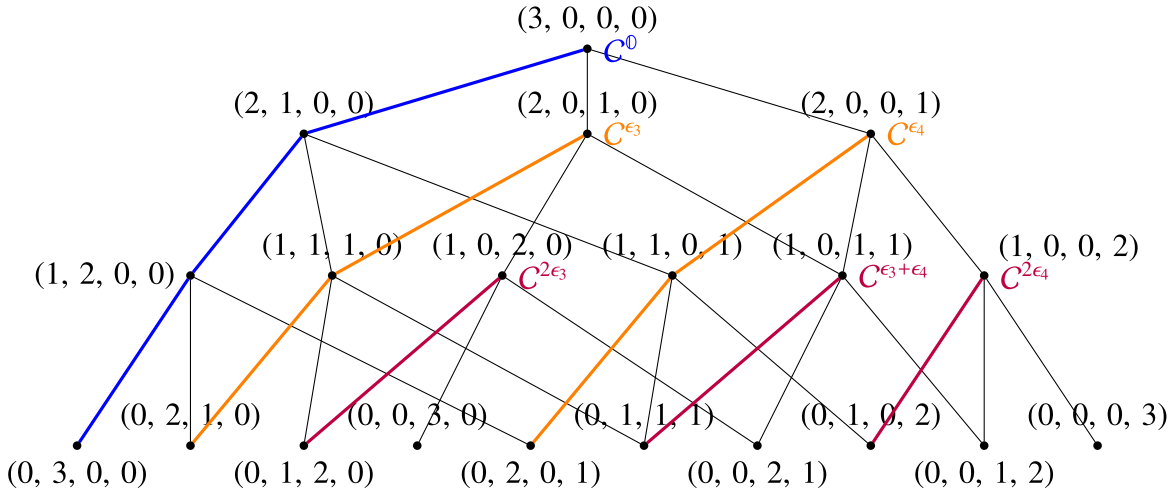


Figure 5.3: Chains for $(\Delta^{\mathbb{Z}}(4,3), \geq_1)$.

Notation 5.0.10. Elements of the symmetric group $\mathfrak{S}_d$ correspond bijectively to chains of $\mathcal{B}_d$ via

$$\sigma \mapsto (\emptyset \subset \{\sigma_1\} \subset \{\sigma_1, \sigma_2\} \subset \cdots \subset \{\sigma_1, \ldots, \sigma_d\}). \tag{5.1}$$

One may also view σ as a word in $[d]$. Set $\sigma|_\ell$ to be the subword $(\sigma_1, \ldots, \sigma_\ell)$ of σ. Then $\sigma|_\ell$ corresponds to a chain of length ℓ in $\mathcal{B}_d$ via

$$\sigma|_\ell \mapsto (\emptyset \subset \{\sigma_1\} \subset \{\sigma_1, \sigma_2\} \subset \cdots \subset \{\sigma_1, \ldots, \sigma_\ell\}).$$

For any two words τ, σ, say that $\tau \subseteq \sigma$ if for all i, there exists some j such that $\tau_i = \sigma_j$.

When referring to the image of $\mathbf{a} \in (\Delta^{\mathbb{Z}}(n,d), \geq_i)$ under a map X_i, say that $\mathbf{a}$ is *labeled by* the element $X_i(\mathbf{a}) \in \mathcal{B}_d$. Given a chain C of $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$, say that C is *labeled by* $\sigma \in \mathfrak{S}_d$ if $X_i(C_k) = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$.

The following construction generalizes work of Lohne in [30]. The idea is to label the maximal chains $\overline{C^{\mathbf{p}}}$ in the order $\prec$ presented in Definition 5.0.8, starting at the bottom of each chain. Every time an element of $\mathbf{a} \in C^{\mathbf{p}}$ is not adjacent to the previously labeled element of the same rank in a specified earlier chain, one must apply a transposition to the word that labeled that chain.

Construction 5.0.11. Adopt notation and hypotheses of Setup 5.0.1, and assume G satisfies conditions (G1)-(G4) of Theorem 5.0.4. Begin by labeling the chain $C^{\mathbb{0}}$ by the word $\sigma^{\mathbb{0}} = (12\ldots d) \in \mathfrak{S}_d$. Label all chains $C^{\mathbf{p}}$ in increasing order with respect to $\prec$ in Definition 5.0.8 as follows. For any $C^{\mathbf{p}}$ with $|\mathbf{p}| = r$, where $1 \le r \le d$, set $m := \min\{\ell \mid \ell \in \operatorname{Supp}(\mathbf{p})\}$. Starting at $k = 0$, apply the transposition $(k, k+1)$ to $\sigma^{\mathbf{p}-e_m}$ for every k such that $C_k^{\mathbf{p}}$ is not adjacent to $C_k^{\mathbf{p}-e_m}$. Let $\sigma^{\mathbf{p}}$ be the resulting word in $\mathfrak{S}_d$ after applying all required transpositions, and label the chain $\overline{C}^{\mathbf{p}}$ by the word $\sigma^{\mathbf{p}}$.

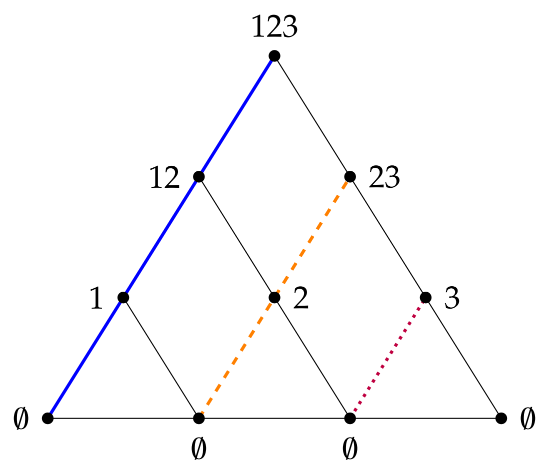


Figure 5.4: Output of Construction 5.0.11 for a graph of linear syzygies.

Example 5.0.12. We apply the algorithm in Construction 5.0.11 to the graph of linear syzygies in Figure 5.4 for $(x, y, z)^3$, which is a subgraph of the one-skeleton of $\mathcal{H}_n^d$ in Figure 2.1. The chains in $(\Delta^{\mathbb{Z}}(3,3), \ge_1)$ are labeled in Figure 5.2. First, set $\sigma^{\mathbb{0}} = (123)$ by construction. To find σ^{e_3}, observe that the rank 1 and rank 2 elements of C^{e_3} are not adjacent to the rank 1 and 2 elements of $C^{\mathbb{0}}$. So $\sigma^{e_3} = (23)(12)\sigma^{\mathbb{0}} = (23)(12)(123) = (23)(213) = (231)$. Similarly, to compute σ^{2e_3}, observe that the rank 1 element of C^{2e_3} is not adjacent to the rank 1 element of C^{e_3}, so $\sigma^{2e_3} = (12)(231) = (321)$.

Remark 5.0.13. Let $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d)$. Then $\mathbf{a}$ appears in the unique chain $C^{\mathbf{p}}$ where $|\mathbf{p}| = d - a_i - a_j$ and $\mathbf{p} = \sum_{\ell \neq i,j} a_\ell e_\ell$. Moreover, observe that $\sigma_k^{\mathbf{p}} = \sigma_k^{\mathbf{p}-e_m}$ for all $k > d - |\mathbf{p}|$ by construction. In particular, every $\mathbf{a} \in (\Delta^{\mathbb{Z}}(n,d), \geq_i)$ is given a unique labeling by Construction 5.0.11, so it is well-defined.

We conclude this section by checking that the rank-preserving maps X_i produced by Construction 5.0.11 are indeed isotone maps and satisfy $(*)$ from Setup 5.0.1 if G satisfies condition (G1)-(G4) in Theorem 5.0.4. The proof follows from a double induction on the chains $C^{\mathbf{p}}$ (following the order $\prec$ from Definition 5.0.8) as well as the rank k on $C^{\mathbf{p}}$.

Proposition 5.0.14. *Adopt notation and hypotheses of Setup 5.0.1, and assume G satisfies conditions (G1)-(G4) in Theorem 5.0.4. Let $\{X_i\}_{i\in[n]}$ be a set of maps from $\Delta^{\mathbb{Z}}(n,d)$ to $\mathcal{B}_d$ as in Construction 5.0.11. Then*

(i) every map $X_i : (\Delta^{\mathbb{Z}}(n,d), \geq_i) \to \mathcal{B}_d$ is isotone, and

(ii) $(\mathbf{c}; s,t)$ is an edge in G if and only if $X_i(\mathbf{c} - e_s) = X_i(\mathbf{c} - e_t)$ for all $i \neq s,t$.

Proof. Proceed by induction on the chains $C^{r,\mathbf{p}}$ and the rank k of the poset. For ease of notation, assume that $i = 1$.

Base case. Construction 5.0.11 labels the chain $C^{\mathbf{0}}$ in an isotone manner by construction. The next chain labeled by the algorithm is $\overline{C}^{e_3}$. By construction, every element $\mathbf{a} \in C^{e_3}$ will satisfy $X_1(\mathbf{a}) = X_1(\mathbf{a} - e_3 + e_2)$ if and only if $\mathbf{a}$ is adjacent to $\mathbf{a} - e_3 + e_2$, and $X_1(\mathbf{a}) \subset X_1(\mathbf{a} + e_1 - e_3)$ by construction, as well.

For any other chain $C^{\mathbf{p}}$, Construction 5.0.11 will always label the bottom element $C_0^{\mathbf{p}}$ with the empty set, so its image under X_1 will always be less than the image of any element above it in $\mathcal{P}_1$ under X_1. Moreover, by Lemma 5.0.5, every edge on the boundary of $\mathcal{T}(n,d)$ is in G, so all elements of rank 0 also satisfy condition (ii).

Induction hypothesis. Assume that all chains preceding $\overrightarrow{C}^{\mathbf{p}}$ have been labeled by Construction 5.0.11, and that all elements in these preceding chains satisfy (i) and (ii) with respect to other elements in chains preceding $\overrightarrow{C}^{\mathbf{p}}$. Moreover, for any $\mathbf{a}' = \overrightarrow{C}_\ell^{\mathbf{p}}$ with $\ell < k$, assume $\mathbf{a}'$ satisfies properties (i) and (ii) with respect to other elements in $\mathcal{P}_1$ in earlier chains.

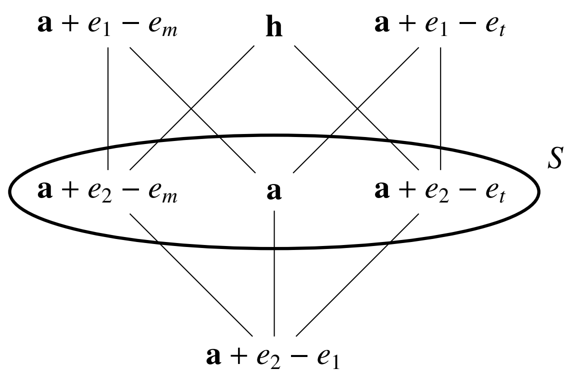


Figure 5.5: The situation in the proof of (i) of Proposition 5.0.14.

Induction Step. Let $\mathbf{a} = \overrightarrow{C}_k^{\mathbf{p}} = \mathbf{c} - e_s$ for $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ and some $s \in [n]$. Set $m := \min\{\ell \mid \ell \in \mathrm{Supp}(\mathbf{a}) \text{ and } \ell \neq 1, 2\}$.

Proof of (i). It suffices to check (i) in the case when $\mathbf{d} = \mathbf{a} + e_1 - e_t$ is a cover of $\mathbf{a}$ in $\mathcal{P}_1$, where $t \in \mathrm{Supp}(\mathbf{a})$. This situation is depicted in Figure 5.5. If $t = 2$ or m, then $X_1(\mathbf{a}) < X_1(\mathbf{a} + e_1 - e_t)$ by construction, so assume $t \neq 2, m$. Observe that $\mathbf{a} + e_1 - e_m$ also covers $\mathbf{a}$ and shares a parent with $\mathbf{a}+e_1-e_t$, namely, $\mathbf{a}+2e_1-e_m-e_t$. If $\mathbf{a}+e_1-e_t$ is adjacent to $\mathbf{a}+e_1-e_m$, then by the induction hypothesis $X_1(\mathbf{a}+e_1-e_t) = X_1(\mathbf{a}+e_1-e_m)$ and so $X_i(\mathbf{a}) < X_i(\mathbf{a} + e_1 - e_t)$.

Assume $\mathbf{a}+e_1-e_t$ is not adjacent to $\mathbf{a}+e_1-e_m$. The element $\mathbf{a}+e_1-e_t$ also covers $\mathbf{a} - e_t + e_2$, and the element $\mathbf{a} + e_1 - e_m$ also covers $\mathbf{a} + e_2 - e_m$. Moreover, $\mathbf{a} + e_2 - e_m$ and $\mathbf{a} + e_2 - e_t$ are both covered by $\mathbf{h} = \mathbf{a} - e_t + e_j - e_m + e_1$. All of the elements in the set $S = \{\mathbf{a}, \mathbf{a} + e_2 - e_m, \mathbf{a} + e_2 - e_t\}$ cover the element $\mathbf{a} - e_1 + e_2$; so by Condition (G2) in Theorem 5.0.4, at least two elements of S must be adjacent. Because of the assumption that $\mathbf{a} + e_1 - e_t$ is not adjacent to $\mathbf{a} + e_1 - e_m$, one has by Condition (G3) that if any pair of element of S are adjacent in G, then all three must be pairwise adjacent. By construction, $X_1(\mathbf{a}) = X_1(\mathbf{a} + e_2 - e_m)$ if and only if they are adjacent in G; and by the induction hypotheses, $X_1(\mathbf{a} + e_2 - e_m) = X_1(\mathbf{a} + e_2 - e_t)$ if and only if they are adjacent in G. Because all three are adjacent, one has $X_1(\mathbf{a}) = X_1(\mathbf{a} + e_2 - e_t) < X_1(\mathbf{a} + e_1 - e_t)$, giving the desired result.

