SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853-3801

TECHNICAL REPORT NO. 1053

July 1993

**A STORAGE MODEL
FOR DATA
COMMUNICATION SYSTEMS**

by

N.U. Prabhu and
António Pacheco¹

¹Research supported by Grant BD/645/90-RM from Junta Nacional de Investigação Científica e Tecnológica. On leave from: Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal.

A STORAGE MODEL FOR DATA COMMUNICATION SYSTEMS

N.U. PRABHU and ANTÓNIO PACHECO*

*School of Operations Research and Industrial Engineering
Cornell University, E&TC Building
Ithaca, NY 14853-3801, USA*

July 15, 1993

Abstract: We consider a storage model where the input and demand are modulated by an underlying Markov chain. Such models arise in data communication systems. The input is a Markov-compound Poisson process and the demand is a Markov linear process. The demand is satisfied if physically possible. We study the properties of the demand and its inverse, which may be viewed as transformed time clocks. We show that the unsatisfied demand is related to the infimum of the net input and that, under suitable conditions, it is an additive functional of the input process. The study of the storage level is based on a detailed analysis of the busy period, using techniques based on infinitesimal generators. The transform of the busy period is the unique solution of a certain matrix-functional equation. Steady state results are also obtained; these are not obvious generalizations of the results for simple storage models. In particular, a generalization of the Pollaczek-Khinchin formula brings new insight.

Keywords: Additive functional, busy period, communication systems, infinitesimal generator, integral equation, manufacturing, Markov-additive processes, Markov-compound Poisson, Markov modulation, matrix-functional equation, Pollaczek-Khinchin formula, storage models.

*Research supported by Grant BD/645/90-RM from Junta Nacional de Investigação Científica e Tecnológica. On leave from: Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal.

1 Introduction

In this paper we analyze a storage process where the input as well as the output are modulated by an underlying Markov chain. Such processes arise frequently from models for data communication systems, which have received considerable attention during the last few years. The specific model studied in this paper was motivated by the pioneering paper by Anick, Mitra and Sondhi [1], who proposed the following model for a multiple source data-handling system. There are N sources of messages, which may be *on* or *off* from time to time. A switch receives these messages at a unit rate from each *on* source and transmits them at a fixed maximum rate c ($1 \leq N < \infty$, $0 < c < \infty$), storing the messages that it cannot transmit in a buffer of infinite capacity. The sources act independently, the durations of *on* and *off* times being independent random variables with exponential densities. Denoting by $J(t)$ the number of *on* sources at time t , these assumptions amount to the statement that $J = \{J(t), t \geq 0\}$ is a birth and death process on the state space $\{0, 1, \dots, N\}$. Also, let us denote by $Z(t)$ the buffer content at time t . The model states that

$$Z(t) = Z(0) + \int_0^t z(Z(s), J(s)) ds, \quad (1)$$

where z is the net input rate given by

$$z(x, j) = \begin{cases} j - c & x > 0 \\ \max(j - c, 0) & x = 0 \end{cases}. \quad (2)$$

The authors derive the steady state distribution of the process (Z, J) , employing matrix-algebraic techniques.

Gaver and Lehoczky [7] investigate a model for an integrated circuit and packet switched multiplexer, in which there are two types of input – data and voice calls. The input of data occurs continuously at a constant rate c_1 (fluid input), and is transmitted at rate c_2 . There are $s + u$ output channels, of which s are reserved for data transmission, while the remaining u are shared by data and voice calls, with calls having priority with preemptive discipline. Calls arrive at a Poisson rate and have independent holding times with exponential density. Calls that find all u channels that service them busy are lost. Data that cannot be transmitted are stored in a buffer of infinite capacity. Let $Z(t)$ be the buffer content and $J(t)$ the number of calls in progress at time t . Then the model states that J is identical with the number in the Erlang loss system $M/M/u/u$, while $Z(t)$ satisfies equation (1) with the net input rate z given by

$$z(x, j) = \begin{cases} c_1 - c_2(s + u - j) & x > 0 \\ \max(c_1 - c_2(s + u - j), 0) & x = 0 \end{cases}. \quad (3)$$

We note that in the two models described above the maximum output rate is constant. However, in practice, such factors as failures of output channels may induce over time a stochastic behaviour on this rate. This is accounted for by Mitra [10], who investigates a fluid model for a system with two groups of machines connected by a buffer. The first group of machines (producers) produces a fluid which is transferred to the buffer and is consumed by a second group of machines (consumers). These machines are *in service* or *in repair* states from time to time and the assumptions for *in service* times and *in repair* times are similar to those on Anick et al. [1] for *on* and *off* periods. The buffer has capacity C ($0 < C \leq \infty$), while the production rate and consumption rate by machine, assumed to be constants, are c_1 and c_2 . We let $J_1(t)$ and $J_2(t)$ denote respectively the number

of *producers* and *consumers* that are *in service* at time t , from the total of N *producers* and M *consumers*. It follows that $J_1 = \{J_1(t) | t \geq 0\}$ and $J_2 = \{J_2(t) | t \geq 0\}$ are independent Markov processes on $\{0, 1, \dots, N\}$ and $\{0, 1, \dots, M\}$, respectively. Again, if we denote by $Z(t)$ the buffer content at time t , the model states that

$$Z(t) = Z(0) + \int_0^t z(Z(s), J_1(s), J_2(s)) ds, \quad (4)$$

where the net input rate z is given by

$$z(x, j_1, j_2) = \begin{cases} \max(c_1 j_1 - c_2 j_2, 0) & x = 0 \\ c_1 j_1 - c_2 j_2 & 0 < x < C \\ \min(c_1 j_1 - c_2 j_2, 0) & x = C \end{cases} . \quad (5)$$

This model is applicable in communication-integrated systems with *producers* being interpreted as sources and *consumers* as channels.

Van Doorn, Jagers and de Wit [17] consider a reservoir (dam) model regulated by a birth and death process J on the state space $\{0, 1, \dots, N\}$ with arbitrary birth and death rates. There is some imprecision in the formulation of the model, but their assumptions seem to imply the equation (1) for the dam content $Z(t)$, with the net input rate z given by

$$z(x, j) = \begin{cases} z_j & x > 0 \\ \max(z_j, 0) & x = 0 \end{cases} . \quad (6)$$

It should be noted that the net input rate of this model is essentially of the same type as (2) and (3). However, the underlying Markov chain makes a weaker presence in this model than in the context of data communication systems as described above.

A model for an access regulator was investigated by Elwalid and Mitra [5]. Here the input of data is regulated by a continuous time Markov chain J with a finite number of states, the input rate being $\lambda(j)$ when J is in state j (the so called Markov modulated fluid source). In addition to a finite data buffer receiving messages (cells), as in Mitra [10], there is a token buffer also of finite size, receiving tokens continuously at rate r . In order to get transmitted, cells have to be combined with tokens. Some cells are lost because of a regulator that controls transmission subject to a peak rate ν .

Models similar to those described here arise also in manufacturing systems. Browne and Zipkin [4] consider an (s, S) inventory system where the demand rate is controlled by an underlying Markov chain J . The model states that the demand in $(0, t]$ is given by

$$D(t) = \int_0^t d(J(s)) ds, \quad (7)$$

The authors study steady state properties of the system. Sevast'yanov [14] analyzes a system with N machines in tandem with buffers in between. Each item has to be processed successively by machines $1, 2, \dots, N$, and items that cannot be processed by a machine are stored in the preceding buffer. The system may be viewed as a tandem storage model modulated by an auxiliary process J indicating the number of machines in operation. With appropriate assumptions, J will turn to be an N -dimensional Markov chain, each component of which indicates whether or not the corresponding machine is in operation. This model and the one by Elwalid and Mitra [5] are simple

examples of storage networks modulated by an underlying Markov chain. Work in this class of models is in progress and will be reported elsewhere. For other references on Markov modulation on communication and manufacturing systems see the papers cited.

The model investigated in this paper is an extension of the model of Anick et al. [1]. We first note that (1) can be written as

$$Z(t) = Z(0) + X(t) - \int_0^t r(Z(s), J(s)) ds, \quad (8)$$

with

$$X(t) = \int_0^t J(s) ds \quad (9)$$

and

$$r(x, j) = \begin{cases} c & x > 0 \\ \min(c, j) & x = 0 \end{cases}. \quad (10)$$

Here $X(t)$ is the total input into the buffer in the time-interval $(0, t]$ and r is the release rate. If we denote by $(X, J) = \{(X, J)(t), t \geq 0\}$ the input process, then it is easily seen that (X, J) is a Markov-additive process (MAP) with the additive component X consisting of a nonnegative drift that depends on the current state of the Markov chain.

In our model we denote by J a Markov chain on a countable state space and assume that the input process (X, J) is an MAP, where, in addition to a nonnegative drift $a(j)$ when J is in state j , X has nonnegative jumps whose rate and size depend on the state of J . This formulation provides for the possibility of two sources of input, one slow source bringing in data in a fluid fashion and the other bringing in packets. As for the demand for transmission of data we assume that it arises at a rate $d(j)$ when the current state of J is j (similar to what happens in the model considered by Mitra [10]). The storage policy is to meet the demand if physically possible, i.e. the output channel is used the most possible. We postulate that the buffer content $Z(t)$ at time t satisfies the integral equation (8) with our input process (X, J) and the release rate r given by

$$r(x, j) = \begin{cases} d(j) & x > 0 \\ \min(d(j), a(j)) & x \leq 0 \end{cases}. \quad (11)$$

We prefer, however, to write (8) slightly differently, as follows. Let X_0 be the part of the input X free of drift, so that

$$X(t) = X_0(t) + \int_0^t a(J(s)) ds. \quad (12)$$

Then (8) becomes

$$Z(t) = Z(0) + X_0(t) + \int_0^t z(Z(s), J(s)) ds, \quad (13)$$

with

$$z(x, j) = \begin{cases} a(j) - d(j) & x > 0 \\ \max(a(j) - d(j), 0) & x \leq 0 \end{cases}. \quad (14)$$

At this stage we make an additional assumption, namely that the demand rate exceeds the input rate from the slow source, so that $d(j) > a(j)$. Thus finally we obtain

$$Z(t) = Z(0) + X_0(t) - \int_0^t 1_{\{Z(s)>0\}} d_1(J(s)) ds, \quad (15)$$

where $d_1(j) = d(j) - a(j) > 0$. Accordingly we may accept (15) as our postulate, changing X_0 to X (free of drift) and d_1 to d .

The following comments are in order concerning our model. In a personal discussion, D. Mitra asked why the input and the demand are modulated by a common Markov chain. Our response to this is that we can formulate an input process (X, J_1) and a demand rate function $d(J_2(t))$, where $J = (J_1, J_2)$ is a two-dimensional Markov chain with possibly dependent components J_1, J_2 , such that J_1, J_2 together modulate the input and the demand. As an example, in Mitra [10] $J = (J_1, J_2)$ is a two-dimensional birth and death process.

A second comment on our model is that the assumption of infinite storage (buffer) capacity may be unrealistic. D. Mitra seems to feel that this may not be necessarily so. In any case, we believe that our model will serve at least as an approximation to the finite capacity case and will in fact provide a reasonable understanding of how data communication networks operate.

Our release rate is time dependent and induces dependence on the storage level through the underlying Markov chain. It is possible to formulate more general release rules among the ones that are a function of the storage level as well the underlying Markov chain. However, considerable analytical difficulties may arise; in particular, some processes of interest may be hard to study or may even not be measurable, and some desirable properties of the model may not hold.

Our final comment is that we have not investigated the computational aspects of our model. However, we believe that numerical procedures can be developed to derive the quantities of interest.

