DYNAMIC PRICING AND SEGMENTATION OPPORTUNITIES THROUGH LOYALTY PROGRAMS

A Thesis

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ABSTRACT

Brands often use loyalty programs to offer customers points, miles, credits, etc... as incentives to create loyalty and drive retention. While loyalty programs are an established marketing research area - pricing, inventory and revenue management questions around the use of loyalty programs remain. Pricing and inventory decisions related to loyalty programs faced by franchisees are of particular interest as brand based loyalty programs are 'funded' by the franchisee and owner profitability and brand loyalty may not always be fully aligned. In this thesis, we build a model which examines dynamic pricing and segmentation opportunities created through the use of loyalty programs. We first analyze improvements in customer segmentation created through the use of loyalty programs and the ability of customers to purchase goods/services with both cash and/or through the use of points/miles. Further, we investigate the role of selling through intermediaries where consumers may not earn loyalty benefits (points) from their inter-mediated purchase. Lastly, we look at the role of dynamic pricing (of both cash and points purchases) as the franchisee looks to meet occupancy thresholds in an effort to increase the value paid to the owner by the brand for product purchases made though redeemed points/miles.

BIOGRAPHICAL SKETCH

Yue Liang received her Bachelor's degree from Florida International University in 2016. After graduation she joined Marriott International as a Revenue Analyst in Hangzhou Cluster Revenue Office. She then got promoted in September 2018 as Revenue Management champion for Hangzhou Marriott Hotel Qianjiang and Courtyard by Marriott hotel Qianjiang. She found this research question when Marriott Bonvoy was launched after Marriott-Starwood merge. In 2019 she pursued research Master degree at Cornell University under the guidance of Professor Chris Anderson and Helen Chun.

This document is dedicated to all Cornell graduate students.

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My sincere thanks also go to Marriott International where I was inspired to have this research question. And the dynamic points inventory control strategy by my boss Mr. David Qi made it a best practice to maximize revenue, which was hugely supported from the General Manager Mr. Peter Pan of Hangzhou Marriott Hotel Qianjiang and his fabulous team. What I learned from Marriott and those wonderful leaders are valuable assets that I cherish forever. And I appreciate the communication with Marriott Bonvoy members Mr. Bobby Hu, Miss Tina Xu, and Mr. Ruolin Miao who expressed their perspectives as Marriott Bonvoy elite members.

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1 Introduction

Loyalty programs are a common marketing effort to increase customer purchase frequency, increase the consumer base and hopefully increase profitability. Loyalty programs come in many different forms from offering every n^{th} item free to the points and miles type programs that are prolific in hospitality today. In many loyalty programs, customers may earn points or miles for a stay, for example Marriott Bonvoy members earn 10 points per dollar spent on eligible hotel stays, with SkyMile members on Delta flights earning 5 miles for each dollar spent on eligible tickets. More recently, consumers can also earn points or miles on non-firm specific purchases through co-branded (e.g. a Delta branded American Express card) credit cards. Co-branded credit cards have become very common across hospitality with US Airlines generating an estimated $4 billion in incremental revenue from the sale of miles to credit card partners (IdeaWorks, 2008).

With accumulated points, either directly through service/product purchases, or indirectly though co-branded credit-cards, customers can then redeem for products (hotel rooms, flights, tickets, cosmetics, etc.) This redemption behavior results in points having some cash-equivalence. The cash value of points will depend on the number required for redemption as well as face value of the item purchased through redemption. Historically, hotels have largely kept the points/stay element of cash-equivalence fairly constant, whereas airline typically have a variable points (miles) / flight model dependent upon demand for the underlying seats. Recently, Marriott and Hilton have championed a dynamic model for points redemption (Grant, 2019). On Marriott's Bonvoy web page (MarriottInternational, 2020), required points to redeem hotels have been dynamically set by three seasons: off, standard and peak and by three activities: PointSavers, Points and Cash+Points.

While loyalty programs may seem to be advantageous and clearly create brand loyalty, their impact to individual firms may not be as clear depending on the nature and structure of the loyalty program. While a centralized firm, like a large airline, may have more direct control over the structure and value of loyalty programs, in decentralized franchiser-franchisee structures (typical of large hotel brands) the franchisee may have less control. For example, an individual branded hotel owner - say the franchisee of a large brand like Marriott or Hilton, will be required to pay the brand for points provided to a customer upon a hotel stay. Similarly, when a guest redeems his/her points for a stay at a branded hotel the brand will need to pay the hotel owner for the redeemed points. Unfortunately, for the hotel owner, what they pay for the points and what they receive are not equivalent, with the difference helping to fund the chains brand building efforts. Under most settings, what the hotel owner receives financially for a redeemed hotel stay will be on a sliding scale dependent upon the occupancy of the hotel on the redeemed night. For low occupancy redemption, the hotel may receive only slightly above its variable cost, receiving a percentage ($< 100\%$) of recent average daily rates (ADR) as the hotel approaches occupancy in the high 90's.

In the following we explore the dynamics of loyalty programs from a pricing standpoint, specially we focus upon how loyalty programs create segmentation opportunities for firms. We then focus on the dynamics of redemption and demand as they relate to owner profitability, illustrating the trade-off between posted prices and variable redemption (points/stay) as they are impacted by the cash equivalents ($ per redeemed stay) that a franchisee receives as the result of occupancy thresholds. While we use lodging as our motivation, the dynamics of our approach hold for any franchiser—franchisee setting where loyalty programs are partially funded by the franchisee.

2 Literature Review

Loyalty programs are one of the most effective ways to establish loyal connections between customers and service firms (Evanschitzky et al., 2012; Yi and Jeon, 2003). Studies have shown that loyalty programs contribute to a more frequent repurchasing behavior from customers, thus bringing more revenue to service firms (Bolton et al., 2000). Loyal customers are willing to pay even when the product rate goes up (Rowley, 2005). It was also stated that loyal customers broadcast positive word of mouth (Bowen and Chen, 2001).

American Airlines pioneered the first loyalty program in 1981 (Berman, 2006). Two years later, InterContinental championed the first hotel loyalty program (Dekay, 2009), followed by Marriott in the same year (Zuo, 2018). At the end of 2018, Marriott Bonvoy had become the largest and most valuable travel program in the lodging industry as Bonvoy had reached nearly 125 million members. Both reward redemption and member contribution percentages reached record levels (Sorenson, 2019). Loyalty programs are one of the most effective ways to establish loyal connections between customers and service firms (Evanschitzky et al., 2012; Yi and Jeon, 2003). Studies have shown that loyalty programs contribute to a more frequent repurchasing behavior from customers, thus bringing more revenue to service firms (Bolton et al., 2000). Loyal customers are willing to pay even when the product rate goes up (Rowley, 2005). It was also stated that loyal customers broadcast positive word of mouth (Bowen and Chen, 2001).

In the eyes of consumers, one of the advantages of loyalty programs is reduced cost or free future purchases. A variety of information sources exist to help consumers maximize the value of the loyalty program points they have accumulated. Online forums (thepointsguy.com, onemileatatime.com, flyertea.com, etc.) provide information enabling consumers to quickly achieve elite membership levels as well as provide guidance on how to maximize the dollar value of their points (Samantha, 2020). Customers often refer to these resources to decide whether to pay by points, by cash or a combination of points and cash (Schlappig, 2020). The success and/or popularity of hospitality related loyalty programs has resulted in frequent flier miles becoming the second largest currency after the US dollar, as approximately 500 billion miles are distributed by 100 million travellers each year . The aggregate number of unredeemed miles is estimated at 8.5 trillion, assuming no new miles are accumulated, it would take nearly 23 years to redeem current outstanding miles (economist, 2002).

While there exists a variety of consumer facing sources on loyalty programs and their value, the academic literature has only recently started to focus on the financial or currency aspects of loyalty programs. Drèze and Nunes (2004) first considered using points as a payment at the beginning of 2000. They introduced several novel currencies (points and flier miles) from loyalty programs such as Marriott Rewards, American Express and Delta SkyMiles, respectively. Such currencies can be functioned as paper money which could be budgeted, saved and spent by consumers as well.

One of the difficulties in valuing rewards programs stems from consumers hesitation in redemption. We do not really know why loyalty program members redeem or why they do not ((Bijmolt et al., 2011). Some consumers hesitate to redeem points because they want more points to reach higher rewards levels and to enjoy more benefits (Peter, 2006). Others may simply have a pathological desire to stockpile points (Saxena et al., 2004). Kwong et al. (2011) have studied factors that impact customers to redeem and indicated that consumers are more willing to spend points when there are easily-anticipated benefits to redeem points. When it is easier to calculate the percentage savings with point purchases, customers will decide to spend points. Stourm et al. (2015) argued that unredeemed points can expire, and they can lose their value if the retailer enters bankruptcy or alters the program rules. Delaying redemption makes customers forgo the time value of money from delayed rewards. They present a model that unites economic, cognitive and psychological motivations for persistent stockpiling in loyalty programs, even though the retailer does not explicitly reward point accumulation. The data shows that redemption behavior is mostly impacted by cognitive and psychological incentives. Their findings also provide insights into how customers are likely to respond to communication strategies, promotions and policy changes.

We have so far discussed about the background of loyalty programs, theoretical and practical studies about reward redemption, plus two more theories related with dynamic redemption pricing. Reward redemption is a benefit that hotels appreciate customers who join their loyalty program after their stays. Customers are motivated by points-pressure, reward behavior and personalized marketing strategies to "unlock" higher membership levels for more privilege benefits. We also consider reward points as a novel currency and mental accounting theory to support this study.

The idea of dynamic redemption rates is related to dynamic pricing, which has been widely implemented in hotels and discussed by many scholars that sellers should adjust pricing based on remaining time and inventory during the selling horizon: i.e. (Ahn et al., 2007; Aviv and Pazgal, 2008). There are limited scholars beginning to address a dynamic redemption pricing but is not yet completely understood. Chung et al. (2019) used a standard dynamic pricing model (Gallego and Van Ryzin, 1994) to study "how reward sales affect prices over a course of selling season."

To our best knowledge, dynamic redemption pricing is believed to be an unexplored area of hotel revenue management. Historically, required number of points to redeem a hotel room is not a function of demand by stay date, but a fixed number set by hotel location and star level. Second, hotels could get compensated by the brand for number of redemption bookings based on occupancy for any given day. Therefore, one typical way for hotels is to push occupancy as high as possible, getting compensated by the brand for loyalty redemption besides regular revenue (Anderson and Xie, 2016).

Perhaps the paper most strongly related to ours is Chung H. and H.S. (2014) where they study dynamic pricing for a service firm where customers can pay via cash (or credit card) as well as redeem rewards points. Similar to our approach customers are heterogeneous in their reservations prices as well as their points balances. Unlike our paper, in their approach customers have a heterogeneous perceived value of a reward points, whereas we assume customers make points versus cash choices to maximize their surplus across their heterogeneous reference price/point balances. Our approach allows for differences across consumers while also providing structural insight into when a firm may engage in different pricing approaches as a function of demand and reimbursement policies set forth by the points issuer.

3 Model Development

We develop a model of a firm selling to loyal strategic consumers, customers are loyal to the firm but strategically decide if they should acquire capacity by paying the posted retail price or through redeeming of points (or miles) accumulated by the firm's loyalty program. Our framework uses seven simple parameters as outlined in Table 1. The firms posts a price P and then sets discount factors δ whereby the price paid of a redemption stay is $\delta_p P$ and the price paid for a restricted purchase, say on an intermediary is $\delta_d P$. Customers have heterogeneous valuations (v) for the firm's product for regular retail (referred to as cash purchases) at price P as well as heterogeneous balances of loyalty program points (γ) used for the purchase of goods or services through the firm's loyalty program at price $\delta_p P$. When products are purchased via loyalty redemption the firm receives a fraction (α) of the redemption stay as a cash equivalent, while customers receive (β) new loyalty program points or credits for retail/cash purchases ($\alpha < \beta$). The firm may choose to differentiate the price of points versus cash purchases uses variable δ to scale the price of a redemption purchase relative to a regular retail purchase, e.g. $P_p = \delta P$. In addition, the firm may choose to offer supply at a discounted price, P_d but without awarding points for future use, e.g. a hotel offering rooms through an opaque or discounted online travel agent channel might also have a discounted cash price where $P_d = \delta_d P$ but the guest does not receive any points or miles for future stays with the brand. We use T as the demand level above which the firms receives a higher redemption reimbursement rate, i.e. with a larger α. We introduce a constraint ζ to regulate steady-state points consumption, i.e. to ensure that sufficient points are generated via cash purchases to fund future points redemption.

Model Notation	
v_i	Customer i's valuation of the product for cash purchases
γ_i	Customer i's points balance
α	Firm conversion factor for points-to-cash
β	Fraction of the retail price P that consumers earn in points
δ	Price ratio for points and inter-mediated selling
T	Demand threshold
ζ	Steady-state points consumption

Table 1: Model Notation

Each customer i looking to acquire service has an independent reference price or valuation v_i for the service provider. Similar to Anderson and Xie (2014), Wang et al. (2009) and Fay (2008) we assume v_i uniformly distributed between 0 and 1, i.e. its density function $f(v_i)$ has value of 1 for $0 \leq v_i \leq 1$ and 0 otherwise.

If the service provider uses regular retail pricing only, and set its price as P. Consumer i has surplus $CS_i = v_i - P$, so only consumers with valuation higher than the price P will purchase. Therefore, we have the expected revenue for the service provider,

$$\pi = \int_P^1 Pf(v_i)dv_i = P(1-P) = P - P^2 \tag{1}$$

Taking the derivative of π with respect to P and setting it to be zero, we can solve for the optimal price: $P^* = \frac{1}{2}$.

Since $\frac{d^2\pi}{dP^2} = -2 < 0$, we substitute$P^*$ back into (1) resulting in (maximized) revenue of $\pi^* = \frac{1}{4}$. Moreover, from (1), it is straightforward to see that the maximum revenue concaves in the prices. Figure 1 summarizes the segmentation (or lack thereof) created by a single posted price.

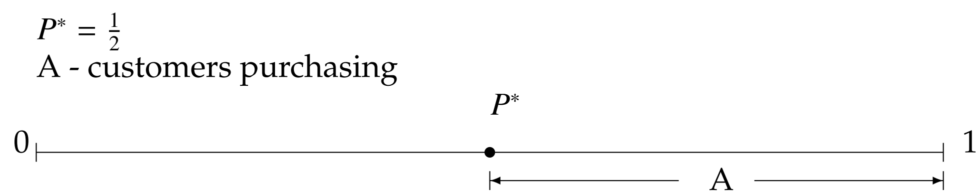


Figure 1: Single Posted Price

The service firm may also allow consumers the option to earn and later redeem loyalty points for capacity instead of having to pay retail. Consumers also have heterogeneous points balances γ_i, and similar to their valuation, it is uniformly distributed between 0 and 1, i.e. its density function $f(\gamma_i)$ has value of 1 for $0 \leq \gamma_i \leq 1$ and 0 otherwise. In addition to using points for product/service purchases, consumers now also earn points when paying retail. Let β represent the fraction of the retail price P that consumers earn in points. A customer now needs to decide if they should pay retail P or pay price P_p using points. For clarity of differentiation we denote someone paying retail prices P as paying with cash versus someone paying with points P_p. Customers decide how (or if) they decide to purchase by maximizing their surplus. A customer may choose to pay cash if $v_i \geq P$, generating a surplus $CS_i = v_i - P + \beta P + \gamma_i$, or they may redeem points if $\gamma_i \geq P_p$ with a surplus $CS_i = v_i + \gamma_i - P_p$. It is obvious that for

$P = P_c = P_p$[1] , if $v_i \geq P$ and the consumer can pay cash for service they will as $CS_i = v_i - P + \beta P + \gamma_i$ is greater than $CS_i = v_i + \gamma_i - P_p$. Figure (2) illustrates the segmentation created through the addition of points redemption, with area A cash purchases, areas B and C having $v_i < P$, but with C having points redemption.

