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**STABILITY OF RANDOM
SETS GENERATED BY
MULTIVARIATE SAMPLES¹**

by

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STABILITY OF RANDOM SETS GENERATED BY MULTIVARIATE SAMPLES

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ABSTRACT. Suppose $\{X_n, n \geq 1\}$ is an iid random sample of random vectors in R^d and let

$$\mathcal{S}_n = \{X_1, \dots, X_n\}$$

be the closed random set induced by the random sample. When does there exist a non-decreasing sequence of non-random compact sets $\{B_n\}$, such that almost surely,

$$\rho(\mathcal{S}_n, B_n) \rightarrow 0, \quad n \rightarrow \infty,$$

where ρ is the Hausdorff metric giving a natural distance measure between two compact sets. When such a monotone sequence exists, we say the sequence of random sets $\{\mathcal{S}_n\}$ is (almost surely) *stable*. Sufficient conditions are given on the underlying distribution guaranteeing stability of the induced random sets and examples are given to show the possibly bizarre nature of the approximating sets.

1. Introduction.

Suppose $\{X_n, n \geq 1\}$ is an iid random sample of random vectors in R^d and let

$$\mathcal{S}_n = \{X_1, \dots, X_n\}$$

be the closed random set induced by the random sample. When does there exist a non-decreasing sequence of non-random compact sets $\{B_n\}$, such that almost surely,

$$(1.1) \quad \rho(\mathcal{S}_n, B_n) \rightarrow 0, \quad n \rightarrow \infty,$$

where ρ is the Hausdorff metric giving a natural distance measure between two compact sets. When such a monotone sequence exists, we say the sequence of random sets $\{\mathcal{S}_n\}$ is (almost surely) *stable*. The stability property of the random sets asserts that the random set $\mathcal{S}_n$ has a decent approximation by a deterministic set B_n and that the approximation gets better as n grows.

If the support of the distribution of the X 's is bounded, then it is elementary to check that $\mathcal{S}_n$ converges in the Hausdorff metric to this support. Thus, we will assume throughout our discussion that the support of the distribution is unbounded.

Proper approximation of the random set induced by a random sample has statistical implications. It may be necessary to estimate the unknown support of a distribution (Moore, 1984; Ripley and Rasson, 1977) based on a sample of size n or the home range of a wildlife species (MacDonald et. al., 1980) based on n sightings of a particular creature. Approximations of random sets have uses in multivariate outlier detection

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and trimming (Barnett, 1976; Eddy, 1982, 1983). In the special case where the population is multivariate normal, Geffroy (1981) has shown that $\mathcal{S}_n$ can be approximated by a growing sequence of ellipsoids and we wish to know in more generality when $\mathcal{S}_n$ is approximable.

A related but easier class of problems considers when $\{\mathcal{S}_n\}$ is *relatively stable*, i.e. when there exist a compact limit set $\mathcal{S}$ and scaling sequences $\{b_n\}$ such that $b_n \rightarrow \infty$ and

$$b_n^{-1}\mathcal{S}_n := \{b_n^{-1}X_1, \dots, b_n^{-1}X_n\} \rightarrow \mathcal{S}$$

as $n \rightarrow \infty$, where convergence is with respect to the Hausdorff metric. In Kinoshita and Resnick (1991), the class of limit sets $\mathcal{S}$ was characterized as polar coordinate transforms of hypographs of upper semi-continuous functions and a necessary and sufficient condition was given for convergence to a particular limit set. See also Davis, Mulrow, Resnick (1988), Eddy (1982), Fisher (1966), Brozius (1991). What makes relative stability far easier to analyze than stability, is that in order for the same normalizing sequence $\{b_n\}$ to be effective in all directions, the tails of the distribution of $\{X_n\}$ must be comparable in all directions. This need not be the case for stability.

In section 3, we consider the one dimensional case where $d = 1$ and $P[X_1 \geq 0] = 1$ and the random variable X_1 is almost surely unbounded. We give a complete solution to the stability problem and find that the random sets $\{\mathcal{S}_n\}$ are stable iff $\{\bigvee_{i=1}^n X_i, n \geq 1\}$ is stable in which case we may take the approximating sets B_n as

$$B_n = [0, b_n] \cap \text{supp}(F)$$

where F is the common distribution of X_n , $\text{supp}(F)$ is the support of F and b_n is the $1 - \frac{1}{n}$ quantile of F , viz,

$$b_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right).$$

We also give an extension of this result to the $d \geq 2$ case where the vector X_1 has its first coordinate restricted to a compact set such as $[0, 1]$ and the second coordinate is unbounded and the two coordinates are independent.

For relative stability, a useful technique was switching to polar coordinates (Kinoshita and Resnick, 1991). This is helpful for stability as well but the transformation is not without difficulties as the polar coordinate transformation is not uniformly bicontinuous. This is discussed in Section 4 where some examples highlight the problems. None-the-less, we consider stability of random sets in polar coordinate space and find that the class of approximating sets is much more complex than one would expect based on experience with relative stability (Kinoshita and Resnick, 1991). For instance, the class of approximating sets sometimes needs to have holes even when the support of the underlying distribution is nicely spread out. This is in contrast to the limit sets in relative stability which are always star shaped (Kinoshita and Resnick, 1991). These matters are discussed in Section 4 and some examples are constructed. Sufficient conditions for stability are given in Section 5 and some examples and concluding discussion is given in Section 6. Some background preliminaries are discussed in Section 2: the Hausdorff metric, upper semi-continuity and hypographs, polar coordinate transformations, stability of maxima and the relation of relative stability to stability.

We have not succeeded in characterizing the class of approximating sets B_n nor can we give necessary and sufficient conditions that $\{B_n\}$ is a suitable approximation. The problem in fully generality seems hard.

2. Preliminaries.

We now consider some items which will be of interest later.

Hausdorff metric: If A is a closed subset of R^d , we define the ϵ -swelling of A as

$$A^\epsilon = \{a : d(a, A) < \epsilon\},$$

where d is the usual Euclidean metric. The Hausdorff metric distance between two compact sets A and B is

$$\rho(A, B) = \inf\{\epsilon : A \subset B^\epsilon, B \subset A^\epsilon\}.$$

When R^d is transformed to a new metric space, we continue to use d and ρ to denote the metrics on the points and closed subsets of the new space.

If $\{A_n\}$ and $\{B_n\}$ are two non-decreasing sequences of compact sets, we have the following pointwise criterion for $\rho(A_n, B_n) \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 2.1. *If $\{A_n\}$ and $\{B_n\}$ are two non-decreasing sequences of compact sets, then as $n \rightarrow \infty$,*

$$\rho(A_n, B_n) \rightarrow 0$$

iff for any $a_n \in A_n$, there exists $b_n \in B_n$ such that $d(a_n, b_n) \rightarrow 0$ and vice versa.

Proof. The necessity of the condition is clear from the definition of the Hausdorff metric. Conversely, suppose the condition holds but $\rho(A_n, B_n)$ does not converge to zero; we will get a contradiction. There must exist $\epsilon > 0$ and a sequence n_k such that

$$\rho(A_{n_k}, B_{n_k}) > \epsilon.$$

Either $A_{n_k} \not\subset B_{n_k}^\epsilon$ or $B_{n_k} \not\subset A_{n_k}^\epsilon$. If, for instance, it happens that infinitely often $A_{n_k} \not\subset B_{n_k}^\epsilon$, then for infinitely many k , there exist $a_{n_k} \in A_{n_k}$ such that

$$d(a_{n_k}, B_{n_k}) > \epsilon/2.$$

For such k , it is impossible to find $b_{n_k} \in B_{n_k}$ such that $d(a_{n_k}, b_{n_k}) \rightarrow 0$ which yields the contradiction. $\square$

As a simple application of this result, suppose we have points $\{x_j, j \geq 1\}$ and $\{y_j, j \geq 1\}$ such that for compact K and a non-decreasing sequence of compacta B_n we have

$$\rho(\{x_1, \dots, x_n\}, K) \rightarrow 0, \quad \rho(\{y_1, \dots, y_n\}, B_n) \rightarrow 0$$

as $n \rightarrow \infty$. Then

$$\rho(\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}, K \cup B_n) \rightarrow 0.$$

The reasoning is as follows: Suppose for every n that

$$a_n \in \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}.$$

If $a_n \in \{x_1, \dots, x_n\}$, then we find $b_n \in K$ such that $d(a_n, b_n)$ is small and if $a_n \in \{y_1, \dots, y_n\}$ we find $b_n \in B_n$. The rest of the verification is similar.

Polar coordinate transformations: A polar coordinate transformation of $R^d \setminus \{0\}$ is a transformation of the form

$$x \mapsto (\|x\|, \|x\|^{-1}x) =: (r, \theta)$$

where $\|x\|$ is some convenient norm on R^d . The unit sphere in R^d in this norm is denoted by $\aleph$:

$$\aleph = \{x \in R^d : \|x\| = 1\}.$$

If we give $R^d \setminus \{0\}$ the topology of R^d relativized to $R^d \setminus \{0\}$, then $\mathcal{F}(R^d \setminus \{0\})$, the closed subsets of $R^d \setminus \{0\}$, is

$$\mathcal{F}(R^d \setminus \{0\}) = \{F^\# \cap (R^d \setminus \{0\}) : F^\# \in \mathcal{F}(R^d)\}.$$

Likewise we give to $(0, \infty) \times \aleph$ the topology relativized from $[0, \infty) \times \aleph$. The closed subsets of $(0, \infty)$ are

$$\mathcal{F}((0, \infty)) = \{F^\# \cap (0, \infty) : F^\# \in \mathcal{F}([0, \infty))\}.$$

Then $x \mapsto (\|x\|, \|x\|^{-1}x)$ is a homeomorphism from $R^d \setminus \{0\} \mapsto (0, \infty) \times \aleph$.

