SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-3801

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Levels in a Capacitated Multi-Echelon
Production-Distribution System

by
James A. Rappold and John A. Muckstadt

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A Computationally Efficient Approach for Determining Inventory Levels in a Capacitated Multi-Echelon Production-Distribution System

James A. Rappold

School of Business

University of Wisconsin, Madison, WI 53706

John A. Muckstadt

School of Operations Research and Industrial Engineering

Cornell University, Ithaca, NY 14853

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Abstract

Planning operations in lean capacitated manufacturing and distribution systems requires careful coordination of information and materials among all activities in the supply chain. These environments are characterized by short production and transportation lead times and by capacitated production facilities in which many products are produced or assembled and distributed. In this paper, we develop a model and a computationally efficient algorithm for setting stocking levels in one type of lean multi-echelon production and distribution system with limited production capacity at the highest echelon.

The system under study is a single product, two-echelon production-inventory system consisting of a production facility, a central warehouse, and M regional distribution centers. There are no fixed costs in the system which affect periodic production and distribution decisions.

Our objective is to determine a system order-up-to level which minimizes the long run average system cost per period. Central to the approach are (1) an inventory allocation model and associated convex cost function designed to allocate a given amount of system inventory across locations, and (2) a characterization of the amount of available system inventory using the shortfall random variable. An exact model must consider the possibility that inventories may be mis-allocated in a given period. By assuming inventory imbalances cannot occur, we develop an approximation model from which we obtain a lower bound on the optimal system order-up-to level. Through an extensive simulation study, we present the quality of our approximation, which on average performed within 0.09% of optimal.

1 Introduction

Planning production in large-scale, lean, capacitated manufacturing and distribution systems requires careful coordination of information and materials among all activities in a supply chain. As companies adopt lean manufacturing philosophies, lead times between successive operations in the supply chain continue to shrink. This compression of time greatly magnifies the relative variation of demand processes and requires careful consideration of finite production capacity when planning production.

Our ultimate goal is to develop a set of unified, computationally efficient models to assist in making production and inventory decisions in large-scale capacity-limited, lean manufacturing/assembly and distribution environments which manage tens of thousands of products across hundreds of stocking locations.

In this paper, we lay the groundwork by determining inventory stocking levels for a single make-to-stock product in a multi-echelon distribution system consisting of a capacitated production facility at the highest echelon, a co-located central warehouse, and M regional distribution centers, which may experience highly variable customer demand. This system is depicted in Figure 1. Based on the ideas developed here, we extend the results to the multiple product environment in Rappold and Muckstadt (1998).

Figure 1: The System Under Study

We assume the system operates in the following manner. At the beginning of a period, each distribution center receives a shipment of inventory sent from the central warehouse at the end of the previous period, and customer demand occurs and is satisfied or backordered at each distribution center. Next, the period's production requirements are established, and production occurs and is available for allocation prior to the end of the current period. Finally, at the period's end, available inventory is allocated and then shipped to the regional distribution centers. These shipments are received at the the distribution centers at the beginning of the next period, and this process repeats indefinitely. We assume there are no fixed costs in this system which affect periodic production and distribution decisions. We also assume that raw materials and other auxiliary resources required for production are readily available.

In this paper, we answer the following three questions:

- How much of the single product should be produced in each period? (Production Control Policy)
- How should inventory be allocated among the central warehouse and the M distribution centers? (Inventory Allocation Policy)
- What should the order-up-to levels be at the central warehouse and at each distribution center? (Safety Stock Policy)

To answer the first question, we use the structural results shown in Federgruen and Zipkin (1986a) and in Aviv and Federgruen (1997) to control production according to a modified base stock policy. Remember that immediately after customer demand occurs at each distribution center, we assume that this demand information is available at the production facility and that the amount produced will either raise the system inventory to a predetermined system order-up-to level or will equal the production capacity. The answers to the remaining questions may be obtained by solving a dynamic programming recursion. Unfortunately, due to the size of the state space, this approach is computationally intractable for all but the most trivial situations. Consequently, we have chosen a different approach which is exact for a single distribution center system ($M = 1$) and which is a close approximation for $M > 1$. The approximation is based on the assumption that to operate optimally, at the end of any period, inventories will not need to be re-allocated among the $M > 1$ distribution centers. Our proposed solution approach works as follows.

First, an inventory allocation model and associated convex cost function is constructed to optimally allocate a given amount of system inventory across all locations. To drastically simplify computational complexity, we ignore the inventory imbalance constraints, then formulate and solve a relaxed allocation problem. Because the imbalance constraints are ignored, this allocation model is an approximation. However, we show subsequently that this relaxation has little impact on overall model performance.

Second, we build on the ideas presented in Tayur (1992) and Roundy and Muckstadt (1997) on the characterization of the state of the system through the inventory shortfall random variable. Specifically, a Markov chain is constructed to model the inventory shortfall of the system and is used to determine a close-to-optimal system order-up-to-level.

Inventory imbalances will occur and could affect production, inventory, and allocation decisions in many environments. However, for the system we examine with short transportation lead times, the impact of imbalances, while they could affect stocking decisions, these imbalances will not significantly affect system costs because of the relative insensitivity of the newsvendor-type cost functions near their optimal values. Furthermore, imbalances will not significantly affect costs since a substantial fraction of system inventory is produced each period and is therefore available for allocation each period. Specifically, we conjecture that for environments in which manufacturing flow times or assembly times and transportation times are short, when mis-distribution occurs, it does not adversely impact overall system cost and performance. We believe that in these environments, the inventory required to protect the system against the risk of insufficient capacity dominates the inventory required to protect the system against the risk and effects of mis-distribution. Through an extensive simulation study, we evaluate the quality of our approximation and the impact of imbalances, and show that on average it performed within 0.09% of optimal.

This paper is organized as follows. In section 2 we review the relevant literature. In section 3 we define notation and specify the model assumptions. In section 4 an inventory allocation model

and an associated cost function are defined. For a single distribution center system ($M = 1$), the model and cost function are exact. For a multiple distribution center system, the cost function is a lower bound on the true system cost. In section 5 we develop an optimization model and an algorithm for determining the system order-up-to level and the individual inventory order-up-to levels for the central warehouse and distribution centers such that the average total system cost per period is minimized over an infinite horizon. In section 6 we present the results of a simulation study designed to assess the impact of inventory imbalance on model performance and to test the overall model effectiveness. In section 7 we summarize the key aspects of the approach and suggest future directions for research. Throughout this research, we use the terms *target inventory level*, *base-stock level*, and *order-up-to level* interchangeably.

2 Literature Review

Uncapacitated multi-echelon inventory systems have been studied extensively for almost forty years. The first work done by Clark and Scarf (1960) considered a system of N locations in series with a fixed ordering cost at the highest echelon. Their work characterized optimal policies and established fundamental ideas such as echelon inventory position and echelon holding costs for decision making. Bessler and Veinott (1966) generalized this work to include general arborescent structures. Research in finite production capacity inventory models began with a fixed production rate, deterministic demand model which was presented in Taft (1918). This was extended to consider finite production capacity in a multiple product system with stochastic demands by Evans (1967), and was modeled as a stochastic dynamic program.

More recently, attention has again been drawn to stochastic inventory models which no longer assume an infinite capacity of supply. Federgruen and Zipkin (1986b,a) examine a discrete time single item, single location inventory problem with no fixed costs and finite production capacity. They establish the optimality of the modified base-stock policy based on the average cost and discounted cost criteria, specified through a single order-up-to level, or critical number. Tayur (1992) shows how to compute this critical number using ideas based on stochastic dam models developed by Prabhu (1980).

Other models have been developed to capture capacity constraints based on queuing models (see Sox et al. (1997)). Güllü and Jackson (1993) formulate this problem in continuous time as a Brownian control problem, which implicitly uses an exponential distribution to model inventory shortfall. Glasserman (1996) considers a multi-product continuous time model in which capacity allocations are made. This work examines heavy traffic situations and the asymptotic behavior of the system when the demand distribution has an exponential tail. These continuous time models, when applied to a multi-product setting, implicitly assume simultaneous production of several products. Under this assumption, adjusting capacity allocations dynamically is not convenient. Because we will subsequently extend our stationary planning model to a finite horizon execution model, we have chosen to construct a discrete time, periodic review model.

Important for the model developed in this paper is the work done by Glasserman (1997), Glasserman and Tayur (1994a), and Roundy and Muckstadt (1997) concerning the characterization of inventory shortfall and its stationary distribution. The work done by Glasserman (1997) provides motivation for approximating the stationary shortfall distribution using a mass exponential

distribution. Roundy and Muckstadt (1997) offer another approximation to the inventory shortfall distribution. Rather than using either of these approaches, we have used a Markov chain approach to model this probability distribution.