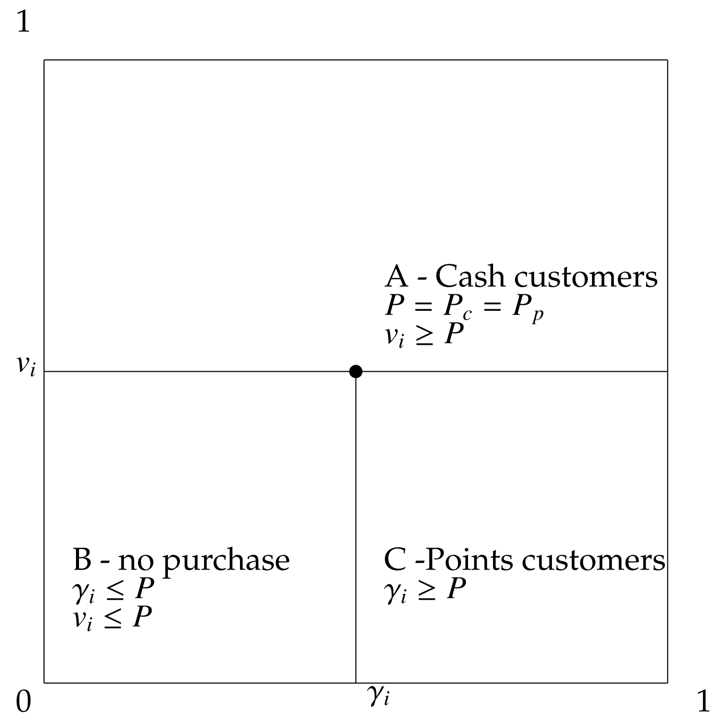


Figure 2: Segmentation through Rewards Redemption

Owing to the segmentation created through redemption, overall demand increases. Demand for cash $D_c = \int_P^1 f(v_i) dv_i = 1 - P$. Cash transactions occur whenever $v_i \geq P$ (with probability $1 - P$). The service firm provides customers with loyalty program points as part of their stay, and is required to 'buy' these points from the brand or franchiser, thus a proportionate cost β is incurred by the service provider. The revenue π to the firm as a function of cash (π_c) is then $\pi_c = (1 - \beta)(P - P^2)$.

[1] In many settings P_c in absolute terms is much different than P, e.g. 15,000 points for a hotel stay, we set $P = P_c$ without any loss of generality as changing P_c would simply require scaling of $f(\gamma)$

Points transaction occur when $\gamma_i \geq P$ and when $v_i \leq P$ (with probability P). As illustrated in Figure 2, consumers only use points if they do not have sufficient cash, which happens with probability P. Thus demand for points purchases $D_p = P\int_P^1 f(v_i)dv_i = P(1-P)$ respectively. Note P has no subscripts as $P = P_c = P_p$. The service provider is compensated for each redemption stay by the brand with factor α, resulting in revenue of αP for redemption $\pi_p = \alpha P^2(1-P)$

Total demand, $D = D_c + D_p = 1 - P^2$, sums up both cash and points demand, resulting in total revenue,

$$\pi = \pi_c + \pi_p = P(1-P)(1-\beta+\alpha P) \tag{2}$$

To find the optimal P, we take the derivative of (2) with respect to P, then set to zero and pick the positive solution. Thus, we get the optimal price,

$$P^* = \frac{-1+\alpha+\beta+\sqrt{1+\alpha+\alpha^2-2\beta-\alpha\beta+\beta^2}}{3\alpha} \tag{3}$$

We plug (3) into (2) for optimal revenue,

$$\pi^* = -\frac{(2+\alpha-2\beta+M)(-1-2\alpha+\beta+M)(-1+\alpha+\beta+M)}{27\alpha^2}, \tag{4}$$

where $M = \sqrt{\alpha+\alpha^2+(-1+\beta)^2-\alpha\beta}$.

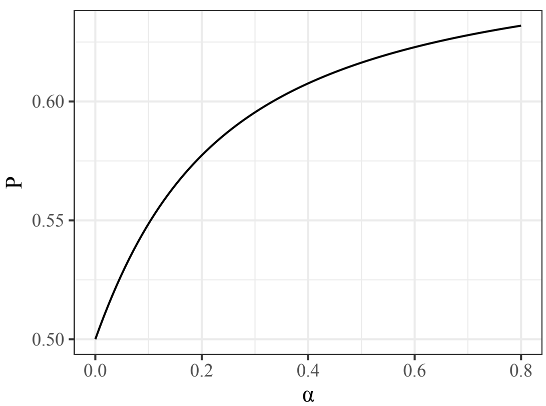
Proposition 1 *When $P = P_c = P_p$, the optimal price strictly increases with respect to α and β.*

Proof. We take the derivatives P^* w.r.t to both α and β for (3):

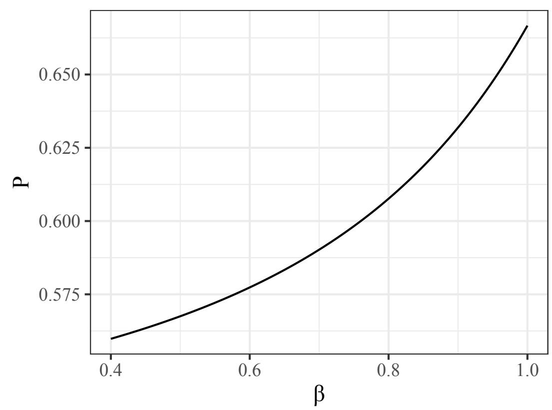
$$\frac{dP^*}{d\alpha} = \frac{(1-\beta)(2+\alpha-2\beta+2\sqrt{\alpha+\alpha^2+(\beta-1)^2-\alpha\beta})}{6\alpha^2\sqrt{\alpha+\alpha^2+(\beta-1)^2-\alpha\beta}} > 0$$

$$\frac{dP^*}{d\beta} = \frac{1 + \frac{2+\alpha-2\beta}{2\sqrt{\alpha+\alpha^2+(\beta-1)^2-\alpha\beta}}}{3\alpha} > 0$$

Example In Figure 3 we illustrate the impacts of α and β upon P^*. We fix $\alpha = 0.4$ to see how P^* changes with respect to β, and $\beta = 0.8$ to see how P^* changes with respect to α. P^* increases with α and β which verifies the above two partial derivatives.



(a) P^* increases with α when $\beta = 0.8$



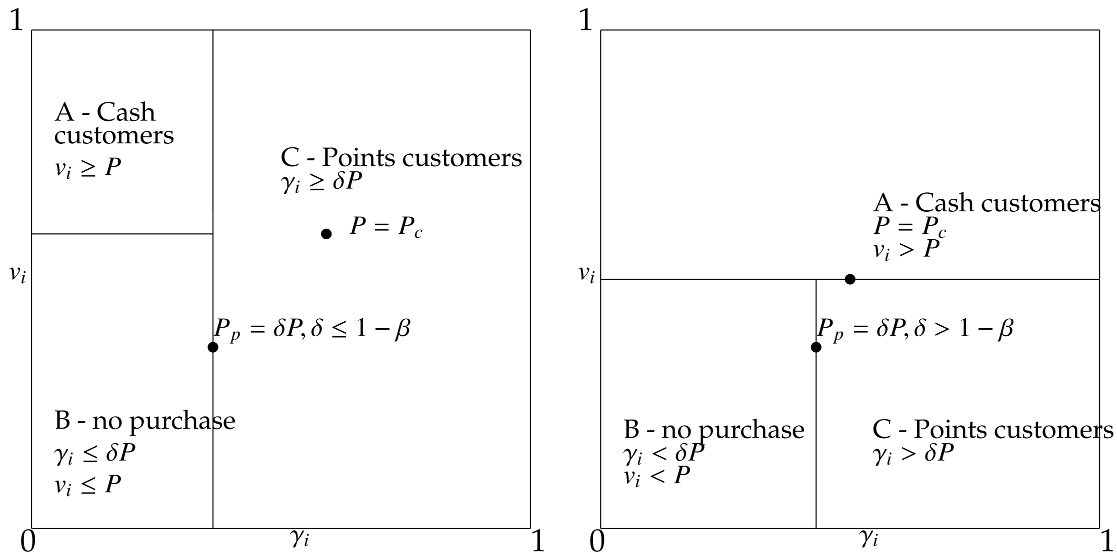
(b) P^* increases with β when $\alpha = 0.4$

Figure 3: P^* changes with α and β at unconstrained prices

3.1 Differential Redemption Pricing

Setting $P = P_c = P_p$, independent of conversion factors and scaling parameters (or distributions) may have a restrictive assumption, as such in the following section we introduce a discount factor δ resulting in $P_p = \delta P$ allowing for differential redemption pricing. The introduction of δ means a points purchase can now take place $\gamma_i \geq \delta P$ with surplus $CS_i = v_i + \gamma_i - \delta P$ having to be compared to $CS_i = v_i - P + \beta P + \gamma_i$ when a consumer is deciding on a payment form.

It is straight forward to show that if $\delta \leq 1-\beta$ then $v_i+\gamma_i-\delta P \geq v_i-P+\beta P+\gamma_i$ and if a customer has sufficient points ($\gamma_i \geq \delta P$) then a points purchase is preferred - even if the customer has $v_i \geq P$. If $\delta \leq 1 - \beta$ then customer segmentation is as illustrated in Figure 4 (a) whereas under larger δ segmentation ($\delta > 1 - \beta$) follows Figure 4 (b).



(a) $\delta \leq 1 - \beta$ (b) $\delta > 1 - \beta$

Figure 4: Segmentation through Differential Pricing

Given the differences between consumer preferences for points versus cash purchases, two demand scenarios arise:

- **Case I**: $\delta < 1 - \beta$, Deep discount
- **Case II:** $\delta \geq 1 - \beta$, Light discount or points premium

3.1.1 Deeply Discounted Redemption Prices

If points (relative to cash) are deeply discounted, customers prefer to redeem points for goods and services over paying cash. Points transactions occur whenever $v_i \geq \delta P$ (with probability $1 - \delta P$) and as a result points demand $D_p = 1 - \delta P$.

Multiplying points demand by firm revenue, $\alpha\delta P$, we get the redemption revenue $\pi_p = \alpha\delta P(1-\delta P)$. Cash transactions occur whenever $v_i \geq P$ and consumers have insufficient points, which happens with probability δP. Thus, cash demand $D_c = \delta P(1-P)$ with price P, resulting revenue $\pi_c = \delta P^2(1-P)$.

Total demand as a summation of cash and points demand,

$$D = 1 - \delta P + \delta P(1-P) = 1 - \delta P^2$$

Firm revenue from points and cash purchases sums up $\delta(1-\beta)P^2(1-P)$ and $\alpha\delta P(1-\delta P)$ respectively,

$$\pi = \delta(1-\beta)P^2(1-P) + \alpha\delta P(1-\delta P) \tag{5}$$

We illustrate it into a constrained optimization programming,

$$\begin{aligned} \max_{P,\delta} \quad & \delta(1-\beta)P^2(1-P) + \alpha\delta P(1-\delta P) \\ \text{s.t.} \quad & \delta \leq 1-\beta \\ & \delta P \leq 1 \\ & P,\delta > 0 \\ & P < 1 \end{aligned}$$

We introduce two Lagrange multipliers λ_1 and λ_2 to rewrite the optimization as

$$L = \delta(1-\beta)P^2(1-P) + \alpha\delta P(1-\delta P) + \lambda_1(1-\delta-\beta) + \lambda_2(1-\delta P) \tag{6}$$

The corresponding KKT conditions,

$$\frac{dL}{dP} = \delta[\alpha - \lambda_2 - 2\alpha\delta P + P(1-\beta)(2-3P)] \tag{6 i}$$

$$\frac{dL}{d\delta} = \lambda_1 + P[\alpha - \lambda_2 - 2\alpha\delta P + P(1-\beta)(1-P)] \tag{6 ii}$$

$$\lambda_1(1-\delta-\beta) = 0 \tag{6 iii}$$

$$\lambda_2(1-\delta P) = 0 \tag{6 iv}$$

Thus we need to discuss all permutations of λ_1 and λ_2 to find the solutions.

I. We assume $\lambda_1 = \lambda_2 = 0$

This assumption satisfies KKT condition (6 iii) and (6 iv)where $1-\delta-\beta > 0$ and $1-\delta P > 0$ and then we plug $\lambda_1 = \lambda_2 = 0$ into KKT conditions (6 i) and (6 ii) and set to zero then get

$$\delta^* = 1 + \frac{1-\beta}{4\alpha}, P^* = \frac{1}{2},$$

However, $\delta^* = 1 + \frac{1-\beta}{4\alpha}$ is greater then 1, while $\delta \leq 1-\beta$ must be smaller than 1. Thus, we could not find any feasible solution when $\lambda_1 = \lambda_2 = 0$.

II. We assume $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

This assumption also satisfies KKT condition (6 iii) and (6 iv)where $1-\delta-\beta > 0$ and $1-\delta P = 0$ and then we plug $\lambda_1 = 0$ into KKT conditions (6 i) and (6 ii)set to zero so we could know $1-P = 2-3P$ thus we can solve for P^* and since $1-\delta P = 0$, δ^* can also be solved.

$$\delta^* = 2, P^* = \frac{1}{2},$$

Obviously this $\delta^* =$ also violates the constraint that $\delta < 1-\beta$. Thus, we also could not find any feasible solution when $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

III. We assume $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

This as well satisfies KKT conditions (6 iii) and (6 iv) where $\delta^* = 1 - \beta$ and $1 - \delta P > 0$. We plug $\delta^* = 1 - \beta$ into (6 i) and set to zero so that we could solve for P^*. There are two solutions for P^* but here we only pick the positive one,

$$P^* = \frac{(1-\beta)(1-\alpha) + M}{3(1-\beta)}$$

where $M = \sqrt{(-1+\beta)[-1+\alpha^2(-1+\beta)+\beta-\alpha(1+2\beta)]}$.

We then plug P^* and δ^* so that we have the optimal revenue

$$\pi^* = \frac{[(1-\beta)(1-\alpha)+M][(1-\beta)(1-\alpha)^2 + M + \alpha(4+2\beta-M)]}{27} \tag{8}$$

IV. We assume $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. This as well satisfies KKT conditions (6 iii) and (6 iv) where $\delta^* = 1 - \beta$ and $1 - \delta P = 0$. Thus,

$$P^* = \frac{1}{1-\beta}$$

However, since $1 - \beta < 1$ so $P^* = \frac{1}{1-\beta} > 1$ violates the constraint that $P < 1$. This is not a feasible solution.

Thus we formulate such lemma.