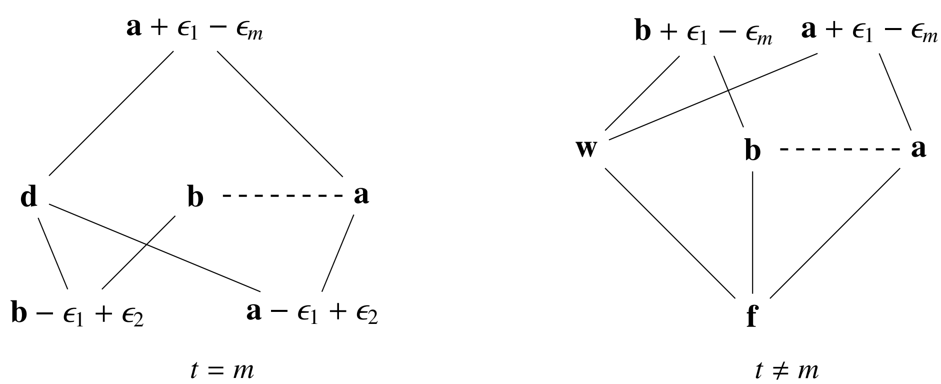


Figure 5.7: Situations in the proof of (ii) of Proposition 5.0.14.

Proof of (ii). It suffices to check (ii) in the case when $\mathbf{b} = \mathbf{c} - e_t$ is in a chain preceding $C^{\mathbf{p}}$. There are two possible cases to check: when $t = m$ (see Figure 5.6a), and when $t \neq m$ (see Figure 5.6b).

- $\mathbf{t} = \mathbf{m}, (\implies)$: Assume $\mathbf{a}$ is adjacent to $\mathbf{b}$. Let $\mathbf{d} = \mathbf{a} - e_m + e_2$. All three elements $\mathbf{a}, \mathbf{b}$, and $\mathbf{d}$ are children of the element $\mathbf{a} + e_1 - e_m$. If $\mathbf{d}$ is adjacent to either $\mathbf{a}$ or $\mathbf{b}$, then all three must be adjacent by (G1). By construction, $X_1(\mathbf{a}) = X_1(\mathbf{d})$, and by induction, $X_1(\mathbf{d}) = X_1(\mathbf{b})$; so $X_1(\mathbf{a}) = X_1(\mathbf{b})$, as desired.

  Now suppose that $\mathbf{d}$ is adjacent to neither $\mathbf{a}$ nor $\mathbf{b}$. The element $\mathbf{a} - e_1 + e_2$ is covered by both $\mathbf{a}$ and $\mathbf{d}$, and the element $\mathbf{b} - e_1 + e_2$ is covered by both $\mathbf{b}$ and $\mathbf{d}$. Therefore, $\mathbf{a} - e_1 + e_2$ must be adjacent to $\mathbf{b} - e_1 + e_2$ by Condition (G3) in Theorem 5.0.4. By the induction hypothesis, $X_1(\mathbf{a} - e_1 + e_2) = X_1(\mathbf{b} - e_1 + e_2)$. In addition, $X_1(\mathbf{a}), X_1(\mathbf{b})$, and $X_1(\mathbf{d})$ must all contain $X_1(\mathbf{a} - e_1 + e_2)$ and be contained in $X_1(\mathbf{a} + e_1 - e_m)$ by induction (for $\mathbf{b}$ and $\mathbf{d}$) and by construction (for $\mathbf{a}$), so there are only two possible options for labels for $\mathbf{a}, \mathbf{b}$, and $\mathbf{d}$. Since $X_1(\mathbf{d}) \neq X_1(\mathbf{a})$ by construction and $X_1(\mathbf{d}) \neq X_1(\mathbf{b})$ by induction, one has that $X_1(\mathbf{a}) = X_1(\mathbf{b})$, as required.

- $\mathbf{t} = \mathbf{m}, (\impliedby)$: Suppose $\mathbf{a}$ is not adjacent to $\mathbf{b}$. By Conditon (G1), at most one of the pair $(\mathbf{a}, \mathbf{b})$ may be adjacent to $\mathbf{d}$. If one is connected to $\mathbf{d}$ and the other is not, then $X_1(\mathbf{a}) \neq X_1(\mathbf{b})$ by construction. Now suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is connected to $\mathbf{d}$. Then $\mathbf{b} - e_1 + e_2$ is not adjacent to $\mathbf{a} - e_1 + e_2$, since otherwise G would violate Condition ((G2)) in Theorem 5.0.4. By construction, $X_1(\mathbf{a}) \cap X_1(\mathbf{d}) = X_1(\mathbf{a} - e_1 + e_2)$, and by induction $X_1(\mathbf{b}) \cap X_1(\mathbf{d}) = X_1(\mathbf{b} - e_1 + e_2)$. Since $X_1(\mathbf{a} - e_1 + e_2) \neq X_1(\mathbf{b} - e_1 + e_2)$ by induction, one has $X_1(\mathbf{a}) \neq X_1(\mathbf{b})$, as desired.

- $\mathbf{t} \neq \mathbf{m}, (\implies)$: Assume $\mathbf{a}$ and $\mathbf{b}$ are adjacent. Observe that if $t \neq m$, then $m \in \mathrm{Supp}(\mathbf{b})$. Consider the element $\mathbf{w} = \mathbf{a} - e_m + e_s = \mathbf{b} - e_m + e_t$. If $\mathbf{w}$ is adjacent to either $\mathbf{a}$ or $\mathbf{b}$, then all three are connected by Conditoin (G1). Now $\mathbf{a}$ and $\mathbf{w}$ fall into the case where $t = m$, so $X_1(\mathbf{a}) = X_1(\mathbf{w})$ and $X_1(\mathbf{w}) = X_1(\mathbf{b})$ by induction, so $X_1(\mathbf{a}) = X_1(\mathbf{b})$.

Now assume that $\mathbf{w}$ is adjacent to neither $\mathbf{a}$ nor $\mathbf{b}$. By Condition (G3), $\mathbf{b} + e_1 - e_m$ must be adjacent to $\mathbf{a} + e_1 - e_m$. Moreover, all of $\mathbf{a}, \mathbf{b}$, and $\mathbf{w}$ cover the element $\mathbf{f} = \mathbf{w} - e_1 + e_m = \mathbf{b} - e_1 + e_t = \mathbf{a} - e_1 + e_s$. By isotonicity of X_1, one has that $X_1(\mathbf{w}) \cap X_1(\mathbf{b}) = X_1(\mathbf{f})$ and $X_1(\mathbf{w}) \cap X_1(\mathbf{a}) = X_1(\mathbf{f})$, so $X_1(\mathbf{a}) = X_1(\mathbf{b})$ as desired.

- $\mathbf{t} \neq \mathbf{m}$, $(\Longleftarrow)$: Assume $\mathbf{a}$ is not adjacent to $\mathbf{b}$. Again consider the element $\mathbf{w} = \mathbf{a} - e_m + e_s = \mathbf{b} - e_m + e_t$. Notice that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{w}$ all share a child $\mathbf{f} = \mathbf{w} - e_1 + e_m = \mathbf{b} - e_1 + e_t = \mathbf{a} - e_1 + e_s$, so by Condition (G2), at least one pair among the three must be adjacent in G. If either $\mathbf{a}$ or $\mathbf{b}$ is adjacent to $\mathbf{w}$, the other cannot be by condition (G1). Combining the induction hypothesis and the argument for the case when $t = m$ above gives the desired result.

$\square$

Part II

Polarizations and Hook Partitions

CHAPTER 6

THE L-COMPLEX IS CELLULAR

6.1 Frames and discrete Morse theory for cellular resolutions

In this section, we recall some important notions on cellular resolutions and Discrete Morse theory for cellular resolutions. For further exposition on frames and cellular resolutions, we refer the reader to [41] and [42]. We will use the terminology of frames to give a convenient framework (pun intended) for defining cellular resolutions. Proposition 6.1.10 will be essential for proving that the L-complexes of Buchsbaum and Eisenbud are cellular. We begin by adopting the following setup:

Setup 6.1.1. Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let M be a monomial ideal in S minimally generated by monomial $m_1, \dots, m_r$. Let L_M denote the set of least common multiples of subsets of $m_1, \dots, m_r$. By convention, $1 \in L_M$ is considered to be the lcm of the empty set.

Definition 6.1.2 (Frame)**.** Adopt notation and hypotheses of Setup 6.1.1. A *frame* (or an r-frame) $\mathbb{U}$ is a complex of finite k-vector spaces with differential ∂ and a fixed basis that satisfies the following conditions:

(1) $U_i = 0$ for $i < 0$ and $i \gg 0$,

(2) $U_0 = k$

(3) $U_1 = k^r$

(4) $\partial(w_j) = 1$ for each basis vector w_j in $U_1 = k^r$.

Given a complex of free modules over some polynomial ring, it is easy to obtain a frame by setting all variables equal to 1. Conversely, given a frame $\mathbb{U}$, one may construct a multigraded complex $\mathbb{G}$ of finitely generated free multigraded S-modules with multidegrees in L_M using the following construction due to Peeva and Velasco [42].

Construction 6.1.3 (Homogenization)**.** Adopt notation and hypotheses of Setup 6.1.1. Let $\mathbb{U}$ be an r-frame. Set

$$G_0 = S \text{ and } G_1 = S(-m_1) \oplus \cdots \oplus S(-m_r).$$

Let $\overline{v}_1, \ldots, \overline{v}_p$ and $\overline{u}_1, \ldots, \overline{u}_q$ be the given bases of U_i and U_{i-1}, respectively. Let $u_1, \ldots, u_q$ be the basis of $G_{i-1} = S^q$ chosen on the previous step of the induction. Introduce $v_1, \ldots, v_p$ that will be a basis of $G_i = S^p$. If

$$\partial(\overline{v}_j) = \sum_{1 \le s \le q} \alpha_{sj} \overline{u}_s$$

with coefficients $\alpha_{sj} \in k$, then set

$$\begin{aligned} \operatorname{mdeg}(v_j) &= \operatorname{lcm}(\operatorname{mdeg}(u_s) \mid \alpha_{sj} \neq 0) \\ G_i &= \bigoplus_{1 \le j \le p} S(-\operatorname{mdeg}(v_j)) \\ d(v_j) &= \sum_{1 \le s \le q} \alpha_{sj} \frac{\operatorname{mdeg}(v_j)}{\operatorname{mdeg}(u_s)} \cdot u_s. \end{aligned}$$

Clearly $\operatorname{coker}(d_1) = S/M$ and the differential d is homogeneous by construction. Call $\mathbb{G}$ the *M-homogenization* of $\mathbb{U}$.

The following simple criterion by Peeva and Velasco [42] determines when a frame supports a graded free resolution of S/M. The abridged version of this result states that exactness can be checked by only considering multihomogeneous strands.

Proposition 6.1.4. *The sequence of modules and homomorphisms $\mathbb{G}$ as in Construction 6.1.3 is a complex. Moreover, if $\mathbb{G}(\leq m)$ is the subcomplex of $\mathbb{G}$ generated by the multihomogeneous basis elements of multidegrees dividing m, then $\mathbb{G}$ is a free multigraded resolution of S/M if and only if for all monomials $1 \neq m \in L_M$, the frame of the complex $G(\leq m)$ is exact.*

A natural source of frames that can be used to support resolutions of monomial ideals are provided by CW-complexes, since the conditions (1) – (4) of Definition 6.1.2 are trivially satisfied.

Notation 6.1.5. Let X be a regular CW-complex, and denote by $X^{(i)}$ the set of i-cells of X and by $X^{(*)} := \bigcup_{i\geq 0} X^{(i)}$ the set of all cells of X. Denote by $C(X;k)$ the augmented oriented cellular chain complex of X over k with

$$C(X;k)_i = \bigoplus_{c\in X^{(i)}} ke_c$$

where e_c denotes the basis element corresponding to the face $c \in X^{(i)}$, and the differential ∂ acts as

$$\partial(e_c) = \sum_{c\geq c'\in X^{(i-1)}} [c,c']e_{c'}$$

where $[c,c']$ is the coefficient in the differential of the cellular homology of X.

With the above notation in mind, we use the language of frames to define cellular resolutions.

Definition 6.1.6 (Cellular Resolution)**.** Adopt notation of Notation 6.1.5. Assume that $|X^{(0)}| = r$ and $M = (m_1, \dots, m_r)$ is a monomial ideal in a polynomial ring S. Label each 0-cell of X by a minimal generator m_i of M. After shifting $C(X;k)$ in homological degree, $C(X;k)[-1]$ is a frame. Denote by $\mathbb{F}_X$ the M-homogenization of $C(X;k)$ as in Construction 6.1.3. The complex $\mathbb{F}_X$ is *supported on X*. The complex $\mathbb{F}_X$ is a *cellular resolution* if it is exact.