The paper is organized as follows. In section 2 we introduce our model and prove the existence and uniqueness of the solution of the basic integral equation. In section 3 we investigate the properties of the actual and unsatisfied demands, and in section 4 the inverse of the actual demand. If we denote by $D(t)$ the total demand in $(0, t]$ and by $B(x)$ its inverse, then the main result is that $\{(D, J)(t), t \geq 0\}$ and $\{(B, J \circ B)(x), x \geq 0\}$ are both MAPs consisting of just a drift. The busy period problem is investigated in section 5. If T denotes the busy period, it follows that $(T, J \circ T)$ is an MAP with both drift and jumps. Section 6 contains some additional notations and results of a technical nature. Transforms of the various processes of interest are obtained in sections 7-9. The steady state behaviour of the model is investigated in section 10.

2 The Model

(a) The Input. The input $\{X(t), t \geq 0\}$ is such that $(X, J) = \{(X, J)(t), t \geq 0\}$ is a Markov-compound Poisson process (MCP) with nonnegative jumps and no drift, and infinitesimal generator given by

$$\mathcal{A}_{(X, J)} f(x, j) = \sum_{k \in \mathcal{E}} \sigma_{jk} \int_0^\infty [f(x + y, k) - f(x, j)] M_{jk} \{dy\}, \quad (16)$$

where $\mathcal{E}$ is a countable set, and $f(x, j)$ is a bounded function on $\mathbb{R}_+ \times \mathcal{E}$ such that for each fixed j , f is continuous and has a bounded continuous derivative $\frac{\partial f}{\partial x}(x, j)$. Thus, jumps in X associated with a transition from state j to state k in J occur at rate σ_{jk} and have distribution M_{jk} . The distributions M_{jk} , $j, k \in \mathcal{E}$ are concentrated on $[0, \infty)$.

(b) *The Demand.* The demand is Markov linear, and a function of the same Markov jump process (MJP) J as the input, namely

$$D(t) = \int_0^t d(J(s)) ds, \quad (17)$$

where d is a positive measurable function. We define d_- and d_+ as follows:

$$0 \leq d_- = \inf_{j \in \mathcal{E}} d(j) \leq \sup_{j \in \mathcal{E}} d(j) = d_+ \leq +\infty. \quad (18)$$

(c) *The Release Policy.* The release rule states that the demand is satisfied if physically possible so that the amount of unsatisfied demand is given by (see Remark 1 and Example 1 below)

$$I(t) = \int_0^t d(J(s)) 1_{\{Z(s) \leq 0\}} ds. \quad (19)$$

The *net input* process is given by $Y(t) = X(t) - D(t)$, and the *output* process is given by $O(t) = D(t) - I(t)$. The underlying MJP J , which modulates the jumps on the input as well as the demand rates has, from (16), infinitesimal generator

$$\mathcal{A}_J f(j) = \lambda_j \sum_{k \in \mathcal{E}} [f(k) - f(j)] p_{jk}, \quad j \in \mathcal{E}, \quad (20)$$

where

$$\Sigma = (\sigma_{jk}) = \Lambda P = (\lambda_j \delta_{jk}) (p_{jk}), \quad (21)$$

with P being a stochastic matrix. We call Σ the *transition rate matrix*, the diagonal matrix Λ the *intensity matrix*, and P the *transition probability matrix* of J . We assume J is non-explosive, which holds, in particular, if $\sup \lambda_j < \infty$. In case J is ergodic, we denote by $\pi = \{\pi_j, j \in \mathcal{E}\}$ its stationary distribution and let $\Pi = \mathbf{1}\pi^T$ where $\mathbf{1}^T = (1, 1, \dots, 1)$, so that

$$\Pi = \left(\lim_{t \rightarrow \infty} P[J(t) = k \mid J(0) = j] \right).$$

We let $\tilde{J} = \{J_n, n \geq 0\}$ be the associated discrete time Markov chain (DTMC) with (stationary) transition probability matrix P . We may then consider $J_n = J(T_{n+1}-)$, ($n \geq 0$) with

$T_n =$ time at which the n -th transition in J occurs.

Having defined the storage process one very important question we need to address is of the following type:

What are the sufficient conditions for the storage process to be a friendly object, e.g. measurable with respect to the σ -field in which we consider the processes X and D ; progressively-measurable with respect to a given filtration of the process (X, D) , etc.?

Theorem 1 *We are given a stochastic process (X, J) , as defined above, on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Consider the integral equation*

$$Z(t) = Z(0) + X(t) - D(t) + \int_0^t d(J(s)) 1_{\{Z(s) \leq 0\}} ds, \quad (22)$$

where $Z(0) \geq 0$. We have ($\mathcal{P}$ -a.s.):

(i) *The integral equation (22) has the unique solution*

$$Z(t) = Z(0) + Y(t) + I(t), \quad (23)$$

where

$$I(t) = \max\{0, -m(t) - Z(0)\}, \quad (24)$$

with

$$m(t) = \inf_{0 \leq \tau \leq t} Y(\tau). \quad (25)$$

(ii) *D and Y are $\mathcal{F}$ -measurable stochastic processes, and Z and I are $(\mathcal{F} \vee \sigma(Z(0)))$ -measurable.*

(iii) *D and I have continuous sample paths, and Y and Z have càdlàg (right-continuous with left-hand limits) sample paths with all jumps being positive.*

(iv) *If in addition we are given a filtration $\{\mathcal{F}_t, t \geq 0\}$ and (X, J) is adapted to the filtration $\{\mathcal{F}_t\}$ then, in addition to the results of (i) – (iii), J, X, D and Y are progressively measurable with respect to $\{\mathcal{F}_t\}$, and I and Z are progressively measurable with respect to $\{\mathcal{F}_t \vee \sigma(Z(0))\}$.*

(v) *If $\mathcal{F}_t^o = \sigma\{(X, J)(s), s \leq t\}$ and $\mathcal{F}_t$ is the proper completion of $\mathcal{F}_t^o$, then D and Y are progressively measurable with respect to $\{\mathcal{F}_t\}$, and I and Z are progressively measurable with respect to the proper completion of $\{\mathcal{F}_t^o \vee \sigma(Z(0))\}$.*

Proof: During the proof, it should be kept in mind that $\mathcal{P}$ -a.s.: (i) (X, J) has càdlàg sample paths, (ii) in a finite time interval, the number of jumps of (X, J) is finite (since J is non-explosive) and (iii) all jumps of X are nonnegative. For simplicity we omit $\mathcal{P}$ -a.s. in the proof.

(i) This proof is based on Prabhu [11]. Let ω , a point in the sample space satisfying (i)-(iii), and $t > 0$ be fixed. From the setting of the theorem and equation (22) we easily see that $Z(t) \geq 0, \forall t$ and for $\tau \in [0, t]$, with $Z(0-) = Z(0)$,

$$Z(t) \geq Z(t) - Z(\tau-) = Y(t) - Y(\tau-) + \int_{\tau}^t d(J(s)) 1_{\{Z(s) \leq 0\}} ds \geq Y(t) - Y(\tau-). \quad (26)$$

If $Z(\tau)$ is never zero for $\tau \in (0, t)$ then from (22) and (26)

$$Z(t) = Z(0) + Y(t) \geq \sup_{0 \leq \tau \leq t} [Y(t) - Y(\tau-)]; \quad (27)$$

otherwise let $t_0 = \max\{\tau : 0 < \tau \leq t, Z(\tau-) = 0\}$ and use (26) and (22) to get

$$Z(t) \geq \sup_{0 \leq \tau \leq t} [Y(t) - Y(\tau-)] \geq [Y(t) - Y(t_0-)] = Z(t). \quad (28)$$

Since $Z(t) \geq Z(0) + Y(t), \forall t \geq 0$, using (27) and (28), it follows easily that

$$Z(t) = \max\{Z(0) + Y(t), \sup_{0 \leq \tau \leq t} [Y(t) - Y(\tau-)]\} = \max\{Z(0) + Y(t), Y(t) - m(t)\}, \quad (29)$$

which with (22) implies that

$$I(t) = \int_0^t d(J(s)) 1_{\{Z(s) \leq 0\}} ds = Z(t) - Z(0) - Y(t) = \max\{0, -m(t) - Z(0)\},$$

as required.

(ii) Since (X, J) is right-continuous it is also $\mathcal{F}$ -measurable by Proposition 1.13 and Remark 1.14 (Chapter 1) in Karatzas and Shreve [9]. The $\mathcal{F}$ -measurability of J implies the $\mathcal{F}$ -measurability of $d(J(t))$ which in turn implies the $\mathcal{F}$ -measurability of D by Remark 4.6(i) (Chapter 1) in Karatzas and Shreve [9], and therefore $Y = X - D$ is also $\mathcal{F}$ -measurable as does $m = \{m(t), t \geq 0\}$ given by (25). Since m is also $(\mathcal{F} \vee \sigma(Z(0)))$ -measurable so are I , by (24), and Z by (23).

(iii) Since J has a finite number of jumps in a finite interval, the continuity of the sample paths of D and I follows. Since X has *càdlàg* non-decreasing sample paths and D has continuous sample paths, $Y = X - D$ has *càdlàg* sample paths with all jumps being positive; this and the fact that I has continuous sample paths imply by (23) that Z has *càdlàg* sample paths with all jumps being positive.

(iv) The proof follows along the same lines of (ii); the fact that (X, J) is progressively measurable with respect to $\{\mathcal{F}_t\}$ follows from the fact that it is adapted to $\{\mathcal{F}_t\}$ and has *càdlàg* sample functions, by Proposition 1.13 (Chapter 1) in Karatzas and Shreve [9].

(v) The statement follows from (iv). $\square$

Remark 1 It follows from Theorem 1 that

$$I(t) = \int_0^t d(J(s)) 1_{\{Z(s)=0\}} ds, \quad (30)$$

since $Z(t) \geq 0, \forall t \geq 0$. If we replace equation (22) by

$$Z(t) = Z(0) + X(t) - D(t) + \int_0^t d(J(s)) 1_{\{Z(s)=0\}} ds, \quad (31)$$

(23) and (24) is also a solution of (31). However, (31) may not have an unique solution as we show in the following. $\square$

Example 1 Consider $\mathcal{E} = \{1\}$, $\Sigma = (0)$, $M_{11}\{0\} = 1$, $d(1) = 1$ and $Z(0) = 0$ a.s.. Note that $X(t) = 0$ and $D(t) = t$, a.s.. Trivially, $Z(t) = -t$ and $I(t) = 0$ is a solution of (31) (in addition to the solution $Z(t) = 0$ and $I(t) = t$, correspondent to (23) and (24)). Note that the uniqueness of the solution (23) and (24) of (31) has been systematically referred to in the literature for models that include this example. $\square$

3 The actual and the unsatisfied demands

We recall the definition of an additive functional of a Markov process (see Blumenthal [3]).

Definition 1 Let $\{R(t), t \geq 0\}$ be a Markov process, $\mathcal{F}_t^o = \sigma\{R(s), s \leq t\}$ and $\mathcal{F}_t$ the proper completion of $\mathcal{F}_t^o$. Then $\{\mathcal{F}_t, t \geq 0\}$ is a right-continuous filtration for $\{R(t), t \geq 0\}$.