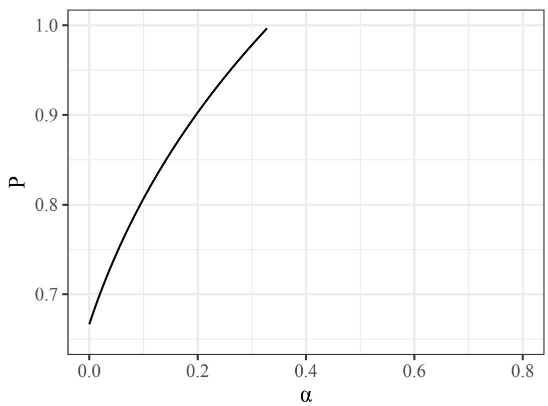
Lemma 1 *When $\delta \leq 1 - \beta$, we can only find the optimum at boundary.*

As with Proposition 1, the optimal price strictly increases with respect to α and β. In this case $\delta = 1 - \beta$ represents a constant and does not change with respect to α but decreases with β. We take derivative of P^* with respect to both α and β to confirm Proposition 1 in the constrained case.

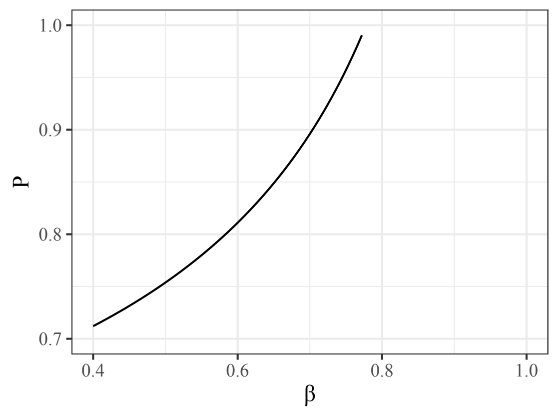
$$\frac{dP^*}{d\alpha} = \frac{1 - \beta - \frac{[-1+2\alpha(-1+\beta)-2\beta](-1+\beta)}{2\sqrt{(-1+\beta)[-1+\alpha^2(-1+\beta)+\beta-\alpha(1+2\beta)]}}}{3(1-\beta)} > 0$$

$$\frac{dP^*}{d\beta} = \frac{\alpha}{2(1-\beta)\sqrt{(-1+\beta)[-1+\alpha^2(-1+\beta)+\beta-\alpha(1+2\beta)]}} > 0$$

Example We still use the example of $\alpha = 0.4$ and $\beta = 0.8$ to show how P changes with respect to both α and β. Similar to the unconstrained case, the boundary optimal price also increases as α and β. But the curves all show a more steep trend compared with the unconstrained optimal price.



(a) P increases with α



(b) P increases with β

Figure 5: P^* changes with α and β at deeply discounted redemption prices

3.1.2 Redemption Premium or Moderate Discount

With moderately discounted points purchases, customers still prefer to use cash if they have enough cash. Thus, cash demand and revenue remain the same with the undifferentiated case.

Though now customers prefer points if they do not have enough cash with probability P, points price now becomes $\alpha\delta P$ after discounting. Thus points demand and revenue now become $D_p = P\int_{\delta P}^{1} f(v_i)dv_i = P(1-\delta P)$ and $\pi_p = \alpha\delta P * P\int_{\delta P}^{1} f(v_i)dv_i = \alpha\delta P^2(1-\delta P)$

Total demand,

$$D = 1 - P + P(1 - \delta P) = 1 - \delta P^2$$

resulting in firm revenue,

$$\pi = (1-\beta)(P - P^2) + \alpha\delta P^2(1 - \delta P) \tag{9}$$

We also write it as a constrained optimization problem,

$$\begin{aligned} \max_{P,\delta} \quad & (1-\beta)(P-P^2) + \alpha\delta P^2(1-\delta P) \\ \text{s.t.} \quad & \delta > 1-\beta \\ & \delta P \leq 1 \\ & P, \delta > 0 \\ & P < 1 \end{aligned}$$

We also transfer this constrained optimization to a new function with two Lagrangian multiplier λ_1 and λ_2,

$$L = (1-\beta)(P - P^2) + \alpha\delta P^2(1 - \delta P) + \lambda_1(\delta + \beta - 1) + \lambda_2(1 - \delta P) \tag{10}$$

The KKT conditions are,

$$\frac{dL}{dP} = 1 + 2P(\alpha\delta - 1) - 3\alpha\delta^2 P^2 + \beta(2P - 1) - \delta\lambda_2 \tag{10 i}$$

$$\frac{dL}{d\delta} = \lambda_1 - P[\lambda_2 - \alpha P(1 - 2\delta P)] \tag{10 ii}$$

$$\lambda_1(\delta + \beta - 1) = 0 \tag{10 iii}$$

$$\lambda_2(1 - \delta P) = 0 \tag{10 iv}$$

Since $\delta < 1 - \beta$, we only need to find internal optimum when $\lambda_1 = 0$, which satisfies KKT condition (10 iii).

I. We assume $\lambda_1 = \lambda_2 = 0$.

To solve for δ^*, we set KKT condition (10 ii) to zero. Then we get $\alpha P^2(1 - 2\delta P) = 0$. Since αP^2 is non-zero, we know that $\delta P = \frac{1}{2}$ and we plug it into KKT condition (10 i) and set to zero then we get,

$$(1-\beta)(1-2P) + \frac{1}{4}\alpha = 0$$

And the optimal price,

$$P^* = \frac{4 + \alpha - 4\beta}{8 - 8\beta}$$

Since $\delta P = \frac{1}{2}$, the optimal discount,

$$\delta^* = \frac{4 - 4\beta}{4 + \alpha - 4\beta}$$

Here we formulate the second proposition.

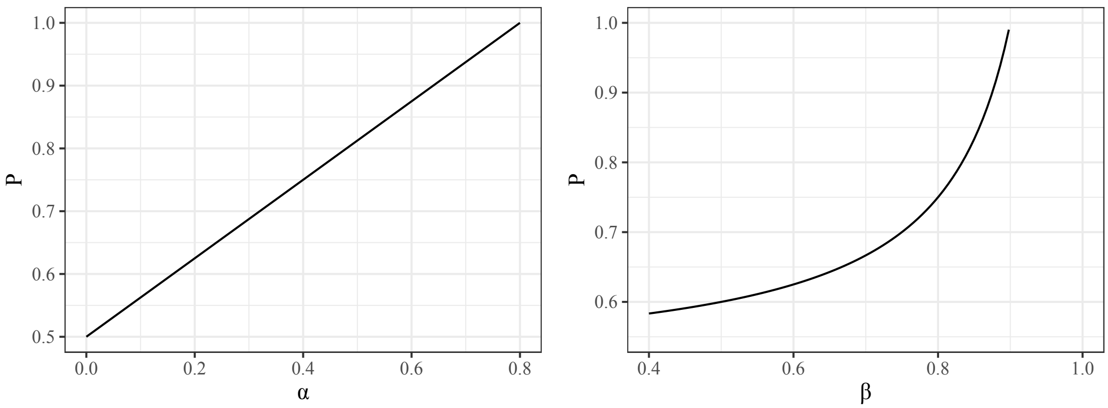
Proposition 2 *When $\delta > 1 - \beta$, the optimal price P and the optimal discount factor δ strictly increase and decrease with respect to α and β.*

Proof. We take derivative of equation (12) and (12) with respect to both α and β.

$$\frac{dP^*}{d\alpha} = \frac{1}{8(1-\beta)} > 0, \frac{dP^*}{d\beta} = \frac{\alpha}{8(1-\beta)^2} > 0$$

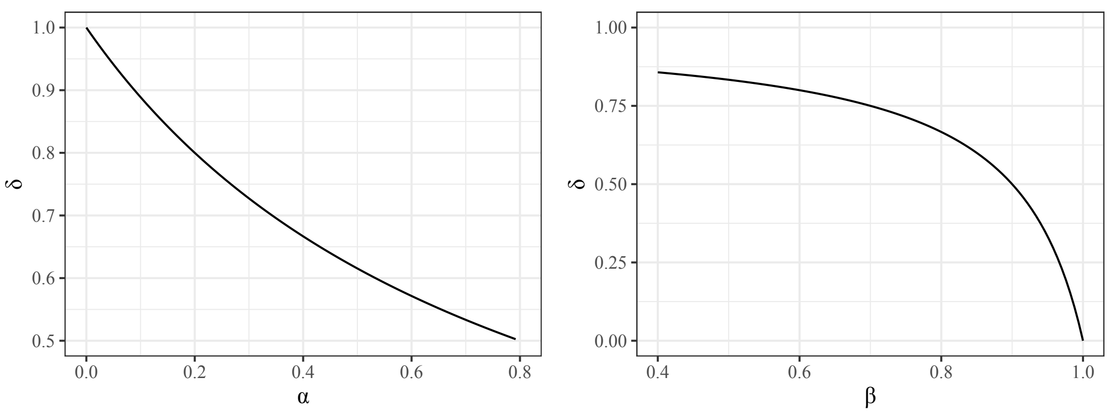
$$\frac{d\delta^*}{d\alpha} = -\frac{4\alpha}{[4(1-\beta)+\alpha]^2} < 0, \frac{d\delta^*}{d\beta} = -\frac{4(1-\beta)}{[4(1-\beta)+\alpha]^2} < 0$$

Example For illustration, we set $\alpha = 0.4$ and $\beta = 0.8$ to see how optimal P and δ change with respect to α and β separately. So from Figure 6 we can tell that similar to unconstrained case, P also increases with both two parameters while δ decreases as shown in Figure 7. It is important to note that δ goes to zero is because firms do not need a very big discount purchase with price.



(a) P^* increases with α (b) P^* increases with β

Figure 6: P^* changes with α and β at redemption premium or moderate discount



(a) δ^* decreases with α (b) δ^* decreases with β

Figure 7: δ^* changes with α and β at redemption premium or moderate discount

We then plug the optimal values back into the original revenue function to obtain,

$$\pi^* = \frac{(4 + \alpha - 4\beta)^2}{64(1 - \beta)} \tag{12}$$

II. We assume $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

This assumption also satisfies KKT condition (10 iii) and (10 iv)where $\delta > 1-\beta$ and $1-\delta P = 0$ and then we plug $\lambda_1 = 0$ into KKT conditions (10 i) and (10 ii) and set to zero so we could know $1-P = 2-3P$ thus we can solve for P^* and since $1-\delta P = 0$, δ^* can also be solved.

$$\delta^* = 2, P^* = \frac{1}{2},$$

Then we can find $\pi^* = \frac{1-\beta}{4}$.

Here we get another lemma under this case.

Lemma 2 *When $\delta > 1-\beta$, we can always find the optimal internal solutions.*

Therefore, based on Lemma 1 and 2, we get the first theorem.

Theorem 1 *To get maximal revenue, firms never deeply discount on points but always discount lightly on points price.*

3.2 Steady-state points consumption

Logically, firms need customers paying with cash and earning points in order to be able to have customers redeem points in the future. We introduce a parameter, ζ, to constrain cash and redemption demand such that $\zeta * D_C \geq D_P$. It should be noted that points demand may exceed cash demand in the event that the firm allows customers to earn points in other forms, e.g. through a co-branded credit card. For the majority of our analysis we assume $\zeta = 1$, i.e. in steady-state (across all customers) points used by customers have to be equal to points issued by the firm.

In this section, we use an alternative solution method for finding optimal values, where instead of using gradients for P and δ, we solve for δ for a fixed P and then perform a one-dimensional search to find the optimal P. We utilize this approach owing to the additional complexity inducted by the steady state points consumption constraint and the inability to find closed-form solutions for both optimal price and discount factor(s). As a result of Theorem 1, we only need to consider the case of redemption premium or moderate discount, i.e. where $\delta > 1 - \beta$ and consumers prefer to use cash over points. An additional constraint, ζ times cash demand should be no less than points demand, $\zeta(1 - P) - P(1 - \delta P) \geq 0$ is added to the formulation.

Thus, the optimization problem becomes,

$$\begin{aligned} \max_{\delta} \quad & (1-\beta)(P-P^2) + \alpha\delta P^2(1-\delta P) \\ \text{s.t.} \quad & \delta \geq 1-\beta \\ & \zeta(1-P) - P(1-\delta P) \geq 0 \\ & \delta P \leq 1 \\ & \delta > 0 \end{aligned}$$

with resulting Lagrangian,

$$L = (1-\beta)(P-P^2) + \alpha\delta P^2(1-\delta P) + \lambda_1(\delta+\beta-1) + \lambda_2[\zeta(1-P) - P(1-\delta P)] + \lambda_3(1-\delta P) \quad (13)$$

The KKT conditions for a fixed P are

$$\frac{dL}{d\delta} = \lambda_1 + P[-\lambda_3 + P(\alpha + \lambda_2 - 2\alpha\delta P)] \quad \text{(13 i)}$$

$$\lambda_1(\delta+\beta-1) = 0 \quad \text{(13 ii)}$$

$$\lambda_2[\zeta(1-P) - P(1-\delta P)] = 0 \quad \text{(13 iii)}$$

$$\lambda_3(1-\delta P) = 0 \quad \text{(13 iii)}$$

A series of cases arise as a function of the two Lagrange multipliers.

I: We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

This satisfies KKT conditions (13 ii), (13 iii) and (13 iii) where $\delta > 1 - \beta$, $\zeta(1 - P) > P(1 - \delta P)$ and $1 - \delta P > 0$. We then plug $\lambda_1 = \lambda_2 = \lambda_3 = 0$ into KKT condition (13 i) and set it to zero then it is fairly easy to have

$$\delta^* = \frac{1}{2P}$$

II: We assume $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$.

This satisfies KKT conditions (13 ii), (13 iii) and (13 iii) where $\delta = 1 - \beta$, $\zeta(1 - P) > P(1 - \delta P)$ and $1 - \delta P > 0$.

$$\delta^* = 1 - \beta$$

III: We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$ and $\lambda_3 = 0$.

This satisfies KKT condition (13 ii), (13 iii) and (13 iii) where $\delta > 1 - \beta$, $\zeta(1 - P) - P(1 - \delta P) = 0$ and $1 - \delta P > 0$.

$$\delta^* = \frac{P - \zeta + P\zeta}{P^2}$$

IV: We assume $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$.

This satisfies KKT condition (13 ii), (13 iii) and (13 iii) where $\delta > 1 - \beta$, $\zeta(1 - P) - P(1 - \delta P) > 0$ and $1 - \delta P = 0$.

$$\delta^* = \frac{1}{P}$$

To simplify, we assume $\zeta = 1$ where points issued equal to points consumed. We now still use the example of $\alpha = 0.4$ and $\beta = 0.8$. Since we do a one-dimensional search for δ, here we do a plot that indicates how optimal results (δ and profit) change as a function of P. In order to see how δ changes value as constraint changes we show the curve of the difference between cash and points demand when $\zeta = 1$. Here δ uses the left first axes while profit and the demand variance uses the right secondary axes. Cash demand first is greater then demand demand and gradually they become the same and the variance equals to 0, which means the ζ constraint changes from un-tight to tight. δ as a function of P, decreases first then when ζ constraint becomes tight, it starts to increase. Profit increases first but then decreases and maximum is at the vertical line where P is approximately 0.678.

