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by

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Geometric Stable Distributions in Banach Spaces

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1 Introduction

It is commonly accepted that stable distributions do provide useful models for many phenomena observed in diverse fields. Useful surveys on the vast literature on stable models are provided by Weron and Weron (1985), Zolotarev (1986), Samorodnitsky and Taqqu (1993). Having as a starting point the works of Robbins (1948a,b) and Renyi (1956), B.V. Gnedenko and his students (cf. Gnedenko and Fahum (1969), Szász (1972), Gnedenko (1983a,b)) examined models in reliability featuring geometric stable (GS) distributions. The setting accepted in those works is the stability property: $X_1 + \cdots + X_T \stackrel{d}{=} a_T X_1$ ($a_T > 0$), T being a geometrically distributed r.v. independent of the i.i.d. sequence $\{X_i\}$. If T is viewed as a moment of extreme change of the fundamentals of a financial asset (or portfolio of assets in the multivariate case), the distribution of X_i 's – a GS-law – can be used to model asset returns data, see Mittnik and Rachev (1989, 1990). The latter successfully competes with the classical α -stable (Paretian) model (Mandelbrot (1963), Fama (1965), DuMouchel (1983)). The idea of using random summation scheme to model stock price changes goes back to Mandelbrot and Taylor (1967) and Clark (1973). A special interest to GS-laws has been recently seen in a number of papers on mathematical reliability theory and queueing theory. The basic example here is the Renyi theorem on geometric summation for i.i.d. random variables, playing the role of a LLN for geometric sums, see Kalashnikov and Vseksvjatskii (1989), Gertsbach (1990), Brown (1990) and the references therein.

We start with a number of examples illustrating the problems considered in this paper.

Example 1. (*Systems with rapid repair*, see Gertsbach (1990), Brown (1990)). Consider an operating unit with a random lifetime X . When an operating unit fails it is instantaneously replaced by a standby identical unit which is available with probability $q = 1 - p$ close to 1.

The survival function of the system is given by $S(t) = P(\sum_{i=1}^{T(p)} X_i > t)$, $t \geq 0$, where X_i 's are i.i.d. copies of X , and $T(p)$ is independent of X_i 's and geometrically distributed with parameter $0 < p < 1$. The usual assumption is that X is exponentially distributed meaning that

$$\sum_{i=1}^{T(p)} X_i \stackrel{d}{=} \frac{1}{p} X, \quad \forall \quad 0 < p < 1.$$

If the law of X is not exactly exponential (but X_i 's are still i.i.d. with finite first moments) one often relies on an approximation of the normalized operating times $S_{T(p)} = X_1 + \cdots + X_{T(p)}$ by an exponentially distributed random variable Y with $EY = EX$. The “difference” between the exact and the approximate distributions goes to zero as $p \rightarrow 0$. The discrepancy between the two can be bounded by

$$\rho(pS_{T(p)}, Y) \leq \frac{\mu_2 p}{\mu_1}$$

where ρ is the Kolmogorov (uniform) distance between the distribution functions and μ_i is the i th moment of X , see Brown (1990). Rachev and Sengupta (1992) give a bound on one discrepancy between the exact and the approximate distributions in the multivariate case ($\mathbf{X}, \mathbf{S}_{T(p)}$ and $\mathbf{Y}$ are random vectors). In this case the deviation is measured in terms of Kantorovich metric

$$\kappa(p\mathbf{S}_{T(p)}, \mathbf{Y}) = \inf\{E\|\mathbf{U} - \mathbf{W}\| : \mathbf{U} \stackrel{d}{=} p\mathbf{S}_{T(p)}, \mathbf{W} \stackrel{d}{=} \mathbf{Y}\}.$$