For capacitated facilities operating in series, Glasserman and Tayur (1994b) examine the stability of the system if each facility operates according to a modified base-stock policy. Stability in this case means that backorders are not unbounded, almost surely. In all of the capacitated models mentioned, a necessary and sufficient condition for system stability, and for the existence of the shortfall distribution, is that average capacity utilization rate be less than one.

Closely related to our problem, Aviv and Federgruen (1997) analyze several attributes of a multi-item capacitated problem in which production of a common component occurs, becomes available after some arbitrary but fixed lead time, and is then allocated to specific finished products. Although their primary attention is given to the case where the common component is not stocked (analogous to central warehouse inventory in our case), they also outline a dynamic program formulation of the problem where stocking is permitted. Our work differs in four respects. First, we have developed a much simpler approach for setting stock levels based on our requirement of computational speed. Second, our allocation model is substantially different, since it is an outgrowth of our experience in the automotive industry in which centralized inventories can have significant economic benefits. Third, our simulation study is considerably richer in the breadth of system configurations. Fourth, our focus is on developing a framework which will be appropriate for lean manufacturing environments.

Although in our model we consider a distribution allocation problem, our purpose is not to develop general distribution allocation models such as those developed by Eppen and Schrage (1981), Jackson (1988), and Erkip et al. (1990). Instead we focus on the specific system described previously with its short production and transportation lead times, which we believe will be attributes found in many real world environments in the future.

3 Notation and Assumptions

A key assumption on which our model is based is that demand random variables are stationary and independent between time periods. The form of the demand distribution and values of the parameters may be unique to each distribution center (DC) and may be positively correlated or independent between DCs. The following notation and variables are used subsequently:

System Parameters

M	Number of distribution centers;
C	Available production capacity per period (in units);
ρ	(= $\frac{\text{expected demand per period}}{\text{available capacity per period}}$) Average capacity utilization rate;

Costs

h_0	Installation holding cost per unit per period at the central warehouse (also called the system holding cost);
h_i	Installation holding cost per unit per period for DC i ;

- b Backorder cost per unit per period at each DC (there is no backorder cost at the central warehouse). For simplicity, each distribution center is assumed to have an identical backorder cost. However, all of the models and algorithms presented can be extended to consider varying backorder costs for each distribution center. We explain the modifications required at the end of section 4;
- h_i^e ($= h_i - h_0 > 0$) The incremental echelon holding cost for DC i ;

Decision Variables

- T The target system order-up-to inventory level;
- $\hat{s}_i$ The optimal order-up-to level for DC i ($i = 1, \dots, M$), as a function of incremental echelon holding costs (h_i^e) and backorder costs;
- $\hat{S}$ ($= \sum_{i=1}^M \hat{s}_i$) The sum of order-up-to levels across DCs when the incremental holding cost, h_i^e , is used to determine DC i 's order-up-to level;
- s_i The optimal order-up-to level for DC i ($i = 1, \dots, M$), as a function of installation holding costs (h_i) and backorder costs;
- S ($= \sum_{i=1}^M s_i$) The sum of order-up-to levels across DCs when installation holding costs (h_i) are used to determine DC i 's order-up-to level;
- x_i The inventory on-hand (or backordered) at DC i after demand has occurred, but before production and allocation;
- y_i The inventory position or stock level at DC i ($i = 1, \dots, M$) after production, allocation, and shipment from the central warehouse (which may be less than s_i or $\hat{s}_i$, depending on the availability of stock);
- x_0 The inventory on-hand at the central warehouse before production and allocation ($x_0 \geq 0$);
- y_0 The stock level at the central warehouse after production, allocation and shipment to the distribution centers ($y_0 \geq 0$);

Random Variables and Probability Distribution Functions

- D_i A random variable representing the one period demand for the single product at distribution center i ;
- $p_i(k)$ ($= \Pr\{D_i = k\} \quad \forall k \geq 0$) The discrete nonnegative probability mass function with finite first and second moments representing the demand at distribution center i ;
- $\bar{D}$ ($= \sum_{i=1}^M D_i$) A random variable representing the one period total system demand;
- V A random variable representing the stationary total system inventory shortfall in a period ($V \geq 0$);
- $\pi(k)$ ($= \Pr\{V = k\} \quad \forall k \geq 0$) The stationary probability distribution of V ;
- A ($= T - V$) A random variable representing the stationary total amount of inventory in the system after production which is available for allocation to distribution centers when $A \geq 0$; the number of backorders in the system when $A < 0$.

4 An Inventory Allocation Model

The decision to stock units in one location over another depends on the available system inventory, expected installation holding costs, incremental echelon holding costs, backorder costs, and expected future demand. The objective of this section is to construct an inventory allocation model that considers these factors.

Let T be the system order-up-to level and C be the per period production capacity. Let x_i denote the inventory level (or backorders) at location $i = 0, \dots, M$, after demand occurs but before production and allocation. Following a modified base stock production policy, after production occurs, there is a total of A units in the system available for allocation, where

$$A = T \wedge \left(\sum_{i=0}^M x_i + C \right). \quad (4.1)$$

The difference $T - A$ is the system inventory shortfall in a period, which we will denote by V . Let y_i be the inventory level (or backorders) at location $i = 0, \dots, M$, after production and allocation such that $\sum_{i=0}^M y_i = A$. For convenience, we will consider x_i and y_i , $i = 0, \dots, M$, to be integer; however, continuous variables may be substituted in all expressions with only slight modifications.

Define the following familiar newsvendor cost function for each distribution center i ,

$$G_i(y_i) = \begin{cases} h_i \sum_{k=0}^{y_i} (y_i - k) p_i(k) + b \sum_{k=y_i}^{\infty} (k - y_i) p_i(k), & \text{for } y_i \geq 0, \\ b [E(D_i) - y_i], & \text{for } y_i < 0. \end{cases} \quad (4.2)$$

Additionally, we define the aggregate expected cost function summed across all distribution centers,

$$G(y_1, \dots, y_M) = \sum_{i=1}^M G_i(y_i). \quad (4.3)$$

We define the first difference and second difference functions of $G_i(\cdot)$, $i = 1, \dots, M$, respectively, as

$$\begin{aligned} \Delta G_i(y_i) &= G_i(y_i + 1) - G_i(y_i) \\ &= \begin{cases} -b + (h_i + b) \cdot \Pr\{D_i \leq y_i\}, & \text{for } y_i \geq 0, \\ -b, & \text{for } y_i < 0, \end{cases} \end{aligned} \quad (4.4)$$

and,

$$\begin{aligned} \Delta^2 G_i(y_i) &= \Delta G_i(y_i + 1) - \Delta G_i(y_i) \\ &= \begin{cases} (h_i + b) \cdot \Pr\{D_i = y_i + 1\}, & \text{for } y_i \geq 0, \\ 0, & \text{for } y_i < 0. \end{cases} \end{aligned} \quad (4.5)$$

Clearly $G_i(y_i)$ is convex in y_i since the second difference function is non-negative, and at $y_i = 0$ we have that $\Delta G_i(-1) = -b \leq -b + (h_i + b) \Pr\{D = 0\} = \Delta G_i(0)$. Furthermore, the integer-valued base-stock level s_i that minimizes $G_i(\cdot)$ is the smallest integer for which $\Delta G_i(\cdot)$ is non-negative. In this case,

$$s_i = \left\{ \text{smallest integer } s : \Pr\{D_i \leq s\} \geq \frac{b}{h_i + b} \right\}. \quad (4.6)$$

Let $S = \sum_{i=1}^M s_i$ be the total system base-stock level which minimizes $G(\cdot)$. By the construction of s_i , it is easily verified that $\Delta G_i(y_i) < 0$ for any $y_i < s_i$, and $\Delta G_i(y_i) \geq 0$ for any $y_i \geq s_i$. From the convexity of $G_i(\cdot)$, we know that $\Delta G_i(y_i) \leq \Delta G_i(y_i + 1)$ for all y_i , and that $G(y_1, \dots, y_M)$ is jointly convex in each y_i , $i = 1, \dots, M$.

We now consider the following two cases: (1) $A \geq 0$, implying there is inventory in the system to be allocated, and (2) $A < 0$, implying the entire system is in backorder for the single product. We define the following continuous cost function, $F(\cdot)$, which depends on A , and on the inventory allocation decisions $y_1, \dots, y_M$,

$$F(A; y_1, \dots, y_M) = \begin{cases} (1) & h_0(A - \sum_{i=1}^M y_i) + G(y_1, \dots, y_M), & \text{for } A \geq 0, \\ (2) & b[\sum_{i=1}^M (E(D_i)) - A], & \text{for } A < 0. \end{cases} \quad (4.7)$$

Case (1) Suppose $A \geq 0$ in a given period. To operate optimally, the best feasible allocation $(y_1, \dots, y_M)$ is determined by solving the following problem (P):

$$(P) \quad \begin{aligned} & \min_{y_1, \dots, y_M} && F(A; y_1, \dots, y_M) \\ & \text{s.t.} && \sum_{i=1}^M y_i \leq A \\ & && y_i \geq x_i, \text{ integer } i = 1, \dots, M. \end{aligned} \quad (4.8)$$

The central warehouse inventory allocation, $y_0 = A - \sum_{i=1}^M y_i$, is held to balance incremental echelon holding costs (h_i^e) with expected backorder costs. The only means by which inventory may decrease is through random customer demand. We will refer to the constraints (4.9) as the *imbalance constraints* since these prevent any deliberate reduction of inventory to correct any mis-allocation of inventory.