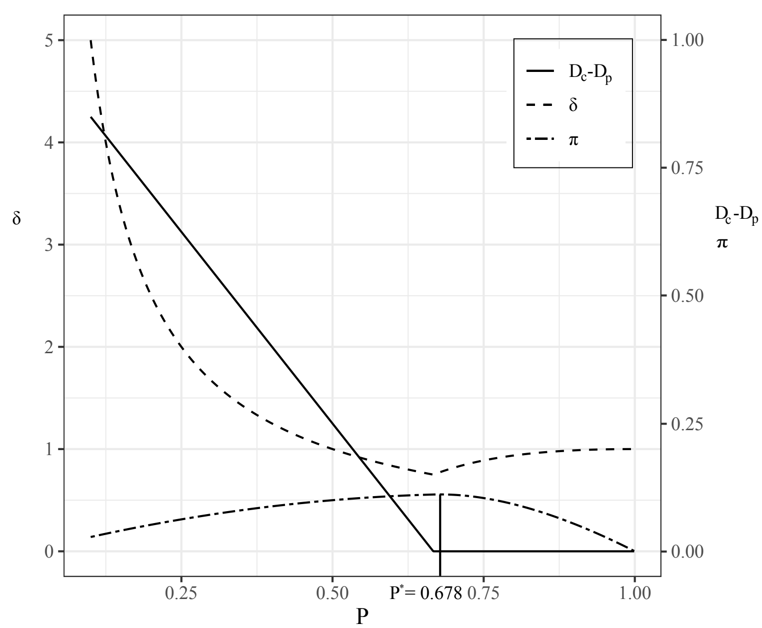


Figure 8: Sensitivity of P at steady-state points consumption

Example In order to show how δ^*, P^*, profit and demand change with respect to α and β, we loop over all possible values of these two variables, find the constrained optimal profit then store P^* and δ^*.

I. Sensitivity of α at steady-state points consumption ($\beta = 0.8$)

	Optimal Values			Profit	Demand		
α	Price (P*)	Point discount (δ^*)	Points price ($\delta^* P^*$)		Total	Cash	Points
0.1	0.562	0.890	0.500	0.063	0.719	0.438	0.281
0.2	0.625	0.800	0.500	0.078	0.688	0.375	0.313
0.3	0.670	0.757	0.507	0.094	0.660	0.330	0.330
0.4	0.678	0.774	0.525	0.111	0.644	0.322	0.322
0.5	0.683	0.785	0.536	0.128	0.634	0.317	0.317
0.6	0.686	0.790	0.542	0.145	0.628	0.314	0.314
0.7	0.689	0.796	0.549	0.162	0.622	0.311	0.311

Table 2: Sensitivity of α at steady-state points consumption ($\beta = 0.8$)

From this table, we can see that with α increases, P also increases. This is consistent with practice as a higher α shows that each room earns a high base rate due to the hotel's high market position. On the other hand, δ shows a trend that increases first then decreases. This is because with lower P, ζ constraint is not tight. Then gradually ζ constraint becomes tight so δ goes up.

Specifically, α at low range indicates a relatively low scale hotels: upper mid-scale or upper scale brands like Courtyard, FourPoints, etc. For these brands, they post low price in market. And since their posted cash price is already relatively low, even though they have the highest δ thus still generate low and same points price which implicates that for these hotels the points price does not change with α. And even if $\alpha = 0.1$ brings a high occupancy than $\alpha = 0.2$, the profit is lower because of posted price P^*. For example, from customers perspective, there is not too much difference for either a Courtyard or FourPoints hotel. They normally have the same hotel category or points price though Best

Available Rates (BAR) slightly differ. And the one with slightly higher BAR has a higher profit even cash demand is lower.

When α goes up, the situation links to higher scale hotels: upper upscale and luxury such as Marriott, Ritz Carlton, etc. All segments has a decreasing demand, while increasing in price, discount factor, and profit. The maximized profit is when α reaches to the highest. With price increases, the occupancy decreases then keep at a stable stage where cash and points demand are equal to 0.31. For example, an 8-category Ritz Carlton requires both highest BAR and points price, its profit still beats any type of other hotels even if they have much higher occupancy.

II. Sensitivity of β at steady-state consumption (α = 0.4)

	Optimal Values			Profit	Demand		
β	Price (P)	Point discount (δ)	Points price (δP)		Total	Cash	Points
0.5	0.600	0.833	0.500	0.180	0.700	0.400	0.300
0.6	0.625	0.800	0.500	0.156	0.688	0.375	0.313
0.7	0.666	0.751	0.500	0.133	0.667	0.334	0.333
0.8	0.678	0.774	0.525	0.111	0.644	0.322	0.322
0.9	0.691	0.800	0.553	0.090	0.618	0.309	0.309

Table 3: Sensitivity of β at steady-state points consumption (α = 0.4)

β as the burning ratio, as it increases, P also increases and δ increases then decreases as well. Maximum profit is when β is the lowest. Occupancy decreases with price increases as well. Since we have α = 0.4 we can take Sheraton hotels but in multiple markets as an example: even though these are all Sheraton hotels, their category may or may not be the same.

Specifically, when β falls in the relatively lower ranges, though price increases as β goes up, with a decreasing δ_p, profit and demand also go down. So here when the price increases, they still share the same hotel category since δ_p needs to decrease. And the lower burning ratio, the less points issued to customers, the higher revenue.

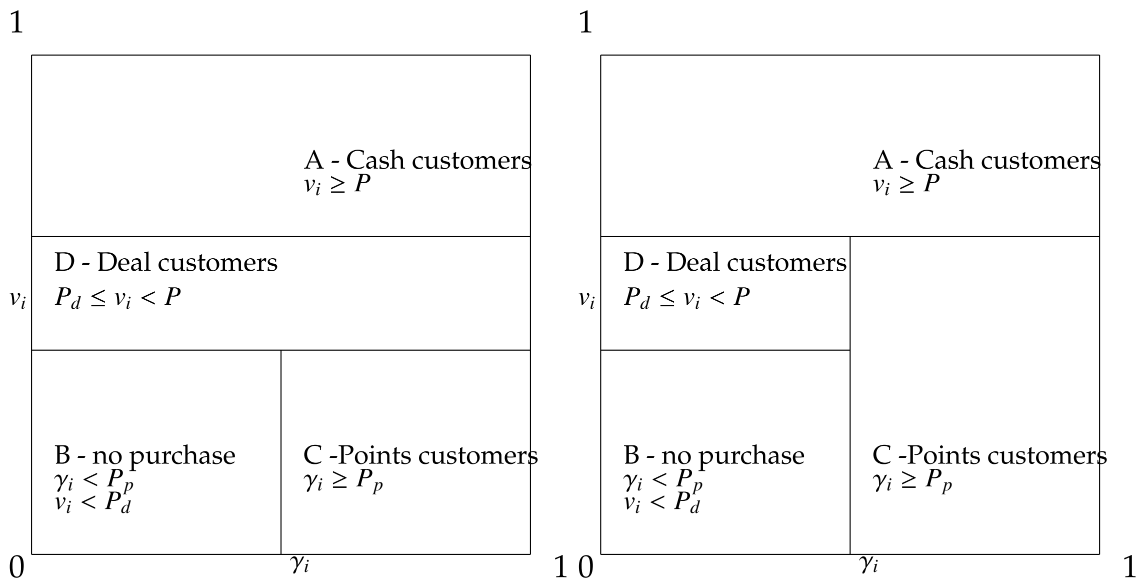
Then when β goes up, though price increases as β goes up, with a decreasing δ_p, profit and demand also go down. Here the same two Sheraton hotels have different categories: the higher β, the higher retail price as well as the higher discount factor to generate a higher points price, thus a higher category. Still, the highest revenue goes to the lowest burning ratio.

3.3 Discounted/Opaque and Inter-mediated Pricing

To increase demand firms may offer capacity at discounted prices. In an effort to further segment customers and reduce dilution capacity offered at discounted prices may be made less desirable through addition of restrictions, e.g. cancellation and refund policies. Given the focus on loyalty programs we utilize a similar restriction to many hotel brands selling rooms via online travel agents where purchasers of rooms on these channels don't receive points with the brand for future stays.

We introduce a discounted deal segment where customers can purchase with at a discounted price $\delta_d P$ while not receiving any points resulting in a surplus of $CS_i = v_i - \delta_d P + \gamma_i$, which we use to compare with cash surplus $CS_i = v_i + \gamma_i - P + \beta P$ and points surplus $CS_i = v_i + \gamma_i - \delta_p P$ where δ_p denotes the discount factor on points price, to compare with previous δ when there is only cash and points segments. For deal segment, we focus on realistic discounting where $\beta + \delta_d > 1$.

As earlier we use this surplus to decide which segment, as a function of v_i and γ_i, a consumer falls within. Since we already know that from Theorem 1, points are never preferred. It is straightforward to show that the key to understand customer preference is to figure out which one is bigger among the two discount factors. Preference order is cash, deal and points when $\delta_p > \delta_d > 1 - \beta$ (illustrated in Figure 9(a) and preference order changes to cash, points and deal when $\delta_d > \delta_p > 1 - \beta$ (illustrated in Figure 9(b).



(a) $\delta_p > \delta_d > 1 - \beta$

(b) $\delta_d > \delta_p > 1 - \beta$

Figure 9: Segmentation through Discounted/Opaque and Inter-mediated Pricing

3.3.1 Discounted Deal Prices Preferred to Points Redemption

When $v_i \geq P$, customers first still prefer cash if they have enough cash with probability $D_c = 1 - P$. Thus, cash segment is the same as unconstrained situation. However, another group of customers may have less cash that only could afford deal price on discounted/opaque channels with probability of $P - \delta_d P$ when $\delta_d P \leq v_i < P$. And if they could not afford deal price but they have enough points, they choose points when $v_i < \delta_d P$ and $\gamma_i \geq P_p$. Otherwise, they leave if $v_i < \delta_d P$ and $\gamma_i < P_p$.

For deal segment, the demand is $D_d = \int_{\delta_d P}^{P} f(v_i) dv_i = P - \delta_d P$. The ADR for each deal room is $\delta_d P$ thus deal revenue is $\pi_d = \delta_d P \int_{\delta_d P}^{P} f(v_i) dv_i = \delta_d P(P - \delta_d P)$. For points segment, since customers could not afford deal price first and customers who could not afford points has a probability of $1 - \delta_p P$ thus points demand is $D_p = \delta_d P \int_{\delta_p P}^{1} f(v_i) dv_i = \delta_d P(P - \delta_p P)$. Each points room now is still worth $\alpha \delta_p P$ so revenue for points segment is $\pi_p = \alpha \delta_p P \delta_d P \int_{\delta_p P}^{1} f(v_i) dv_i = \alpha \delta_p P * \delta_d P(1 - \delta_p P)$.

To sum up all demands of three segments, total demand D

$$D = 1 - \delta_p \delta_d P^2$$

To sum up all demands of three segments, total revenue

$$\pi = P(1 - P)(1 - \beta) + \delta_d P(P - \delta_d P) + \alpha \delta_p P * \delta_d P(1 - \delta_p P) \tag{15}$$

To solve the optimal solutions, we form this optimization program,

$$\begin{aligned}
\max_{\delta_p, \delta_d} \quad & P(1 - P)(1 - \beta) + \delta_d P(P - \delta_d P) + \alpha \delta_p P * \delta_d P(1 - \delta_p P) \\
\text{s.t.} \quad & \delta_d \geq 1 - \beta \\
& \delta_p \geq \delta_d \\
& \zeta(1 - P) - \delta_d P(1 - \delta_p P) \geq 0 \\
& \delta_d \leq 1 \\
& \delta_p P \leq 1 \\
& \delta_d, \delta_p > 0
\end{aligned}$$

As earlier, we introduce three Lagrange multipliers λ_1, λ_2, λ_3, λ_4 and λ_5 to rewrite the objective function,

$$\begin{aligned} L = & P(1-P)(1-\beta) + \delta_d P(P - \delta_d P) + \alpha\delta_p P * \delta_d P(1-\delta_p P) \\ & + \lambda_1(\delta_d + \beta - 1) \\ & + \lambda_2(\delta_p - \delta_d) \\ & + \lambda_3(\zeta(1-P) - \delta_d P(1-\delta_p P)) \\ & + \lambda_4(1-\delta_d) \\ & + \lambda_5(1-\delta_p P) \end{aligned} \tag{16}$$

The KKT conditions are,

$$\frac{dL}{d\delta_p} = \lambda_2 + P[-\lambda_5 + \delta_d P(\alpha + \lambda_3 - 2\alpha\delta_p P)] \tag{16 i}$$

$$\frac{dL}{d\delta_d} = -\lambda_4 + \lambda_1 - \lambda_2 + P[P - 2\delta_d P + \lambda_3(\delta_p P - 1) + \alpha P(\delta_p - \delta_p P^2)] \tag{16 ii}$$

$$\lambda_1(\delta_d + \beta - 1) = 0 \tag{16 iii}$$

$$\lambda_2(\delta_p - \delta_d) = 0 \tag{16 iv}$$

$$\lambda_3(\zeta(1-P) - \delta_d P(1-\delta_p P)) = 0 \tag{16 v}$$

$$\lambda_4(1-\delta_d) = 0 \tag{16 vi}$$

$$\lambda_5(1-\delta_p P) = 0 \tag{16 vii}$$

Similar to our solution approach in 3.2, we do a one-dimensional search for δ given P represents any fixed number and the solutions are,

I. We assume $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption satisfies KKT condition (16 iii), (16 iv), (16 v) (16 vi) and (16 vii) where $\delta_d + \beta - 1 = 0, \delta_p - \delta_d = 0$, $\zeta(1-P) - \delta_d P(1-\delta_p P) > 0$, $1 - \delta_d > 0$ and

$1 - \delta_p P > 0$ thus

$$\delta_p^* = \delta_d^* = 1 - \beta$$

II. We assume $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption satisfies KKT condition (16 iii), (16 iv), (16 v) (16 vi) and (16 vii) where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We then plug $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\delta_d + \beta - 1 = 0$ into KKT condition (16 i) and set it to zero then we get

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = 1 - \beta$$

III. We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ and $\lambda_5 = 0$.

This assumption satisfies KKT condition (16 iii), (16 iv), (16 v), (16 vi) and (16 vii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d = 0$ and $1 - \delta_p P > 0$. We then plug $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = 0$ and $1 - \delta_d = 0$ into KKT condition (16 i) and set it to zero then we get

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = 1$$

IV. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We first plug $\lambda_2 = \lambda_3 = \lambda_5 = 0$ into KKT condition (16 i) and set it to zero so that we could solve for δ_p^*. Then we plug it as well as $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ into KKT condition (16 ii) and set it to zero to get δ_d^*. Thus,

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = \frac{\alpha + 4P}{8P}$$

V. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_5 \neq 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$ and $1-\delta_p P = 0$. We first could know $\delta_p^* = \frac{1}{P}$. Thus according to it and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ we know δ_d^* by setting KKT condition (16 ii) to zero.

$$\delta_p^* = \frac{1}{P}, \delta_d^* = \frac{1}{2}$$

VI. We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ and $\lambda_5 \neq 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d = 0$ and $1-\delta_p P = 0$. Thus,

$$\delta_p^* = \frac{1}{P}, \delta_d^* = 1$$

VII. We assume $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_5 \neq 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d >$ and $1-\delta_p P = 0$. Thus,

$$\delta_p^* = \frac{1}{P}, \delta_d^* = 1 - \beta$$

VIII. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$ $\lambda_3 = \lambda_4 = 0$ and $\lambda_5 \neq 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$ and $1-\delta_p P = 0$. Thus,

$$\delta_p^* = \delta_d^* = \frac{1}{P}$$

IX. We assume $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii) where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We plug $\delta_d + \beta - 1 = 0$ into $\zeta(1-P) - \delta_d P(1-\delta_p P) = 0$ to solve for δ_p^*. Thus,

$$\delta_p^* = \frac{P(1-\beta+\zeta)-\zeta}{(1-\beta)P^2}, \delta_d^* = 1-\beta$$

X. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii) where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We plug $\delta_p = \delta_d$ into $\zeta(1-P) - \delta_d P(1-\delta_p P) = 0$ to solve for δ_p^*. Thus,

$$\delta_p^* = \delta_d^* = \frac{1 - \sqrt{1+4\zeta(-1+P)}}{2P}$$

or

$$\delta_p^* = \delta_d^* = \frac{1 + \sqrt{1+4\zeta(-1+P)}}{2P}$$

XI. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We plug these equations into KKT condition (16 i) and (16 ii) and set them to zero so that we could get two equations to solve the two equal discount factors. Solving them simultaneously and thus,

$$\delta_p^* = \delta_d^* = \frac{-1 + \alpha + \sqrt{1 + \alpha^2 - 2\alpha + 3\alpha P}}{3\alpha P}$$

XII. We assume $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$ and $1-\delta_p P > 0$. We plug $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5 = 0$ into KKT condition (16 i) and (16 ii) and set them to zero, as well as $\zeta(1-P) = \delta_d P(1-\delta_p P)$ so that we could solve both discount factors. Let $M = 9\alpha^2\zeta(P-1)P^6$ and $N = \sqrt{3\alpha^3 P^{12}(27\alpha(P-1)^2\zeta^2 + P^3)}$,

$$\delta_p^* = \frac{1}{P} - \frac{P^2}{[3(M+N)]^{\frac{1}{3}}} + \frac{(M+N)^{\frac{1}{3}}}{3^{\frac{2}{3}}\alpha p^3}$$

$$\delta_d^* = \frac{\frac{P}{3} + \frac{\alpha P^6}{[3(M+N)]^{\frac{2}{3}}} + \frac{(M+N)^{\frac{2}{3}}}{3^{\frac{4}{3}}\alpha P^4}}{2P}$$

XII. We assume $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, $\lambda_4 \neq 0$ and $\lambda_5 = 0$.

This assumption also satisfies KKT condition (16 iii), (16 iv) and (16 v), (16 vi) and (16 vii)where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d = 0$ and $1-\delta_p P > 0$. Thus,

$$\delta_p^* = \frac{P - \zeta + p\zeta}{P^2}, \delta_d^* = 1$$

We still draw a plot to see the impacts of P under this case. δ_p is the left first axes while δ_d, profit and the demand variance uses the right secondary axes. Cash demand first is greater then demand demand and gradually they become the same and the variance equals to 0, which means the ζ constraint changes from un-tight to tight. δ_d shows a deceasing trend while when ζ constraint becomes tight the slope becomes deeper. δ_p, similar to section 3.2, decreases first and then increases when ζ constraint becomes tight. Profit increases first but then decreases and maximum is at the vertical line where P is approximately 0.939.

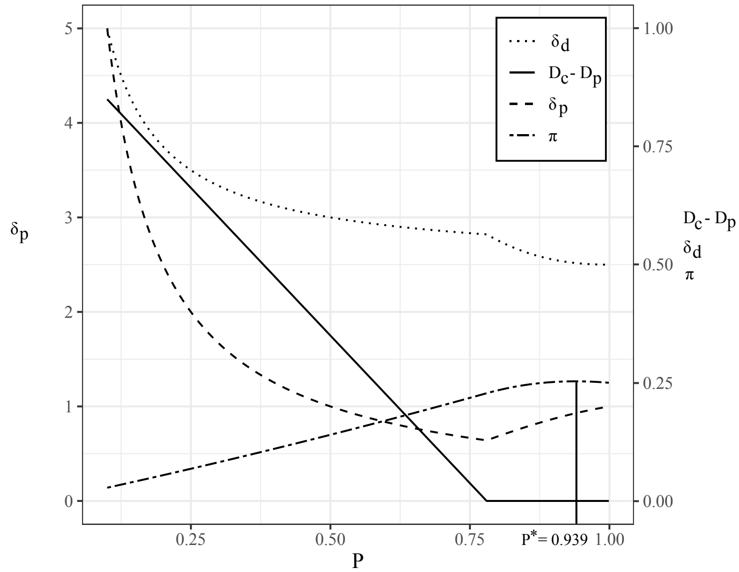


Figure 10: Sensitivity of P when preference is cash, deal and points

Now we loop over all possible value of α and β to find maximal outputs as well.

I. Sensitivity of α when preference is cash, deal and points (β = 0.8)

	Optimal Values					Profit	Demand			
α	Price (P)	Point discount (δ_p)	Deal discount (δ_d)	Points price ($\delta_p P$)	Deal price ($\delta_d P$)		Total	Cash	Deal	Points
0.1	0.999	0.999	0.500	0.998	0.500	0.250	0.501	0.001	0.499	0.001
0.2	0.999	0.999	0.500	0.998	0.500	0.250	0.501	0.001	0.499	0.001
0.3	0.999	0.999	0.500	0.998	0.500	0.250	0.501	0.001	0.499	0.001
0.4	0.939	0.928	0.504	0.871	0.473	0.253	0.588	0.061	0.466	0.061
0.5	0.908	0.883	0.511	0.802	0.464	0.260	0.628	0.092	0.444	0.092
0.6	0.887	0.851	0.520	0.755	0.461	0.268	0.652	0.113	0.426	0.113
0.7	0.872	0.830	0.531	0.723	0.463	0.277	0.665	0.128	0.409	0.128

Table 4: Sensitivity of α when preference is cash, deal and points (β = 0.8)

With one more distribution channel added, all optimum stay at a steady state first when α in lower values. In order to get highest revenue to cover such low cash-points conversion, such Courtyard or FourPoints post a very high retail price and points price to lower points demand. So they post a good price on deal to drive more volume. The other possible reason is that competitors in market which has the same burning ratio post a very high price thus even though Courtyard hotels drop rate, there is still the same demand but with less profit. Sometimes revenue managers post a high price: i.e. only sell expensive suites cash prices just to stop customers redeem points, which on the other hand verifies the importance of dynamic points pricing.

Then gradually with α increases, the highest revenue also goes to highest α. But we need to notice that the price is decreasing while profit increases. For example, a JW Marriott hotel has a high category because of their excellent performance last year. With good reputation, they have a very high cash and points demand but they need to reduce deal demand to drive revenue.

II. Sensitivity of β when preference is cash, deal and points (α = 0.4)

	Optimal Values					Profit	Demand			
β	Price (P)	Point discount (δ_p)	Deal discount (δ_d)	Points price ($\delta_p P$)	Deal price ($\delta_d P$)		Total	Cash	Deal	Points
0.5	0.837	0.757	0.532	0.634	0.445	0.284	0.718	0.163	0.392	0.163
0.6	0.864	0.808	0.521	0.698	0.450	0.271	0.686	0.136	0.414	0.136
0.7	0.897	0.864	0.511	0.775	0.459	0.261	0.644	0.103	0.438	0.103
0.8	0.939	0.928	0.504	0.871	0.473	0.253	0.588	0.061	0.466	0.061
0.9	0.999	0.999	0.500	0.998	0.500	0.250	0.501	0.001	0.499	0.001

Table 5: Sensitivity of β when preference is cash, deal and points (α = 0.4)

β as the burning ratio here shows the opposite trend with α as the earning ratio: prices, discount factors and deal demand increase and other outputs decrease. The highest β = 0.9 (i.e. Ritz Carlton) generates the lowest profit while the highest profit goes the the lowest β = 0.5 (i.e. Sheraton). The increasing β brings with increasing price as category increases. For example, the best strategy for the Ritz Carlton is to post a very high cash price while focus on deal segment. However, the Sheraton posts a relatively low price which attracts the most cash demand thus pushing to the highest revenue. So we could conclude that under these cases and this market, it might be vise not to open very luxury hotels.

3.3.2 Moderately Discounted Points Redemption Preferred to Deal Prices

Still, customers prefer cash segment first if they have enough cash when $v_i \geq P$. And they don't have enough cash they pay points next when $v_i < P$ and $\gamma_i \geq \delta_p P$. Thus the probability is the same with redemption premium or moderate discount situation in cash and points only case. But after these two options, if customers do not have enough points but have less money to afford the deal price, then they pay via discounted/opaque channels when $\delta_d P \leq v_i < P$ and $\gamma_i < \delta_p P$. Last, when $v_i < \delta_d P$ and $\gamma_i < \delta_p P$, customers leave.

Demands for cash and points are $D_c = 1 - P$ and $D_p = P(1 - \delta_p P)$. Revenue for cash and points is $\pi_c = P(1 - P)(1 - \beta)$ and $\pi_p = \alpha\delta_p P * P(1 - \delta_p P)$. For deal segment, since customers could not afford points segment with probability of $\delta_p P$, and the probability that customers could afford deal but could not afford cash is $P - \delta_d P$. Thus, deal demand is $D_d = \delta_p P \int_{\delta_d P}^{P} f(v_i) dv_i = \delta_p P(P - \delta_d P)$. Deal revenue is $\pi_d = \delta_d P \delta_p P \int_{\delta_d P}^{P} f(v_i) dv_i = \delta_d P * \delta_p P(P - \delta_d P)$.

We sum up three demands and total demand is

$$D = 1 - P + P(1 - \delta_p P) + \delta_p P(P - \delta_d P) = 1 - \delta_p \delta_d P^2$$

Total firm revenue is

$$\pi = P(1 - P)(1 - \beta) + \alpha\delta_p P * P(1 - \delta_p P) + \delta_d P * \delta_p P(P - \delta_d P) \tag{18}$$

So the optimization program changes to,

$$
\begin{array}{lrcl}
\max\limits_{\delta_p,\delta_d} & \multicolumn{3}{l}{P(1-P)(1-\beta)+\alpha\delta_p P * P(1-\delta_p P)+\delta_d P * \delta_p P(P-\delta_d P)} \\
\text{s.t.} & \delta_p & \geq & 1-\beta \\
 & \delta_d & \geq & \delta_p \\
 & \zeta(1-P)-P(1-\delta_p P) & \geq & 0 \\
 & \delta_d & \leq & 1 \\
 & \delta_p P & \leq & 1 \\
 & \delta_p,\delta_d & > & 0
\end{array}
$$

Similarly we also introduce three Lagrange multipliers λ_1, λ_2 and λ_3 to rewrite the objective function,

$$
\begin{aligned}
L = {} & P(1-P)(1-\beta)+\alpha\delta_p P * P(1-\delta_p P)+\delta_d P * \delta_p P(P-\delta_d P) \\
& + \lambda_1(\delta_p+\beta-1) \\
& + \lambda_2(\delta_d-\delta_p) \\
& + \lambda_3(\zeta(1-P)-P(1-\delta_p P)) \\
& + \lambda_4(1-\delta_d) \\
& + \lambda_5(1-\delta_p P)
\end{aligned}
\tag{19}
$$

The KKT conditions are,

$$\frac{dL}{d\delta_p} = \lambda_1 - \lambda_2 + P[-\lambda_5 + P(\alpha + \lambda_3 - 2\alpha\delta_p P + \delta_d P(1 - \delta_d)] \quad \text{(19 i)}$$

$$\frac{dL}{d\delta_d} = -\lambda_4 + \lambda_2 + \delta_p P^3(1 - 2\delta_d) \quad \text{(19 ii)}$$

$$\lambda_1(\delta_p + \beta - 1) = 0 \quad \text{(19 iii)}$$

$$\lambda_2(\delta_d - \delta_p) = 0 \quad \text{(19 iv)}$$

$$\lambda_3(\zeta(1 - P) - P(1 - \delta_p P)) = 0 \quad \text{(19 v)}$$

$$\lambda_4(1 - \delta_d) = 0 \quad \text{(19 vi)}$$

$$\lambda_5(1 - \delta_p P) = 0 \quad \text{(19 vii)}$$

We also do one-dimensional search here. For fixed P and δ_p that deal revenue (the only part that is a function of δ_d) is a function of $\delta_d P(P - \delta_d P)$ and we also have $\delta_d \geq \delta_p$. Thus, deal revenue can be maximized only when the two discount factors are equal where KKT condition (19 iv) satisfies $\delta_d \neq 0$.

Thus, the constrained optimization solutions are,

I. We assume $\lambda_1 \neq 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption satisfies KKT conditions (19 iii), (19 iv), (19 v), (19 vi) and (19 vii) where $\delta_p = 1 - \beta$, $\delta_p = \delta_d$, $\zeta(1 - P) > 1 - \delta_p P$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$ thus

$$\delta_p^* = \delta_d^* = 1 - \beta$$

II. We assume $\lambda_1 = \lambda_3 = \lambda_4 = 0$ and $\lambda_5 \neq 0$.

This assumption satisfies KKT conditions (19 iii), (19 iv), (19 v), (19 vi) and (19 vii) where $\delta_p = 1 - \beta$, $\delta_p = \delta_d$, $\zeta(1 - P) > 1 - \delta_p P$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$

thus

$$\delta_p^* = \delta_d^* = \frac{1}{P}$$

III. We assume $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

This assumption satisfies KKT conditions (19 iii), (19 iv), (19 v), (19 vi) and (19 vii) where $\delta_p > 1 - \beta$, $\delta_p = \delta_d$, $\zeta(1 - P) > 1 - \delta_p P$, $1 - \delta_d > 0$ and $1 - \delta_p P > 0$. We plug $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ into KKT condition (19 i) and (19 ii) and set them to zero so that we could get two equations to solve λ_2 and $\delta_p = \delta_d$. Thus,

$$\delta_p^* = \delta_d^* = \frac{P - \alpha P - \sqrt{P(3\alpha + P - 2\alpha P + \alpha^2 P)}}{3P}$$

or

$$\delta_p^* = \delta_d^* = \frac{P - \alpha P + \sqrt{P(3\alpha + P - 2\alpha P + \alpha^2 P)}}{3P}$$

IV. We assume $\lambda_1 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = 0$.

This assumption satisfies KKT conditions (19 iii), (19 iv), (19 v), (19 vi) and (19 vii) where $\delta_p > 1 - \beta$, $\delta_p = \delta_d$, $\zeta(1 - P) = 1 - \delta_p P$, $1 - \delta_d > 0$ and $1 - \delta_p P \geq 0$. Thus we plug $\delta_p = \delta_d$ into $\zeta(1 - P) = P(1 - \delta_p P)$ and we get

$$\delta_p^* = \delta_d^* = \frac{P - \zeta + P\zeta}{P^2}$$

We also draw the plot to the sensitivity of P when the preference is cash, points and deal. Since we already know that the optimum is always when the two discount factors are the same so there are only one discount factor curve shown. It first starts at 0 which is because there are only two-segment and we do not recommend deal segment here. Gradually with prices increases profit increases first then decrease where the optimal profit is when $P^* = 0.682$.

