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Complications with Stochastic Volatility Models

by

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Abstract

We show a class of stochastic volatility price models for which the most natural candidates for martingale measures are only strictly local martingale measures, contrary to what is usually assumed in the finance literature. We also show the existence of martingale measures, however, and give explicit examples.

1 Introduction

In the mathematical finance literature there is a growing interest in the study of incomplete markets, and among the efforts in understanding their properties the stochastic volatility models have become increasingly popular: see for example Hull and White (87), Scott (87), Wiggins (87), Johnson and Shanno (87), Stein and Stein (91), Heston (93), Dupire (92), Hofmann, Platen and Schweizer (92) among others. They all start by imposing certain dynamics for a “volatility process” and construct asset prices as stochastic-exponentials of integrals of this volatility with respect to a Brownian motion.

If we want to apply results from arbitrage-pricing theory, which characterizes *reasonable* price functionals as discounted expected values with respect to certain probability measures (see Harrison and Kreps (79) and Harrison and Pliska (81)), to these models, the first property we need to verify is the existence of at least one probability measure equivalent to the original and under which prices of all traded assets, relative to a given one called the *numeraire*, become martingales (this is called a martingale measure). The existence of this measure is a sufficient condition to prevent *arbitrage opportunities*: strategies to trade and make money with zero initial cost and no risk of loss, and obviously inconsistent with economic equilibrium. A proof of the fact that the existence of this measure is also necessary can be found in Dalang, Morton and Willinger (90) or more general in Schachermayer (94) in the case when the time index set is discrete. For continuous time processes see Delbaen (92) and Delbaen and Schachermayer (94a) and references therein, which give appropriate definitions of *arbitrage* in continuous time for bounded price processes.

In this paper we define a class of models which includes several widely known in the finance literature as particular cases and show that, surprisingly, in some cases the most natural candidates for martingale measures are **local** martingale measures but **not** martingale measures. On the other hand, we show that, at least when coefficients are constant, there still exists a martingale measure, so *arbitrage opportunities* don't exist and the models are consistent with economic equilibrium.

2 Model

We begin with a fairly general stochastic volatility framework that captures several of the known models. Start with a 2-dimensional Brownian motion $B_t = (B_t^{(1)}, B_t^{(2)})$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ that satisfies the usual conditions, with $\mathcal{F}_0$ consisting only of sets with probability zero or one.

Suppose there exist diffusion processes (S_t, v_t) on $(\Omega, \mathcal{F}, P)$ (see Remark 2.2), $v_t \geq 0$ a.s., that satisfy the equations:

$$(1) \quad \begin{aligned} dS_t &= S_t v_t^\alpha (\sigma_t^{(1)} dB_t^{(1)} + \sigma_t^{(2)} dB_t^{(2)}) + S_t \mu_t(\omega) dt, & S_0 &= x \\ dv_t &= v_t (a_t^{(1)} dB_t^{(1)} + a_t^{(2)} dB_t^{(2)}) + \rho_t (L_t - v_t) dt, & v_0 &= 1 \end{aligned}$$

Here α is a positive constant, $\sigma_t = (\sigma_t^{(1)}, \sigma_t^{(2)})$ and $a_t = (a_t^{(1)}, a_t^{(2)})$ are deterministic vectors defining the direction and strength of the diffusion component increments of S_t and v_t respectively. S_t is the price of a given asset at time t , with expected rate of return μ_t and random volatility $v_t^\alpha |\sigma_t|$. v_t is the process driving the volatility function which in turn is a diffusion with linear coefficients and a mean reversion drift term.

If we let $\alpha = 1/2$ we obtain the model proposed by Hull and White (87) and further studied by Dupire (92). If $\alpha = 1$ we get the model suggested by Wiggins (87) and an example from Eisenberg and Jarrow (94).

Remark 2.1. Observe that α is almost a redundant parameter. Its effect can be embedded in the drift and diffusion coefficients of v_t if we assume a slightly more general form of the drift function. We keep α and the simple form of the drift to emphasize the relationship with existing models.

Notation: From now on we will write $\sigma_t dB_t$ to represent $\sigma_t^{(1)} dB_t^{(1)} + \sigma_t^{(2)} dB_t^{(2)}$ and $a_t dB_t$ is defined similarly. For any vectors $x, y \in \mathbb{R}^2$ let $|x| = \sqrt{x_1^2 + x_2^2}$ denote the Euclidean norm and $(x \cdot y) = x_1 y_1 + x_2 y_2$ the inner product.