For A compact in R^d , we will abuse notation and write $T(A)$ or TA when we mean $T(A \setminus \{0\})$.

The advantage of the polar coordinate transformation is that the θ -coordinate is restricted to the compact set $\mathbb{N}$.

Upper semi-continuous functions: The space of non-negative upper semi-continuous functions on the space $\mathbb{N}$ is denoted $US(\mathbb{N})$ and has its usual topology (Vervaat, 1988) generated by sub-basis sets of the form

$$\{f \in US(\mathbb{N}) : f^\vee(K) < x\}, \quad \{f \in US(\mathbb{N}) : f^\vee(G) > x\}$$

where K is compact and G is open in $\mathbb{N}$, and

$$f^\vee(K) = \sup_{x \in K} f(x).$$

If $l \in US(\mathbb{N})$ then

$$\text{hypo } l := \{(r, \theta) : 0 < r \leq l(\theta), \theta \in \mathbb{N}\}.$$

A basic fact is that l is upper semi-continuous iff $\text{hypo } l$ is closed in $(0, \infty) \times \mathbb{N}$. If A is a closed subset of $(0, \infty) \times \mathbb{N}$, then we define the *hypograph hull* of A as

$$\text{hypo } A = \text{hypo } l$$

where

$$(2.1) \quad l(\theta) = \begin{cases} \sup\{r > 0 : (r, \theta) \in A\}, & \text{if } \{r > 0 : (r, \theta) \in A\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

If A is bounded as a subset of $[0, \infty) \times \mathbb{N}$, then l is upper semi-continuous since $\text{hypo } l$ is closed in $(0, \infty) \times \mathbb{N}$. To check this, suppose

$$(r_n, \theta_n) \in \text{hypo } A = \{(r, \theta) : 0 < r \leq r', \text{ for some } (r', \theta) \in A\},$$

and that $(r_n, \theta_n) \rightarrow (r, \theta)$. We must verify that $(r, \theta) \in \text{hypo } A$. There exists $(r'_n, \theta_n) \in A$ such that $r_n \leq r'_n$. Since $\{(r'_n, \theta_n)\}$ is compact in $[0, \infty) \times \mathbb{N}$, along some subsequence $\{n'\}$ we have a subsequential limit,

$$(r'_{n'}, \theta_{n'}) \rightarrow (r', \theta) \in A.$$

Since

$$0 < r = \lim_{n' \rightarrow \infty} r_{n'} \leq \lim_{n' \rightarrow \infty} r'_{n'} = r',$$

we have $r' > 0$. Then $(r', \theta) \in A$ and since $r \leq r'$, we have $(r, \theta) \in \text{hypo } A$ and $\text{hypo } A$ is closed.

For considering necessary conditions for stability, it is easier to deal with hypographs, rather than arbitrary closed sets and the following is useful.

Proposition 2.2. *Suppose A_n and $A_n^\#$ are closed subsets of $(0, \infty) \times \mathbb{N}$. If*

$$\rho(A_n, A_n^\#) \rightarrow 0,$$

then

$$\rho(\text{hypo } A_n, \text{hypo } A_n^\#) \rightarrow 0.$$

Proof. It is convenient to define the metric on $(0, \infty) \times \mathbb{N}$ as

$$d((r, \theta), (r', \theta')) := |r - r'| \vee \|\theta - \theta'\|.$$

Given $\epsilon > 0$, we have for all large n that

$$A_n \subset (A_n^\#)^\epsilon, \quad A_n^\# \subset (A_n)^\epsilon.$$

Suppose $(r, \theta) \in \text{hypo } A_n = \text{hypo } l_n$, where l_n is defined as in (2.1). Then $(l_n(\theta), \theta) \in A_n$ and since $A_n \subset (A_n^\#)^\epsilon$ there exists a point $(r^\#, \theta^\#) \in A_n^\#$ such that

$$(2.2) \quad |l_n(\theta) - r^\#| \vee \|\theta - \theta^\#\| < \epsilon.$$

If $r^\# > r$, then $(r, \theta^\#) \in \text{hypo } A_n^\#$ and $d((r, \theta^\#), (r, \theta)) < \epsilon$. If $r^\# \leq r$, then $(r^\#, \theta^\#) \in A_n^\# \subset \text{hypo } A_n^\#$ and

$$|r^\# - r| \leq |r^\# - l_n(\theta)| < \epsilon,$$

from (2.2) so that $d((r, \theta), (r^\#, \theta^\#)) < \epsilon$. For either case, we conclude that $(r, \theta) \in \text{hypo } A_n$, implies $(r, \theta) \in (\text{hypo } A_n^\#)^\epsilon$. Interchanging the roles of A_n and $A_n^\#$ shows how to complete the proof. $\square$

An obvious corollary, is that if the polar transformed version of (1.1) holds, namely

$$\rho(\{(R_j, \Theta_j), j \leq n\}, TB_n) \rightarrow 0,$$

where T is a polar coordinate transformation, then

$$\rho(\cup_{j=1}^n (0, R_j] \times \{\Theta_j\}, \text{hypo } TB_n) \rightarrow 0.$$

If $d = 2$, this last convergence says a family of n lines is getting close to TB_n .

Stability of maxima: Suppose $\{R_i, i \geq 0\}$ are iid non-negative random variables with common distribution F . Define the sample maxima

$$M_n = \bigvee_{i=1}^n R_i, \quad n \geq 1.$$

Call $\{M_n\}$ (almost surely) *stable* iff there exist centering constants $b_n \rightarrow \infty$ such that almost surely

$$(2.3) \quad M_n - b_n \rightarrow 0, \quad n \rightarrow \infty.$$

Note that (2.3) holds iff

$$e^{M_n - b_n} = e^{M_n} / e^{b_n} = \frac{\bigvee_{i=1}^n e^{R_i}}{e^{b_n}} \rightarrow 1$$

so $\{M_n\}$ is stable iff $\{e^{M_n}\}$ is relatively stable and from this follows some characterizations. (See Kinoshita and Resnick, 1991; Resnick and Tomkins, 1973; Barndorff-Nielsen, 1963; Gnedenko, 1943; de Haan, 1970.)

Proposition 2.3. *Suppose $\{R_i, i \geq 0\}$ are iid non-negative random variables with common distribution F . Then $\{M_n\}$ is stable iff for every $\delta > 0$*

$$(2.4) \quad \int_1^\infty \frac{F(dx)}{1 - F(x - \delta)} < \infty.$$

In this case we may choose the normalizing sequence $\{b_n\}$ as

$$b_n = \left(\frac{1}{1 - F} \right)^\leftarrow (n)$$

and $\{b_n\}$ satisfies

$$(2.5) \quad b_{nk} - b_n \rightarrow 0$$

as $n \rightarrow \infty$ for any $k \geq 0$.

Proof. Since $\{e^{M_n}/e^{b_n}\}$ is relatively stable, the results follow from the references given before the statement of the proposition. For (2.5), note that from relative stability, $\{e^{b_n}\}$ is a slowly varying sequence and then (2.5) follows. $\square$

It is fairly easy to see that (2.4) holds if $-\log P[R_1 > \log x]$ is regularly varying with index $\alpha > 0$. So, for example, (2.4) holds for the distribution function

$$F(x) = 1 - \exp\{-e^{\alpha x}\}, \quad x > 0.$$

It may also be checked that (2.4) holds for distributions of the form

$$F(x) = 1 - e^{-x^\beta}, \quad x > 0, \beta > 1.$$

When confusion will not result, we will sometimes abuse terminology and say that a distribution function satisfying (2.4) is *stable*.

Some comparisons of stability and relative stability: Stability for maxima implies relative stability of maxima which is easily seen by dividing in (2.3) by b_n . This relationship of stability and relative stability is not true in general for random sets and we present some simple observations on this relationship. As before, T is a polar coordinate transformation.

Proposition 2.4. *Suppose that as $n \rightarrow \infty$, we have for some $l \in US(\mathbb{N})$, $b_n \rightarrow \infty$, and non-decreasing sequence $\{B_n\}$ of compacta in R^d , that*

$$(2.6) \quad \rho(T(b_n^{-1}\mathcal{S}_n), \text{hypo } l) \rightarrow 0$$

and

$$(2.7) \quad \rho(T(\mathcal{S}_n), T(B_n)) \rightarrow 0.$$

If we define $l_n \in US(\mathbb{N})$ by

$$\text{hypo } T(B_n) = \text{hypo } l_n,$$

then

$$(2.8) \quad b_n^{-1}l_n \rightarrow l,$$

in $US(\mathbb{N})$,

$$(2.9) \quad \rho(T(b_n^{-1}B_n), \text{hypo } l) \rightarrow 0,$$

and

$$(2.10) \quad \rho(\text{hypo } T(b_n^{-1}\mathcal{S}_n), \text{hypo } l) \rightarrow 0.$$

Remark: The form of the limit in (2.6) comes from the result in Kinoshita and Resnick (1991) which shows that the class of limits after a polar coordinate transformation is the class of hypographs of functions in $US(\mathbb{N})$.