A family $\{A(t), t \geq 0\}$ of functions with values in $[0, \infty]$ is called an additive functional of $\{R(t), t \geq 0\}$ provided:

- (i) $t \rightarrow A(t), t \geq 0$ is non-decreasing, right-continuous and $A(0) = 0$, a.s. for all initial distributions.
- (ii) For each $t, A(t) \in \mathcal{F}_t$.
- (iii) For each $s, t \geq 0, A(t+s) = A(t) + A(s) \circ \phi_t$, a.s. for any initial distribution, where ϕ is a shift operator,

$$\phi_t : \Omega \rightarrow \Omega \text{ such that } (\phi_t(\omega))(r) = \omega(r+t), r \geq 0. \quad \square$$

Remark 2 With the setting of Definition 1, a functional A of $\{R(t), t \geq 0\}$ of the form

$$A(t) = \int_0^t f(R(s)) ds, \quad (32)$$

where f is a bounded non-negative measurable function, is called a *classical* additive functional by Blumenthal [3], but as remarked in Blumenthal and Getto [2] (p.151), A is still an additive functional of R even when f is not bounded provided $A(t)$ is finite for all t a.s.. $\square$

Lemma 1 The demand process $\{D(t), t \geq 0\}$ has the following properties:

- (i) D is a continuous additive functional of the MJP J .
- (ii) D has strictly increasing sample paths, for all initial distributions.
- (iii) If $d_- > 0$, then $D(\infty) = \lim_{t \rightarrow \infty} D(t) = \infty$, for all initial distributions.
- (iv) $d_- \leq \frac{D(t)}{t} \leq d_+$, for all $t > 0$.
- (v) If J is ergodic, then $\lim_{t \rightarrow \infty} \frac{D(t)}{t} = \sum_{j \in \mathcal{E}} d(j) \pi_j$, a.s..
- (vi) If J has a persistent state, then $\lim_{t \rightarrow \infty} D(t) = \infty$, a.s..

Proof: (i) Since J is non-explosive, J has a finite number of transitions in finite time a.s. and this implies that $D(t) < \infty$ a.s., $\forall t \geq 0$; the statement now follows from Remark 2.

(ii) Since $d(j) > 0, \forall j \in \mathcal{E}$, then all sample paths of $\{D(t), t \geq 0\}$ are strictly increasing regardless of the initial distribution and the result follows.

(iii) The statement follows since, if $d_- > 0$, for any $t \geq 0$ and any initial distribution,

$$D(t) = \int_0^t d(J(s)) ds \geq \int_0^t d_- ds = d_- t.$$

(iv) Since for all $t > 0$, $d_-t \leq D(t) \leq d_+t$, the statement follows.

(v) d is a positive measurable function on $\mathcal{E}$, so that, if J is ergodic,

$$\begin{aligned} \frac{D(t)}{t} &= \frac{1}{t} \int_0^t d(J(s)) ds = \frac{1}{t} \int_0^t \sum_{j \in \mathcal{E}} d(j) 1_{\{J(s)=j\}} ds \\ &= \sum_{j \in \mathcal{E}} \frac{d(j)}{t} \int_0^t 1_{\{J(s)=j\}} ds \rightarrow \sum_{j \in \mathcal{E}} d(j) \pi_j, \text{ as } t \rightarrow \infty, \text{ a.s..} \end{aligned}$$

(vi) Given a persistent state $j \in \mathcal{E}$ (and since $d(j) > 0$),

$$D(t) = \int_0^t d(J(s)) ds \geq \int_0^t d(j) 1_{\{J(s)=j\}} ds \rightarrow \infty, \text{ as } t \rightarrow \infty, \text{ a.s.,}$$

and the statement follows. $\square$

The demand process can be identified as a random clock, its speed at t being $d(J(t))$, and with transformed time scale $[0, D(\infty))$. Lemma 1 (iii) and (vi) gives sufficient conditions for the transformed time scale to be infinite a.s.; we note the condition $d(j) > 0, \forall j \in \mathcal{E}$ is not a sufficient one (in general) as the following example shows.

Example 2 Suppose $\mathcal{E} = \{1, 2, 3, \dots\}$, $d(j) = j^{-2}$ and Σ , the intensity matrix of J , is such that for $j = 1, 2, \dots$, $\sigma_{j,j+1} = 1$; $\sigma_{jk} = 0$ otherwise. Then, for any initial distribution,

$$E(D(\infty)) \leq \sum_{j=1}^{\infty} j^{-2} < \infty,$$

thus $D(\infty) < \infty$ a.s.. $\square$

Since D is an additive functional of J , the process (D, J) is clearly an MAP and so is (Y, J) . In Theorem 2 we give the infinitesimal generators of these two processes; in fact we are able to conclude that they are MCPs.

Theorem 2 *The processes (D, J) and (Y, J) are MCPs; their infinitesimal generators are given by:*

$$\mathcal{A}_{(D,J)} f(x, j) = d(j) \frac{\partial}{\partial x} f(x, j) + \sum_{k \neq j} \sigma_{jk} [f(x, k) - f(x, j)], \quad (33)$$

$$\mathcal{A}_{(Y,J)} f(x, j) = -d(j) \frac{\partial}{\partial x} f(x, j) + \sum_{k \in \mathcal{E}} \sigma_{jk} \int_0^{\infty} [f(x+y, k) - f(x, j)] M_{jk}\{dy\}. \quad (34)$$

Proof: For $A \in \mathcal{B}([0, +\infty])$ and $j, k \in \mathcal{E}$, let $P_{jk}(A; t) = P\{D(t) \in A, J(t) = k \mid J(0) = j\}$ then,

$$P_{jk}(A; h) = \delta_{jk}(1 - \lambda_j h) 1_{\{h d(j) \in A\}} + \int_0^h \sigma_{jk} e^{-\lambda_j \tau} e^{-\lambda_k(h-\tau)} 1_{\{\tau d(j) + (h-\tau)d(k) \in A\}} d\tau + o(h),$$

and consequently

$$\begin{aligned}
\mathcal{A}_{(D,J)} f(x,j) &= \lim_{h \rightarrow 0+} \left[\frac{1}{h} \sum_{k \in \mathcal{E}} \int_0^{\infty+} [f(x+y,k) - f(x,j)] P_{jk}(\{dy\}; h) \right] \\
&= d(j) \lim_{h \rightarrow 0+} \left[\int_0^{\infty+} \frac{f(x+y,j) - f(x,j)}{h d(j)} (1 - \lambda_j h) 1_{\{h d(j) \in dy\}} \right] \\
&\quad + \sum_{k \in \mathcal{E}} \int_0^{\infty+} [f(x+y,k) - f(x,j)] \sigma_{jk} \\
&\quad \lim_{h \rightarrow 0+} \left[\frac{e^{-\lambda_k h}}{h} \int_0^h e^{-(\lambda_j - \lambda_k) \tau} d\tau 1_{\{\tau d(j) + (h-\tau)d(k) \in dy\}} \right] \\
&= d(j) \frac{\partial}{\partial x} f(x,j) + \sum_{k \neq j} \sigma_{jk} [f(x,k) - f(x,j)].
\end{aligned}$$

This shows that (D, J) is an MCPP with drift function $d(j)$, $j \in \mathcal{E}$ and no jumps. Thus (Y, J) is an MCPP with drift function $-d(j)$, $j \in \mathcal{E}$, and the same jumps as (X, J) so that by (16) the infinitesimal generator of (Y, J) is as given in (34). $\square$

We give in Theorem 3 some properties of the unsatisfied demand and of the infimum of the net input, which is naturally related to the unsatisfied demand by Theorem 1.

Theorem 3 *The unsatisfied demand $I = \{I(t), t \geq 0\}$ is a continuous additive functional of (Z, J) , moreover, if $Z(0) = 0$ a.s., then I is a continuous additive functional of (X, J) .*

The negative of the infimum of the net input process $-m$ (m is given by (25)) is a continuous additive functional of (X, J) .

Proof: Since (Z, J) is a Markov process, the fact that the unsatisfied demand I is a continuous additive functional of (Z, J) follows from the definition of the unsatisfied demand, as given in (19), and Remark 2.

Suppose now $Z(0) = 0$ a.s., then (since $X(0) = 0$ a.s.) $\sigma\{(X, J)(s), s \leq t\}$ is equal to $\sigma\{(Z, J)(s), s \leq t\}$ and I becomes a continuous additive functional of (X, J) . The rest of the theorem follows trivially since, by Theorem 1, $I(t) = -m(t)$ a.s. in case $Z(0) = 0$ a.s.. $\square$

4 The inverse of the demand

It is of interest to study the right-continuous inverse of the demand process $D(t)$, i.e. for $x \geq 0$ we want to know the amount of time needed for the demand to exceed x viewed as a stochastic process. Denoting this process as B we thus have

$$B(x) = \inf\{t : D(t) > x\}. \quad (35)$$

We study below some of properties of the right-continuous inverse of an additive functional of a Markov process. The following result for inverses of additive functionals is partially mentioned by Blumenthal [3] but seems to call for a detailed proof.

Lemma 2 *If $\{A(t), t \geq 0\}$ is an additive functional of the Markov process $\{R(t), t \geq 0\}$ and $\tilde{A}(x)$ is its right-continuous inverse, $\tilde{A}(x) = \inf\{t : A(t) > x\}$, then:*

- (i) $\tilde{A}$ has non-decreasing and a.s. right-continuous sample paths.
- (ii) If A has continuous and strictly increasing sample paths a.s. then (a.s.):
 - (a) $\tilde{A}(x) = \infty$ for $x \in [A(\infty), \infty)$.
 - (b) $\tilde{A}(0) = 0$ and $\tilde{A}$ has continuous and strictly increasing sample paths on $[0, A(\infty))$.
- (iii) For each x , $\tilde{A}(x)$ is a stopping time.
- (iv) For any $t \in [0, \infty)$, a.s., $A(t) = \inf\{x : \tilde{A}(x) > t\}$.

Proof: (i) If $x_1 \leq x_2$, then

$$\tilde{A}^*(x_2) = \{t : A(t) > x_2\} \subseteq \{t : A(t) > x_1\} = \tilde{A}^*(x_1),$$

so the indexed sets $\tilde{A}^*(x)$ are non-increasing on x , and

$$\tilde{A}(x_1) = \inf \tilde{A}^*(x_1) \leq \inf \tilde{A}^*(x_2) = \tilde{A}(x_2).$$

This shows that $\tilde{A}(x)$ is non-decreasing. In the following we refer only to non-decreasing sample paths of A . Let $\{x_n\}$ be a decreasing sequence with limit $x \geq 0$; since $\{\tilde{A}(x_n)\}$ is a non-increasing sequence bounded below by $\tilde{A}(x)$ the sequence converges in $[0, +\infty]$ and

$$\lim \tilde{A}(x_n) \geq \tilde{A}(x). \quad (36)$$

Now note that for $y \geq 0$, $\tilde{A}^*(y)$ is either $(\tilde{A}(y), +\infty)$ or $[\tilde{A}(y), +\infty)$, so

$$(\tilde{A}(y), +\infty) \subseteq \tilde{A}^*(y) \subseteq [\tilde{A}(y), +\infty), \forall y \geq 0.$$

Consequently

$$(\tilde{A}(x), +\infty) \subseteq \tilde{A}^*(x) = \bigcup_n \tilde{A}^*(x_n) \subseteq \bigcup_n [\tilde{A}(x_n), +\infty) \subseteq [\lim \tilde{A}(x_n), +\infty),$$

which implies that $\lim \tilde{A}(x_n) \leq \tilde{A}(x)$. In view of (36) it follows that

$$\lim \tilde{A}(x_n) = \tilde{A}(x).$$

Therefore $\tilde{A}$ has right-continuous sample paths almost surely.