To avoid having to solve (P) dynamically via a dynamic program, define the relaxed allocation problem (RP):

$$(RP) \quad \begin{aligned} & \min_{y_1, \dots, y_M} && F(A; y_1, \dots, y_M) \\ & \text{s.t.} && \sum_{i=1}^M y_i \leq A \\ & && y_i \geq 0, \text{ integer } i = 1, \dots, M. \end{aligned} \quad (4.10)$$

By relaxing the imbalance constraints in (P), we effectively allow transshipments to occur instantaneously at no cost in the event that an inventory imbalance occurs. Thus, by relaxing these constraints, we no longer need to solve a dynamic programming problem, but rather the much simpler relaxed problem (RP). Although in practice we will not allow transshipments, we will use (RP) to compute a lower bound on the optimal system order-up-to-level, T . In section 6, we will simulate operating a logistics system according to (P) while using the policies computed via (RP). Observe that for $M = 1$, the problems (P) and (RP) are equivalent since there is no possibility of inventory imbalance. Thus for $M = 1$, (RP) is an exact cost function and for $M > 1$ the solution to (RP) yields a lower bound on the true system costs. Note that if demand is positively correlated

across the distribution centers, Erkip et al. (1990) show that the likelihood of inventory imbalances is reduced.

Next, examine (RP). Since $G(\cdot)$ is convex, it follows that $F(\cdot)$ is convex in A . Relaxing the constraints yields the integer-valued order-up-to levels based on incremental holding costs for each distribution center, $\hat{s}_i$, which are determined as

$$\hat{s}_i = \left\{ \text{smallest integer } s : \Pr\{D_i \leq s\} \geq \frac{h_0 + b}{h_i + b} = \frac{h_0 + b}{h_i^e + h_0 + b} \right\}. \quad (4.12)$$

This result can be shown by taking first differences of the objective function. Let $\hat{S} = \sum_{i=1}^M \hat{s}_i$ be the sum of the base-stock levels which minimize $F(\cdot)$. Recall that each s_i is computed based on the installation holding cost rates (h_i). Observe that for $h_0 \geq 0$, $(b)/(h_i + b) \leq (h_0 + b)/(h_i + b)$, and therefore, $s_i \leq \hat{s}_i$ for all $i = 1, \dots, M$, and $S \leq \hat{S}$. That is, the desired target inventory for each distribution center based on installation costs will be less than or equal to the maximum desired allocation quantity, which is computed based on incremental echelon costs. The following lemma shows that for any level of system inventory $A \geq 0$, a distribution center i will never stock more than $\hat{s}_i$ units of inventory.

Lemma 1 *For all $A \geq 0$ and integer, $F(A; y_1, \dots, y_M)$ is minimized by $y_i \leq \hat{s}_i$, $i = 1, \dots, M$.*

Proof: Consider the following first and second difference functions with respect to y_i ,

$$\begin{aligned} \Delta F(y_i) &= F(A; y_1, \dots, (y_i + 1), \dots, y_M) - F(A; y_1, \dots, y_i, \dots, y_M) \\ &= -(h_0 + b) + (h_i + b) \cdot \Pr\{D_i \leq y_i\}, \\ \Delta^2 F(y_i) &= (h_i + b) \cdot \Pr\{D_i = y_i + 1\}. \end{aligned} \quad (4.13)$$

By construction, $\hat{s}_i$ is the largest non-negative integer for which $\Delta F(\cdot) \leq 0$. Consequently, for $y_i > \hat{s}_i$, $\Delta F(y_i) > 0$. This implies that $y_i \leq \hat{s}_i$ for all $i = 1, \dots, M$, are necessary conditions for optimality. $\square$

Since we assume that production lead times are zero, under the relaxed allocation problem (RP) if there is some distribution center for which h_i^e is zero, then the central warehouse will never hold any inventory. Intuitively, if it doesn't cost any more to hold stock at the distribution centers than at the central warehouse, it will be more beneficial in terms of system costs (and distribution center managers' customer service measures) to hold the stock at the distribution centers. Note that this gives no indication of how much inventory should be in the system. Rather, it indicates how to best allocate the available inventory under the assumption that inventory imbalances do not occur. Observe that $\hat{s}_i$ is sensitive to the values of h_i^e and b . In general, $\hat{s}_i$ will be large for small h_i^e and large b , depending on the form of the demand process. Specifically, if $p_i(k) > 0 \quad \forall k = 0, 1, 2, \dots$, then $\hat{s}_i \rightarrow \infty$ as $h_i^e \rightarrow 0$ or $b \rightarrow \infty$. Thus, we will assume $h_i^e > 0$, for all i , so that the allocation problem is not trivial. This implies that holding costs are more expensive at the distribution centers than at the central warehouse. This assumption is reasonable not only because distribution centers tend to be located in more expensive operating areas than are central warehouses, but also because h_i^e reflects the value added associated with handling, storing, reporting, and transporting activities performed by an additional link in the supply chain.

Case (2) Suppose that $A < 0$ and therefore that the entire system is in a backorder state. Practically speaking, this case is of little relevance since changes to the system would have been made to avoid $A < 0$, such as adding overtime capacity, outsourcing production, etc. However, in determining T it is important to assign a cost to the event $A < 0$.

Since explicitly allocating backorders in a given period is not a natural solution, we will approximate the expected system cost and temporarily depart from our requirement that y_i be integer for all $i = 1, \dots, M$. Let $\bar{D} = \sum_{i=1}^M D_i$ be a random variable representing the total system demand per period. For $A < 0$, we define:

$$\begin{aligned} F(A; y_1, \dots, y_M) &= b[E(\bar{D}) - A] \\ \text{where } y_i &:= A \cdot \frac{E(D_i)}{E(\bar{D})} \quad i = 1, \dots, M \\ y_0 &= 0. \end{aligned} \tag{4.14}$$

We denote by $L(A)$ the minimum one period expected cost of allocating the available system inventory A , across the M distribution centers and the central warehouse:

$$L(A) = \min_{y_1, \dots, y_M} F(A; y_1, \dots, y_M). \tag{4.15}$$

Recall $\hat{S}$ is the sum of the unconstrained optimal solutions to (RP) . Equivalently, $L(A)$ can be stated as follows:

$$L(A) = \begin{cases} (1) \quad h_0(A - \hat{S}) + G(\hat{s}_1, \dots, \hat{s}_M) & \text{for } A \geq \hat{S}, \\ \quad \text{where } y_i = \hat{s}_i, \quad i = 1, \dots, M, \\ \quad y_0 = A - \hat{S}, \\ (2) \quad \min_{y_1, \dots, y_M} G(y_1, \dots, y_M) & \text{for } 0 \leq A < \hat{S} \\ \quad \text{s.t.} \quad \sum_{i=1}^M y_i = A, \\ \quad y_i \geq 0, \text{ integer}, i = 1, \dots, M, \\ \quad y_0 = 0, \\ (3) \quad b \cdot [E(\bar{D}) - y_i] & \text{for } A < 0, \\ \quad \text{where } y_i = A \cdot \left[\frac{E(D_i)}{E(\bar{D})} \right], \quad i = 1, \dots, M, \\ \quad y_0 = 0. \end{cases} \tag{4.16}$$

By definition, $L(A)$ is continuous and is linear for $A < 0$ and for $A > \hat{S}$. For any possible level of available system stock A , we can efficiently obtain an allocation of inventory across all locations. If $A \geq \hat{S}$ or $A < 0$, the allocation decisions are obvious and predetermined. If $0 \leq A < \hat{S}$, the allocation decisions are determined using a greedy algorithm. Note that if $0 \leq A < \hat{S}$, $y_0 = 0$, and the cost term associated with holding inventory at the central warehouse is $h_0(A - \sum_{i=1}^M y_i) = h_0 \cdot y_0 = 0$. Our assumption that imbalance does not occur implies that y_i may not be at least as large as x_i , the stock remaining on hand at location i at the end of the previous period. By ignoring this constraint, we consequently underestimate the expected costs and hence the total inventory required.

Theorem 2 $L(A)$ is convex for all A .

