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TECHNICAL REPORT NO. 1210

May 1998

**Sample Correlation Behavior
for the Heavy Tailed
General Bilinear Process**

by

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S. Resnick and E. van den Berg were partially supported by NSF Grant DMS-97-04982 at Cornell University.

SAMPLE CORRELATION BEHAVIOR FOR THE HEAVY TAILED GENERAL BILINEAR PROCESS

SIDNEY RESNICK AND ERIC VAN DEN BERG

ABSTRACT. In this paper, we consider the class of general bilinear models given by

$$X_t = Z_t + \sum_{i=1}^p \phi_i X_{t-i} + \sum_{j=1}^q \theta_j Z_{t-j} + \sum_{i=1}^m \sum_{j=1}^l b_{ij} X_{t-i} Z_{t-j}, \quad t \in \mathbb{Z},$$

where $\{Z_t\}$ is an i.i.d. sequence of heavy tailed noise variables, and where $\prod_{j=1}^l b_{1j} \neq 0$. By means of a point process analysis, we show that the sample correlation function converges in distribution to a nondegenerate random variable. Thus standard model selection and fitting tools when applied to nonlinear heavy tailed models will be misleading. Also, consistency of the Hill estimator as an estimate of the tail index for this class of bilinear models is proved.

1. INTRODUCTION

Currently an important topic in time series analysis is how to deal with data which exhibit features like long range dependence, nonlinearity and heavy tails. Many datasets from fields such as telecommunications, finance and economics appear to be compatible with the assumption of heavy tailed marginals. Examples include file lengths, CPU time to complete a job, call holding times, interarrival times between packets in a network and lengths of on/off cycles. (See [9, 10, 25])

A pivotal question is how to fit models to such data. In the traditional setting of stationary time series with finite variance, every purely nondeterministic process can be represented as a linear process driven by an uncorrelated input sequence. For such processes, the autocorrelation function (ACF) can be well approximated by that of a finite order ARMA(p, q) model. In particular, one can choose an autoregressive (AR) model of order p , such that the ACF's of the two models agree for lags $1, \dots, p$. So from a second order point of view, linear models are sufficient for data analysis.

For stationary time series with infinite variance $\{X_t\}$, the class of linear models does not appear to be sufficiently rich and flexible for modeling purposes. If $\{X_t\}$ is the linear process MA(∞),

$$X_t = \sum_{j=1}^{\infty} \psi_j Z_{t-j},$$

where $\{\psi_j\}$ satisfies a summability condition, then one can still define an analogue of the ACF in terms of the coefficients $\{\psi_j\}$ of the linear filter, namely

$$\rho(h) = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+h}}{\sum_{j=0}^{\infty} \psi_j^2}.$$

AMS 1991 subject classifications. Primary 60F05; secondary 62M10.

Key words and phrases. heavy tails, time series, bilinear process, sample correlation, model selection, point processes .

S. Resnick and E. van den Berg were partially supported by NSF Grants DMS-97-04982 at Cornell University.

The sample ACF, defined for heavy tailed data as

$$\hat{\rho}_H(h) = \frac{\sum_{t=1}^{n-h} X_t X_{t+h}}{\sum_{t=1}^n X_t^2}, \quad h = 1, 2, \dots$$

actually has a number of desirable properties for $\text{MA}(\infty)$, such as consistency ($\hat{\rho}_H(h) \rightarrow_P \rho(h)$), and a reasonably fast rate of convergence. (See ([6],[7].) On the other hand, if the model is nonlinear, then it is not clear what, if anything, $\hat{\rho}(h)$ converges to. See e.g. [4] for a discussion. Davis and Resnick ([8]) have shown that for the simple bilinear model

$$X_t = cX_{t-1}Z_{t-1} + Z_t, \quad t \in \mathbb{Z}$$

the sample ACF converges to a nondegenerate random variable depending on the lag. Recently, limit behavior of the sample ACF of stable moving averages has been described in [18]. The main result of our current paper shows that for the class of general bilinear models given by ([12, 24])

$$X_t = Z_t + \sum_{i=1}^p \phi_i X_{t-i} + \sum_{j=1}^q \theta_j Z_{t-j} + \sum_{i=1}^m \sum_{j=1}^l b_{ij} X_{t-i} Z_{t-j}, \quad t \in \mathbb{Z}$$

under the condition that $\prod_{j=1}^l b_{1j} \neq 0$, the sample correlation function also converges to a nondegenerate random variable depending on the lag. As a consequence, other model fitting and diagnostic tools which depend on the sample ACF, such as the Akaike Information Criterion (AIC) for identifying the order of an AR model, and the Yule-Walker estimates for fitting an AR model will not converge to constants either, but will converge in distribution to nondegenerate random variables. This indicates how failure to account for nonlinearities can be quite misleading. Further discussion is contained in [11].

Here we show an example of the erratic behavior of the sample ACF for general bilinear processes. We simulated three independent samples, ($\text{test}_i, i = 1, 2, 3$) of size 100,000 from the bilinear process

$$(1.1) \quad X_t = 0.02X_{t-1}Z_{t-1} + 0.02X_{t-1}Z_{t-2} + Z_t, \quad t \in \mathbb{Z}$$

where $\{Z_t\}$ are iid Pareto random variables,

$$P[Z_1 > x] = 1/x, \quad x > 1.$$

The results of the simulations are shown in Figure 1, which graphs the heavy tailed ACF for $\text{test}_i, i = 1, 2, 3$. The graphs look rather different, reflecting the fact that we are in effect sampling three times from the nondegenerate limit distribution of the heavy tailed ACF. If one tried to fit the data with a linear model, the chosen model would be quite different for each sample. This underlines that failure to account for nonlinearity means there is great potential to be misled in the sorts of models one tries to fit.

Section 2 of this paper discusses a matrix-vector description of the general bilinear time series, which makes this model suitable for further analysis. Section 3 deals with tail behavior of X_t and a suitable approximation to it, which is used in Section 4 to give a detailed point process analysis of asymptotic properties of the general bilinear process. Section 5 derives the convergence of the sample ACF as a corollary to the point process results of Section 4. In the last section, consistency of the Hill estimator for the given class of bilinear processes is proved.

FIGURE 1. Heavy tailed ACF for three bilinear samples

2. THE GENERAL BILINEAR TIME SERIES IN MATRIX-VECTOR FORM

A real valued stationary time series $\{X_t, t \in \mathbb{Z}\}$ is called a general bilinear time series if it satisfies the following equations:([12], [24])

$$(2.1) \quad X_t = Z_t + \sum_{i=1}^p \phi_i X_{t-i} + \sum_{j=1}^q \theta_j Z_{t-j} + \sum_{i=1}^m \sum_{j=1}^l b_{ij} X_{t-i} Z_{t-j}, \quad t \in \mathbb{Z}$$

where $\{Z_t\}$ is an iid sequence of random variables, ϕ_i, θ_i, b_{ij} are nonrandom parameters and p, q, m and l are known non-negative integers. Without loss of generality, we can assume that $p = m$ in (2.1). We define the $p \times p$ matrix function χ by

$$(2.2) \quad \chi(u_1, \dots, u_l) = \begin{pmatrix} \phi_1 + \sum_{j=1}^l b_{1j} u_j & \phi_2 + \sum_{j=1}^l b_{2j} u_j & \dots & \dots & \phi_p + \sum_{j=1}^l b_{pj} u_j \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Now define $B(t-1) = \chi(Z_{t-1}, \dots, Z_{t-l})$ and let the $p \times (1+q)$ matrix Θ be

$$(2.3) \quad \Theta := \begin{pmatrix} 1 & \theta_1 & \dots & \theta_{q-1} & \theta_q \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Define also $\mathbf{Z}(t) = (Z_t, Z_{t-1}, \dots, Z_{t-q})'$, and $\mathbf{X}(t) = (X_t, X_{t-1}, \dots, X_{t-p+1})'$. Then from (2.1) and the definitions of $\mathbf{X}(t)$, Θ , $\mathbf{Z}(t)$, and $B(t-1)$ we get

$$(2.4) \quad \mathbf{X}(t) = \Theta \mathbf{Z}(t) + B(t-1) \mathbf{X}(t-1).$$

We assume throughout this paper that $\{Z_n, -\infty < n < \infty\}$ are i.i.d. random variables satisfying:

$$(2.5) \quad P[|Z_1| > x] = x^{-\alpha} L(x), \quad \alpha > 0, x > 0$$

$$(2.6) \quad \lim_{x \rightarrow \infty} \frac{P[Z_1 > x]}{P[|Z_1| > x]} = r,$$

$$(2.7) \quad \lim_{x \rightarrow \infty} \frac{P[Z_1 < -x]}{P[|Z_1| > x]} = 1 - r,$$

where L is slowly varying at infinity, and $0 \leq r \leq 1$.

For $l \geq 1$, $0 < \alpha \leq l+1$, assume

$$(2.8) \quad \sum_{i=1}^p \left(|\phi_i|^{\frac{\alpha}{(l+1)}} + \sum_{j=1}^l |b_{ij}|^{\frac{\alpha}{(l+1)}} E(|Z_1|^{\frac{l\alpha}{(l+1)}})^{\frac{1}{l}} \right) < 1.$$

For $l \geq 1$, $\alpha > l+1$, assume

$$(2.9) \quad \sum_{i=1}^p \left(|\phi_i| + \sum_{j=1}^l |b_{ij}| E(|Z_1|^{\frac{l\alpha}{(l+1)}})^{\frac{(l+1)}{l\alpha}} \right) < 1$$

□

With an eye on the regular variation results we prove later, we show that conditions (2.8) or (2.9) are in fact strong enough to ensure that

$$(2.10) \quad \sum_{n=1}^{\infty} E \|B(t-1) \cdots B(t-n)\|^{\frac{\alpha}{l+1}} < \infty.$$

Here $\|\cdot\|$ denotes the L_∞ -matrix norm, which can be written for a given $m \times n$ matrix A as

$$(2.11) \quad \|A\| = \max_{1 \leq i \leq m} \sum_{j=1}^n |A_{ij}|,$$

where A_{ij} denotes the element in the i th row and j th column of A . For any $m \times p$ matrix A and $p \times n$ matrix B , the L_∞ -norm satisfies

$$(2.12) \quad \|AB\| \leq \|A\| \|B\|.$$

We will see in Corollary 3.5 that $|X_t|$ has regularly varying tail probabilities with index $-\alpha/(l+1)$, so for infinite variance analysis, the range $0 < \alpha/(l+1) < 2$ is of most interest. In the following, we will write $M_n := \prod_{i=1}^n B(t-i)$ for $B(t-1) \cdots B(t-n)$. Note that the elements in the first row of $B(t-i)$ are affine transformations of $Z_{t-i}, \dots, Z_{t-i-l+1}$. The random variable $Z_{t-i-l+1}$ occurs only in the l consecutive matrices $B(t-i), \dots, B(t-i-l+1)$. In particular, Z_{t-n} occurs only in

$B(t - (n - l + 1)), \dots, B(t - n)$. Multiplying the B matrices, we see that M_n contains only powers of Z_i 's less than or equal to $l \wedge n$.