Proof. We begin by showing (2.9). Denote $T(X_j) = (R_j, \Theta_j)$. From (2.6), if $(rl(\theta), \theta) \in \text{hypo } l$, $(0 < r \leq 1)$, then there exist $k(n) \leq n$ such that

$$(b_n^{-1} R_{k(n)}, \Theta_{k(n)}) \rightarrow (rl(\theta), \theta).$$

From (2.7) and Proposition 2.1, there exist $(s_n, \omega_n) \in TB_n$ such that

$$d((s_n, \omega_n), (R_{k(n)}, \Theta_{k(n)})) \rightarrow 0.$$

Then $R_{k(n)} - s_n \rightarrow 0$ and therefore

$$\frac{R_{k(n)}}{b_n} - \frac{s_n}{b_n} \rightarrow 0.$$

Thus

$$d((b_n^{-1} s_n, \omega_n), (rl(\theta), \theta)) \rightarrow 0.$$

Conversely, suppose for some subsequence $\{n'\}$ that

$$(b_{n'}^{-1} s_{n'}, \omega_{n'}) \in T(b_{n'}^{-1} B_{n'})$$

converges. We need to show it converges to a limit in hypo l . Since $(s_{n'}, \omega_{n'}) \in T(B_{n'})$, we have from Proposition 2.1, that there exists $(R(n'), \Theta(n')) \in T\mathcal{S}_{n'}$ such that

$$d((s_{n'}, \omega_{n'}), (R(n'), \Theta(n'))) \rightarrow 0.$$

Then

$$d((b_{n'}^{-1} s_{n'}, \omega_{n'}), (b_{n'}^{-1} R(n'), \Theta(n'))) \rightarrow 0$$

as well. Thus $(b_{n'}^{-1} R(n'), \Theta(n'))$ has a limit and because of (2.6), this limit must be in hypo l . We therefore conclude that the limit of $(b_{n'}^{-1} s_{n'}, \omega_{n'})$ must converge to something in hypo l . This is sufficient for showing (2.9). (Cf. Matheron, 1975.)

From (2.9), we know from Proposition 2.2, that

$$\rho(\text{hypo } b_n^{-1} l_n, \text{hypo } l) \rightarrow 0.$$

This gives (2.8) by Vervaat (1989), Section 7. We get (2.10) from Proposition 2.2. $\square$

It seems reasonable that if (2.6) and (2.7) in Proposition 2.4 are satisfied, then (2.9) can be strengthened to:

$$\rho(T(\mathcal{S}_n), b_n \text{hypo } l) \rightarrow 0.$$

The following example shows that this is not true in general.

Example 2.1.

Consider the iid sequence $\{(R_j, \Theta_j), j \geq 1\}$, where $\Theta_n \sim U[0, 1]$ and $F_{R|\Theta}$, the conditional distribution of R_n given Θ_n , is given for $x > 0$ by

$$F_{R|\Theta}(x) = \begin{cases} 1 - \exp\{-x^2\}, & \text{if } \theta < \frac{1}{2} \\ 1 - \exp\{-x^4\}, & \text{if } \theta > \frac{1}{2} \end{cases}.$$

Set $b_n = (\log n)^{\frac{1}{2}}$, and define

$$l_n(\theta) = \begin{cases} (\log n)^{\frac{1}{2}}, & \text{if } \theta < \frac{1}{2} \\ (\log n)^{\frac{1}{4}}, & \text{if } \theta > \frac{1}{2}. \end{cases}$$

Conditional on knowing $1_{[\Theta_n < \frac{1}{2}]}$, we have R_n and Θ_n are independent, and therefore we have by Corollary 3.2 proved below in Section 3, that

$$\rho(\{(R_j, \Theta_j), j \leq n\}, \text{hypo } l_n) \rightarrow 0.$$

If $l(\theta)$ is defined by

$$l(\theta) = \begin{cases} 1, & \text{if } \theta < \frac{1}{2} \\ 0, & \text{if } \theta > \frac{1}{2} \end{cases}$$

then it is not hard to verify that

$$\rho(\{(\frac{R_j}{b_n}, \Theta_j), j \leq n\}, \text{hypo } l) \rightarrow 0$$

where $\text{hypo } l = \{(r, \theta) : 0 < r \leq 1, 0 \leq \theta \leq 1/2\}$. Since

$$b_n \text{hypo } l = \{(r, \theta) : 0 < r \leq \sqrt{\log n}, 0 \leq \theta \leq 1/2\}$$

it is clear that although

$$\rho(\{(R_j, \Theta_j), 1 \leq j \leq n\}, \text{hypo } l_n) \rightarrow 0,$$

and

$$\rho(\{(b_n^{-1} R_j, \Theta_j), 1 \leq j \leq n\}, \text{hypo } l) \rightarrow 0,$$

it is **not** true that

$$\rho(\{(R_j, \Theta_j), 1 \leq j \leq n\}, b_n \text{hypo } l)$$

converges to zero.

We now present a comparison of stability and relative stability in the particular case that $T B_n = \text{hypo } b_n l$, where $b_n \rightarrow \infty$ and $l \in US(\mathbb{N})$.

Proposition 2.5. *If*

$$\rho(T(\mathcal{S}_n), \text{hypo } b_n l) \rightarrow 0,$$

then

$$\rho(T(b_n^{-1} \mathcal{S}_n), \text{hypo } l) \rightarrow 0,$$

as $n \rightarrow \infty$.

Remark: An example, to be given later, shows the result is false unless $T(B_n)$ is of the form $\text{hypo } b_n l$.

Proof. This is straightforward. Given $\epsilon > 0$, for all large n we have $b_n > 1$ and

$$\text{hypo } b_n l \subset (T \mathcal{S}_n)^\epsilon.$$

For $0 < \delta \leq 1$, $(\delta l(\theta), \theta) \in \text{hypo } l$ and therefore $(b_n \delta l(\theta), \theta) \in \text{hypo } b_n l$. There exists $(R_{j(n)}, \Theta_{j(n)}) \in T \mathcal{S}_n$ such that

$$\|\theta - \Theta_{j(n)}\| \vee |b_n \delta l(\theta) - R_{j(n)}| < \epsilon,$$

so that

$$\|\theta - \Theta_{j(n)}\| \vee |\delta l(\theta) - \frac{R_{j(n)}}{b_n}| < \epsilon,$$

and therefore

$$(\delta l(\theta), \theta) \in (T(b_n^{-1} \mathcal{S}_n))^\epsilon.$$

This implies

$$\text{hypo } l \subset (T(b_n^{-1} \mathcal{S}_n))^\epsilon.$$

Conversely, suppose $T\mathcal{S}_n \subset (\text{hypo } b_n l)^\epsilon$. For any (R_j, Θ_j) , there exists $0 < \delta \leq 1$, such that $(b_n l(\theta)\delta, \theta) \in \text{hypo } b_n l$ and

$$\|\Theta_j - \theta\| \bigvee |R_j - b_n l(\theta)\delta| < \epsilon.$$

Therefore,

$$\|\Theta_j - \theta\| \bigvee \left| \frac{R_j}{b_n} - l(\theta)\delta \right| < \epsilon$$

which implies

$$\left(\frac{R_j}{b_n}, \Theta_j \right) \in (\text{hypo } l)^\epsilon$$

and therefore

$$T^{-1}(b_n^{-1}\mathcal{S}_n) \subset (\text{hypo } l)^\epsilon. \quad \square$$

3. One dimensional stability and extensions.

Let $\{R_j, j \geq 1\}$ be iid non-negative random variables with common distribution function F . Denote the support of F by $\text{supp}(F)$. Suppose $\text{supp}(F)$ is unbounded and set $\mathcal{S}_n = \{R_1, \dots, R_n\}$.

Theorem 3.1. *The sequence of sets $\{\mathcal{S}_n\}$ is (a.s.) stable iff $\{M_n = \bigvee_{i=1}^n R_i, n \geq 1\}$ is a.s. stable, in which case*

$$\rho(\mathcal{S}_n, [0, b_n] \cap \text{supp}(F)) \rightarrow 0,$$

where as usual

$$b_n = \left(\frac{1}{1-F} \right)^{\leftarrow} (n).$$

Proof. The only if part is easy. Given $\rho(\mathcal{S}_n, B_n) \rightarrow 0$, then $M_n \in \mathcal{S}_n$ and by Proposition 2.1, there exists $a_n \in B_n$ such that $|M_n - a_n| \rightarrow 0$. Since $M_n \rightarrow \infty$, so also $a_n \rightarrow \infty$ and therefore $\{M_n\}$ is stable.