Proof: From the definition of $L(A)$, it is clear that for $0 \leq A < \hat{S}$, $L(A)$ is convex in A due to the convexity of $G(\cdot)$. For $A > \hat{S}$, $L(A)$ is also convex since as A increases in this region, $L(A)$ increases linearly at a rate h_0 , because $G(\hat{s}_1, \dots, \hat{s}_M)$ is a constant. For $A < 0$, as A decreases, $L(A)$ increases linearly at a rate b . What remains to be shown for the convexity of $L(A)$ for all $A \geq 0$ is that at $A = \hat{S}$ and at $A = 0$, the left-hand first difference is less than or equal to the right-hand first difference. Define $\Delta L(A) = L(A+1) - L(A)$. At $A = \hat{S}$, the left-hand first difference is given by

$$\begin{aligned} \Delta L(\hat{S} - 1) &= L(\hat{S}) - L(\hat{S} - 1) \\ &= G(\hat{s}_1, \dots, \hat{s}_M) - G(\hat{s}_1, \dots, (\hat{s}_i - 1), \dots, \hat{s}_M), \quad \text{for some } 1 \leq i \leq M, \\ &= -b + (h_i + b) \cdot \Pr\{D_i \leq \hat{s}_i - 1\} \\ &< h_0 \quad \text{by the definition of } \hat{s}_i. \end{aligned}$$

The right-hand first difference is given by $\Delta L(\hat{S}) = L(\hat{S}+1) - L(\hat{S}) = h_0 \geq 0$. Hence, $\Delta L(\hat{S} - 1) \leq \Delta L(\hat{S})$. Similarly, at $A = 0$, the left-hand first difference is given by

$$\begin{aligned} \Delta L(-1) &= L(0) - L(-1) \\ &= G(0, \dots, 0) - b[E(\bar{D}) + 1] \\ &= -b. \end{aligned}$$

The right-hand first difference is given by $\Delta L(0) = L(1) - L(0) = \Delta G(0, \dots, 1, \dots, 0) = -b + (h_i + b) \Pr\{D_i \leq 0\}$ for some location $i = 1, \dots, M$. Hence, $\Delta L(-1) \leq \Delta L(0)$ and $L(A)$ is convex for all A . $\square$

Figure 2: An Example of the Inventory Allocation Cost Function, $L(A)$

From the definition of s_i in (4.6), for $A \leq S \leq \hat{S}$, $G(y_1, \dots, y_M)$ is decreasing in y_i , and hence $L(A)$ is decreasing. For $S < A \leq \hat{S}$, due to the equality constraint in (4.16), there will be at least

one distribution center i for which $y_i > s_i$. From the definition of s_i , $G(\cdot)$ will be increasing, and hence $L(A)$ will be increasing. Table 1 and Figure 2 summarize and illustrate the key facts about the inventory allocation cost function $L(A)$.

To summarize, for $M = 1$, $L(A)$ is exact and an optimal allocation of inventory is achievable each period. For $M > 1$, $L(A)$ is constructed from the relaxed problem (RP) and hence is a lower bound on the true system cost.

Table 1: Key Attributes of $L(A)$

Value of A	Behavior of $L(A)$ and y_i
$A < 0$	<ul style="list-style-type: none"> • $y_i = A \cdot E[D_i]/E(\bar{D}) < 0$, for all $i = 1, \dots, M$ • $y_0 = 0$ • $\Delta L(A) = -b$; $\Delta^2 L(A) = 0$
$0 \leq A < S$	<ul style="list-style-type: none"> • $y_i \leq s_i$, for all $i = 1, \dots, M$ • $y_0 = 0$ • $\Delta L(A) < 0$; $\Delta^2 L(A) \geq 0$
$A = S$	<ul style="list-style-type: none"> • $y_i = s_i$, for all $i = 1, \dots, M$ • $y_0 = 0$ • $\Delta L(A) \geq 0$; $\Delta^2 L(A) \geq 0$
$S < A < \hat{S}$	<ul style="list-style-type: none"> • $s_i \leq y_i \leq \hat{s}_i$, for all $i = 1, \dots, M$ • $y_0 = 0$ • $0 < \Delta L(A) < h_0$; $\Delta^2 L(A) \geq 0$
$A = \hat{S}$	<ul style="list-style-type: none"> • $y_i = \hat{s}_i$, for all $i = 1, \dots, M$ • $y_0 = 0$ • $\Delta L(A) = h_0$; $\Delta^2 L(A) \geq 0$
$A > \hat{S}$	<ul style="list-style-type: none"> • $y_i = \hat{s}_i$, for all $i = 1, \dots, M$ • $y_0 = A - \hat{S}$ • $\Delta L(A) = h_0$; $\Delta^2 L(A) = 0$

5 Determining the Target Inventory Level

Our objective in this section is to determine the system order-up-to level T^* which minimizes the long run average cost per period. Central to our approach is the inventory shortfall random variable as discussed extensively in Roundy and Muckstadt (1997) and Tayur (1992). Let V_t be the system inventory shortfall random variable in period t , $\bar{D}$ be the total system demand random variable, and C be the production capacity per period. Then,

$$\begin{aligned} V_t &= [V_{t-1} + \bar{D} - C]^+ \\ V_0 &= 0. \end{aligned}$$

Tayur (1992) shows that a stationary distribution for V_t , denoted V , exists if and only if $E(\bar{D}) < C$. Note that $V_t \geq 0$, $\forall t \geq 0$ and that V_t does not depend on the order-up-to level T .

Since the demand process is assumed to be independent between periods, $\{V_t\}_{t \geq 0}$ can be modeled as a Markov chain with a countably infinite state space $\{0, 1, 2, \dots\}$, according to the following transition probabilities:

$$P_{ij} = \Pr\{V_t = j | V_{t-1} = i\} = \begin{cases} \Pr\{\bar{D} \leq C - i\} & \text{if } j = 0, i \leq C \\ \Pr\{\bar{D} = C + j - i\} & \text{if } j > 0, i \leq C + j \\ 0 & \text{otherwise.} \end{cases} \quad (5.17)$$

It is easily verified that the Markov chain is irreducible and aperiodic. Furthermore, since we assumed that $E(\bar{D}) < C$, the chain is positive recurrent and possesses a unique stationary probability distribution. Let $\pi(k) = \Pr\{V = k\}$, $\forall k \geq 0$, be this stationary probability distribution. Given T and the system inventory shortfall V_t , the amount of system inventory is $A_t = T - V_t$ in period t . Hence in the limit, $\Pr\{A = n\} = \Pr\{V = T - n\}$, $n \in \{T, T - 1, T - 2, \dots\}$.

The amount of inventory available for allocation following production is the state of the system, which determines the expected cost incurred based on the current allocation decision. Thus, given V , and therefore A , we expect to incur a cost $L(A)$. This cost depends only on the state of the system A and not on previous or future states because we have assumed that imbalance cannot occur and that demand is independent from period to period.

Recall that the strong law of large numbers for Markov chains states that the long run average cost per period converges almost surely to its expectation. In our case this means that for any initial distribution of inventory shortfall V , we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N L(T - V_t) &= \sum_{k=0}^{\infty} L(T - k) \cdot \pi(k) \quad \text{a.s.} \\ &= E_V[L(T - V)]. \end{aligned} \quad (5.18)$$

See Resnick (1992) for details.

We use (5.18) to determine T^* ; that is, we propose to find T^* by solving:

$$\begin{aligned} \min_T H(T) &= E_V[L(T - V)] = E_V[L(A)] \\ &= \sum_{k=0}^{\infty} L(T - k) \cdot \Pr\{V = k\}. \end{aligned} \quad (5.19)$$

Theorem 3 *The optimal system order-up-to level T^* which minimizes $H(T)$ in (5.19) is greater than or equal to $S = \sum_{i=1}^M s_i$. That is, $T^* \geq S$.*

Proof: Define the first and second difference functions of H at T as,

$$\begin{aligned}\Delta H(T) &= H(T+1) - H(T) \\ &= \sum_{k=0}^{\infty} L(T+1-k) \Pr\{V=k\} - \sum_{k=0}^{\infty} L(T-k) \Pr\{V=k\} \\ &= \sum_{k=0}^{\infty} \Delta L(T-k) \Pr\{V=k\}, \text{ and} \\ \Delta^2 H(T) &= \sum_{k=0}^{\infty} \Delta^2 L(T-k) \Pr\{V=k\},\end{aligned}\tag{5.20}$$

where $\Delta L(k) = L(k+1) - L(k)$, and $\Delta^2 L(k) = \Delta L(k+1) - \Delta L(k)$, are as defined earlier. From Theorem 2, we know that

$$\begin{aligned}\Delta L(k) &< 0, & \text{for } k < S, \\ \Delta L(k) &\geq 0, & \text{for } k \geq S, \\ \Delta^2 L(k) &\geq 0, & \text{for } k \geq 0, \\ \Delta^2 L(k) &= 0, & \text{for } k < 0,\end{aligned}$$