We now prove (2.10). By Jensen's inequality, for each $j = 1, \dots, l$, we can bound $E|Z_1|^{\frac{j\alpha}{l+1}}$ by $(E|Z_1|^{\frac{l\alpha}{l+1}})^{j/l}$. (Take $X = |Z|^{\frac{l\alpha}{l+1}}$, $s = j/l$, consider EX^s). First, assume $0 < \alpha \leq l + 1$, and define

$$\tilde{\phi}_i := (|\phi_i|^{\frac{\alpha}{l+1}} + \sum_{j=1}^l |b_{ij}|^{\frac{\alpha}{l+1}} E(|Z_1|^{\frac{l\alpha}{l+1}})^{\frac{j}{l}}), \quad i = 1, \dots, p$$

By the triangle inequality:

$$\begin{aligned} E\|M_n\|^{\frac{\alpha}{l+1}} &\leq E\left(\sum_{i=1}^p \sum_{j=1}^p |(M_n)_{ij}|^{\frac{\alpha}{l+1}}\right) \\ &\leq \sum_{i=1}^p \sum_{j=1}^p E|(M_n)_{ij}|^{\frac{\alpha}{l+1}} \end{aligned}$$

Let $\tilde{\chi}(u_1, \dots, u_l)$ be $\chi(u_1, \dots, u_l)$ with $|\phi_i|^{\frac{\alpha}{l+1}}$ replacing ϕ_i and $|b_{ij}|^{\frac{\alpha}{l+1}}$ replacing b_{ij} . Let $u = (E|Z_1|^{\frac{l\alpha}{l+1}})^{1/l}$, and define $\tilde{B} = \tilde{\chi}(u, \dots, u)$. Note that the top row of $\tilde{B}$ is $(\tilde{\phi}_i, i = 1, \dots, p)$ and let $\tilde{M}_n = \tilde{B}^n$. Recall $0 < \alpha \leq l + 1$. Then we can use the triangle inequality, the bound $E|Z_1|^{\frac{j\alpha}{l+1}} \leq u^{j/l}$ obtained above, and the fact that the highest power of any Z_i appearing in M_n is bounded by $l \wedge n$, to get

$$\begin{aligned} E\|M_n\|^{\frac{\alpha}{l+1}} &\leq \sum_{i=1}^p \sum_{j=1}^p (\tilde{M}_n)_{ij} \\ &\leq p\|\tilde{M}_n\| = p\|\tilde{B}^n\|. \end{aligned}$$

So we have:

$$\sum_{n=1}^{\infty} E\left\|\prod_{i=1}^n B(t-i)\right\|^{\frac{\alpha}{l+1}} \leq p \sum_{n=1}^{\infty} \|\tilde{B}^n\|.$$

The series $\sum_{n=1}^{\infty} \|\tilde{B}^n\|$ converges if $\sum_{i=1}^p \tilde{\phi}_i < 1$. To see this, we argue as follows: If $\sum_{i=1}^p \tilde{\phi}_i = s < 1$, then since $\tilde{B}$ is nonnegative, we can write, (with $\mathbf{e}_i$ denoting the unit basis elements in $\mathbb{R}^p$, considered as column vectors):

$$\begin{aligned} \|\tilde{B}\| &= \max_{1 \leq i \leq p} \mathbf{e}_i' \tilde{B} (1, \dots, 1)' \\ &= \max\left(\sum_{i=1}^p \tilde{\phi}_i, 1, \dots, 1\right) \\ &\leq \max(s, 1, \dots, 1), \\ \|\tilde{B}^2\| &= \max_i \mathbf{e}_i' \tilde{B}^2 (1, 1, \dots, 1)' \\ &\leq \max_i \mathbf{e}_i' \tilde{B} (s, 1, \dots, 1)' \\ &= \max(\tilde{\phi}_1 s + \sum_{i=2}^p \tilde{\phi}_i, s, 1, \dots, 1) \end{aligned}$$

$$\leq \max(s, s, 1, \dots, 1).$$

Similarly:

$$\begin{aligned}\|\tilde{B}^p\| &\leq \max(s, \dots, s) = s, \\ \|\tilde{B}^{pk}\| &\leq \max(s^k, \dots, s^k) = s^k.\end{aligned}$$

With $\tilde{B}^0 = I$ by convention and $C = \sum_{n=0}^{p-1} \|\tilde{B}^n\|$ we can therefore write, using (2.12):

$$(2.13) \quad \sum_{n=1}^{\infty} E \left\| \prod_{i=1}^n B(t-i) \right\|^{\frac{\alpha}{l+1}} \leq p \sum_{n=1}^{\infty} \|\tilde{B}^n\|$$

$$(2.14) \quad \leq pC \sum_{k=0}^{\infty} \|\tilde{B}^{pk}\|$$

$$(2.15) \quad \leq pC \sum_{k=0}^{\infty} s^k < \infty.$$

We now verify (2.10) under the condition $\alpha > l + 1$. Define

$$\sum_{i=1}^p \tilde{\phi}_i := \sum_{i=1}^p (|\phi_i| + \sum_{j=1}^l |b_{ij}| E(|Z_1|^{\frac{l\alpha}{l+1}})^{\frac{l+1}{l\alpha}})$$

in this case and $\tilde{B}$ as before, but with the new $\tilde{\phi}_i$'s in its top row. We can use Minkowski's inequality to derive

$$(2.16) \quad E \|M_n\|^{\frac{\alpha}{l+1}} \leq (p \|\tilde{B}^n\|)^{\frac{\alpha}{l+1}}.$$

As before

$$\|\tilde{B}^p\| \leq s := \sum_{i=1}^p \tilde{\phi}_i < 1$$

by (2.9), and

$$\|\tilde{B}^{pk}\| \leq s^k,$$

so again

$$(2.17) \quad \sum_{n=1}^{\infty} E \left\| \prod_{i=1}^n B(t-i) \right\|^{\frac{\alpha}{l+1}} \leq \sum_{n=1}^{\infty} (p \|\tilde{B}^n\|)^{\frac{\alpha}{l+1}}$$

$$(2.18) \quad \leq C_1 \sum_{k=0}^{\infty} \|\tilde{B}^{pk}\|^{\frac{\alpha}{l+1}}$$

$$(2.19) \quad \leq C_1 \sum_{k=0}^{\infty} s^{k \frac{\alpha}{l+1}}$$

$$(2.20) \quad = C_1 \sum_{k=0}^{\infty} (s^{\frac{\alpha}{l+1}})^k < \infty$$

□

$$\leq \max(s, s, 1, \dots, 1).$$

Similarly:

$$\begin{aligned}\|\tilde{B}^p\| &\leq \max(s, \dots, s) = s, \\ \|\tilde{B}^{pk}\| &\leq \max(s^k, \dots, s^k) = s^k.\end{aligned}$$

With $\tilde{B}^0 = I$ by convention and $C = \sum_{n=0}^{p-1} \|\tilde{B}^n\|$ we can therefore write, using (2.12):

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As before

$$\|\tilde{B}^p\| \leq s := \sum_{i=1}^p \tilde{\phi}_i < 1$$

by (2.9), and

$$\|\tilde{B}^{pk}\| \leq s^k,$$

so again

$$(2.17) \quad \sum_{n=1}^{\infty} E \left\| \prod_{i=1}^n B(t-i) \right\|^{\frac{\alpha}{l+1}} \leq \sum_{n=1}^{\infty} (p \|\tilde{B}^n\|)^{\frac{\alpha}{l+1}}$$

$$(2.18) \quad \leq C_1 \sum_{k=0}^{\infty} \|\tilde{B}^{pk}\|^{\frac{\alpha}{l+1}}$$

$$(2.19) \quad \leq C_1 \sum_{k=0}^{\infty} s^{k \frac{\alpha}{l+1}}$$

$$(2.20) \quad = C_1 \sum_{k=0}^{\infty} (s^{\frac{\alpha}{l+1}})^k < \infty$$

□

It is now easy to check that

$$\begin{aligned}
 X_t &= \mathbf{e}'_1 \{ \Theta \mathbf{Z}(t) + \sum_{n=1}^{\infty} [\prod_{j=1}^n B(t-j)] \Theta \mathbf{Z}(t-n) \} \\
 (2.21) \quad &= \mathbf{e}'_1 \Theta \mathbf{Z}(t) + \sum_{n=1}^{\infty} X_t^{(n)} = \sum_{n=0}^{\infty} X_t^{(n)}
 \end{aligned}$$

is a well defined stationary process, satisfying (2.1). Here we have

$$(2.22) \quad X_t^{(n)} = \mathbf{e}'_1 \prod_{j=1}^n B(t-j) \Theta \mathbf{Z}(t-n), \quad n \geq 1, \quad X_t^{(0)} = \mathbf{e}'_1 \Theta \mathbf{Z}(t).$$

The conditions (2.8) or (2.9) are not necessary for strict stationarity, but are sufficient for the regular variation analysis of the tail of the distribution of X_t . See [15],[16].

3. APPROXIMATIONS AND TAIL WEIGHTS

Assume that

$$(3.1) \quad b_1^* := \prod_{j=1}^l b_{1j} \neq 0$$

and set

$$(3.2) \quad Y_t^{(l)} = b_1^* Z_{t-l}^{l+1}$$

$$(3.3) \quad Y_t^{(n)} = \mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^* Z_{t-n}^{l+1}, \quad n > l.$$

We will see in Section 4 (cf. Theorem 4.1) that for point process asymptotics, $Y_t^{(n)}$ is an adequate approximation to $X_t^{(n)}$ and that X_t can be approximated by $\sum_{n \geq l} Y_t^{(n)}$. Roughly, this results from the fact that what is relevant in the products of Z_i 's making up $X_t^{(n)}$ is the Z_i with highest power. From the discussion following (2.12) we know that for any n , $\prod_{i=1}^n B(t-i)$ only contains powers of Z_i less than or equal to $l \wedge n$. In fact, assuming $n \geq l$, $\mathbf{e}'_1 \prod_{i=n-l+1}^n B(t-i) \mathbf{e}_1$ contains exactly one term with a Z_i of power l , namely $b_1^* Z_{t-n}^l$. Now from (2.22) and (2.3) it follows that for $n \geq l$

$$(3.4) \quad X_t^{(n)} = \mathbf{e}'_1 \prod_{j=1}^n B(t-j) \Theta \mathbf{Z}(t-n)$$

$$(3.5) \quad = \mathbf{e}'_1 \prod_{j=1}^n B(t-j) \mathbf{e}_1 (Z_{t-n} + \sum_{i=1}^q \theta_i Z_{t-n-i}).$$

$$(3.6)$$

It is not hard to see that in the above expression exactly one Z_i has maximum power $l+1$, namely Z_{t-n} . All terms containing Z_{t-n}^{l+1} are collected in $Y_t^{(n)}$ as defined in (3.3).

We now begin with a series of lemma's about the tail behavior of $Y_t^{(n)}$ as well as sums of these variables, which will be of use in later sections.

Lemma 3.1. *Given random variables $Y_1, \dots, Y_k$. Suppose F is a distribution function such that $1 - F$ is regularly varying with index $-\alpha$ and*

$$(3.7) \quad \lim_{x \rightarrow \infty} \frac{P(Y_i > x)}{1 - F(x)} = p_i c_i, \quad i = 1, \dots, k, \quad 0 \leq p_i \leq 1,$$

$$(3.8) \quad \lim_{x \rightarrow \infty} \frac{P(Y_i < -x)}{1 - F(x)} = (1 - p_i) c_i,$$

$$(3.9) \quad \lim_{x \rightarrow \infty} \frac{P(|Y_i| > x, |Y_j| > x)}{1 - F(x)} = 0, \quad i \neq j.$$

Then

$$(3.10) \quad \lim_{x \rightarrow \infty} \frac{P(|\sum_{i=1}^k Y_i| > x)}{1 - F(x)} = \sum_{i=1}^k c_i.$$

Proof. The proof is analogous to that of Lemma 2.1 in [8]. Define a_n such that

$$n(1 - F(a_n)) \rightarrow 1, \quad n \rightarrow \infty,$$

and define for $i = 1, \dots, k$:

$$\nu_i(x) = (p_i x^{-\alpha} 1_{(x>0)} + (1 - p_i)(-x)^{-\alpha} 1_{(x \leq 0)}) dx.$$

Then the definition of a_n yields vague convergence

$$nP\left(\frac{Y_i}{a_n} \in \cdot\right) \rightarrow_v c_i \nu_i(dx)$$

in the space of measures on $[-\infty, \infty]/\{\mathbf{0}\}$. Given a locally compact Hausdorff topological space $\mathbb{E}$ (e.g. $\mathbb{R}^d$), and $x \in \mathbb{E}, A \subset \mathbb{E}$, define the point measure

$$\epsilon_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $k = 2$. The assumptions imply

$$nP(a_n^{-1}(Y_1, Y_2) \in (dx, dy)) \rightarrow_v c_1 \nu_1(dx) \epsilon_0(dy) + c_2 \epsilon_0(dx) \nu_2(dy).$$

in the space of measures on $[-\infty, \infty]^2/\{\mathbf{0}\}$. Take $X_{n,j} = a_n^{-1}(Y_1^{(j)}, Y_2^{(j)})$, $j = 1, \dots, n$, where $(Y_1^{(j)}, Y_2^{(j)})$ are all i.i.d. copies of (Y_1, Y_2) . Then we have (see Proposition 3.21 in [20])

$$\sum_{j=1}^n \epsilon_{X_{n,j}} \Rightarrow \sum_{t=1}^{\infty} \epsilon_{(c_1 j'_t, 0)} + \sum_{t=1}^{\infty} \epsilon_{(0, c_2 j''_t)}$$

where $\Rightarrow$ denotes weak convergence in the space of Radon point measures on $[-\infty, \infty]^2/\{\mathbf{0}\}$, and $\{j'_t\}, \{j''_t\}$ are the points of two independent Poisson processes with mean measures ν_1 and ν_2 respectively. In the following, we will denote a Poisson process with mean measure μ by $\text{PRM}(\mu)$. The map $T : (x, y) \mapsto |x + y|$ is vaguely continuous, so by Proposition 3.18 in [20]:

$$\sum_{j=1}^n \epsilon_{(a_n^{-1}|Y_1^{(j)} + Y_2^{(j)}|)} \Rightarrow \sum_{t=1}^{\infty} \epsilon_{(c_1 |j'_t|)} + \sum_{t=1}^{\infty} \epsilon_{(c_2 |j''_t|)}.$$