(ii) We consider only continuous and strictly increasing sample paths of A such that $A(0) = 0$ (for which the associated sample paths of $\tilde{A}$ are non-decreasing and right-continuous) and keep the notation used in (i). If $A(\infty) < x < \infty$ then clearly $\tilde{A}(x) = \infty$ and by right-continuity also $\tilde{A}(A(\infty)) = \infty$; this proves (a). Trivially $\tilde{A}(0) = 0$ since $A(t) > 0, \forall t > 0$, thus to prove (b) we need to show that $\tilde{A}$ is left-continuous and strictly increasing on $\mathcal{S} = [0, A(\infty))$. We have, $\tilde{A}^*(y) = (\tilde{A}(y), +\infty), \forall y \in \mathcal{S}$. Let $\{x_n\}$ be a non-negative increasing sequence with limit $x \in \mathcal{S}$, then $\{\tilde{A}(x_n)\}$ is a non-decreasing sequence bounded above by $\tilde{A}(x) < \infty$, so the sequence converges on $[0, +\infty)$ and

$$\lim \tilde{A}(x_n) \leq \tilde{A}(x). \quad (37)$$

Consequently

$$[\tilde{A}(x), +\infty) = \{t : A(t) \geq x\} = \bigcap_n \{t : A(t) > x_n\} = \bigcap_n (\tilde{A}(x_n), +\infty) \supseteq (\lim \tilde{A}(x_n), +\infty),$$

which implies that $\lim \tilde{A}(x_n) \geq \tilde{A}(x)$. Again, in view of (37) it follows that

$$\lim \tilde{A}(x_n) = \tilde{A}(x).$$

This proves that $\tilde{A}(x)$ is left-continuous and thus continuous on $\mathcal{S}$. Suppose, with the same conditions, that $\tilde{A}(x)$ is not strictly increasing on $\mathcal{S}$, that is

$$\exists y < x < A(\infty) \text{ such that } \tilde{A}(y) = \tilde{A}(x).$$

Then $(\tilde{A}(y), +\infty) = (\tilde{A}(x), +\infty)$ or equivalently

$$\{t : A(t) > y\} = \{t : A(t) > x\}.$$

Since $A(0) = 0$, this contradicts the fact that $A(t)$ is continuous and strictly increasing. Thus $\tilde{A}(x)$ is strictly increasing on $\mathcal{S}$. We have thus proved that $\tilde{A}$ has continuous and strictly increasing sample paths on $[0, A(\infty))$ a.s..

(iii) Since for any $t, x \geq 0$

$$\{\tilde{A}(x) \leq t\} = \bigcap_n \left\{ \sup_{0 \leq \tau \leq t + \frac{1}{n}} A(\tau) > x \right\} \in \mathcal{F}_{t+} = \mathcal{F}_t,$$

$\tilde{A}(x)$ is a stopping time.

(iv) We prove the result for non-decreasing and right-continuous sample paths of A . For $t \geq 0$, let

$$\bar{A}(t) = \inf\{x : \tilde{A}(x) > t\},$$

then, using the fact that A and $\tilde{A}$ are non-decreasing and right-continuous and the definitions of $\tilde{A}$ and $\bar{A}$,

$$x < (>) \bar{A}(t) \Rightarrow \tilde{A}(x) \leq (>) t \Rightarrow x \leq (\geq) A(t).$$

This shows that $\bar{A}(t) = A(t)$, $\forall t \geq 0$, which proves the statement. $\square$

Theorem 4 *For all initial distributions, the inverse of the demand has the following properties:*

(i) $B(0) = 0$ and B has continuous and strictly increasing sample paths on $[0, D(\infty))$ and $B(x) = \infty$ for $x \in [D(\infty), \infty)$.

(ii) For each x , $B(x)$ is a stopping time.

(iii) For any $t \in [0, \infty)$,

$$D(t) = \inf\{x : B(x) > t\}. \tag{38}$$

(iv) $\frac{1}{d_+} \leq \frac{B(x)}{x} \leq \frac{1}{d_-}$, for all $x > 0$.

(v) If J is ergodic and $d_+ < \infty$, then $\lim_{x \rightarrow \infty} \frac{B(x)}{x} = \left[\sum_{j \in \mathcal{E}} d(j) \pi_j \right]^{-1}$, a.s..

(vi) $\{(B, J \circ B)(x), 0 \leq x < D(\infty)\}$ is an MAP with infinitesimal generator:

$$\mathcal{A}_{(B, J \circ B)} f(x, j) = \frac{1}{d(j)} \frac{\partial}{\partial x} f(x, j) + \sum_{k \neq j} \frac{\sigma_{jk}}{d(j)} [f(x, k) - f(x, j)]. \quad (39)$$

Proof: The statements (i), (ii) and (iii) follow in the same way as Lemma 2 (ii)-(iv) by Lemma 1 (i) and (ii).

(iv) It suffices to prove the result for $0 < d_- \leq d_+ < +\infty$; if so, given $x > 0$ and arbitrary t_1, t_2 such that $t_2 > x/d_-$ and $t_1 < x/d_+$,

$$D(t_2) \geq t_2 d_- > x \text{ and } D(t_1) \leq t_1 d_+ < x.$$

Now, by the definition of $B(x)$, we have $B(x) \leq t_2$ and $B(x) \geq t_1$. This gives the desired result since t_1, t_2 are arbitrary.

(v) From Lemma 1(v),

$$\lim_{t \rightarrow \infty} \frac{D(t)}{t} = \sum_{j \in \mathcal{E}} d(j) \pi_j, \text{ a.s.,}$$

and since, by Lemma 1(i) and (ii), D has continuous and strictly increasing sample paths, then for all $x > 0$,

$$B(x) = t \Leftrightarrow D(t) = x.$$

Since by (iv) $\lim_{x \rightarrow \infty} B(x) = \infty$, this gives

$$\lim_{x \rightarrow \infty} \frac{B(x)}{x} = \lim_{t \rightarrow \infty} \frac{t}{D(t)} = \frac{1}{\sum_{j \in \mathcal{E}} d(j) \pi_j}.$$

(vi) The statement is a consequence of the results obtained in Example 3 below. $\square$

Remark 3 If we let J^B be the MJP with intensity matrix $\Sigma^B = D^{-1} \Lambda P$, where $D = (\delta_{jk} d(j))$, then (up to the explosion time $D(\infty)$ and by (39)) $(B, J \circ B)$ is equivalent to the MCPP (B, J^B) with drift $[d(j)]^{-1}$, $j \in \mathcal{E}$, and no jumps. Note that if J is ergodic and $d_+ < \infty$ then J^B is also ergodic and has stationary distribution π^B given by:

$$\pi_j^B = \frac{\pi_j d(j)}{\sum_{k \in \mathcal{E}} \pi_k d(k)};$$

this would lead to Theorem 4 (v) using a procedure similar to the one used to obtain Lemma 1 (v). Since the infinitesimal generator of a process characterizes the process, using Theorem 2, we obtain

$$B(x) = \int_0^x \frac{1}{d(J^B(y))} dy, \text{ for } x < D(\infty).$$

As was done for the demand, we can identify the inverse of the demand as a random clock with $[d(J^B(y))]^{-1}$ being its time speed at y . $\square$

5 The busy period process

The busy period is defined as the time the storage system would become empty if the initial level was $x(\geq 0)$. Denoting this time as $T(x, k)$, when the initial state of J is k , we have

$$T(x, k) = \inf \{t : Z(t) = 0\} \quad \text{on} \quad \{Z(0) = x, J(0) = k\}, \quad (40)$$

for $x \in \mathbb{R}_+$, $k \in \mathcal{E}$, with the convention that the infimum of an empty set is $+\infty$. We are also interested in the state of J at the time of emptiness, namely $J \circ T(x, k)$. If $T(x, k) = \infty$ we define $J \circ T = \Delta$. Of special interest is the situation when the initial level x has the distribution M_{jk} with $j \in \mathcal{E}$. This busy period may represent (for example) the busy period initiated by the input arriving with a transition from state j to state k in J . In this case, we shall call this the busy period initiated by the transition j to k and denote it as T_{jk} . Let

$$G_{jk}(A, l) = P\{(T_{jk}, J \circ T_{jk}) \in A \times \{l\}\} \quad (41)$$

be the distribution of $(T_{jk}, J \circ T_{jk})$. We have

$$G_{jk}(A, l) = \int_0^{\infty+} P\{(T, J \circ T)(x, k) \in A \times \{l\}\} M_{jk}\{dx\}, \quad (42)$$

for any Borel subset A of $[0, \infty]$, $j, k \in \mathcal{E}$ and $l \in \mathcal{E} \cup \Delta$. Note that T_{jk} has possibly an atom at (∞, Δ) . We let $G_{jkl}(A) = G_{jk}(A, l)$ and for simplicity we attach the atom at (∞, Δ) (with value $G_{jk\Delta}\{\infty\}$) to the measure G_{jkj} , so that for $j, k, l \in \mathcal{E}$ with j, k fixed,

$$\sum_{l \neq j} G_{jkl}([0, \infty)) + G_{jkj}([0, +\infty)) = 1. \quad (43)$$

We next consider the process $(T, J \circ T) = \{(T, J \circ T)(x), x \geq 0\}$ where $T(x)$ is now defined as $T(x, J(0))$ with $J(0)$ having an arbitrary distribution. Thus we are studying the effect of the initial storage level, regardless of the initial state of J . We have the following.

Lemma 3 *$(T, J \circ T)$ is a MAP with infinitesimal generator given by*

$$\mathcal{A}_{(T, J \circ T)} f(x, j) = \frac{1}{d(j)} \left[\frac{\partial}{\partial x} f(x, j) + \sum_{l \in \mathcal{E}} \int_0^{\infty+} [f(x+y, l) - f(x, j)] \sum_{k \in \mathcal{E}} \sigma_{jk} G_{jkl}\{dy\} \right], \quad (44)$$

where $G_{jkl}\{\infty\} = 0$ for $l \neq j$.

Proof: On the set $\{J(0) = j\}$, $j \in \mathcal{E}$, consider for $A \in \mathcal{B}([0, +\infty))$ the events

$$C(j, h; A, l) = \{T(h) \in A, J \circ T(h) = l\},$$

$$H(j, h; A, l) = \left\{ \begin{array}{l} T(h) \in A, J \circ T(h) = l \text{ and on } [0, B(h)] \\ \text{there are no transitions in } J \end{array} \right\},$$

and for $k \in \mathcal{E}$,

$$E_k(j, h; A, l) = \left\{ \begin{array}{l} T(h) \in A, J \circ T(h) = l \text{ and the first transition in } J \\ \text{is to state } k \text{ and occurs before time } B(h) \end{array} \right\}.$$

It is easily seen that

$$\begin{aligned}
P[C(j, h; A, l)] &= \sum_{k \in \mathcal{E}} P[E_k(j, h; A, l)] + P[H(j, h; A, l)] \\
&= \sum_{k \in \mathcal{E}} \int_0^{\frac{h}{d(j)}} \sigma_{jk} e^{-\lambda_j \tau} \int_0^{\infty+} \sum_{m \in \mathcal{E}} G_{jkm} \{A - \tau - ds\} P[C(m, h - \tau d(j); ds, l)] d\tau \\
&\quad + \delta_{jl} \left[1 - \lambda_j \frac{h}{d(j)} \right] 1_{\{\frac{h}{d(j)} \in A\}} + o(h).
\end{aligned}$$

