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TECHNICAL REPORT NO. 1048

March 1993

POSSIBLE SAMPLE PATHS
OF SELF-SIMILAR
 α -STABLE PROCESSES¹

by

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¹This research was supported by the NSA grant 92G-116.

POSSIBLE SAMPLE PATHS OF SELF-SIMILAR α -STABLE PROCESSES ^{*†‡}

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March 9, 1993

Abstract

We describe all possibilities for sample path properties of H -self-similar α -stable processes with stationary increments for all pairs (α, H) , completing the picture presented by Kôno and Maejima (1991).

Running title: Sample paths of self-similar stable processes

*This research was supported by the NSA grant 92G-116.

†**AMS 1980 Subject Classification:** Primary 60G17, 60E07

‡**Keyword and phrases:** Stable processes, self-similarity, sample path properties

1 Introduction

A stochastic process $\{X(t), t \in [0, \infty)\}$ (or $t \in R$) is called H -self-similar if for every $c > 0$, $\{X(ct), t \in [0, \infty)\}$ and $\{c^H X(t), t \in [0, \infty)\}$ have the same finite-dimensional distributions. H -self-similar stochastic processes *with stationary increments* (or H -sssi processes) have been studied extensively in the last 10 years. This interest partly stems from questions in theoretical physics, but also from the relationship between self-similarity and long-range dependence. We refer the reader to Taqqu (1986) for more information. In particular, in the *heavy-tailed case* (i.e. when the process $\{X(t), t \in [0, \infty)\}$ does not have finite variance), the natural way of defining long-range dependence is via self-similar processes. α -stable H -sssi processes with $0 < \alpha < 2$ are among the most appealing of such heavy-tailed H -self-similar processes, and, as a result, they drew much attention in the literature. We restrict ourselves in this paper to symmetric processes, and we remind the reader that a stochastic process $\{X(t), t \in T\}$ is symmetric α -stable (SaS) when every linear combination of its coordinates is a one-dimensional SaS random variable, i.e. has characteristic function of the form $\exp(-\sigma|\theta|^\alpha)$.

Parameters (α, H) of a non-degenerate SaS H -sssi processes are well known to be restricted to the region $\mathcal{P} = \{0 < \alpha \leq 2, 0 < H \leq 1 \vee 1/\alpha\}$. Various parts of the feasible region are important for different aspects of probabilistic modeling. Kôno and Maejima (1991) discussed possible sample paths of SaS H -sssi processes with parameters in different parts of $\mathcal{P}$, discussing, in particular, whether, for a given choice of (α, H) , one can have a sample continuous or nowhere bounded process. We reproduce here their results presented in a graphical form.

In the region B_2 all S α S H -sssi processes are sample continuous, and the same is true for the regions A_1 and A_2 (the processes in the latter two regions are Fractional Brownian motions). The regions A_2 , B_3 and B_4 contain both sample continuous processes (e.g. sub-Gaussian S α S processes based on Fractional Brownian motions) and nowhere bounded processes (e.g. Linear Fractional Stable motions for the regions A_2 and B_4 , and Log-fractional Stable motions for the region B_3). A reader unfamiliar with the terminology is referred to Kôno and Maejima (1991). Finally, in the region C_1 the only S α S H -sssi processes are S α S Lévy motions, which are neither sample continuous nor nowhere bounded.

The question of the possible sample paths of S α S H -sssi processes in C_2 has remained open, and the purpose of this note is to show that any choice of parameters (α, H) in C_2 allows both continuous sample paths and nowhere bounded sample paths.

2 A process in C_2 with continuous sample paths

We use the process X_{12} of Kôno and Maejima (1991), which is based on a particular family of H -sssi processes due to Kesten and Spitzer (1979). Let $U_1, U_2, \dots$ be a sequence of (say) symmetric i.i.d. integer valued random variables in the domain of normal attraction of a symmetric θ -stable law, $1 < \theta \leq 2$, and let $\xi(1), \xi(2), \dots$ be a sequence of i.i.d. real random variables in the domain of normal attraction of a β -stable law, $0 < \beta \leq 2$. The two sequences are independent. Set $S_k = U_1 + \dots + U_k$, and let $W_n = \sum_{k=1}^n \xi(S_k)$. Then the sequence of stochastic processes $\{D_n(t) = n^{-\delta} W_{nt}, t \in [0, \infty)\}$ (with W_s defined, for a non-integer s , by linear interpolation) converges, as $n \rightarrow \infty$, weakly in $C[0, \infty)$, to a sample continuous δ -self-similar process $\{\Delta(t), t \geq 0\}$ with stationary increments, where $\delta = 1 - 1/\theta + 1/\theta\beta$ (Kesten and Spitzer (1979)). It is not difficult to show that for every $t > 0$

