THE QQ—ESTIMATOR AND HEAVY TAILS

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ABSTRACT. A common visual technique for assessing goodness of fit and estimating location and scale is the qq–plot. We apply this technique to data from a Pareto distribution and more generally to data generated by a distribution with a heavy tail. A procedure for assessing the presence of heavy tails and for estimating the parameter of regular variation is discussed which can supplement other standard techniques such as the Hill plot.

1. Introduction.

A graphical technique called the qq-plot is a commonly used method of visually assessing goodness of fit and of estimating location and scale parameters. The method is standard and ubiquitious in various forms. See for example Rice (1988) and Castillo (1988). The method is based on the following simple observation: If

$$U_{1,n} \leq U_{2,n} \leq \dots U_{n,n}$$

are the order statistics from n iid observations which are uniformly distributed on $[0,1]$, then by symmetry

$$E(U_{i+1,n} - U_{i,n}) = \frac{1}{n+1}$$

and hence

$$EU_{i,n} = \frac{i}{n+1}.$$

Thus since $U_{i,n}$ should be close to its mean $i/(n+1)$, the plot of $\{(i/(n+1), U_{i,n}), 1 \leq i \leq n\}$ should be roughly linear. Now suppose that

$$X_{1,n} \leq X_{2,n} \leq \dots X_{n,n}$$

are the order statistics from an iid sample of size n which we suspect comes from a particular continuous distribution G. If our suspicion is correct, the plot of $\{(i/(n+1), G(X_{i,n})), 1 \leq i \leq n\}$ should be approximately linear and hence also the plot of $\{G^{\leftarrow}(i/(n+1), X_{i,n}), 1 \leq i \leq n\}$ should be approximately linear. Note $G^{\leftarrow}(i/(n+1))$ is a theoretical quantile and $X_{i,n}$ is the correponding quantile of the empirical distribution function and hence the name *qq-plot*.

Suppose we suspect the data comes from a location–scale family

$$G_{\mu,\sigma}(x) = G_{0,1}(\frac{x-\mu}{\sigma})$$

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where μ, σ are unknown. The plot of

$$\{(G^{\leftarrow}_{\mu,\sigma}(i/(n+1), X_{i,n}), 1 \le i \le n\}$$

should be approximately a line through 0 of slope 1 and since

$$G^{\leftarrow}_{\mu,\sigma}(y) = \sigma G^{\leftarrow}_{0,1}(y) + \mu$$

the plot of

$$\{(G^{\leftarrow}_{0,1}(i/(n+1), X_{i,n}), 1 \le i \le n\}$$

should be approximately a line of slope σ and intercept μ. Thus visually we can assess the goodness of fit of the location–scale family and provide estimates of μ, σ.

The relevance of this technique to heavy tails is the following. Suppose we suspect that $Z_{1,n} \le \cdots \le Z_{n,n}$ are the order statistics from a random sample from a Pareto family indexed by its shape parameter $\alpha > 0$:

$$F_\alpha(x) = 1 - x^{-\alpha}, \quad x \ge 1.$$

Then of course for $y > 0$

$$G_{0,\alpha}(y) := P[\log Z_1 > y] = e^{-\alpha y}$$

and the plot of

$$\{(G^{\leftarrow}_{0,1}(\frac{i}{n+1}), \log Z_{i,n}), 1 \le i \le n\} = \{(-\log(1 - \frac{i}{n+1}), \log Z_{i,n}), 1 \le i \le n\}$$

should be approximately a line with intercept 0 and slope α^{-1}.

If $\{(x_i, y_i), 1 \le i \le n\}$ are n points in the plane, a standard textbook calculation yields that the slope of the least squares line through these points is

$$SL(\{(x_i, y_i), 1 \le i \le n\}) = \frac{\bar{S}_{xy} - \bar{x}\bar{y}}{\bar{S}_{xx} - \bar{x}^2} \tag{1.1}$$

where as usual

$$S_{xy} = \sum_{i=1}^n x_i y_i, \quad S_{xx} = \sum_{i=1}^n x_i^2$$

and "bar" indicates average. Thus for the Pareto example, if we set

$$x_i = -\log(1 - \frac{i}{n+1}), \quad y_i = \log Z_{n,i},$$

then an estimator of α^{-1} is

$$\widehat{\alpha^{-1}} = \frac{\sum_{i=1}^n - \log(\frac{i}{n+1})\{n \log Z_{n-i+1,n} - \sum_{j=1}^n \log Z_{n-j+1,n}\}}{n\sum_{i=1}^n(-\log(\frac{i}{n+1}))^2 - (\sum_{i=1}^n - \log(\frac{i}{n+1}))^2}, \tag{1.2}$$

which we call the *qq–estimator*.

In Figure 1.1 we present a qq–plot of a random sample of size 1000 from the Pareto distribution with $\alpha = 1$. The qq–estimator gives an estimate of 0.9722397.

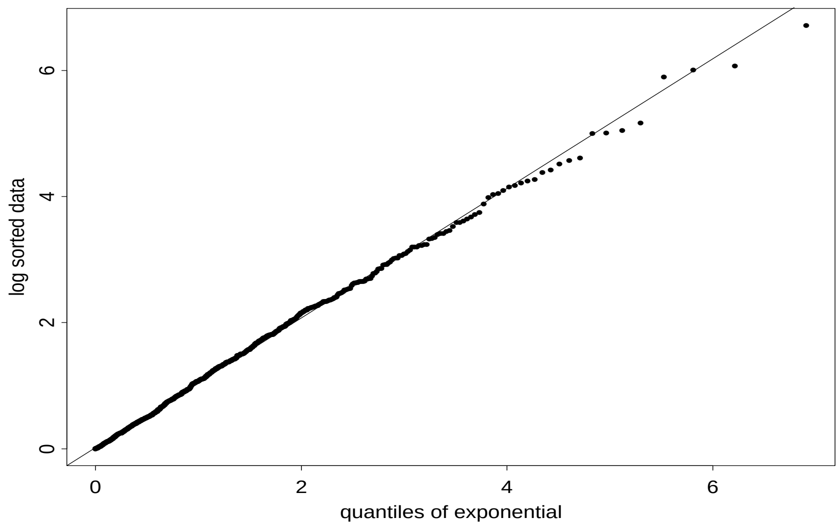

Figure 1.1

Because of the frequency with which the qq–plot is used, we sought to investigate the properties of the plot and the estimator when the underlying distribution was not exactly Pareto. If one wishes to drop the Pareto assumption but stay within the heavy tail class, one can assume the distribution is only Pareto from some point on (see, for example Feigin, Resnick and Starica (1994)) but we decided to make the more general and less ad-hoc assumption that tails were regularly varying. So we suppose we have a random sample $Z_1, \ldots, Z_n$ from a distribution F satisfying

$$1 - F(x) \sim x^{-\alpha} L(x), \quad (x \to \infty) \tag{1.3}$$

where L is a slowly varying function satisfying

$$\lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1.$$

We now modify (1.2) to make it suitable for the regularly varying case. Here is the rationale for the modification. Observe that since $1 - F$ is regularly varying, we have for large t.

$$\frac{1 - F(tx)}{1 - F(t)} = P[\frac{Z_1}{t} > x | Z_1 > t] \approx x^{-\alpha}. \tag{1.4}$$

Now choose $k = k(n) \to \infty$ such that $k/n \to 0$. Then the $k + 1$st largest order statistic $Z_{n-k,n}$ satisfies $Z_{n-k,n} \xrightarrow{P} \infty$ as $n \to \infty$. This follows, for example, from Smirnov, (1952). Conditional on $Z_{n-k,n}$ we have that $Z_{n,n}, Z_{n-1,n}, \ldots, Z_{n-k+1,n}$ have the distribution of the order statistics from a random sample of size k from a distribution concentrating on $(Z_{n-k,n}, \infty)$ of the form $F(\cdot)/(1 - F(Z_{n-k,n})$. Thus conditional on $Z_{n-k,n}$,

$$\left(\frac{Z_{n-k+i,n}}{Z_{n-k,n}}, i = 1, \ldots, k \right)$$

behave like the order statistics from a sample of size k from the distribution concentrating on $(1, \infty)$ with tail

$$\frac{1 - F(Z_{n-k,n} x)}{1 - F(Z_{n-k,n})} \approx x^{-\alpha}$$

where the approximation follows from (1.4). So in the notation of (1.1) it is reasonable to define the qq–estimator based on the upper k order statistics to be

$$\widehat{\alpha^{-1}} = \widehat{\alpha^{-1}}(k) = SL(\{(-\log(1-\frac{i}{k+1}), \log(\frac{Z_{n-k+i,n}}{Z_{n-k,n}}), 1 \le i \le k\}). \tag{1.5}$$

Some modest simplification of (1.4) is possible if we note the following readily checked properties of the SL function: For any real numbers a, b we have

$$SL(\{(x_i, y_i), 1 \le i \le n\}) = SL(\{(x_i + a, y_i + b), 1 \le i \le n\}). \tag{1.6}$$

Thus (1.4) simplifies to

$$\widehat{\alpha^{-1}} = SL(\{(-\log(1-\frac{i}{k+1}), \log Z_{n-k+i,n}), 1 \le i \le k\}). \tag{1.7}$$

In practice we would make a qq–plot of all the data and choose k based on visual observation of the portion of the graph which looked linear. Then we would compute the slope of the line through the chosen upper k order statistics and the corresponding exponential quantiles. Choosing k is an art as well as a science and the estimate of α is usually rather sensitive to the choice of k. Alternatively, one can plot $\{(k, \widehat{\alpha^{-1}}(k)), 1 \le k \le n\}$ and look for a stable region of the graph as representing the true value of α^{-1}. This is analogous to what is done with the Hill estimator of α^{-1}

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \log(\frac{Z_{n-k+i,n}}{Z_{n-k,n}}). \tag{1.8}$$

Choosing k is the Achilles heel of all these procedures; the difficulty can sometimes be lessened by smoothing. See Resnick and Starica (1995).