We also impose the following restrictions on the time-dependent coefficients:

$$\begin{aligned} \sigma : \mathbb{R}^+ &\rightarrow \mathbb{R}^2, & 0 < |\sigma_t| < M & \text{ for all } t \\ a : \mathbb{R}^+ &\rightarrow \mathbb{R}^2, & 0 < |a_t| < M & \text{ for all } t \\ \rho : \mathbb{R}^+ &\rightarrow \mathbb{R}, & 0 \leq \rho_t < M & \text{ for all } t \\ L : \mathbb{R}^+ &\rightarrow \mathbb{R}, & 0 < L_t < M & \text{ for all } t \\ \mu : \mathbb{R}^+ \times \Omega &\rightarrow \mathbb{R} & \mathcal{F}_t\text{-adapted and a.s. locally bounded.} \\ \alpha &> 0. \end{aligned}$$

for some $M \in \mathbb{R}^+$.

Remark 2.2. There exists exactly one solution to (1). v_t should satisfy the 1-dimensional stochastic differential equation

$$dv_t = v_t |a_t| d\bar{B}_t + \rho_t (L_t - v_t) dt, \quad v_0 = 1$$

with respect to the Brownian motion

$$\overline{B}_t = \int_0^t \frac{a_u^{(1)} dB_u^{(1)} + a_u^{(2)} dB_u^{(2)}}{\sqrt{(a_u^{(1)})^2 + (a_u^{(2)})^2}}$$

and the drift and diffusion coefficients are Lipschitz continuous with linear growth, so we can apply Itô's existence theorem to find v_t (unique by Lipschitz continuity) and obtain S_t as the unique exponential semimartingale of $\int v_t^\alpha \sigma_t dB_t + \int \mu_t dt$ starting at x (see Elliot (82) Theorem 13.5). We will show later that $v_t > 0$ for all $t \in \mathbb{R}^+$ (see Proposition 4.1).

Let $a_t^\perp$ be a vector in $\mathbb{R}^2$ orthogonal to a_t , with the same magnitude, and

$$\delta_t = \frac{\mu_t}{v_t^\alpha (a_t^\perp \cdot \sigma_t)} a_t^\perp$$

if a_t and σ_t are not parallel for any $t \in \mathbb{R}^+$ then δ_t is finite and locally bounded a.s. Denote

$$G_t = \exp\left(-\int_0^t \delta_t dB_t + \frac{1}{2} \int_0^t \delta_t^2 dt\right)$$

Assumption 2.3. $|(a_t \cdot \sigma_t)| < |a_t| |\sigma_t|$ for all t (the vectors are not parallel) and the exponential local martingale G_t is in fact a martingale.

Assumption 2.3 holds if, for example, δ_t is a.s. uniformly bounded on $[0, T]$ for all $T \in \mathbb{R}^+$. We can then define the new measure Q on $(\Omega, \mathcal{F}_t)$ for every $t \in \mathbb{R}^+$ as

$$Q(A) = P(1_A G_t) \quad \text{for all } A \in \mathcal{F}_t$$

and Girsanov's Theorem (Karatzas and Shreve (88) Theorem 3.5.1) asserts that under Q the process

$$W_t = B_t + \int_0^t \delta_t dt$$

is a Brownian motion; therefore we can obtain the processes S_t, v_t under the measure Q as the only solutions to the equations

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma_t dW_t, & S_0 &= x \\ dv_t &= v_t a_t dW_t + \rho_t (L_t - v_t) dt, & v_0 &= 1 \end{aligned}$$

3 Results

We will study the simplest case, ignoring the effect of interest rates and assuming all processes are time homogeneous.

Assumption 3.1. *We suppose that the spot interest rate is zero and that the coefficients are constant in time: $\sigma_t \equiv \sigma$, $a_t \equiv a$, $\rho_t \equiv \rho$ and $L_t \equiv L$, for all t .*

Our main theorem can be stated as follows

Theorem 3.2. *Let W_t be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, Q)$, and let (S_t, v_t) satisfy*

$$\begin{aligned} dS_t &= S_t v_t^\alpha (\sigma^{(1)} dW_t^{(1)} + \sigma^{(2)} dW_t^{(2)}), & S_0 &= x \\ dv_t &= v_t (a^{(1)} dW_t^{(1)} + a^{(2)} dW_t^{(2)}) + \rho(L - v_t)dt, & v_0 &= 1 \end{aligned}$$

Then S_t is a martingale under Q if and only if $(a \cdot \sigma) \leq 0$.

Remark 3.3. The explicit form of the drift coefficient for v_t is not essential to the proof; the same result can be obtained for most drifts which are linear or bounded functions of v_t . Even for arbitrary bounded drift we obtain one of the implications as the following corollary shows.

Corollary 3.4. *Let W_t^b be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, Q^b)$, and (S_t, v_t) satisfy*

$$(2) \quad \begin{aligned} dS_t &= S_t v_t^\alpha (\sigma^{(1)} dW_t^{b(1)} + \sigma^{(2)} dW_t^{b(2)}), & S_0 &= x \\ dv_t &= v_t (a^{(1)} dW_t^{b(1)} + a^{(2)} dW_t^{b(2)}) + \rho(L - v_t)dt + b_t(\omega)dt, & v_0 &= 1 \end{aligned}$$

Assume $(a \cdot \sigma) > 0$ and for all $T \in \mathbb{R}^+$ b_t/v_t is bounded on $[0, T] \times \Omega$. Then S_t is a strictly local martingale under Q^b .