Definition 6.1.7 (Face Multidegrees, Subcomplex $X_{\leq m}$)**.** Let $M = (m_1, \ldots, m_r)$ be a monomial ideal in a polynomial ring S, and let X be a regular CW-complex with 0-cells labeled by the generators of M. The multidegree of each vertex of X is given by its monomial label. Define a face c to have multidegree

$$\operatorname{mdeg}(c) = \operatorname{lcm}(m_i \mid m_i \in c).$$

By convention, $\operatorname{mdeg}(\emptyset) = 1$. Define the following subcomplexes of X:

$$X_{\leq m} := \{c \in X \mid \operatorname{mdeg}(c) \text{ divides } m\}$$
$$X_{<m} := \{c \in X \mid \operatorname{mdeg}(c) \text{ strictly divides } m\}.$$

The following Proposition is an immediate consequence of Proposition 6.1.4 combined with the notation and hypotheses introduced in Definition 6.1.7.

Proposition 6.1.8. *Let $M = (m_1, \ldots, m_r)$ be a monomial ideal in a polynomial ring S, and let X be a regular CW-complex with 0-cells labeled by the minimal generators of M. The complex $\mathbb{F}_X$ from Definition 6.1.6 is a free resolution of S/M if and only if for all multidegrees $1 \neq m \in L_M$, the complex $X_{\leq m}$ is acyclic over k.*

Next, we introduce some of the basic machinery of discrete Morse theory for cellular resolutions. Discrete Morse theory was developed by Forman in [17] to extend the ideas from Morse theory in differential geometry to CW complexes. The interested reader is encouraged to consult Forman's survey paper [18] for further reading on discrete Morse theory. The application of discrete Morse theory to the study of cellular resolutions was first explored by Batzies and Welker in [5] as a method of "cutting down" a large cellular resolution in such a manner that the resulting subcomplex is also a cellular resolution.

Construction 6.1.9 (Discrete Morse Function, Acyclic Matching, Critical Cells)**.** Adopt Notation 6.1.5. Let G_X be the directed graph on the set of cells of X whose set E_X of edges is given by $c \to c'$ for $c' \subseteq c$ and $\dim(c') = \dim(c) - 1$. A *discrete Morse function* arises from a set $A \subseteq E_X$ of edges in G_X satisfying:

1. each cell occurs in at most one edge of A, and
2. the graph G_X^A with edge set

$$E_X^A := (E_X \setminus A) \cup \{c' \to c \mid c \to c' \in A\}$$

   is acyclic (i.e., it does not contain a directed cycle).

Such a set $A \subseteq E_X$ is called an *acyclic matrching* of G_X. A cell of X is *A-critical* with respect to A if it is not contained in any edge of A. An acyclic matching is *homogeneous* if $c \to c' \in A$ implies that $\mathrm{mdeg}(c) = \mathrm{mdeg}(c')$.

The proof of the following proposition can be found in the appendix of [5], and shows that acyclic matchings can be used to induce acyclic subcomplexes that are also supported on cell complexes.

Proposition 6.1.10. *Let X be a regular CW-complex which supports a free resolution of a monomial ideal M, and let A be a homogeneous acyclic matching of G_X. Then there is a (not necessarily regular) CW-complex X_A whose i-cells are in one-to-one correspondence with the A-critical i-cells of X such that X_A is homotopy equivalent to X.*

Moreover, X_A inherits a multigrading from X, and for any multidegree α and restriction $A_{\leq\alpha}$ of A to $X_{\leq\alpha}$, one has

$$X_{\leq\alpha} \simeq (X_{\leq\alpha})_{A_{\leq\alpha}} \cong (X_A)_{\leq\alpha}.$$

In particular, X_A also supports a cellular resolution of the ideal M.

Definition 6.1.11 (Morse Complex)**.** The complex X_A of Proposition 6.1.10 is called the *Morse complex* of X for the matching A.

Remark 6.1.12. The explicit construction of the Morse complex X_A from an acyclic matching A is quite technical, and therefore is not included here. The interested reader is encouraged to consult the appendix of [5] for more details.

6.2 Background on L-complexes

The goal of this section is to introduce the L-complexes of Buchsbaum and Eisenbud and to make clear our conventions on Young tableaux. For further details on Schur modules and their use in the construction of free resolutions, one may consult Weyman's book [51].

The following notation will be used throughout this section and the rest of this chapter.

Notation 6.2.1. Let R be a polynomial ring over a field k. Let F be a free R-module of rank n with basis $f_1, \dots, f_n$. Denote by $S_d(F)$ the dth symmetric power of F, and by $\bigwedge^n F$ the nth exterior power of F. Let $J = \{j_1 < j_2 < \cdots < j_k\} \subset [n]$. Define

$$f_J := f_{j_1} \wedge \cdots \wedge f_{j_k} \in \bigwedge^k F.$$

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $\sum_i \alpha_i = d$, set

$$f^\alpha := f_1^{\alpha_1} f_2^{\alpha_2} \dots f_n^{\alpha_n} \in S_d(F).$$

Setup 6.2.2. Let F denote a free R-module of rank n, and $S = S(F)$ the symmetric algebra on F with the standard grading. Define a complex

$$\cdots \longrightarrow \bigwedge^{a+1} F \otimes_R S_{b-1} \xrightarrow{\kappa_{a+1,b-1}} \bigwedge^a F \otimes_R S_b \xrightarrow{\kappa_{a,b}} \cdots$$

where the maps $\kappa_{a,b}$ are defined as the composition

$$\begin{aligned}\bigwedge^{a} F \otimes_R S_b &\to \bigwedge^{a-1} F \otimes_R F \otimes_R S_b \\ &\to \bigwedge^{a-1} F \otimes_R S_{b+1}\end{aligned}$$

where the first map is comultiplication in the exterior algebra and the second map is the standard module action (where we identify $F = S_1(F)$). Define

$$L_b^a(F) := \ker \kappa_{a,b}.$$

Let $\psi : F \to R$ be a morphism of R-modules with $\mathrm{im}(\psi)$ an ideal of grade n. Let $\mathrm{kos}^{\psi} : \bigwedge^i F \to \bigwedge^{i-1} F$ denote the standard Koszul differential; that is, the composition

$$\begin{aligned}\bigwedge^{i} F &\to F \otimes_R \bigwedge^{i-1} F \quad \text{(comultiplication)} \\ &\xrightarrow{\psi\otimes 1} R \otimes_R \bigwedge^{i-1} F \cong \bigwedge^{i-1} F \quad \text{(module action).}\end{aligned}$$

Explicitly, if $J = \{j_1 < \cdots < j_k\}$, then

$$\mathrm{kos}^{\psi}(f_J) = \sum_{i\in[k]} (-1)^i \psi(f_{j_i}) \cdot f_{J\setminus j_i}.$$

Definition 6.2.3 (L-Complex)**.** Adopt notation and hypotheses of Setup 6.2.2. Define the complex

$$L(\psi, b): \ 0 \longrightarrow L_b^{n-1} \xrightarrow{\mathrm{kos}^{\psi}\otimes 1} \cdots \xrightarrow{\mathrm{kos}^{\psi}\otimes 1} L_b^0 \xrightarrow{S_b(\psi)} R \longrightarrow 0$$

where $\mathrm{kos}^{\psi} \otimes 1 : L_b^a(F) \to L_b^{a-1}$ is induced by making the following diagram commute:

$$\begin{tikzcd} \bigwedge^a F \otimes S_b(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] & \bigwedge^{a-1} F \otimes S_b(F) \\ L_b^a(F) \arrow[u] \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] & L_b^{a-1}(F) \arrow[u] \end{tikzcd}$$

The following Proposition shows that the L-complexes constitute a minimal free resolution of powers of complete intersections in general.

Proposition 6.2.4. *Let $\psi : F \to R$ be an R-module homomorphism from a free module F of rank n such that the image $\operatorname{im}(\psi)$ is an ideal of grade n. Then the complex $L(\psi, b)$ of Definition 6.2.3 is a minimal free resolution of $R/\operatorname{im}(\psi)^b$*

We also have (see Proposition 2.5(c) of [9], or just use Proposition 6.2.6)

$$\operatorname{rank}_R L_b^a(F) = \binom{n+b-1}{a+b}\binom{a+b-1}{a}.$$

Moreover, using the notation and language of Chapter 2 of [51], $L_b^a(F)$ is the Schur module $L_{(a+1,1^{b-1})}(F)$. This allows us to identify a standard basis for such modules.

Notation 6.2.5. We use the English convention for partition diagrams. That is, the partition $(3, 2, 2)$ corresponds to the diagram

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

A Young tableau is *standard* if it is strictly increasing in both the columns and rows. It is *semistandard* if it is strictly increasing in the columns and nondecreasing in the rows.

Proposition 6.2.6. *Adopt notation and hypotheses as in Setup 6.2.2. Then a basis for $L_b^a(F)$ is represented by all Young tableaux of the form*

$$\begin{array}{|c|c|c|c|}\hline i_0 & j_1 & \cdots & j_{b-1} \\ \hline i_1 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} i_a & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

with $i_0 < \cdots < i_a$ and $i_0 \leq j_1 \leq \cdots \leq j_{b-1}$.

Proof. See Proposition 2.1.4 of [51] for a more general statement. $\square$

Remark 6.2.7. Adopt notation and hypotheses of Setup 6.2.2. Let F have basis $f_1, \dots, f_n$. In the statement of Proposition 6.2.6, we think of a tableau as representing the element

$$\kappa_{a+1,b-1}(f_{i_0} \wedge \cdots \wedge f_{i_a} \otimes f_{j_1} \cdots f_{j_{b-1}}) \in \bigwedge^a F \otimes S_b(F).$$

We will often write $f_{i_1} \wedge \cdots \wedge f_{i_{a+1}} \otimes f_{j_1} \cdots f_{j_{b-1}} \in L_b^a(F)$, with the understanding that we are identifying $L_b^a(F)$ with the cokernel of $\kappa_{a+2,b-2} : \bigwedge^{a+2} F \otimes S_{b-2}(F) \to \bigwedge^{a+1} F \otimes S_{b-1}(F)$.

The following Observation is sometimes referred to as the *shuffling* or *straightening* relations satisfied by tableaux in the Schur module $L_b^a(F)$.

Observation 6.2.8. Any tableau of the form

$$T = \begin{array}{|c|c|c|c|}\hline i_0 & j_1 & \cdots & j_{b-1} \\ \hline i_1 \\ \cline{1-1} \vdots \\ \cline{1-1} i_a \\ \cline{1-1}\end{array}$$

with $j_1 \leq \cdots \leq j_{b-1}$ viewed as an element in $L_b^a(F)$ with $b \geq 2$ may be rewritten as a linear combination of other tableaux in the following way:

$$T = \sum_{k=0}^{a} (-1)^k \begin{array}{|c|c|c|c|c|}\hline j_1 & i_k & j_2 & \cdots & j_{b-1} \\ \hline i_0 \\ \cline{1-1} \vdots \\ \cline{1-1} \widehat{i_k} \\ \cline{1-1} \vdots \\ \cline{1-1} i_a \\ \cline{1-1}\end{array}.$$

Notice that if $i_0 > j_1$ and $i_0 < \cdots < i_a$, then this rewrites T as a linear combination of semistandard tableaux after reordering the row into ascending order.

6.3 The L-complex is cellular

In this section, we apply discrete Morse theory to the so-called *hypersimplex resolution* (see Definition 6.3.2) of $\mathfrak{m}^d$ in a novel way to obtain a CW-complex which supports the L-complex with the exact basis elements described in Proposition 6.2.6. In particular, Proposition 6.3.10 implies that the L-complex of Buchsbaum and Eisenbud is CW-cellular. While Batzies and Welker had previously obtained a minimal cellular resolution of $\mathfrak{m}^d$ by finding an acyclic matching on the hypersimplex resolution in [5], the minimal resolution they obtain is instead isomorphic to the Eliahou-Kervaire resolution of $\mathfrak{m}^d$.

Notation 6.3.1. The notation $\Delta(n,d)$ will denote the dilated $(n-1)$-simplex $d\cdot\Delta^{n-1}$; that is,

$$\Delta(n,d) = d\cdot\Delta^{n-1} := \left\{(y_1,\dots,y_n)\in\mathbb{R}^n \mid \sum_{i=1}^{n} y_i = d, y_i \geq 0 \text{ for } i=1,\dots,n\right\}.$$

Definition 6.3.2 (Hypersimplicial Complex)**.** Let $\mathcal{H}_n^d$ be the polytopal CW-complex with the underlying space $\Delta(n,d)$, with CW-complex stucture induced by intersection with the cubical CW-complex structure on $\mathbb{R}^d$ given by the integer lattice $\mathbb{Z}^d$. That is, the closed cells of $\mathcal{H}_n^d$ are given by all hypersimplices

$$\begin{aligned} C_{\mathbf{a},J} &:= \Delta(n,d)\cap\left\{(y_1,\dots,y_n)\in\mathbb{R}^n \mid y_i = a_i \text{ for } i\in[d]\setminus J \text{ and } y_j\in[a_j,a_j+1] \text{ for } j\in J\right\} \\ &= \operatorname{conv}\left(\mathbf{a}+\sum_{j\in J}\ell_j\epsilon_j \mid \ell_j\in\{0,1\}, \sum_{j\in J}\ell_j = d-|\mathbf{a}|\right) \end{aligned}$$

with $\mathbf{a}\in\mathbb{N}^n$, $J\subset[n]$, $|\mathbf{a}| := \sum_{i\in[n]} a_i$, ϵ_i the ith unit vector in $\mathbb{R}^n$, either subject to the conditions $|a| = d$ and $J=\emptyset$ (these are the 0-cells), or the condition $1\leq d-|\mathbf{a}|\leq |J|-1$. The CW-complex $\mathcal{H}_n^d$ is multigraded by setting $\operatorname{lcm}(C_{\mathbf{a},J}) := \mathbf{a}+\sum_{j\in J}\epsilon_j$. Call $\mathcal{H}_n^d$ the *hypersimplicial complex*.