In Section 2 we prove weak consistency of $\widehat{\alpha^{-1}}$ under the regular variation assumption (1.3). Section 3 concentrates on proving asymptotic normality under a second order strengthening of (1.3). Section 4 contains some additional examples and comments.

2. Consistency of the qq–estimator.

In this Section we prove the weak consistency of the qq-estimator. In view of (1.5), (1.6) and (1.8) we may write the estimator $\widehat{\alpha^{-1}}$ as

$$\widehat{\alpha^{-1}} = \frac{\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\log(\frac{Z_{n-k+i,n}}{Z_{n-k,n}}) - \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))H_k}{\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))^2 - (\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))^2} \tag{2.1}$$

where the Hill estimator H_k was defined in (1.8).

Theorem 2.1. *Suppose $k = k(n) \to \infty$ in such a way that as $n \to \infty$ we have $k/n \to 0$. Suppose $Z_1, \dots, Z_n$ are a random sample from F, a distribution with regularly varying tail satisfying (1.3). Then the qq–estimator $\widehat{\alpha^{-1}}$ given in (2.1) is weakly consistent for $1/\alpha$:*

$$\widehat{\alpha^{-1}} \xrightarrow{P} \alpha^{-1}$$

as $n \to \infty$.

Proof. Write the denominator in (2.1) as

$$\frac{1}{k}S_{xx} - (\frac{1}{k}S_x)^2$$

where as $n \to \infty$

$$\frac{1}{k}S_{xx} = \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))^2 \sim \int_0^1 (-\log x)^2 dx$$
$$= \int_0^\infty y^2 e^{-y} dy = 2 \tag{2.2}$$

and

$$\frac{1}{k}S_{x} = \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1})) \sim \int_0^1 (-\log x) dx$$
$$= \int_0^\infty y e^{-y} dy = 1. \tag{2.3}$$

Furthermore as $n \to \infty$

$$\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))H_k \sim \int_0^1 (-\log x) dx H_k \xrightarrow{P} \alpha^{-1}$$

by the weak consistency of the Hill estimator (Mason, 1982). So for consistency of the qq–estimator it suffices to show

$$A_n := \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\log(\frac{Z_{n-k+i,n}}{Z_{n-k,n}}) \xrightarrow{P} \frac{2}{\alpha}. \tag{2.4}$$

We do this by using Potter's inequalities (Bingham, Goldie and Teugels, 1986) and Renyi's representation of order statistics (Resnick, 1992, page 439). Potter's inequalities take the following form: Since $1/(1-F)$ is regularly varying with index α, the inverse $U = (1/(1-F))^{\leftarrow}$ is regularly varying with index $1/\alpha$ and for $\epsilon > 0$, there exists $t_0 = t_0(\epsilon)$ such that if $y \geq 1$ and $t \geq t_0$

$$(1-\epsilon)y^{\alpha^{-1}-\epsilon} \leq \frac{U(ty)}{U(t)} \leq (1+\epsilon)y^{\alpha^{-1}+\epsilon}. \tag{2.5}$$

We now rephrase this in terms of the function

$$R = -\log(1-F) = \log U^{\leftarrow}.$$

Then $U = R^{\leftarrow} \circ \log$ and taking logarithms in (2.5) and then converting from a multiplicative to an additive form yields that

$$\log(1-\epsilon) + (\alpha^{-1}-\epsilon)y \leq \log R^{\leftarrow}(s+y) - \log R^{\leftarrow}(s) \leq \log(1+\epsilon) + (\alpha^{-1}+\epsilon)y \tag{2.6}$$

for $s \geq \log t_0$, and $y \geq 0$.

The reason for introducing the R function is that if $E_1, E_2, \dots E_n$ are iid unit exponentially distributed random variables then

$$(Z_1, Z_2, \dots, Z_n) \stackrel{d}{=} (R^{\leftarrow}(E_j), j = 1, \dots, n).$$

The Renyi representation (Resnick, 1992, Lemma 5.11.1) states that if $E_{1,n} \leq E_{2,n} \leq \dots \leq E_{n,n}$ are the order statistics associated to $E_1, E_2, \dots E_n$, then

$$(E_{1,n}, E_{2,n} - E_{1,n}, \dots, E_{n,n} - E_{n-1,n}) \stackrel{d}{=} (\frac{E_n}{n}, \frac{E_{n-1}}{n-1}, \dots, \frac{E_n}{1}). \tag{2.7}$$

For A_n we have

$$\begin{aligned}
A_n =& \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\log(\frac{Z_{n-k+i,n}}{Z_{n-k,n}})\\
\stackrel{d}{=}& \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\log(\frac{R^{\leftarrow}(E_{n-k+i,n})}{R^{\leftarrow}(E_{n-k,n})})\\
=& \frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\,(\log(R^{\leftarrow}(E_{n-k,n}+(E_{n-k+i,n}-E_{n-k,n}))-\log R^{\leftarrow}(E_{n-k,n}))
\end{aligned}$$

and applying the Potter inequality (2.6) we have the upper bound

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))\left(\log(1+\epsilon)+(\alpha^{-1}+\epsilon)(E_{n-k+i,n}-E_{n-k,n})\right)\\
=&(1+o(1))\log(1+\epsilon)+(\alpha^{-1}+\epsilon)\frac{1}{k}\sum_{i=1}^{k}(-\log(1-\frac{i}{k+1}))(E_{n-k+i,n}-E_{n-k,n}),
\end{aligned}$$

where we have applied (2.3). Of course a similar lower bound is obtained using the other half of the Potter inequalities. From the Renyi representation, for $i=1,\dots,k$

$$\begin{aligned}
E_{n-k+i,n}-E_{n-k,n} =& \sum_{j=n-k+1}^{n-k+i}(E_{j,n}-E_{j-1,n})\\
\stackrel{d}{=}& \sum_{j=n-k+1}^{n-k+i}\frac{E_{n-j+1}}{n-j+1}\\
=& \sum_{j=k-i+1}^{k}\frac{E_j}{j}.
\end{aligned}$$

Thus (2.4) will be proven provided we show

$$\frac{1}{k}\sum_{i=1}^{k}(-\log(\frac{i}{k+1}))\sum_{j=i}^{k}\frac{E_j}{j}\xrightarrow{P}2. \tag{2.8}$$

For the proof of 2.8 we need two simple lemmas.

Lemma 2.2. *We have that*

$$\frac{1}{k}\sum_{j=1}^{k}\log j=-1+\log k+\frac{1}{k}\{\log\sqrt{2\pi}+\frac{1}{2}\log k\}+o(1). \tag{2.9}$$

Proof of Lemma 2.2. From Stirling's formula

$$k!\sim e^{-k}k^{k+1/2}\sqrt{2\pi}$$

so that

$$\log k!-(-k+(k+\frac{1}{2})\log k+\log\sqrt{2\pi})\to 0.$$

Since

$$\frac{1}{k}\log k!=\frac{1}{k}\sum_{j=1}^{k}\log j,$$

the result follows. □

Lemma 2.3. *If $\{E_j, j \geq 1\}$ are iid unit exponentially distributed random variables*

$$\frac{1}{k}\sum_{j=1}^{k} E_j \log j = o_p(1) + (-1 + \log k + \frac{1}{k}\{\log\sqrt{2\pi} + \frac{1}{2}\log k\}.$$

Proof of Lemma 2.3. By the classical Kolmogorov convergence criterion, the series

$$\sum_{j=1}^{\infty} \frac{\log j}{j}(E_j - 1)$$

converges a.s. since taking variances yields

$$\sum_{j=1}^{\infty}\left(\frac{\log j}{j}\right)^2 < \infty.$$

Thus by Kronecker's Lemma (see, for example, Port, 1994)

$$\frac{1}{k}\sum_{j=1}^{k}(E_j - 1)\log j \to 0$$

almost surely as $k \to \infty$. Thus $\frac{1}{k}\sum_{j=1}^{k}(E_j - 1)\log j = o_p(1)$ and thus

$$\frac{1}{k}\sum_{j=1}^{k} E_j \log j = o_p(1) + \frac{1}{k}\sum_{j=1}^{k}\log j.$$