Example 2. (*Regenerative processes*, see Smith (1958), Iglehart and Shedler (1980), Kalashnikov (1989)). Consider a regenerative process

$$z(t) = \xi_i(t - S_{i-1}) \quad \text{if} \quad S_{i-1} \leq t < S_i$$

where S_i is a sum of positive i.i.d. X_i 's and $(\xi_i(t), t \in (0, X_i)), i = 1, 2, \dots$ are i.i.d. random cycles ($\xi_i(t)$ is a random process on the random interval $(0, X_i)$). In each cycle an event A (a “break-down”) may occur with a small probability p . Then the first break-down for the process Z occurs at $\tau = S_{T(p)-1} + y_{T(p)}$, where $y_{T(p)}$ is the waiting time from the beginning of cycle $T(p)$ until A occurs. Again the common approach to the problem of estimating the d.f. of τ is based on using the appropriate limit distribution to approximate the law of $S_{T(p)-1}$. The rate of convergence to the limit determines the quality of approximation.

Example 3. (*Pareto capital model*, see Ziemba (1979), Klebanov, Melamed and Rachev (1990), Mittink and Rachev (1990) and the references therein). Starting with initial capital portfolio of d assets $(c^{(1)}, \dots, c^{(d)})$, the value of the portfolio after n periods of time is $\mathbf{P}_n = (c^{(1)} \prod_{i=1}^n X_i^{(1)}, \dots, c^{(d)} \prod_{i=1}^n X_i^{(d)})$, where $\mathbf{X}_i = (X_i^{(1)}, \dots, X_i^{(d)})$, $i = 1, 2, \dots$ are positive i.i.d. random vectors. The process continues until a fundamental change of the market conditions occurs, after which further investment stops being profitable. Under natural circumstances this happens at a geometric moment $T(p)$. As $p \rightarrow 0$, $\mathbf{P}_{T(p)}^{p^{1/\alpha}}$ ($0 < \alpha \leq 2$), under some regularity conditions, converges in distribution to a Pareto distributed random vector $\mathbf{P}_0$. Since the exact joint distribution of $\mathbf{P}_{T(p)}$ —the values of the assets in the portfolio at the terminal time $T(p)$ —is hard to evaluate, an approximation by $\mathbf{P}_0^{-p^{1/\alpha}}$ is frequently used. The quality of approximation (in the uniform distance)

$$\rho(\mathbf{P}_{T(p)}, \mathbf{P}_0^{-p^{1/\alpha}}) \xrightarrow{p \rightarrow 0} 0$$

is again based on the bounds for $\kappa(p^{1/\alpha} \sum_{i=1}^{T(p)} (\log c^{(1)} X_i^{(1)}, \dots, \log c^{(d)} X_i^{(d)}), (\log P_0^{(1)}, \dots, \log P_0^{(d)}))$.

Example 4. (*The process of stock-price changes*, see Mandelbrot and Taylor (1967), Clark (1973)). Mandelbrot and Taylor (1967) modelled the process of stock returns by a symmetric α -stable motion (L). For a given period $t \in [0, T]$, $L(t)$ is viewed as a subordinated process $L(t) = X(\tau(t))$. The $X(\tau)$, $\tau > 0$, is a mean zero Brownian motion with variance σ^2 , and the directing process $\tau(t)$ is a strictly increasing $\frac{\alpha}{2}$ -stable motion with d.f.

$$\phi_{\tau(t)}(\theta) = \exp\{-\gamma t |\theta|^{\alpha/2} [1 - i(\theta/|\theta|) \tan(\frac{\pi\alpha}{4})]\}.$$

In Mandelbrot and Taylor (1967) (see also Clark (1973)) $\tau(t)$ is interpreted as the cumulative volume (or number) of transactions up to physical time t , and $X(\tau)$, $\tau > 0$, is the stock price on a “time scale” measured in the volume of transactions. Then $L(t) = X(\tau(t))$ is a symmetric α -stable motion with ch.f.

$$\phi_{L(t)}(\theta) = \exp\{-\gamma(\sigma^2/2)^{\alpha/2} (\cos \frac{\pi\alpha}{4})^{-\alpha/2} t |\theta|^\alpha\}.$$