The limit point process is $\text{PRM}((|c_1|^\alpha + |c_2|^\alpha) \alpha x^{-\alpha-1} dx)$. Thus by Proposition 3.21 of [20], we have

$$nP(a_n^{-1}(|Y_1 + Y_2|) \in dx) \rightarrow_v \mu(dx)$$

where $\mu(dx) = (|c_1|^\alpha + |c_2|^\alpha)\alpha x^{-\alpha-1}dx$, proving the statement for $k = 2$. The statement for general k follows by induction. More about point processes follows in Section 4. $\square$

Lemma 3.2. Suppose $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7). Let $W_t^{(n)} := \mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i)\mathbf{e}_1 b_1^*$. Then for the variables $\{Y_t^{(n)}, n \geq l\}$ we have as $x \rightarrow \infty$

$$(3.11) \quad P[Y_t^{(n)} > x] \sim aE(|W_t^{(n)}|^{\frac{\alpha}{l+1}})P[|Z_1|^{l+1} > x]$$

and

$$(3.12) \quad P[Y_t^{(n)} < -x] \sim (1-a)E(|W_t^{(n)}|^{\frac{\alpha}{l+1}})P[|Z_1|^{l+1} > x]$$

where

$$a := \frac{rE((W_t^{(n)})^{\alpha/(l+1)}1_{(W_t^{(n)} \geq 0)}) + (1-r)E((-W_t^{(n)})^{\alpha/(l+1)}1_{(W_t^{(n)} < 0)})}{E(|W_t^{(n)}|^{\alpha/(l+1)})}$$

Furthermore, for all fixed $j > n \geq l$, as $x \rightarrow \infty$

$$(3.13) \quad \frac{P[|Y_t^{(n)}| > x]}{P[|Y_t^{(j)}| > x]} \rightarrow c_{nj} := \frac{E(|W_t^{(n)}|^{\frac{\alpha}{l+1}})}{E(|W_t^{(j)}|^{\frac{\alpha}{l+1}})}$$

and

$$(3.14) \quad \frac{P[|Y_t^{(n)}| > x, |Y_t^{(j)}| > x]}{P[|Y_t^{(j)}| > x]} \rightarrow 0.$$

Proof. The proof follows closely the argument of [8]. A generalization of a result of Breiman due to Cline ([3], page 78,79, see also [2],[19]), says that if ξ is a random variable satisfying (2.5), (2.6) and (2.7), and η is another random variable independent of ξ and satisfying $E(|\eta|^\gamma) < \infty$ for some $\gamma > \alpha$, then

$$P[\eta\xi > x] \sim aE|\eta|^\alpha P[|\xi| > x], \quad x \rightarrow \infty,$$

and

$$P[\eta\xi < -x] \sim (1-a)E|\eta|^\alpha P[|\xi| > x], \quad x \rightarrow \infty,$$

where

$$a = \frac{rE(\eta^\alpha 1_{(\eta \geq 0)}) + (1-r)E((- \eta)^\alpha 1_{(\eta < 0)})}{E(|\eta|^\alpha)}.$$

Now for $n \geq l$, $Y_t^{(n)}$ is defined in (3.2) and (3.3) and $E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i)\mathbf{e}_1 b_1^*|)^\gamma < \infty$ for $\alpha/(l+1) < \gamma < \alpha/l$. So the Cline result gives the first and second parts of the lemma. The third part of the lemma, (3.13), now follows easily. For the last part, (3.14), note that:

$$\begin{aligned} & P[|Y_t^{(n)}| > x, |Y_t^{(j)}| > x] \\ &= P[|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i)\mathbf{e}_1 b_1^*||Z_{t-n}|^{l+1} > x, |\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i)\mathbf{e}_1 b_1^*||Z_{t-j}|^{l+1} > x] \end{aligned}$$

and letting $\|B\|$ denote the L_∞ -matrix norm of the matrix B , we get the bound

$$\leq P[\|\prod_{i=1}^{n-l} B(t-i)\| \|b_1^*\| |Z_{t-n}|^{l+1} > x, \|\prod_{i=1}^{j-l} B(t-i)\| \|b_1^*\| |Z_{t-j}|^{l+1} > x]$$

$$\begin{aligned}
&\leq P\left[\left\|\prod_{i=1}^{n-l} B(t-i)\right\| \|b_1^*\| |Z_{t-n}|^{l+1} > x, \right. \\
&\quad \left. \left\|\prod_{i=1}^{n-l} B(t-i)\right\| \left\|\prod_{i=n-l+1}^{j-l} B(t-i)\right\| \|b_1^*\| |Z_{t-j}|^{l+1} > x\right] \\
&:= P[A|Z_{t-n}|^{(l+1)} > x, AC|Z_{t-j}|^{l+1} > x].
\end{aligned}$$

Now by Pratt's lemma ([17]), the ratio in (3.14) converges to 0 as $x \rightarrow \infty$. □

Combining Lemma 3.1 and Lemma 3.2, we obtain:

Corollary 3.1. *If $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7), then*

$$(3.15) \quad P\left[\left|\sum_{n=l}^k Y_t^{(n)}\right| > x\right] \sim \sum_{n=l}^k E(|e'_1| \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*) P[|Z_1|^{l+1} > x]$$

as $x \rightarrow \infty$. In fact also

$$(3.16) \quad P\left[\sum_{n=l}^k |Y_t^{(n)}| > x\right] \sim \sum_{n=l}^k E(|e'_1| \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*) P[|Z_1|^{l+1} > x]$$

and

$$(3.17) \quad P\left[\sum_{n=l}^k \left\|\prod_{i=1}^{n-l} B(t-i)\right\| \|b_1^*\| |Z_{t-n}|^{l+1} > x\right] \sim \sum_{n=l}^k E\left(\left\|\prod_{i=1}^{n-l} B(t-i)\right\| \|b_1^*\|^{\frac{\alpha}{l+1}}\right) P[|Z_1|^{l+1} > x]$$

as $x \rightarrow \infty$.

Proof. The first statement comes from applying Lemma 3.1 to $Y_t^{(n)}$, $n \geq l$, the second statement comes from applying the same Lemma to $|Y_t^{(n)}|$, $n \geq l$, and the last statement follows after applying Lemma 3.1 to $\tilde{Y}_t^{(n)} = \left\|\prod_{i=1}^{n-l} B(t-i)\right\| \|b_1^*\| |Z_{t-n}|^{l+1}$, $n \geq l$. □

The next Corollaries deal with the case where the number of summands is infinite, for $|Y_t^{(n)}|$, $n \geq l$, and $\tilde{Y}_t^{(n)}$, $n \geq l$ respectively.

Corollary 3.2. *If $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7), then*

$$(3.18) \quad \frac{P[\sum_{n=l}^{\infty} |Y_t^{(n)}| > x]}{P[|Z_1|^{l+1} > x]} \sim \sum_{n=l}^{\infty} E(|e'_1| \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*)$$

as $x \rightarrow \infty$.

Proof. The proof follows the argument of [3], outlined in [20], p. 228. For notational convenience, we again use

$$W_t^{(n)} := e'_1 \prod_{j=1}^{n-l} B(t-j) e_1 b_1^*,$$

so that

$$|Y_t^{(n)}| = |W_t^{(n)}| |Z_{t-n}|^{l+1}.$$

For any $k \geq l$, we have

$$P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x\right] \geq P\left[\sum_{j=l}^k |Y_t^{(j)}| > x\right].$$

Applying Corollary 3.1 gives:

$$\liminf_{x \rightarrow \infty} \frac{P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} \geq \liminf_{x \rightarrow \infty} \frac{P\left[\sum_{j=l}^k |Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} = \sum_{j=l}^k E|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}.$$

Letting $k \rightarrow \infty$ we get:

$$(3.19) \quad \liminf_{x \rightarrow \infty} \frac{P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} \geq \sum_{j=l}^{\infty} E|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}.$$

To finish the proof, we proceed as follows (see [20], p.228). Decompose the event $[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x]$ according to whether $[\vee_{j=l}^{\infty} |Y_t^{(j)}| > x]$ or $[\vee_{j=l}^{\infty} |Y_t^{(j)}| \leq x]$. This gives:

$$\begin{aligned} P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x\right] &= P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x, \vee_{j=l}^{\infty} |Y_t^{(j)}| > x\right] + P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x, \vee_{j=l}^{\infty} |Y_t^{(j)}| \leq x\right] \\ &\leq P\left[\bigcup_{j=l}^{\infty} \{|Y_t^{(j)}| > x\}\right] + P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| 1_{\{|Y_t^{(j)}| \leq x\}} > x, \vee_{j=l}^{\infty} |Y_t^{(j)}| \leq x\right] \\ &\leq \sum_{j=l}^{\infty} P\left[|Y_t^{(j)}| > x\right] + P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| 1_{\{|Y_t^{(j)}| \leq x\}} > x\right] \end{aligned}$$

Then by Markov's inequality,

$$\frac{P\left[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} \leq \frac{\sum_{j=l}^{\infty} P\left[|Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} + \sum_{j=l}^{\infty} \frac{E|W_t^{(j)}| |Z_{t-j}|^{l+1} 1_{\{|W_t^{(j)}| |Z_{t-j}|^{l+1} \leq x\}}}{x P\left[|Z_1|^{l+1} > x\right]} = I + II.$$

To handle I , an upper bound similar to the one that makes Cline's and Breiman's result ([3, 2]) work, obtained e.g. using the Karamata representation, allows us to interchange sum and limit:

$$\lim_{x \rightarrow \infty} \frac{\sum_{j=l}^{\infty} P\left[|Y_t^{(j)}| > x\right]}{P\left[|Z_1|^{l+1} > x\right]} = \sum_{j=l}^{\infty} E|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}.$$

Considering II , assume temporarily that $0 < \alpha < l + 1$. By Lemma 3.2, the distribution tail of $|Y_t^{(j)}| = |W_t^{(j)}| |Z_{t-j}|^{(l+1)}$ is regularly varying with index $-\alpha/(l+1)$. Now

$$\begin{aligned} &\frac{E|W_t^{(j)}| |Z_{t-j}|^{l+1} 1_{\{|W_t^{(j)}| |Z_{t-j}|^{l+1} \leq x\}}}{x P\left[|Z_1|^{l+1} > x\right]} \\ &= \frac{E|W_t^{(j)}| |Z_{t-j}|^{l+1} 1_{\{|W_t^{(j)}| |Z_{t-j}|^{l+1} \leq x\}}}{x P\left[|W_t^{(j)}| |Z_{t-j}|^{l+1} > x\right]} \cdot \frac{P\left[|W_t^{(j)}| |Z_{t-j}|^{l+1} > x\right]}{P\left[|Z_1|^{(l+1)} > x\right]} \\ &= \frac{E|Y_t^{(j)}| 1_{\{|Y_t^{(j)}| \leq x\}}}{x P\left[|Y_t^{(j)}| > x\right]} \cdot \frac{P\left[|W_t^{(j)}| |Z_{t-j}|^{l+1} > x\right]}{P\left[|Z_1|^{l+1} > x\right]} \end{aligned}$$

From an integration by parts and Karamata's theorem, it is seen as in [20], p.229, that

$$\frac{E|Y_t^{(j)}|1_{|Y_t^{(j)}| \leq x}}{xP[|Y_t^{(j)}| > x]} \rightarrow \left(1 - \frac{\alpha}{(l+1)}\right)^{-1} - 1 = \frac{\alpha}{(l+1)}\left(1 - \frac{\alpha}{(l+1)}\right)^{-1}$$

Since $P[|Z_1|^{l+1} > x]$ is regularly varying with index $-\alpha/(l+1)$, we can use its Karamata representation and (3.11) and (3.12) to obtain that for sufficiently large x , and some constant $k > 0$,

$$\frac{P[|Y_t^{(j)}| > x]}{P[|Z_1|^{l+1} > x]} = \frac{P[|W_t^{(j)}||Z_{t-j}|^{l+1} > x]}{P[|Z_1|^{l+1} > x]} \leq kE(|W_t^{(j)}|^{\frac{\alpha}{l+1}})$$

Combining these observations, we conclude, for sufficiently large x :

$$\frac{E|W_t^{(j)}||Z_{t-j}|^{l+1}1_{|W_t^{(j)}||Z_{t-j}|^{l+1} \leq x}}{xP[|Z_1|^{l+1} > x]} \leq k_1E(|W_t^{(j)}|^{\frac{\alpha}{l+1}})$$

for some constant $k_1 > 0$. This bound is summable, because of (2.13). Therefore we conclude:

$$\limsup_{x \rightarrow \infty} \sum_{j=l}^{\infty} \frac{E|W_t^{(j)}||Z_{t-j}|^{l+1}1_{|W_t^{(j)}||Z_{t-j}|^{l+1} \leq x}}{xP[|Z_1|^{l+1} > x]} \leq k_1 \sum_{j=l}^{\infty} E(|W_t^{(j)}|^{\frac{\alpha}{l+1}}).$$

So

$$(3.20) \quad \limsup_{x \rightarrow \infty} \frac{P[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x]}{P[|Z_1|^{l+1} > x]} \leq k_2 \sum_{j=l}^{\infty} E(|W_t^{(j)}|^{\frac{\alpha}{l+1}})$$

for some constant k_2 .