An appeal to Lemma 2.2 finishes the proof. □

Proof of (2.8). We now complete the proof of Theorem 2.1 by verifying (2.8). For what follows set

$$\bar{E}_k = \frac{1}{k}\sum_{j=1}^{k} E_j.$$

Then for (2.8) we have

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{k}(-\log(\frac{i}{k+1}))\sum_{j=i}^{k}\frac{E_j}{j} \\
&=\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{j}\sum_{i=1}^{j}(-\log(\frac{i}{k+1})\right)E_j \\
&=\frac{1}{k}\sum_{j=1}^{k}\left(\log(k+1) - \frac{1}{j}\sum_{i=1}^{j}\log i\right)E_j \\
&=\log(k+1)\bar{E}_k - \frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{j}\sum_{i=1}^{j}\log i\right)E_j
\end{aligned}$$

and applying Lemma 2.2 we get

$$
\begin{aligned}
&= \log(k+1)\bar{E}_k - \frac{1}{k}\sum_{j=1}^{k}\left(-1+\log j + O(\frac{\log j}{j})\right)E_j \\
&= \log(k+1)\bar{E}_k + \bar{E}_k - \frac{1}{k}\sum_{j=1}^{k}(\log j)E_j - \frac{1}{k}\sum_{j=1}^{k}O(\frac{\log j}{j})E_j .
\end{aligned}
$$

The last term on the right is $o_p(1)$. Applying Lemma 2.3 we get

$$
\begin{aligned}
&= \log(k+1)\bar{E}_k + \bar{E}_k - \left(o_p(1) - 1 + \log k + O(\frac{\log k}{k}\right) \\
&= \log(k+1)\bar{E}_k - \log k + (\bar{E}_k + 1) + o_p(1) \\
&= 2 + \log(k+1)\bar{E}_k - \log k + o_p(1).
\end{aligned}
$$

It remains to show that

$$
\log(k+1)\bar{E}_k - \log k \xrightarrow{P} 0.
$$

This is easy since

$$
\begin{aligned}
\log(k+1)\bar{E}_k - \log k &= \log(k+1)(\bar{E}_k - 1) + \log(k+1) - \log k \\
&= \frac{\log(k+1)}{\sqrt{k}}\sqrt{k}(\bar{E}_k - 1) + o(1) \\
&\xrightarrow{P} 0,
\end{aligned}
$$

by the Central Limit Theorem. This completes the proof. □

3. Asymptotic normality of the qq-estimator.

We continue to suppose that $\{Z_n, n \geq 1\}$ is iid with common distribution F such that $1-F$ is regularly varying at infinity of order $-\alpha$, with $\alpha > 0$. $Z_{1,n} < Z_{2,n} < \ldots < Z_{n,n}$ are the order statistics of $Z_1, \ldots, Z_n$. We investigate the limit distribution of the qq-estimator $\widehat{\alpha^{-1}}$ defined in (2.1). Because of (1.6) we can write

$$
\widehat{\alpha^{-1}} = \frac{\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\left\{k\log\left(Z_{n-i+1,n}\right) - \sum_{j=1}^{k}\log\left(Z_{n-j+1,n}\right)\right\}}{k\sum_{i=1}^{k}\left(-\log\left(\frac{i}{k+1}\right)\right)^2 - \left(\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\right)^2}. \tag{3.1}
$$

For this study, we will suppose that a second order regular variation condition holds for the non-decreasing function U, where recall

$$
U(t) = \left(\frac{1}{1-F}\right)^{\leftarrow}(t), \; t > 0.
$$

The condition is as follows. Set $\gamma = \alpha^{-1}$. We suppose there exists $\rho \leq 0$ and a function $0 < A(t) \to 0$ such that for all $x > 1$

$$
\frac{\frac{U(tx)}{U(t)} - x^{\gamma}}{A(t)} \quad \to \quad cx^{\gamma}\left(\frac{x^{\rho}-1}{\rho}\right) \quad (t \to \infty) \tag{3.2}
$$

for some $c \in \mathbb{R}$. If $\rho = 0$, interpret $(x^{\rho}-1)/\rho$ as $\log x$. Necessarily $A(\cdot)$ is regularly varying of index ρ and U is regularly varying of index γ. The form of the limit is discussed in de Haan and Stadmüller (1994); see also Geluk and de Haan (1987). By means of Vervaat's Lemma (Vervaat, 1972), (3.2) may be inverted and expressed in terms of $1-F$ to obtain

$$\frac{\frac{1-F(sx)}{1-F(s)}-x^{-\alpha}}{A\left(\frac{1}{1-F(s)}\right)} \quad \rightarrow \quad c\alpha x^{-\alpha}\left(\frac{1-x^{\rho\alpha}}{|\rho|}\right), \tag{3.3}$$

as $s \rightarrow \infty$.

As an example, suppose

$$1-F(x)=x^{-\alpha}+cx^{-\alpha^1}, \quad x>1$$

where $c>0,\ \alpha'>\alpha>0$. Then

$$A(s) \sim \alpha^{-1}c(\frac{\alpha'}{\alpha}-1)s^{1-\alpha'/\alpha},\ s \rightarrow \infty$$

and $\rho = 1-\frac{\alpha'}{\alpha}$. As another example, consider the Cauchy distribution with density

$$F'(x)=\frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$$

and distribution function

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\arctan x, \quad x \in \mathbb{R}.$$

Then $F^{\leftarrow}(y)=\tan\left(\pi\left(y-1/2\right)\right)$ so $U(x)=F^{\leftarrow}(1-1/x)=\cot(\pi/x)$ and from a power series expansion of $\cot(z)$ we find

$$U(x)=\frac{x}{\pi}\left(1-\frac{\pi^2}{3x^2}+0\left(\frac{1}{x^2}\right)\right).$$

This gives as $t \rightarrow \infty$

$$\frac{U(tx)}{U(t)}-x \sim x \cdot \frac{2\pi^2}{3t^2}\left(\frac{1-x^{-2}}{2}\right)$$

and with

$$A(t)=\frac{2\pi^2}{3t^2}$$

we get

$$\frac{\frac{U(tx)}{U(t)}-x}{A(t)} \rightarrow x\left(\frac{1-x^{-2}}{2}\right)$$

so that $\rho=-2$ and $\alpha=\gamma=1$.

We also need to assume a condition which restricts the growth of $k=k(n)$. We assume

$$k \rightarrow \infty,\ k/n \rightarrow 0,\ \sqrt{k}\,A(n/k) \rightarrow 0. \tag{3.4}$$

Note this condition depends on the underlying (unknown) distribution F since the function A depends on F. This condition is commonly used in the literature. See for example Dekkers and de Haan (1989).

We now state the result of this section.

Theorem 3.1. *If (3.2) and (3.4) hold, then*

$$\sqrt{k}\left(\widehat{\alpha^{-1}}-\alpha^{-1}\right) \xrightarrow{d} N\left(0,\, 2\alpha^{-2}\right). \tag{3.5}$$

Remark: The asymptotic variance of $\sqrt{k}\left(\widehat{\alpha^{-1}}-\alpha^{-1}\right)$ is thus $2\alpha^{-2}$. In contrast, the Hill estimator $H_{k,n}=\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Z_{n-i,n}}{Z_{n-k,n}}\right)$ satisfies under (3.2) and (3.4)

$$\sqrt{k}\left(H_{k,n}-\alpha^{-1}\right) \xrightarrow{d} N\left(0,\, \alpha^{-2}\right)$$

and hence has an asymptotic variance of α^{-2}. However, the Hill estimator exhibits considerable bias in certain circumstances and thus asymptotic variance is not a good criterion for superiority.

Proof. We first use (3.2) to obtain inequalities. Following what has become a standard method, which is described, for example, in lecture notes of de Haan (1991), we observe that since $A(t)\to 0$, (3.2) implies

$$x^{-\gamma}\frac{U(tx)}{U(t)}-1\to 0$$

as $t\to\infty$ and hence

$$\begin{aligned}\frac{\log U(tx)-\log U(t)-\gamma\log x}{A(t)} &= \frac{\log\left(x^{-\gamma}\frac{U(tx)}{U(t)}\right)}{A(t)}\\ &= \frac{\log\left(1+\left(x^{-\gamma}\frac{U(tx)}{U(t)}-1\right)\right)}{A(t)}\\ &\sim \frac{x^{-\gamma}\frac{U(tx)}{U(t)}-1}{A(t)}\to c\left(\frac{x^{\rho}-1}{\rho}\right).\end{aligned}$$

Thus $V(t):=\log U(t)-\gamma\log t$ satisfies

$$\frac{V(tx)-V(t)}{A(t)} \to c\left(\frac{x^{\rho}-1}{\rho}\right).$$

From Geluk and de Haan, 1987, page 16ff, the convergence is locally uniform in x.