The index of stability α can be assumed to be greater than 1. (See Akgiray and Booth (1988), where stable models were fit to the data on 200 stocks, and for all stocks the index α exceeds 1.) Following the main idea of Mandelbrot (1963) and Fama (1965) on stability of the stock price changes, we view L as a random element of the Banach space $L^p([0, T])$, $1 < p < \alpha$, satisfying

$$n^{-1/\alpha}(L_1 + \dots + L_n) \stackrel{d}{=} L$$

for any $n \in \mathbb{N}$ and i.i.d. copies $L_1, \dots, L_n$ of L . Extending the Mandelbrot model to accommodate the possibility of market crashes, assume that for any period of length T there is a small probability $p > 0$ with which a new information affecting future fundamentals of the underlying assets may become available. Then the above “stability” property should be properly replaced by

$$p^{1/\alpha}(L_1 + \dots + L_{T(p)}) \stackrel{d}{=} L$$

where $T(p)$ is geometrically distributed and independent of L_i 's. Then the stock price process at the terminal period has the same law as $p^{-1/\alpha}L(t)$, $t \in [0, T]$ and we refer to the law of L as a *geometric stable law*. If the above equality is replaced by the more realistic $p^{1/\alpha}(L_1 + \dots + L_{T(p)}) \xrightarrow[p \rightarrow 0]{d} L$, we face again the problem of determining the quality of approximation (i.e. the rate of convergence) resulting from using the limiting GS law when $p \in (0, 1)$.

The above examples demonstrate both the need to understand the structure of geometric stable random vectors and the rate of convergence in their domain of attraction. The structure of GS laws is best seen through their membership in the class of geometric infinitely divisible laws. Although univariate geometric stable and infinitely divisible laws have been studied in, among others, Szász

(1972), Klebanov, Maniya and Melamed (1984), Sakalauskas (1985) and Korolev (1989), very little is known about the multivariate and, especially, infinite dimensional case.

We start, therefore, with studying geometric infinitely divisible laws and GS laws on Banach spaces (Section 2 of the paper). In Section 3 we derive Berry-Essen type estimates for the rate of convergence in the domain of attraction of GS laws, extending the results of Kalashnikov and Vsekhvatskii (1989) and Brown (1990) on the rate of convergence in the univariate Renyi theorem.

2 Geometric infinitely divisible and geometric stable measures on Banach space

In the sequel, B stands for a Banach space, B' is its topological dual, and $\mathcal{P}(B)$ is the set of Radon probability measures (shortly, laws) on B . The characteristic functional (ch.f.) of $\mu \in \mathcal{P}(B)$ is

$$\hat{\mu}(t) = \int_B \exp\{i \langle x, t \rangle\} \mu(dx), \quad t \in B'.$$

The set of infinitely divisible (ID) laws is denoted by $\mathcal{ID}(B)$. $\mu \in \mathcal{P}(B)$ is strictly stable if for any $n \in \mathbf{N}$ there exist $a_n > 0$ such that $\hat{\mu}(t)^n = \hat{\mu}(a_n t)$, $t \in B'$ (see Araujo and Giné (1980), Linde (1983), Weron (1984), Samorodnitsky and Taqqu (1993) and the references therein for all necessary facts on ID and stable laws on Banach spaces).

The law μ is said to be geometrically infinitely divisible (GID) if for any $p \in (0, 1)$ there exists a law $\nu = \nu_p$ such that

$$(2.1) \quad \hat{\mu} = p\hat{\nu}/(1 - q\hat{\nu}),$$

where $q = 1 - p$. Clearly, in terms of B -valued r.v.'s, (2.1) is equivalent to

$$(2.2) \quad Y \stackrel{d}{=} \sum_{i=1}^{T(p)} Y_i^{(p)}$$

with Y having distribution μ , and $Y_i^{(p)}$ being i.i.d. with common distribution ν and independent of the geometric r.v. $T(p)$. The random variable $T(p)$ is, distributionally, "of size p^{-1} ", that is,

$$pT(p) \xrightarrow{w} A \quad \text{as } p \rightarrow 0,$$

where A has the standard (mean 1) exponential distribution. Therefore, one expects the series in the right hand side of (2.2) to be, distributionally, of the same size as the series $\sum_{i=1}^{[p^{-1}]} Y_i^{(p)}$. The exact relation between series of these types is given in the following theorem, quoted from Rosinski ((1976), the idea

of which goes back to Gnedenko's transfer theorem (see Gnedenko and Fahum (1969)).