We are now able to obtain the infinitesimal generator of the process $(T, J \circ T)$,

$$\begin{aligned}
\mathcal{A}_{(T, J \circ T)} f(x, j) &= \lim_{h \rightarrow 0+} \left[\frac{1}{h} \sum_{l \in \mathcal{E}} \int_0^{\infty+} [f(x + y, l) - f(x, j)] P[C(j, h; dy, l)] \right] \\
&= \frac{1}{d(j)} \lim_{h \rightarrow 0+} \left[\int_0^{\infty+} \frac{f(x + y, j) - f(x, j)}{h/d(j)} \left(1 - \lambda_j \frac{h}{d(j)} \right) 1_{\{\frac{h}{d(j)} \in dy\}} \right] \\
&\quad + \sum_{l \in \mathcal{E}} \int_0^{\infty+} [f(x + y, l) - f(x, j)] \sum_{k \in \mathcal{E}} \frac{\sigma_{jk}}{d(j)} \\
&\quad \lim_{h \rightarrow 0+} \left[\frac{d(j)}{h} \int_0^{\frac{h}{d(j)}} e^{-\lambda_j \tau} \int_0^{y-\tau} \sum_{m \in \mathcal{E}} G_{jkm} \{dy - \tau - ds\} P[C(m, h - \tau d(j); ds, l)] d\tau \right] \\
&= \frac{1}{d(j)} \frac{\partial}{\partial x} f(x, j) + \sum_{l \in \mathcal{E}} \int_0^{\infty+} [f(x + y, l) - f(x, j)] \sum_{k \in \mathcal{E}} \frac{\sigma_{jk}}{d(j)} G_{jkl} \{dy\},
\end{aligned}$$

which is the desired result. $\square$

Example 3 As an application of Lemma 3 we derive the infinitesimal generator of the inverse of the demand process $\{(B, J \circ B)(x), x \geq 0\}$ which coincides with the storage busy period process if no input arrives to the system, i.e.

$$M_{jk} = \varepsilon_0, \quad \forall j, k \in \mathcal{E},$$

where ε_0 is a distribution concentrated at the origin. This implies that for $A \in \mathcal{B}([0, +\infty])$,

$$G_{jkl}(A) = \begin{cases} 0 & \text{if } k \neq l \\ \varepsilon_0(A) & \text{if } k = l \end{cases}, \quad (45)$$

so that the generator of $\{(B, J \circ B)(x), x \geq 0\}$ is given by

$$\mathcal{A}_{(B, J \circ B)} f(x, j) = \frac{1}{d(j)} \left[\frac{\partial f}{\partial x}(x, j) + \sum_{l \neq j} \sigma_{jl} [f(x, l) - f(x, j)] \right],$$

thus proving Theorem 4 (vi). $\square$

We consider for $j \in \mathcal{E}$ the random variable $(\tilde{T}, J \circ \tilde{T})_j$, where $\tilde{T}_j$ represents the duration of the storage busy period initiated by a new input if at present J is in state j . Denote by $\tilde{G}_j$ its

distribution and by p_{jl}^* the probability that this busy period ends in state l . Our setting implies that for $A \in \mathcal{B}([0, \infty])$, $l \in \mathcal{E}$

$$\tilde{G}_{jl}(A) \equiv \tilde{G}_j(A, \{l\}) = \sum_{k \in \mathcal{E}} p_{jk} G_{jkl}(A), \quad (46)$$

and

$$p_{jl}^* = P\{J \circ \tilde{T}_j = l\} = \tilde{G}_{jl}([0, \infty]). \quad (47)$$

In terms of the definitions above, the infinitesimal generator of $(T, J \circ T)$ (given by (44)) can be expressed, with $\tilde{G}_{jl}\{\infty\} = 0$ for $l \neq j$, as follows:

$$\mathcal{A}_{(T, J \circ T)} f(x, j) = \frac{1}{d(j)} \frac{\partial f}{\partial x}(x, j) + \frac{\lambda_j}{d(j)} \sum_{l \in \mathcal{E}} \int_0^{\infty+} [f(x+y, l) - f(x, j)] \tilde{G}_{jl}\{dy\}. \quad (48)$$

We elaborate a bit more to increase the interpretability of (48). Suppose the present state of the MJP J is j . We are now able to consider the (conditional) distribution of the length of the busy period initiated by a new input given that at the end of the same busy period the Markov chain J is in a given state l . We denote such a variable by T_{jl}^* and its distribution by M_{jl}^* and for completeness we set $T_{jl}^* = 0$ if $p_{jl}^* = 0$. Thus, for $A \in \mathcal{B}([0, \infty])$, $j, l \in \mathcal{E}$,

$$M_{jl}^*(A) = \begin{cases} \frac{\tilde{G}_{jl}(A)}{\tilde{G}_{jl}([0, \infty])} & p_{jl}^* \neq 0 \\ 1_{\{0 \in A\}} & p_{jl}^* = 0 \end{cases}, \quad (49)$$

where we note that

$$p_{jl}^* M_{jl}^*(A) = \tilde{G}_{jl}(A). \quad (50)$$

We define for $j, l \in \mathcal{E}$,

$$\lambda_j^* = \frac{\lambda_j}{d(j)}, \quad \sigma_{jl}^* = \lambda_j^* p_{jl}^*, \quad (51)$$

and let J^* be a MJP on the state space $\mathcal{E}$ with intensity matrix $\Sigma^* = D^{-1} \Lambda P^*$, where $P^* = (p_{jk}^*)$ and $D = (\delta_{jk} d(j))$. As a direct consequence of Lemma 3, or even more immediately from (48) we have:

Theorem 5 $(T, J \circ T)$ is a MAP whose generator is given by

$$\begin{aligned} \mathcal{A}_{(T, J \circ T)} f(x, j) &= \frac{1}{d(j)} \frac{\partial f}{\partial x}(x, j) + \int_0^{\infty+} [f(x+y, j) - f(x, j)] \sigma_{jj}^* M_{jj}^*\{dy\} \\ &\quad + \sum_{l \neq j} \int_0^{\infty} [f(x+y, l) - f(x, j)] \sigma_{jl}^* M_{jl}^*\{dy\}. \end{aligned} \quad (52)$$

Thus (T, J^*) is an MCP with drift function $[d(j)]^{-1}$, $j \in \mathcal{E}$, and jumps with distribution M_{jl}^* occurring at rate σ_{jl}^* . $\square$

Theorem 5 provides insights that lead to a more meaningful probabilistic interpretation of the results obtained in Lemma 3. Among other facts, it is easy to see that the motion of the process (T, J^*) is related with the motion of the process $(T, J \circ T)$ (as it should) but the perspective from which we see the busy period evolving is quite different in the two processes. Some comments are in order.

Remark 4 The transitions between states of $\mathcal{E}$ occurring during a busy period initiated by a new input in $(T, J \circ T)$ are eliminated in the process (T, J^*) . In addition, the jumps in the additive component of the process (T, J^*) (the busy period) are lengths of busy periods initiated by new arrivals. $\square$

Remark 5 Time is rescaled; when J^* is in state j , time is measured in units $[d(j)]^{-1}$ if referred to the Markov chain J . Thus the busy period process increases locally at rate equal to the reciprocal of the present demand rate $[d(j)]^{-1}$. In particular, the busy period local timing is only a function of the present state of the MJP J^* . $\square$

Remark 6 With respect to the length of busy periods initiated by new arrivals, the actual values of the jump rates $\{\lambda_j, j \in \mathcal{E}\}$ and of the demand rates $\{d(j), j \in \mathcal{E}\}$ are immaterial; these quantities impact the length of busy periods initiated by new arrivals only through the ratios $\{\lambda_j^* = \frac{\lambda_j}{d(j)}, j \in \mathcal{E}\}$. This expresses a relativity property of the jump rates and demand rates. Nevertheless the demand rates impact the length of busy periods (locally) as a function of their absolute values in terms of the drift of the busy period process. $\square$

As shown in Theorem 5, the storage busy period process (T, J^*) is an MCPP, thus a relatively simple process. Nevertheless, at this point, we cannot make direct use of this property since we do not know the distributions M_{jl}^* of the jumps, nor the values of the probabilities p_{jl}^* . In the following our aim will be to get a better description of the objects we have just mentioned.

6 Some more notations and results

In this section we mainly introduce some more notations and results that we use in the following sections. The reader may prefer to proceed directly to the next section now and return whenever the need arises.

Definition 2 If for $j, l \in \mathcal{E}$, $A_{jl}\{dx\}$ is a finite measure on $[0, \infty]$ and $\theta > 0$, then we denote by $A(\theta) = (A_{jl}(\theta))$ the Laplace transform matrix of the matrix measure $A = (A_{jl}\{dx\})$,

$$A(\theta) = \left(\int_0^\infty e^{-\theta x} A_{jl}\{dx\} \right). \quad (53)$$

For completeness, if $A = (A_{jl})$ is a matrix of nonnegative numbers we consider that A_{jl} is an atomic measure concentrated solely at the origin to which it assigns measure A_{jl} , so that $A(\theta) \equiv A$. $\square$

Definition 3 We let

$$S = \{f = (f_{jk}), j, k \in \mathcal{E} \text{ such that } f_{jk} : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is bounded and continuous}\},$$

and we define a norm in S ,

$$\|f\| = \sup_{k \in \mathcal{E}} \sum_{j \in \mathcal{E}} \|f_{jk}\|,$$

where $\|f_{jk}\| = \sup_{x \geq 0} |f_{jk}(x)|$. Then $(S, \|\cdot\|)$ is a Banach space. We define, for $f \in S$ and $s > 0$,

$$\mathcal{L}(f, s) = \left(\int_0^\infty e^{-st} f_{jk}(t) dt \right). \quad \square$$

Definition 4 Suppose J is a pure jump process on a countable state space $\mathcal{E}$, that V and W are nonnegative process such that (V, J) and (W, J) are adapted to a given filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$, and that for all $x \geq 0$, $T(x)$ is an $\mathcal{F}$ -stopping time. We define (with the operator E denoting the usual expectation and A being $\mathcal{F}_0$ -measurable) the following types of expectation matrices:

$$E_J[W(t)] = (E[W(t); J(t) = k | J(0) = j]),$$

$$E_J[W(t)|A] = (E[W(t); J(t) = k | A, J(0) = j]),$$

$$E_J[T(x)] = (E[T(x); J \circ T(x) = k | J(0) = j]).$$

In a similar way we define for $\theta, \theta_1, \theta_2 > 0$ the following types of Laplace transform matrices:

$$\psi_W(s; \theta) = \mathcal{L} \left(E_J \left[e^{-\theta W(\cdot)} \right], s \right) = \left(\int_0^\infty e^{-st} E \left[e^{-\theta W(t)}; J(t) = k | J(0) = j \right] dt \right),$$

$$\psi_{(W,S)}(s; \theta_1, \theta_2) = \mathcal{L} \left(E_J \left[e^{-\theta_1 W(\cdot) - \theta_2 S(\cdot)} \right], s \right);$$

obvious definitions are considered for the conditional case (e.g. $\psi_{W|A}(s; \theta)$). $\square$

With $D = (d(j)\delta_{jk})$, we define for $\theta > 0$

$$\Phi(\theta) = \Lambda [I - \Gamma(\theta)], \quad \Phi^*(\theta) = D^{-1} \Lambda [I - \Gamma^*(\theta)], \quad (54)$$

where,

$$\Gamma(\theta) = (p_{jk} M_{jk}(\theta)), \quad \Gamma^*(\theta) = (p_{jk}^* M_{jk}^*(\theta)); \quad (55)$$

in addition, we let

$$\Psi(\theta) = \Phi(\theta) + D\theta, \quad \Psi^*(\theta) = \Phi^*(\theta) + D^{-1}\theta. \quad (56)$$

Suppose J is an MJP on a countable state space $\mathcal{E}$ with intensity matrix Σ , and (W, J) is an MCPP with nonnegative jumps whose infinitesimal generator is given by

$$\mathcal{A}_{(W,J)} f(x, j) = d(j) \frac{\partial f}{\partial x}(x, j) + \sum_{k \in \mathcal{E}} \sigma_{jk} \int_0^\infty [f(x+y, k) - f(x, j)] M_{jk}\{dy\}. \quad (57)$$

The distribution measures M_{jk} are thus concentrated on $[0, \infty)$. We assume that $\{d(j)\}$ and $\{\lambda_j = \sum_{k \in \mathcal{E}} \sigma_{jk}\}$ are uniformly bounded.