Remark 3.5. Observe that, when the filtration $\mathcal{F}_t$ is generated by the Brownian motion B_t , Girsanov's theorem implies that for every $T \in \mathbb{R}^+$ and every martingale measure Q^b on $\mathcal{F}_T$ equivalent to Q there exist b_t and W_t^b such that (S_t, v_t) satisfy (2) on $[0, T]$. The condition b_t/v_t being bounded means Q^b is "close" to Q . Many interesting measures, for example, the *minimal martingale measure* introduced by Hofmann, Platen and Schweizer (92), if it exists, should satisfy this condition, as we will show in the next section.

This result has very important economic consequences: if S_t is a positive strictly local martingale Fatou's lemma implies it is a strict supermartingale. Thus

price functionals obtained as conditional expected values with respect to Q^b give systematically lower than market value prices for the stock. This is not an indication of *arbitrage opportunities*, however, even when there are enough instruments traded to complete the market: Fatou's lemma implies again that any investor who wants to take advantage of the mispricing must keep a portfolio reaching arbitrarily negative values before expiration. Besides the practical problem that such plan will very soon reach the point where "your friendly broker calls for extra margin," in the theory of continuous trading it is necessary to prohibit this kind of strategies in order to obtain economic equilibrium (see Harrison and Pliska (81) for the details).

In the extreme case when σ is parallel to a all local martingale measures coincide on the filtration generated by the stock and volatility processes and, using the same techniques that appear here, it is possible to show that this measure is not a martingale measure for the stock S_t whenever $(a \cdot \sigma) > 0$. Other examples of unique strictly local martingale measures appeared in Delbaen and Schachermayer (94b). Even in this case there are no *arbitrage opportunities* whose values remain bounded below or, more generally, there is no free-lunch-with-vanishing-risk, as defined and proved in Delbaen and Schachermayer (94a) as long as there exists at least one local martingale measure.

However, there still exist martingale measures whenever σ and a are not parallel, as shown in the following theorem

Theorem 3.6. *Let W_t be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, Q)$, and (S_t, v_t) solve*

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dW_t, & S_0 &= x \\ dv_t &= v_t a dW_t + \rho(L - v_t)dt, & v_0 &= 1 \end{aligned}$$

For every $T \in \mathbb{R}^+$ there exist a probability measure Q^a equivalent to Q on $\mathcal{F}_T$ with respect to which S_t is a martingale on $[0, T]$. If $(a \cdot \sigma) > 0$ we can find Q^a under which S_t satisfies

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dW_t^a, & S_0 &= x \\ dv_t &= v_t a dW_t^a + (a \cdot \sigma) \left(\frac{\rho L}{(a \cdot \sigma)} - v_t^{1+\alpha} \right) dt, & v_0 &= 1 \end{aligned}$$

on $t \in [0, T]$, where W_t^a is a Brownian motion under Q^a .

4 Proofs

Notation: We will use the same symbol $(Q, Q^a, Q^b, \dots)$ to indicate both probability and expected value, for example, $Q(A)$ when A is a set means the

probability of A whereas $Q(X)$ when X is a random variable means the expected value of X ; this makes the formulas simpler and clearer when there are several measures on the same space.

We will show first that $v_t \geq 0$ a.s. for all t , so v_t^α is defined, as promised in the statement of the model.

Proposition 4.1. *Suppose v_t satisfies*

$$dv_t = v_t a_t dW_t + \rho_t(L_t - v_t)dt, \quad v_0 = 1$$

then $v_t > 0$ a.s. for all $t \in \mathbb{R}^+$.

Proof: Let $\tilde{v}_t$ be the unique solution to the SDE

$$d\tilde{v}_t = \tilde{v}_t a_t dW_t - \rho_t \tilde{v}_t dt, \quad \tilde{v}_0 = 1$$

then $\tilde{v}_t$ is a semimartingale exponential given by

$$\tilde{v}_t = \exp \left(\int_0^t a_u dW_u - \int_0^t \left(\rho_u + \frac{1}{2} |a_u|^2 \right) dt \right)$$

and the comparison theorem (see for example Ikeda and Watanabe (89) VI, Theorem 1.1) implies $v_t \geq \tilde{v}_t > 0$ for all t . $\square$