Theorem 2.1 *Let $X_{ni}, n \geq 1, i \geq 1$, be an array of B -valued Radon random variables, such that for each $n = 1, 2, \dots$ $X_{n1}, X_{n2}, \dots$ are i.i.d. Let $N_1, N_2, \dots$ be a sequence of positive integer-valued random variables, independent of the array of the X 's. For $n \geq 1$ and $k \geq 1$ set $S_k^{(n)} = \sum_{i=1}^k X_{ni}$.*

(i) Suppose that there is a sequence of positive integers $k_1, k_2, \dots$, a probability measure $\mu \in \mathcal{P}(B)$ and a real valued random variable A such that

$$(2.3) \quad S_{k_n}^{(n)} \xrightarrow{w} S \sim \mu \text{ as } n \rightarrow \infty$$

and

$$(2.4) \quad \frac{N_n}{k_n} \xrightarrow{w} A \text{ as } n \rightarrow \infty.$$

Then

$$S_{N_n}^{(n)} \xrightarrow{w} \tilde{S} \sim \nu \text{ as } n \rightarrow \infty,$$

where

$$\hat{\nu}(t) = \int_0^\infty (\hat{\mu}(t))^s F_A(ds) \text{ for every } t \in B',$$

F_A being the distribution function of A .

(ii) The sequence of the laws of $S_{N_n}^{(n)}, n = 1, 2, \dots$ is relatively compact if and only if there is a sequence of positive integers $k_1, k_2, \dots$ such that the sequences of the laws of $S_{k_n}^{(n)}, n = 1, 2, \dots$ and $\frac{N_n}{k_n}, n = 1, 2, \dots$ are relatively compact in $\mathcal{P}(B)$ and $\mathcal{P}(R)$ accordingly.

Theorem 2.1 is very useful in the proof of the following result on the structure of geometric infinitely divisible laws.

Theorem 2.2 *(i) $\mu \in \mathcal{GID}(B)$ implies $\mu \in \mathcal{ID}(B)$.*

(ii) The class $\mathcal{GID}(B)$ is closed under weak convergence.

(iii) Let $\mu \in \mathcal{GID}(B)$. Then $\hat{\nu}(t) = \exp(1 - 1/\hat{\mu}(t)), t \in B'$, is the ch.f. of a law $\nu \in \mathcal{ID}(B)$. Conversely, if $\nu \in \mathcal{ID}(B)$, then $\hat{\mu}(t) = \frac{1}{1 - \log \hat{\nu}(t)}, t \in B'$, is the ch.f. of a law $\mu \in \mathcal{GID}(B)$.

(iv) A functional $\hat{\mu} : B' \rightarrow C$ is the ch.f. of a $\mu \in \mathcal{GID}(B)$ if and only if

$$\begin{aligned} \hat{\mu}(t) &= 1/[1 - \int_B (e^{i\langle x, t \rangle} - 1 - i\langle x, t \rangle 1_{\{\|x\| \leq 1\}}) \lambda(dx) \\ &\quad + \frac{1}{2} \langle Rt, t \rangle + i\langle x_0, t \rangle], t \in B', \end{aligned}$$

where R is the covariance operator of a Gaussian probability measure on B , λ is a Lévy measure, and $x_0 \in B$.

(v) $\mu \in \mathcal{GID}(B)$ if and only if there are laws $\nu_p \in \mathcal{P}(B)$, $0 < p < 1$, such that

$$\sum_{i=1}^{T(p)} Y_i^{(p)} \xrightarrow{w}, \text{ as } p \rightarrow 0,$$

where $Y_i \sim \mu$, $Y_i^{(p)}, i = 1, 2, \dots$ are i.i.d. with common law ν_p , independent of the geometric random variable $T(p)$.