If $\alpha \geq l+1$ we proceed as in [20], p.229. Pick $\gamma \in (\frac{\alpha}{l+1}, \frac{\alpha}{l})$. Define $W = \sum_{j=l}^{\infty} |W_t^{(j)}|$, and $p_j = |W_t^{(j)}|/W$. By Jensen's inequality

$$\begin{aligned} \left(\sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{l+1}\right)^{\gamma} &= W^{\gamma} \left(\sum_{j=l}^{\infty} p_j |Z_{t-j}|^{l+1}\right)^{\gamma} \\ &\leq W^{\gamma} \sum_{j=l}^{\infty} p_j |Z_{t-j}|^{(l+1)\gamma} \\ &= W^{\gamma-1} \sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{(l+1)\gamma} \end{aligned}$$

Then

$$\begin{aligned} \frac{P[\sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{l+1} > x]}{P[|Z_1|^{l+1} > x]} &= \frac{P[(\sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{l+1})^{\gamma} > x^{\gamma}]}{P[|Z_1|^{(l+1)\gamma} > x^{\gamma}]} \\ &\leq \frac{P[W^{\gamma-1} \sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{(l+1)\gamma} > x^{\gamma}]}{P[|Z_1|^{(l+1)\gamma} > x^{\gamma}]} \end{aligned}$$

We can use the preceding case to obtain

$$(3.21) \quad \limsup_{x \rightarrow \infty} \frac{P[\sum_{j=l}^{\infty} |W_t^{(j)}||Z_{t-j}|^{l+1} > x]}{P[|Z_1|^{l+1} > x]} \leq k_3 \sum_{j=l}^{\infty} E((W^{\gamma-1} \sum_{j=l}^{\infty} |W_t^{(j)}|)^{\frac{\alpha}{(l+1)\gamma}}) < \infty$$

which is similar to (3.20). To check that the above series indeed converges, we can argue as follows.

$$\sum_{j=l}^{\infty} E((W^{\gamma-1}|W_t^{(j)}|)^{\frac{\alpha}{(l+1)\gamma}}) = \sum_{j=l}^{\infty} E(W^{\frac{\alpha}{l+1}(1-\frac{1}{\gamma})}|W_t^{(j)}|^{\frac{\alpha}{l+1}\frac{1}{\gamma}})$$

By Hölder's inequality, with $1/q = 1/\gamma$ etc., this is

$$\begin{aligned} &\leq \sum_{j=l}^{\infty} (E(W^{\frac{\alpha}{l+1}}))^{1-\frac{1}{\gamma}} (E(|W_t^{(j)}|^{\frac{\alpha}{l+1}}))^{\frac{1}{\gamma}} \\ &= \left(E\left(\sum_{i=l}^{\infty} |W_t^{(i)}|^{\frac{\alpha}{l+1}}\right) \right)^{1-\frac{1}{\gamma}} \sum_{j=l}^{\infty} (E(|W_t^{(j)}|^{\frac{\alpha}{l+1}}))^{\frac{1}{\gamma}}. \\ &:= AB. \end{aligned}$$

Consider A: By Minkowski's inequality:

$$\left(E\left(\sum_{i=l}^{\infty} |W_t^{(i)}|^{\frac{\alpha}{l+1}}\right) \right)^{(1-\frac{1}{\gamma})} \leq \left(\sum_{i=l}^{\infty} E(|W_t^{(i)}|^{\frac{\alpha}{l+1}})^{\frac{l+1}{\alpha}} \right)^{\frac{\alpha}{l+1}(1-\frac{1}{\gamma})}$$

which is by (2.17):

$$\begin{aligned} &= \left(\sum_{n=0}^{\infty} p \|\tilde{B}^n\| \right)^{\frac{\alpha}{l+1}(1-\frac{1}{\gamma})} \\ &\leq \left(C_1 \sum_{k=0}^{\infty} s^k \right)^{\frac{\alpha}{l+1}(1-\frac{1}{\gamma})} < \infty \end{aligned}$$

by (2.9). Again using (2.17), it follows that $B < \infty$ as well. So indeed (3.21) is valid.

The proof is finished as follows: For any $\epsilon > 0$:

$$\frac{P[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x]}{P[|Z_1|^{l+1} > x]} \leq \frac{P[\sum_{j=l}^m |Y_t^{(j)}| > (1-\epsilon)x]}{P[|Z_1|^{l+1} > x]} + \frac{P[\sum_{j=m+1}^{\infty} |Y_t^{(j)}| > \epsilon x]}{P[|Z_1|^{l+1} > x]}.$$

From Corollary 3.1 and (3.20), it follows that

$$\limsup_{x \rightarrow \infty} \frac{P[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x]}{P[|Z_1|^{l+1} > x]} \leq (1-\epsilon)^{-\frac{\alpha}{(l+1)}} \sum_{j=l}^m E(|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}) + \epsilon^{-\frac{\alpha}{(l+1)}} k_2 \sum_{j=m+1}^{\infty} E(|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}).$$

Now let $m \rightarrow \infty$, and then send $\epsilon \rightarrow 0$, to obtain

$$\limsup_{x \rightarrow \infty} \frac{P[\sum_{j=l}^{\infty} |Y_t^{(j)}| > x]}{P[|Z_1|^{l+1} > x]} \leq \sum_{j=l}^{\infty} E(|W_t^{(j)}|^{\frac{\alpha}{(l+1)}}).$$

This and (3.19) complete the proof. □

Corollary 3.3. *If $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7), then*

$$(3.22) \quad \frac{P[\sum_{n=l}^{\infty} \|\prod_{i=1}^{n-l} B(t-i)\| b_1^* |Z_{t-n}|^{l+1} > x]}{P[|Z_1|^{l+1} > x]} \sim \sum_{n=l}^{\infty} E(\|\prod_{i=1}^{n-l} B(t-i)\| \|b_1^*\|)^{\frac{\alpha}{l+1}}$$

as $x \rightarrow \infty$.

Proof. This can be proved in a completely analogous manner to the previous Corollary, with the new definition

$$\tilde{W}_t^{(n)} := \left\| \prod_{i=1}^{n-l} B(t-i) \right\| |b_1^*|$$

replacing $W_t^{(n)}$. □

Next, we derive a bound on $|X_t^{(n)}|$ and use it to bound X_t . Recall from (2.21) and (2.22), that $X_t = \sum_{n=0}^{\infty} X_t^{(n)}$ with

$$X_t^{(n)} = \mathbf{e}_1' \prod_{j=1}^n B(t-j) \Theta \mathbf{Z}(t-n), \quad n \geq 1, \quad X_t^{(0)} = \mathbf{e}_1' \Theta \mathbf{Z}(t).$$

In fact, from the definition of the B -matrices, it follows that

$$\prod_{j=1}^n B(t-j) \Theta \mathbf{Z}(t-n) = (X_t^{(n)}, X_{t-1}^{(n-1)}, \dots, X_{t-p+1}^{(n-p+1)})'.$$

If we set

$$(3.23) \quad R_t^{(n)} = X_t^{(n)} - Y_t^{(n)}, \quad n \geq l,$$

then for $n \geq l$ can write:

$$\begin{aligned} X_t^{(n)} &= \mathbf{e}_1' \prod_{j=1}^n B(t-j) \Theta \mathbf{Z}(t-n) \\ &= \mathbf{e}_1' \prod_{j=1}^{n-l} B(t-j) \prod_{i=1}^l B(t-(n-l)-i) \Theta \mathbf{Z}(t-(n-l)-l) \\ &= \mathbf{e}_1' \prod_{j=1}^{n-l} B(t-j) (b_1^* Z_{t-n}^{l+1} + R_{t-(n-l)}^{(l)}, X_{t-(n-l)-1}^{(l-1)}, \dots, X_{t-(n-l)-p+1}^{(l-p+1)})'. \end{aligned}$$

The last equation uses (3.2). We can now bound $|X_t^{(n)}|$ by

$$(3.24) \quad |X_t^{(n)}| \leq \left\| \prod_{j=1}^{n-l} B(t-j) \right\| (|b_1^*| |Z_{t-n}|^{l+1} + |R_{t-(n-l)}^{(l)}| + \sum_{i=1}^{p-1} |X_{t-(n-l)-i}^{(l-i)}|).$$

From the discussion after (2.11), it follows that $\prod_{j=1}^{n-l} B(t-j)$ is independent of Z_{t-n} . However, $R_{t-(n-l)}^{(l)} + \sum_{i=1}^{p-1} X_{t-(n-l)-i}^{(l-i)}$ is a sum of products of Z_i 's, containing $Z_{t-n}^j, j = 1, \dots, l$ in some of its terms. This complicates the tail analysis of infinite sums of $X_t^{(n)}$'s. To obtain a more useful bound on $|X_t^{(n)}|$, we proceed as follows. In $b_1^* Z_{t-n}^{l+1} + R_{t-(n-l)}^{(l)} + \sum_{i=1}^{p-1} X_{t-(n-l)-i}^{(l-i)}$, written as a sum of products of Z_i 's, replace each term by its absolute value. In the resulting expression, replace each $|Z_{t-n}|$ by $(1 + |Z_{t-n}|)$. Next, for each individual term of the modified sum of products, identify the power of $(1 + |Z_{t-n}|)$ in that term, say p . (Note that for the first term $p = l + 1$, and for each further term we have $0 \leq p \leq l$.) Now multiply the term by $(1 + |Z_{t-n}|)^{l+1-p}$. After this step, every

term of the modified expression contains $(1 + |Z_{t-n}|)^{l+1}$, so we can factor out $(1 + |Z_{t-n}|)^{l+1}$. Call the remaining factor R_{t-n}^* . From the above operations, we obtain the following bound for $|X_t^{(n)}|$:

$$(3.25) \quad |X_t^{(n)}| \leq \left\| \prod_{i=1}^{n-l} B(t-i) \right\| R_{t-n}^* (1 + |Z_{t-n}|)^{l+1}.$$

By construction, $\left\| \prod_{i=1}^{n-l} B(t-i) \right\| R_{t-n}^*$ is independent of Z_{t-n} , and R_{t-n}^* is a sum of products of powers of $|Z_i|$'s less than or equal to l . The bound (3.25) will be used to analyze the tail behavior for $|X_t|$. In the next two Corollaries, we relate the tail behavior of $|X_t|$ to that of $Y_t^{(n)}$, $n \geq l$.

Corollary 3.4. *If $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7) then*

$$(3.26) \quad P\left[\sum_{n=0}^k X_t^{(n)} > x\right] \sim \sum_{n=l}^k E(|e_1' \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*|)^{\frac{\alpha}{l+1}} P[|Z_1|^{l+1} > x]$$

as $x \rightarrow \infty$.