Then Theorem 3.2 is an immediate consequence of the following two results

Lemma 4.2. *If (S_t, v_t) satisfies*

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dW_t, & S_0 &= x \\ dv_t &= v_t a_t dW_t + \rho(L - v_t)dt, & v_0 &= 1 \end{aligned}$$

then S_t is a supermartingale and for every $T \in \mathbb{R}^+$

$$Q(S_T) = S_0 Q(\hat{v}_t \text{ doesn't explode on the interval } [0, T])$$

where $\hat{v}_t$ is the unique solution up to an explosion time to the SDE

$$d\hat{v}_t = \hat{v}_t a_t dW_t + \rho(L - \hat{v}_t)dt + \hat{v}_t^{\alpha+1} (a \cdot \sigma)dt \quad \hat{v}_0 = 1.$$

Lemma 4.3. *The (unique) solution to the equation*

$$d\hat{v}_t = \hat{v}_t a_t dW_t + \rho(L - \hat{v}_t)dt + \hat{v}_t^{\alpha+1} (a \cdot \sigma)dt \quad \hat{v}_0 = 1.$$

explodes to $+\infty$ in finite time with positive probability if and only if $(a \cdot \sigma) > 0$

Proof of Theorem 3.2:

S_t is a martingale if and only if it is a supermartingale and $Q(S_t) = S_0$ for all $t \in \mathbb{R}^+$, and by Lemmas 4.2 and 4.3 this happens if and only if $(a \cdot \sigma) \leq 0$. $\square$

Proof of Lemma 4.2:

S_t is the integral of a predictable process with respect to a local martingale, so it is again a local martingale, and it is also positive because it is a solution to the equation of the semimartingale-exponential of $\int v_t^\alpha \sigma dW_t$ (see for example Karatzas and Shreve (88) example 3.3.9). Define a sequence of stopping times

$$\tau_n = \inf \left\{ t \in \mathbb{R}^+ : \int_0^t v_t^{2\alpha} |\sigma|^2 dt \geq n \right\}$$

then $S_t^{(n)} = S_{t \wedge \tau_n}$ is a local martingale under Q for $n = 1, 2, \dots$. Define $Z_T^{(n)} = \int_0^{t \wedge \tau_n} v_t^\alpha \sigma dW_t$. Then $S_t^{(n)}$ is the semimartingale-exponential of $Z_t^{(n)}$ started at x ,

$$dS_t^{(n)} = S_t^{(n)} dZ_t^{(n)} \quad S_0 = x.$$

and $\langle Z^{(n)} \rangle_t \leq n$ for all t , so $S_t^{(n)}$ is a positive martingale for $n = 1, 2, \dots$ and the sequence $(\tau_n)_{n=1,2,\dots}$ reduces S_t . Then Fatou's Lemma implies, for $0 \leq u < t$,

$$\begin{aligned} Q(S_t | \mathcal{F}_u) &= Q(\liminf_{n \rightarrow \infty} S_t^{(n)} | \mathcal{F}_u) \\ &\leq \liminf_{n \rightarrow \infty} Q(S_t^{(n)} | \mathcal{F}_u) \\ &= S_u. \end{aligned}$$

so S_t is a supermartingale under Q .

Now fix $T \in \mathbb{R}^+$ and define the new probability measure Q_n on $(\Omega, \mathcal{F}_T)$ as

$$Q_n(A) = \frac{1}{S_0} Q(S_T^{(n)} 1_A) \quad \text{for all } A \in \mathcal{F}_T$$

then the Lebesgue Dominated convergence Theorem and Girsanov's Theorem imply that for every set Γ in the Borel sigma field on the space of continuous paths in $[0, T]$,

$$\begin{aligned} Q(S_T 1_{\{W \in \Gamma\}}) &= \lim_{n \rightarrow \infty} Q(S_T^{(n)} 1_{\{W \in \Gamma, \tau_n \geq T\}}) \\ &= S_0 \lim_{n \rightarrow \infty} Q_n(1_{\{W \in \Gamma, \tau_n \geq T\}}) \\ &= S_0 \lim_{n \rightarrow \infty} Q(1_{\{\widehat{W} \in \Gamma, \widehat{\tau}_n \geq T\}}) \end{aligned}$$

where $\widehat{W}_t$ and $\widehat{v}_t$ are defined as the (unique in law) solutions to the following stochastic differential equations

$$\begin{aligned} d\widehat{W}_t &= dW_t + \widehat{v}_t^\alpha \sigma dt, & \widehat{W}_0 &= 0 \\ d\widehat{v}_t &= \widehat{v}_t a dW_t + \rho(L - \widehat{v}_t) + \widehat{v}_t^{\alpha+1} (a \cdot \sigma) dt & \widehat{v}_0 &= 1. \end{aligned}$$

and $\hat{\tau}_n$ is given by

$$\hat{\tau}_n = \inf \left\{ t \in \mathbb{R}^+ : \int_0^t \hat{v}_s^{2\alpha} |\sigma|^2 ds \geq n \right\}$$