Suppose for concreteness that $c>0$ and $\rho<0$. Then (Geluk and de Haan, 1987) $V(\infty)=\lim_{t\to\infty}V(t)$ exists and $V(\infty)-V(t)\sim\frac{c}{|\rho|}A(t)\in RV_{\rho}$. Applying Potter's inequalities, given $\varepsilon>0$, there exists $t_0=t_0(\varepsilon)$ such that for $t\ge t_0$ and $x\ge 1$

$$(1-\varepsilon)x^{\rho-\varepsilon} \le \frac{V(\infty)-V(tx)}{V(\infty)-V(t)} \le (1+\varepsilon)x^{\rho+\varepsilon}$$

whence

$$(1-\varepsilon)x^{\rho-\varepsilon}-1 \le \frac{V(t)-V(tx)}{V(\infty)-V(t)} \le (1+\varepsilon)x^{\rho+\varepsilon}-1$$

and so

$$\left(1-(1+\varepsilon)x^{\rho+\varepsilon}\right)c_1A(t)\le\log U(tx)-\log U(t)-\gamma\log x\le\left(1-(1-\varepsilon)x^{\rho-\varepsilon}\right)c_2A(t). \tag{3.6}$$

where $c_2 = (1+\varepsilon)c/|\rho|$ and $c_1 = (1-\varepsilon)c/|\rho|$. Similar inequalities hold if either $c < 0$ or $\rho = 0$ and we proceed with the proof assuming that (3.6) holds. Note that (3.6) can be rewritten in terms of $F^{\leftarrow}$ as

$$\begin{aligned}(1-(1+\varepsilon)x^{\rho+\varepsilon})c_1 A(t) &\le \log F^{\leftarrow}\left(1-\frac{1}{tx}\right) - \log F^{\leftarrow}(1-\frac{1}{t}) - \gamma \log x \\ &\le (1-(1-\varepsilon)\,x^{\rho-\varepsilon})c_2 A(t). \end{aligned} \tag{3.6'}$$

Recalling (2.1) and the notation

$$S_x = \sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right) \text{ and } S_{xx} = \sum_{i=1}^{k}\left(-\log\left(\frac{i}{k+1}\right)\right)^2,$$

we may write

$$\widehat{\alpha^{-1}} = \frac{\frac{1}{k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\Delta_{i,k}}{D}$$

with

$$\Delta_{i,k} = \sum_{j=1}^{k}(\log Z_{n-i+1,n} - \log Z_{n-j+1,n})$$

and

$$D = \frac{1}{k}S_{xx} - \left(\frac{1}{k}S_x\right)^2 \to 1, \text{ as } n\to\infty.$$

In fact, we have that

$$\sqrt{k}\left(\widehat{\alpha^{-1}} - \frac{1}{k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\Delta_{i,k}\right) \xrightarrow{P} 0 \quad (n\to\infty). \tag{3.7}$$

To see this, observe that

$$\sqrt{k}\left|\widehat{\alpha^{-1}} - \frac{1}{k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\Delta_{i,k}\right| = \sqrt{k}\left|\frac{1-D}{D}\right|\left|\frac{1}{k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\Delta_{i,k}\right|.$$

From Section 2, $k^{-2}\sum_{j=1}^{k} -\log\left(\frac{i}{k+1}\right)\Delta_{i,k}$ is stochastically bounded and since $D\to 1$, (3.7) will be proved provided we show

$$|1-D\,| = O\left(\frac{\log^2 k}{k}\right). \tag{3.8}$$

We have

$$1-D = 1 - \frac{1}{k}\sum_{i=1}^{k}\log^2 i + \frac{2\log(k+1)}{k}\sum_{i=1}^{k}\log i - \log^2(k+1) + \frac{1}{k}\left(\frac{1}{k}\sum_{i=1}^{k}\log i - \log(k+1)\right)^2.$$

Since

$$\frac{1}{k}\int_1^k \log^2 u du \le \frac{1}{k}\sum_{1}^{k}\log^2 i \le \frac{1}{k}\int_1^{k+1}\log^2 u du$$

and

$$\frac{1}{k}\int_k^{k+1}(\log u)^2 du \sim \frac{\log^2 k}{k}$$

and for $x \geq 1$

$$\int_1^x \log^2 u du = 2(x-1) + x\log x(\log x - 2),$$

we find that

$$\frac{1}{k}\sum_1^k \log^2 i = 2 + (\log k)\left((\log k) - 2\right) + O\left(\frac{\log^2 k}{k}\right). \tag{3.9}$$

The combination of (3.9) and Lemma 2.2 yield (3.8) and hence (3.7).

Let $U_1, U_2, \ldots$ be independent uniform random variables on $[0,1]$ and let q_n denote the left continuous uniform quantile function, defined by

$$q_n(t) = \inf\{s : F_n(s) \geq t\}$$

where F_n is the uniform empirical distribution; that is

$$q_n(t) = \begin{cases} 0, & \text{for } t = 0 \\ U_{i,n}, & \text{for } \frac{i-1}{n} < t \leq \frac{i}{n} \text{ and } 1 \leq i \leq n. \end{cases}$$

By using

$$Z_{n-i+1,n} \stackrel{d}{=} F^{\leftarrow}(U_{n-i+1,n}) \stackrel{d}{=} F^{\leftarrow}(1-U_{i,n}),$$

we have

$$\Delta_{i,k} \stackrel{d}{=} \sum_{j=1}^k \left(\log F^{\leftarrow}(1-U_{i,n}) - \log F^{\leftarrow}(1-U_{j,n})\right) = \Delta_{j\geq i} - \Delta_{j<i}$$

with

$$\Delta_{j\geq i} := \sum_{j=i}^k \left(\log F^{\leftarrow}(1-U_{i,n}) - \log F^{\leftarrow}(1-U_{j,n})\right)$$

and

$$\Delta_{j<i} := \sum_{j=1}^{i-1}\left(\log F^{\leftarrow}(1-U_{j,n}) - \log F^{\leftarrow}(1-U_{i,n})\right),$$

and by applying (3.6) to $\Delta_{j\geq i}$ and $\Delta_{j<i}$, we get that for all $\varepsilon > 0$, there exists t_0 such that on the set $[1/U_{k,n} > t_0]$ (which, since $U_{k,n} \stackrel{P}{\to} 0$ is a set whose probability converges to 1)

$$Inf_{i,k} \leq \Delta_{i,k} - \gamma\sum_{j=1}^k \log\left(\frac{U_{j,n}}{U_{i,n}}\right) < Sup_{i,k} \tag{3.10}$$

where

$$\begin{aligned} Sup_{i,k} = &\sum_{j=i}^k \left\{\left(1-(1-\varepsilon)\left(\frac{U_{j,n}}{U_{i,n}}\right)^{\rho-\varepsilon}\right)c_2 A\left(\frac{1}{U_{j,n}}\right)\right\} \\ &- \sum_{j=1}^{i-1}\left\{\left(1-(1+\varepsilon)\left(\frac{U_{i,n}}{U_{j,n}}\right)^{\rho+\varepsilon}\right)c_1 A\left(\frac{1}{U_{i,n}}\right)\right\} \end{aligned}$$

and

$$
\begin{aligned}
Inf_{i,j} =& \sum_{j=i}^{k} \left\{ \left(1 - (1+\varepsilon) \left(\frac{U_{j,n}}{U_{i,n}} \right)^{\rho+\varepsilon} \right) c_1 A \left(\frac{1}{U_{j,n}} \right) \right\} \\
& - \sum_{j=1}^{i-1} \left\{ \left(1 - (1-\varepsilon) \left(\frac{U_{i,n}}{U_{j,n}} \right)^{\rho-\varepsilon} \right) c_2 A \left(\frac{1}{U_{i,n}} \right) \right\}.
\end{aligned}
$$

Therefore

$$
\Delta_{ik} \stackrel{d}{=} \gamma \sum_{j=1}^{k} \log \left(\frac{U_{j,n}}{U_{i,n}} \right) + e_{i,k}, \tag{3.11}
$$

$e_{i,k}$ being an error term such that $Inf_{i,k} < e_{i,k} < Sup_{i,k}$. Because of (3.7) and (3.11) we thus have

$$
\begin{aligned}
\widehat{\alpha^{-1}} & \stackrel{d}{=} \frac{1}{k^2} \sum_{i=1}^{k} - \log \left(\frac{i}{k+1} \right) \Delta_{i,k} + O_p \left(\frac{1}{\sqrt{k}} \right) \\
& = \frac{1}{\alpha} D_n + e_k^{(1)} + O_p \left(\frac{1}{\sqrt{k}} \right),
\end{aligned} \tag{3.12}
$$

where

$$
e_k^{(1)} := \frac{1}{k^2} \sum_{i=1}^{k} - \log \left(\frac{i}{k+1} \right) e_{i,k}
$$

and

$$
D_n := \frac{1}{k^2} \sum_{i=1}^{k} - \log \left(\frac{i}{k+1} \right) \sum_{j=1}^{k} \log \left(\frac{U_{j,n}}{U_{i,n}} \right).
$$

Now

$$
\begin{aligned}
D_n =& \frac{1}{k^2} \sum_{i=1}^{k-1} - \log \left(\frac{i+1}{k+1} \right) \sum_{j=1}^{k-1} \log \left(\frac{U_{j+1,n}}{U_{i+1,n}} \right) \\
& + \frac{\log(k+1)}{k^2} \sum_{j=1}^{k} \log \left(\frac{U_{j,n}}{U_{1,n}} \right) + \frac{1}{k^2} \sum_{i=2}^{k} - \log \left(\frac{i}{k+1} \right) \log \left(\frac{U_{1,n}}{U_{i,n}} \right).
\end{aligned}
$$

But

$$
\frac{\log(k+1)}{k^2} \sum_{j=1}^{k} \log \left(\frac{U_{j,n}}{U_{1,n}} \right) + \frac{1}{k^2} \sum_{i=2}^{k} - \log \left(\frac{i}{k+1} \right) \log \left(\frac{U_{1,n}}{U_{i,n}} \right) = \frac{1}{k^2} \sum_{i=2}^{k} \log i \log \left(\frac{U_{i,n}}{U_{1,n}} \right).
$$

We show

$$
\frac{1}{k^2} \sum_{i=2}^{k} \log i \log \frac{U_{i,n}}{U_{1,n}} = O_p \left(\frac{1}{\sqrt{k}} \right). \tag{3.13}
$$