Suppose $T < S$. Then $\Delta H(T) < 0$ since $\Delta L(x) < 0$ for all $x < S$. Thus for any $T < S$, decreasing T increases the expected cost function $H(T)$. Hence, the value of T^* which minimizes $H(T)$ must be at least as large as S . That is, $T^* \geq S$. $\square$

Theorem 4 *$H(T)$ is convex in T .*

Proof: We will show that the second difference of $H(\cdot)$ is non-negative for all T . Recall for all $A < 0$, $\Delta^2 L(A) = 0$, since for $A < 0$, $L(A)$ is linear by construction. Thus, we have that

$$\begin{aligned}\Delta^2 H(T) &= \sum_{k=0}^{\infty} \Delta^2 L(T-k) \Pr\{V=k\} \\ &= \sum_{k=0}^T \Delta^2 L(T-k) \Pr\{V=k\} + \sum_{k=T+1}^{\infty} \Delta^2 L(T-k) \Pr\{V=k\} \\ &= \sum_{k=0}^T \Delta^2 L(T-k) \Pr\{V=k\} \\ &\geq 0. \quad \square\end{aligned}\tag{5.21}$$

We will exploit this convexity to determine the optimal system order-up-to level T^* by using another greedy algorithm. From Theorem 3 and its proof, we know that $T^* \geq S$, and that for

$A < 0$, $\Delta L(A) = -b$ is constant. For $T > S$, we have that,

$$\begin{aligned}\Delta H(T) &= \sum_{k=0}^{\infty} \Delta L(T-k) \Pr\{V=k\} \\ &= \sum_{k=0}^T \Delta L(T-k) \Pr\{V=k\} - b \Pr\{V \geq T+1\}.\end{aligned}$$

Thus,

$$T^* = \left\{ \text{smallest integer } T \geq S : \Delta H(T) \geq 0 \right\}. \quad (5.22)$$

To summarize, T^* can be computed very quickly by using $L(A)$ and conditioning on the stationary system shortfall random variable V . To compute the distribution of V , it is possible to numerically solve the Markov chain (5.17) or use an approximation of the type described in Roundy and Muckstadt (1997) or Tayur (1992). For $M = 1$, T^* is exact. For $M > 1$, because we have assumed imbalance cannot occur and $L(A)$ is a lower bound on the true system cost, T^* is a lower bound on the true optimal order-up-to level.

6 Empirical Results

A comprehensive simulation study was conducted in order to test the inventory imbalance assumption, to measure the accuracy of the cost approximation, and to understand the implications of operating the system described according to the policies developed thus far.

By ignoring the imbalance constraints in (P) , we are able to gain tremendous computational efficiency, which was the primary motivation for formulating and solving the relaxed problem. In this section, we demonstrate the results of operating according to the policies developed thus far. While the order-up-to levels will be computed using (RP) , on a periodic basis the actual allocations will be determined by solving (P) , *i.e.*, by recognizing the existence of initial inventories at each distribution center each period.

6.1 Simulation Experiments

A total of 486 different system configurations were evaluated. For each configuration, ten independent replications were simulated of 100,000 renewals each, for a total of over 1 billion simulated periods of activity. We define a renewal in this system to occur when the system inventory is restored to T , or equivalently, when $V_t = 0$. The complete simulation results for each system configuration are presented in the appendix.

The factors under study are shown in Table 2. One key factor in the evaluation is the number of DCs in the system. We examined systems having 1, 5, 10, 25, and 50 distribution centers in order to understand how the number of DCs affects the appropriateness of the imbalance assumption.

Random demands were generated from negative binomial distributions. The demand process variance, or VTMR (Variance to Mean Ratio), represents the demand variance at each location

Table 2: Controllable System Factors of Interest

System Factor	Abbreviation	Factor Levels
Number of DCs	N	1, 5, 10, 25, 50
System Utilization, $\bar{D}/C$	ρ	81.4%, 90%, 95%
Demand process variance	VTMR	1.01, 2, 5
Backorder cost ratio	b/h	9, 90
Incremental holding cost ratio	h_0/h	.10, .50, .90
Distribution of demand across DCs	Max:Min	1, 10, 100
Target inventory order-up-to level (TIL)	T	(Computed)

and the demand variance for the entire system. Three VTMR values were examined as indicated in Table 2. The minimum VTMR value yields a probability distribution that closely matches a Poisson distribution. Our objective was to examine the cases where demand processes are more variable than Poisson processes. We wanted to measure how well the approximation model performed in more highly variable situations.

Total average system demand per period was identical over all scenarios tested and was equal to 171 units per period. Production capacity per period was adjusted to vary the average system utilization rate.

To examine the effect of identical versus non-identical DC demand, we varied the average demand among the DCs in different proportions in each case. This distribution of demand, as shown in Table 2, is characterized through the *Max:Min ratio*. The Max:Min ratio is defined as the ratio of the highest expected demand at a DC to the lowest expected demand at a DC. A Max:Min ratio of 1 corresponds to the identical retailer case, while a Max:Min ratio of 100 implies that the largest DC's expected demand per period is 100 times the expected demand per period of the smallest DC. Again, the average total system demand per period remained fixed.

There were two components which determined the system cost structure, namely the echelon holding cost h_i^e and the backorder cost b . In all cases, the individual distribution center holding costs h_i were identical and set to a value of \$1 per unit per period. The echelon holding cost was varied by changing the value of h_0/h , where $h = h_i = \$1$ is the value of the distribution center holding cost. The higher the value of h_0/h , the smaller the echelon costs. The backorder cost was varied by changing the value of b/h . The higher the value of b/h , the greater the effect of backorders. Our objective was to understand the implications of various system cost structures on model performance and appropriateness.

The performance measures of interest are listed in Table 3. The expected cost is the theoretical cost computed based on the relaxed problem (*RP*). The *%CostError* measures the accuracy of the theoretical cost compared with the empirical actual cost. This is our primary measure of the quality of our approach. The *%UnitsImbalance* measures the fraction of units that were in imbalance

Table 3: System Performance Measures

Measure	Description
Expected Cost	The value of $H(T)$.
%CostError	The value of $\frac{(ActualCost - ExpectedCost)}{ActualCost}$.
%UnitsImbalance	The value of $\frac{Total\ Number\ of\ units\ in\ imbalance}{Total\ number\ of\ units\ demanded}$.
Actual Fill	The actual fill rate obtained.

during the simulation. This measure helps to identify those situations where the inventory imbalance assumption is most appropriate. The Actual Fill measures the fraction of demand satisfied immediately from stock.

6.2 Highlights of Results

Observe from the results table in the appendix that the average overall cost error was 0.09% and varied between +4.2% and -1.1%. For the majority of the scenarios, the model performed exceptionally well. Specifically, the cost errors were less than 0.2% in 90% of all scenarios examined. The inventory levels computed from the model were able to account for the finite production capacity and ignoring the risk of imbalances did not significantly increase average costs.

The model performed less well when ρ was high and VTMR was high because the system cannot respond effectively to large demand spikes due to finite capacity. Also, when h_0/h is close to 1, less inventory is held in the system and most of the inventory is held at the distribution centers because it is relatively more costly to carry inventory. This increases the risk of inventory imbalances and hence increases model error. Despite these worst case scenarios, in all cases the model error was no greater than 4.2%.

6.3 Base Stock Levels and System Performance

Although increasing the number of distribution centers in a logistics system may afford a manufacturer proximity to its clients, its impact on overall system performance and cost may be detrimental. Even under a low capacity utilization rate, T increases dramatically as M increases due to additional safety stock requirements and average system fill rate performance is uniformly lower, albeit slightly lower.

As the difference in average demand per period increases the risk of imbalance increases; however, this is not necessarily detrimental to system performance since (a) the actual inventories are lower than if imbalance risk was explicitly considered, and (b) fill rates in most cases did not decrease. Therefore, assuming identical retailers when in fact they are not identical provides an upper bound on T^* .

While T is a function of both demand variance and capacity utilization, in general, T is substantially more sensitive to VTMR than to ρ . On average, T increased 11% due to an increase in ρ , and increased 65% due to an increase in VTMR. See Table 4 and the Appendix. This observation is important since as companies continue to adopt just-in-time (JIT) manufacturing philosophies and lead times between successive operations decrease, relative demand variability increases dramatically. See Muckstadt (1997) for details and examples.