The existence of a weak solution $(\widehat{W}_t, \hat{v}_t)$ to these equations up to an explosion time $\hat{\tau}_\infty = \lim_{n \rightarrow \infty} \hat{\tau}_n$ follows from the continuity of the drift and diffusion coefficients (see Ikeda-Watanabe(89) IV.Theorem 2.3); we even get pathwise uniqueness on $\{(x, y) : y > 0\}$ because the coefficients of the SDE are locally Lipschitz (see Ikeda-Watanabe (89) IV.Theorem 3.1), so strong solutions always exist and are unique up to time $\hat{\tau}_\infty$. Applying Girsanov's theorem we obtain that the process

$$W_t^{(n)} = W_t - \int_0^t 1_{\{u \leq \tau_n\}} v_u^\alpha \sigma du$$

is a Brownian motion under Q_n and W_t and v_t satisfy

$$\begin{aligned} dW_t &= dW_t^{(n)} + 1_{\{t \leq \tau_n\}} v_t^\alpha \sigma dt, & W_0 &= 0 \\ dv_t &= v_t a dW_t^{(n)} + \rho(L - v_t) dt + 1_{\{t \leq \tau_n\}} v_t^{\alpha+1} (a \cdot \sigma) dt & v_0 &= 1. \end{aligned}$$

which is the same as the previous equation up to time τ_n , and because $\{W \in \Gamma, \tau_n \geq T\} \in \mathcal{F}_{\tau_n}$ the identity follows.

Now let $\Gamma = C[0, T]$ to get

$$\begin{aligned} Q(S_T) &= S_0 \lim_{n \rightarrow \infty} Q(1_{\{\hat{\tau}_n \geq T\}}) \\ &= S_0 Q(\hat{\tau}_n \geq T \text{ for some } n) \\ &= S_0 Q(\hat{v}_t \text{ doesn't explode until after time } T) \quad \square \end{aligned}$$

Proof of Lemma 4.3:

We will apply Feller's test to $\hat{v}_t$ to check whether it explodes in finite time. Notice the test is applicable because $\hat{v}_t$ is a 1-dimensional Ito diffusion with respect to the Brownian motion $(a/|a|)B_t$ and the volatility and drift coefficients are locally bounded above and away from zero on the interval $(0, \infty)$. (see for example Karatzas and Shreve (88) section 5.5). The careful reader might notice that, in the statement of Feller's test, explosion means that the process escapes from the interval it is defined on, $(0, \infty)$ for $\hat{v}_t$, whereas here by explosion we mean that the process tends to $+\infty$. However, we will show it's impossible that $\hat{v}_t$ reaches zero in finite time, but at this point we should be aware of the distinction.

First we need to calculate the *scale function*, for $x \in (0, \infty)$:

$$\begin{aligned} p(x) &= \int_1^x \exp \left(-2 \int_1^y \frac{(\rho L - \rho z - (a \cdot \sigma) z^{1+\alpha})}{z^2 |a|^2} dz \right) dy \\ &= C \int_1^x \exp \left(\frac{2\rho L}{|a|^2} y^{-1} + \frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma) y^\alpha}{|a|^2 \alpha} \right) dy \end{aligned}$$

$$p'(x) = C \exp \left(\frac{2\rho L}{|a|^2} x^{-1} + \frac{2\rho}{|a|^2} \log(x) - \frac{2(a \cdot \sigma) x^\alpha}{|a|^2 \alpha} \right)$$

so

$$\begin{aligned} p(0+) &= -C \int_0^1 \exp \left(\frac{2\rho L}{|a|^2} y^{-1} + \frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma) y^\alpha}{|a|^2 \alpha} \right) dy \\ &\sim - \int_0^1 \exp \left(\frac{2\rho L}{|a|^2} y^{-1} \right) dy \\ &= - \int_1^\infty \frac{\exp \left(\frac{2\rho L}{|a|^2} u \right)}{u^2} dy \\ &= \begin{cases} -\infty & \text{if } \rho > 0 \\ > -\infty & \text{if } \rho = 0 \end{cases} \end{aligned}$$

$$\begin{aligned} p(+\infty) &= C \int_1^\infty \exp \left(\frac{2\rho L}{|a|^2} y^{-1} + \frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma) y^\alpha}{|a|^2 \alpha} \right) dy \\ &\sim \int_1^\infty \exp \left(-\frac{2(a \cdot \sigma) y^\alpha}{|a|^2 \alpha} \right) dy \\ &= \begin{cases} +\infty & \text{if } (a \cdot \sigma) \leq 0 \\ < +\infty & \text{if } (a \cdot \sigma) > 0 \end{cases} \end{aligned}$$

Now we will distinguish 2 cases:

CASE 1: $(a \cdot \sigma) \leq 0$.