We have

$$
\begin{aligned}
0 \leq & \sqrt{k}\frac{1}{k^2}\sum_{i=2}^{k}\log i \log\frac{U_{i,n}}{U_{1,n}} \\
\leq & \log\frac{U_{k,n}}{U_{1,n}}\ \frac{\sqrt{k}}{k^2}\sum_{i=2}^{k}\log i \\
\sim & \log\frac{U_{k,n}}{U_{1,n}}\left(\frac{\log k}{\sqrt{k}}\right) = \log\left(\frac{k\cdot\frac{n}{k}U_{k,n}}{nU_{1,n}}\right)\frac{\log k}{\sqrt{k}} \\
= & \frac{(\log k)^2}{\sqrt{k}} + \log\left(\frac{n}{k}U_{k,n}\right)\frac{\log k}{\sqrt{k}} - \log nU_{1,n}\frac{\log k}{\sqrt{k}} \\
\xrightarrow{P} & 0
\end{aligned}
$$

since $\frac{n}{k}U_{k,n} \xrightarrow{P} 1$ and $\{nU_{1,n}\}$ is convergent in distribution. We conclude that

$$
D_n \stackrel{d}{=} \frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) + O_p\left(\frac{1}{\sqrt{k}}\right). \tag{3.14}
$$

But it is also true that

$$
\begin{aligned}
\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) \stackrel{d}{=} & \log\left(\frac{1-U_{n-j,n}}{1-U_{n-i,n}}\right) \\
= & \log\left(\frac{1-q_n(1-j/n)}{1-q_n(1-i/n)}\right) \\
= & \log\left(\frac{1-q_n(1-j/n)}{j/n}\right) - \log\left(\frac{1-q_n(1-i/n)}{i/n}\right) + \log\left(\frac{j}{i}\right).
\end{aligned}
$$

Moreover, by definition of the uniform quantile process, $\beta_n(t) = \sqrt{n}(q_n(t)-t)$, we have, for $0 < s \leq 1$

$$
\begin{aligned}
\log\left(\frac{1-q_n(1-s)}{s}\right) = & \log\left(\frac{1-(1-s)-n^{-1/2}\beta_n(1-s)}{s}\right) \\
= & \log\left(1-\frac{\beta_n(1-s)}{s\sqrt{n}}\right) \\
= & -\frac{\beta_n(1-s)}{s\sqrt{n}} + R_n^{(1)}(s).
\end{aligned}
$$

Then

$$
\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) \stackrel{d}{=} \frac{1}{\sqrt{n}}\left(\frac{\beta_n(1-i/n)}{i/n} - \frac{\beta_n(1-j/n)}{j/n}\right) + \log\left(\frac{j}{i}\right) + R_n^{(2)}(i,j)
$$

with

$$
R_n^{(2)}(i,j) := R_n^{(1)}(j/n) - R_n^{(1)}(i/n)
$$

and by using the result of M. Csörgő, S. Csörgő, Horváth, and Mason (1986), namely $\beta_n(t) = -B_n(t) + n^{-1/2+\nu}O_p(t^\nu)$ with $0 < \nu < 1/2$, where $\{B_n(s), 0 \leq s \leq 1\}$ is a sequence of Brownian Bridges, we obtain

$$
\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) \stackrel{d}{=} \frac{1}{\sqrt{n}}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right) + \log\left(\frac{j}{i}\right) + R_n^{(3)}(i,j)
$$

with

$$R_n^{(3)}(i,j) := R_n^{(2)}(i,j) + \frac{1}{n^{1-\nu}}\left(\frac{O_p((i/n)^\nu)}{i/n} - \frac{O_p((j/n)^\nu)}{j/n}\right).$$

This gives us

$$D_n \stackrel{d}{=} \alpha M_n + \frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log(\frac{j}{i}) + R_n^{(4)} \tag{3.16}$$

with

$$M_n := \frac{1}{\alpha k^2\sqrt{n}}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right)$$

and

$$R_n^{(4)} := \frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1} R_n^{(3)}(i,j)$$

but also

$$R_n^{(4)} = D_n - \frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log\left(\frac{j}{i}\right) - \alpha M_n.$$

Therefore, (3.12) and this last expression provide

$$\begin{aligned}\widehat{\alpha^{-1}} =& M_n + \frac{1}{\alpha k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log\left(\frac{j}{i}\right) + R_n^{(4)} + e_k^{(1)}\\ =& M_n + \frac{1}{\alpha} + R_k + \frac{1}{\alpha}R_n^{(4)} + e_k^{(1)}\end{aligned}$$

with

$$R_k = \frac{1}{\alpha}\left(\frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log\left(\frac{j}{i}\right) - 1\right).$$

To analyze R_k, we note

$$\begin{aligned}&\frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\log\left(\frac{j}{i}\right)\\ =&\frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\sum_{j=1}^{k-1}\log\left(\frac{j}{i}\right)\\ =&\frac{1}{k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\sum_{j=1}^{k-1}\log(j/k) + \frac{k-1}{k^2}\sum_{i=1}^{k-1}\log\left(\frac{i+1}{k}\right)\log(i/k)\\ \sim&\int_{2/k}^{1} -\log x\,dx\int_{1/k}^{1-\frac{1}{k}}\log x\,dx + \frac{k-1}{k}\int_{1/k}^{1-\frac{1}{k}}\log\left(x+\frac{1}{k}\right)\log x\,dx\\ =&1 + O\left(\frac{\log^2 k}{k}\right).\end{aligned}$$

Then $R_k = \alpha^{-1}O((\log^2 k)/k)$ and $\sqrt{k}R_k = 0(1)$ as $k \to \infty$. Finally, we get from (3.12), (3.14), (3.16)

$$\hat{\alpha^{-1}} = \alpha^{-1} + \Pi_n + E_n + 0_p\left(\frac{1}{\sqrt{k}}\right) \tag{3.17}$$

with

$$M_n := \frac{1}{\alpha k^2\sqrt{n}}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right) \tag{3.18}$$

and

$$E_n = \frac{1}{\alpha}R_n^{(4)} + e_k^{(1)} = E_{n1} + E_{n2},$$

where from (3.16)

$$E_{n1} = \frac{1}{\alpha}R_n^{(4)} = \frac{1}{\alpha k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\left\{\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) - \log\left(\frac{j}{i}\right)\right\} - M_n$$

and $E_{n2} = e_k^{(1)}$ is given after (3.12). We now analyze the behavior of each term in (3.17).

Part 1: Behavior of $\sqrt{k}M_n$: Recall

$$\begin{aligned} M_n :=& \frac{1}{\alpha k^2\sqrt{n}}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right) \\ =& \frac{1}{\alpha k^2\sqrt{n}}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\sum_{j=1}^{k-1}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right). \end{aligned}$$

By definition of the Brownian Bridge, B_n satisfies

$$\left\{\frac{B_n(1-t)}{t}, 0 < t \le 1\right\} \stackrel{d}{=} \left\{W(1) - \frac{W(t)}{t}, 0 < t \le 1\right\},$$

where W is a standard Wiener process. So

$$M_n \stackrel{d}{=} \frac{1}{\alpha k^2}\sqrt{n}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\sum_{j=1}^{k-1}\left(\frac{W(j/n)}{j/n} - \frac{W(i/n)}{i/n}\right).$$

Since $W(c\cdot) \stackrel{d}{=} \sqrt{c}W(\cdot)$ for $c > 0$, we get by writing

$$\frac{W(j/n)}{j/n} = \frac{W(\frac{j}{k}k/n)}{j/k \cdot k/n} \stackrel{d}{=} \frac{\sqrt{k/n}}{k/n}\frac{W(j/k)}{j/k} = \sqrt{n/k}\frac{W(j/k)}{j/k}$$

that

$$M_n \stackrel{d}{=} \frac{1}{\alpha k^2\sqrt{k}}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\sum_{j=1}^{k-1}\left(\frac{W(j/k)}{j/k} - \frac{W(i/k)}{i/k}\right)$$

and thus

$$\begin{aligned}
\sqrt{k}M_n \stackrel{d}{=} &\frac{1}{\alpha}\left(\frac{1}{k}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\right)\left(\frac{1}{k}\sum_{j=1}^{k-1}\frac{W(j/k)}{j/k}\right) \\
&-\frac{1}{\alpha}\left(\frac{k-1}{k}\right)\frac{1}{k}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\frac{W(i/k)}{i/k} \\
\to &\frac{1}{\alpha}\left(\int_0^1 -\log u du\right)\left(\int_0^1 \frac{W(s)}{s}ds\right) \\
&-\frac{1}{\alpha}\int_0^1 -\log u\frac{W(u)}{u}du \\
= &\frac{1}{\alpha}\left\{\int_0^1 \frac{W(s)}{s}ds - \int_0^1 (-\log u)\frac{W(u)}{u}du\right\} \\
= &\frac{1}{\alpha}\int_0^1 (1+\log u)\frac{W(u)}{u}du =: Z.
\end{aligned}$$

So Z is a Gaussian random variable with mean 0 and variance

$$\text{Var}Z = \frac{2}{\alpha^2}\int_{0<s<u<1}(1+\log u)(1+\log s)\frac{s\wedge u}{us}duds = \frac{2}{\alpha^2},$$

the integral being performed via Mathematica. Thus we can conclude

$$\sqrt{k}M_n \xrightarrow{d} N\left(0, 2/\alpha^2\right). \tag{3.19}$$

Part 2: Behavior of $\sqrt{k}E_n$:

We will show that $\sqrt{k}E_n = O_p(1)$ as $k \to \infty$, $n \to \infty$ and $k/n \to 0$. For this, we will split E_n into two terms and use different methods on each. Let $E_n := E_{n1} + E_{n2}$, with

$$E_{n1} := \frac{1}{\alpha k^2}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k+1}\right)\sum_{j=1}^{k-1}\left[\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) - \log\left(\frac{j}{i}\right) - \frac{1}{\sqrt{n}}\left(\frac{B_n(1-\frac{j}{n})}{\frac{j}{n}} - \frac{B_n(1-\frac{i}{n})}{\frac{i}{n}}\right)\right]$$

and

$$E_{n2} := \frac{1}{\alpha k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)e_{i,k},$$

where

$$Inf_{i,k} < e_{i,k} < sup_{i,k}.$$

We first consider E_{n1} and follow the method developed in Csörgő, Deheuvels and Mason (1985). We have

$$E_{n1} = \frac{1}{\alpha k}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)R_{i,k}$$

where

$$R_{i,k} := \frac{1}{k}\sum_{j=1}^{k-1}\left\{\log\left(\frac{U_{j+1,n}}{U_{i+1,n}}\right) - \log(j/i) - \frac{1}{\sqrt{n}}\left(\frac{B_n(1-j/n)}{j/n} - \frac{B_n(1-i/n)}{i/n}\right)\right\}.$$

Using again (3.15), we get

$$R_{i,k} \stackrel{d}{=}$$
$$\frac{1}{k}\sum_{j=1}^{k-1}\left\{\left(\log\left(\frac{1-q_n(1-j/n)}{j/n}\right)-\frac{1}{\sqrt{n}}\frac{B_n(1-j/n)}{j/n}\right)-\left(\log\left(\frac{1-q_n(1-i/n)}{i/n}\right)-\frac{1}{\sqrt{n}}\frac{B_n(1-i/n)}{i/n}\right)\right\}$$

and then

$$\begin{aligned} E_{n1} =& \frac{1}{\alpha k}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\frac{1}{k}\sum_{j=1}^{k-1}\left(\log\left(\frac{1-q_n(1-j/n)}{j/n}\right)-\frac{1}{\sqrt{n}}\frac{B_n(1-j/n)}{j/n}\right) \\ & -\frac{1}{\alpha k}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)\left(\log\left(\frac{1-q_n(1-i/n)}{i/n}\right)-\frac{1}{\sqrt{n}}\frac{B_n(1-i/n)}{i/n}\right). \end{aligned}$$

Define

$$c_k = k^{-1}\sum_{i=1}^{k-1} -\log\left(\frac{i+1}{k}\right)$$

so that $c_k \to 1$ and

$$\alpha E_{n1} = \frac{1}{k}\sum_{i=1}^{k-1}\left(c_k - \log\left(\frac{i+1}{k}\right)\right)\left[\log\left(\frac{1-q_n(1-\frac{i}{n})}{i/n}\right)-\frac{1}{\sqrt{n}}\frac{B_n(1-\frac{i}{n})}{i/n}\right].$$

Write

$$\begin{aligned} \log\left(\frac{1-q_n(1-\frac{i}{n})}{\frac{i}{n}}\right) &= \log\left(1-\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right) \\ &= \frac{-\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}} + g\left(\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right) \end{aligned}$$

where $g(u) = u + \log(1-u)$. So

$$\begin{aligned} \alpha\left|E_{n1}\right| \leq& \frac{1}{k}\sum_{i=1}^{k-1}\left(c(k)-\log\left(\frac{i+1}{k}\right)\right)\left|\frac{-\beta_n(1-\frac{i}{n})-B_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right| \\ & + \frac{1}{k}\sum_{i=1}^{k-1}\left(c(k)-\log\left(\frac{i+1}{k}\right)\right)\left|g\left(\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right)\right| \\ =& \left|E'_{n1}\right| + \left|E''_{n1}\right|. \end{aligned}$$

From Lemma 10 of Csörgő, Deheuvels and Mason (1985), we have for $0<\nu<1/2$

$$\sup_{\frac{1}{n+1}\leq\zeta\leq 1}\left|\frac{-\beta(1-\zeta)-B_n(1-\zeta)}{\zeta^{-\nu+1/2}}\right| = O_p\left(n^{-\nu}\right).$$

Thus

$$
\begin{aligned}
|E'_{n1}| \leq & \frac{1}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\left|\frac{-\beta_n(1-\frac{i}{n})-B_n(1-\frac{i}{n})}{(\frac{i}{n})^{-\nu+1/2}}\right|\left(\frac{i}{n}\right)^{-\nu+1/2}/\frac{i}{n}\sqrt{n} \\
\leq & \frac{O_p(n^{-\nu})}{n^{-\nu}}\frac{1}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\left(\frac{i}{k}\right)^{-\nu-1/2}k^{-\nu-1/2} \\
= & O_p(1)k^{-\nu-1/2}\int_0^1(2-\log u)u^{-\nu-1/2}du \\
= & O_p(k^{-\nu-1/2}).
\end{aligned}
$$

Therefore

$$\sqrt{k}E'_{n1} = O_\rho(k^{-\nu}) \xrightarrow{d} 0.$$

For E''_{n1} we have that

$$
\begin{aligned}
g\left(\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right) = & \log\left(\frac{1-q_n(1-\frac{i}{n})}{i/n}\right)+\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}} \\
= & \log\left(\frac{1-q_n(1-\frac{i}{n})}{i/n}\right)+\frac{q_n(1-\frac{i}{n})-(1-\frac{i}{n})}{i/n} \\
= & \log\left(\frac{i}{n}+\left(1-\frac{i}{n}-q_n\left(1-\frac{i}{n}\right)\right)\right)-\log\frac{i}{n}+q_n\left(1-\frac{i}{n}\right)-\left(1-\frac{i}{n}\right)
\end{aligned}
$$

and from a two-term Taylor expansion, this is

$$= -\frac{\left(1-\frac{i}{n}-q_n\left(1-\frac{i}{n}\right)\right)^2}{2\Theta_n^2\left(\frac{i}{n}\right)} = \frac{-\beta_n^2\left(1-\frac{i}{n}\right)}{2n\Theta_n^2\left(\frac{i}{n}\right)}$$

where

$$\min\left\{\frac{i}{n}, 1-q_n\left(1-\frac{i}{n}\right)\right\} \leq \Theta_n\left(\frac{i}{n}\right) \leq \max\left\{\frac{i}{n}, 1-q_n\left(1-\frac{i}{n}\right)\right\}.$$

From Lemma 13, page 1069 of Csörgő, Deheuvels and Mason (1985), we obtain the following: given $\varepsilon > 0$, there exists n_0 and $\rho > 1$ such that $n \geq n_0$ implies

$$P(A_n(\rho)) > 1-\varepsilon$$

where

$$A_n(\rho) = \bigcap_{\frac{1}{n+1}\leq\zeta\leq 1}\left[\frac{\zeta}{\rho} \leq 1-q_n(1-\zeta) \leq \rho\zeta\right].$$

Thus on $A_n(\rho)$ we observe that for $n \geq n_0$

$$\frac{i}{\rho n} \leq \frac{i}{n}\wedge\frac{i}{\rho n} \leq \Theta_n\left(\frac{i}{n}\right) \leq \frac{i}{n}\vee\rho\left(\frac{i}{n}\right) = \rho\frac{i}{n}.$$

Therefore on $A_n(\rho)$

$$\begin{aligned}|E''_{n1}| =&\frac{1}{k}\sum_{i=1}^{k-1}\left(c(k)-\log\left(\frac{i+1}{k}\right)\right)\left|g\left(\frac{\beta_n(1-\frac{i}{n})}{\frac{i}{n}\sqrt{n}}\right)\right|\\ \leq&\frac{1}{k}\sum_{i=1}^{k-1}\left(c(k)-\log\left(\frac{i+1}{k}\right)\right)\frac{\beta_n^2(1-\frac{i}{n})}{2n\Theta_n^2(\frac{i}{n})}\\ \leq&\frac{1}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\rho^2\frac{\beta_n^2(1-\frac{i}{n})}{2n\left(\frac{i}{n}\right)^2}.\end{aligned}$$

It follows that for any $\xi > 0$

$$\begin{aligned}P\left[\sqrt{k}\,|E''_{n1}|>\right] \leq& P(A_n(\rho)^c)+P\left[\frac{\sqrt{k}}{k}\sum_{i=1}^{k-1}\left((2-\log\left(\frac{i+1}{k}\right)\right)\rho^2\frac{\beta_n^2(1-\frac{i}{n}}{2n(\frac{i}{n})^2}>\xi\right]\\ \leq&\varepsilon+P\left[\frac{\sqrt{k}}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\rho^2\frac{\beta_n^2(1-\frac{i}{n}}{2n(\frac{i}{n})^2}>\xi\right].\end{aligned}$$

Now from Lemma 14, page 1070 of Csörgő, Deheuvels and Mason (1985), for every $\nu \in (0, \frac{1}{2})$ we have

$$\sup_{\frac{1}{n}\leq\zeta\leq1}\left|\frac{\beta_n(1-\zeta)}{\sqrt{n}\zeta^{1-\nu}}\right| = O_p\left(n^{-\nu}\right).$$