Table 4: T as a function of VTMR and ρ

	VTMR = 1.01	VTMR = 5	Average
$\rho = 81.4\%$	264	420	342
$\rho = 95.0\%$	272	486	379
Average	268	443	

6.4 System Cost Structures

As $h_0/h \rightarrow 1$, or equivalently $h_i^e \rightarrow 0$ on average for $i = 1, \dots, M$, it becomes relatively more expensive to hold inventory in the system because the minimum cost associated with bringing an additional unit into the system is at least h_0 . Thus the system will stock less. Additionally, as $h_0/h \rightarrow 1$, there is less economic incentive to stock inventory centrally, thus the inventory is pushed out to the distribution centers. Both of these two effects associated with $h_0/h \rightarrow 1$ increase the risk of inventory imbalance thus model error in any period. The model was least accurate in the case where the demand variance-to-mean ratio was 5 and $h_0/h = 0.90$.

Similarly, as $h_0/h \rightarrow 0$, or equivalently h_i^e increases on average for $i = 1, \dots, M$, it becomes cheaper to hold inventory centrally and thus provides an incentive to bring more units into the system to raise the system order-up-to level. This greatly helps reduce the risk of inventory imbalances and thus improves the performance of the model.

The relative accuracy of the model did not change over the range of b/h values tested in the experiment. Since this range was extreme, we conjecture that the model's validity is quite good for realistic values of this ratio.

7 Conclusions

In this paper, we developed a computationally efficient approach for planning inventory levels in a lean capacitated manufacturing and distribution system for a single item. The approximation model is based on developing the probability distribution of the shortfall random variable and observing that the distribution of the available inventory in a period is determined directly from this distribution. By ignoring the possibility of inventory imbalance, we were able to construct a lower

bound on the true cost based on a newsvendor-like cost function to allocate available inventory. The algorithms are suitable for very large scale systems. Through a comprehensive simulation study, this lower bound was shown to be within 0.09% of optimal. In a companion paper Rappold and Muckstadt (1998), we address stocking decisions in a multiple product, multiple location system.

8

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					N	Max:Min								25				50				Grand Total
					1	5				10				15				1				
Utilization	VTMR	ho/h	b/h	Data	1	5	3	4	205	1	3	4	222	250	300	213						
0.9	1.01	0.1	9	Average of TIL	189	210	207	205		230	222	222		250	300	213						
				Average of ExpectedCost	\$24.05	\$53.55	\$50.78	\$49.28		\$76.80	\$72.57	\$70.04	\$124.01	\$178.11	\$58.61							
				Average of %CostError	0.088%	-0.022%	0.034%	-0.039%		0.006%	0.001%	0.043%	-0.098%	-0.066%	0.014%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.000%		0.000%	0.000%	0.002%	0.000%	0.005%	0.000%							
				Average of ActualFill	99.61%	99.12%	99.10%	99.08%		98.93%	98.71%	98.85%	97.34%	97.11%	99.02%							
			90	Average of TIL	209	244	239	237		270	268	266	350	401	250							
				Average of ExpectedCost	\$36.74	\$82.79	\$78.44	\$76.01		\$119.40	\$113.17	\$109.44	\$197.64	\$289.71	\$91.11							
				Average of %CostError	-0.135%	0.037%	-0.149%	0.045%		-0.029%	-0.046%	0.060%	-0.041%	-0.161%	-0.032%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.014%		0.000%	0.003%	0.003%	0.000%	0.004%	0.003%							
				Average of ActualFill	99.97%	99.93%	99.93%	99.93%		99.87%	99.89%	99.90%	99.88%	99.61%	99.91%							
		0.5	9	Average of TIL	189	210	207	205		230	222	222	250	300	213							
				Average of ExpectedCost	\$24.05	\$53.55	\$50.78	\$49.28		\$76.80	\$72.57	\$70.04	\$124.01	\$178.11	\$58.61							
				Average of %CostError	0.025%	-0.036%	-0.018%	-0.071%		-0.008%	-0.025%	0.044%	-0.098%	-0.059%	-0.014%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.000%		0.000%	0.000%	0.001%	0.000%	0.005%	0.000%							
				Average of ActualFill	99.61%	99.12%	99.10%	99.08%		98.93%	98.71%	98.85%	97.34%	97.10%	99.02%							
			90	Average of TIL	203	244	239	237		270	268	266	350	400	249							
				Average of ExpectedCost	\$37.16	\$82.79	\$78.44	\$76.01		\$119.40	\$113.17	\$109.44	\$197.64	\$289.72	\$91.17							
				Average of %CostError	-0.016%	0.040%	0.054%	-0.066%		-0.038%	0.029%	0.029%	-0.109%	-0.176%	0.002%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.014%		0.000%	0.003%	0.003%	0.000%	0.005%	0.003%							
				Average of ActualFill	99.96%	99.93%	99.93%	99.93%		99.87%	99.89%	99.90%	99.88%	99.61%	99.91%							
		0.9	9	Average of TIL	189	210	207	205		230	222	222	250	300	213							
				Average of ExpectedCost	\$24.05	\$53.55	\$50.78	\$49.28		\$76.80	\$72.57	\$70.04	\$124.01	\$178.11	\$58.61							
				Average of %CostError	-0.064%	-0.006%	0.018%	-0.058%		0.008%	-0.012%	0.017%	-0.097%	-0.059%	-0.015%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.000%		0.000%	0.000%	0.002%	0.000%	0.005%	0.000%							
				Average of ActualFill	99.61%	99.12%	99.10%	99.08%		98.93%	98.71%	98.85%	97.34%	97.10%	99.02%							
			90	Average of TIL	203	244	239	237		270	268	266	350	400	249							
				Average of ExpectedCost	\$37.16	\$82.79	\$78.44	\$76.01		\$119.40	\$113.17	\$109.44	\$197.64	\$289.72	\$91.17							
				Average of %CostError	0.242%	0.020%	0.029%	-0.089%		-0.016%	-0.006%	0.091%	-0.108%	-0.177%	0.035%							
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.014%		0.000%	0.003%	0.003%	0.000%	0.005%	0.003%							
				Average of ActualFill	99.96%	99.93%	99.93%	99.93%		99.87%	99.89%	99.90%	99.88%	99.61%	99.91%							
2	0.1	9	Average of TIL	210	230	227	226		250	248	246	300	350	236								
			Average of ExpectedCost	\$35.77	\$79.08	\$75.10	\$72.62		\$113.77	\$108.13	\$104.02	\$187.45	\$276.33	\$87.03								
			Average of %CostError	-0.069%	0.010%	0.036%	-0.007%		-0.058%	0.022%	0.033%	0.098%	0.046%	-0.003%								
			Average of %UnitsImbalance	0.000%	0.000%	0.002%	0.035%		0.000%	0.005%	0.021%	0.004%	0.061%	0.009%								