Remark. In part (iii) of the theorem, $\log \hat{\nu}(t)$ stands for the uniquely determined continuous branch of the logarithm; see Prop. 1.9.1 of Linde (1983).

Proof of Theorem 2.2.

(i) Let $\mu \in \mathcal{GID}(B)$ and $Y \sim \mu$. By (2.2) for every $0 < p < 1$

$$Y \stackrel{d}{=} \sum_{i=1}^{T(p)} Y_i^{(p)} \stackrel{d}{=} \tilde{Y}_1^{(p)} + \sum_{i=1}^{T(p)-1} Y_i^{(p)},$$

where $\tilde{Y}_1^{(p)}, Y_1^{(p)}, Y_2^{(p)}, \dots$ are i.i.d. random variables with normal law ν_p independent of $T(p)$. Observe, further, that $T(p) - 1$ is a real valued i.d. random variable such that for every $n \geq 1$

$$T(p) - 1 \stackrel{d}{=} N_{1,n}^{(p)} + \dots + N_{n,n}^{(p)},$$

where $N_{1,n}^{(p)}, \dots, N_{n,n}^{(p)}$ are i.i.d. nonnegative integer valued random variables. Let now $n \geq 1$ be fixed. We have

$$\begin{aligned} Y &\stackrel{d}{=} \tilde{Y}_1^{(p)} + \sum_{j=1}^n \sum_{i=1}^{N_{j,n}^{(p)}} Y_{i,j}^{(p)} \\ (2.5) \quad &:= \tilde{Y}_1^{(p)} + \sum_{j=1}^n Z_j^{(p)} := \tilde{Y}_1^{(p)} + S^{(p)} \end{aligned}$$

where $\tilde{Y}_1^{(p)}, Y_{i,j}^{(p)}, i \geq 1, j \geq 1$ are i.i.d. random variables with common law ν_p , independent of $N_{1,n}^{(p)}, \dots, N_{n,n}^{(p)}$. It follows from Theorem 2.1 that the family $\{\nu_p, 0 < p < 1\}$ is relatively compact, and it follows from (2.1) that $\hat{\nu}_p(t) \xrightarrow{p \rightarrow 0} 1$ for every $t \in B'$.

Therefore, $\tilde{Y}_1^{(p)} \xrightarrow{w} 0$ as $p \rightarrow 0$. It follows then by Prop. 2.2.2 of Linde (1983) that the family of laws of $S^{(p)}$, $0 < p < 1$ is relatively compact as well.

Another application of Theorem 2.1 shows now that the family of the laws of $Z_1^{(p)}$, $0 < p < 1$ is relatively compact. Choose now a sequence $p_k \downarrow 0$ such that $Z_1^{(p_k)} \xrightarrow[k \rightarrow \infty]{w} Z_1$, a B -valued Radon random variable. It follows that from (2.5) that $Y \stackrel{d}{=} Z_1 + \dots + Z_n$, where $Z_1, \dots, Z_n$ are i.i.d., and since this is true for every $n \geq 1$, the law of Y is infinitely divisible.

(ii) Let $\mu_k \in \mathcal{GID}(B)$, $k = 1, 2, \dots$, $X_k \sim \mu_k$, $k = 1, 2, \dots$, and $X_k \xrightarrow[k \rightarrow \infty]{w} X \sim \mu$. Fix a $p \in (0, 1)$. By (2.1), the functional

$$(2.6) \quad \hat{\nu}_k^{(p)} = \frac{\hat{\mu}_k}{p + q\hat{\mu}_k}$$

is, for any $k = 1, 2, \dots$, the ch.f. of a probability law $\nu_k^{(p)}$. Let $Y_i^{(k,p)}$, $i = 1, 2, \dots$ be a sequence of i.i.d. random variables with common law $\nu_k^{(p)}$. Since $\hat{\mu}_k \xrightarrow[k \rightarrow \infty]{w} \mu$, it follows that

$$(2.7) \quad \hat{\nu}_k^{(p)} \xrightarrow[k \rightarrow \infty]{} \hat{\nu}^{(p)} = \frac{\hat{\mu}}{p + q\hat{\mu}}.$$

It follows then from (2.6) that $\hat{\mu} = \frac{p\hat{\nu}^{(p)}}{1 - q\hat{\nu}^{(p)}}$, and so $\mu \in \mathcal{GID}(B)$ if $\hat{\nu}^{(p)}$ is the ch.f. of a probability law. Because of (2.7), the latter will follow once it is established that the sequence $\nu_k^{(p)}$, $k = 1, 2, \dots$ is relatively compact.