Pick $\nu\varepsilon\left(\frac{1}{4},\frac{1}{2}\right)$ and then

$$\begin{aligned}P\left[\sqrt{k}\,|E''_{n1}|>\xi\right] \leq&\varepsilon+P\left[\rho^2\sqrt{k}\frac{1}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\frac{\beta_n^2\left(1-\frac{i}{n}\right)}{n\left(\frac{i}{n}\right)^{2(1-\nu)}}\frac{\left(\frac{i}{n}\right)^{2(1-\nu)}}{\left(\frac{i}{n}\right)^2}>\xi\right]\\ =&\varepsilon+P\left[O_p(1)k^{-2\nu}\sqrt{k}\frac{1}{k}\sum_{i=1}^{k-1}\left(2-\log\left(\frac{i+1}{k}\right)\right)\left(\frac{i}{k}\right)^{-2\nu}>\xi\right]\\ =&\varepsilon+\rho\left[O_p\left(k^{-2\nu+1/2}\right)>\xi\right].\end{aligned}$$

Since $-2\nu+\frac{1}{2}<0$, as a consequence of $\nu>\frac{1}{4}$, and since $\varepsilon>0$ is arbitrary, we get

$$\sqrt{k}\,|E''_{n1}| \xrightarrow{P} 0$$

as desired.

We now consider the remainder E_{n2} using properties of regular variation. Recall that

$$\frac{1}{\alpha k^2}\sum_{i=1}^{k}-\log\left(\frac{i}{k-1}\right)Inf_{i,k} < E_{n2} < \frac{1}{\alpha k^2}\sum_{i=1}^{k}-\log\left(\frac{i}{k+1}\right)Sup_{i,k}. \tag{3.21}$$

and consider the upper bound B_k of E_{n2}. We prove that $\sqrt{k}B_k \xrightarrow{P} 0$ as $k\to\infty$. Let $B_k := \frac{1}{\alpha}\left(B_k^{(1)}+B_k^{(2)}\right)$, with

$$B_k^{(1)} := \frac{1}{k^2}\sum_{i=1}^{k}-\log\left(\frac{i}{k+1}\right)\sum_{j=i}^{k}\left(1-(1-\varepsilon)\left(\frac{U_{j,n}}{U_{i,n}}\right)^{\rho-\varepsilon}\right)c_2A\left(\frac{1}{U_{j,n}}\right)$$

and

$$B_k^{(2)} := \frac{1}{k^2}\sum_{i=1}^{k}\log\left(\frac{i}{k+1}\right)\sum_{j=1}^{i-1}\left(1-(1+\varepsilon)\left(\frac{U_{i,n}}{U_{j,n}}\right)^{\rho+\varepsilon}\right)c_1 A\left(\frac{1}{U_{i,n}}\right).$$

We first show $\sqrt{k}B_k^{(2)} \xrightarrow{P} 0$. By Potter's inequalities (Bingham, Goldie, Teugels, 1987), for given $\varepsilon > 0$, there exists t_0 such that for $t \geq t_0$ and all $x \geq 1$

$$(1-\delta)x^{\rho-\varepsilon} \leq \frac{A(tx)}{A(t)} \leq (1+\delta)x^{\rho+\varepsilon}.$$

Therefore for $i = 1, \dots, k$

$$\begin{aligned} A\left(\frac{1}{U_{i,n}}\right) &= A\left(\frac{1}{U_{k+1,n}}\cdot\frac{U_{k+1,n}}{U_{i,n}}\right) \\ &\leq \left(\frac{U_{k+1,n}}{U_{i,n}}\right)^{\rho+\delta} A\left(\frac{1}{U_{k+1,n}}\right) \end{aligned}$$

on the set $\left[U_{k+1,n} \leq t_0^{-1}\right]$, which is a set whose probability approaches 1 as $n \to \infty$. So it suffices to prove

$$\begin{aligned} &\sqrt{k}\frac{1}{k^2}\sum_{i=1}^{k}\left|\log\frac{i}{k+1}\right|\left|\sum_{j=1}^{i-1}\right| 1-(1+\varepsilon)\left(\frac{U_{i,n}}{U_{j,n}}\right)^{\rho+\varepsilon}\left| A\left(\frac{1}{U_{i,n}}\right)\right. \\ &= \sqrt{k}A\left(\frac{1}{U_{k+1,n}}\right)\frac{1}{k^2}\sum_{i=1}^{k}-\log\frac{i}{k+1}\sum_{j=1}^{i-1}\left|1+(1+\varepsilon)\left(\frac{U_{i,n}}{U_{j,n}}\right)^{\rho+\varepsilon}\right|\left(\frac{U_{k+1,n}}{U_{i,n}}\right)^{\rho+\varepsilon} \xrightarrow{P} 0. \end{aligned}$$

Now $U_{k+1,n}/\frac{k}{n} \xrightarrow{P} 1$ implies by regular variation that $A\left(\frac{1}{U_{k,n}}\right)/A(n/k) \xrightarrow{P} 1$ so since $\sqrt{k}A(n/k) \to 0$ is assumed in (3.4), it remains to show that

$$\frac{1}{k}\sum_{i=1}^{k}-\log\frac{i}{k+1}\left(\frac{U_{k+1,n}}{U_{i,n}}\right)^{\rho+\varepsilon} + \frac{1}{k^2}\sum_{i=1}^{k}-\log\frac{i}{k+1}\sum_{j=1}^{i-1}\left(\frac{U_{k+1,n}}{U_{j,n}}\right)^{\rho+\varepsilon} = A + B$$

is stochastically bounded. Set $\Gamma_i = E_1 + \cdots + E_i, i \geq 1$ where $\{E_n, n \geq 1\}$ is iid with $P[E_1 > x] = e^{-x}, x > 0$. Then

$$(U_{1,n}, \cdots, U_{nn}) \stackrel{d}{=} \left(\frac{\Gamma_1}{\Gamma_{n+1}}, \cdots, \frac{\Gamma_n}{\Gamma_{n+1}}\right).$$

Thus A has the same distribution as

$$\frac{1}{k}\sum_{i=1}^{k}\left(-\log\frac{i}{k+1}\right)\left(\frac{\Gamma_{k+1}}{\Gamma_i}\right)^{\rho+\varepsilon}.$$

Since $\Gamma_i \sim i$ a.s. as $i \to \infty$, it is now clear that A is stochastically bounded. B is handled similarly after observing that

$$\begin{aligned} B &\leq \left(\frac{1}{k}\sum_{i=1}^{k}-\log\frac{i}{k+1}\right)\left(\frac{1}{k}\sum_{j=1}^{k}\left(\frac{U_{k+1,n}}{U_{j,n}}\right)\right)^{\rho+\varepsilon} \\ &\stackrel{d}{=} \left(\frac{1}{k}\sum_{i=1}^{k}-\log\left(\frac{i}{k+1}\right)\right)\left(\frac{1}{k}\sum_{j=1}^{k}\frac{1}{(\Gamma_j|\Gamma_k)}\rho+\varepsilon\right) \end{aligned}$$

and

$$\frac{1}{k}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right) \sim \int_0^1 -\log u du$$

and

$$\frac{1}{k}\sum_{j=1}^{k}\frac{1}{(\Gamma_j|\Gamma_k)^{\rho+\varepsilon}} \leq (\text{const})\ \frac{1}{k}\sum_{j=1}^{k}\frac{1}{(j/k)^{\rho+\varepsilon}} \sim \text{const} \int_0^1 u^{-(\rho+\varepsilon)}du < \infty.$$

We now deal with $B_k^{(1)}$. Again, using Potter's inequalities we get

$$\begin{aligned} A\left(\frac{1}{U_{j,n}}\right) =& A\left(\frac{1}{U_{k,n}}\cdot\frac{U_{k,n}}{U_{j,n}}\right) \\ \leq& A(1/U_{k,n})\left(\frac{U_{k,n}}{U_{j,n}}\right)^{\rho+\varepsilon}(1+\varepsilon) \end{aligned}$$

on $[U_{k,n}\leq t_0]$. Also $A(1/U_{k,n})/A(n/k) \xrightarrow{P} 1$. So to show $\sqrt{k}B_k^{(1)} \xrightarrow{P} 0$ it suffices to show

$$\frac{1}{k^2}\sum_{i=1}^{k} -\log\left(\frac{i}{k+1}\right)\sum_{j=i}^{k}\left(\frac{U_{k,n}}{U_{j,n}}\right)^{\rho+\varepsilon} \tag{3.23}$$

is stochastically bounded. The expression in (3.23) is bounded above by

$$\begin{aligned} &\left(\frac{1}{k}\sum_{i=1}^{k} -\log\frac{i}{k+1}\right)\left(\frac{1}{k}\sum_{j=1}^{k}\left(\frac{U_{k,n}}{U_{j,n}}\right)^{\rho+\varepsilon}\right) \\ &\qquad \stackrel{d}{=} (1+o(1))\,\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\Gamma_j/\Gamma_k}\right)^{\rho+\varepsilon} \end{aligned}$$

which is stochastically bounded as checked when dealing with B of $B_k^{(2)}$. 194z So we can conclude that $\sqrt{k}B_k \xrightarrow{P} 0$ as $k\to\infty$. Of course, we proceed in exactly the same way for the lower bound of E_{n2}. Those two results on the upper and lower bounds imply that $\sqrt{k}E_{n2} \xrightarrow{P} 0$ as $k\to\infty$, which completes the proof of $\sqrt{k}E_n = 0_p(1)$ as $k\to\infty$. □

4. Concluding remarks and examples.

The qq estimator is easy to implement and we give some examples of its use. Write $\hat{\alpha}(k)$ for $1/\widehat{\alpha^{-1}}$ when k upper order statistics are used in the estimation of α. We then make a plot of $\{(k,\hat{\alpha}(k)), 1\leq k\leq n\}$ and compare it with the corresponding Hill plot $\{(k, H_{k,n}^{-1}), 1\leq k\leq n\}$.