Lemma 4 *The process W with generator given by equation (57) has the following Laplace transform matrix*

$$E_J \left[e^{-\theta W(t)} \right] = e^{-t\Psi_W(\theta)} = e^{-t[\Phi_W(\theta) + D_W\theta]}, \quad (58)$$

where $\Psi_W = \Psi$, $\Phi_W = \Phi$, $D_W = D$, so that

$$\Psi_W(\theta) = \Lambda [I - \Gamma(\theta)] + D\theta. \quad (59)$$

Proof: The result follows similarly to Theorem 19 in Prabhu [13]. $\square$

Remark 7 The Laplace transform matrix of the additive component of an MCPP with nonnegative jumps is a direct extension (to a matrix-like form) of the Laplace transform of a compound-Poisson process, as it is easily seen from equations (58) and (59).

The presence of the matrices Λ and D in equations (58) and (59) expresses the fact that the (compound) arrival rate and the drift of the additive component of an MCPP depend on the state of the underlying Markov chain, as opposed to the compound Poisson case where they are fixed constants. $\square$

Remark 8 Assume $d_+ < \infty$, $s, \theta > 0$, and let $Q = \Sigma - \Lambda$ and 0 denote the null process $0(t) \equiv 0$ then, using Lemma 4, we have:

$$\pi(t) = (P[J(t) = k | J(0) = j]) = E_J \left[e^{-\theta 0(t)} \right] = e^{tQ}, \quad (60)$$

$$\hat{\pi}(s) = \int_0^\infty e^{-st} \pi(t) dt = \psi_0(s; \cdot) = [sI - Q]^{-1}, \quad (61)$$

$$E_J \left[e^{-\theta X(t)} \right] = e^{-t\Phi(\theta)}, \quad \psi_X(s; \theta) = [sI + \Phi(\theta)]^{-1}, \quad (62)$$

$$E_J \left[e^{-\theta D(t)} \right] = e^{-t[D\theta - Q]}, \quad \psi_D(s; \theta) = [sI + D\theta - Q]^{-1} \quad (63)$$

and,

$$E_J \left[e^{-\theta Y(t)} \right] = e^{-t[\Phi(\theta) - D\theta]}, \quad \psi_Y(s; \theta) = [sI - D\theta + \Phi(\theta)]^{-1}. \quad (64)$$

$\square$

The following is a result on the generator of a operator semigroup, which is Proposition 2.5 (Chapter 1) of Golstein [8].

Lemma 5 *Let W be a Banach space and $B(W)$ be the space of all bounded linear operators from W to W . If $\eta \in B(W)$, then*

$$\alpha = \left\{ \alpha(t) = e^{t\eta} \doteq \sum_{n=0}^{\infty} \frac{(t\eta)^n}{n!}; t \in \mathbb{R}_+ \right\} \quad (65)$$

is a semigroup satisfying:

$$\| \alpha(t) - I \| \rightarrow 0, \quad \text{as } t \rightarrow 0+. \quad (66)$$

Moreover, η is the generator of α . Conversely, if α is a semigroup satisfying (66), then the generator η of α belongs to $B(W)$ and $\alpha(t) = e^{t\eta}$. $\square$

Next we give a definition that is very useful for the remainder of the paper.

Definition 5 If $-\eta$ is the generator of a semigroup as described in Lemma 5 and $A(\theta)$ is a Laplace transform matrix as described in Definition 2, we define

$$A \circ \eta = \left(\int_0^\infty \sum_{k \in \mathcal{E}} A_{jk} \{dx\} (e^{-x\eta})_{kl} \right). \quad \square$$

Remark 9 Note that if A is a matrix of nonnegative numbers, then $A \circ \eta = A$. Very important is the fact that, if in Definition 5 A is such that $(A_{jl}[0, \infty])$ is a substochastic matrix and $-\eta$ is the generator of a contraction semigroup, then $A \circ \eta$ is a contraction map from S to S . $\square$

7 Laplace transform of the busy period

The process (T, J^*) is an MCPP with nonnegative jumps and positive drift. We now use the definitions and results of the previous section to derive the Laplace transform matrix of T , but in order to use Lemma 4 we need to add some additional conditions to the storage process, which we call the *standard conditions*,

$$0 < d_- \leq d_+ < \infty \quad \text{and} \quad \lambda_+ = \sup_{j \in \mathcal{E}} \lambda_j < \infty. \quad (67)$$

In the rest of the paper we assume implicitly that the *standard conditions* hold.

Theorem 6 *The Laplace transform matrix of the storage busy period is given by*

$$E_J \left[e^{-\theta T(x)} \right] = e^{-x\Psi^*(\theta)}, \quad (68)$$

where as previously defined,

$$\Psi^*(\theta) = \Phi^*(\theta) + D^{-1}\theta = D^{-1}\Lambda [I - \Gamma^*(\theta)] + D^{-1}\theta. \quad (69)$$

Moreover, and more importantly, Γ^* satisfies the functional equation

$$\Gamma^*(\theta) = \Gamma \circ \Psi^*(\theta). \quad (70)$$

so that

$$\Gamma^*(\theta) = \Gamma \circ \left(D^{-1}\theta + D^{-1}\Lambda [I - \Gamma^*(\theta)] \right), \quad (71)$$

and

$$D\Psi^*(\theta) = \theta I + \Phi \circ \Psi^*(\theta). \quad (72)$$

Proof: From the conditions of the theorem, Theorem 5 and since (as a consequence of the standard conditions) the rates $\{\lambda_j^* = \frac{\lambda_j}{d(j)}\}$ and $\{[d(j)]^{-1}\}$ are uniformly bounded, we conclude by Lemma 4 that $E_J \left[e^{-\theta T(x)} \right] = e^{-x\Psi_T(\theta)}$, where

$$\Psi_T(\theta) = \Phi_T(\theta) + D_T\theta = D^{-1}\Lambda [I - \Gamma^*(\theta)] + D^{-1}\theta = \Psi^*(\theta)$$

and

$$\begin{aligned}\Gamma^*(\theta) &= \left(\int_0^\infty \sum_{k \in \mathcal{E}} p_{jk} M_{jk} \{dx\} E \left[e^{-\theta T(x)}; J \circ T(x) = l | J(0) = k \right] \right) \\ &= \left(\int_0^\infty \sum_{k \in \mathcal{E}} p_{jk} M_{jk} \{dx\} \left(e^{-x \Psi^*(\theta)} \right)_{kl} \right) = \Gamma \circ \Psi^*(\theta).\end{aligned}$$

Therefore, equations (68), (69), and (70) follow, and (71) is an immediate consequence of equations (69) and (70). From (69), (70) and Remark 9, we have

$$D\Psi^*(\theta) = \theta I + \Lambda[I - \Gamma] \circ \Psi^*(\theta) = \theta I + \Phi \circ \Psi^*(\theta). \quad \square$$

As (69) shows, to study the distribution of the busy period initiated by a given initial storage level all we need to study is the distribution of the busy period initiated by new arrivals (which may or may not be associated with a transition on J). Such information is given in our study by $\Gamma^*(\theta)$. We show in Theorem 7 how $\Gamma^*(\theta)$ may be computed.

Theorem 7 *There exists a unique bounded continuous solution $\Gamma^*(\theta)$ of the functional equation (71). Moreover, if for $A \in S$ we define*

$$\Psi^*(A(\theta)) = D^{-1}\theta + D^{-1}\Lambda[I - A(\theta)], \quad (73)$$

and let $\Gamma_0^*(\theta) = 0$ say, and

$$\Gamma_{n+1}^*(\theta) = \Gamma \circ \Psi^*(\Gamma_n^*(\theta)), \quad n = 0, 1, 2, \dots,$$

then

$$\Gamma^*(\theta) = \lim_{n \rightarrow \infty} \Gamma_n^*(\theta).$$

Proof: In view of (73), (71) can be written as

$$\Gamma^*(\theta) = \Gamma \circ \Psi^*(\Gamma^*(\theta)),$$

which is a fixed point equation. Moreover, by Remark 9, $\Gamma \circ \Psi^*$ is a contraction map from S to S . The rest of the proof now follows from a Banach fixed point theorem such as Theorem 3.8.2 in Friedman [6]. $\square$

Remark 10 Consider an $M/G/1$ system with arrival rate λ and let G^* and B^* be the Laplace-Stieltjes transforms of the service time and busy period initiated by a new arrival, respectively. Let $\phi = \lambda(1 - G^*)$ and $E[e^{-sT(x)}] = e^{-x\eta(s)}$, where $T(x)$ is the duration of a busy period initiated by a given amount of work x . It should be noted that equation (71) is a generalization of Takács functional equation for the $M/G/1$ busy period initiated by a new input,

$$B^*(s) = G^*(s + \lambda[1 - B^*(s)]),$$

whereas equation (72) is a generalization of the functional equation associated with the busy period initiated by a given amount of work,

$$\eta(s) = s + \phi(\eta(s)),$$

as given in Prabhu [11]. $\square$

Remark 11 From Theorem 6 and since $(B, J \circ B)$ is equal to $(T, J \circ T)$ if there is no input (i.e. the distribution measures M_{jk} are probability measures concentrated at the origin. In this case $\Phi(\theta) \equiv \Lambda - \Sigma = -Q$ and, by (69), $\Psi^*(\theta) = D^{-1}(\theta I - Q)$), it follows that

$$E_J \left[e^{-\theta B(x)} \right] = e^{-x D^{-1}[\theta I - Q]}, \quad \psi_B(s; \theta) = \left[sI + D^{-1}(\theta I - Q) \right]^{-1}. \quad \square$$

8 The unsatisfied demand and demand rejection rate

We note that $I(t)$ is the local time at zero of the storage level, where the clock has variable speed; the clock speed being $d(j)$ when J is in state j . We let $\zeta(t)$ be the demand rejection rate at time t , so that

$$\zeta(t) = d(J(t)) 1_{\{Z(t)=0\}}, \quad (74)$$

and

$$I(t) = \int_0^t \zeta(s) ds. \quad (75)$$

Theorem 8 For $\theta, s > 0$, we have:

$$\mathcal{L} \left(E_J \left[e^{-\theta I(\cdot)} \zeta(\cdot) | Z(0) = x \right], s \right) = e^{-x \Psi^*(s)} [\theta I + \Psi^*(s)]^{-1}. \quad (76)$$

Proof: We have, using the fact that $\zeta(t) = 0$ for $t < T(x)$,

$$\mathcal{L} \left(E_J \left[e^{-\theta I(\cdot)} \zeta(\cdot) | Z(0) = x \right], s \right) = \left(E \int_{T(x)}^{\infty} \left[e^{-st - \theta I(t)} \zeta(t); J(t) = k | Z(0) = x, J(0) = j \right] dt \right)$$