Proof. We have, from (3.23), that for any $\epsilon > 0$:

$$\frac{P[\sum_{n=0}^k X_t^{(n)} > x]}{P[|Z_1|^{l+1} > x]} \leq \frac{P[\sum_{n=l}^k Y_t^{(n)} > (1-\epsilon)x]}{P[|Z_1|^{l+1} > x]} + \frac{P[\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)} > \epsilon x]}{P[|Z_1|^{l+1} > x]} := I + II.$$

Note that $X_t^{(n)}$, $n = 0, \dots, l-1$ and $R_t^{(n)}$, $n = l, \dots, m$ all are finite sums of products of Z_i 's, with highest power of an individual Z_i occurring in any term $\leq l$. This follows from the fact that all terms containing Z_t^{l+1} are collected in $Y_t^{(n)}$ as seen in the discussion following (3.3). Therefore, using the regular variation of the Z_i 's, we may check that $II \rightarrow 0$ as follows. Pick $0 < \delta < 1$. By a similar argument as the one given for (3.25), we can bound $|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}|$ by

$$|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \tilde{R}_{t-n} (1 + |Z_{t-n}|)^{l+\delta},$$

where $\tilde{R}_{t-n}$ is independent of Z_{t-n} , and is a sum of products of powers of $|Z_i|$'s less than or equal to l . Since $P(1 + |Z_1| > x)/P(|Z_1| > x) \rightarrow 1$ as $x \rightarrow \infty$, it follows from Cline's result that

$$P[\tilde{R}_{t-n} (1 + |Z_{t-n}|)^{l+\delta} > \epsilon x] \sim E(\tilde{R}_{t-n}^{\frac{\alpha}{l+\delta}}) \epsilon^{\frac{\alpha}{l+\delta}} P[|Z_1|^{l+\delta} > x] := K_1(\epsilon) P[|Z_1|^{l+\delta} > x].$$

So

$$\begin{aligned} II &:= \frac{P[\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)} > \epsilon x]}{P[|Z_1|^{l+1} > x]} \leq \frac{P[\tilde{R}_{t-n} (1 + |Z_{t-n}|)^{l+\delta} > \epsilon x]}{P[|Z_1|^{l+1} > x]} \\ &\sim \frac{K_1(\epsilon) P[|Z_1|^{l+\delta} > x]}{P[|Z_1|^{l+1} > x]} \\ &\sim K_1(\epsilon) x^{\frac{\alpha}{l+1} - \frac{\alpha}{l+\delta}} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned}$$

since $0 < \delta < 1$. By Corollary 3.1,

$$I \sim (1-\epsilon)^{\frac{\alpha}{l+1}} \sum_{n=l}^k E(|e_1' \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

Now let $\epsilon \downarrow 0$ to obtain:

$$(3.27) \quad \limsup_{x \rightarrow \infty} \frac{P[|\sum_{n=0}^k X_t^{(n)}| > x]}{P[|Z_1|^{l+1} > x]} \leq \sum_{n=l}^k E(|e'_1 \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

On the other hand, for all $\epsilon > 0$, we can also write

$$\frac{P[|\sum_{n=0}^k X_t^{(n)}| > x]}{P[|Z_1|^{l+1} > x]} \geq \frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \epsilon x]}{P[|Z_1|^{l+1} > x]}.$$

We claim that

$$\frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \epsilon x]}{P[|Z_1|^{l+1} > x]} \sim \frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]}.$$

To prove this, we need to show that

$$\frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]} \rightarrow 1 \text{ as } x \rightarrow \infty.$$

Note that

$$\begin{aligned} 1 &\geq \frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]} \\ &= 1 - \frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| > \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]} \\ &\geq 1 - \frac{P[|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| > \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]}. \end{aligned}$$

The claim is proved when we show that

$$\frac{P[|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| > \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Now by Corollary 3.1

$$P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x] \sim (1+\epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^k E(|e'_1 \prod_{i=1}^{n-l} B(t-i) e_1 b_1^*|)^{\frac{\alpha}{l+1}} P[|Z_1|^{l+1} > x] := K_2(\epsilon) P[|Z_1|^{l+1} > x].$$

Thus

$$\begin{aligned} \frac{P[|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| > \epsilon x]}{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]} &\sim \frac{1}{K_2(\epsilon)} \frac{P[|\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| > \epsilon x]}{P[|Z_1|^{l+1} > x]} \\ &= \frac{1}{K_2(\epsilon)} II \rightarrow 0 \text{ as } x \rightarrow \infty. \end{aligned}$$

The proof of the Corollary is finished as follows:

$$\frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x, |\sum_{n=0}^{l-1} X_t^{(n)} + \sum_{n=l}^k R_t^{(n)}| \leq \epsilon x]}{P[|Z_1|^{l+1} > x]} \sim \frac{P[|\sum_{n=l}^k Y_t^{(n)}| > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]}$$

$$\sim (1 + \epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^k E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}$$

by Corollary 3.1. Let $\epsilon \downarrow 0$ again, to get

$$\liminf_{x \rightarrow \infty} \frac{P[|\sum_{n=0}^k X_t^{(n)}| > x]}{P[|Z_1|^{l+1} > x]} \geq \sum_{n=l}^k E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

This and (3.27) complete the proof. $\square$

Corollary 3.5. *If $\{Z_t\}$ satisfies (2.5), (2.6), and (2.7), then*

$$(3.28) \quad P[|X_t| > x] \sim \sum_{n=l}^{\infty} E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}} P[|Z_1|^{l+1} > x].$$

Hence $|X_t|$ has regularly varying tail probability with index $-\alpha/(l+1)$.

Proof. Note that $X_t = \sum_{n=0}^{\infty} X_t^{(n)}$. For any integer m and $\epsilon > 0$, we can write

$$\begin{aligned} \frac{P[|X_t| > x]}{P[|Z_1|^{l+1} > x]} &\leq \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1-\epsilon)x]}{P[|Z_1|^{l+1} > x]} + \frac{P[|\sum_{n=m+1}^{\infty} X_t^{(n)}| > \epsilon x]}{P[|Z_1|^{l+1} > x]} \\ &:= I + II. \end{aligned}$$

Without loss of generality, let $m \geq l$. By Corollary 3.4, we know

$$I \sim (1-\epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^m E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

For II, we use the triangle inequality followed by (3.25). This gives:

$$\frac{P[|\sum_{n=m+1}^{\infty} X_t^{(n)}| > \epsilon x]}{P[|Z_1|^{l+1} > x]} \leq \frac{P[\sum_{n=m+1}^{\infty} \|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^* (1 + |Z_{t-n}|)^{l+1} > \epsilon x]}{P[|Z_1|^{l+1} > x]}.$$

Now $P(1 + |Z_1| > x)/P(|Z_1| > x) \rightarrow 1$ as $x \rightarrow \infty$, and using (2.13) it is easily seen that $\sum_{n=l}^{\infty} E(\|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^*)^{\frac{\alpha}{l+1}} < \infty$. Hence we can argue as in Corollary 3.2, with $\tilde{W}_t^{(n)} = \|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^*$ replacing $W_t^{(n)}$ etc., to get

$$\frac{P[\sum_{n=m+1}^{\infty} \|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^* (1 + |Z_{t-n}|)^{l+1} > \epsilon x]}{P[|Z_1|^{l+1} > x]} \sim \epsilon^{\frac{-\alpha}{l+1}} \sum_{n=m+1}^{\infty} E(\|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^*)^{\frac{\alpha}{l+1}}$$

as $x \rightarrow \infty$. If we let $m \rightarrow \infty$, and then $\epsilon \downarrow 0$, we obtain:

$$(3.29) \quad \limsup_{x \rightarrow \infty} \frac{P[|X_t| > x]}{P[|Z_1|^{l+1} > x]} \leq \sum_{n=l}^{\infty} E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

To finish the proof of the Corollary, select any integer m and any $\epsilon > 0$. We can then write

$$\begin{aligned} \frac{P[|X_t| > x]}{P[|Z_1|^{l+1} > x]} &\geq \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x, |\sum_{n=m+1}^{\infty} X_t^{(n)}| \leq \epsilon x]}{P[|Z_1|^{l+1} > x]} \\ &= \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x, |\sum_{n=m+1}^{\infty} X_t^{(n)}| \leq \epsilon x]}{P[\sum_{n=0}^m X_t^{(n)} > (1+\epsilon)x]} \frac{P[\sum_{n=0}^m X_t^{(n)} > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]} \end{aligned}$$

$$\begin{aligned}
&= \left(1 - \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x, |\sum_{n=m+1}^\infty X_t^{(n)}| > \epsilon x]}{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x]} \right) \\
&\quad \times \frac{P[\sum_{n=0}^m X_t^{(n)} > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]} \\
(3.30) \quad &\geq \left(1 - \frac{P[|\sum_{n=m+1}^\infty X_t^{(n)}| > \epsilon x]}{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x]} \right) \frac{P[\sum_{n=0}^m X_t^{(n)} > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]}.
\end{aligned}$$

By the triangle inequality and (3.25),

$$\begin{aligned}
P[|\sum_{n=m+1}^\infty X_t^{(n)}| > \epsilon x] &\leq P[\sum_{n=m+1}^\infty \|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^* (1 + |Z_{t-n}|)^{l+1} > \epsilon x] \\
&\sim \epsilon^{\frac{-\alpha}{l+1}} \sum_{n=m+1}^\infty E(\|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^*)^{\frac{\alpha}{l+1}} P[|Z_1|^{l+1} > x]
\end{aligned}$$

as $x \rightarrow \infty$, as seen above. By Corollary 3.4

$$P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x] \sim (1+\epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^m E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}} P[|Z_1|^{l+1} > x]$$

as $x \rightarrow \infty$. Dividing the last two results, we see that as $x \rightarrow \infty$

$$\begin{aligned}
&\frac{P[\sum_{n=m+1}^\infty \|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^* (1 + |Z_{t-n}|)^{l+1} > \epsilon x]}{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x]} \\
&\rightarrow \frac{\epsilon^{\frac{-\alpha}{l+1}} \sum_{n=m+1}^\infty E(\|\prod_{i=1}^{n-l} B(t-i)\| R_{t-n}^*)^{\frac{\alpha}{l+1}}}{(1+\epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^m E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}}.
\end{aligned}$$

As $m \rightarrow \infty$ the righthand side converges to 0. Therefore, for any integer m and any $\epsilon > 0$, as $x \rightarrow \infty$

$$\begin{aligned}
\frac{P[|X_t| > x]}{P[|Z_1|^{l+1} > x]} &\geq \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x, |\sum_{n=m+1}^\infty X_t^{(n)}| \leq \epsilon x]}{P[|Z_1|^{l+1} > x]} \\
&\sim (1 - \eta_m) \frac{P[|\sum_{n=0}^m X_t^{(n)}| > (1+\epsilon)x]}{P[|Z_1|^{l+1} > x]} \\
&\sim (1 - \eta_m)(1+\epsilon)^{\frac{-\alpha}{l+1}} \sum_{n=l}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}
\end{aligned}$$

where $\eta_m \rightarrow 0$ as $m \rightarrow \infty$. Let $m \rightarrow \infty$ and $\epsilon \downarrow 0$ to obtain

$$(3.31) \quad \liminf_{x \rightarrow \infty} \frac{P[|X_t| > x]}{P[|Z_1|^{l+1} > x]} \geq \sum_{n=l}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{n-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}.$$

Combining this result and (3.29) completes the proof of (3.28). $\square$

As a consequence of (3.28),

$$\lim_{x \rightarrow \infty} \frac{P(|X_t| > tx)}{P(|X_t| > x)} = t^{-\frac{\alpha}{l+1}}$$

so that $P[|X_t| > x]$ is regularly varying with index $-\alpha/(l+1)$.

4. POINT PROCESS ANALYSIS

In this section, we investigate the limit behavior of a sequence of point processes associated with the stated bilinear time series model (2.1).

The object of interest in this section is the sequence of point processes based on the points $\{a_n^{-(l+1)} X_t, t = 1, \dots, n\}$, where a_n is the $1 - n^{-1}$ quantile of $|Z_1|$, i.e.

$$(4.1) \quad a_n = \inf\{x : P[|Z_1| > x] < n^{-1}\}.$$

Here are the necessary definitions and notation concerning point processes:

For a locally compact Hausdorff topological space $\mathbb{E}$, we let $M_p(\mathbb{E})$ be the space of Radon point measures on $\mathbb{E}$. This means $m \in M_p(\mathbb{E})$ is of the form

$$m = \sum_{i=1}^{\infty} \epsilon_{x_i},$$

where $x_i \in \mathbb{E}$ are the locations of the point masses of m and ϵ_{x_i} denotes the point measure

$$\epsilon_{x_i}(A) = \begin{cases} 1 & \text{if } x_i \in A, \\ 0 & \text{if } x_i \notin A. \end{cases}$$

where $A \subset \mathbb{E}$. We assume that all measures in $M_p(\mathbb{E})$ are Radon, which means that for any $m \in M_p(\mathbb{E})$ and any compact $K \subset \mathbb{E}$, $m(K) < \infty$. On the space $M_p(\mathbb{E})$ we use the vague metric $\rho(\cdot, \cdot)$. Its properties are discussed for example in [20], Section 3.4 and [14]. Note that a sequence of measures $m_n \in M_p(\mathbb{E})$ converge vaguely to $m_0 \in M_p(\mathbb{E})$ if for any continuous function $f : \mathbb{E} \mapsto [0, \infty)$ with compact support we have $m_n(f) \rightarrow m_0(f)$ where $m_n(f) = \int_{\mathbb{E}} f dm_n$. The non-negative functions with compact support will be denoted by $C_K^+(\mathbb{E})$.