Let $J = (j_0 < \cdots < j_r)$ and use the notation $J_v := J \setminus \{j_v\}$. Then the differential of $\mathcal{H}_n^d$ is given by

$$\partial(C_{\mathbf{a},J}) = \begin{cases} \sum_{v=0}^{r}(-1)^v(C_{\mathbf{a},J_v} - C_{\mathbf{a}+\epsilon_v,J_v}) & \text{if } 2 \leq d - |\mathbf{a}| \leq |J| - 2 \\ \sum_{v=0}^{r}(-1)^v C_{\mathbf{a},J_v} & \text{if } 1 = d - |\mathbf{a}| \leq |J| - 2 \\ \sum_{v=0}^{r}(-1)^v C_{\mathbf{a}+\epsilon_j,J_v} & \text{if } 2 \leq d - |\mathbf{a}| = |J| - 1 \\ C_{\mathbf{a}+e_{j_0},\emptyset} - C_{a+e_{j_1},\emptyset} & \text{if } d - |\mathbf{a}| = 1 \\ 0 & \text{if } |\mathbf{a}| = d. \end{cases}$$

Observation 6.3.3. The one skeleton of $\mathcal{H}_d^n$ is exactly $\mathcal{T}(n,d)$ from Notation 2.4.1 in the previous chapter.

6.3.4. *Adopt notation and hypotheses of Setup 6.2.2. If $J = (j_0 < j_1 < \cdots < j_r)$ and $\mathbf{a} \in \mathbb{N}^d$, note that every r-dimensional cell $C_{\mathbf{a},J}$ corresponds to the element $f_J \otimes f^{\mathbf{a}}$ in $\bigwedge^{r+1} F \otimes S_{|a|}(F)$. These elements, in turn, can be represented as hook tableaux with strictly increasing columns and weakly increasing rows:*

$$C_{\mathbf{a},J} \quad \longleftrightarrow \quad f_J \otimes f^{\mathbf{a}} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|} \hline j_0 & 1^{a_1} & \dots & n^{a_n} \\ \hline j_1 & \multicolumn{3}{c}{} \\ \cline{1-1} j_2 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} j_r & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

We will implicitly use this correspondence to refer to cells $C_{\mathbf{a},J}$ of $\mathcal{H}_n^d$ as tableaux or elements of $\bigwedge^{r+1} F \otimes S_{|a|}(F)$. The $\mathfrak{m}^d$-homogenization (see Construction 6.1.3) of $\mathcal{H}_n^d[-1]$

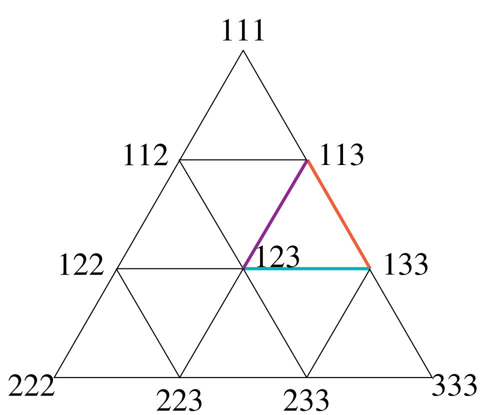
therefore corresponds to the double complex in Figure 6.1 where the maps κ and $\mathrm{kos}^{\psi} \otimes 1$ are as defined in Setup 6.2.2.

$$
\begin{tikzcd}[column sep=small]
 & & & & & 0 \arrow[d] & & \\
 & & & & 0 \arrow[r] \arrow[d] & \bigwedge^n F \arrow[d, "\kappa"] & & \\
 & & & \iddots \arrow[r] \arrow[d] & \vdots \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \vdots \arrow[d, "\kappa"] & & \\
 & & 0 \arrow[r] \arrow[d] & \cdots \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \bigwedge^5 F \otimes S_{d-3} \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \bigwedge^4 F \otimes S_{d-3}(F) \arrow[d, "\kappa"] & & \\
 & 0 \arrow[r] \arrow[d] & \bigwedge^n F \otimes S_{d-2}(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \cdots \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \bigwedge^4 F \otimes S_{d-2}(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] \arrow[d, "\kappa"] & \bigwedge^3 F \otimes S_{d-2}(F) \arrow[d, "\kappa"] & & \\
0 \arrow[r] & \bigwedge^n F \otimes S_{d-1}(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"'] & \cdots \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"'] & \bigwedge^4 F \otimes S_{d-1}(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"'] & \bigwedge^3 F \otimes S_{d-1}(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"'] & \bigwedge^2 F \otimes S_{d-1}(F) \arrow[d, "\kappa"] \arrow[dr, bend left] & & \\
 & & & & & F \otimes S_d(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] & S_d(F) \arrow[r, "\mathrm{kos}^{\psi}\otimes 1"] & 0
\end{tikzcd}
$$

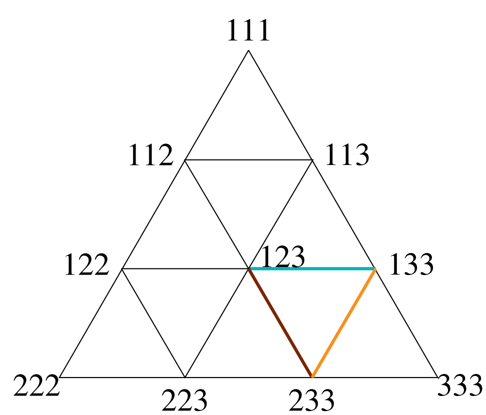
Figure 6.1: Double complex supported on $\mathcal{H}_n^d$.

Example 6.3.5. The edge with vertices $(1, 1, 1)$ and $(1, 0, 2)$ in blue-green in Figure 6.2a and 6.2b corresponds to the cell $C_{(1,0,1),23}$, which has the following image in the double complex of Figure 6.1:

$$
\begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \mapsto x_2 \cdot \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline\end{array} - x_3 \cdot \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}.
$$



The cell $C_{(1,0,1),123}$ in $\mathcal{H}_3^3$.



The cell $C_{(0,0,1),123}$ in $\mathcal{H}_3^3$.

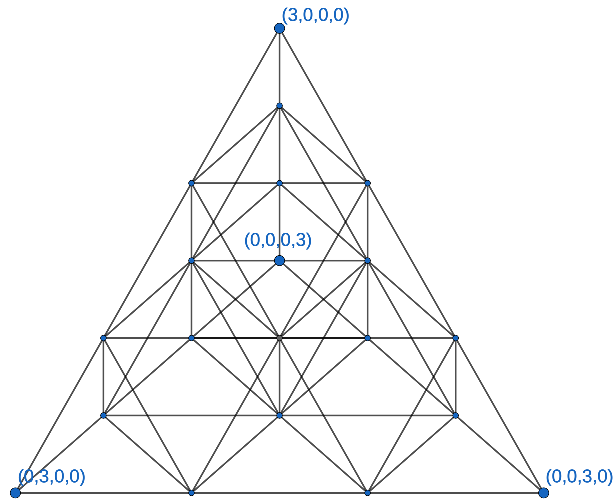


Figure 6.3: The one-skeleton of $\mathcal{H}_4^3$.

The "up-simplex" corresponding to the cell $C_{(1,0,1),123}$ colored in Figure 6.2a has image

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} \mapsto x_1\cdot\begin{array}{|c|c|c|}\hline 2&1&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} - x_2\cdot\begin{array}{|c|c|c|}\hline 1&1&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} + x_3\cdot\begin{array}{|c|c|c|}\hline 1&1&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}.$$

Finally, the "down-simplex" of Figure 6.2b corresponding to the cell $C_{(0,0,1),123}$ has image

$$\begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&1&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&3&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}.$$

The following simple observation turns out to be critical for applications in Section 7.1.

Observation 6.3.6. All elements of $\bigwedge^2 F \otimes S_{d-1}$ contained in the one-skeleton of a "down-triangle" are related by a straightening relation as in Observation 6.2.8.

Example 6.3.7. Figure 6.3 depicts the one-skeleton of $\mathcal{H}_4^3$. In this case, the hypersimplices which appear as maximal cells are not solely simplices: there are four octahedra in the complex, which correspond to the four possible elements of $\bigwedge^4 F \otimes F$. The image of any one of these octahedra in the double complex of Figure 6.1 is a linear combination of eight tableaux, corresponding to the eight faces of the octahedron: four "up-triangles" coming from its image under κ, and four "down-triangles" corresponding to its image under $\mathrm{kos}^{\psi} \otimes 1$.

Proposition 6.3.8 (see [5]). *Let $\mathfrak{m}^d = (x_1, \ldots, x_n)^d \subset k[x_1, \ldots, x_n]$. Then $\mathcal{H}_n^d$ defines a multigraded cellular free resolution of $\mathfrak{m}^d$.*

Batzies and Welker [5] use discrete Morse theory to show that the Eliahou–Kervaire resolution for powers of the graded maximal ideal is cellular. We apply their techniques to obtain a minimal cellular resolution isomorphic to the L-complex as in Definition 6.2.3. To do this, we find an acyclic matching on $\mathcal{H}_n^d$ distinct from the one in [5] which has a corresponding Morse complex which supports the L-complex.

Proposition 6.3.9. *Let $\mathcal{H}_n^d$ be as in Definition 6.3.2. Consider the matching $C_{\mathbf{a},J} \to C_{\mathbf{a}+\epsilon_{\min J}, J \setminus \min J}$, where*

1. $\mathbf{a} \in \mathbb{N}^n$,
2. $J \subset [n]$ *is such that* $2 \leq d - |\mathbf{a}| \leq |J| - 1$, *and*
3. $\min J \leq \min \mathbf{a}$.

Then this is an acyclic homogeneous matching A on $\mathcal{H}_n^d$ as in Construction 6.1.9. Moreover, if $\widetilde{\mathcal{H}}_n^d = (\mathcal{H}_n^d)_A$ denotes the corresponding Morse complex, then the A-critical cells of $\mathcal{H}_n^d$ are:

1. *the* 0*-cells* $C_{\mathbf{a},\emptyset}$, *where* $\mathbf{a} \in \mathbb{N}^n \cap \Delta(n, d)$, *and*

2. *the cells* $C_{\mathbf{a},J}$ *such that* $\min J \leq \min \mathbf{a}$ *and* $|\mathbf{a}| = d - 1$.

Proof. A is a matching because cells $C_{\mathbf{a},J}$ on the left hand side must satisfy $\min J \leq \min \mathbf{a}$ while the cells $C_{\mathbf{a}',J'}$ on the right hand side must satisfy $\min J' > \min \mathbf{a}'$. Suppose, seeking contradiction, that A contains a cycle. Observe that $\operatorname{mdeg}(C_{\mathbf{a},J})$ must be weakly decreasing along every directed edge of the cycle, so in particular it must be constant. Observe that every element $C_{\mathbf{a},J}$ at the head of an edge directed upwards in E^A satisfies $\min J \leq \min \mathbf{a}$, but then every element at the head of an arrow pointing "down" from one of these elements must have the same $\min J$. But this element $C_{\mathbf{a}',J'}$ must also be at the tail of some other element of E^A pointing upwards, so it must also satisfy that $\min J' < \min \mathbf{a}$, which is a contradiction. □

We conclude this section with the main result of this chapter, which states that the L-complex from Definition 6.2.3 is supported on a CW-complex.

Proposition 6.3.10. *Let* $\widetilde{\partial}$ *denote the differential of the Morse complex* $\widetilde{H}^d_n = (H^d_n)_A$ *and* $\widetilde{C}_{\mathbf{a},J}$ *the cell in* $\widetilde{H}^d_n$ *corresponding to the A-critical cell* $C_{\mathbf{a},J}$ *of* $\mathcal{H}^d_n$*. Then* $\widetilde{\mathcal{H}}^d_n$ *supports a minimal linear cellular resolution of a power of the graded maximal ideal which is isomorphic to the L-complex from Definition 6.2.3.*

Proof. The A-critical cells of $\mathcal{H}^d_n$ are exactly the 0-cells $C_{\mathbf{a},\emptyset}$ for $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d) \cap \mathbb{N}^n$ and all cells $C_{\mathbf{a},J}$ such that $\min J \leq \min \mathbf{a}$ and $|\mathbf{a}| = d - 1$; in particular, the critical cells correspond to exactly those standard hook tableaux which are basis elements of the modules in the L-complex. Let $C_{\mathbf{a},J}$ be a critical cell with $p = \min J$ and $q = \min\{i \mid i \in \operatorname{supp}(\mathbf{a})\}$. Then the differential ∂ of $\mathcal{H}^d_n$ applied to $C_{\mathbf{a},J}$ has at most one nonstandard tableau in its image, which would be $C_{\mathbf{a},J_p}$. This element is matched with $C_{\mathbf{a}-\epsilon_q,J_p\cup q}$, which has an image under ∂ consisting of standard tableaux of the same shape and multidegree as $C_{\mathbf{a},J_p}$, and potentially some tableaux corresponding to elements which are matched with elements one dimension lower and therefore do not appear in $\widetilde{\partial}$. In particular, after homogenization, $\widetilde{\partial}$ corresponds exactly to the differential of the L-complex in Definition 6.2.3. $\square$

CHAPTER 7

POLARIZATIONS VIA HOOK PARTITIONS

This chapter is based on joint work with Keller VandeBogert.