Conversely, suppose that $\{M_n\}$ is stable. As in Section 1, define the sets B_n by,

$$B_n = [0, b_n] \cap \text{supp}(F)$$

where $b_n = F^{\leftarrow}(1 - n^{-1})$. Fix $\epsilon > 0$ and let $\delta > 0$ be arbitrary. Since $\{b_n, n \geq 1\}$ satisfies (2.5), there exists N such that for all $n \geq N$,

$$(3.1) \quad b_{n+1} - b_n < \epsilon/2.$$

By the stability of $\{M_n\}$, we have that with probability 1, $\mathcal{S}_n \subset B_n^\epsilon$ for n sufficiently large. Also by the stability of $\{M_n\}$, there exists $N_1(\delta) > N$, such that

$$(3.2) \quad P[|M_n - b_n| < \epsilon/2, \forall n \geq N_1(\delta)] \geq 1 - \delta.$$

Now define the sets A_1 and A_2 by

$$A_1 = [0, b_{N_1(\delta)}] \cap \text{supp}(F)$$

and

$$A_2 = [b_{N_1(\delta)}, \infty) \cap \text{supp}(F).$$

Note that if

$$|M_n - b_n| < \epsilon/2, \quad \forall n \geq N_1(\delta),$$

then we have that for all $n > N_1(\delta)$,

$$A_2 \cap B_n = [b_{N_1(\delta)}, b_n] \cap B_n \subset \mathcal{S}_n^\epsilon,$$

since by (3.1), any $y \in A_2 \cap B_n$ is within $\epsilon/2$ of b_k for some $k \geq N_1(\delta)$, and by (3.2) any such b_k is within $\epsilon/2$ of $M_k \in \mathcal{S}_n$.

It is easy to verify that there exists $N_2(\delta)$ such that for all $n > N_2(\delta)$,

$$A_1 \cap B_n \subset \mathcal{S}_n^\epsilon,$$

since $[0, b_{N_1(\delta)}] \cap \text{supp}(F)$ will get filled by repeatedly sampling the R 's. Thus

$$P[B_n \subset \mathcal{S}_n^\epsilon, \forall n > N_1(\delta) \vee N_2(\delta)] \geq 1 - 2\delta.$$

It follows that $\rho(\mathcal{S}_n, B_n) \rightarrow 0$ a.s. $\square$

We now apply Theorem 3.1 to characterize higher dimensional stability when the underlying measure is a product measure, one of whose factors concentrates on a compact set.

Corollary 3.2. *Let $\{(R_j, \Theta_j), j \geq 1\}$ be iid random elements of $(0, \infty) \times \mathbb{R}$, where $\mathbb{R}$ is a compact subset of a finite dimensional Euclidean space and for each j , R_j and Θ_j are independent. Let the support of the distribution of Θ_1 be $\mathbb{R}_1$ and the support of the distribution of R_1 be supp and suppose supp is unbounded. There exists a non-decreasing sequence of closed sets A_n , bounded in $[0, \infty) \times \mathbb{R}$ such that*

$$\rho(\{(R_j, \Theta_j), 1 \leq j \leq n\}, A_n) \rightarrow 0,$$

iff $\{M_n = \vee_{i=1}^n R_i, n \geq 1\}$ is relatively stable. In this case, we may take A_n as:

$$(3.3) \quad A_n = (0, b_n] \cap \text{supp} \times \mathbb{R}_1,$$

where as usual

$$b_n = \left(\frac{1}{P[R_1 > \cdot]} \right)^{\leftarrow}(n).$$

Proof. Suppose that $\{M_n\}$ is stable. Let $\epsilon > 0$ be given. Since $\mathbb{R}_1$ is compact, we may find m neighborhoods $I_k, 1 \leq k \leq m$ which cover $\mathbb{R}_1$ with the diameter of each I_k being at most ϵ . Define

$$A_n^{(k)} = (0, b_n] \cap \text{supp} \times I_k$$

so that $\cup_{k=1}^m A_n^{(k)} = A_n$. Fix k and suppose without loss of generality that

$$p_k := P[\Theta_1 \in I_k] > 0.$$

Define $N_n^{(k)}$ to be the number of the first n observations that are in $(0, \infty) \times I_k$; i.e.

$$N_n^{(k)} = \sum_{i=1}^n 1_{[\Theta_i \in I_k]}$$

and define the indices where the Θ 's fall in I_k by

$$\begin{aligned} \nu_k(1) &= \inf\{j : \Theta_j \in I_k\}, \\ \nu_k(2) &= \inf\{j > \nu_k(1) : \Theta_j \in I_k\}, \\ &\vdots \quad \quad \quad \vdots \end{aligned}$$

etc. Then $\{R_{\nu_k(i)}, i \geq 1\}$ is an iid sequence, each of whose members has the distribution of R_1 . Hence by Theorem 3.1 we have

$$\rho(\{R_{\nu_k(i)}, 1 \leq i \leq n\}, (0, b_n] \cap \text{supp}) \rightarrow 0.$$

Therefore, since $N_n^{(k)} \rightarrow \infty$ we have

$$(3.4) \quad \rho(\{R_{\nu_k(i)}, 1 \leq i \leq N_n^{(k)}\}, (0, b_{N_n^{(k)}}] \cap \text{supp}) \rightarrow 0.$$

Since $N_n^{(k)}/n \rightarrow p_k > 0$, we have from (2.5) that

$$b_{N_n^{(k)}} - b_n = b_{[n \cdot N_n^{(k)}/n]} - b_n \rightarrow 0$$

and thus

$$(3.5) \quad \rho((0, b_{N_n^{(k)}}] \cap \text{supp}, (0, b_n] \cap \text{supp}) \rightarrow 0.$$

We therefore conclude from (3.4) and (3.5) that

$$\rho(\{R_j : \Theta_j \in I_k, 1 \leq j \leq n\}, [0, b_n] \cap \text{supp}) \rightarrow 0.$$

So for given ϵ and sufficiently large n we have

$$\begin{aligned} & \{(R_j, \Theta_j), 1 \leq j \leq n\} \\ &= \cup_{k=1}^m \{(R_j, \Theta_j) : \Theta_j \in I_k, 1 \leq j \leq n\} \\ &\subset \cup_{k=1}^m ((0, b_n] \cap \text{supp} \times I_k)^\epsilon \\ &\subset (\cup_{k=1}^m (0, b_n] \cap \text{supp} \times I_k)^\epsilon \\ &= ((0, b_n] \cap \text{supp} \times \mathbb{R}_1)^\epsilon. \end{aligned}$$

Also

$$\begin{aligned} & (0, b_n] \cap \text{supp} \times \mathbb{R}_1 \\ &= \cup_{k=1}^m ((0, b_n] \cap \text{supp} \times I_k) \\ &\subset \cup_{k=1}^m \{(R_j, \Theta_j) : \Theta_j \in I_k, 1 \leq j \leq n\}^{2\epsilon} \\ &\subset (\cup_{k=1}^m \{(R_j, \Theta_j) : \Theta_j \in I_k, 1 \leq j \leq n\})^{2\epsilon} \\ &= \{(R_j, \Theta_j), 1 \leq j \leq n\}^{2\epsilon}. \end{aligned}$$

This completes the proof in one direction. To show the implication in the other direction, let

$$b_n = \sup\{r : (r, \theta) \in A_n\}.$$

Then,

$$|M_n - b_n| \leq \rho(\{(R_j, \theta_j), 1 \leq j \leq n\}, A_n),$$

which implies that the $\{M_n\}$ is stable. $\square$

4. Remarks on the shape of approximating sets and the polar coordinate transformation.

Suppose $\{\mathbf{X}_j, j \geq 1\}$ are iid random vectors in R^d . If there exist constants $b_n \rightarrow \infty$ such that for some compact set S

$$\rho(\{b_n^{-1}\mathbf{X}_1, \dots, b_n^{-1}\mathbf{X}_n\}, S) \rightarrow 0,$$

then as verified in Kinoshita and Resnick (1991), S must be star shaped. This means that if $\mathbf{x} \in S$, then for any $0 < \delta < 1$ we have $\delta\mathbf{x} \in S$. After a polar coordinate transformation T we get for some $l \in US(\aleph)$ that

$$\rho(T(\{b_n^{-1}\mathbf{X}_1, \dots, b_n^{-1}\mathbf{X}_n\}), \text{hypo } l) \rightarrow 0.$$

So the class of limit sets for relative stability is easy to describe: In cartesian coordinates the limit sets are star shaped and in polar coordinates they are hypographs of non-negative upper semi continuous functions on $\aleph$. However, for stability the situation is more complicated. We now give an example where approximating sets in polar coordinates are not hypographs, nor is there any sequence of hypographs which can serve as approximating sets.

Example 4.1.

Suppose $\{(R_j, \Theta_j), j \geq 1\}$ are iid on $E := (0, \infty) \times [0, 4]$ where R_n and Θ_n are independent. Furthermore, suppose Θ_n is uniformly distributed on $[0, 4]$, and that R_n has distribution function

$$P[R_n \leq x] = 1 - e^{-x^2}, \quad x \geq 0.$$

This distribution satisfies the integral condition of Proposition 2.3. Therefore, Corollary 3.2 implies that with $b_n = \sqrt{\log n}$ and $A_n = (0, b_n] \times [0, 4]$,

$$(4.1) \quad \rho(\{(R_j, \Theta_j), j \leq n\}, A_n) \rightarrow 0$$

almost surely.

We will construct our example by applying a convergence preserving transformation to (4.1).