To this end note that the sequence μ_k , $k = 1, 2, \dots$ is tight. For a given $\epsilon > 0$ choose a compact set $K \subset B$ such that $P(X_k \notin K) \leq \epsilon$, $k = 1, 2, \dots$. Since (2.1) and (2.2) are equivalent, we conclude that for any $k = 1, 2, \dots$

$$\begin{aligned} pP(Y_1^{(k,p)} \notin K) &\leq \sum_{n=1}^{\infty} pq^{n-1} P\left(\sum_{i=1}^n Y_i^{(k,p)} \notin K\right) \\ &= P(X_k \notin K) \leq \epsilon, \end{aligned}$$

and so the sequence $\nu_k^{(p)}$, $k = 1, 2, \dots$ is, indeed, tight.

(iii) If $\mu \in \mathcal{GID}(B)$, then $\hat{\nu}_n = \frac{\hat{\mu}}{1/n + (1-1/n)\hat{\mu}}$ is the ch.f. of a probability law ν_n for every $n = 1, 2, \dots$. Let $X_i^{(n)}$, $i = 1, 2, \dots$ be a sequence of i.i.d. random variables with common law ν_n . Since, obviously,

$$\exp\{1 - 1/\hat{\mu}\} = \lim_{n \rightarrow \infty} \exp\{n(\hat{\nu}^{(n)} - 1)\},$$

Theorem 4.9 of Araujo and Giné (1980) applies. That is, to show that $\exp\{1 - 1/\hat{\mu}\}$ is the ch.f. of an ID law, it is enough to check that the sequence of the laws of

$$W_n = \sum_{i=1}^n X_i^{(n)}, n = 1, 2, \dots$$

is tight. However, for any $n = 1, 2, \dots$

$$Z \stackrel{d}{=} \sum_{i=1}^{T(1/n)} X_i^{(n)},$$

where $Z \sim \mu$. Thus, Theorem 2.1 applies, proving tightness of the laws of $W_n, n = 1, 2, \dots$.

Conversely, let $\nu \in \mathcal{ID}(B)$. Denote

$$\hat{\nu}_n = \frac{1}{1 - n(\hat{\nu}^{1/n} - 1)}, \quad n = 1, 2, \dots$$

Observe that for every $p \in (0, 1]$

$$\frac{\hat{\nu}_n}{p + q\hat{\nu}_n} = \frac{1}{1 + np(1 - \hat{\nu}^{1/n})} = \frac{1}{1 + np} \sum_{k=0}^{\infty} \left(\frac{np}{1 + np} \right)^k \hat{\nu}^{k/n}$$

and so, first, $\hat{\nu}_n$ is the ch.f. of a probability law ν_n (take $p = 1$) and, second, $\nu_n \in \mathcal{GID}(B)$. Since, obviously, $\hat{\nu}_n \xrightarrow{n \rightarrow \infty} 1/(1 - \log \hat{\nu})$, and the class $\mathcal{GID}(B)$ is closed under weak convergence, we may conclude that $1/(1 - \log \hat{\nu})$ is the ch.f. of a GID law provided we prove relative compactness of the family $\nu_n, n = 1, 2, \dots$. To this end, let $Y_i^{(n)}, i = 1, 2, \dots$ be a sequence of i.i.d. random variables with common law $\nu^{1/n}$, $V_n \sim \nu_n$ and $Y \sim \nu$. Then, for every $n = 1, 2, \dots$

$$V_n \stackrel{d}{=} \sum_{i=1}^{T(1/(n+1))-1} Y_i^{(n)}, Y \stackrel{d}{=} \sum_{i=1}^n Y_i^{(n)}.$$

Now Theorem 2.1 applies.