Here we have $p(+\infty) = +\infty$ so $\hat{v}_t$ never explodes to $+\infty$ in finite time (see Karatzas and Shreve (88) Proposition 5.5.22). Depending on the value of ρ we get

$$\text{If } \rho = 0 \quad \text{then} \quad \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = 0 \quad \sup_{0 \leq t < \hat{\tau}_\infty} \hat{v}_t < \infty$$

$$\text{If } \rho > 0 \quad \text{then} \quad \inf_{0 \leq t < \hat{\tau}_\infty} \hat{v}_t = 0 \quad \sup_{0 \leq t \leq \hat{\tau}_\infty} \hat{v}_t = \infty \quad \hat{\tau}_\infty = +\infty \text{ a.s.}$$

CASE 2: $(a \cdot \sigma) > 0$.

Again depending on the value of ρ we get two cases

$$\text{If } \rho = 0 \text{ then } 0 < Q \left(\lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = \infty \right) = 1 - Q \left(\lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = 0 \right) < 1.$$

$$\text{If } \rho > 0 \text{ then } \inf_{0 \leq t < \hat{\tau}_\infty} \hat{v}_t > 0 \quad \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = +\infty \text{ a.s.}$$

To see if $Q(\hat{\tau}_\infty < \infty) > 0$ we study the behavior of the following function at the endpoints

$$\begin{aligned} u(x) &= \int_1^x p'(y) \int_1^y \frac{2}{p'(z)|a|^2 z^2} dz dy \\ &= \frac{2}{|a|^2} \int_1^x \frac{1}{p'(z)z^2} \int_z^x p(y) dy dz \\ &= \frac{2}{|a|^2} \int_1^x \frac{\exp \left(\frac{-2\rho L}{|a|^2} z^{-1} - \frac{2\rho}{|a|^2} \log(z) + \frac{2(a \cdot \sigma)z^\alpha}{|a|^2 \alpha} \right)}{|a|^2 z^2} \\ &\quad \cdot \int_z^x \exp \left(\frac{2\rho L}{|a|^2} y^{-1} + \frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma)y^\alpha}{|a|^2 \alpha} \right) dy dz \end{aligned}$$

where the second equation is justified by Fubini's Theorem.

Now when $x > 1$

$$\begin{aligned} u(x) &\leq \frac{2}{|a|^2} \int_1^x \frac{\exp \left(\frac{-2\rho}{|a|^2} \log(z) + \frac{2(a \cdot \sigma)z^\alpha}{|a|^2 \alpha} \right)}{|a|^2 z^2} \int_z^x \exp \left(\frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma)y^\alpha}{|a|^2 \alpha} \right) dy dz \\ &\sim \frac{2}{|a|^2} \int_1^x \frac{\exp \left(\frac{(a \cdot \sigma)z^\alpha}{|a|^2 \alpha} \right)}{|a|^2 z^2} \int_z^x \exp \left(\frac{-(a \cdot \sigma)y^\alpha}{|a|^2 \alpha} \right) dy dz \end{aligned}$$

because the function $g(x) = \frac{2\rho}{|a|^2} \log(x) - \frac{(a \cdot \sigma)x^\alpha}{|a|^2 \alpha}$ is decreasing on $x \geq M$ for some $M \in \mathbb{R}^+$; so

$$\left(\frac{2\rho}{|a|^2} \log(y) - \frac{(a \cdot \sigma)y^\alpha}{|a|^2 \alpha} \right) - \left(\frac{2\rho}{|a|^2} \log(z) - \frac{(a \cdot \sigma)z^\alpha}{|a|^2 \alpha} \right) \leq 0$$

on $M \leq z \leq y$. Let $\beta = (a \cdot \sigma)/|a|^2$ and n be the smallest positive integer greater than or equal to $\gamma = (1 - \alpha)/\alpha$. Using integration by parts several times we obtain the following majorant for the inner integral above,

$$\begin{aligned}
\int_z^x e^{-(\beta/\alpha)y^\alpha} dy &= \int_{z^\alpha}^{x^\alpha} e^{-(\beta/\alpha)u} \frac{u^{(1-\alpha)/\alpha}}{\alpha} du \\
&\leq \int_{z^\alpha}^\infty e^{-(\beta/\alpha)u} \frac{u^\gamma}{\alpha} du \\
&= \sum_{k=0}^{n-1} \frac{\gamma(\gamma-1)\dots(\gamma-k+1)\alpha^k}{\beta^{k+1}} e^{-(\beta/\alpha)z^\alpha} z^{1-\alpha(1+k)} \\
&\quad + \int_{z^\alpha}^\infty \frac{\gamma(\gamma-1)\dots(\gamma-n+1)\alpha^{n-1}}{\beta^n} e^{-(\beta/\alpha)u} u^{\gamma-n} du \\
&\leq \sum_{k=0}^{n-1} \frac{\gamma(\gamma-1)\dots(\gamma-k+1)\alpha^k}{\beta^{k+1}} e^{-(\beta/\alpha)z^\alpha} z^{1-\alpha(1+k)} \\
&\quad + z^{1-\alpha(1+n)} \int_{z^\alpha}^\infty \frac{\gamma(\gamma-1)\dots(\gamma-n+1)\alpha^{n-1}}{\beta^n} e^{-(\beta/\alpha)u} du
\end{aligned}$$