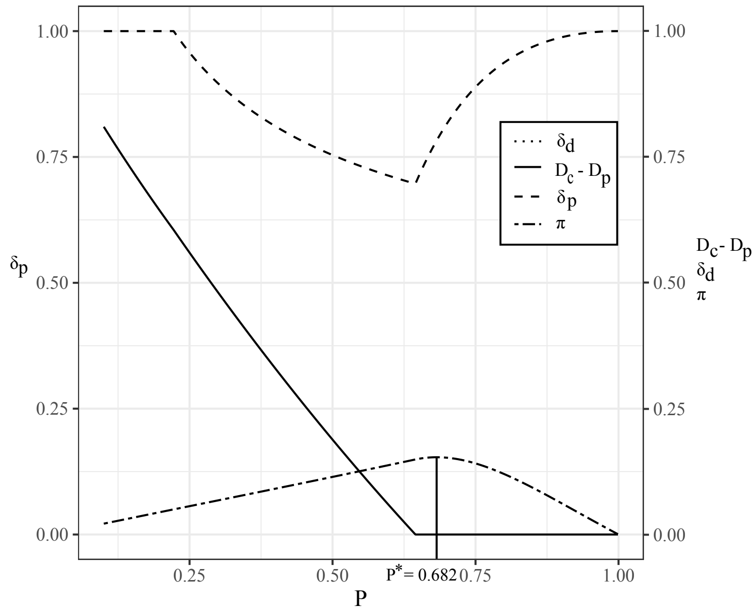


Figure 11: Sensitivity of P when the preference is cash, points and deal

Now we loop over all possible value of α and β to find maximal profits then store the optimal price and discount factors.

I. Sensitivity of α when preference is cash, points and deal ($\beta = 0.8$)

	Optimal Values					Profit	Demand			
α	Price (P)	Point discount (δ_p)	Deal discount (δ_d)	Points price ($\delta_p P$)	Deal price ($\delta_d P$)		Total	Cash	Deal	Points
0.1	0.671	0.760	0.760	0.510	0.510	0.103	0.740	0.329	0.082	0.329
0.2	0.676	0.770	0.770	0.521	0.521	0.120	0.729	0.324	0.081	0.324
0.3	0.679	0.777	0.777	0.527	0.527	0.137	0.722	0.321	0.080	0.321
0.4	0.682	0.783	0.783	0.534	0.534	0.154	0.715	0.318	0.079	0.318
0.5	0.684	0.787	0.787	0.538	0.538	0.170	0.711	0.316	0.079	0.316
0.6	0.686	0.790	0.790	0.542	0.542	0.188	0.706	0.314	0.078	0.314
0.7	0.688	0.794	0.794	0.547	0.547	0.205	0.701	0.312	0.077	0.312

Table 6: Sensitivity of α when preference is cash, points and deal ($\beta = 0.8$)

Here we notice that the optimal profit under each α satisfies that cash demand equals to points demand. And even though deal price is lower it generates the lowest demand. As α increases, there is not too much price increases along with demand decrease but the highest $\alpha = 0.7$ i.e. JW Marriott) generates almost double profit of the lowest $\alpha = 0.1$ (i.e. FourPoints). For example in this market, this JW Marriott and FourPoints don't have too much price gap but since JW has the highest revenue for each points room, it has much higher revenue than the FourPoints.

II. Sensitivity of β when preference is cash, points and deal ($\alpha = 0.4$)

	Optimal Values					Profit	Demand			
β	Price (P)	Point discount (δ_p)	Deal discount (δ_d)	Points price ($\delta_p P$)	Deal price ($\delta_d P$)		Total	Cash	Deal	Points
0.5	0.666	0.748	0.748	0.498	0.498	0.219	0.751	0.334	0.083	0.334
0.6	0.671	0.760	0.760	0.510	0.510	0.197	0.740	0.329	0.082	0.329
0.7	0.676	0.770	0.770	0.521	0.521	0.175	0.729	0.324	0.081	0.324
0.8	0.682	0.783	0.783	0.534	0.534	0.154	0.715	0.318	0.079	0.318
0.9	0.688	0.794	0.794	0.547	0.547	0.132	0.701	0.312	0.077	0.312

Table 7: Sensitivity of β when preference is cash, points and deal ($\alpha = 0.4$)

Still, the optimal profit under each α satisfies that cash demand equals to points demand. The higher β burns more cost to issue points thus the lowest revenue goes to the highest $\beta = 0.9$.

Numerically from the table examples under these two preference cases we could tell that preference with cash, deal and points generates higher profit than the preference with cash, points and deal.This is because δ_d wants to be no less than δ_p: i.e. we only get that segmentation because we constraint it to appear. So here we introduce the second theorem.

Theorem 2 *Preference with cash, deal and points generates higher profit than the preference with cash, points and deal.*

3.4 Demand Dependent Redemption Reimbursement

While in the previous section we have shown that differential pricing may have limited financial impact, firms may choose to deploy more active pricing in the event of demand and supply imbalances. As discussed previously, when firms are part of a larger brand there may be a transaction loss between the value a firm pays for loyal program points it provides to customers and the payments it receives (from the brand) upon redemption by customers. As an illustration, most large hotel brands typically compensate a franchisee anywhere from 25-50% of their ADR (average daily rate) for a redemption stay if the hotel is a moderate demand levels (i.e. the room might have otherwise gone unoccupied). The compensation (% of ADR) to the franchisee increases with occupancy reflecting the increasing opportunity cost of the redemption stay.

In this section we consider a setting where a firm receives α_l for redeemed goods/services when demand is low, i.e. when demand is below a certain threshold T ($D < T$) and α_h otherwise. Specifically we look a firm who may choose to lower price P, and/or discount points in an attempt to raise demand above the redemption threshold T to increase the level redemption to α_h.

Owing to earlier results we do not need to look at all segmentation. Linking back to section 3.3 we illustrate the implication of dynamic pricing strategy when deal segment is added. For the two segmentation introduced in 3.3.2 when preference is cash, points, and deal please note that we don't consider this segmentation as displayed in Figure (9b) according to Theorem 2.

3.4.1 Cash and Points only

The updated optimization problem now becomes,

$$\begin{aligned}
\max_{\delta} \quad & (1-\beta)(P-P^2) + \alpha\delta P^2(1-\delta P) \\
\text{s.t.} \quad & \delta \geq 1-\beta \\
& \zeta(1-P) \geq P(1-\delta P) \\
& 1-\delta P^2 \geq T \\
& \delta P \leq 1 \\
& \delta > 0
\end{aligned}$$

Similarly we also introduce three Lagrange multipliers λ_1, λ_2, λ_3 and λ_4 to rewrite the objective function,

$$\begin{aligned}
L = {} & (1-\beta)(P-P^2) + \alpha\delta P^2(1-\delta P) \\
& + \lambda_1(\delta+\beta-1) \\
& + \lambda_2[\zeta(1-P) - P(1-\delta P)] \\
& + \lambda_3(1-\delta P^2 - T) \\
& + \lambda_4(1-\delta P)
\end{aligned} \tag{21}$$

The KKT conditions are,

$$\frac{dL}{d\delta} = \lambda_1 + P^2(\alpha - \lambda_2 + \lambda_3 - 2\alpha\delta P) \tag{21 i}$$

$$\lambda_1(\delta+\beta-1) = 0 \tag{21 ii}$$

$$\lambda_2(\zeta(1-P) - P(1-\delta P)) = 0 \tag{21 iv}$$

$$\lambda_3(1-\delta P^2 - T) \tag{21 iii}$$

$$\lambda_4(1-\delta P) = 0 \tag{21 v}$$

We also do one dimensional search for δ and P given P fixed. Then we take derivative of the L function with respect to δ. And the solutions are,

I. We assume $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

This assumption satisfies KKT conditions (21 ii),(21 iii), (21 iv) and (21 v) where $\delta = 1 - \beta$, $\zeta(1 - P) > P(1 - \delta P)$, $1 - \delta P^2 > T$ and $1 - \delta P > 0$, thus

$$\delta^* = 1 - \beta$$

II. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$, and $= \lambda_3 = \lambda_4 = 0$.

This assumption satisfies KKT conditions (21 ii),(21 iii), (21 iv) and (21 v) where $\delta > 1 - \beta$, $\zeta(1 - P) = P(1 - \delta P)$, $1 - \delta P^2 > T$, and $1 - \delta P > 0$, thus

$$\delta^* = \frac{P - \zeta + P\zeta}{P^2}$$

III. We assume $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = 0$.

This assumption satisfies KKT conditions (21 ii),(21 iii), (21 iv) and (21 v) where $\delta > 1 - \beta$, $1 - \delta P^2 = T$, $\zeta(1 - P) > P(1 - \delta P)$, and $1 - \delta P > 0$, thus

$$\delta^* = \frac{1 - T}{P^2}$$

IV. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$.

This assumption satisfies KKT conditions (21 ii),(21 iii), (21 iv) and (21 v) where $\delta > 1 - \beta$, $\zeta(1 - P) > P(1 - \delta P)$, $1 - \delta P^2 > T$, and $1 - \delta P > 0$. We plug$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. into KKT conditions (21 i) and set to zero thus

$$\delta^* = \frac{1}{2P}$$

V. We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 \neq 0$.

This assumption satisfies KKT conditions (21 ii),(21 iii), (21 iv) and (21 v) where $\delta > 1-\beta$, $\zeta(1-P) > P(1-\delta P)$, $1-\delta P^2 > T$, and $1-\delta P = 0$. Thus

$$\delta^* = \frac{1}{P}$$

Here we need to clarify that the key result that we focus here is when $\delta^* = \frac{1-T}{P^2}$ where we have the Lagrangian for the demand constraint and this constraint is binding. All other formulas look the same as what we have for section 3.2 with the the demand constraint is not binding.

With demand constraint added, the sensitivity of P shows a different curve compared with section 3.2 when there is no demand constraint. Here we notice that there is a maximum value of $P = 0.650$ could attribute to the feasible optimal values, where cash demand equals to points demand which indicates a tight ζ constraint. Both discount factor and demand variance decrease and the profit increases until the optimum found at $P^* = 0.650$.

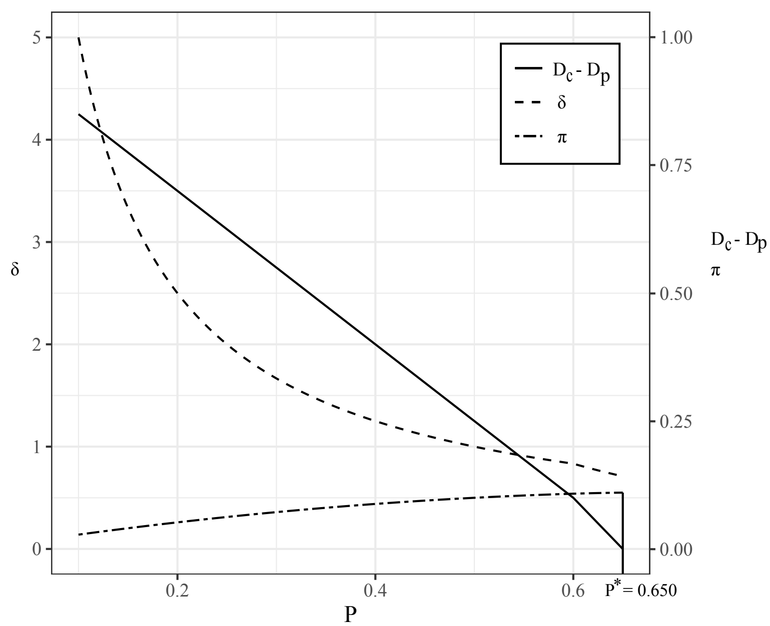


Figure 12: Sensitivity of P at cash and points only model ($\alpha = 0.4, \beta = 0.8$)

From 3.2 we already know that when $\alpha = 0.4$ and $\beta = 0.8$ the optimal demand is approximately 0.644. Thus we pick $T > 0.644 = 0.7$ as the starting demand threshold that hotels can receive redemption reimbursement if they push up points occupancy thus at the high α region.

Sensitivity of T at cash and points only model ($\alpha = 0.4, \beta = 0.8$)

	Optimal Values			Profit	Demand		
T	Price (P*)	Point discount (δ^*)	Points price ($\delta^* P^*$)		Total	Cash	Points
0.7	0.650	0.710	0.462	0.110	0.700	0.350	0.350
0.8	0.599	0.557	0.334	0.101	0.800	0.401	0.399
0.9	0.535	0.349	0.187	0.082	0.900	0.465	0.435
	0.678	0.774	0.525	0.111	0.644	0.322	0.322

Table 8: Sensitivity of T at cash and points only model ($\alpha = 0.4, \beta = 0.8$)

As T increases, all outputs keep at a stable stage where cash demand equals to points demand and profit gets to maximum. However, when T reaches to a higher level at 0.7, profit start to decrease even though occupancy increases.

We also add one row of the results in section 3.2 when there is no demand constrained. Thus it is clear to show that unconstrained profit is always higher then when we have the demand constraint. Compared with section 3.2 where the demand constraint is not discussed, we need to figure out the how profit changes if hotels switch from α_l to α_h. We take $\alpha = 0.2$ as an example. When $\alpha = 0.2$ the optimal profit is approximately 0.078 while here with the same α and when $T = 0.8$ the profit now becomes lower as around 0.075. Thus to achieve the same profit hotels switch to α_h to get more compensation.

Thus we draw the plot of T to find the variance between α_h and α_l, between the optimal profits, discount factors. Since we want to find the α_h and according to the curves below, we could see that we decrease the new price and discount factor to get the same profit level. Here we have the difference between constrained optimal and unconstrained optimal P and δ on the left axes and $\alpha_h - \alpha_l$ as the right axes. As T goes up, the α variance curve shows a deeper increase. However in reality, hotel brands like Marriott shows a steady increase starting from when T is 0.85 and linear increase by tiers of T. Other most hotel brands typically only have two tiers: one is above 0.9 and the other is between 0.95 to 1. This can be our management insight for hotel brands to make compensation policy for redemption rooms for hotels.