For $n \geq 2$, let T_n be the open set of points of E inside the triangle with vertices at $(2n, 1)$, $(2n, 3)$ and $(2, 2n+1)$; that is,

$$T_n = \{(r, \theta) \in E : 1 < \theta < 3; 2n < r < 2n+1 - |\theta - 2|\}.$$

Set $T = \cup_{n=2}^{\infty} T_n$. Define the function $\phi : E \rightarrow E$, by

$$\phi((r, \theta)) = \begin{cases} (r, \theta), & \text{if } (r, \theta) \in T^c \\ (r-2, \theta), & \text{if } (r, \theta) \in T. \end{cases}$$

Consider now the iid sample $\{(R'_j, \Theta'_j) = \phi(R_n, \Theta_n), j \leq n\}$ and set $A'_n = \phi(A_n)$. Although ϕ is not continuous, we will verify that almost surely

$$(4.2) \quad \rho(\{(R'_j, \Theta'_j), j \leq n\}, A'_n) \rightarrow 0.$$

We need two lemmas to help with the verification of (4.2).

Lemma 4.1. *Suppose $0 < \epsilon < 1$. If $s \in \{(R_j, \Theta_j), j \leq n\}$, $\alpha \in A_n$, and $d(s, \alpha) < \epsilon$, then there exists $\alpha' \in A'_n$ such that $d(\phi(s), \alpha') < \epsilon$.*

Proof. There are three cases to be considered.

- (1) If $s, \alpha \in T^c$ then set $\alpha' = \alpha$. If $s, \alpha \in T$, then set $\alpha' = \phi(\alpha)$.
- (2) Suppose $s \in T$ and $\alpha \in T^c$. Denote the coordinates of s by (R, Θ) and the coordinates of α by (r, θ) . Since $s \in T$, we have $R \geq 4$. So we may assume that $r \geq 2$. Since $\alpha \in A_n \cap T^c$, we have that $\alpha' := (r-2, \theta) \in A'_n$. Since $d(\phi(s), \alpha') = d(s, \alpha)$, we are done.

- (3) Suppose that $s \in T^c$ and $\alpha \in T$. Continue to suppose that $s = (R, \Theta)$ and $\alpha = (r, \theta)$. By the definition of A_n , if $R \leq r$ and $s \in T^c$ then $\phi(s) = s$. Thus, $s \in A'_n$, and the Lemma is true with $\alpha' = s$. Now assume that $R > r$. If also $\Theta \notin [1, 3]$, then we may set $\alpha' = (b_n \wedge R, \Theta)$ and then $\alpha' \in A'_n$ and

$$\begin{aligned} d(\alpha', \phi(s)) &= d((b_n \wedge R, \Theta), (R, \Theta)) \\ &= |b_n \wedge R - R| \\ &= \begin{cases} 0, & \text{if } b_n \geq R, \\ |b_n - R| \leq |r - R| \leq \epsilon, & \text{if } b_n < R. \end{cases} \end{aligned}$$

Thus, for the rest of the discussion we will assume that $s \notin T, \alpha \in T, R > r, \Theta \in [1, 3]$ and for concreteness we assume $\Theta \in [1, 2]$. Define

$$\alpha' := (r, \Theta - (R - r)).$$

This is the point α of the triangle moved left by an amount $R - r$. Since A_n is the hypograph of a constant function and $0 < R - r < \epsilon$, we have $\alpha' \in A_n$. We wish to show $\alpha' \in A'_n$. First we demonstrate that $\alpha' \notin T$ and that the left shift of α actually moves α to the new position α' which is outside the triangle. Suppose $\alpha \in T_k$ for some fixed k . Since $s \notin T$,

$$(4.3) \quad R > 2k + 1 - |\Theta - 2| = 2k - 1 + \Theta.$$

In order to show $\alpha' \notin T$, we need to show

$$(4.4) \quad r > 2k + 1 - |\Theta - (R - r) - 2| = 2k - 1 + \Theta - R + r.$$

Subtracting r from both sides we see that (4.4) holds iff

$$0 > 2k - 1 + \Theta - R;$$

that is, iff

$$R > 2k - 1 + \Theta.$$

Since this is (4.3) we conclude that $\alpha \in A_n, \alpha' \notin T$, and hence $\alpha' \in A'_n$. Since

$$\begin{aligned} d(\alpha', \phi(s)) &= d(\alpha', s) \\ &= |R - r| \vee |\Theta - (\Theta - (R - r))| \\ &= |R - r| \vee |R - r| \leq \epsilon, \end{aligned}$$

the desired conclusion is obtained. If $2 \leq \Theta \leq 3$, then set

$$\alpha' := (r, \Theta + (R - r))$$

and apply a similar argument. The proof of lemma 4.1 is complete. $\square$

The proof of (4.2) requires one more lemma.

Lemma 4.2. *If $\rho(\{(R_j, \Theta_j), j \leq n\}, A_n) < \epsilon$, and $\alpha \in A_n$, there exists $s' \in \{(R'_j, \Theta'_j), j \leq n\}$ such that $d(s', \phi(\alpha)) < 13\epsilon$.*

Proof. As in the previous lemma, by the hypothesis of this lemma, there exists $s \in \{(R_j, \Theta_j), j \leq n\}$ such that $d(s, \alpha) < \epsilon$. Now

$$\begin{aligned} d(\phi(\alpha), \phi(s)) &\leq d(\phi(\alpha), \alpha) + d(\alpha, s) + d(s, \phi(s)) \\ &\leq 2 + \epsilon + 2 = 4 + \epsilon. \end{aligned}$$

So without loss of generality, we may assume $\epsilon < 1/3$. (If $\epsilon > 1/3$, so that $3\epsilon > 1$, we would already have

$$\begin{aligned} d(\phi(\alpha), \phi(s)) &\leq 4 + d(\alpha, s) \\ &\leq 12\epsilon + \epsilon = 13\epsilon \end{aligned}$$

and there would be nothing else to prove.

As in the previous lemma, there are three cases to be considered.

- (1) If $s, \alpha \in T$ then set $s' = \phi(s)$. If $s, \alpha \in T^c$ then set $s' = s$.
- (2) Suppose that $s \in T$ and $\alpha \in T^c$. In order to verify this case, we will show that there exist $\hat{s} \in \{(R_j, \Theta_j), j \leq n\}$ such that $d(\alpha, \hat{s}) < 3\epsilon$ and $\hat{s} \in T^c$. Continue to suppose that $\alpha = (r, \theta)$ and $\epsilon < \frac{1}{3}$. Since $d(s, \alpha) < \epsilon$, and $s \in T$ it must be that $\frac{2}{3} \leq \theta \leq \frac{9}{3}$. Also since $s \in T$, there exists k such that $s \in T_k$. If $r \leq 2k$ (i.e. r lies below the base of the triangle T_k) then set $\hat{\alpha} = (r - 2\epsilon, \theta)$, otherwise define $\hat{\alpha}$ by

$$\hat{\alpha} = \begin{cases} (r, \theta - 2\epsilon), & \text{if } \theta < 2, \\ (r, \theta + 2\epsilon), & \text{if } \theta \geq 2. \end{cases}$$

The point $\hat{\alpha}$ is the point α shifted by the amount 2ϵ in the appropriate direction; the goal is to ensure

$$(4.5) \quad \{\hat{\alpha}\}^\epsilon \cap T = \emptyset.$$

In each of the above cases, since A_n is the hypograph of a constant function we have that $\hat{\alpha} \in A_n$.

By the hypothesis of the lemma, there exists $\hat{s} \in \{(R_j, \Theta_j), j \leq n\}$ such that $d(\hat{s}, \hat{\alpha}) < \epsilon$. By (4.5) we have that $\hat{s} \notin T$, which implies that $\hat{s} = \phi(\hat{s})$. Using the fact that $d(\alpha, \hat{\alpha}) = 2\epsilon$ and the triangle inequality we have that

$$d(\hat{s}, \phi(\alpha)) = d(\hat{s}, \alpha) < 3\epsilon.$$

So set $s' = \hat{s}$ and we are done.

- (3) Suppose $s \in T^c$ and $\alpha \in T$. By the proof of the third case in Lemma 4.1 there exists $\alpha' \in A_n \cap T^c$ such that $d(s, \alpha') < \epsilon$. This implies that $d(\alpha, \alpha') < 2\epsilon$. With $\alpha' = (r', \theta')$, set $\hat{\alpha} =: (r' - 2, \theta')$. This is the point α' moved downward by the amount 2, so

$$d(\hat{\alpha}, \phi(\alpha)) = d(\alpha', \alpha) < 2\epsilon.$$

By the hypothesis there exists $\hat{s} \in \{(R_j, \Theta_j), j \leq n\}$ such that $d(\hat{s}, \hat{\alpha}) < \epsilon$. Now since $\hat{\alpha} \in T^c$ one of the first two cases of the lemma must apply to $\hat{s}$ and $\hat{\alpha}$ so there exists $s' \in \{(R'_j, \Theta'_j), j \leq n\}$ such that $d(s', \hat{\alpha}) < 3\epsilon$. Hence we have that $d(s', \phi(\alpha)) < 5\epsilon$. $\square$

Since $\rho(\{(R_j, \Theta_j), j \leq n\}, A_n) \rightarrow 0$ a.s., we have by Lemmas 4.1 and 4.2 that almost surely

$$\rho(\{(R'_j, \Theta'_j), j \leq n\}, A'_n) \rightarrow 0.$$

We now wish to establish that if $\{\tilde{A}_n, n \geq 1\}$ is a sequence of hypographs of non-negative upper semicontinuous functions, then $\{\tilde{A}_n, n \geq 1\}$ cannot serve as approximating sets for $\{(R'_j, \Theta'_j), j \leq n\}$. We may find integers k and n such that $k \geq 2$, and

$$2k + 1 \leq b_n \leq 2k + 2.$$

In this case, $A'_n = \phi(A_n)$ has a triangular cutout. If $\tilde{A}_n$ is a hypograph of an upper semicontinuous function, then $\rho(A'_n, \tilde{A}_n) \geq \frac{1}{2}$. So by the triangle inequality, $\{\tilde{A}_n\}$ can not be an approximating sequence. This completes the discussion of Example 4.1.