Now we carry out the transformation $I(t) = \tau$. We have $t = T(x + \tau) = T(x) + T(\tau) \circ \phi_{T(x)}$ (where ϕ is a shift operator as in Definition 1) and $d\tau = dI(t) = \zeta(t)dt$. Therefore the right hand side of the last expression becomes

$$\begin{aligned} & \left(E \int_0^{\infty} \left[e^{-sT(x) - s[T(\tau) \circ \phi_{T(x)}] - \theta \tau}; J(t) = k | Z(0) = x, J(0) = j \right] d\tau \right) \\ &= \left(\sum_{l \in \mathcal{E}} \left(E_J \left[e^{-sT(x)} \right] \right)_{jl} \left(\int_0^{\infty} e^{-\theta \tau} E_J \left[e^{-sT(\tau)} \right] d\tau \right)_{lk} \right) \\ &= \left(\sum_{l \in \mathcal{E}} \left(e^{-x \Psi^*(s)} \right)_{jl} \left(\int_0^{\infty} e^{-\tau [\theta I + \Psi^*(s)]} d\tau \right)_{lk} \right) \\ &= \left(\sum_{l \in \mathcal{E}} \left(e^{-x \Psi^*(s)} \right)_{jl} \left([\theta I + \Psi^*(s)]^{-1} \right)_{lk} \right) \\ &= e^{-x \Psi^*(s)} [\theta I + \Psi^*(s)]^{-1} \quad \square \end{aligned}$$

Corollary 1 For $\theta, s > 0$, we have:

$$\mathcal{L} \left(E_J [\zeta(\cdot) | Z(0) = x], s \right) = e^{-x \Psi^*(s)} [\Psi^*(s)]^{-1}, \quad (77)$$

and

$$\mathcal{L} \left(E_J \left[1_{\{Z(\cdot)=0\}} | Z(0) = x \right], s \right) = e^{-x \Psi^*(s)} [\Psi^*(s)]^{-1} D^{-1}. \quad (78)$$

Proof: By letting $\theta \rightarrow 0$ in Theorem 8, (77) follows. If $J(t) = k$, $\zeta(t) = d(k) 1_{\{Z(t)=0\}}$, thus

$$\begin{aligned}\mathcal{L}(E_J[\zeta(\cdot)|Z(0)=x], s) &= \mathcal{L}\left(E_J\left[1_{\{Z(\cdot)=0\}}|Z(0)=x\right] D, s\right) \\ &= \mathcal{L}\left(E_J\left[1_{\{Z(\cdot)=0\}}|Z(0)=x\right], s\right) D;\end{aligned}$$

(78) follows from this and (77). $\square$

9 The storage level and unsatisfied demand

In this section we study the process $((Z, I), J)$; we are thus interested in the storage level and unsatisfied demand processes along with their interaction. We start this study with a lemma.

Lemma 6 *We have for $\theta_1, \theta_2 > 0$*

$$e^{-\theta_1 Z(t) - \theta_2 I(t)} = e^{-\theta_1 [Z(0) + Y(t)]} - (\theta_1 + \theta_2) \int_0^t e^{-\theta_1 [Y(t) - Y(\tau)] - \theta_2 I(\tau)} \zeta(\tau) d\tau. \quad (79)$$

Proof: For $\theta > 0$, we have,

$$\theta \int_0^t e^{-\theta I(\tau)} \zeta(\tau) d\tau = \left[-e^{-\theta I(\tau)} \right]_0^t = 1 - e^{-\theta I(t)}. \quad (80)$$

Since $Z(t) = Z(0) + Y(t) + I(t)$ and using (80), we have for $\theta_1, \theta_2 > 0$,

$$\begin{aligned}e^{-\theta_1 Z(t) - \theta_2 I(t)} &= e^{-\theta_1 [Z(0) + Y(t)]} e^{-(\theta_1 + \theta_2) I(t)} \\ &= e^{-\theta_1 [Z(0) + Y(t)]} \left[1 - (\theta_1 + \theta_2) \int_0^t e^{-(\theta_1 + \theta_2) I(\tau)} \zeta(\tau) d\tau \right] \\ &= e^{-\theta_1 [Z(0) + Y(t)]} - (\theta_1 + \theta_2) \int_0^t e^{-\theta_1 [Y(t) - Y(\tau)] - \theta_2 I(\tau)} \zeta(\tau) d\tau. \quad \square\end{aligned}$$

Theorem 9 *We have for $s, \theta_1, \theta_2 > 0$,*

$$\psi_{(Z, I)|Z(0)=x}(s; \theta_1, \theta_2) = \{e^{-\theta_1 x} I - (\theta_1 + \theta_2) e^{-x \Psi^*(s)} [\theta_2 I + \Psi^*(s)]^{-1}\} \psi_Y(s; \theta_1), \quad (81)$$

where, as given by (64), $\psi_Y(s; \theta_1) = [sI - D\theta_1 + \Phi(\theta_1)]^{-1}$.

Proof: Using Lemma 6 and since (Y, J) is independent of $Z(0)$, $\psi_{(Z, I)|Z(0)=x}(s; \theta_1, \theta_2)$ is equal to

$$\psi_{x+Y}(s; \theta_1) - (\theta_1 + \theta_2) \mathcal{L}\left(E_J\left[\int_0^{(\cdot)} e^{-\theta_1 [Y(\cdot) - Y(\tau)] - \theta_2 I(\tau)} \zeta(\tau) d\tau | Z(0) = x\right], s\right). \quad (82)$$

Now, note that

$$\psi_{x+Y}(s; \theta_1) = e^{-\theta_1 x} \psi_Y(s; \theta_1), \quad (83)$$

and since $Y(t) - Y(\tau)$ and $(I, \zeta)(\tau)$ are independent as are (Y, J) and $Z(0)$, by interchanging the order of integration in the expression defining the second term in (82), we get

$$\begin{aligned}
& \left[\mathcal{L} \left(E_J \left[\int_0^{(\cdot)} e^{-\theta_1 [Y(\cdot) - Y(\tau)] - \theta_2 I(\tau)} \zeta(\tau) d\tau \mid Z(0) = x \right], s \right) \right]_{jk} \\
&= \sum_{l \in \mathcal{E}} \left[\mathcal{L} \left(E_J \left[e^{-\theta_2 I(\cdot)} \zeta(\cdot) \mid Z(0) = x \right], s \right) \right]_{jl} \\
&\quad \int_{\tau}^{\infty} e^{-s(t-\tau)} E \left[e^{-\theta_1 [Y(t) - Y(\tau)]}; J(t) = k \mid J(\tau) = l \right] dt \\
&= \sum_{l \in \mathcal{E}} \left(e^{-x \Psi^*(s)} [\theta_2 I + \Psi^*(s)]^{-1} \right)_{jl} [\psi_Y(s; \theta_1)]_{lk} \\
&= \left[e^{-x \Psi^*(s)} [\theta_2 I + \Psi^*(s)]^{-1} \psi_Y(s; \theta_1) \right]_{jk}, \tag{84}
\end{aligned}$$

by Theorem 8. The statement follows from (82), (83) and (84). $\square$

Corollary 2 For $s, \theta > 0$, we have:

$$\psi_{Z|Z(0)=x}(s; \theta) = \{e^{-\theta x} I - \theta e^{-x \Psi^*(s)} [\Psi^*(s)]^{-1}\} \psi_Y(s; \theta), \tag{85}$$

and

$$\psi_{I|Z(0)=x}(s; \theta) = \left\{ I - e^{-x \Psi^*(s)} \left[I + \theta^{-1} \Psi^*(s) \right]^{-1} \right\} \hat{\pi}(s). \tag{86}$$

Proof: (85) and (86) follow by letting $\theta_2 \rightarrow 0$ and $\theta_1 \rightarrow 0$, respectively, in equation (9). $\square$

If at time 0 the storage unit is empty, Theorem 9 and Corollary 2 characterize the process $((Y - m, -m), J)$, where m is the infimum of the net input. Corollary 3 gives this characterization.

Corollary 3 For $s, \theta_1, \theta_2 > 0$, we have,

$$\psi_{(Y-m, -m)}(s; \theta_1, \theta_2) = \{I - (\theta_1 + \theta_2) [\theta_2 I + \Psi^*(s)]^{-1}\} \psi_Y(s; \theta_1), \tag{87}$$

$$\psi_{Y-m}(s; \theta) = \{I - \theta [\Psi^*(s)]^{-1}\} \psi_Y(s; \theta), \tag{88}$$

and

$$\psi_{-m}(s; \theta) = \{I - \theta [\theta I + \Psi^*(s)]^{-1}\} \hat{\pi}(s). \quad \square \tag{89}$$

The unsatisfied demand up to time t , $I(t)$, is zero if and only if the storage facility is nonempty on $[0, t]$. This fact is used to deduce Theorem 10 which characterizes the distribution of $I(t)$ as t goes to infinity.

Theorem 10 If J is ergodic and $x, y \geq 0$, then

$$\left(\lim_{t \rightarrow \infty} P[I(t) \leq y, J(t) = k \mid Z(0) = x, J(0) = j] \right) = \left[I - e^{-(x+y) \Psi^*(0)} \right] \Pi. \tag{90}$$

Proof: For any $j, k \in \mathcal{E}$ and $t, x, y \geq 0$, we have:

$$P(I(t) \leq y, J(t) = k \mid Z(0) = x, J(0) = j) = P(I(t) = 0, J(t) = k \mid Z(0) = x + y, J(0) = j).$$

From this, Corollary 2, and a Tauberian theorem, we obtain

$$\begin{aligned} \left(\lim_{t \rightarrow \infty} P[I(t) \leq y, J(t) = k \mid Z(0) = x, J(0) = j] \right) &= \lim_{s \rightarrow 0+} \left\{ s \psi_{1_{\{I(\cdot) \leq y\}} \mid Z(0)=x}(s; \theta) \right\} \\ &= \lim_{s \rightarrow 0+} \left\{ s \psi_{1_{\{I(\cdot)=0\}} \mid Z(0)=x+y}(s; \theta) \right\} = \lim_{s \rightarrow 0+} \left\{ s \lim_{\theta \rightarrow \infty} \psi_{I \mid Z(0)=x+y}(s; \theta) \right\} \\ &= \lim_{s \rightarrow 0+} \left[s \lim_{\theta \rightarrow \infty} \left\{ I - e^{-(x+y)\Psi^*(s)} \left[I + \theta^{-1} \Psi^*(s) \right]^{-1} \right\} \hat{\pi}(s) \right] \\ &= \left[I - e^{-(x+y)\Psi^*(0)} \right] \lim_{s \rightarrow 0+} \{ s \hat{\pi}(s) \} = \left[I - e^{-(x+y)\Psi^*(0)} \right] \Pi. \quad \square \end{aligned}$$

10 The steady state

In this section we study the storage process in steady state when a *congestion measure* ρ is less than one. We assume the state space $\mathcal{E}$ of the MJP J is finite but some of our results may hold for countable $\mathcal{E}$. Note that, since $\mathcal{E}$ is finite, the *standard conditions* hold. In addition, if $\tilde{J}$ is irreducible, we may assume without loss of generality that $\tilde{J}$ is aperiodic (by introducing self-transitions in J that bring no input to the storage system).

Let $\bar{d}$ be the *demand rate* and $\bar{\alpha}$ be the *offered load*,

$$\bar{d} = \sum_{k \in \mathcal{E}} \pi_k d(k), \quad \bar{\alpha} = \sum_{k \in \mathcal{E}} \pi_k \alpha_k, \quad (91)$$

where α_k is the offered load when J is in state k ,

$$\alpha_k = \lambda_k \sum_{l \in \mathcal{E}} p_{kl} \beta_{kl}, \quad \beta_{kl} = \int_0^\infty x M_{kl}\{dx\}, \quad (92)$$

β_{kl} being the average value of a jump associated with a transition from state k to state l in J , which we assume finite. We now define the *congestion measure* ρ as

$$\rho = \frac{\bar{\alpha}}{\bar{d}}. \quad (93)$$

Lemma 7 *If $\rho < 1$, $\mathcal{E}$ is finite and $\tilde{J}$ is irreducible (and aperiodic without loss of generality), then*

- (i) $\frac{Y(t)}{t} \rightarrow \bar{\alpha} - \bar{d} (< 0)$ a.s., for all initial distributions.
- (ii) $\inf\{t > 0 : Z(t) = 0\} < \infty$ a.s., for all initial distributions.
- (iii) For all $j \in \mathcal{E}$, $\tilde{T}_j < \infty$ a.s. .

