SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-3801

TECHNICAL REPORT NO. 1019

August 1992

CRITICAL TIME OF THE LOGNORMAL DISTRIBUTION

by

Dong Shang Chang¹

¹Department of Business Administration, National Central University,
Chungli, Taiwan, R.O.C.

ABSTRACT

The critical time is the time point as the failure rate starts to decrease and also as the mean residual lifetime starts to increase¹. The estimated critical time is useful for determining the duration of a burn-in process. The method for estimating the critical time of the failure rate for lognormal lifetime distribution is discussed. A single time censored data is used as a example for illustration.

1. INTRODUCTION

The lognormal distribution has been used widely as a lifetime distribution model. The hazard function of lognormal increases from zero at time 0 to a maximum at the critical time, and then decreases back down to zero with increasing time. For many products, the failure rate does not go to zero with increasing time. Nevertheless, the model may adequately represent most of the range of life, especially early life. The feature of lognormal hazard function suggests that the mean residual lifetime can be increased if burn-in is conducted beyond the critical time, of course, provided that in practice the critical time should not exceed certain fraction of the mean life. Thus the estimation of the critical time can help us determine whether a burn-in is worth to do and if so when is the ideal cut-off point.

In this paper the author first investigates the property of the critical time related to the parameters of lognormal distribution. Then using the maximum likelihood estimators of its parameters and their asymptotic confidence intervals, the point estimate and confidence interval for the critical time are obtained. Finally, a type-I censored data is used as a illustrative example.

Notations

μ, σ	the mean and standard deviation of lognormal distribution
$\hat{(\cdot)}$	the estimate of $(\cdot)$
MLE	maximum likelihood estimate
$\phi(\cdot), \Phi(\cdot)$	the pdf and cdf of standard normal distribution
$h(t; \cdot)$	hazard function of lognormal distribution
$g'(\cdot)$	the first derivative of function $g(\cdot)$
$\log(\cdot)$	natural log base e of $(\cdot)$
t^*	critical time of failure rate

$(\underline{\cdot}), (\overline{\cdot})$ lower and upper confidence intervals of $(\cdot)$

α_1, α_2 arbitrary number, $0 \leq \alpha_i \leq 1, i=1, 2$

2. CRITICAL TIME AND ITS PROPERTIES

Let T be a random variable such that $\log T$ is normally distributed with unknown mean μ and standard deviation σ . The hazard function for T is given as follow:

$$h(t; \mu, \sigma) = \frac{\phi(z)}{t\sigma \cdot \Phi(-z)}, \quad \text{for } t \geq 0 \quad (1)$$

where $z = (\log(t) - \mu)/\sigma$.

Define $f(t; \mu, \sigma) = \phi(z)/t\sigma$, from eq.(1) we have

$$f'(t; \mu, \sigma) = -p(t; \mu, \sigma) \cdot f(t; \mu, \sigma)$$

where

$$p(t; \mu, \sigma) = \frac{1}{t} + \left(\frac{\log(t) - \mu}{\sigma} \right) \cdot \frac{1}{\sigma t} \quad (2)$$

It follows that

$$\begin{aligned} h'(t; \mu, \sigma) &= \frac{\Phi(-z) \cdot f'(t; \mu, \sigma) + f^2(t; \mu, \sigma)}{\Phi^2(-z)} \\ &= h(t; \mu, \sigma) \cdot [-p(t; \mu, \sigma) + h(t; \mu, \sigma)] \end{aligned} \quad (3)$$

then the solution to eq.(3)

$$h(t; \mu, \sigma) - p(t; \mu, \sigma) = 0 \quad (4)$$

is the critical time t^* of the lognormal hazard function.

For $\sigma \leq 0.2$, the hazard function $h(t; \mu, \sigma)$ increases and is much like that of a normal distribution, i.e., the critical time is too high for its practical use. For the purpose of burn-in, the evaluation of critical time is useful only when σ is large. Given $\mu=0$, we solve the critical time by eq.(4) for various values of σ ranging from 0.3 to 1.28. The resulting critical time is depicted in figure 1. It can be seen that the critical time is a monotonic decreasing function of σ . Similarly, given $\sigma=1$, we observe that the critical time is a monotonic increasing function of μ as shown in figure 2.

3. ESTIMATION OF THE CRITICAL TIME

By the invariance principle of MLE (see Zehna²), the MLE of the critical time is then the solution of eq.(4) where μ and σ are replaced by the MLE $\hat{\mu}$ and $\hat{\sigma}$ respectively.

Suppose that the $100 \cdot (1 - \alpha_1)\%$ confidence interval of μ and the $100 \cdot (1 - \alpha_2)\%$ confidence interval of σ are given by

$$[\underline{\mu}, \bar{\mu}] \text{ and } [\underline{\sigma}, \bar{\sigma}]$$

respectively. We have the following result.