7.1 Hook tableaux and polarizations

The goal of this section is to provide a dictionary between the notation and terminology introduced in Chapter 2 and the Schur modules appearing in the L-complexes of Section 6.2. More precisely, we give a new combinatorial characterization of all polarizations of powers of the graded maximal ideal in terms of generating sets of the Schur module $L_d^1(F)$. The main result of this section is Theorem 7.1.6, which shows that the spanning tree condition of Theorem 2.5.1 is equivalently asking that the Young tableaux canonically associated to the linear syzygy edges form a basis for the associated Schur module.

The actual dictionary for translating between the different aforementioned frameworks is given by Proposition 7.1.2; these results will be employed in Section 7.2 to extend the results in Chapter 2 to the case of restricted powers.

Notation 7.1.1. Fix integers n and d, and let $S = k[x_1, \dots, x_n]$ be a polynomial over a field k. Let $\check{X}_i = \{x_{i1}, \dots, x_{id}\}$ be a set of variables, and let $\tilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ be a polynomial ring in the union of all these variables. Denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of S.

Let $\Delta(n, d)$ be the dilated $(n-1)$-simplex from Definition 6.3.1. Denote by $\Delta^{\mathbb{Z}}(n, d) = \Delta(n, d) \cap \mathbb{Z}^n$ the set of lattice points of the dilated simplex $d\Delta_{n-1}$, i.e., the set of tuples $\mathbf{a} = (a_1, \dots, a_n)$ of non-negative integers with $\sum_i^n a_i = d$. Denote by $\mathcal{T}(n,d) = \mathcal{T}(n,d)$ the one-skeleton of the hypersimplicial complex $\mathcal{H}_n^d$ from Definition 6.3.2.

Let $L_b^a(F)$ be the Schur module defined in Setup 6.2.2.

Proposition 7.1.2. *Adopt notation and hypotheses of Notation 7.1.1. Then:*

(*a*) *There exists a bijection $\psi_{n,d}$ from $\Delta^{\mathbb{Z}}(n,d)$ to $S_d(F)$.*

(*b*) *For any pair $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$ and $R \subseteq \mathrm{Supp}(\mathbf{c})$ such that $|R| = t$, the complete subgraph $D_R(\mathbf{c})$ (see Definition 2.4.12) corresponds to a unique element of $\bigwedge^t F \otimes S_{d-t+1}(F)$.*

(*c*) *There exists a bijection $\theta_{n,d}$ from the edges of $\Delta^{\mathbb{Z}}(n,d)$ to the elements of $\bigwedge^2 F \otimes S_{d-1}(F)$.*

Proof. For (a), the map

$$\psi_{n,d} : \Delta^{\mathbb{Z}}(n,d) \to S_d(F) \tag{7.1}$$

such that $\psi_{n,d}(\mathbf{a}) = f^{\mathbf{a}}$ gives the desired bijection.

For (b), Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$. If $R \subseteq \mathrm{Supp}(\mathbf{c})$, define

$$\omega_{n,d}(\mathbf{c}, R) := f_R \otimes \psi_{n,d-|R|+1}(\mathbf{c}_R) \tag{7.2}$$

where $\mathbf{c}_R = \mathbf{c} - \sum_{i \in R} \epsilon_i$. In particular, if $R = \mathrm{Supp}(\mathbf{c})$, then the map

$$\omega_{n,d} : \Delta^{\mathbb{Z}}(n,d+1) \to \bigwedge^t F \otimes S_{d-t-1}(F) \tag{7.3}$$

such that $\omega_{n,d}(\mathbf{c}) = f_{\mathrm{Supp}(\mathbf{c})} \otimes \psi_{n,d-t+1}(\mathbf{c}')$ is a bijection between the down-triangles of $\mathcal{T}(n,d)$ and $\bigwedge^t F \otimes S_{d-t-1}(F)$, where $\mathbf{c}' = \mathbf{c} - \sum_{i \in \mathrm{Supp}(\mathbf{c})} \epsilon_i$ and $t = |\mathrm{Supp}(\mathbf{c})|$.

For (c), If $(\mathbf{c}; i, j)$ is an edge in $\mathcal{T}(n,d)$, then the map

$$\theta_{n,d}(\mathbf{c}; i, j) := f_i \wedge f_j \otimes \psi_{n,d-1}(\mathbf{c} - \epsilon_i - \epsilon_j) \tag{7.4}$$

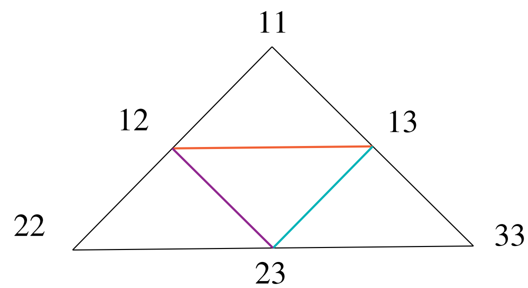


Figure 7.1: $\mathcal{T}(3,2)$ with vertex labels in $S_2(F)$.

gives a bijection between edges of $\mathcal{T}(n,d)$ and elements of $\bigwedge^2 F \otimes S_{d-1}(F)$. $\square$

7.1.3. *Let* $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ *be a down-triangle and let* $R \subseteq \mathrm{Supp}(\mathbf{c})$*. Set* $T = \omega(\mathbf{c}, R) \in \bigwedge^{|R|} F \otimes S_{d-|R|-1}(F)$*. The tableaux appearing in the image* $\kappa_{|R|,n-|R|+1}(T)$ *(where* $\kappa_{a,b}$ *is defined in Setup 6.2.2) correspond exactly to the down-triangles* $D_P(\mathbf{c})$ *such that* $|P| = |R| - 1$*. In particular, as noted in Observation 6.3.6, if* $|R| = 3$*,* $\kappa_{3,n-2}(T)$ *corresponds to a linear combination of the labels appearing on the three edges of* $D_R(\mathbf{c})$*.*

Example 7.1.4. Figure 7.1 depicts $\mathcal{T}(3,2)$ with vertex labels in $S_2(F)$ where F is a free module with basis elements $\{f_1, f_2, f_3\}$. The unique maximal down-triangle in this graph corresponds to $\mathbf{c} = (1,1,1) \in \Delta^{\mathbb{Z}}(3,3)$, or, via the map in part (b) of Proposition 7.1.2, $f_1 \wedge f_2 \wedge f_3 \in \bigwedge^3 F$. Applying the map $\kappa_{3,0}$ from Definition 6.2.3 gives exactly the tableaux corresponding to the edges of the down-triangle $\mathbf{c}$:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \xmapsto{\kappa_{3,0}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Definition 7.1.5. Let $\chi = \{X_i\}$ denote a set of isotone maps

$$X_i : (\Delta^{\mathbb{Z}}(n,d), \geq_i) \to \mathcal{B}_d$$

as in Construction 2.4.10. Let LS_χ be the set of linear syzygy edges after applying χ to the generators of $\mathfrak{m}^d$ as in Definition 2.4.11. Denote by $\mathrm{tab}(\chi)$ the set of tableaux in $\bigwedge^2 F \otimes S_{d-1}(F)$ associated to the edges LS_χ via the correspondence in 6.3.4.

Theorem 7.1.6. *Adopt notation and hypotheses of Setup 7.1.1. Let $\chi = \{X_i\}$ denote a set of isotone maps as in Construction 2.4.10. Then the following are equivalent:*

1. *The elements of* $\mathrm{tab}(\chi)$ *span the module $L_d^1(F)$.*
2. *For every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$,* $\mathrm{LS}(\mathbf{c})$ *contains a spanning tree of the complete down-graph $D(\mathbf{c})$.*
3. *The set of isotone maps $X_1, \ldots, X_n$ determine a polarization of $(x_1, \ldots, x_n)^d$.*

Proof. Note that (2) $\Longleftrightarrow$ (3) is Theorem 2.5.1.

(2) $\Longrightarrow$ (1): Take $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ and let $\mathbf{a} = \mathbf{c} - \epsilon_i$ and $\mathbf{b} = \mathbf{c} - \epsilon_j$ for some $i, j \in \mathrm{Supp}(\mathbf{c})$. If $LS(\mathbf{c})$ contains a spanning tree, then for any two vertices $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{c}$, there exists a path in $LS(\mathbf{c})$ connecting them. It suffices to show that for any $\mathbf{a}$ and $\mathbf{b}$ in $D(\mathbf{c})$, $\theta_{n,d}(\mathbf{c}; i, j)$ is in the span of $\mathrm{tab}(\mathfrak{X})$.

Proceed by induction on k, the number of edges in the shortest path from $\mathbf{a}$ to $\mathbf{b}$. If $k = 1$, then the tableau corresponding to the edge between $\mathbf{a}$ and $\mathbf{b}$ is in $\mathrm{tab}(\mathfrak{X})$. Now assume that for any two vertices $\mathbf{c} - \epsilon_i$ and $\mathbf{c} - \epsilon_j$ such that the shortest path in $LS(\mathbf{c})$ between them is length k, the tableau $\theta_{n,d}(\mathbf{c}; i, j)$ is a linear combination of elements in $\mathrm{tab}(\mathfrak{X})$. Let $\mathbf{a}$ and $\mathbf{b}$ be two vertices of $D(\mathbf{c})$ such that the shortest path between them is length $k+1$, i.e., there is a set of vertices

$$\mathbf{a} = \mathbf{d}^1, \ldots, \mathbf{d}^{k+2} = \mathbf{b}$$

such that each $\mathbf{d}^j$ is a vertex in $D(\mathbf{c})$ and each pair $(\mathbf{d}^j, \mathbf{d}^{j+1})$ is connected by an edge in $\mathrm{LS}(\mathbf{c})$. The length of the shortest path between $\mathbf{a}$ and $\mathbf{d}^{k+1}$ is k, so by the induction hypothesis, the tableau labeling the edge between them is spanned by the elements of $\mathrm{tab}(\mathfrak{X})$. If $\mathbf{a} = \mathbf{c} - \epsilon_{i_1}, \mathbf{b} = \mathbf{c} - \epsilon_{i_2}$, and $\mathbf{d}^{k+1} = \mathbf{c} - \epsilon_{i_3}$, set $R = \{i_1, i_2, i_3\}$ and consider the "smaller" down-triangle $D_R(\mathbf{c})$ (see Definition 2.4.12). By Observation **??**, the edge between $\mathbf{a}$ and $\mathbf{b}$ is a linear combination of the other two edges of $D_R(\mathbf{c})$, which in turn have been shown to be linear combinations of elements of $\mathrm{tab}(\mathfrak{X})$, hence proving the claim.

(1) $\Longrightarrow$ (2): Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ be a complete down-graph in $\mathcal{T}(n, d)$. It suffices to show that for any two vertices $\mathbf{a} = \mathbf{c} - \epsilon_i$ and $\mathbf{b} = \mathbf{c} - \epsilon_j$, there exists a path from $\mathbf{a}$ to $\mathbf{b}$ by edges labeled by tableaux in $\mathrm{tab}(\mathfrak{X})$. Since $\mathrm{tab}(\mathfrak{X})$ must contain a basis of the module $L^1_d(F)$, one may assume that $\mathrm{tab}(\mathfrak{X})$ is a basis itself. Suppose $\theta_{n,d}(\mathbf{c}; i, j)$ is not in $\mathrm{tab}(\mathfrak{X})$. The result follows from the following claims:

(i) There exists some $R = \{i, j, \ell_1\} \subseteq \mathrm{Supp}(\mathbf{c})$ such that $D_R(\mathbf{c})$ has at least one edge labeled by an element of $\mathrm{tab}(\mathfrak{X})$.

(ii) Suppose $(\mathbf{c}; i, \ell_1)$ is the unique edge of $D_R(\mathbf{c})$ labeled by an element of $\mathrm{tab}(\mathfrak{X})$. Then there exists some $P = \{\ell_1, \ell_2, j\} \subseteq \mathrm{Supp}(\mathbf{c})$ such that at least one edge of $D_P(\mathbf{c})$ has a label appearing in $\mathrm{tab}(\mathfrak{X})$ and $\ell_2 \neq i$.

(iii) Let $\Gamma(\mathbf{c})$ be the subgraph of $D(\mathbf{c})$ with edges labeled by elements of $\mathrm{tab}(\mathfrak{X})$. If $\Gamma(\mathbf{c})$ contains a cycle, then it corresponds to a linearly dependent subset of $\mathrm{tab}(\mathfrak{X})$.

To see (i), suppose no tableaux corresponding to edges in any possible $D_R(\mathbf{c})$ are in $\mathrm{tab}(\mathfrak{X})$. Then no tableaux in any of the possible straightening relations containing $\theta_{n,d}(\mathbf{c}; i, j)$ coming from the image of any of the $\omega_{n,d}(\mathbf{c}, R)$ under $\kappa_{3,n-2}$ appear in $\mathrm{tab}(\mathfrak{X})$. Hence, $\theta_{n,d}(\mathbf{c}; i, j)$ is not in the span of $\mathrm{tab}(\mathfrak{X})$.