			Average of ActualFill	99.40%	98.68%	98.70%	98.74%		97.83%	98.04%	98.16%	96.43%	94.16%	98.44%								
		90	Average of TIL	241	286	281	278		330	322	317	430	557	298								
			Average of ExpectedCost	\$54.98	\$125.49	\$119.57	\$116.00		\$184.62	\$176.26	\$170.57	\$317.77	\$490.01	\$140.73								
			Average of %CostError	-0.196%	-0.099%	0.087%	0.105%		-0.025%	0.032%	0.000%	-0.152%	-0.160%	-0.016%								
			Average of %UnitsImbalance	0.000%	0.000%	0.001%	0.048%		0.000%	0.017%	0.025%	0.004%	0.060%	0.013%								
			Average of ActualFill	99.95%	99.89%	99.88%	99.89%		99.82%	99.82%	99.82%	99.61%	99.32%	99.86%								
0.5	9	Average of TIL	199	230	226	224		250	248	246	300	350	234									
		Average of ExpectedCost	\$36.78	\$79.08	\$75.11	\$72.63		\$113.77	\$108.13	\$104.02	\$187.45	\$276.33	\$87.17									
		Average of %CostError	0.023%	-0.010%	0.030%	-0.005%		-0.030%	-0.001%	0.065%	0.098%	-0.076%	0.010%									
		Average of %UnitsImbalance	0.000%	0.000%	0.002%	0.041%		0.000%	0.005%	0.021%	0.004%	0.061%	0.010%									
		Average of ActualFill	99.35%	98.69%	98.68%	98.71%		97.83%	98.04%	98.16%	96.43%	94.18%	98.43%									
	90	Average of TIL	226	280	274	272		330	322	316	430	557	293									
		Average of ExpectedCost	\$59.58	\$125.68	\$119.87	\$116.32		\$184.62	\$176.26	\$170.57	\$317.77	\$490.01	\$141.48									
		Average of %CostError	0.231%	-0.006%	0.075%	0.113%		-0.028%	0.015%	-0.053%	0.106%	-0.165%	0.048%									
		Average of %UnitsImbalance	0.000%	0.000%	0.002%	0.078%		0.000%	0.017%	0.027%	0.004%	0.060%	0.018%									
		Average of ActualFill	99.94%	99.88%	99.88%	99.89%		99.82%	99.82%	99.82%	99.60%	99.32%	99.86%									
0.9	9	Average of TIL	199	230	226	224		250	248	246	300	350	234									
		Average of ExpectedCost	\$36.78	\$79.08	\$75.11	\$72.63		\$113.77	\$108.13	\$104.02	\$187.45	\$276.33	\$87.17									
		Average of %CostError	0.052%	0.023%	-0.016%	0.047%		-0.018%	-0.005%	0.050%	0.099%	-0.076%	0.019%									
		Average of %UnitsImbalance	0.000%	0.000%	0.002%	0.041%		0.000%	0.005%	0.021%	0.004%	0.061%	0.010%									
		Average of ActualFill	99.35%	98.68%	98.68%	98.71%		97.83%	98.04%	98.16%	96.43%	94.18%	98.43%									
	90	Average of TIL	223	280	274	272		330	322	316	430	557	292									
		Average of ExpectedCost	\$59.87	\$125.68	\$119.87	\$116.32		\$184.62	\$176.26	\$170.57	\$317.77	\$490.01	\$141.52									
		Average of %CostError	0.041%	0.040%	0.179%	0.154%		-0.026%	-0.018%	-0.018%	0.107%	-0.163%	0.049%									
		Average of %UnitsImbalance	0.000%	0.000%	0.002%	0.078%		0.000%	0.017%	0.027%	0.004%	0.060%	0.018%									
		Average of ActualFill	99.93%	99.88%	99.87%	99.89%		99.82%	99.82%	99.82%	99.60%	99.32%	99.85%									
5	0.1	9	Average of TIL	271	294	291	289		320	313	309	377	450	300								
			Average of ExpectedCost	\$62.08	\$135.09	\$128.59	\$123.57		\$195.49	\$185.60	\$177.06	\$325.79	\$482.43	\$149.13								
			Average of %CostError	-0.301%	0.027%	0.037%	0.117%		-0.017%	0.026%	0.119%	0.050%	0.120%	0.003%								
			Average of %UnitsImbalance	0.000%	0.000%	0.016%	0.048%		0.004%	0.055%	0.143%	0.155%	0.650%	0.045%								
			Average of ActualFill	98.95%	97.51%	97.53%	97.65%		95.90%	96.10%	96.36%	92.23%	87.45%	97.00%								
		90	Average of TIL	339	401	394	389		475	463	452	661	924	424								
			Average of ExpectedCost	\$96.91	\$223.16	\$213.67	\$206.39		\$336.47	\$322.26	\$309.17	\$603.68	\$963.14	\$254.79								
			Average of %CostError	-0.435%	-0.013%	-0.050%	0.137%		0.062%	0.114%	0.160%	0.103%	0.059%	-0.002%								
			Average of %UnitsImbalance	0.000%	0.000%	0.013%	0.058%		0.003%	0.083%	0.241%	0.109%	0.623%	0.063%								
			Average of ActualFill	99.91%	99.76%	99.77%	99.77%		99.61%	99.63%	99.63%	99.15%	98.57%	99.71%								
0.5	9	Average of TIL	234	274	269	265		310	302	297	375	450	281									
		Average of ExpectedCost	\$71.71	\$136.85	\$130.59	\$125.85		\$195.88	\$186.12	\$177.68	\$325.80	\$482.43	\$151.54									
		Average of %CostError	-0.097%	-0.055%	0.138%	0.230%		-0.047%	0.105%	0.159%	0.050%	0.010%	0.061%									
		Average of %UnitsImbalance	0.000%	0.000%	0.056%	0.116%		0.003%	0.160%	0.217%	0.168%	0.643%	0.086%									
		Average of ActualFill	98.63%	97.40%	97.46%	97.50%		96.06%	96.14%	96.34%	92.24%	87.48%	96.93%									
	90	Average of TIL	299	370	363	357		460	446	435	661	924	398									
		Average of ExpectedCost	\$118.94	\$227.61	\$218.57	\$211.73		\$337.48	\$323.35	\$310.36	\$603.70	\$963.14	\$260.39									
		Average of %CostError	-0.489%	0.198%	0.324%	0.870%		-0.035%	0.094%	0.241%	-0.045%	0.022%	0.168%									
		Average of %UnitsImbalance	0.000%	0.000%	0.085%	0.473%		0.009%	0.160%	0.445%	0.167%	0.623%	0.172%									
		Average of ActualFill	99.88%	99.74%	99.76%	99.75%		99.62%	99.62%	99.63%	99.20%	98.58%	99.70%									
0.9	9	Average of TIL	228	274	269	265		310	302	297	375	450	281									
		Average of ExpectedCost	\$72.41	\$136.85	\$130.59	\$125.85		\$195.88	\$186.12	\$177.68	\$325.80	\$482.43	\$151.64									
		Average of %CostError	-0.168%	0.041%	0.088%	0.095%		-0.040%	0.122%	0.137%	0.051%	0.012%	0.039%									
		Average of %UnitsImbalance	0.000%	0.000%	0.056%	0.116%		0.009%	0.160%	0.217%	0.169%	0.646%	0.086%									
		Average of ActualFill	98.49%	97.39%	97.47%	97.51%		96.06%	96.14%	96.34%	92.24%	87.48%	96.91%									
	90	Average of TIL	286	370	363	357		460	446	435	661	924	396									
		Average of ExpectedCost	\$128.53	\$227.61	\$218.57	\$211.73		\$337.48	\$323.35	\$310.36	\$603.70	\$963.14	\$261.74									
		Average of %CostError	-1.124%	-0.030%	0.124%	1.060%		-0.038%	0.005%	0.318%	-0.042%	0.056%	0.044%									
		Average of %UnitsImbalance	0.000%	0.000%	0.085%	0.474%		0.009%	0.159%	0.448%	0.168%	0.626%	0.173%									
		Average of ActualFill	99.85%	99.75%	99.76%	99.75%		99.62%	99.62%	99.63%	99.20%	98.58%	99.69%									
Total Average of TIL					230	271	266	264		310	303	299	394	497	281							
Total Average of ExpectedCost					\$56.53	\$117.24	\$111.83	\$108.19		\$171.25	\$163.18	\$156.92	\$292.73	\$446.62	\$131.31							
Total Average of %CostError					-0.133%	0.009%	0.057%	0.147%		-0.021%	0.025%	0.083%	-0.002.									