Theorem 1 (a) For complete data, the $100 \cdot (1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2)\%$ level confidence interval for the critical time is given by the interval

$$[\underline{t}^*, \bar{t}^*] = [t^*(\underline{\mu}, \bar{\sigma}), t^*(\bar{\mu}, \underline{\sigma})], \quad (5)$$

(b) For censored data, the confidence level for the interval of critical time derived from eq.(5) is $100 \cdot (1 - \alpha_1 - \alpha_2)\%$.

Proof: (a) By the definition of confidence interval in Bickel and Doksum³, we only need to show that

$$P[t^*(\underline{\mu}, \bar{\sigma}) \leq t^*(\mu, \sigma) \leq t^*(\bar{\mu}, \underline{\sigma})] \geq 1 - \alpha_1 - \alpha_2 - \alpha_1 \alpha_2.$$

From the concept of probability, we know that if set $A \subseteq B$ then $P(A) \leq P(B)$. Now we have

$$\inf_{\Omega} t^*(\mu, \sigma) \leq t^*(\mu, \sigma) \leq \sup_{\Omega} t^*(\mu, \sigma) \text{ whenever } \underline{\mu} \leq \mu \leq \bar{\mu}, \underline{\sigma} \leq \sigma \leq \bar{\sigma}.$$

where $\Omega = \{ (\mu, \sigma): \underline{\mu} \leq \mu \leq \bar{\mu}, \underline{\sigma} \leq \sigma \leq \bar{\sigma} \}$. The latter is a subset of the former. Thus

$$\begin{aligned} & P[\inf_{\Omega} t^*(\mu, \sigma) \leq t^*(\mu, \sigma) \leq \sup_{\Omega} t^*(\mu, \sigma)] \\ & \geq P[\underline{\mu} \leq \mu \leq \bar{\mu}, \underline{\sigma} \leq \sigma \leq \bar{\sigma}] \\ & = 1 - \alpha_1 - \alpha_2 - \alpha_1 \alpha_2 \text{ since } \hat{\mu} \text{ and } \hat{\sigma} \text{ are independent for complete data.} \end{aligned}$$

It is readily seen that the critical time t^* is monotone increasing in μ and monotone decreasing in σ . Hence it follows that

$$\inf_{\Omega} t^*(\mu, \sigma) = t^*(\underline{\mu}, \bar{\sigma}) \text{ and } \sup_{\Omega} t^*(\mu, \sigma) = t^*(\bar{\mu}, \underline{\sigma}).$$

(b) Since $(\hat{\mu}, \hat{\sigma})$ from censored data is dependent, the confidence level of the interval of critical time is $100 \cdot (1 - \alpha_1 - \alpha_2)\%$.

□

4. EXAMPLE

There is a single time censored example given in Nelson⁴ on the locomotive control failure. The data consist of $n=96$ controls and $r=59$ controls ran 135,000 miles without failure. The MLE of the lognormal parameters are $\hat{\mu}=5.117$ and $\hat{\sigma}=0.7055$ (base natural e logs). For $\alpha_1=\alpha_2=0.05$, the approximate 95% confidence intervals of μ and σ are $[4.9116, 5.3224]$ and $[0.5436, 0.9156]$, respectively. The hazard function is depicted in figure 3.

The MLE for the critical time is obtained by solving eq.(4) where μ and σ are replaced by $\hat{\mu}$ and $\hat{\sigma}$ as $\hat{t}^*=202.1$ thousand miles. We obtain the 90% confidence interval of the critical time by using $\underline{\mu}$ and $\bar{\sigma}$ to compute $\underline{t}^*$; and using $\bar{\mu}$ and $\underline{\sigma}$ to compute $\bar{t}^*$ as in eq.(5). The resulting interval is given by

$$[\underline{t}^*, \bar{t}^*] = [103.58, 335.95].$$

Since the mean time to failure is 213.98 thousand miles, using the estimated critical time as the burn-in time period might be impractical.

5. CONCLUSION

The estimation of the critical time can help reliability analysts determine the duration of a burn-in process for increasing the mean residual lifetime. The approach for constructing the confidence interval of the critical time for an lognormal distribution from the confidence interval of its parameters is developed. We make no assumptions regarding

the type of data for using this approach, it could be based on various truncated or censored sample. The same procedure also could be applied to obtain the estimation of the critical time for other lifetime model, such as Inverse Gaussian distribution and Birnbaum—Saunders distribution.

REFERENCES

1. Chhikara R. S. and Folks J. L. (1977), " The inverse Gaussian as a lifetime model ", Technometrics, Vol. 19, pp. 461—468.
2. Zehna P. W. (1966), " Invariance of maximum likelihood estimation ", Ann. Math. Stats., Vol. 37, pp. 755.
3. Bickel P. J. and Doksum K. A. (1977), Mathematical Statistics: Basic ideas and selected topics, Holden—Day Inc..
4. Nelson W. (1982), Applied Life Data Analysis, John Wiley & Sons, Inc..

Figure 1: Critical time as a function of σ , given $\mu=0$.

Figure 2: Critical time as a function of μ , given $\sigma=1$.

Figure 3: Hazard function for locomotive control example.