For (ii), observe that the image of $\omega_{n,d}(\mathbf{c}, R)$ under $\kappa_{3,n-2}$ gives that $\theta_{n,d}(\mathbf{c}; i, j)$ is in the span of $\theta_{n,d}(\mathbf{c}; i, \ell_1)$ and $\theta_{n,d}(\mathbf{c}; j, \ell_1)$. By assumption, $\theta_{n,d}(\mathbf{c}; i, \ell_1) \in \mathrm{tab}(\mathfrak{X})$. Suppose every other $D_P(\mathbf{c})$ with $|P| = 3$ containing $\theta_{n,d}(\mathbf{c}; j, \ell_1)$ has no edges labeled by tableaux in $\mathrm{tab}(\mathfrak{X})$. Then $\theta_{n,d}(\mathbf{c}; j, \ell_1)$ is not in the span of $\mathrm{tab}(\mathfrak{X})$, contradicting the assumption that $\mathrm{tab}(\mathfrak{X})$ spans $L^1_d(F)$.

To check (iii), proceed by induction on the length k of the cycle. In the base case where $k = 3$, the cycle forms the edges of a down-triangle $D_R(\mathbf{c})$ where $|R| = 3$. The three tableaux labeling the edges of $D_R(\mathbf{c})$ make up the straightening relation from the image of $\omega_{n,d}(\mathbf{c}, R)$ under $\kappa_{3,n-2}$, implying they are linearly dependent. This contradicts the assumption that $\mathrm{tab}(\mathfrak{X})$ forms a basis. Now assume that any cycle of length k in $\Gamma(\mathbf{c})$ gives a linearly dependent subset of $\mathrm{tab}(\mathfrak{X})$, and suppose there is a cycle $\mathbf{a}^1, \ldots, \mathbf{a}^{k+1}$ such that the edges $(\mathbf{a}^j, \mathbf{a}^{j+1})$ and $(\mathbf{a}^1, \mathbf{a}^k)$ are labeled by tableaux in $\mathrm{tab}(\mathfrak{X})$ and each $\mathbf{a}^j = \mathbf{c} - \epsilon_{i_j}$. Let $R = \{i_1, i_k, i_{k+1}\}$. By the induction hypothesis, $\theta_{n,d}(\mathbf{c}; i_1, i_k)$ is equal to a linear combination of tableaux $\theta_{n,d}(\mathbf{c}; i_k, i_{k+1})$ and $\theta_{n,d}(\mathbf{c}; i_1, i_{k+1})$; but also by the induction hypotheses, it is a linear combination of tableaux of the form $\theta_{n,d}(\mathbf{c}; i_j, i_{j+1})$ where $1 \leq j \leq k$. Therefore, the cycle corresponds to a linearly dependent subset of $\mathrm{tab}(\mathfrak{X})$.

With claims (i)-(iii) established, iterate the following process. Choose a triangle $D_R(\mathbf{c})$ such that $R = \{i, j, \ell\}$ and some edge of $D_R(\mathbf{c})$ is labeled by an element of $\mathrm{tab}(\mathfrak{X})$. If both $\theta_{n,d}(\mathbf{c}; i, \ell_1)$ and $\theta_{n,d}(\mathbf{c}; \ell_1, j)$ are in $\mathrm{tab}(\mathfrak{X})$, the claim follows. Suppose that only one of these tableaux are in $\mathrm{tab}(\mathfrak{X})$; without loss of generality, assume that $\theta_{n,d}(\mathbf{c}; i, \ell) \in \mathrm{tab}(\mathfrak{X})$. By (ii), there is some triangle $D_P(\mathbf{c})$ such that $P = \{\ell_1, \ell_2, j\}$ and at least one edge of $D_P(\mathbf{c})$ has a label appearing in $\mathrm{tab}(\mathfrak{X})$. If both $\theta_{n,d}(\mathbf{c}; \ell_1, \ell_2)$ and $\theta_{n,d}(\mathbf{c}; \ell_2, j)$ appear in $\mathrm{tab}(\mathfrak{X})$, then the claim follows. Otherwise, repeat this process. This process must terminate by (iii), giving the desired path.

$\square$

7.2 Cellular resolutions and polarizations of restricted powers of the graded maximal ideal

In this section, we extend the results from Chapter 2 and Section 7.1 to the case of so-called *restricted powers* of the graded maximal ideal. This class of ideals comes from bounding the multidegrees appearing in the generators of $\mathfrak{m}^d$. We show in Proposition 7.2.6 that these ideals also have a minimal, linear, cellular resolution arising as a subcomplex of the L-complex. In addition, we give two combinatorial characterizations of polarizations of this class of ideals: one in terms of their graphs of linear syzygies, and one in terms of spanning sets of a submodule of the associated Schur module. These characterizations are given by Theorem 7.2.8.

Setup 7.2.1. Let I be a monomial ideal in a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k with generating set $\mathcal{G}(I) = (m_1, \dots, m_r)$. Let $\mathbf{u} \in I$ be a monomial, and define the monomial ideal $I_{\leq \mathbf{u}}$ to be the ideal generated by monomials $\{m_i \mid m_i \text{ divides } \mathbf{u}\}$.

Let d_i be the highest power of x_i that appears in a minimal generator of I. Set $\check{X}_i = \{x_{i1}, \dots, x_{id_i}\}$ for all i, and define the polynomial ring $\widetilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ in the union of all these variables. Observe that $\widetilde{S}$ has a $\mathbb{Z}^n$-grading induced by the first indices of the variables in $\widetilde{S}$.

Let $\widetilde{I}$ be a polarization of I. Define $\widetilde{I}_{\leq \mathbf{u}}$ to be generated by those elements of $\widetilde{I}$ with $\mathbb{Z}^n$-multidegree bounded above by $\mathbf{u}$.

The following useful proposition was observed in [22].

Proposition 7.2.2. *Let I be a monomial ideal in a polynomial ring S. Fix a multihomogeneous basis of a multigraded free resolution $\mathbb{F}_I$ of S/I. Denote by $\mathbb{F}_I(\leq \mathbf{u})$ the subcomplex of $\mathbb{F}$ generated by multihomogeneous basis elements of multidegrees dividing $\mathbf{u}$.*

(1) *The subcomplex $\mathbb{F}_I(\leq \mathbf{u})$ is a multigraded free resolution of $S/(I_{\leq \mathbf{u}})$.*

(2) *If $\mathbb{F}_I$ is a minimal multigraded free resolution of S/I, then $\mathbb{F}_I(\leq \mathbf{u})$ is independent of the choice of basis.*

(3) *If $\mathbb{F}_I$ is a minimal multigraded free resolution of S/I, then the resolution $\mathbb{F}_I(\leq \mathbf{u})$ is also minimal.*

Proposition 7.2.3. *Adopt Setup 7.2.1. Then $\widetilde{I}_{\leq \mathbf{u}}$ is a polarization of $I_{\leq \mathbf{u}}$.*

Proof. Let $\widetilde{\mathbb{F}}$ be a free resolution of $\widetilde{I}$. Then $\widetilde{\mathbb{F}}$ is a multigraded resolution with respect to the $\mathbb{Z}^n$-multigrading where each variable $x_{i,j}$ in $\check{X}_i$ has multidegree e_i. By the definition of a polarization, $\widetilde{\mathbb{F}} \otimes \widetilde{S}/\sigma \cong \mathbb{F}$, where σ is a regular $\widetilde{S}/\widetilde{I}$-sequence of variable differences and $\mathbb{F}$ is a minimal free resolution of I. By Proposition 7.2.2, both $\mathbb{F}(\leq \mathbf{u})$ and $\widetilde{\mathbb{F}}(\leq \mathbf{u})$ are minimal multigraded free resolutions of $I_{\leq \mathbf{u}}$ and $\widetilde{I}_{\leq \mathbf{u}}$, respectively; in particular, one has that

$$\widetilde{\mathbb{F}}(\leq \mathbf{u}) \otimes \widetilde{S}/\sigma \cong \mathbb{F}(\leq \mathbf{u}). \tag{7.5}$$

It remains to check that σ is indeed a regular sequence on $\widetilde{S}/\widetilde{I}_{\leq \mathbf{u}}$. This follows immediately from the string of isomorphisms:

$$H_\bullet(\widetilde{\mathbb{F}}(\leq \mathbf{u}) \otimes \widetilde{S}/\sigma) \cong \operatorname{Tor}_\bullet^{\widetilde{S}}(\widetilde{S}/\widetilde{I}_{\leq \mathbf{u}}, \widetilde{S}/\sigma) \cong H_\bullet(\widetilde{S}/\widetilde{I}_{\leq \mathbf{u}} \otimes K(\sigma)_\bullet),$$

where $K(\sigma)_\bullet$ denotes the Koszul complex on σ. Since $\widetilde{\mathbb{F}}(\leq \mathbf{u}) \otimes \widetilde{S}/\sigma \cong \mathbb{F}(\leq \mathbf{u})$ is acyclic, it follows that $H_{>0}(\widetilde{S}/\widetilde{I} \otimes K(\sigma)_\bullet) = 0$, so σ is regular on $\widetilde{S}/\widetilde{I}$ by, for instance, [41, Theorem 14.7]. $\square$

We apply these results to extend results on cellular resolutions and polarizations from powers of the maximal ideal to so-called *restricted powers* of the graded maximal ideal. This terminology conforms with that of [21] and [22].

Definition 7.2.4 (Restricted Powers)**.** Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal in a polynomial ring S over a field k. For any vector $\mathbf{u} \in \mathbb{N}^n$, define the *restricted power* of $\mathfrak{m}$ to be $\mathfrak{m}^{\mathbf{d}}(\leq \mathbf{u})$ to be the ideal generated by $(\mathbf{x}^{\mathbf{a}} \mid \sum_i^n a_i = d$ and $a_i \leq u_i$ for all $i \in [n])$.

Setup 7.2.5. Let $S = k[x_1, \ldots, x_n]$ be a polynomial over a field k. Denote by $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S. For $\mathbf{u} \in \mathbb{N}^n$, let $\mathfrak{m}^d(\leq \mathbf{u})$ be the restricted power of the graded maximal ideal as in Definition 7.2.4. Let $\check{X}_i = \{x_{i1}, \ldots, x_{id}\}$ be a set of variables, and let $\widetilde{S} = k[\check{X}_1, \ldots, \check{X}_n]$ be a polynomial ring in the union of all these variables. Let $\mathcal{B}_d(\leq u_i)$ be the *truncated Boolean poset* on $[d]$ with elements of rank at most u_i in $\mathcal{B}_d$.

Let $\mathcal{H}_n^d$ be the hypersimplicial complex from Definition 6.3.2, and let $\mathcal{H}_n^d(\leq \mathbf{u})$ be the induced subcomplex with cells $\{C_{\mathbf{a},J} \mid \mathrm{mdeg}(C_{\mathbf{a},J}) \leq \mathbf{u}\}$. Denote by $\Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d)$ and $\mathcal{T}_{\leq \mathbf{u}}(n, d)$ the 0-skeleton and 1-skeleton of $\mathcal{H}_n^d(\leq \mathbf{u})$, respectively. We also use the notation $\Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d)$ for $\Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d)$.

Let $L^a_{b,\leq \mathbf{u}}(F)$ be the Schur module defined in Setup 6.2.2 restricted to multidegrees $\leq \mathbf{u}$, where the multidegree of an element $f_J \otimes f^\alpha \in \bigwedge F^a \otimes S_b(F)$ is defined to be $\alpha + \sum_{j \in J} \epsilon_j$, where ϵ_j is the j'th unit vector in $\mathbb{N}^n$.

Proposition 7.2.6. *Adopt notation and hypotheses of Setup 7.2.5. Then:*

1. *The induced subcomplex $\mathcal{H}_n^d(\leq \mathbf{u})$ supports a polyhedral cellular resolution of $\mathfrak{m}^d(\leq \mathbf{u})$.*

2. *The induced subcomplex* $\widetilde{\mathcal{H}}_n^d(\leq \mathbf{u})$ *supports a minimal CW cellular resolution of* $\mathfrak{m}^d(\leq \mathbf{u})$ *which is isomorphic to a subcomplex of the L-complex.*

In particular, $\mathfrak{m}^d(\leq \mathbf{u})$ *has a linear minimal free resolution.*

Proof. Apply Proposition 7.2.2. □

Corollary 7.2.7. *Adopt notation and hypotheses of Setup 7.2.1 and let* $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$*. The ideal generated by all squarefree monomials of a given degree in* S *has a non-minimal resolution supported on the polyhedral cell complex* $\mathcal{H}_n^d(\leq \mathbb{1})$*, and it has a minimal free resolution supported on the CW-complex* $\widetilde{\mathcal{H}_n^d}(\leq \mathbb{1})$*, , the restriction of the Morse complex from Proposition 6.3.9.*

Moreover, one can extend the characterizations of polarizations of powers of the graded maximal ideal in Theorem 7.1.6 to restricted powers of the maximal ideal. Observe that all the definitions in Chapter 2 work in this context, exchanging $\Delta^{\mathbb{Z}}(n, d)$ with $\Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d)$ and exchanging $\mathcal{B}_d$ with $\mathcal{B}_d(\leq u_i)$ as required.