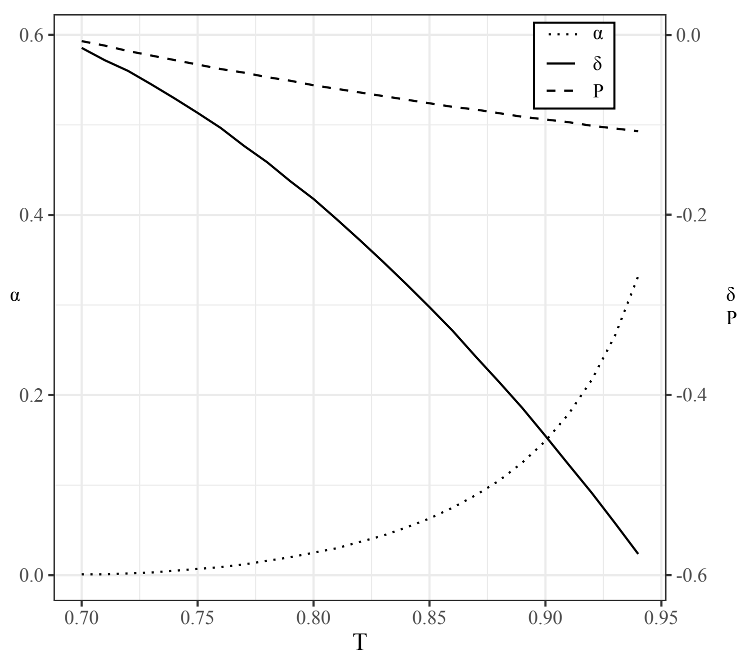


Figure 13: Dynamic Redemption Reimbursement at cash and points only model ($\alpha = 0.2, \beta = 0.8$)

3.4.2 Discounted Deal Prices Preferred to Points Redemption

The new optimization problem now changes to

$$
\begin{aligned}
\max_{\delta_p,\delta_d} \quad & P(1-P)(1-\beta) + \delta_d P(P-\delta_d P) + \alpha\delta_p P * \delta_d P(1-\delta_p P) \\
\text{s.t.} \quad & \begin{array}{rcl}
\delta_d & \geq & 1-\beta \\
\delta_p & \geq & \delta_d \\
\zeta(1-P) - \delta_d P(1-\delta_p P) & \geq & 0 \\
\delta_d & \leq & 1 \\
\delta_p P & \geq & 1 \\
1-\delta_p\delta_d P^2 & \geq & T \\
\delta_d & > & 0
\end{array}
\end{aligned}
$$

Thus, we introduce four Lagrange multiplier λ_1, λ_2, λ_3, and λ_4 here to solve for the optimal solution,

$$
\begin{aligned}
L = {} & P(1-P)(1-\beta) + \delta_d P(P-\delta_d P) + \alpha\delta_p P * \delta_d P(1-\delta_p P) \\
& + \lambda_1(\delta_d + \beta - 1) \\
& + \lambda_2(\delta_p - \delta_d) \\
& + \lambda_3(\zeta(1-P) - \delta_d P(1-\delta_p P)) \\
& + \lambda_4(1-\delta_d) \\
& + \lambda_5(1-\delta_p P) \\
& + \lambda_6(1-\delta_p\delta_d P^2 - T)
\end{aligned}
\tag{23}
$$

The KKT conditions are,

$$\frac{dL}{d\delta_p} = \lambda_2 + P[-\lambda_5 + \delta_d P(\alpha + \lambda_3 - 2\alpha\delta_p P - \lambda_6)] \tag{23 i}$$

$$\frac{dL}{d\delta_d} = -\lambda_4 + \lambda_1 - \lambda_2 + P[P - 2\delta_d P + \lambda_3(\delta_p P - 1) + \alpha P(\delta_p - \delta_p P^2) - \lambda_6\delta_p P] \tag{23 ii}$$

$$\lambda_1(\delta_d + \beta - 1) = 0 \tag{23 iii}$$

$$\lambda_2(\delta_p - \delta_d) = 0 \tag{23 iv}$$

$$\lambda_3(\zeta(1-P) - \delta_d P(1 - \delta_p P)) = 0 \tag{23 v}$$

$$\lambda_4(1 - \delta_d) = 0 \tag{23 vi}$$

$$\lambda_5(1 - \delta_p P) = 0 \tag{23 vii}$$

$$\lambda_6(1 - \delta_p\delta_d P^2 - T) = 0 \tag{23 viii}$$

The constrained optimization solutions are,

I. We assume $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 = 0$, $\delta_p - \delta_d = 0$, $\zeta(1-P) - \delta_d P(1 - \delta_p P) > 0$, $1 - \delta_d > 0$, $1 - \delta_p P > 0$ and $1 - \delta_p\delta_d P^2 > T$. Thus

$$\delta_p^* = \delta_d^* = 1 - \beta$$

II. We assume $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$, $1 - \delta_p P > 0$ and $1 - \delta_p\delta_d P^2 > T$. We then plug $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ and $\delta_d + \beta - 1 = 0$ into KKT condition (23 i) and set it to zero then we get

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = 1 - \beta$$

III. We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ and $\lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d = 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. We then plug $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = 0$ and $1-\delta_d = 0$ into KKT condition (23 i) and set it to zero then we get

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = 1$$

IV. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. We first plug $\lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = 0$ into KKT condition (23 i) and set it to zero so that we could solve for the optimal δ_p. Then we plug it as well as $\lambda_1 = \lambda_4 0$ into KKT condition (23 ii) and set it to zero to get the optimal δ_d. Thus,

$$\delta_p^* = \frac{1}{2P}, \delta_d^* = \frac{\alpha + 4P}{8P}$$

V. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 \neq 0$ and $\lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 \geq 0$, $\delta_p \geq \delta_d$, $\zeta(1-P) \geq \delta_d P(1-\delta_p P)$, $1-\delta_d \geq 0$, $1-\delta_p P = 0$ and $1-\delta_p\delta_d P^2 \geq T$. Thus according to it and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ we know δ_d^*.

$$\delta_p^* = \frac{1}{P}, \delta_d^* = \frac{1}{2}$$

VI. We assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$, $\lambda_5 \neq 0$ and $\lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d = 0$, $1-\delta_p P = 0$ and $1-\delta_p\delta_d P^2 > T$. Thus,

$$\delta_p^* = \frac{1}{P}, \delta_d^* = 1$$

VII. We assume $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 \neq 0$ and $\lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P = 0$ and $1-\delta_p\delta_d P^2 > T$. Thus,

$$\delta_p^* = \frac{1}{P}, \delta_d^* = 1-\beta$$

VIII. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$ $\lambda_3 = \lambda_4 = 0$, $\lambda_5 \neq 0$ and $\lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P = 0$ and $1-\delta_p\delta_d P^2 > T$. Thus,

$$\delta_p^* = \delta_d^* = \frac{1}{P}$$

VIII. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 \neq 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P = 0$ and $1-\delta_p\delta_d P^2 = T$. Thus,

$$\delta_p^* = \frac{1}{P}, \delta_d^* = \frac{1-T}{P}$$

IX. We assume $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. We plug $\delta_d+\beta-1 = 0$ into $\zeta(1-P)-\delta_d P(1-\delta_p P) = 0$ to solve for δ_p^*. Thus,

$$\delta_p^* = \frac{P(1-\beta+\zeta)-\zeta}{(1-\beta)P^2}, \delta_d^* = 1-\beta$$

X. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. We plug $\delta_p = \delta_d$ into $\zeta(1-P) - \delta_d P(1-\delta_p P) = 0$ thus

$$\delta_p^* = \delta_d^* = \frac{1-\sqrt{1+4\zeta(-1+P)}}{2P}$$

or

$$\delta_p^* = \delta_d^* = \frac{1+\sqrt{1+4\zeta(-1+P)}}{2P}$$

XI. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. We plug these into KKT condition (23 i) and (23 ii) and set them to zero so that we could solve both discount factors. Solving them simultaneously and thus,

$$\delta_p^* = \delta_d^* = \frac{-1+\alpha+\sqrt{1+\alpha^2-2\alpha+3\alpha P}}{3\alpha P}$$

XII. We assume $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1 - P) = \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$, $1 - \delta_p P > 0$ and $1 - \delta_p \delta_d P^2 > T$. We plug them into KKT condition (23 i) and (23 ii) and set to zero so that we could solve both discount factors. Let $M = 9\alpha^2\zeta(P - 1)P^6$ and $N = \sqrt{3\alpha^3 P^{12}(27\alpha(P - 1)^2\zeta^2 + P^3)}$,

$$\delta_p^* = \frac{1}{P} - \frac{P^2}{[3(M + N)]^{\frac{1}{3}}} + \frac{(M + N)^{\frac{1}{3}}}{3^{\frac{2}{3}}\alpha p^3}$$

$$\delta_d^* = \frac{\frac{P}{3} + \frac{\alpha P^6}{[3(M+N)]^{\frac{2}{3}}} + \frac{(M+N)^{\frac{2}{3}}}{3^{\frac{4}{3}}\alpha P^4}}{2P}$$

XIII. We assume $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p = \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$, $1 - \delta_p P > 0$ and $1 - \delta_p \delta_d P^2 = T$. We plug $\delta_p = \delta_d$ into $1 - \delta_p \delta_d P^2 = T$ and get

$$\delta_p^* = \delta_d^* = \frac{\sqrt{1 - T}}{P}$$

XIV. We assume $\lambda_1 = 0\lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$, $\lambda_5 = 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d = 0$, $1 - \delta_p P > 0$ and $1 - \delta_p \delta_d P^2 = T$. Thus,

$$\delta_p^* = \frac{1 - T}{P^2}, \delta_d^* = 1$$

XV. We assume $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 = 0$, $\delta_p > \delta_d$, $\zeta(1 - P) > \delta_d P(1 - \delta_p P)$, $1 - \delta_d > 0$,

$1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 = T$. We plug $\delta_d = 1-\beta$ into $1-\delta_p\delta_d P^2 = T$ and get

$$\delta_p^* = \frac{T-1}{(\beta-1)P^2}, \delta_d^* = 1-\beta$$

XVI. We assume $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$ and $\lambda_4 = \lambda_5 = 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 = T$. Thus,

$$\delta_p^* = \frac{T-1}{P[T-1+\zeta(P-1)]}, \delta_d^* = \frac{1-T+\zeta(1-P)}{P}$$

XVII. We assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_6 \neq 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) > \delta_d P(1-\delta_p P)$, $1-\delta_d > 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 = T$. We plug these into KKT condition (23 i) and (23 ii) and set them to zero so that we solve both discount factors. Let $J = \sqrt{3\alpha^3 P^{12}(P^3 + 27\alpha(T-1)^2)}$ and $K = 9\alpha^2 P^6(T-1)$

$$\delta_p^* = \frac{-3^{(\frac{2}{3})}\alpha P^5 + 3^{(\frac{1}{3})}(J-K)^{\frac{2}{3}}}{3\alpha P^3 (J-K)^{\frac{1}{3}}}$$

$$\delta_d^* = \frac{6 + \frac{6*3^{\frac{1}{3}}\alpha P^5}{(J-K)^{\frac{2}{3}}} + \frac{2*3^{(\frac{2}{3})}(J-K)^{\frac{2}{3}}}{\alpha P^5}}{36}$$

XVIII. We assume $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$ and $\lambda_4 \neq 0$ and $\lambda_5 = \lambda_6 = 0$.

This assumption satisfies KKT condition (23 iii), (23 iv) (23 v), (23 vi), (23 vii) and (23 viii) where $\delta_d + \beta - 1 > 0$, $\delta_p > \delta_d$, $\zeta(1-P) = \delta_d P(1-\delta_p P)$, $1-\delta_d = 0$, $1-\delta_p P > 0$ and $1-\delta_p\delta_d P^2 > T$. Thus,

$$\delta_p^* = \frac{P-\zeta+\zeta P}{P^2}, \delta_d^* = 1$$

Here we need to clarify that when we have the Lagrangian for the demand constraint and when this constraint is binding, we get five new results as a function of T which we should pay attention to. All other formulas look the same as what we have for section 3.3.1 with the the demand constraint is not binding.

Now we also draw a plot to see how P impacts on optima discount factors, profit and how the cash and points demand variance changes. With demand constraint added, the overall trend of all curves are still similar to section 3.3.1 while since we add the demand constraint, the optimal price decreases compared with section 3.3.1 which now becomes lower at $P^* = 0.884$.

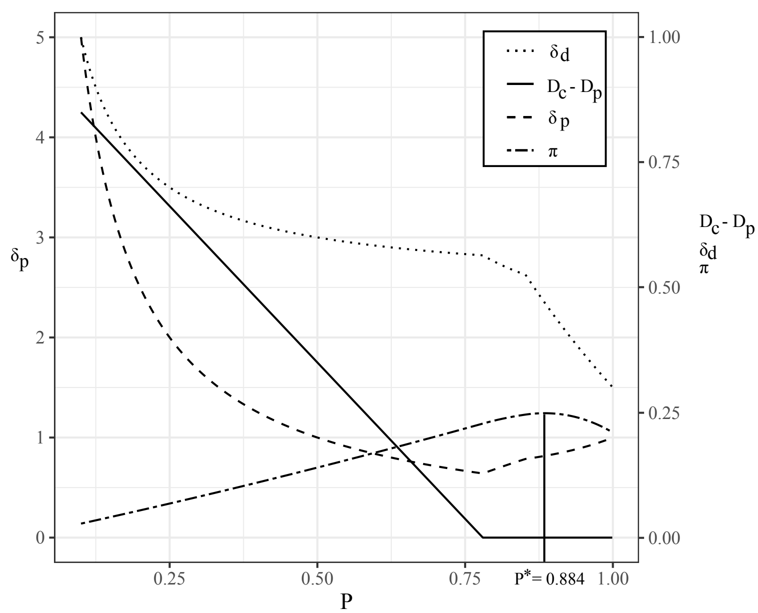


Figure 14: Sensitivity of P when preference is cash, deal and points with demand constraint ($\alpha = 0.4, \beta = 0.8$)

Now we loop over all possible value of T to find maximal profits then store the best feasible price, discount factors, occupancy, and demand of cash, deal and points segments.

Sensitivity of T when preference is cash, deal and points ($\alpha = 0.4, \beta = 0.8$)

	Optimal Values					Profit	Demand			
T	Price (P)	Point discount (δ_p)	Deal discount (δ_d)	Points price ($\delta_p P$)	Deal price ($\delta_d P$)		Total	Cash	Deal	Points
0.6	0.933	0.918	0.857	0.501	0.467	0.253	0.600	0.067	0.466	0.067
0.7	0.884	0.816	0.721	0.471	0.416	0.249	0.700	0.116	0.468	0.116
0.8	0.836	0.657	0.549	0.435	0.364	0.235	0.800	0.164	0.472	0.164
0.9	0.786	0.405	0.318	0.399	0.314	0.209	0.900	0.214	0.472	0.214
	0.939	0.928	0.504	0.871	0.473	0.253	0.588	0.061	0.466	0.061

Table 9: Sensitivity of T when preference is cash, deal and points ($\alpha = 0.4, \beta = 0.8$)

From section 3.3.1 we already know that optimal demand is 0.588 then we pick $T > 0.588 = 0.6$ as the starting occupancy to receive redemption reimbursement. Thus in order to get the same profit, firms need to dynamically price lower, pushing up to a higher occupancy to α_h for more points redemption reimbursement.

We also look into the different between α_h and α_l. We still use the example of $\alpha = 0.2$ and $\beta = 0.8$. Still, the left axes represents the difference of P, δ_p and δ_d while the right axes is the difference between α_h and α_l. The overall trend of all variance curves are similar to section 3.4.1 while less deep as deal segment added. But still as T increases, α should not be linearly increasing per current points redemption reimbursement policy.

4 Summary and Discussions

In order to understand how customers make purchase decisions, we first start with fixed and unconstrained model, then we add constraint on points discount and found that firms never offer huge discounts on points, but they do offer a

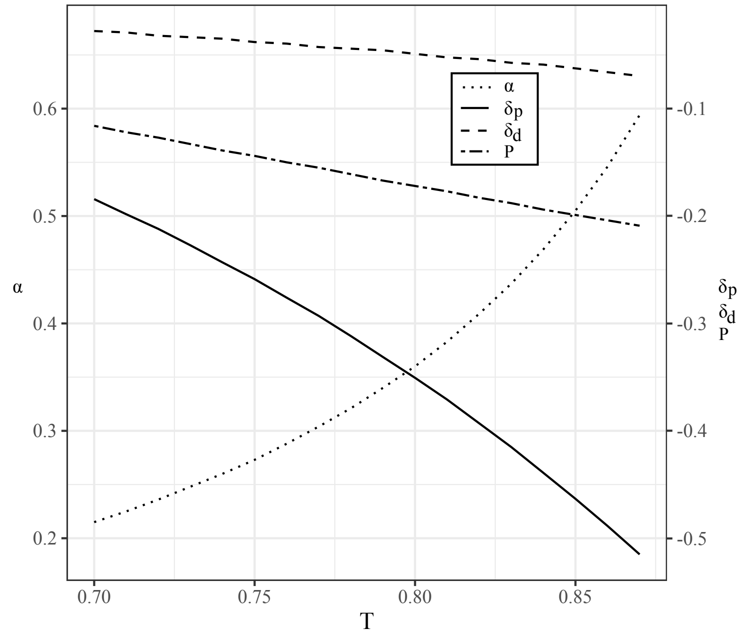


Figure 15: Dynamic Redemption Reimbursement when preference is cash, deal and points ($\alpha = 0.2, \beta = 0.8$)

moderate discount on the price of points for profit maximum. Next we introduce a steady-state points consumption and we observe a higher profits which indicates that firms should value the co-brand credit partners. Continuing we explore how profit changes with one more segment on inter-mediate channels. With different preference we figure out that preference for cash, deal, and points yields a bigger profit than preference for cash, points and deal. Last we add demand constraint in order to investigate whether profit can be increased when firms push up occupancy to achieve the demand threshold for more redemption reimbursement.