$$E|\Delta(t)|^p < \infty \text{ for every } 0 < p < \beta.$$

We now construct a S α S H -sssi process with a given (α, H) in C_2 as follows. Choose a $\beta \in (\alpha, 1)$ such that

$$\theta = \frac{1-\beta}{\beta} \frac{1}{H-1} \in (1, 2].$$

Let $\{\Delta(t), t \geq 0\}$ be the corresponding Kesten-Spitzer process with all continuous sample paths, living, say, on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Letting M be a S α S random measure on $(\Omega_1, \mathcal{F}_1)$ with control measure P_1 , we now define

$$X(t) = \int_{\Omega_1} \Delta(t, \omega_1) M(d\omega_1), \quad t \geq 0. \quad (2.1)$$

Then $\{X(t), t \geq 0\}$ is a S α S H -sssi process. Since $\{\Delta(t), t \geq 0\}$ has continuous sample paths and $0 < \alpha < 1$, sample continuity of $\{X(t), t \geq 0\}$ will follow once we show that

$$E_1 \sup_{0 \leq t \leq 1} |\Delta(t)|^\alpha < \infty. \quad (2.2)$$

In terms of the approximating processes it is enough, of course, to assume that $\xi(x)$'s are themselves symmetric β -stable, and then to show that for every $0 < p < \beta$ there is a constant $C < \infty$ such that for every $n \geq 1$

$$E \sup_{0 \leq t \leq 1} |D_n(t)|^p \leq C. \quad (2.3)$$

With $N_i(x)$ being the number of times the random walk $\{S_k, k \geq 1\}$ visits x up to time i , we define for each $n \geq 1$ and $x \in \mathcal{Z}$ a random vector $\mathbf{N}^{(n,x)} \in C[0,1]$ by $\mathbf{N}^{(n,x)}(i/n) = N_i(x)$ for $i = 0, 1, \dots, n$, and with linear interpolation between these points. Then

$$\begin{aligned} E \sup_{0 \leq t \leq 1} |D_n(t)|^p &= n^{-p\delta} E \sup_{0 \leq t \leq 1} \left| \sum_{x \in \mathcal{Z}} \xi(x) N_{[nt]}(x) \right|^p \\ &\leq n^{-p\delta} E \left\| \sum_{x \in \mathcal{Z}} \xi(x) \mathbf{N}^{(n,x)} \right\|^p. \end{aligned} \quad (2.4)$$

Write the expectation in the right hand side of (2.4) as $E_{\mathbf{N}} E_{\xi}$. For fixed $\{\mathbf{N}^{(n,x)}, x \in \mathcal{Z}\}$ the vector $\mathbf{Y} = \sum_{x \in \mathcal{Z}} \xi(x) \mathbf{N}^{(n,x)}$ is an S β S random vector in $C[0,1]$ with spectral measure given by

$$\begin{aligned} \Gamma_{\mathbf{Y}} &= \sum_{x \in \mathcal{Z}} \|\mathbf{N}^{(n,x)}\|^\beta \frac{1}{2} \left(\delta_{\{\mathbf{N}^{(n,x)}/\|\mathbf{N}^{(n,x)}\|\}} + \delta_{\{-\mathbf{N}^{(n,x)}/\|\mathbf{N}^{(n,x)}\|\}} \right) \\ &= \sum_{x \in \mathcal{Z}} N_n(x)^\beta \frac{1}{2} \left(\delta_{\{\mathbf{N}^{(n,x)}/\|\mathbf{N}^{(n,x)}\|\}} + \delta_{\{-\mathbf{N}^{(n,x)}/\|\mathbf{N}^{(n,x)}\|\}} \right). \end{aligned}$$