so there are positive constants C_i such that

$$\begin{aligned}
u(x) &\sim C_1 \int_1^x \frac{\sum_{k=0}^n z^{1-\alpha-\alpha k}}{z^2} dz \\
&\leq C_2 \int_1^\infty z^{-(1+\alpha)} dz \\
&< \infty
\end{aligned}$$

for $x > 1$, so we get $u(\infty) < \infty$ and thus Feller's test implies $\hat{v}_t$ explodes in finite time with positive probability when $(a \cdot \sigma) > 0$.

We can check the value of u at zero as well; if $\rho > 0$ then $p(0+) = -\infty$ so $u(0+) = +\infty$ (see Karatzas and Shreve (88) Problem 5.5.27) and if $\rho = 0$ then for $x < 1$,

$$\begin{aligned}
u(x) &= \frac{2}{|a|^2} \int_x^1 \frac{e^{(\beta/\alpha)z^\alpha}}{z^2} \int_x^z e^{-(\beta/\alpha)y^\alpha} dy dz \\
&\geq \frac{2}{|a|^2} e^{-(|\beta|/\alpha)} \int_x^1 \frac{1}{z^2} \int_x^z dy dz \\
&= e^{-(|\beta|/\alpha)} \left(-\ln(x) + x \left(1 - \frac{1}{x} \right) \right) \\
&\rightarrow +\infty \quad \text{as } x \rightarrow 0+.
\end{aligned}$$

so in any case $u(0+) = +\infty$ and this with the previous result on $p(x)$ imply (see Proposition 5.5.32 in Karatzas and Shreve (88)) that $Q\{\hat{\tau}_\infty < \infty\} = 1$ if $\rho > 0$ and $0 < Q\{\hat{\tau}_\infty < \infty\} < 1$ if $\rho = 0$.

Our next step is to show that the event $\{\hat{\tau}_\infty < \infty, \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = 0\}$ has probability zero. Define $\tilde{v}_t$ as the unique solution to

$$d\tilde{v}_t = \tilde{v}_t a dW_t - \rho \tilde{v}_t dt, \quad \tilde{v}_0 = 1.$$

then

$$\tilde{v}_t = \exp \left(aW_t - \left(\rho + \frac{1}{2}|a|^2 \right) t \right)$$

and the comparison theorem (see for example Ikeda and Watanabe (89) VI, Theorem 1.1) implies that if $(a \cdot \sigma) > 0$ then $\hat{v}_t \geq \tilde{v}_t$ a.s., so $Q\{\hat{\tau}_\infty < \infty, \lim_{t \rightarrow \hat{\tau}_\infty} \tilde{v}_t = 0\} = 0$ implies $Q\{\hat{\tau}_\infty < \infty, \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = 0\} = 0$; therefore

$$\begin{aligned} & Q \left\{ \hat{\tau}_\infty < \infty, \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = \infty \right\} \\ &= Q \{ \hat{\tau}_\infty < \infty \} - Q \left\{ \hat{\tau}_\infty < \infty, \lim_{t \rightarrow \hat{\tau}_\infty} \hat{v}_t = 0 \right\} \\ &= Q \{ \hat{\tau}_\infty < \infty \} \\ &> 0. \end{aligned}$$

and $\hat{v}_t$ explodes to $+\infty$ in finite time with positive probability. $\square$

Now we are ready to prove the corollary

Proof of Corollary 3.4:

Suppose $(a \cdot \sigma) > 0$ and S_t is a martingale under Q^b . Fix $T \in \mathbb{R}^+$ and define the new equivalent measure $\tilde{Q}^b$ on $\mathcal{F}_T$ as

$$\tilde{Q}^b(A) = \frac{Q^b(1_A S_T)}{S_0} \quad \text{for all } A \in \mathcal{F}_T$$

Girsanov's theorem states that $\tilde{W}_t^b = W_t^b - \int_0^t 1_{\{u \leq T\}} v_u^\alpha \sigma du$ is a Brownian motion under $\tilde{Q}^b$ and v_t satisfies

$$dv_t = v_t a d\tilde{W}_t^b + \rho(L - v_t)dt + b_t dt + 1_{\{t \leq T\}} v_t^{\alpha+1} (a \cdot \sigma) dt, \quad v_0 = 1.$$

and because b_t/v_t is locally bounded (uniformly in ω) the process

$$M_t = \exp \left(- \int_0^t \left(\frac{b_t}{v_t |a|^2} \right) a d\tilde{W}_t^b - \frac{1}{2} \int_0^t \frac{|b_t|^2}{|v_t|^2 |a|^2} dt \right)$$