Thus we build a model for customer segmentation: who prefer cash, points or deal segment under which situations. Then with all above constraints added how preference changes. After understanding customer segmentation, we formulate constrained optimization programming step by step, and introduce multiple Lagrange multipliers to solve for the best pricing strategy: how should we price on each distribution channel.

The model contains four given parameters: α as the earning ratio represents the cash value of each points redemption room that the hotel receives from the brand. On the other hand β as the burning ratio specifies the cost that the hotel buy points from the brand to be issued to customers who purchase rooms via cash. We need to note that there are multiple ways that customers can require points and the most typical example is by credit card consumption. Thus we introduce ζ as the scale to regulate cash demand should be no less than points demand: i.e. $\zeta = 1$ means points issued are exactly the points consumed. The last parameter T is the demand threshold given by the brand. This is because when market demand is higher, each hotel room could have been sold at incredibly high cash rate in order to maximize revenue. However, loyalty programs require hotels to satisfy loyal customers' redemption needs. Thus, when the occupancy reaches to such T, the brand gives the hotel redemption reimbursement as "compensation" reward revenue.

The very first step is to understand customer surplus which is further used to compare to understand customer decision making. Cash customers cost cash for each room while as loyalty program reward they could receive points from the hotel. Thus their cash balance decreases and points balance increases. Points customers only burn points and there is no change for their cash balance. Similar to cash segment, deal customers still pay some cash via the inter-mediated channels but they could not receive any points rewarded because they do not buy via direct channels. Thus, their cash balance decreases and no change for points balance. This logic is vitally important and the first step for us to understand segmentation demand. With demand formulated, we multiple the price of each segment and sum them up to get the total revenue. Without any loss of generality we scale the prices to be all the same P and we solve for the discount factors for deal and points segment then by multiplying P we could know their prices as well.

First of all, we start with the fixed setting which is typically how hotels are doing now when there is no real-time dynamics for points price. Each year hotels receives a category to define their points price: A category-one Marriott hotel only needs 7,500 points while a category-8 hotel is worth 85,000 points. This is fairly easy to solve as customers who are "richer" than the cash price will buy cash and then if they could not afford the cash and if they are "richer" than points price they buy points. Otherwise if they are "poor" in either cash or points balance, they leave and such demands go to other hotels. Marriott start to introduce the idea of dynamic points pricing back to the merge with Starwood, it can only allow hotels to choose from three seasons: Off, standard and peak. However, this could not help hotels to monitor points redemption inventory control in real time, we introduce a constraint that allows points price to decrease. Thus, with the lower points, will customers "richer" again and will the lost demand come back?

Second, if we decide to discount points price, attract more demand and increase revenue, we need to think about how "attractive" should such promotion be: all of a sudden drop an 8-CAT Ritz Carlton deeply the to be 7,500 or moderately? So in this step we are trying to figure out the discount factor δ and points price dropped from P to δP. Cash customers have the same surplus while points customers now burn less points thus we compare the new points surplus with cash. A deep discount indeed could bring more points demand but each points room still generates lower revenue than cash room and it might hurt the inventory should have been sold to cash customers as they might switch to pay points. Mathematically we also prove that there is no interior solution here. Then with a moderate discount, cash customers don't switch and more points rooms sold plus more cash converted per points room. Thus we conclude that firms never deeply discount on points but always discount lightly on points price. However, what if customers have multiple access to points (credit cards) so that their points balance are not steady? Then we add a constraint ζ to regulate their points balance.

In section 3.2 we still introduce Lagrange multipliers to translate into a constrained optimization programming but we use a new method to solve for the optimal profit. In stead of solving both price and discount factor, we only solve for δ given a fixed P which indicates all possible values from zero to one. This is due to the added complexity introduced by the steady state points consumption constraint, as well as the difficulties to find closed-form solutions for both optimal pricing and discount factor. Thus, here P becomes a parameter and all solutions of δ now is a function of P, α, β and ζ. In order to display how P from zero to one impacts δ, as well as other outputs: profit and the cash/points demand variance, we loop over P in three decimals from 0.001 to 0.999 with the example of $\alpha = 0.4, \beta = 0.8$ and $\zeta = 1$. As constraints move from not tight to tight, curves change accordingly which results in the profit increases then decreases such that we find the optimal profit then we store this P.

We also want to find the optimal profit for any given set of α and β so we also loop over all possible values of α and β in one decimal from 0.1 to 0.9, find the optimal profit then store for the corresponding price and discount factor then we analyze how α and β impact all the optimal results.

If the steady-state demand constraint is tight, thus hotels are generating more cash sales than hotels plan to do optimally, if in absence of the constraint. So we can relax the constraint by allowing points to be accumulated not just through cash sales, but through co-branded credit cards. This indicates that ζ can be lower then hotels generate a higher profit. Plus, the brand could make money from selling points to those credit cards. However, hotels make less profit. We need to be cautious here because the profit we are trying to maximize is for the owners, not for the brand. So owners are better-off as we relax the constraint, but now there are going to be more points stays. We can get from Lagrangian for ζ as the marginal effect for per unit increase to understand how much the profit changes once relaxing the constraint.

Hotels not only have their own direct channels, they also post price on intermediated channels like OTAs (Expedia, Booking.com, etc) as a tradition for many years. Logically it is very easy to understand that with one more segment the revenue will be higher than there are only two segments. Revenue managers are forced and required to keep the rate parity of all inter-mediate channels, where they show the same price as official website in order to drive direct bookings and reduce commission. But there are multiple channels that allow customers to enjoy a lower discount when they are "qualified" to do so: i.e. a customer is at a very high membership at Expedia, and when s/he logs into his Expedia account there are always more attractive deal price than the official website. Another example coule be the typical Opaque price situation. This new segment by nature has a lower price than official website P thus we define its price as $\delta_d P$ where δ_p denotes its discount factor. Then we adjust points price under this setting as $\delta_p P$ to differentiate with each other.

From section 3.1 we understand that points will never be preferred so we already know that the first preference will always be cash so our focus is whether customers prefer deal, points or points, deal. Similarly our first step is still comparing surplus: among deal and points which one is more attractive to customers? To translate to mathematics: which segment generates a higher surplus? We find that preference order is cash, deal and points when $\delta_p > \delta_d > 1-\beta$ and preference order changes to cash, points and deal when $\delta_d > \delta_p > 1-\beta$. Similar to the approach in section 3.2 we also loop P to see its sensitivity under each preference order and also loop over α and β to find the numerical optimum.

For cash, points and deal preference order, we find that those two discount factors are optimal when they are equal, and this is because..... By checking profit of these two preference orders, we get another theorem that preference with cash, deal and points generates higher profit than the preference with cash, points and deal.

Now we come to the most interesting part where we introduce the demand threshold T. We compare occupancy (total demand) with it to see if hotels can receive redemption reimbursement. With more cash rewarded to each points room, we introduce the idea of α_h and α_l as the cash converted now is higher thus hotels switch from α_l region to α_h region. We first figure out when there are only two segments: cash and points then we also solve with deal segment added. We also use the same methods as in section 3.2 while we also draw plots to check the impacts of T that whether the switch of α attributes to a higher profit.

We also create a table shows the optimal results as a summary of the numerical results and how they change in optimal values as we increase segmentation, as constraints are either binding (B) or not binding (NB). "NUM" under P means we are using one-dimensional search by assuming P is also a fixed number.

	Constraints						Equations		
	Discount	Steady-state	Demand	$\delta_p - \delta_d$	$1-\delta_d$	$1-\delta_p P$	P	δ_p	δ_d
Cash and Points	NB					NB	$\frac{4+\alpha-4\beta}{8-8\beta}$	$\frac{4-4\beta}{4+\alpha-4\beta}$	
	NB					B	$\frac{1}{2}$	2	
	NB	NB				NB	NUM	$\frac{1}{2P}$	
	B	NB				NB	NUM	$1-\beta$	
	NB	B				NB	NUM	$\frac{P-\zeta+P\zeta}{P^2}$	
	NB	NB				B	NUM	$\frac{1}{P}$	
	B	NB	NB			NB	NUM	$1-\beta$	
	NB	B	NB			NB	NUM	$\frac{P-\zeta+P\zeta}{P^2}$	
	NB	NB	B			NB	NUM	$\frac{1-T}{P^2}$	
	NB	NB	NB			NB	NUM	$\frac{1}{2P}$	
	NB	NB	NB			B	NUM	$\frac{1}{P}$	
Cash, Deal and Points	B	NB		NB	B	NB	NUM	$1-\beta$	$1-\beta$
	B	NB		NB	NB	NB	NUM	$\frac{1}{2P}$	$1-\beta$
	NB	NB		NB	B	NB	NUM	$\frac{1}{2P}$	1
	NB	NB		NB	NB	NB	NUM	$\frac{1}{2P}$	$\frac{\alpha+4P}{8P}$
	NB	NB		NB	NB	B	NUM	$\frac{1}{P}$	$\frac{1}{2}$
	NB	NB		B	B	B	NUM	$\frac{1}{P}$	1
	B	NB		NB	NB	B	NUM	$\frac{1}{P}$	$1-\beta$
	NB	NB		B	NB	B	NUM	$\frac{1}{P}$	$\frac{1}{P}$
	B	B		NB	NB	NB	NUM	$\frac{P(1-\beta+\zeta)-\zeta}{(1-\beta)P^2}$	$1-\beta$
	NB	B		B	NB	NB	NUM	$\frac{1-\sqrt{1-4\zeta(-1+P}}{2P}$	$\frac{1-\sqrt{1-4\zeta(-1+P}}{2P}$
	NB	B		B	NB	NB	NUM	$\frac{1-\sqrt{1+4\zeta(-1+P}}{2P}$	$\frac{1-\sqrt{1+4\zeta(-1+P}}{2P}$
	NB	NB		B	NB	NB	NUM	$\frac{-1+\alpha+\sqrt{1+\alpha^2-2\alpha+3\alpha P}}{3\alpha P}$	$\frac{-1+\alpha+\sqrt{1+\alpha^2-2\alpha+3\alpha P}}{3\alpha P}$
	NB	B		NB	NB	NB	NUM	$\frac{1}{P}-\frac{P^2}{[3(M+N)]^{\frac{1}{3}}}+\frac{(M+N)^{\frac{1}{3}}}{3^{\frac{2}{3}}\alpha p^3}$	$\frac{\frac{P}{3}+\frac{\alpha P^6}{[3(M+N)]^{\frac{2}{3}}}+\frac{(M+N)^{\frac{2}{3}}}{3^{\frac{4}{3}}\alpha P^4}}{2P}$ 1
	NB	B		NB	B	NB	NUM	$\frac{P-\zeta+p\zeta}{P^2}$	1
	B	NB	NB	NB	B	NB	NUM	$1-\beta$	$1-\beta$
	B	NB	NB	NB	NB	NB	NUM	$\frac{1}{2P}$	$1-\beta$
	NB	NB	NB	NB	B	NB	NUM	$\frac{1}{2P}$	1
	NB	NB	NB	NB	NB	NB	NUM	$\frac{1}{2P}$	$\frac{\alpha+4P}{8P}$
	NB	NB	NB	NB	NB	B	NUM	$\frac{1}{P}$	$\frac{1}{2}$
	NB	NB	NB	B	B	B	NUM	$\frac{1}{P}$	1
	B	NB	NB	NB	NB	B	NUM	$\frac{1}{P}$	$1-\beta$
	NB	NB	NB	B	NB	B	NUM	$\frac{1}{P}$	$\frac{1}{P}$
	B	B	NB	NB	NB	NB	NUM	$\frac{P(1-\beta+\zeta)-\zeta}{(1-\beta)P^2}$	$1-\beta$
	NB	B	NB	B	NB	NB	NUM	$\frac{1-\sqrt{1-4\zeta(-1+P}}{2P}$	$\frac{1-\sqrt{1-4\zeta(-1+P}}{2P}$
	NB	B	NB	B	NB	NB	NUM	$\frac{1-\sqrt{1+4\zeta(-1+P}}{2P}$	$\frac{1-\sqrt{1+4\zeta(-1+P}}{2P}$
	NB	NB	NB	B	NB	NB	NUM	$\frac{-1+\alpha+\sqrt{1+\alpha^2-2\alpha+3\alpha P}}{3\alpha P}$	$\frac{-1+\alpha+\sqrt{1+\alpha^2-2\alpha+3\alpha P}}{3\alpha P}$
	NB	B	NB	NB	NB	NB	NUM	$\frac{1}{P}-\frac{P^2}{[3(M+N)]^{\frac{1}{3}}}+\frac{(M+N)^{\frac{1}{3}}}{3^{\frac{2}{3}}\alpha p^3}$	$\frac{\frac{P}{3}+\frac{\alpha P^6}{[3(M+N)]^{\frac{2}{3}}}+\frac{(M+N)^{\frac{2}{3}}}{3^{\frac{4}{3}}\alpha P^4}}{2P}$ 1
	NB	B	NB	NB	B	NB	NUM	$\frac{P-\zeta+p\zeta}{P^2}$	1
	NB	NB	B	NB	NB	B	NUM	$\frac{1}{P}$	$\frac{1-T}{P}$
	NB	NB	B	B	NB	NB	NUM	$\frac{\sqrt{1-T}}{P}$	$\frac{\sqrt{1-T}}{P}$
	NB	NB	B	NB	B	NB	NUM	$\frac{\sqrt{1-T}}{P^2}$	1
	B	NB	B	NB	NB	NB	NUM	$\frac{T-1}{(\beta-1)P^2}$	$1-\beta$
	NB	NB	B	NB	B	NB	NUM	$\frac{\sqrt{1-T}}{P^2}$	1
	NB	B	B	NB	NB	NB	NUM	$\frac{T-1}{P[T-1+\zeta(P-1)]}$	$\frac{1-T+\zeta(1-P)}{P}$
	NB	NB	B	NB	NB	NB	NUM	$\frac{-3^{(\frac{2}{3})}\alpha P^5+3^{(\frac{1}{3})}(J-K)^{\frac{2}{3}}}{3\alpha P^3(J-K)^{\frac{1}{3}}}$	$\frac{6+\frac{6*3^{\frac{1}{3}}\alpha P^5}{(J-K)^{\frac{2}{3}}}+\frac{2*3^{(\frac{2}{3})}(J-K)^{\frac{2}{3}}}{\alpha P^5}}{36}$ 2

Table 10: Summary Results

[1] $M = 9\alpha^2\zeta(P-1)P^6$, $N = \sqrt{3\alpha^3 P^{12}(27\alpha(P-1)^2\zeta^2 + P^3)}$

[2] $J = \sqrt{3\alpha^3 P^{12}(P^3 + 27\alpha(T-1)^2)}$, $K = 9\alpha^2 P^6(T-1)$

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