Theorem 7.2.8. *Adopt notation and hypotheses of Setup 7.2.1. Let* $\mathfrak{X} = \{X_i\}_{i\in[n]}$ *denote a set of rank-preserving isotone maps*

$$X_i : (\Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d), \leq_i) \to \mathcal{B}_d(\leq u_i)$$

as in Construction 2.4.10. Denote by $\operatorname{tab}(\mathfrak{X})$ *be the set of tableaux in* $\bigwedge^2 F \otimes S_{d-1}(F)$ *associated to the linear syzygy edges in* $\mathcal{T}_{\leq \mathbf{u}}(n, d)$ *after applying the isotone maps in* $\mathfrak{X}$ *to its vertices. Then the following are equivalent:*

1. *The elements of* $\operatorname{tab}(\mathfrak{X})$ *span the module* $L^1_{d,\leq \mathbf{u}}(F)$*.*
2. *For every* $\mathbf{c} \in \Delta^{\mathbb{Z}}_{\leq \mathbf{u}}(n, d+1)$*,* $\mathrm{LS}(\mathbf{c})$ *contains a spanning tree of the complete down-graph* $D(\mathbf{c})$*.*

3. *The set of isotone maps $X_1, \dots, X_n$ determine a polarization of* $\mathfrak{m}^d(\leq \mathbf{u})$.

Proof. (1) $\implies$ (2): The proof is identical to that of Theorem 7.1.6.

(2) $\implies$ (3): This follows from Proposition 7.2.3.

(3) $\implies$ (1): Suppose the set of isotone maps $\mathfrak{X} = \{X_i\}_{i\in[n]}$ determine a polarization $\mathfrak{m}^d\widetilde{(\leq \mathbf{u})}$ of $\mathfrak{m}^d(\leq \mathbf{u})$. Let $\widetilde{\mathbb{F}}$ be a (not necessarily minimal) free resolution of $\mathfrak{m}^d\widetilde{(\leq \mathbf{u})}$ with linear syzygies corresponding to the set of linear syzygy edges induced by $\mathfrak{X}$. Let $\mathbb{F} = \widetilde{\mathbb{F}} \otimes \widetilde{S}/\sigma$, be the depolarization of $\widetilde{\mathbb{F}}$, where σ is a regular sequence of variable differences. Then $\mathbb{F}$ is a free resolution of $\mathfrak{m}^d(\leq \mathbf{u})$ with linear syzygies which are in bijection with $\mathrm{tab}(\mathfrak{X})$. Since $\mathbb{F}$ must be a free resolution of $\mathfrak{m}^d(\leq \mathbf{u})$, $\mathrm{tab}(\mathfrak{X})$ must span $L^1_{d,\leq \mathbf{u}}(F)$.

□

Part III

Polarizations and Triangulations of Root Polytopes

CHAPTER 8

TRIANGULOIDS AND POLARIZATIONS

Triangulations arise naturally in a wide variety of contexts, including optimization, combinatorics, algebra, topology, and computer science. In particular, triangulations of a product of simplices $\Delta^{d-1} \times \Delta^{n-1}$ have been shown to be in bijection with various objects, including:

- fine mixed subdivisions of the dilated simplex $d \cdot \Delta^{n-1}$ (see [45] or [27]);
- tropical oriented matroids (see [4] or [39]);
- tropical pseudohyperplane arrangements (see [26]);
- matching ensembles (see [6] or [40]);
- ...and more (see [43])

More generally, for any bipartite graph G contained in the complete bipartite graph $K_{n,d}$, Postnikov introduced in [43] the notion of a *root polytope* Q_G, which specializes to a product of simplices in the case when $G = K_{n,d}$. He showed that every triangulation of root polytope gives rise to a fine mixed subdivision of a *generalized permutahedron* P_G. Triangulations of root polytopes have arisen in the contexts of pipe dream complexes (see [35]), noncrossing alternating trees (see [33] and [34]), and subword complexes (see [12]).

However, there was no general theory characterizing *all* triangulations of root polytopes until the work of Galashin, Nenashev, and Postnikov in [20]. They introduce the notion of a *trianguloid*, which is a collection of bipartite graphs $\mathcal{G} = \{G_a\}$ (one for each lattice point a of a certain trimmed generalized permutathedron) satisfying four axioms. It turns out that these trianguloids are in bijection with triangulations of root polytopes. Galashin, Nenashev, and Postnikov represent these "trianguloids" as graphs with multiple colored edges between the lattice points of $d \cdot \Delta^{n-1}$, and their conditions are quite technical.

In this chapter, we provide a novel perspective: the bipartite graphs G_a in a trianguloid can be written as monomials, and the axioms for the graphs give rise to conditions on these monomials. We show in Theorem 8.1.4 that in the case where the root polytope is a product of simplices $\Delta^{n-1} \times \Delta^{d-1}$, three of the trianguloid axioms are equivalent to the condition that the monomials generate a polarization of a power of the graded maximal ideal. In particular, we show that the most technical axiom for a trianguloid, called the *hexagon axiom*, is equivalent to the spanning tree condition for a polarization in Theorem 2.5.1.

8.1 Trianguloids are polarizations

Trianguloids were introduced by Galashin, Nenashev, and Postnikov in [20] as a manner of axiomatizing triangulations of root polytopes. In particular, they show that trianguloids are in bijection with triangulations of $\Delta^{n-1} \times \Delta^{d-1}$, and that these ideas can be extended to encompass triangulations of all root polytopes Q_G corresponding to a bipartite graph G. In this section, we relate their axioms for trianguloids to the combinatorial characterization of polarizations of powers of the graded maximal ideal in Theorem 2.5.1.

The following phrasing of the definition of a trianguloid in the case of $\Delta^{n-1} \times \Delta^{d-1}$ can be found in [31].

Definition 8.1.1 (Trianguloid)**.** Let $\mathcal{G} = \{G_a \mid a \in \Delta^{\mathbb{Z}}(n,d)\}$ be a collection of bipartite graphs on $L \sqcup R$ in bijection with the lattice points $\Delta^{\mathbb{Z}}(n,d)$. Let $\mathcal{N}_a(v)$

denote the neighborhood of v in G_a. The collection $\mathcal{G}$ is a *pre-trianguloid* if it satisfies the following axioms:

(T1) The graph G_a has right degree vector a,

(T2) Each graph has no isolated left nodes,

(T3) For $a, a' \in \Delta^{\mathbb{Z}}(n,d)$ where $a' = a + e_p - e_q$, one has $\mathcal{N}_a(r_p) \subset \mathcal{N}_{a'}(r_p)$.

$\mathcal{G}$ is a *trianguloid* if it is a pre-trianguloid which satisfies the following *hexagon axiom*:

(T4) Let $c \in \Delta^{\mathbb{Z}}(n, d-2)$ and $i, j, k \in [d]$ be distinct such that $\mathcal{N}_{c+e_i+e_j}(r_j) \neq \mathcal{N}_{c+e_j+e_k}(r_j)$. Then

$$\mathcal{N}_{c+e_i+e_j}(r_i) = \mathcal{N}_{c+e_i+e_k}(r_i), \quad \mathcal{N}_{c+e_i+e_k}(r_k) = \mathcal{N}_{c+e_j+e_k}(r_k).$$

The definition of a trianguloid turns out to be eerily similar to characterizations of polarizations of powers of the graded maximal ideal from earlier chapters. We recall the Theorem 2.5.1 from Chapter 2, now phrasing it axiomatically to demonstrate the parallel with the axiomitization of triangulations of root polytopes above.

Theorem 8.1.2. *Let G be a subgraph of $\mathcal{T}(n,d)$. Let $\mathcal{B}_n$ be the Boolean poset on $[n]$. Suppose that a set of maps $\{X_i\}_{1\leq i\leq d}$ where*

$$X_i : (\Delta^{\mathbb{Z}}(n,d), \leq_i) \to \mathcal{B}_n$$

exists on the vertices of G such that the following properties hold:

(P1) The maps X_i are rank-preserving.

(P2) *The maps X_i are isotone.*

(P3) $(\mathbf{c}; i, j)$ *is an edge in G if and only if $X_p(\mathbf{c} - e_i) = X_p(\mathbf{c} - e_j)$ for all $p \neq i, j$.*

For any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n, d)$, let $m_i(\mathbf{a}) = \prod_{j \in X_i(\mathbf{a})} x_{ij}$ and $m(\mathbf{a}) = \prod_{i=1}^{n} m_i(\mathbf{a})$. Let J be the ideal in $\widetilde{S}$ generated by the monomials $m(\mathbf{a})$. Then J is a polarization of $(x_1, \ldots, x_n)^d \subset S$ if and only if the graph G satisfies the following property:

(P4) *For every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, the linear syzygy edges $\mathrm{LS}(\mathbf{c})$ contain a spanning tree for the down-graph $D(\mathbf{c})$.*

We will now prove the main theorem of this chapter: namely, that three of the trianguloid axioms are equivalent to polarizations of powers of the graded maximal ideal. The primary difficulty is showing that the "hexagon axiom" (T4) in Definition 8.1.1 is equivalent to the "spanning tree condition" (P4) in Theorem 8.1.2. The fact that (P4) implies (T4) follows from Lemma 2.4.15. The proof technique to show that (T4) implies (P4) models the proof of the forward direction of Theorem 2.5.1.

Notation 8.1.3. Let $\mathcal{G} = \{G_a \mid a \in \Delta^{\mathbb{Z}}(n, d)\}$ be a collection of bipartite graphs on $L \sqcup R$ in bijection with the lattice points $\Delta^{\mathbb{Z}}(n, d)$. Denote by $E(G_a)$ the edge set $(i, j) \in [d] \times [n]$ of the bipartite graph G_a. Define $I(\mathcal{G}) \subset \widetilde{S}$ be the ideal

$$I(\mathcal{G}) := \left(\prod_{(i,j) \in E(G_a)} x_{ij} \mid G_a \in \mathcal{G} \right).$$

Theorem 8.1.4. *Let $\mathcal{G} = \{G_a \mid a \in \Delta^{\mathbb{Z}}(n, d)\}$ be a collection of bipartite graphs on $L \sqcup R$ in bijection with the lattice points $\Delta^{\mathbb{Z}}(n, d)$. Let $I(\mathcal{G})$ be the ideal of $\mathcal{G}$ as in Notation 8.1.3. Then $I(\mathcal{G}) \subset \widetilde{S}$ is a polarization of $(x_1, \ldots, x_n)^d \subset S$ if and only if $\mathcal{G}$ satisfies axioms (T1), (T3), and (T4) in the definition of a trianguloid (Definition 8.1.1).*

Proof. It is straightforward to verify from the definitions that (T1) is equivalent to (P1) and that (T3) is equivalent to to (P2). Moreover, the hexagon axiom (T4) follows directly from Lemma 2.4.15 and (P3).

It remains to show that the hexagon axiom (T4) together with (T1) and (T3) imply the spanning tree condition (P4); that is, we wish to show that for any $\mathbf{c} \in \Delta^{\mathbb{Z}}(n+1)$, there is a path of linear syzygy edges inside the simplex $(\mathbf{c} - e_i \mid i \in \operatorname{Supp} \mathbf{c})$ from $\mathbf{c} - e_v$ to $\mathbf{c} - e_w$ for any $v, w \in \operatorname{Supp}(\mathbf{c})$. For ease of notation, assume that $\operatorname{Supp}(\mathbf{c}) = [d]$, although the following argument works for arbitrary $\operatorname{Supp}(\mathbf{c})$.

By (T3), every $\mathcal{N}_{\mathbf{c}-e_i}(r_i) \subset \mathcal{N}_{\mathbf{c}-e_j}(r_i)$ for all $i, j \in \operatorname{Supp}(\mathbf{c})$. Define the bipartite graph $H_{\mathbf{c}-e_i}$ to have the edge set

$$E(G_{\mathbf{c}-e_i}) \setminus \bigcup_{j \in [n]} \bigcup_{\ell \in \mathcal{N}_{\mathbf{c}-e_j}(r_j)} (\ell, r_j).$$

Denote by $\widetilde{\mathcal{N}}_{\mathbf{c}-e_i}(r_p)$ the neighborhood of r_p in $H_{\mathbf{c}-e_i}$. Define the *distance* between $\mathbf{c} - e_w$ and $\mathbf{c} - e_v$ to be

$$\operatorname{dist}(\mathbf{c} - e_w, \mathbf{c} - e_v) := \text{the number of elements } k \text{ such that } \widetilde{\mathcal{N}}_{\mathbf{c}-e_v}(r_k) \neq \widetilde{\mathcal{N}}_{\mathbf{c}-e_w}(r_k).$$

Observe that this distance is always greater than or equal to 2 for distinct $\mathbf{v}, \mathbf{w}$, since they themselves are always included. If $\operatorname{dist}(\mathbf{c} - e_w, \mathbf{c} - e_v) = 2$, then there exists a linear syzygy edge between $\mathbf{c} - e_v$ and $\mathbf{c} - e_w$.

Suppose, seeking contradiction, that one may partition the vertices of $\mathbf{c}$ into V_1 and V_2 such that there is no linear syzygy edge between any vertex in V_1 and any vertex in V_2. Choose $\mathbf{c} - e_v \in V_1$ and $\mathbf{c} - e_w \in V_2$ which minimizes $\operatorname{dist}(\mathbf{c} - e_v, \mathbf{c} - e_w)$; call this distance dist. For ease of notation, let $v = 1$ and let $w = 2$. Then $H_{\mathbf{c}-e_1}$ and $H_{\mathbf{c}-e_2}$ are depicted in Figure 8.2. Because v and w cannot be connected by assumption, $\operatorname{dist} \geq 3$. Suppose, again for ease of notation, that $i_p \neq j_p$ for $p \in [3, \operatorname{dist}]$ and $i_p = j_p$ for all $p \in [\operatorname{dist}, n]$.