We will work in $(0, \infty) \times \mathbb{N}$ since it is much easier to consider the stability problem when one coordinate is restricted to a compact set. However, stability of $\{(R_j, \Theta_j), j \leq n\} = \{T(X_j), j \leq n\}$ does not necessarily mean stability of $\{X_j, j \leq n\}$. The difficulty is illustrated in the next example.

Example 4.2.

Let $E = R_+^2$ and $\|\cdot\|$ be the Euclidean norm. Define the sequences of sets $\{A_n\}$ and $\{B_n\}$ by

$$B_n = \{x \in E : \|x\| \leq 2^n\}$$

and

$$A_n = \{(\frac{ri}{2^n}, r(1 - \frac{i^2}{4^n})^{\frac{1}{2}}), 0 \leq i \leq 2^n, r = \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{4^n}{2^n}\}.$$

Define the polar coordinate transformation T by,

$$T(x) = (\|x\|, \frac{x^{(1)}}{\|x\|})$$

where $x = (x^{(1)}, x^{(2)})$. Then we have that

$$T(\frac{ri}{2^n}, r(1 - \frac{i^2}{4^n})^{\frac{1}{2}}) = (r, \frac{i}{2^n})$$

and therefore

$$T(A_n) = \{(\frac{r}{2^n}, \frac{j}{2^n}), 1 \leq r \leq 4^n, 1 \leq j \leq 2^n\}$$

and $T(B_n) = (0, 2^n] \times [0, 1]$. If d is defined on $T(E)$ by,

$$d(x, y) = \|x^1 - y^1\| \vee \|x^2 - y^2\|$$

then it is easy to see that

$$\rho(T(A_n), T(B_n)) = \frac{1}{2^{n+1}} \rightarrow 0,$$

but we will show that it is not true that in Cartesian coordinates

$$(4.6) \quad \rho(A_n, B_n) \rightarrow 0.$$

For fixed n consider the point β_n , in B_n , given by $\beta_n = 2^n(\frac{3}{2^{n+1}}, (1 - \frac{9}{4^{n+1}})^{\frac{1}{2}})$. Since $A_n \subset B_n$,

$$\begin{aligned} \rho(A_n, B_n) &\geq d(\beta_n, A_n) = d(\beta_n, 2^n(\frac{1}{2^n}, (1 - \frac{2}{2^n})^{\frac{1}{2}})) \\ &\geq 2^n(\frac{3}{2^{n+1}} - \frac{1}{2^n}) = \frac{1}{2}. \end{aligned}$$

So $\rho(A_n, B_n)$ does not converge to 0.

Although $T : R^d \setminus \{0\} \mapsto (0, \infty) \times \mathbb{N}$ is a homeomorphism, it is not uniformly bicontinuous, as would be necessary if stability in one space were to be equivalent to stability in the second space. The following observation emphasizes this point.

Proposition 4.3. *Let T be a bijection of $R^d \setminus \{0\}$ and $(0, \infty) \times \aleph$. Then $\rho(TA_n, TB_n) \rightarrow 0$ implies $\rho(A_n, B_n) \rightarrow 0$ for any compact subsets A_n, B_n of R^d , iff T^{-1} is uniformly continuous.*

Note if T is a polar coordinate transformation, T^{-1} is not uniformly continuous.

Proof. Suppose T^{-1} is uniformly continuous. For any compacta A_n, B_n suppose $\rho(TA_n, TB_n) \rightarrow 0$. Then if $a_n \in A_n$, there exists $b_n \in B_n$ such that $d(T(a_n), T(b_n)) \rightarrow 0$ and since T^{-1} is uniformly continuous we have $d(a_n, b_n) \rightarrow 0$. Similarly, if $b_n \in B_n$, there exists $a_n \in A_n$ such that $d(a_n, b_n) \rightarrow 0$, and therefore we have $\rho(A_n, B_n) \rightarrow 0$.

Conversely, if T^{-1} is not uniformly continuous, then there exist $a_n, b_n \in R^d \setminus \{0\}$ and $d(T(a_n), T(b_n)) \rightarrow 0$, but $d(a_n, b_n) \not\rightarrow 0$. Let $A_n = \{a_n\}, B_n = \{b_n\}$ to finish.

5. Sufficient conditions for stability.

We now present some sufficient conditions for stability. It is easier to deal with hypographs of upper semi-continuous functions. Therefore, the sufficient conditions that we present will in fact be for stability of hypo $\mathcal{S}_n$ and we will investigate when hypo $\mathcal{S}_n$ can be approximated by a sequence of monotone approximating sets which are hypographs.

We begin with simple criteria.

Proposition 5.1. *Suppose $\aleph$ is a compact metric space and that $f_n, g_n \in US(\aleph)$. Define*

$$A_n = \text{hypo } f_n, \quad B_n = \text{hypo } g_n.$$

The following are equivalent.

(1)

$$\lim_{n \rightarrow \infty} \rho(A_n, B_n) = 0.$$

(2) *For all $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for all $n \geq n_0$ and all compact $K \subset \aleph$*

$$\begin{aligned} f_n^\vee(K) &\leq g_n^\vee(K^\epsilon) + \epsilon \\ g_n^\vee(K) &\leq f_n^\vee(K^\epsilon) + \epsilon. \end{aligned}$$

(3) *Given a compact $K \subset \aleph$ and any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon, K)$ such that for all $n \geq n_0(\epsilon, K)$*

$$\begin{aligned} f_n^\vee(K) &\leq g_n^\vee(K^\epsilon) + \epsilon \\ g_n^\vee(K) &\leq f_n^\vee(K^\epsilon) + \epsilon. \end{aligned}$$

(4) *For all $\epsilon > 0$ and any compact $K \subset \aleph$,*

$$\begin{aligned} \limsup_{n \rightarrow \infty} f_n^\vee(K) - g_n^\vee(K^\epsilon) &\leq \epsilon \\ \limsup_{n \rightarrow \infty} g_n^\vee(K) - f_n^\vee(K^\epsilon) &\leq \epsilon. \end{aligned}$$

Remark: The second equivalence is similar to Proposition 2.1 of Khanna and Roy (1991).

Proof. (1) $\rightarrow$ (2) : Given $\rho(A_n, B_n) \rightarrow 0$, and given $\epsilon > 0$, there exists n_0 such that for $n \geq n_0$ we have

$$A_n \subset B_n^\epsilon, \quad B_n \subset A_n^\epsilon.$$

For any $K \subset \aleph$, there exists $x_n \in K$ such that

$$f_n^\vee(K) = f_n(x_n).$$

Then if $n \geq n_0$ we have

$$(f_n(x_n), x_n) \in A_n \subset B_n^\epsilon.$$

Therefore, there exists $(\theta_n g(y_n), y_n) \in B_n$ such that $0 \leq \theta_n \leq 1$ and

$$|x_n - y_n| \vee |f_n(x_n) - \theta_n g(y_n)| \leq \epsilon.$$

So

$$\begin{aligned} f_n^\vee(K) &= f_n(x_n) \leq \theta_n g(y_n) + \epsilon \\ &\leq g(y_n) + \epsilon \end{aligned}$$

and because $|x_n - y_n| \leq \epsilon$ and $x_n \in K$ we get

$$f_n^\vee(K) \leq g_n^\vee(K^\epsilon) + \epsilon,$$

as desired. The second inequality is obtained by symmetry.

(2) $\rightarrow$ (3) : Obvious.

(3) $\rightarrow$ (4) : Obvious.