(iv) $\int_0^t 1_{\{Z(s)=0\}} ds \rightarrow \infty$ a.s. as $t \rightarrow \infty$, for all initial distributions.

(v) For all $j, k \in \mathcal{E}$, $p_{jk} > 0 \Rightarrow p_{jk}^* > 0$.

(vi) The DTMC with transition probability matrix $P^* = (p_{jk}^*)$ is persistent nonnull and we denote its limit distribution by π^* .

(vii) The MJP J° with intensity matrix ΛP^* is persistent nonnull and has limit distribution π° , with

$$\pi_k^\circ = \frac{\frac{\pi_k^*}{\lambda_k}}{\sum_{l \in \mathcal{E}} \frac{\pi_l^*}{\lambda_l}}. \quad (94)$$

(viii) $(Z, J)(t) \xrightarrow{\mathcal{D}} (Z_\infty, J_\infty)$ as $t \rightarrow \infty$, for all initial distributions, where J_∞ is the stationary version of J .

(ix) $\frac{I(t)}{t} \rightarrow \bar{d} - \bar{\alpha}$ a.s., for all initial distributions.

Proof: (i) Since Y can be seen as an accumulated reward process of the ergodic MJP J the result follows immediately from a limit theorem for this type of processes.

(ii) Suppose $Z(0) = x \geq 0$, and let $T_x = \inf\{t \geq 0 : Z(t) = 0\}$. We have,

$$Y(s) \leq -x \Rightarrow T_x \leq s. \quad (95)$$

The statement follows from (i) and (95).

(iii) The statement is an immediate consequence of (ii).

(iv) From (ii) and (iii), the storage level process returns an infinite number of times to zero a.s.; moreover, since each visit to the zero level has a duration which is stochastically greater or equal to an exponential random variable with mean $[\sup_{j \in \mathcal{E}} \lambda_j]^{-1}$, the statement follows.

(v) If $p_{jk} > 0$, we have

$$p_{jk}^* \geq p_{jk} \int_0^\infty e^{-\lambda_k \frac{x}{d(k)}} M_{jk}\{dx\} > 0.$$

(vi) From (v) and the fact that $\tilde{J}$ is irreducible and aperiodic, it follows that the DTMC with transition probability matrix P^* is irreducible and aperiodic, and therefore persistent nonnull, since $\mathcal{E}$ is finite.

(vii) The statement follows directly from (vi) and the relation between the stationary distribution of an ergodic MJP and its associated DTMC.

(viii) The statement follows from the relation between limit properties of our storage process and the embedded Markov-Lindley process $Z_n = Z(T_{n+1}-)$, as given by Theorem 1 of Van Doorn and Regterschot [16] and the limit properties of this Markov-Lindley process as given in Theorem 4 of Prabhu and Tang [15].

(ix) From (viii), $\frac{Z(t)}{t} \rightarrow 0$, a.s.. From this, (i), and the fact that $Z(t) = Z(0) + Y(t) + I(t)$ the statement follows. $\square$

Remark 12 Since our aim is to study the steady state behaviour of the storage process, and using Lemma 7 (ii) and (viii), we may without loss of generality assume $Z(0) = 0$ a.s.. From Lemma 7 (iv), J° can be identified as J on the time set $\mathcal{T}^\circ$,

$$\mathcal{T}^\circ = \{t \geq 0 : Z(t) = 0\}, \quad (96)$$

and we will assume J° has time set $\mathcal{T}^\circ$. J° gives thus the evolution of J when all busy periods are removed from consideration. $\square$

Theorem 11 *If $\rho < 1$, $\mathcal{E}$ is finite and $\tilde{J}$ is irreducible (and aperiodic without loss of generality), then*

$$(i) \quad P(Z_\infty = 0) = (1 - \rho) \frac{\sum_{k \in \mathcal{E}} \pi_k d(k)}{\sum_{k \in \mathcal{E}} \pi_k^\circ d(k)}. \quad (97)$$

(ii) *For all initial distributions,*

$$\frac{I(t)}{t} \longrightarrow (1 - \rho) \sum_{k \in \mathcal{E}} \pi_k d(k) = (1 - \rho) \bar{d}, \text{ a.s..} \quad (98)$$

(iii) *With ζ_∞ being the limit distribution of $\zeta(t)$ and $k \in \mathcal{E}$, we have*

$$P(Z_\infty = 0, J_\infty = k) = P(Z_\infty = 0) \pi_k^\circ, \quad (99)$$

$$E[\zeta_\infty 1_{\{J_\infty = k\}}] = P(Z_\infty = 0) \pi_k^\circ d(k). \quad (100)$$

(iv) *With (from (iii)) $E_J[\zeta_\infty] = P(Z_\infty = 0) \Pi^\circ D$, where $\Pi^\circ = \mathbf{1} \pi^\circ$, we have*

$$E_J[e^{-\theta Z_\infty}] = E_J[\zeta_\infty] \theta [D\theta - \Phi(\theta)]^{-1}. \quad (101)$$

Proof: (i) With $\mathcal{T}^\circ$ defined by (96) and $\mathcal{T}_t^\circ = [0, t] \cap \mathcal{T}^\circ$, we have, by Remark 12 and Lemma 7 (vii),

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{I(t)}{\int_0^t 1_{\{Z(u)=0\}} du} &= \lim_{t \rightarrow \infty} \frac{\int_{\mathcal{T}_t^\circ} d(J(u)) du}{\int_{\mathcal{T}_t^\circ} 1 du} \\ &= \lim_{s \rightarrow \infty} \frac{\int_0^s d(J^\circ(u)) du}{\int_0^s 1 du} = \sum_{k \in \mathcal{E}} \pi_k^\circ d(k), \text{ a.s..} \end{aligned} \quad (102)$$

From Lemma 7 (viii), and for all initial distributions,

$$\frac{1}{t} \int_0^t 1_{\{Z(u)=0\}} du \rightarrow P(Z_\infty = 0), \text{ a.s..} \quad (103)$$

From Lemma 7 (ix), (102), (103) and the fact that

$$\frac{I(t)}{t} = \frac{I(t)}{\int_0^t 1_{\{Z(u)=0\}} du} \times \frac{\int_0^t 1_{\{Z(u)=0\}} du}{t}, \quad (104)$$

we get

$$\bar{d} - \bar{\alpha} = P(Z_\infty = 0) \sum_{k \in \mathcal{E}} \pi_k^\circ d(k),$$

from which the result follows.

(ii) The statement is an immediate consequence of Lemma 7 (ix).

(iii) We have,

$$P(Z_\infty = 0, J_\infty = k) = P(Z_\infty = 0) P(J_\infty = k \mid Z_\infty = 0) = P(Z_\infty = 0) \pi_k^\circ,$$

and (99) holds; (100) follows directly from (99).

(iv) From (85) and by a Tauberian theorem, we have

$$\begin{aligned} E_J [e^{-\theta Z_\infty}] &= \lim_{s \rightarrow 0+} \psi_{Z|Z(0)=0}(s; \theta) = \left\{ \lim_{s \rightarrow 0+} s [\Psi^*(s)]^{-1} \right\} \theta [D\theta - \Phi(\theta)]^{-1} \\ &= E_J [\zeta_\infty] \theta [D\theta - \Phi(\theta)]^{-1}, \end{aligned}$$

where the last inequality follows from (77) and a Tauberian theorem. $\square$

Remark 13 Note that (101) is a generalization of the Pollaczek-Khinchin transform formula for the steady-state waiting time in queue in the $M/G/1$ system that with the notation of Remark 10, $\rho = \lambda/\mu$ ($1/\mu$ being the expected value of the service time distribution), and W^* being the Laplace transform of the waiting time in queue, is given by:

$$W^*(\theta) = (1 - \rho) \frac{\theta}{\theta - \lambda [1 - G^*(\theta)]}. \quad (105)$$

In particular, (101) and (105) show that in the Pollaczek-Khinchin formula $(1 - \rho)$ stands for the limit demand rejection rate. Moreover, from (100), (97) and (98), the average demand rejection rate is $(1 - \rho) \bar{d}$ which is (obviously) the limit rate of increase of $I(t)$. As (97) shows the limit probability of the storage system being empty may differ substantially from $(1 - \rho)$. $\square$

References

- [1] D. Anick, D. Mitra and M.M. Sondhi, Stochastic Theory of a Data-Handling System with Multiple Sources, *Bell System Tech. J.*, 61 (1982), 1871-1894.
- [2] R.M. Blumenthal and R.K. Gettoor, *Markov Processes and Potential Theory* (Academic Press, New York, 1968).
- [3] R.M. Blumenthal, *Excursions of Markov Processes* (Birkäuser, Boston, 1992).
- [4] S. Browne, P. Zipkin, Inventory Models with Continuous, Stochastic Demands, *Annals Appl. Prob.*, 1 (1991), 419-435.
- [5] A.I. Elwalid and D. Mitra, Analysis and Design of Rate-Based Congestion Control of High Speed Networks, I: Stochastic Fluid Models, Access Regulation, *Queueing Systems Theory Appl.*, 9 (1991), 29-64.

- [6] A. Friedman, *Foundations of Modern Analysis* (Dover, New York, 1982).
- [7] D.P. Gaver and J.P. Lehoczky, Channels that Cooperatively Service a Data Stream and Voice Messages, *IEEE Trans. Commun.*, 30 (1982), 1153-1162.
- [8] J.A. Goldstein, *Semigroups of Linear Operators and Applications* (Oxford University Press, New York, 1985).
- [9] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus* (Springer-Verlag, New York, 1988).
- [10] D. Mitra, Stochastic Theory of a Fluid Model of Producers and Consumers Coupled by a Buffer, *Adv. Appl. Prob.*, 20 (1988), 646-676.
- [11] N.U. Prabhu, *Stochastic Storage Processes* (Springer-Verlag, New York, 1980).
- [12] N.U. Prabhu and Y. Zhu, Markov-Modulated Queueing Systems, *Queueing Systems Theory Appl.*, 5 (1989), 215-245.
- [13] N.U. Prabhu, Markov Renewal and Markov-Additive Processes – A Review and Some New Results, *Proceedings of KAIST Mathematics Workshop*, Volume 6, (B.D. Choi and J.W. Yim, Eds), Korea Advanced Institute of Science and Technology, Taejon (1991) , 57-94.
- [14] B.A. Sevast'yanov, Influence of a Storage Bin Capacity on the Average Standstill Time of a Production Line, *Theory Prob. Appl.*, 7 (1962), 429-438.
- [15] N.U. Prabhu and L.C. Tang, Markov-Modulated Single Server Queueing Systems, *To appear in J. Appl. Prob.*.
- [16] E.A. Van Doorn and G.J.K. Regterschot, Conditional Pasta, *Opns. Res. Lett.*, 7 (1988), 229-232.
- [17] E.A. Van Doorn, A.A. Jagers and J.S.J. de Wit, A Fluid Reservoir Regulated by a Birth-Death Process, *Stochastic Models*, 4 (1988), 457-472.