Figure 1 is a simple comparison of the Hill plot and the qq–plot of estimates of α for 1000 observations from a Pareto distribution where $\alpha = 1$. Examining the qq–plot shows an estimate of about .98. The qq–plot seems to be a bit less volatile than the Hill plot.

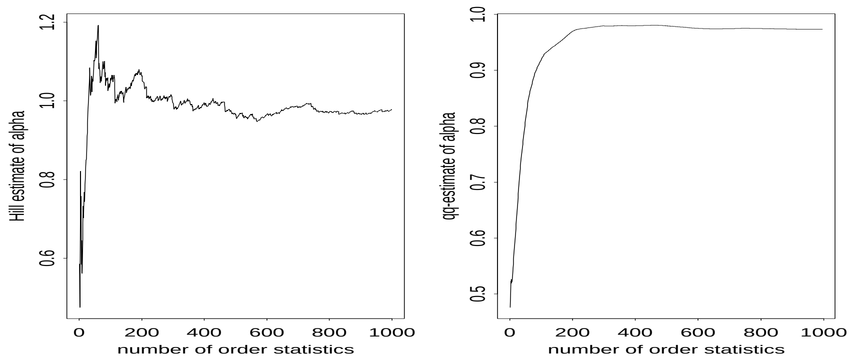

Figure 4.1

Next, we consider an example of a set of real data which exhibits large values and seems to be generated by a sequence of independent random variables. The data set represents the interarrival times between packets generated and sent to a host by a terminal during a logged-on session. The terminal hooks to a network through a host and communicates with the network sending and receiving packets. The length of the periods between two consecutive packets received by the host were recorded as our data. The total length of the data is 783. Figure 4.2 is a time series plot of our data set showing indications of heavy tails.

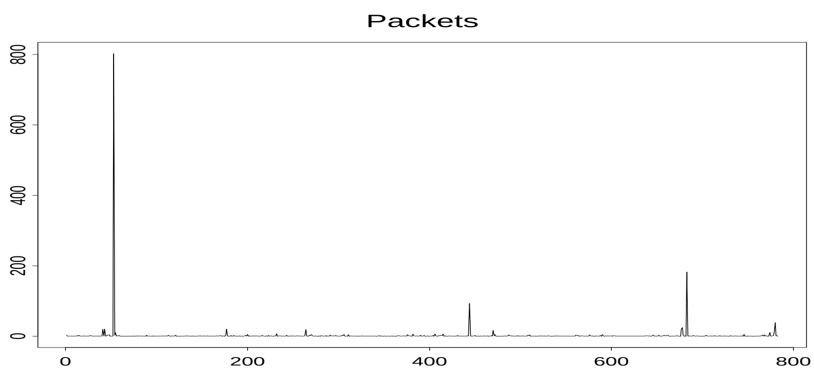

Figure 4.2

To assess the appropriateness of applying either the Hill estimator or the qq–estimator, both of which are designed for independent data, we next examine the classical acf function and a modification called the heavy tailed acf:

$$\rho_{HEAV}(h) = \frac{\sum_{i=1}^{n-h} Z_i Z_{i+h}}{\sum_{i=1}^{n} Z_i^2}.$$

The classical acf function applies a mean correction which is not be appropriate in the heavy tailed case. Plots which vary little from 0 are exploratory evidence of independence; as seen in Figure 4.3, this is the case with this data.

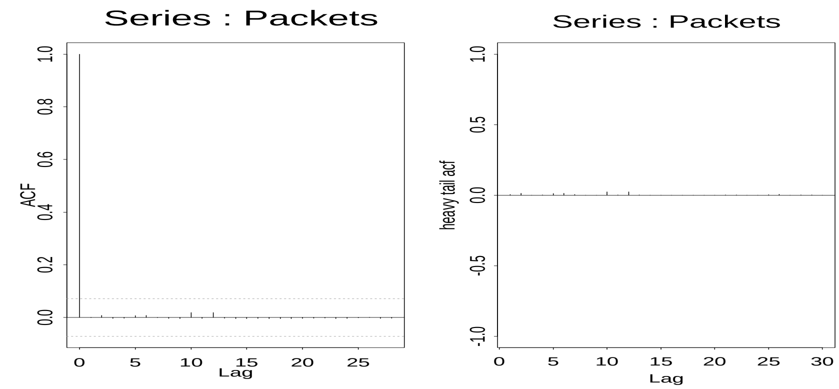

Figure 4.3

Finally, in Figure 4.4 we display the Hill and qq plots. The Hill plot is somewhat inconclusive. The qq–plot indicates a value of about .97 A smoothed version of the Hill plot (Resnick and Starica (1995)) yields a value of about 1.1.

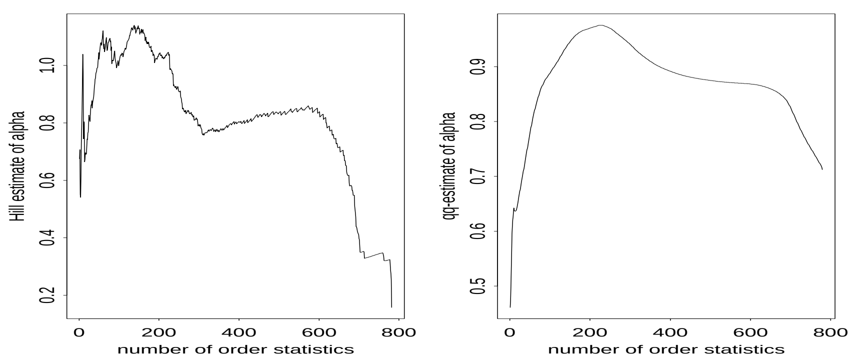

Figure 4.4

The Hill estimator, being optimized for the Pareto distribution, is likely to exhibit considerable bias when the distribution has a regularly varying tail. Consider 5000 observations from the distribution F with tail

$$1 - F(x) \sim x^{-1}(\log x)^5. \tag{4.1}$$

The logarithm in the tail fools both estimators as shown in Figure 4.5. Remember the correct answer is 1.

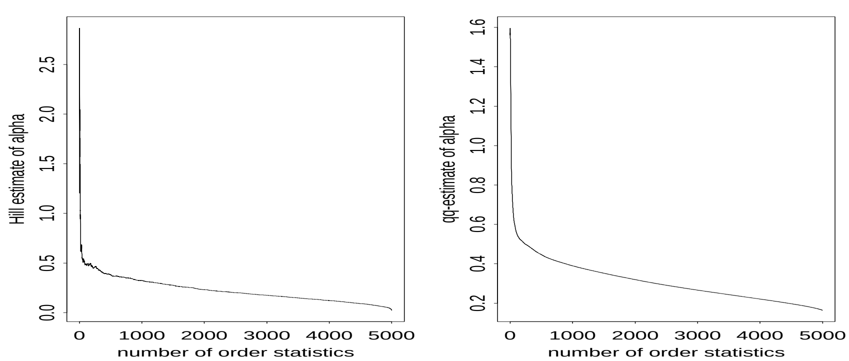

Figure 4.5

Our last plot, Figure 4.6, continues to consider the data from the tail in (4.1) and exhibits a static qq–plot for a fixed value of k, namely $k = 500$. The plot shows many upper order statistics of the log sorted data deviating significantly from the least squares line plotted through the upper 500 log sorted order statistics. One of the advantages of qq–plotting over the Hill estimator is that the residuals contain information which potentially can be utilized to combat the bias in the estimates when the tail is not Pareto. We hope in the near future to develop this into a technique which will be useful in bias correction.

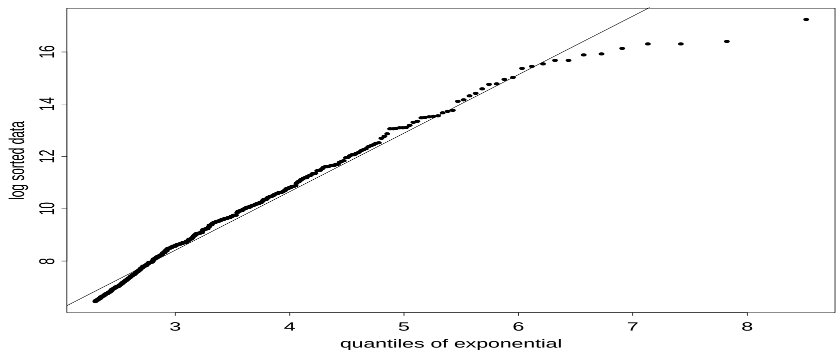

Figure 4.6

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