A Poisson process on $\mathbb{E}$ with mean measure μ will be denoted by $\text{PRM}(\mu)$. Of primary interest in our applications is the case when $\mathbb{E}_m = [-\infty, \infty]^m \setminus \{\mathbf{0}\}$, where compact subsets are closed subsets of $[-\infty, \infty]^m$ which are bounded away from $\mathbf{0}$.

The point process convergence in Proposition 3.1 of [8] underpins the main results of this section. For the readers convenience, we state the result below:

Proposition 4.1. ([8]) *Suppose the marginal distribution F of the iid sequence $\{Z_t\}$ satisfies (2.5), (2.6) and (2.7) and m is a fixed positive integer. Suppose further that $\sum_{s=1}^{\infty} \epsilon_{j_s}$ is $\text{PRM}(\mu)$, where $\mu(dx) = \alpha(rx^{-\alpha-1}1_{[x>0]} + (1-r)(-x)^{-\alpha-1}1_{[x<0]})dx$ and $\{U_{kl}, U'_{kl}, k \geq 1, l \geq 1\}$ are iid with distribution F . If $\mathbf{e}_i \in [-\infty, \infty]$ denotes the basis element of $\mathbb{R}^m$ with i th component equal to 1 and the rest zero and $\mathbb{E}_m = [-\infty, \infty]^m \setminus \{\mathbf{0}\}$, then*

$$\begin{aligned} \sum_{t=1}^n \epsilon_{(a_n^{-1}(Z_{t-i}, i=1, \dots, m), Z_{t-j}, j=1, \dots, m)} &\Rightarrow \sum_{s=1}^{\infty} \epsilon_{(j_s \mathbf{e}_1, \text{sgn}(j_s) \infty, U'_{s1}, \dots, U'_{s, m-1})} \\ &+ \sum_{s=1}^{\infty} \epsilon_{(j_s \mathbf{e}_2, U_{s1}, \text{sgn}(j_s) \infty, U'_{s1}, \dots, U'_{s, m-2})} \\ &+ \dots + \sum_{s=1}^{\infty} \epsilon_{(j_s \mathbf{e}_m, U_{s, m-1}, \dots, U_{s1}, \text{sgn}(j_s) \infty)} \end{aligned}$$

in $M_p(\mathbb{E}_m \times [-\infty, \infty]^m)$.

For $k = l, \dots, m$, consider the following point processes defined on the space $\mathbb{E}_1 := [-\infty, \infty] \setminus \{0\}$ by

$$I_n^k = \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)}}$$

where $Y_t^{(k)}$ is defined as in (3.3). We first establish the joint convergence of $(I_n^{(l)}, \dots, I_n^{(m)})$ in $M_p^{m-l+1}(\mathbb{E}_1)$:

Proposition 4.2. *Under the assumptions of Proposition 4.1, we have*

$$(I_n^{(l)}, \dots, I_n^{(m)}) \Rightarrow (I^{(l)}, \dots, I^{(m)})$$

in $M_p^{m-l+1}(\mathbb{E}_1)$, where $I^{(k)} = \sum_{s=1}^{\infty} \epsilon_{j_s^{l+1} b_1^*(\mathbf{e}_1'(B_{s,1} \dots B_{s,k-l}) \mathbf{e}_1)}$ and

$$B_{s,r} = \chi(U_{s,r}, \dots, U_{s,r+l-1})$$

where χ is defined as in (2.2).

Proof. The proof follows Proposition 3.2 in [8]: For $k \in \{l, \dots, m\}$, the restriction

$$g(x_1, \dots, x_m, u_1, \dots, u_m) = (x_k, u_1, \dots, u_{k-1})$$

is a continuous mapping from $\mathbb{E}_m \times [-\infty, \infty]^m$ into $\mathbb{E}_1 \times [-\infty, \infty]^{k-1}$ with the property that $g^{-1}(K)$ is compact for every compact $K \subset \mathbb{E}_1 \times [-\infty, \infty]^{k-1}$. Therefore, by Proposition 3.18, page 148 of [20], this mapping induces a continuous mapping from $M_p(\mathbb{E}_m \times [-\infty, \infty]^m)$ into $M_p(\mathbb{E}_1 \times [-\infty, \infty]^{k-1})$ and hence applying the mapping to the convergence in Proposition 4.1 we get

$$(4.2) \quad \tilde{I}_n^k := \sum_{t=1}^n \epsilon_{(a_n^{-1} Z_{t-k}, Z_{t-1}, \dots, Z_{t-k+1})} \Rightarrow \tilde{I}^k := \sum_{s=1}^{\infty} \epsilon_{(j_s, U_{s,k-1}, \dots, U_{s,1})},$$

where the convergence is joint in $k = l, \dots, m$. If M and $-M$ are continuity points of F , then this convergence also holds for these point processes restricted to the set $\mathbb{E}_1 \times [-M, M]^{k-1}$:

$$(4.3) \quad \tilde{I}_n^{k,M}(\cdot) := \tilde{I}_n^k(\cdot \cap (\mathbb{E}_1 \times [-M, M]^{k-1})) \Rightarrow \tilde{I}^{k,M}(\cdot) := \tilde{I}^k(\cdot \cap (\mathbb{E}_1 \times [-M, M]^{k-1}))$$

jointly for $k = l, \dots, m$ in $M_p^{m-l+1}(\mathbb{E}_1 \times [-\infty, \infty]^{k-1})$.

Now consider the mapping (for $k > l$):

$$f_k(x, u_1, \dots, u_{k-1}) = \begin{cases} b_1^* x^{(l+1)} (\mathbf{e}_1' B_1 \dots B_{k-l} \mathbf{e}_1), & \text{if } \bigvee_{i=1}^{k-1} |u_i| < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Here $B_r = \chi(u_r, \dots, u_{r+l-1})$. Let χ^* be the matrix function χ (see (2.2)) with all constants ϕ_i and b_{ij} , $i = 1, \dots, p$, $j = 1, \dots, l$ replaced by their absolute values. Let $B_M^* = \chi^*(M, \dots, M)$. If K is the compact set in $\mathbb{E}_1$ given by $\{x : |x| \geq b\}$ then

$$f_k^{-1}(K) \cap (\mathbb{E}_1 \times [-M, M]^{k-1}) \subset \{x : |x| > (b / (b_1^* \mathbf{e}_1' (B_M^*)^{k-1} \mathbf{e}_1))^{1/(l+1)}\} \times [-M, M]^{k-1}$$

which is compact in $\mathbb{E}_1 \times [-\infty, \infty]^{k-1}$. It follows that $f_k^{-1}(K)$ restricted to $\mathbb{E}_1 \times [-M, M]^{k-1}$ is compact for any compact subset K of $\mathbb{E}_1$, and since f_k is continuous on the support of $\tilde{I}^{k,M}$ a.s., we have by Corollary 1.2 of [19] and (4.3) that

$$(\tilde{I}_n^{l,M} \circ f_l^{-1}, \dots, \tilde{I}_n^{m,M} \circ f_m^{-1}) \Rightarrow (\tilde{I}^{l,M} \circ f_l^{-1}, \dots, \tilde{I}^{m,M} \circ f_m^{-1})$$

in $M_p^{m-l+1}(\mathbb{E}_1)$. Since the point processes $I^{(k)} = \tilde{I}^k \circ f_k^{-1}$ are well defined Poisson processes, we have as $M \rightarrow \infty$

$$(\tilde{I}^{l,M} \circ f_l^{-1}, \dots, \tilde{I}^{m,M} \circ f_m^{-1}) \rightarrow (I^{(l)}, \dots, I^{(m)})$$

pointwise in the vague metric a.s. Now $I_n^{(k)} = \tilde{I}_n^k \circ f_k^{-1}$, so by Theorem 4.2 in [1], the conclusion of the Proposition will follow if we show that for each k and any $\eta > 0$,

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P[\rho(\tilde{I}_n^{k,M} \circ f_k^{-1}, \tilde{I}_n^k \circ f_k^{-1}) > \eta] = 0.$$

By the form of the metric ρ , it suffices to show for any $h \in C_K^+(\mathbb{E}_1)$ that

$$(4.4) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P[|\tilde{I}_n^{k,M} \circ f_k^{-1}(h) - \tilde{I}_n^k \circ f_k^{-1}(h)| > \eta] = 0.$$

For some $\delta > 0$, the support of h is contained in the set $G_\delta = \{x : |x| > \delta\}$, so the above probability is bounded by $P[\tilde{I}_n^k(G_{\delta'} \times K_M^c) \geq 1]$, where $\delta' = (\delta/b_1^* \mathbf{e}_1'(B^*)_M^{k-1} \mathbf{e}_1)^{1/(l+1)}$ and K_M^c is the complement of $K_M := [-M, M]^{k-1}$. Using (4.2), this probability converges as $n \rightarrow \infty$ to

$$P[\tilde{I}^k(G_{\delta'} \times K_M^c) \geq 1]$$

which goes to 0 as $M \rightarrow \infty$. □

Proposition 4.3. *On the space $M_p(\mathbb{E}_{m-l+1})$ with vague metric ρ_{m-l+1} ,*

$$\rho_{m-l+1}\left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)})}, \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}\right) \xrightarrow{P} 0.$$

Proof. Because of the definition of ρ_{m-l+1} (cf. Proposition 3.17 in [20]), it suffices to show that for all bounded rectangles in $\mathbb{E}_{m-l+1}$,

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)})}(B) - \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}(B) \xrightarrow{P} 0.$$

To check this, we follow the same lines of reasoning as in [5], Proposition 2.1. Let

$$B = (b_l, c_l] \times \dots \times (b_m, c_m]$$

be a bounded rectangle in $\mathbb{E}_{m-l+1} = [-\infty, \infty]^{m-l+1} \setminus \{\mathbf{0}\}$. Then either B is bounded away from each of the coordinate axes or intersects exactly one in an interval. (See page 181 in [5]). If B is bounded away from each of the coordinate axes, then

$$E\left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)})}(B)\right) = nP(a_n^{-(l+1)}(Y_1^{(l)}, \dots, Y_1^{(m)}) \in B) \rightarrow 0$$

by Lemma 3.2, and of course also

$$E\left(\sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}(B)\right) = 0.$$

In the other case, suppose $0 \in (b_i, c_i]$, $i \neq i'$, and $0 \notin (b_{i'}, c_{i'}]$. We can write:

$$(4.5) \quad \sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)})}(B) \leq \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(i')}}((b_{i'}, c_{i'}]) = \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}(B).$$

But, also by Lemma 3.2:

$$\begin{aligned} E\left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)})}(B)\right) &= nP(a_n^{-(l+1)}(Y_1^{(l)}, \dots, Y_1^{(m)}) \in B) \\ &= nP(a_n^{-(l+1)}(0, \dots, 0, Y_1^{(i')}, \dots, 0) \in B) + o(1) \end{aligned}$$

$$= E\left(\sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}(B)\right) + o(1).$$

So in both cases:

$$E\left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} (Y_t^{(l)}, \dots, Y_t^{(m)})}(B) - \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(k)} \mathbf{e}_k}(B)\right) \rightarrow 0,$$

which because of (4.5) implies the convergence in the proposition. $\square$

Theorem 4.1. Suppose $\{X_t\}$ is the bilinear process given by (2.1) with $b_1^* := \prod_{j=1}^l b_{1j} \neq 0$ and the parameters satisfying (2.8) or (2.9). Assume the marginal distribution F of the iid noise $\{Z_t\}$ satisfies (2.5), (2.6) and (2.7). Let a_n be given by (4.1). Let $\sum_{i=1}^\infty \epsilon_{j_s}$ be PRM(μ) with $\mu(dx) = \alpha(rx^{-\alpha-1}1_{[x>0]} + (1-r)(-x)^{-\alpha-1}1_{[x<0]})dx$ and $\{U_{s,k}, s \geq 1, k \geq 1\}$ are i.i.d. with distribution F . Define $B_{s,r} = \chi(U_{s,r}, \dots, U_{s,r+l-1})$ and

$$W_{s,k} = \begin{cases} \mathbf{e}'_1 B_{s,1} \cdots B_{s,k-l} \mathbf{e}_1, & \text{if } k > l \\ 1, & \text{if } k = l \\ 0, & \text{if } k < l. \end{cases}$$

Then

(i) In $M_p(\mathbb{E}_1)$:

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} X_t} \Rightarrow \sum_{k=l}^\infty \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* W_{s,k}}.$$