Since $C[0,1]$ is of Rademacher type 1, and $0 < p < \beta < 1$, we conclude that there is a constant $C_1 < \infty$ such that

$$E_{\xi} \|\mathbf{Y}\|^p \leq C_1 \left(\Gamma_{\mathbf{Y}}(B) \right)^{p/\beta} = C_1 \left(\sum_{x \in \mathcal{Z}} N_n(x)^\beta \right)^{p/\beta},$$

where B is the unit ball in $C[0,1]$. Therefore, for every $n \geq 1$,

$$E \sup_{0 \leq t \leq 1} |D_n(t)|^p \leq C_1 n^{-p\delta} E \left(\sum_{x \in \mathcal{Z}} N_n(x)^\beta \right)^{p/\beta}. \quad (2.5)$$

Let $K_n = \max_{i \leq n} |S_i|$. Then $N_n(x) = 0$ for every $|x| > K_n$. Therefore, Hölder's inequality ($0 < \beta < 1$) gives

$$\left(\sum_{x \in \mathcal{Z}} N_n(x)^\beta \right)^{1/\beta} \leq \left(2K_n + 1 \right)^{-1+1/\beta} \sum_{x=-K_n}^{K_n} N_n(x) = \left(2K_n + 1 \right)^{-1+1/\beta} n. \quad (2.6)$$

It is well known (and easy to show) that for every $0 < r < \theta$ there is a constant $C^{(r)} < \infty$ such that for every $n \geq 1$

$$EK_n^r \leq C^{(r)} n^{r/\theta}. \quad (2.7)$$

Now (2.3) follows from (2.5), (2.6) and (2.7) with $r = p(-1 + 1/\beta)$.

3 A process in C_2 with nowhere bounded sample paths

Given an (α, H) in C_2 we let

$$\alpha_1 = \left(\frac{\alpha}{H}\right)^{1/2} \in (\alpha, \alpha^{1/2}] \subset (0, 1), \quad (3.1)$$

$$H_1 = (\alpha H)^{1/2} \in [\alpha^{1/2}, 1) \subset (0, 1). \quad (3.2)$$

Let $\{Y(t), t \geq 0\}$ be a symmetric Linear Fractional 1-stable Motion with parameter of self-similarity H_1 . Note that this process is not sample bounded on every interval of positive length. Let further $\{Z(t), t \in R\}$ be an increasing strictly α_1 -stable Lévy motion, independent of $\{Y(t), t \geq 0\}$. The two processes are assumed to be defined on probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ accordingly, and we choose separable and measurable versions of the two processes. Define (on the product probability space)

$$W(t) = Z(Y(t)), t \geq 0.$$

It follows from Corollary 6.3 of Vervaat (1985) that $\{W(t), t \geq 0\}$ is $(H_1/\alpha_1) = H$ -sssi. Further, for every $0 \leq a < b < \infty$ we have $\sup_{a \leq t \leq b} |Y(t)| = \infty$ a.s.. Therefore, for P_1 -almost every ω_1 there is, say, a countable set $t_1, t_2, \dots$ in (a, b) such that $|Y(t_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Therefore,

$$P_2\left(\sup_{a \leq t \leq b} |Z(Y(t))| = \infty\right) \geq P_2\left(\sup_{s \in \{Y(t_n), n \geq 1\}} |Z(s)| = \infty\right) = 1.$$

Therefore,

$$P_1 \times P_2\left(\sup_{a \leq t \leq b} |W(t)| = \infty\right) = 1. \quad (3.3)$$

Since $\alpha < \alpha_1$, we conclude immediately that for every $t > 0$ we have $E_1 \times E_2 |W(t)|^\alpha < \infty$.

Let now again M be a SaS random measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ with control measure $P_1 \times P_2$. Define

$$X(t) = \int_{\Omega_1 \times \Omega_2} W(t, (\omega_1, \omega_2)) M(d(\omega_1, \omega_2)), t \geq 0.$$

Then $\{X(t), t \geq 0\}$ is an S α S H -sssi process, and since by (3.3) $E_1 \times E_2 \sup_{a \leq t \leq b} |W(t)|^\alpha = \infty$ for every $0 \leq a < b < \infty$, it follows that $\{X(t), t \geq 0\}$ is nowhere bounded.

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