				N	Max/Min				10				25				50				Grand Total		
				1	5	1	3	4	21	1	3	4	21	1	3	4	21	1	3	4	21		
Utilization	VTMR	hoh	bh	Data																			
0.95	1.01	0.1	9	Average of TIL	211	223	222	221	236	233	232	262	306	227									
				Average of ExpectedCost	\$26.26	\$55.24	\$52.52	\$51.00	\$77.91	\$74.05	\$71.46	\$125.46	\$179.15	\$60.23									
				Average of %CostError	-0.091%	-0.008%	-0.056%	0.119%	0.012%	-0.017%	0.021%	0.008%	0.015%	-0.003%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.029%	0.000%	0.001%	0.011%	0.000%	0.014%	0.006%									
				Average of ActualFill	99.60%	99.04%	99.14%	99.21%	98.78%	98.71%	98.75%	97.26%	96.95%	98.99%									
				Average of TIL	238	263	260	258	288	284	281	350	418	270									
			90	Average of ExpectedCost	\$39.77	\$84.95	\$80.74	\$78.33	\$121.37	\$115.04	\$111.31	\$198.15	\$291.40	\$93.31									
				Average of %CostError	0.125%	0.190%	0.110%	-0.045%	0.057%	-0.015%	0.094%	-0.038%	-0.064%	0.071%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.008%	0.000%	0.002%	0.010%	0.001%	0.004%	0.003%									
				Average of ActualFill	99.96%	99.93%	99.93%	99.93%	99.86%	99.90%	99.90%	99.85%	99.60%	99.91%									
				Average of TIL	196	215	213	212	230	228	226	260	300	218									
			0.5	Average of ExpectedCost	\$29.66	\$55.92	\$53.28	\$51.77	\$78.09	\$74.31	\$71.85	\$125.58	\$179.29	\$61.13									
				Average of %CostError	-0.214%	0.065%	0.033%	0.158%	0.040%	0.000%	0.074%	0.012%	-0.030%	0.022%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.063%	0.000%	0.007%	0.020%	0.001%	0.024%	0.013%									
				Average of ActualFill	99.44%	99.06%	99.10%	99.13%	98.64%	98.75%	98.78%	97.63%	96.83%	98.95%									
				Average of TIL	222	250	246	245	280	274	272	350	412	258									
			90	Average of ExpectedCost	\$48.07	\$87.27	\$83.39	\$81.14	\$121.85	\$116.03	\$112.42	\$198.15	\$292.40	\$96.93									
				Average of %CostError	0.349%	0.021%	0.101%	0.182%	0.111%	0.043%	0.171%	-0.008%	-0.035%	0.136%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.031%	0.000%	0.005%	0.025%	0.001%	0.023%	0.009%									
				Average of ActualFill	99.95%	99.92%	99.92%	99.93%	99.89%	99.89%	99.90%	99.85%	99.64%	99.91%									
				Average of TIL	195	215	213	212	230	228	226	260	300	218									
			0.9	Average of ExpectedCost	\$29.81	\$55.92	\$53.28	\$51.77	\$78.09	\$74.31	\$71.85	\$125.58	\$179.29	\$61.15									
				Average of %CostError	-0.142%	0.019%	0.012%	0.173%	0.040%	0.013%	0.116%	0.013%	-0.030%	0.032%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.063%	0.000%	0.007%	0.021%	0.001%	0.024%	0.013%									
				Average of ActualFill	99.43%	99.07%	99.10%	99.13%	98.64%	98.75%	98.77%	97.63%	96.83%	98.95%									
				Average of TIL	217	250	246	244	280	274	272	350	412	257									
			90	Average of ExpectedCost	\$51.62	\$87.27	\$83.39	\$81.14	\$121.85	\$116.03	\$112.42	\$198.15	\$292.40	\$96.43									
				Average of %CostError	0.024%	-0.005%	0.207%	0.082%	0.050%	0.061%	0.240%	-0.008%	-0.027%	0.092%									
				Average of %UnitsImbalance	0.000%	0.000%	0.000%	0.034%	0.000%	0.005%	0.025%	0.001%	0.023%	0.009%									
				Average of ActualFill	99.94%	99.92%	99.92%	99.92%	99.89%	99.89%	99.90%	99.85%	99.64%	99.91%									
				Average of TIL	252	265	263	262	279	276	274	316	364	269									
2	2	0.1	9	Average of ExpectedCost	\$40.08	\$82.01	\$78.73	\$76.29	\$116.87	\$111.27	\$107.17	\$189.65	\$278.21	\$90.51									
				Average of %CostError	-0.122%	0.015%	0.061%	0.103%	0.012%	0.007%	0.074%	0.026%	-0.018%	0.019%									
				Average of %UnitsImbalance	0.000%	0.001%	0.001%	0.018%	0.000%	0.005%	0.027%	0.006%	0.081%	0.008%									
				Average of ActualFill	99.35%	98.66%	98.67%	98.71%	97.75%	97.95%	98.07%	96.22%	93.93%	98.38%									
				Average of TIL	303	331	329	326	367	361	357	458	584	343									
			90	Average of ExpectedCost	\$61.30	\$129.29	\$124.41	\$120.92	\$188.45	\$180.21	\$174.58	\$320.36	\$492.06	\$145.34									
				Average of %CostError	0.555%	-0.054%	0.175%	0.166%	-0.005%	0.037%	0.178%	0.031%	0.052%	0.144%									
				Average of %UnitsImbalance	0.000%	0.001%	0.000%	0.016%	0.000%	0.006%	0.021%	0.008%	0.147%	0.008%									
				Average of ActualFill	99.94%	99.89%	99.88%	99.88%	99.82%	99.82%	99.82%	99.69%	99.41%	99.86%									
				Average of TIL	222	240	238	236	261	258	255	304	351	246									
			0.5	Average of ExpectedCost	\$50.26	\$85.65	\$82.70	\$80.52	\$118.47	\$112.90	\$109.14	\$190.37	\$278.75	\$94.32									
				Average of %CostError	0.122%	-0.051%	-0.079%	0.041%	0.082%	0.015%	0.045%	0.024%	0.112%	0.025%									
				Average of %UnitsImbalance	0.000%	0.004%	0.006%	0.065%	0.000%	0.031%	0.070%	0.013%	0.159%	0.026%									
				Average of ActualFill	99.03%	98.39%	98.47%	98.47%	97.70%	97.86%	97.91%	96.02%	93.58%	98.19%									
				Average of TIL	271	303	300	297	343	336	332	450	584	316									
			90	Average of ExpectedCost	\$83.00	\$138.30	\$134.01	\$130.87	\$192.42	\$184.32	\$178.99	\$320.54	\$492.06	\$154.11									
				Average of %CostError	-0.258%	0.264%	0.065%	0.512%	0.068%	-0.047%	0.406%	0.067%	0.259%	0.144%									
				Average of %UnitsImbalance	0.000%	0.005%	0.003%	0.068%	0.000%	0.052%	0.276%	0.013%	0.148%	0.058%									
				Average of ActualFill	99.91%	99.86%	99.87%	99.86%	99.80%	99.81%	99.81%	99.66%	99.41%	99.84%									
				Average of TIL	214	240	238	236	261	258	255	304	351	245									
			0.9	Average of ExpectedCost	\$52.76	\$85.65	\$82.70	\$80.52	\$118.47	\$112.90	\$109.14	\$190.37	\$278.75	\$94.67									
				Average of %CostError	-0.228%	0.021%	-0.042%	0.047%	0.083%	-0.011%	0.097%	0.025%	0.112%	-0.003%									
				Average of %UnitsImbalance	0.000%	0.004%	0.006%	0.065%	0.000%	0.031%	0.070%	0.013%	0.159%	0.026%									
				Average of ActualFill	98.82%	98.38%	98.46%	98.47%	97.70%	97.87%	97.90%	96.02%	93.58%	98.16%									
				Average of TIL	260	299	295	292	343	336	332	450	584	312									
			90	Average of ExpectedCost	\$95.42	\$139.22	\$135.14	\$132.19	\$192.42	\$184.32	\$178.99	\$320.54	\$492.06	\$156.31									
				Average of %CostError	0.017%	0.056%	0.191%	0.860%	0.088%	0.080%	0.615%	0.077%	0.391%	0.262%									
				Average of %UnitsImbalance	0.000%	0.012%	0.011%	0.187%	0.000%	0.052%	0.276%	0.013%	0.148%	0.075%									
				Average of ActualFill	99.88%	99.86%	99.86%	99.85%	99.80%	99.81%	99.81%	99.66%	99.41%	99.83%									
				Average of TIL	379	389	388	387	405	400	398	444	495	394									
5	5	0.1	9	Average of ExpectedCost	\$73.14	\$142.88	\$138.48	\$133.57	\$204.12	\$194.43	\$186.09	\$332.44	\$487.40	\$158.65									
				Average of %CostError	0.024%	0.303%	-0.081%	0.045%	0.159%	-0.047%	0.085%	0.087%	0.047%	0.082%									
				Average of %UnitsImbalance	0.000%	0.003%	0.007%	0.021%	0.003%	0.028%	0.055%	0.078%	0.463%	0.021%									
				Average of ActualFill	98.79%	97.40%	97.43%	97.55%	95.78%	96.04%	96.30%	92.21%	87.11%	96.89%									
				Average of TIL	502	534	531	528	582	573	566	733	970	551									
			90	Average of ExpectedCost	\$113.38	\$233.89	\$227.49	\$220.42	\$347.37	\$333.45	\$320.68	\$610.93	\$967.40	\$267.72									
				Average of %CostError	-0.281%	-0.048%	0.082%	-0.058%	0.191%	0.247%	0.140%	0.190%	0.081%	0.038%									
				Average of %UnitsImbalance	0.000%	0.002%	0.004%	0.019%	0.001%	0.027%	0.081%	0.058%	0.680%	0.026%									
				Average of ActualFill	99.90%	99.75%	99.76%	99.76%	99.60%	99.62%	99.63%	99.14%	98.63%	99.70%									
				Average of TIL	303	321	319	317	345	339	335	409	478	328									
			0.5	Average of ExpectedCost	\$107.10	\$161.53	\$157.87	\$153.94	\$214.29	\$205.59	\$198.44	\$335.41	\$488.59	\$176.29									
				Average of %CostError	0.435%	0.352%	0.017%	0.530%	0.344%	0.482%	1.492%	0.375%	0.165%	0.520%									
				Average of %UnitsImbalance	0.000%	0.024%	0.054%	0.161%	0.021%	0.276%	0.561%	0.309%	1.080%	0.164%									
				Average of ActualFill	97.91%	96.85%	96.94%	96.95%	95.58%	95.70%	95.60%	92.22%	87.32%	96.37%									
				Average of TIL	422	457	454	450	513	503	495	699	950	478									
			90	Average of ExpectedCost	\$181.08	\$270.70	\$265.41	\$259.78	\$366.25	\$353.50	\$342.27	\$614.33	\$967.85	\$301.61									
				Average of %CostError	-0.271%	0.814%	-0.788%	2.048%	0.317%	0.094%	1.200%	0.333%	0.178%	0.490%									
				Average of %UnitsImbalance	0.000%	0.020%	0.029%	0.157%	0.011%	0.187%	0.532%	0.289%	1.017%	0.141%									
				Average of ActualFill	99.81%	99.69%	99.72%	99.68%	99.58%	99.59%	99.57%	99.21%	98.55%	99.65%									
				</																			

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