(ii) In $M_p(\mathbb{E}_{h+1})$,

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} (X_t, X_{t-1}, \dots, X_{t-h})} \Rightarrow \sum_{k=l}^\infty \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* (W_{s,k}, W_{s,k-1}, \dots, W_{s,k-h})}.$$

Proof. (i) Propositions 4.2 and 4.3 imply that

$$(4.6) \quad \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} (Y_t^{(l)}, \dots, Y_t^{(m)})} \Rightarrow \sum_{k=1}^m \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* W_{s,k} \mathbf{e}_k}$$

on $M_p(\mathbb{E}_{m-l+1})$. Now the map

$$(y_l, \dots, y_m) \mapsto \sum_{k=l}^m y_k$$

induces a continuous map from $M_p(\mathbb{E}_m) \mapsto M_p(\mathbb{E}_1)$ and so by the continuous mapping theorem applied to the above convergence, we get

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} \sum_{k=l}^m Y_t^{(k)}} \Rightarrow \sum_{k=l}^m \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* W_{s,k}}$$

in $M_p(\mathbb{E}_1)$. As $m \rightarrow \infty$,

$$\sum_{k=l}^m \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* W_{s,k}} \rightarrow \sum_{k=l}^\infty \sum_{s=1}^\infty \epsilon_{j_s^{(l+1)} b_1^* W_{s,k}}$$

pointwise in the vague metric and so by Theorem 4.2 in [1], it suffices to show that for any $\eta > 0$ and $f \in C_K^+(\mathbb{E}_1)$,

$$(4.7) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left[\left|\sum_{t=1}^n f(a_n^{-(l+1)}) \sum_{k=l}^m Y_t^{(k)} - \sum_{t=1}^n f(a_n^{-(l+1)}) X_t\right| > \eta\right] = 0.$$

To prove (4.7), we write

$$P[a_n^{-(l+1)} \bigvee_{t=1}^n \left|\sum_{k=l}^m Y_t^{(k)} - X_t\right| > \eta] \leq P[a_n^{-(l+1)} \bigvee_{t=1}^n (|\sum_{k=0}^{l-1} X_t^{(k)} + \sum_{k=l}^m R_t^{(k)}| + |\sum_{k=m+1}^{\infty} X_t^{(k)}| > \eta)].$$

So

$$\begin{aligned} P[a_n^{-(l+1)} \bigvee_{t=1}^n \left|\sum_{k=l}^m Y_t^{(k)} - X_t\right| > \eta] &\leq nP[a_n^{-(l+1)} |\sum_{k=0}^{l-1} X_t^{(k)} + \sum_{k=l}^m R_t^{(k)}| > \eta/2] \\ &\quad + nP[a_n^{-(l+1)} |\sum_{k=m+1}^{\infty} X_t^{(k)}| > \eta/2] \\ &:= I + II. \end{aligned}$$

First consider I: The same argument as in Corollary 3.4 works here: note that $X_t^{(k)}, k = 0, \dots, l-1$ and $R_t^{(k)}, k = l, \dots, m$ all are finite sums of products of Z_i 's, with highest power of an individual Z_i occurring in any term $\leq l$. This follows from the fact that all terms containing Z_{t-k}^{l+1} are collected in $Y_t^{(k)}$ as seen in the discussion following (3.3). Therefore, as in Corollary 3.4, it is easy to check that

$$nP[a_n^{-(l+1)} (\sum_{k=0}^{l-1} |X_t^{(k)}| + \sum_{k=l}^m |R_t^{(k)}|) > \eta/2] \rightarrow 0$$

as $n \rightarrow \infty$.

To handle II, we can write, by Corollary 3.5:

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} nP[a_n^{-(l+1)} |\sum_{k=m+1}^{\infty} X_t^{(k)}| > \eta/2] &= \lim_{m \rightarrow \infty} (\eta/2)^{-\frac{\alpha}{l+1}} \sum_{k=m+1}^{\infty} E(|e'_1 \prod_{i=1}^{k-l} B(t-i) e_1 b_1^*|)^{\frac{\alpha}{l+1}} \\ &= 0. \end{aligned}$$

The rest of the proof of (4.7) is now identical to the argument given for (2.11) in [5], with this last result substituting for Lemma 2.3 of [5].

(ii) Using a slight modification to the arguments given in Propositions 4.2 and 4.3, we get:

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} Y_t^{(l)}} \Rightarrow \sum_{s=1}^{\infty} \epsilon_{j_s^{(l+1)} b_1^* W_{s,l}}$$

and

$$\sum_{t=1}^n \epsilon_{b_n^{-(l+1)} (Y_t^{(k)}, Y_{t-1}^{(k-1)})} \Rightarrow \sum_{s=1}^{\infty} \epsilon_{j_s^{(l+1)} b_1^* (W_{s,k}, W_{s,k-1})},$$

$k = l+1, \dots, m$, jointly. Similarly, we can show:

$$\rho_{m-l+1} \left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)} (Y_t^{(l)}, \dots, Y_t^{(m)}, Y_{t-1}^{(l)}, \dots, Y_{t-1}^{(m-1)})}, \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)} (Y_t^{(k)} e_k + Y_{t-1}^{(k-1)} e_{m+k-1})} \right) \xrightarrow{P} 0.$$

Combining these results, we obtain the point process convergence result

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(l)}, \dots, Y_t^{(m)}, Y_{t-1}^{(l)}, \dots, Y_{t-1}^{(m-1)})} \Rightarrow \sum_{k=l}^m \sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(Y_t^{(k)} \mathbf{e}_k + Y_{t-1}^{(k-1)} \mathbf{e}_{m+k-1})}$$

in $M_p(\mathbb{E}_{2(m-l)+1})$, where the $\mathbf{e}_i$ are the unit basis elements in $\mathbb{R}^{2(m-l)+1}$. Then, using the continuous mapping of $M_p(\mathbb{E}_{2(m-l)+1}) \mapsto M_p(\mathbb{E}_2)$ induced by the function

$$(x_l, \dots, x_m, u_l, \dots, u_{m-1}) \mapsto \left(\sum_{k=l}^m x_k, \sum_{k=l}^{m-1} u_k \right)$$

we obtain

$$\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(\sum_{k=l}^m Y_t^{(k)}, \sum_{k=l}^{m-1} Y_{t-1}^{(k)})} \Rightarrow \sum_{k=l}^m \sum_{s=1}^{\infty} \epsilon_{j_s^{(l+1)} b_1^*(W_{s,k}, W_{s,k-1})}.$$

The rest of the proof of (ii) is analogous to that given for (i). $\square$

5. SAMPLE ACF

By applying continuous functionals to the basic convergence result of Theorem 4.1, the limiting behavior of a number of statistics can be easily derived. Most interestingly, we can describe the limiting behavior of the vector of heavy tailed sample correlations $\{\hat{\rho}_H(l), l = 1, \dots, h\}$ for integers $h = 1, 2, \dots$. Recall $\hat{\rho}_H(l)$ was defined to be

$$\hat{\rho}_H(l) = \frac{\sum_{t=1}^{n-l} X_t X_{t+l}}{\sum_{t=1}^n X_t^2}.$$

In [5, 6, 7] it was shown that for a heavy tailed MA(∞) process, the sample ACF is a consistent estimate of a model ACF defined in terms of the coefficients of the linear filter. That is, the sample ACF converges to a constant. In contrast, we find for the bilinear process, that sample correlations converge in distribution to non-degenerate limit random variables depending on the lag.

Theorem 5.1. *Suppose $\{X_t\}$ is the bilinear process (2.1), with $b_1^* := \prod_{j=1}^l b_{1j} \neq 0$ and the parameters satisfying (2.8) or (2.9). Assume the marginal distribution F of the i.i.d. noise $\{Z_t\}$ satisfies (2.5), (2.6) and (2.7) with $0 < \alpha < 2(l+1)$. Then we have for any $h = 1, 2, \dots$ that*

$$(\hat{\rho}_H(r), r = 1, \dots, h) \Rightarrow (L_i, i = 1, \dots, h)$$

in $\mathbb{R}^h$, where in the notation of Theorem 4.1

$$L_i = \frac{\sum_{s=1}^{\infty} \sum_{k=1}^{\infty} j_s^{2(l+1)} W_{s,k} W_{s,k-i}}{\sum_{s=1}^{\infty} \sum_{k=1}^{\infty} j_s^{2(l+1)} W_{s,k}^2}$$

$i = 1, \dots, h$.

Proof. Theorem 4.1 (ii) implies:

$$(5.1) \quad \left(\sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(X_t, X_{t-r})}, r = 1, \dots, h \right) \Rightarrow \left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \epsilon_{j_s^{(l+1)} b_1^*(W_{s,k}, W_{s,k-r})}, r = 1, \dots, h \right)$$

in $M_p^h(\mathbb{E}_2)$. For simplicity, we consider convergence of a single component in the above equation, but in the end it should be obvious that joint convergence occurs.

For convenience we focus on the first component convergence in (5.1):

$$(5.2) \quad \sum_{t=1}^n \epsilon_{a_n^{-(l+1)}(X_t, X_{t-1})} \Rightarrow \sum_{s=1}^{\infty} \sum_{k=l}^{\infty} \epsilon_{j_s^{(l+1)} b_1^*(W_{s,k}, W_{s,k-1})}.$$

Pick $\delta > 0$ and restrict the state space to

$$\mathbb{E}_\delta = \{(x_1, x_2) \in \mathbb{E}_2 : |x_1| \vee |x_2| > \delta\}$$

to obtain

$$\sum_{t=1}^n \epsilon_{a_n^{(l+1)}(X_t, X_{t-1})}(\cdot \cap \mathbb{E}_\delta) \Rightarrow \sum_{s=1}^{\infty} \sum_{k=l}^{\infty} \epsilon_{j_s^{(l+1)} b_1^*(W_{s,k}, W_{s,k-1})}(\cdot \cap \mathbb{E}_\delta).$$

Let $W_{s,k}^* = b_1^* W_{s,k}$. Because the state space has been compactified by restriction, we may apply the functional which multiplies components to obtain (see e.g. [20], Proposition 3.18)

$$\sum_{t=2}^n 1_{[|X_t| \vee |X_{t-1}| > a_n^{(l+1)} \delta]} \epsilon_{a_n^{-2(l+1)}(X_t, X_{t-1})} \Rightarrow \sum_{s=1}^{\infty} \sum_{k=l}^{\infty} 1_{[|j_s^{(l+1)} W_{s,k}^*| \vee |j_s^{(l+1)} W_{s,k-1}^*| > \delta]} \epsilon_{j_s^{2(l+1)} W_{s,k}^* W_{s,k-1}^*}.$$

Each point process on the previous line has only a finite number of points, so applying the summation functional gives

$$\begin{aligned} \gamma_{n,\delta}(1) &:= \sum_{t=2}^n 1_{[|X_t| \vee |X_{t-1}| > a_n^{(l+1)} \delta]} a_n^{-2(l+1)}(X_t, X_{t-1}) \\ &\Rightarrow \gamma_{\infty,\delta}(1) := \sum_{s=1}^{\infty} \sum_{k=l}^{\infty} 1_{[|j_s^{(l+1)} W_{s,k}^*| \vee |j_s^{(l+1)} W_{s,k-1}^*| > \delta]} j_s^{2(l+1)} W_{s,k}^* W_{s,k-1}^*. \end{aligned}$$

We claim

$$\gamma_{n,0}(1) \Rightarrow \gamma_{\infty,0}(1)$$

in $\mathbb{R}$. To prove this, we have to check (by [1], Theorem 4.2):

$$(5.3) \quad \gamma_{\infty,\delta}(1) \Rightarrow \gamma_{\infty,0}(1) \quad \text{as } \delta \downarrow 0,$$

and

$$(5.4) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P[|\gamma_{n,\delta}(1) - \gamma_{n,0}(1)| > \eta] = 0.$$

To verify (5.3), it is sufficient to check that the series

$$(5.5) \quad \sum_{s=1}^{\infty} \sum_{k=l}^{\infty} j_s^{2(l+1)} W_{s,k}^* W_{s,k-1}^* = \sum_{s=1}^{\infty} j_s^{2(l+1)} C_s$$

is absolutely convergent, where $C_s = \sum_{k=l}^{\infty} W_{s,k}^* W_{s,k-1}^*$. Since $\alpha/(2(l+1)) < 1$, we have by the triangle inequality

$$(5.6) \quad E|C_s|^{\frac{\alpha}{2(l+1)}} \leq (|b_1^*|)^{\frac{\alpha}{l+1}} \sum_{k=l}^{\infty} E(|W_{s,k}| |W_{s,k-1}|)^{\frac{\alpha}{2(l+1)}}$$

Now

$$\begin{aligned} |W_{s,k}| &\leq \|B_{s,1} \cdots B_{s,k-l}\| \\ &\leq \|B_{s,1} \cdots B_{s,k-l-1}\| \|B_{s,k-l}\|. \end{aligned}$$

By Cauchy-Schwarz, we have

$$E|W_{s,k} W_{s,k-1}|^{\frac{\alpha}{2(l+1)}} \leq (E|W_{s,k}|^{\frac{\alpha}{l+1}})^{\frac{1}{2}} (E|W_{s,k-1}|^{\frac{\alpha}{l+1}})^{\frac{1}{2}}$$

$$\leq (p\|\tilde{B}^{k-l-1}\|)^{2\frac{1}{2}(\frac{\alpha}{l+1}\vee 1)} \cdot (p\|\tilde{B}\|)^{\frac{1}{2}(\frac{\alpha}{l+1}\vee 1)} = M_1\|\tilde{B}^{k-l-1}\|^{(\frac{\alpha}{l+1}\vee 1)}, \quad k > l.$$

Now, by (2.13)–(2.17), there are constants $M_2 > 0$ and $0 < a < 1$ such that

$$E|C_s|^{\frac{\alpha}{2(l+1)}} \leq M_2 \sum_{k'=0}^{\infty} a^{k'} < \infty.$$

The independence of the C_s over s together with the above inequalities imply that $\sum_{s=1}^{\infty} \epsilon_{j_s^{2(l+1)}|C_s|}$ is PRM(μ), with intensity measure $\mu[x, \infty) = E(|C_1|^{\alpha/(2(l+1))})x^{-\alpha/(2(l+1))}$ and hence has absolutely summable points a.s. (see [19] and [5], p.192).