Now consider the vertex $\mathbf{c} - e_3$. By (T4), one has that $\mathcal{N}_{\mathbf{c}-e_1}(r_3) \neq \mathcal{N}_{\mathbf{c}-e_2}(r_3)$ implies that $\mathcal{N}_{\mathbf{c}-e_3}(r_2) = \mathcal{N}_{\mathbf{c}-e_1}(r_2)$ and $\mathcal{N}_{\mathbf{c}-e_3}(r_1) = \mathcal{N}_{\mathbf{c}-e_2}(r_1)$. Then $H_{\mathbf{c}-e_3}$ is the graph depicted in Figure 8.3. But now, each k_ℓ in Figure 8.3 must be equal to either i_ℓ or j_ℓ.

Suppose that $\mathcal{N}_{\mathbf{c}-e_3}(r_p) \neq \mathcal{N}_{\mathbf{c}-e_1}(r_p)$ **and** $\mathcal{N}_{\mathbf{c}-e_3}(r_p) \neq \mathcal{N}_{\mathbf{c}-e_2}(r_p)$. Then by (T4), one has

$$\mathcal{N}_{\mathbf{c}-e_p}(r_3) = \mathcal{N}_{\mathbf{c}-e_1}(r_3) \quad \text{and} \quad \mathcal{N}_{\mathbf{c}-e_p}(r_3) = \mathcal{N}_{\mathbf{c}-e_2}(r_3).$$

Together, these imply that $\mathcal{N}_{\mathbf{c}-e_p}(r_3) = \mathcal{N}_{\mathbf{c}-e_1}(r_3) = \mathcal{N}_{\mathbf{c}-e_2}(r_3)$, which is a contradiction. So now one has that

$$\operatorname{dist}(\mathbf{c} - e_1, \mathbf{c} - e_3) < \operatorname{dist}(\mathbf{c} - e_1, \mathbf{c} - e_2), \quad \operatorname{dist}(\mathbf{c} - e_2, \mathbf{c} - e_3) < \operatorname{dist}(\mathbf{c} - e_1, \mathbf{c} - e_2)$$

which contradicts dist being the minimal distance between V_1 and V_2, regardless of whether $\mathbf{c} - e_3$ is in V_1 or V_2. □

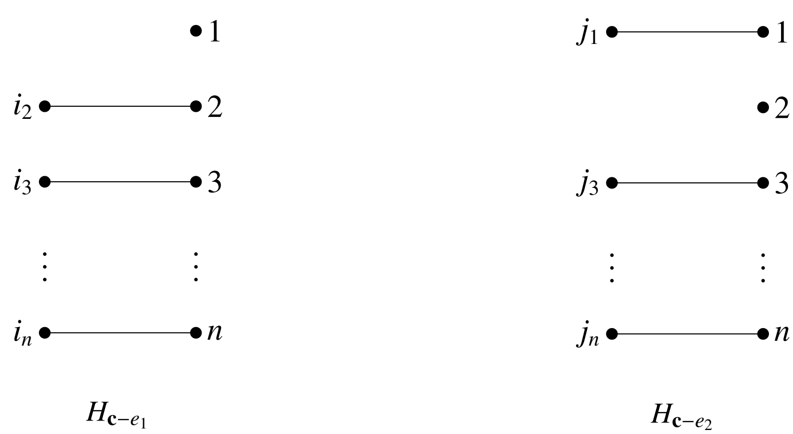


Figure 8.2: Graphs $H_{\mathbf{c}-e_i}$ in the proof of Theorem 8.1.4.

In particular, we have the following corollary.

Corollary 8.1.5. *Adopt Notation 8.1.3 and let $\mathcal{G}$ be a trianguloid. Then $I(\mathcal{G})$ is a polarization of a power of the graded maximal ideal.*

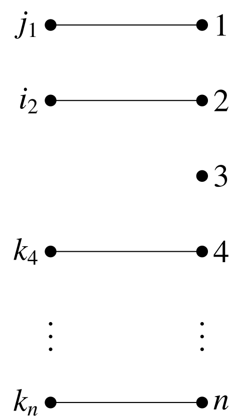


Figure 8.3: $H_{\mathbf{c}-e_3}$ in the Proof of Theorem 8.1.4.

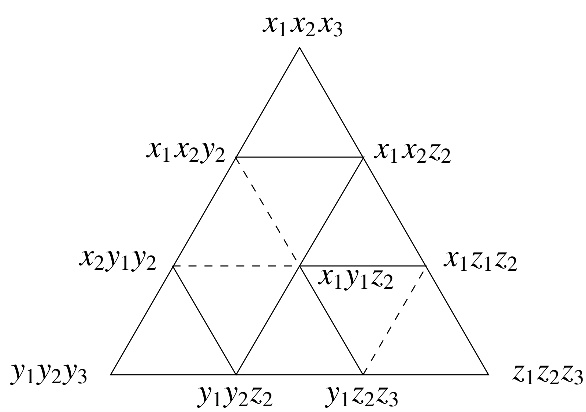


Figure 8.4: A polarization which does not give a mixed subdivision.

As the following example shows, the converse of the above corollary does not hold.

Example 8.1.6. Figure 8.4 depicts the same polarization of $(x, y, z)^3$ from Example 2.5.2. This polarization does not give rise to a mixed subdivision. In particular, note the cell with five vertices which is not a mixed cell. One can verify that it is not possible to give this polarization a labeling up to symmetry which also satisfies (T2) by studying closely the possible vertex labels of the unmixed cell.

BIBLIOGRAPHY

[1] Ayah Almousa, Gunnar Fløystad, and Henning Lohne. Polarizations of powers of graded maximal ideals. *arXiv preprint arXiv:1912.03898*, 2019.

[2] Klaus Altmann, Mina Bigdeli, Jürgen Herzog, and Dancheng Lu. Algebraically rigid simplicial complexes and graphs. *Journal of Pure and Applied Algebra*, 220(8):2914–2935, 2016.

[3] Annetta Aramova, Jürgen Herzog, and Takayuki Hibi. Shifting operations and graded Betti numbers. *Journal of Algebraic Combinatorics*, 12(3):207–222, 2000.

[4] Federico Ardila and Mike Develin. Tropical hyperplane arrangements and oriented matroids. *Mathematische Zeitschrift*, 262(4):795–816, 2009.

[5] Ekkehard Batzies and Volkmar Welker. Discrete Morse theory for cellular resolutions. *Journal fur die Reine und Angewandte Mathematik*, 543:147–168, 2002.

[6] David Bernstein and Andrei Zelevinsky. Combinatorics of maximal minors. *Journal of Algebraic Combinatorics*, 2(2):111–121, 1993.

[7] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier Sci. B. V., Amsterdam, 1995.

[8] Heinz Bruggesser and Peter Mani. Shellable decompositions of cells and spheres. *Mathematica Scandinavica*, 29(2):197–205, 1971.

[9] David A Buchsbaum and David Eisenbud. Generic free resolutions and a family of generically perfect ideals. *Advances in Mathematics*, 18(3):245–301, 1975.

[10] Alessio DAlì, Gunnar Fløystad, and Amin Nematbakhsh. Resolutions of co-letterplace ideals and generalizations of Bier spheres. *Transactions of the American Mathematical Society*, 371(12):8733–8753, 2019.

[11] John A Eagon and Victor Reiner. Resolutions of Stanley-Reisner rings and Alexander duality. *Journal of Pure and Applied Algebra*, 130(3):265–275, 1998.

[12] Laura Escobar and Karola Mészáros. Subword complexes via triangulations of root polytopes. *Algebraic Combinatorics*, 1(3):395–414, 2018.

[13] Sara Faridi. Monomial ideals via square-free monomial ideals. *Lecture Notes in Pure and Applied Mathematics*, 244:85–114, 2005.

[14] Gunnar Fløystad. The colorful Helly theorem and colorful resolutions of ideals. *Journal of Pure and Applied Algebra*, 215(6):1255–1262, 2011.

[15] Gunnar Fløystad. Poset ideals of p-partitions and generalized letterplace and determinantal ideals. *Acta Mathematica Vietnamica*, 44(1):213–241, 2019.

[16] Gunnar Fløystad, Bjørn Møller Greve, and Jürgen Herzog. Letterplace and co-letterplace ideals of posets. *Journal of Pure and Applied Algebra*, 221(5):1218–1241, 2017.

[17] Robin Forman. Morse theory for cell complexes. *Advances in mathematics*, 134(1):90–145, 1998.

[18] Robin Forman. A user's guide to discrete Morse theory. *Séminaire Lotharingien de Combinatoire [electronic only]*, 48:B48c–35, 2002.

[19] Ralf Fröberg. A study of graded extremal rings and of monomial rings. *Mathematica Scandinavica*, pages 22–34, 1982.

[20] Pavel Galashin, Gleb Nenashev, and Alexander Postnikov. Trianguloids and triangulations of root polytopes. *arXiv preprint arXiv:1803.06239*, 2018.

[21] Vesselin Gasharov. Green and Gotzmann theorems for polynomial rings with restricted powers of the variables. *Journal of Pure and Applied Algebra*, 130(2):113–118, 1998.

[22] Vesselin Gasharov, Takayuki Hibi, and Irena Peeva. Resolutions of a-stable ideals. *Journal of Algebra*, 254(2):375–394, 2002.

[23] Robin Hartshorne. Connectedness of the Hilbert scheme. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 29(1):7–48, 1966.

[24] Jürgen Herzog. A generalization of the Taylor complex construction. *Communications in Algebra*, 35(5):1747–1756, 2007.

[25] Jürgen Herzog and Takayuki Hibi. Monomial ideals. In *Monomial Ideals*. Springer, 2011.

[26] Silke Horn. A topological representation theorem for tropical oriented matroids. *Journal of Combinatorial Theory, Series A*, 142:77–112, 2016.

[27] Birkett Huber, Jörg Rambau, and Francisco Santos. The Cayley trick, lifting subdivisions and the bohne-dress theorem on zonotopal tilings. *Journal of the European Mathematical Society*, 2(2):179–198, 2000.

[28] Gil Kalai et al. Algebraic shifting. In *Computational commutative algebra and combinatorics*, pages 121–163. Mathematical Society of Japan, 2002.

[29] Gil Kalai and Roy Meshulam. A topological colorful Helly theorem. *Advances in Mathematics*, 191(2):305–311, 2005.

[30] Henning Lohne. The many polarizations of powers of maximal ideals. *arXiv preprint arXiv:1303.5780*, 2013.

[31] Georg Loho and Ben Smith. Matching fields and lattice points of simplices. *Advances in Mathematics*, 370:107232, 2020.

[32] Peter McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17(2):179–184, 1970.

[33] Karola Mészáros. Root polytopes, triangulations, and the subdivision algebra, I. *Transactions of the American Mathematical Society*, 363(8):4359–4382, 2011.

[34] Karola Mészáros. Root polytopes, triangulations, and the subdivision algebra, II. *Transactions of the American Mathematical Society*, 363(11):6111–6141, 2011.

[35] Karola Mészáros. Pipe dream complexes and triangulations of root polytopes belong together. *SIAM Journal on Discrete Mathematics*, 30(1):100–111, 2016.

[36] Satoshi Murai. Generic initial ideals and squeezed spheres. *Advances in Mathematics*, 214(2):701–729, 2007.

[37] Satoshi Murai. Spheres arising from multicomplexes. *Journal of Combinatorial Theory, Series A*, 118(8):2167–2184, 2011.

[38] Uwe Nagel and Victor Reiner. Betti numbers of monomial ideals and shifted skew shapes. *the electronic journal of combinatorics*, pages R3–R3, 2009.

[39] Suho Oh and Hwanchul Yoo. Triangulations of $\Delta^{n-1} \times \Delta^{d-1}$ and tropical oriented matroids. In *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, pages 717–728. Discrete Mathematics and Theoretical Computer Science, 2011.

[40] Suho Oh and Hwanchul Yoo. Matching ensembles (extended abstract). In *In 27th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2015)*, pages 667–672. Discrete Mathematics and Theoretical Computer Science, 2015.

[41] Irena Peeva. *Graded syzygies*, volume 14. Springer Science & Business Media, 2010.

[42] Irena Peeva and Mauricio Velasco. Frames and degenerations of monomial resolutions. *Transactions of the American Mathematical Society*, pages 2029–2046, 2011.

[43] Alexander Postnikov. Permutohedra, associahedra, and beyond. *International Mathematics Research Notices*, 2009(6):1026–1106, 2009.

[44] Gian-Carlo Rota and Joel A Stein. Standard basis in supersymplectic algebras. *Proceedings of the National Academy of Sciences of the United States of America*, 86(8):2521, 1989.

[45] Francisco Santos. The Cayley trick and triangulations of products of simplices. *Contemporary Mathematics*, 374:151–178, 2005.

[46] Ludwig Schläfli. *Theorie der vielfachen Kontinuität*, volume 38. Zürcher & Furrer, 1901.

[47] Philip William Schwartau. *Liaison addition and monomial ideals*. PhD thesis, Brandeis University, 1982.

[48] Ali Soleyman Jahan. Prime filtrations and Stanley decompositions of squarefree modules and Alexander duality. *Manuscripta Math.*, 130(4):533–550, 2009.

[49] Richard P Stanley. Balanced Cohen–Macaulay complexes. *Transactions of the American Mathematical Society*, 249(1):139–157, 1979.

[50] Richard P Stanley. Enumerative combinatorics volume 1 second edition. *Cambridge studies in advanced mathematics*, 2011.

[51] Jerzy Weyman. *Cohomology of vector bundles and syzygies*, volume 149. Cambridge University Press, 2003.

[52] Kohji Yanagawa. Alternative polarizations of Borel fixed ideals. *Nagoya Mathematical Journal*, 207:79–93, 2012.