(4) $\rightarrow$ (1) : The set of all compact subsets of $\mathbb{N}$, $\mathcal{K}(\mathbb{N})$, is a compact metric space when metrized by the Hausdorff metric (Vervaat, 1988) and therefore is totally bounded: Given ϵ , there is a finite ϵ -net $\{K_1, \dots, K_m\}$ such that for any K there is K_i closeby; i.e. $\rho(K, K_i) < \epsilon$. Given this ϵ , there exist n_i such that if $n \geq n_i$ then

$$\begin{aligned} f_n^\vee(K_i) &\leq g_n^\vee(K_i^\epsilon) + 2\epsilon \\ g_n^\vee(K_i) &\leq f_n^\vee(K_i^\epsilon) + 2\epsilon. \end{aligned}$$

Now suppose $n \geq \vee_{i=1}^m n_i$ and that $x_n \in A_n = \text{hypo } f_n$. Then x_n can be written as

$$x_n = (\delta_n f(a_n), a_n),$$

for some $a_n \in \mathbb{N}$ and $0 \leq \delta_n \leq 1$. Set $K = \{a_n\}$ and for some i_0 we have $K \subset K_{i_0}$ so that

$$\begin{aligned} f_n(a_n) &\leq f_n^\vee(\{a_n\}) \leq f_n^\vee(K_{i_0}) \\ &\leq g_n^\vee((\overline{K_{i_0}})^\epsilon) + 2\epsilon \\ &\leq g_n^\vee(K_{i_0}^{2\epsilon}) + 2\epsilon. \end{aligned}$$

There exists $a'_n \in K_{i_0}^{2\epsilon}$ such that

$$g_n^\vee(K_{i_0}^{2\epsilon}) \leq g_n^\vee(\{a'_n\}) + \epsilon.$$

Therefore,

$$f_n(a_n) \leq g_n^\vee(\{a'_n\}) + 3\epsilon.$$

Also $(g_n(a'_n), a'_n) \in B_n = \text{hypo } g_n$ so that $\delta_n g_n(a'_n), a'_n) \in B_n$. Since

$$|a_n - a'_n| < 2\epsilon$$

we get

$$x_n = (\delta_n f_n(a_n), a_n) \in B_n^{3\epsilon}.$$

Therefore, $A_n \subset B_n^{3\epsilon}$ if $n \geq n_0$ and a symmetric argument provides the reverse inequality, thus finishing the proof. $\square$

From these criteria emerge the following sufficient condition.

Corollary 5.2. Suppose $\aleph$ is a compact metric space and that $f_n, g_n \in US(\aleph)$. Continue to suppose

$$A_n = \text{hypo } f_n, \quad B_n = \text{hypo } g_n.$$

Let $\mathcal{K} \subset \mathcal{K}(\aleph)$ be a class of compact subsets of $\aleph$ such that for any $K \in \mathcal{K}(\aleph)$ and any $\epsilon > 0$, there exists $K_0 \in \mathcal{K}$ (depending on K and ϵ) such that

$$(5.1) \quad K \subset K_0 \subset K^\epsilon.$$

If for all $K_0 \in \mathcal{K}$ we have

$$(5.2) \quad f_n^\vee(K_0) - g_n^\vee(K_0) \rightarrow 0,$$

as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \rho(A_n, B_n) = 0.$$

Proof. Let K be an arbitrarily chosen compact set and suppose $\epsilon > 0$ is given. Choose K_0 as in (5.1). For sufficiently large n we have

$$f_n^\vee(K_0) \leq g_n^\vee(K_0) + \epsilon, \quad g_n^\vee(K_0) \leq f_n^\vee(K_0) + \epsilon.$$

Therefore, for such large n ,

$$\begin{aligned} f_n^\vee(K) &\leq f_n^\vee(K_0) \leq g_n^\vee(K_0) + \epsilon \\ &\leq g_n^\vee(K^\epsilon) + \epsilon. \end{aligned}$$

A similar inequality with the roles of f_n and g_n reversed can be obtained and thus by Proposition 5.1, we get $\lim_{n \rightarrow \infty} \rho(A_n, B_n) = 0$. $\square$

From this we have sufficient conditions for stability of hypographs.

Theorem 5.3. Let $\{\mathbf{X}_n, n \geq 1\}$ be iid random variables in R^d and suppose after a polar coordinate transformation these variables are expressed as $\{(R_n, \Theta_n), n \geq 1\}$. Let $\aleph = \{x \in R^d : \|x\| = 1\}$ be the unit sphere. For any compact set $K \subset \aleph$, define the quantile function

$$b_n(K) = \left(\frac{1}{P[R_1 > \cdot | \Theta_1 \in K]} \right)^{-} (n).$$

Let $\mathcal{K}$ be a class of compact subsets of $\aleph$ such that (5.1) is true. Suppose for some sequence $\{l_n\} \subset US(\aleph)$ we have for every $K \in \mathcal{K}$ such that $P[\Theta_1 \in K] > 0$ that:

(1) the distribution

$$P[R_1 \in \cdot | \Theta_1 \in K]$$

is stable and

(2)

$$l_n^\vee(K) - b_n(K) \rightarrow 0.$$

If $P[\Theta_1 \in K] = 0$, for $K \subset \mathcal{K}$, suppose

$$l_n^\vee(K) \rightarrow 0.$$

Then almost surely, we have

$$\lim_{n \rightarrow \infty} \rho(\text{hypo } \{(R_j, \Theta_j), 1 \leq j \leq n\}, \text{hypo } l_n) \rightarrow 0.$$

Proof. Define the upper semi-continuous function f_n on $\mathbb{R}$ by

$$f_n(\theta) = \sum_{j=1}^n R_j 1_{\{\Theta_j\}}(\theta).$$

If $P[\Theta_1 \in K] > 0$, and $K \in \mathcal{K}$, then from the stability assumption

$$f_n^\vee(K) = \bigvee_{\Theta_j \in K} R_j = b_n(K) + o(1) = l_n^\vee(K) + o(1),$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$ almost surely. This follows because $\{R_j : \Theta_j \in K\}$ is iid with common distribution $P[R_1 \in \cdot | \Theta_1 \in K]$.

If $P[\Theta_1 \in K] = 0$, then $f_n^\vee(K) = 0$ and thus

$$f_n^\vee(K) - l_n^\vee(K) = -l_n^\vee(K) \rightarrow 0.$$

The desired result now follows from the previous Corollary 5.2. $\square$

A possible way to get l_n from $b_n(\cdot)$ is discussed in Section 6.

In case R_1 and Θ_1 are independent (for example, if spherical symmetry is present), the condition that the conditional distribution of R_1 given $\Theta_1 \in K$ be stable reduces to stability of the distribution of R_1 and we recover part of Corollary 3.2. Another special case is obtained if we suppose $l_n = b_n l$ where $b_n \rightarrow \infty$ and $l \in US(\mathbb{R})$. This is discussed next.

Corollary 5.4. Let $\{\mathbf{X}_n, n \geq 1\}$ be iid random variables in R^d and suppose after a polar coordinate transformation these variables are expressed as $\{(R_n, \Theta_n), n \geq 1\}$. Let $\mathbb{R}$, $b_n(K)$, be as in Theorem 5.3 and let $b_n = b_n(\mathbb{R})$. Define for $l \in US(\mathbb{R})$

$$\mathcal{K}_l = \{\overline{\{a\}^r} : a \in \mathbb{R}, r > 0, l^\vee(\{a\}^r) = l^\vee(\overline{\{a\}^r})\}.$$

For every $K \in \mathcal{K}_l$ such that $P[\Theta_1 \in K] > 0$, suppose we have

(1) the distribution

$$P[R_1 \in \cdot | \Theta_1 \in K]$$

is stable and

(2)

$$(5.3) \quad b_n(K) - b_n l^\vee(K) \rightarrow 0,$$

If $P[\Theta_1 \in K] = 0$, suppose

$$l^\vee(K) = 0.$$

Then almost surely, we have

$$\lim_{n \rightarrow \infty} \rho(\text{hypo } \{(R_j, \Theta_j), 1 \leq j \leq n\}, b_n \text{ hypo } l) \rightarrow 0.$$

Proof. The class $\mathcal{K}_l$ satisfies (5.1) (cf. Vervaat, 1988, page 26). The result then follows from Theorem 5.3. $\square$

The condition (5.3) is a tail balancing condition which describes how the quantities $P[R_1 > x | \Theta_1 \in K]$ compare for large x when the set of directions K varies. Compare this with condition (4.6) of Kinoshita and Resnick (1991).

6. Additional comments and examples.

Intrinsic characterization of approximating sets: Theorem 5.3 suggests that the function l_n defining the hypograph approximating set should be related to a quantile of the conditional distribution of R given Θ . How can we make this more precise?

Recall that $\{\mathbf{X}_n, n \geq 1\}$ is an iid random sequence in R^d and that after a polar coordinate transformation these variables can be expressed as $\{(R_n, \Theta_n), n \geq 1\}$. $\mathbb{N}$ is the unit sphere $\{x \in R^d : \|x\| = 1\}$. For any compact set $K \subset \mathbb{N}$, define

$$b_t(K) = \left(\frac{1}{P[R_1 > \cdot | \Theta_1 \in K]} \right)^{\leftarrow} (t).$$

The second condition of Theorem 5.3 is that for some $l_n \in US(\mathbb{N})$

$$(6.1) \quad l_n^\vee(K) - b_n(K) \rightarrow 0.$$

Suppose the joint density of R_1, Θ_1 is so smooth that for any $a \in \mathbb{N}$

$$(6.2) \quad P[R_1 > r | \Theta_1 \in \{a\}^\epsilon] \rightarrow P[R_1 > r | \Theta_1 = a],$$

as $\epsilon \rightarrow 0$, where the right side of (6.2) is a fixed version of the conditional distribution of R_1 . (What we have in mind is the situation where the joint density of (R_1, Θ_1) exists and so does the conditional density.) From (6.2) we get

$$H_\epsilon(r) := \frac{1}{P[R_1 > r | \Theta_1 \in \{a\}^\epsilon]} \rightarrow H(r) := \frac{1}{P[R_1 > r | \Theta_1 = a]}.$$

Now H_ϵ is a family of non-decreasing functions converging to a limit; thus the inverses also converge (Resnick, 1987, page 5; Billingsley, 1986). Thus we get as $\epsilon \rightarrow 0$ that

$$\begin{aligned} b_t(\{a\}^\epsilon) &:= \left(\frac{1}{P[R_1 > \cdot | \Theta_1 \in \{a\}^\epsilon]} \right)^{\leftarrow} (t) \\ &\rightarrow \left(\frac{1}{P[R_1 > \cdot | \Theta_1 = a]} \right)^{\leftarrow} (t) \\ &=: b_t(a). \end{aligned}$$

The idea now is to hope the following is true: For compact K in a sufficiently rich class $\mathcal{K}$ satisfying (5.1)

$$(6.3) \quad \bigvee_{a \in K} b_t(a) - b_t(K) \rightarrow 0$$

as $t \rightarrow \infty$. If (6.3) holds, then we define

$$l_n^\vee(K) = \bigvee_{a \in K} b_n(a)$$

and then l_n can be defined as the sup-derivative (Vervaat, 1988) of $l_n^\vee$. This characterizes l_n intrinsically in terms of the distribution.