It remains to check (5.4). We mimic the argument given in [5], p.193. The probability in (5.4) is bounded by

$$P[a_n^{-2(l+1)} \sum_{t=1}^n |X_t X_{t-1}| 1_{[|X_t| \vee |X_{t-1}| \leq a_n^{(l+1)} \delta]} > \eta] \leq \frac{n}{a_n^{2(l+1)}} E|X_2 X_1| 1_{[|X_2| \vee |X_1| \leq b_n^{(l+1)} \delta]} / \eta$$

which by Cauchy's inequality is dominated by

$$\frac{n}{a_n^{2(l+1)}} E(|X_1|^2 1_{[|X_1| \leq a_n^{(l+1)} \delta]}) / \eta$$

Since $P[|X_1| > x]$ is regularly varying with index $-\alpha/(l+1) \in (-2, 0)$ (see Corrolary 3.5), we get by Karamata's theorem that

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \frac{n}{a_n^{2(l+1)}} E(|X_1|^2 1_{[|X_1| \leq a_n^{(l+1)} \delta]}) / \eta = 0,$$

proving (5.4). We have now shown that $\gamma_{n,0}(1) \Rightarrow \gamma_{\infty,0}(1)$ and in fact, examining the above proof shows that

$$(5.7) \quad (\gamma_{n,0}(0), \gamma_{n,0}(1)) \Rightarrow (\gamma_{\infty,0}(0), \gamma_{\infty,0}(1)),$$

where $\gamma_{n,0}(0) = \sum_{t=1}^n X_t^2 / a_n^{2(l+1)}$. Dividing the second component by the first in the above equation yields the first component convergence given in the statement of the Theorem for $h = 1$. The case $h > 1$ follows analogously. $\square$

6. HILL ESTIMATOR

Given a time series $\{X_t, -\infty < t < \infty\}$ with regularly varying tail probabilities, a key question is how to estimate the tail index. A popular estimator often used for this purpose is Hill's estimator ([13]), which is defined for positive time series as follows: For $1 \leq i \leq n$, write $X_{(i)}$ for the i -th largest value of $X_1, X_2, \dots, X_n$. Hill's estimator based on the observations $X_1, \dots, X_n$ is

$$H_{k,n}^X = \frac{1}{k} \sum_{i=1}^k \log \frac{X_{(i)}}{X_{(k+1)}},$$

where $k < n$ is the number of order statistics used in the estimation. We prove next, that for the general bilinear model, the Hill estimator gives a consistent estimate of $(l+1)/\alpha$.

Theorem 6.1. *Suppose $\{X_t\}$ is the bilinear process (2.1), where the marginal distribution F of the i.i.d. noise $\{Z_t\}$ satisfies (2.5), (2.6) and (2.7) and the parameters b_{ij}, ϕ_i and θ_j satisfy (2.8) or (2.9). Continue to assume $b_1^* := \prod_{j=1}^l b_{1j} \neq 0$. Let $k(n) \rightarrow \infty, n/k \rightarrow \infty$. Then*

$$H_{k,n}^{|X|} := \frac{1}{k} \sum_{i=1}^k \log \frac{|X|_{(i)}}{|X|_{(k+1)}} \xrightarrow{P} \frac{l+1}{\alpha},$$

where $\{|X|_{(i)}, i = 1, \dots, n\}$ are the order statistics of $\{|X_t|, t = 0, \dots, n\}$.

Proof. We apply Proposition 2.2 of [22]. Define $\tilde{a}_n$ such that

$$\frac{n}{k}P(X_1 > \tilde{a}_n) \rightarrow 1$$

as $n \rightarrow \infty$. For $m \geq 1$, define $X_{n,i}^{(m)} := |\sum_{j=0}^m X_i^{(j)} / \tilde{a}_n|$. Let $X_{n,i} := |\sum_{j=0}^\infty X_i^{(j)} / \tilde{a}_n|$. By Corollary 3.4 and Corollary 3.5, we have as $x \rightarrow \infty$

$$\frac{n}{k}P(X_{n,i}^{(m)} > x) \rightarrow \frac{\sum_{j=l}^m E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}}{\sum_{j=l}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}} x^{-\frac{\alpha}{l+1}}.$$

We can define the measures $\nu^{(m)}$ of Proposition 2.2 in [22] by

$$\nu^{(m)}((x, \infty]) := \frac{\sum_{j=l}^m E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}}{\sum_{j=l}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}} x^{-\frac{\alpha}{l+1}},$$

so that as $m \rightarrow \infty$, $\nu^{(m)} \rightarrow_\nu \nu$, where $\nu((x, \infty]) = x^{-\alpha/(l+1)}$. By similar reasoning as in Corollary 3.5, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{n}{k} P(|X_{n,1}^{(m)} - X_{n,1}| > \epsilon) &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{n}{k} P(|\sum_{j=m+1}^\infty X_1^{(j)} / \tilde{a}_n| > \epsilon) \\ &= \lim_{m \rightarrow \infty} \frac{\sum_{j=m+1}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}}{\sum_{j=l}^\infty E(|\mathbf{e}'_1 \prod_{i=1}^{j-l} B(t-i) \mathbf{e}_1 b_1^*|)^{\frac{\alpha}{l+1}}} \epsilon^{-\frac{\alpha}{l+1}} = 0. \end{aligned}$$

The conditions of Proposition 2.2 of [22] are verified, so consistency of the Hill estimator is proved. $\square$

The Hill plot is the plot of

$$(k, \frac{1}{H_{k,n}^X}), 1 \leq k < n).$$

In theory, for the bilinear process the Hill plot should have a stable regime at height roughly $\alpha/(l+1)$. However, in our samples called $\text{test}_i, i = 1, 2, 3$, discussed in the introduction, this region is hard to find, as can be seen in Figure 2. (Note that $\alpha/(l+1) = 1/3$).

To counteract the high volatility of the Hill plot, Resnick and Starica ([23]) developed a smoothing technique yielding the *smooHill* plot: Pick an integer u (usually 2 or 3) and define

$$\text{smooH}_{k,n} = \frac{1}{(u-1)k} \sum_{j=k+1}^{uk} H_{j,n}.$$

In the i.i.d. case when a second order regular variation condition holds, the asymptotic variance of $\text{smooH}_{k,n}$ is less than of the Hill estimator.

As an alternative to the Hill plot, it is sometimes useful to display the Hill estimation values as:

$$(\theta, H_{\lceil n^\theta \rceil, n}), 0 \leq \theta \leq 1,$$

where we write $\lceil y \rceil$ for the smallest integer greater or equal to $y \geq 0$. This alternative display (*AltHill*) is sometimes revealing since the initial order statistics get shown more clearly and cover a bigger portion of the displayed space. See [21]. Figure 3 shows the combined results for sample test_1 , where the smoothing parameter u is 2. A combination of the *AltHill* and *SmooHill* plots (*AltSmooHill*) seems to assist most in identifying the correct tail index.

FIGURE 2. Hill plot for three bilinear samples

REFERENCES

- [1] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968.
- [2] L. Breiman. On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.*, 10:323–331, 1965.
- [3] D. Cline. *Estimation and linear prediction for regression, autoregression and ARMA with infinite variance data*. PhD thesis, Department of Statistics, Colorado State University, Ft. Collins, CO 80521 USA, 1983.
- [4] J. Cohen, S. Resnick, and G. Samorodnitsky. Sample correlations of infinite variance time series models: an empirical and theoretical study. Available at <http://www.orie.cornell.edu/trlist/trlist.html> as TR1202.ps.Z, 1997.
- [5] R.A. Davis and S. Resnick. Limit theory for moving averages of random variables with regularly varying tail probabilities. *Annals of Probability*, 13:179–195, 1985.
- [6] R.A. Davis and S. Resnick. More limit theory for the sample correlation function of moving averages. *Stochastic Processes and their Applications*, 20:257–279, 1985.
- [7] R.A. Davis and S. Resnick. Limit theory for the sample covariance and correlation functions of moving averages. *Annals of Statistics*, 14:533–558, 1986.
- [8] R.A. Davis and S. Resnick. Limit theory for bilinear processes with heavy tailed noise. *Annals of Applied Probability*, 6(4):1191–1210, 1996.
- [9] D. Duffy, A. McIntosh, M. Rosenstein, and W. Willinger. Analyzing telecommunications data from working common channel signaling subnetworks. In *Proceedings of the 25th Interface*, San Diego, CA, 1993.
- [10] D. Duffy, A. McIntosh, M. Rosenstein, and W. Willinger. Statistical analysis of ccsn/ss7 traffic data from working ccs subnetworks. *IEEE Journal on Selected Areas in Communications*, 12:544–551, 1994.
- [11] P. Feigin and S. Resnick. Pitfalls of fitting autoregressive models with emphasis on heavy-tailed time series. Unpublished manuscript. Available at <http://www.orie.cornell.edu/trlist/trlist.html> as TR1163.ps.Z, 1996.
- [12] C.W.J. Granger and A. Anderson. *An Introduction to Bilinear Time Series Models*. Vandenhoeck and Ruprecht, Gottingen, 1978.
- [13] B. Hill. A simple approach to inference about the tail of a distribution. *Annals of Statistics*, 3:1163–1174, 1975.

FIGURE 3. Combined plot for sample test1

- [14] O. Kallenberg. *Random Measures*. Akademie-Verlag, Berlin, third edition, 1983.
- [15] J. Liu. On the existence of a general multiple bilinear time series. *Journal of Time Series Analysis*, 10:341–355, 1989.
- [16] J. Liu. A note on causality and invertibility of a general bilinear time series model. *Advances in Applied Probability*, 22:247–250, 1990.
- [17] J. Pratt. On interchanging limits and integrals. *Annals of Mathematical Statistics*, 31:74–77, 1960.
- [18] S. Resnick and F. Xue. How misleading can sample acf's of stable moving averages be? (very!), 1998. Technical Report 1209, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY 14853.
- [19] Sidney Resnick. Point processes, regular variation and weak convergence. *Advances in Applied Probability*, 18:66–138, 1986.
- [20] Sidney Resnick. *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York, 1987.
- [21] Sidney Resnick. Heavy tail modeling and teletraffic data. *Annals of Statistics*, 25:1805–1869, 1997.
- [22] Sidney Resnick and Catalin Starica. Tail index estimation for dependent data, 1996. Available at <http://www.orie.cornell.edu/trlist/trlist.html> as TR1174.ps.Z. To appear: *Ann. Appl. Probability*.
- [23] Sidney Resnick and Catalin Starica. Smoothing the hill estimator. *Journal of Applied Probability*, 29:271–293, 1997.
- [24] T. Subba Rao and M. M. Gabr. *An Introduction to Bispectral Analysis and Bilinear Time Series Models*, volume 24 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1984.
- [25] W. Willinger, M. Taqqu, R. Sherman, and D. Wilson. Self-similarity through high-variability: Statistical analysis of ethernet lan traffic at the source level. *IEEE/ACM Transactions on Networking*, 5(1):71–96, 1997.

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