is a martingale under $\tilde{Q}^b$ and defining the equivalent measure Q^* with

$$\frac{dQ^*}{d\tilde{Q}^b} = M_T$$

we see $W_t^* = \widetilde{W}_t^b + \int_0^t 1_{\{u \leq T\}} \left(\frac{b_u}{v_u |a|^2} \right) adu$ is a Brownian motion under Q^* and v_t satisfies

$$dv_t = v_t adW_t^* + \rho(L - v_t)dt + v_t^{\alpha+1}(a \cdot \sigma)dt, \quad v_0 = 1.$$

on $[0, T]$ for arbitrary T ; which is a contradiction because we already proved that the only solution to this equation explodes in finite time with positive probability when $(a \cdot \sigma) > 0$. $\square$

One important martingale measure is the so called *minimal equivalent martingale measure* Q^* (see Hofmann, Platen and Schweizer(92)) under which all local martingales with respect to the original measure P orthogonal to S_t remain local martingales under Q^* . If Q^* exists we should get the equations

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dW_t^*, & S_0 &= x \\ dv_t &= v_t adW_t^* + \rho(L - v_t) - \frac{v_t(a \cdot \sigma)}{|\sigma|^2 v_t^\alpha} \mu_t dt, & v_0 &= 1 \end{aligned}$$

where W_t^* is a Brownian motion under Q^* . Then from Corollary 3.4 we obtain

Corollary 4.4. *Let W_t be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P)$, and (S_t, v_t) satisfy equation (1). If μ_t/v_t^α is locally bounded (uniformly in ω) then S_t doesn't admit a minimal martingale measure.*

Proof of Theorem 3.6:

In light of the previous results, the only case we still need to verify is $(a \cdot \sigma) > 0$. Let $\widetilde{Q}^a$ be given by

$$\frac{d\widetilde{Q}^a}{dQ} = \exp \left(- \int_0^T \frac{v_t^\alpha(a \cdot \sigma)}{(a \cdot \sigma^\perp)} \sigma^\perp dB_t - \frac{1}{2} \int_0^T \frac{v_t^{2\alpha}(a \cdot \sigma)^2 |\sigma^\perp|^2}{(a \cdot \sigma^\perp)^2} dt \right)$$

Observe that $\left(a \cdot \frac{(a \cdot \sigma)}{(a \cdot \sigma^\perp)} \sigma^\perp \right) = (a \cdot \sigma) > 0$ so the fact that the right hand side is a martingale in T is proven the same way as we did for S_t when $(a \cdot \sigma) \leq 0$. Then Girsanov's theorem implies that under $\widetilde{Q}_t^a$ the process $\widetilde{B}_t^a = B_t + \int_0^t \frac{v_t^\alpha(a \cdot \sigma)}{(a \cdot \sigma^\perp)} \sigma^\perp dt$ is a Brownian motion. Then do a new measure change with

$$\frac{dQ^a}{d\widetilde{Q}^a} = \exp \left(- \frac{\rho}{(a \cdot \sigma^\perp)} \sigma^\perp \widetilde{B}_T^a - \frac{1}{2} \frac{\rho^2 |\sigma^\perp|^2}{(a \cdot \sigma^\perp)^2} T \right)$$

and get the new Brownian Motion $B_t^a = \widetilde{B}_t^a + \int_0^t \frac{\rho}{(a \cdot \sigma^\perp)} \sigma^\perp dt$ under Q_t^a . Now

(S_t, v_t) satisfy

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dB_t^a, & S_0 &= x \\ dv_t &= v_t a dB_t^a + (a \cdot \sigma) \left(\frac{\rho L}{(a \cdot \sigma)} - v_t^{1+\alpha} \right), & v_0 &= 1 \end{aligned}$$

and following exactly the same argument as in Lemma 4.2 we obtain that S_t is a martingale under Q^a iff $\hat{v}_t$ doesn't explode to $+\infty$ in finite time, where $\hat{v}_t$ satisfies

$$d\hat{v}_t = \hat{v}_t a d\hat{B}_t + \rho L dt, \quad v_0 = 1$$

this is an equation with linear coefficients in $\hat{v}_t$, so it always has a nonexploding solution. $\square$

5 Conclusion

We have shown that for a class of models with stochastic volatility that is used in practice for derivatives pricing, the most natural candidates for martingale measures are only strictly local martingale measures. There exist martingale measures, however, as long as the diffusion coefficients for the stock and the volatility are not parallel.

The same techniques can be used to study the kind of models that appear in Hull and White (88), Johnson-Shanno (87) and Scott (92) where the volatility follows a process of constant elasticity of variance, and the limiting case where the volatility is given as an Ornstein-Uhlenbeck process, as proposed in Scott (87), Stein and Stein (92) and Heston (93). The results depend on the coefficient for the elasticity of variance, and they will appear somewhere else.

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