We need the following lemma to make this argument more precise.

Lemma 6.1. *Given U_1, U_2 , monotone functions on R such that for all $x > 0$*

$$(6.4) \quad U_2^{+-}(t+x) - U_2^{+-}(t) \rightarrow 0$$

as $t \rightarrow \infty$. Then

$$U_2^-(t) - U_1^-(t) \rightarrow 0$$

iff

$$\lim_{t \rightarrow \infty} \frac{U_1(t+x)}{U_2(t)} = \begin{cases} \infty, & \text{if } x > 0. \\ 0, & \text{if } x < 0. \end{cases}$$

Proof. We have

$$\frac{e^{U_2^-(t)}}{e^{U_1^-(t)}} \rightarrow 1$$

iff

$$\frac{U_2(\log t)}{U_1(\log tx)} \rightarrow \begin{cases} 0, & \text{if } x > 1, \\ \infty, & \text{if } x < 1 \end{cases}$$

from Proposition 3.5 of Kinoshita and Resnick, 1991. Changing variables gives the desired result. $\square$

We may now state and prove the characterization.

Proposition 6.2. *Suppose the assumptions of Theorem 5.3 hold. Suppose for all $a \in \mathbb{N}$, as $\epsilon \rightarrow 0$ we have*

$$(6.2) \quad P[R_1 > r | \Theta_1 \in \{a\}^\epsilon] \rightarrow P[R_1 > r | \Theta_1 = a]$$

for all $r > 0$ and $a \in \mathbb{N}$. If for $K \in \mathcal{K}$ we have as $s \rightarrow \infty$

$$(6.5) \quad \frac{P[R_1 > s + y | \Theta_1 \in K]}{\bigvee_{a \in K} P[R_1 > s + y | \Theta_1 = a]} \rightarrow \begin{cases} 0, & \text{if } y > 0, \\ \infty, & \text{if } y < 0, \end{cases}$$

then

$$(6.6) \quad \bigvee_{a \in K} b_t(a) - b_t(K) \rightarrow 0$$

and we may define the upper semi-continuous function l_n appearing in Theorem 5.3 as the sup-derivative of the sup-measure

$$l_n^V(K) := \bigvee_{a \in K} b_t(a).$$

Proof. Set $U_2(t) = b_t(K)$ so that (6.4) is satisfied and set $U_1(t) = \bigvee_{a \in K} b_t(a)$. The equivalence of (6.5) and (6.6) follows from Lemma 6.1. $\square$

In the special case that $l_n = b_n l$ we may define l intrinsically as follows: Suppose (6.2) and (6.5) hold in addition to the assumptions of Theorem 5.3. Then (6.6) holds. If in addition

$$(6.7) \quad \lim_{t \rightarrow \infty} b_t(a)/b_t \rightarrow \tilde{l}(a)$$

for all $a \in \mathbb{N}$ and furthermore

$$(6.8) \quad b_t(a) - b_t \tilde{l}(a) \rightarrow 0,$$

locally uniformly in a , then we may define the upper semi-continuous function l as the sup-derivative of the sup-measure $\tilde{l}^V$. To check this, note that (6.6) still holds. Because of (6.8) we get

$$\bigvee_{a \in K} b_t(a) - b_t \tilde{l}^V(K)$$

and the result follows.

We will illustrate the use of Proposition 6.2 with the following class of random variables. Let $f(\theta), \theta \in [0, 1]$ be a continuous strictly positive function. Consider the iid sequence $\{(R_j, \Theta_j), j \geq 1\}$, where $\Theta_n \sim U[0, 1]$ and $F_{R|\Theta}(\cdot)$, the conditional distribution of R_n given Θ_n , is given for $x > 0$ by

$$F_{R|\Theta}(x|\theta) = 1 - \exp(-f(\theta)x^2).$$

Lemma 6.3. *For any compact set K the distribution*

$$P[R_1 \in \cdot | \Theta_1 \in K]$$

is stable.

Proof. For a compact set K , let $F_K(\cdot)$ be the distribution function for R_n given $\Theta_n \in K$. Set $M = f^\vee(K)$ and $m = f^\wedge(K)$. Now,

$$F_K(dx) = \frac{1}{P(K)} \int_K 2xf(\theta)e^{-x^2f(\theta)}d\theta$$

and

$$\begin{aligned} 1 - F_K(x - \delta) &= \frac{1}{P(K)} \int_K e^{-(x-\delta)^2f(\theta)}d\theta \\ &\geq e^{(2x\delta - \delta^2)m} \frac{1}{2MxP(K)} \int_K 2xf(\theta)e^{-x^2f(\theta)}d\theta \\ &= e^{(2x\delta - \delta^2)m} \frac{1}{2Mx} F_K(dx). \end{aligned}$$

Therefore

$$\int_1^\infty \frac{F_K(dx)}{1 - F_K(x - \delta)} \leq \int_1^\infty 2Mxe^{-2x\delta m + \delta^2 m} dx < \infty,$$

which verifies that $F_K(\cdot)$ is stable.

Lemma 6.3 verifies that for this class of random variables, condition (1) of Theorem 5.3 is satisfied. To verify condition (2), one must first come up with a suitable upper semi-continuous function $l_n(\theta)$.

Example 6.1.

Let $f(\theta)$ be given by

$$f(\theta) = 1 + \theta, \quad \theta \in [0, 1].$$

In this case, $b_t(\theta)$ is given by

$$b_t(\theta) = \sqrt{\log(t)/(1 + \theta)}$$

Define the class of compact sets $\mathcal{K}$ by,

$$\mathcal{K} = \{\{\theta\}^r : \theta \in [0, 1], r > 0\}.$$

The function suggested by Proposition 6.2 would be $l_n(\theta) = b_n l(\theta)$, where $l(\theta) = 1/(f(\theta))^{\frac{1}{2}}$ and $b_n = \sqrt{\log n}$. If $K \in \mathcal{K}$, then K has the form $K = [a - r, a + r]$. In order to verify (2) of Theorem 5.3 for this choice of $l_n(\theta)$, we need to verify that

$$l_n^\vee(K) - b_n(K) \rightarrow 0.$$

We have that

$$\begin{aligned} F_K(l_n^\vee(K)) &= F_K(b_n/(1 + a - r)^{\frac{1}{2}}) \\ &= \frac{1}{2r} \int_{a-r}^{a+r} 1 - \exp(-(1 + \theta)[b_n/(1 + a - r)^{\frac{1}{2}}]^2) d\theta \\ &\geq \frac{1}{2r} \int_{a-r}^{a+r} 1 - \exp(-[b_n]^2) d\theta \\ &= 1 - 1/n. \end{aligned}$$

Hence, $b_n(K) \leq l_n^V(K)$ for any n . Now let $\epsilon > 0$. We want to show that for n large enough, $b_n(K) \geq l_n^V(K) - \epsilon$. In order to simplify the notation, let

$$x_n = l_n^V(K) - \epsilon = b_n/(1 + a - r)^{\frac{1}{2}} - \epsilon.$$

$$\begin{aligned} F_K(x_n) &= \frac{1}{2r} \int_{a-r}^{a+r} 1 - \exp(-(1 + \theta)[x_n]^2) d\theta \\ &= 1 - \frac{1}{(x_n)^2} \exp(-(1 + a - r)[x_n]^2)(1 - \exp(-2r[x_n]^2)) \\ &= 1 - (1/n) \left[\frac{1}{(x_n)^2} \exp(-(1 + a - r)[-2\epsilon b_n/(1 + a - r)^{\frac{1}{2}} + \epsilon^2])(1 - \exp(-2r[x_n]^2)) \right] \end{aligned}$$

For n sufficiently large,

$$1 - (1/n) \left[\frac{1}{(x_n)^2} \exp(-(1 + a - r)[-2\epsilon b_n/(1 + a - r)^{\frac{1}{2}} + \epsilon^2])(1 - \exp(-2r[x_n]^2)) \right] < 1 - 1/n,$$

which by the continuity of $F_K(\cdot)$ implies that $b_n(K) > l_n^V(K) - \epsilon$. This together with Lemma 6.3 gives us that by Theorem 5.3 we have that almost surely

$$\lim_{n \rightarrow \infty} \rho(\text{hypo } \{(R_j, \Theta_j), 1 \leq j \leq n\}, \text{hypo } l_n) \rightarrow 0.$$

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