(iv) This is immediate corollary of (iii) and the Lévy-Khinchin formula for ID laws (cf. Weron (1984)).

(v) If $\mu \in \mathcal{GID}(B)$ then one can construct the laws ν_p by (2.1). Conversely, suppose that there are laws $\nu_p, 0 < p < 1$ such that

$$\sum_{i=1}^{T(p)} Y_i^{(p)} \xrightarrow{w} Y \text{ as } p \rightarrow 0,$$

where $Y \sim \mu$ and $Y_i^{(p)}, i = 1, 2, \dots$ are i.i.d. random variables with common law ν_p . Fix $0 < p < 1$. For a $0 < p' < p$, let $T_j(p'/p), j = 1, 2, \dots$ be i.i.d. $Ge(p'/p)$ random variables, independent of everything else around. Note that

$$T(p') \stackrel{d}{=} 1 + \sum_{j=1}^{T(p)} (T_j(p'/p) - 1),$$

and so

$$(2.8) \quad \sum_{i=1}^{T(p')} Y_i^{(p')} \stackrel{d}{=} \tilde{Y}_1^{(p')} + \sum_{j=1}^{T(p)} \sum_{i=1}^{T_j(p')-1} Y_{i,j}^{(p')},$$

where $\tilde{Y}_{i,j}^{(p')}$, $Y_{i,j}^{(p')}$, $j = 1, 2, \dots$ are i.i.d. random variables with common law $\nu_{p'}$, independent of $T(p)$ and $T_j(p'/p)$, $j = 1, 2, \dots$. Let μ_p and μ'_p be the laws of $\sum_{i=1}^{T(p)} Y_i^{(p)}$ and $\sum_{i=1}^{T(p)-1} Y_i^{(p)}$ accordingly. Observe that the family μ'_p , $0 < p < 1$ is tight because the family μ_p , $0 < p < 1$ is, and so Proposition 2.2.2 of Linde (1983) shows that the family ν_p , $0 < p < 1$ is tight, and then, obviously, by computing the ch.f.'s we obtain $\mu_p \xrightarrow{w} \delta_0$ as $p \rightarrow 0$. It follows now from (2.8) that

$$(2.9) \quad \sum_{j=1}^{T(p)} \sum_{i=1}^{T_j(p')-1} Y_{i,j}^{(p')} \xrightarrow{w} Y \text{ as } p' \rightarrow 0.$$

Denote by $\sigma_{p',p}$ the law of $\sum_{i=1}^{T_j(p')-1} Y_{i,j}^{(p')}$. The argument of the proof of part (ii) shows that the family $\sigma_{p',p}$, $0 < p' < p$, is tight. That is, there is a law ν'_p and a sequence $p'_n \downarrow 0$ such that $\sigma_{p'_n,p} \xrightarrow{w} \nu'_p$ as $n \rightarrow \infty$. Obviously, ν'_p satisfies (2.1), and so ν is, indeed, in $\mathcal{GITD}(B)$. $\square$

We move next to discuss an important subclass of GID laws on B : that of geometric stable laws.

Recall that a law μ on B is said to be geometrically stable (GS) if for any $p \in (0, 1)$ there exist $a = a(p) > 0$ such that

$$(2.10) \quad \hat{\mu}(t) = \frac{p\hat{\mu}(at)}{1 - q\hat{\mu}(at)}, \text{ for all } t \in B'.$$

In other words,

$$(2.11) \quad a(p) \sum_{i=1}^{T(p)} Y_i \stackrel{d}{=} Y_1 \text{ for all } 0 < p < 1,$$

where Y_i 's are i.i.d. r.v.'s with distribution μ and $T(p)$ is an independent of them geometric r.v. We denote the class of all GS laws by $\mathcal{GS}(B)$. Clearly $\mathcal{GS}(B) \subset \mathcal{GITD}(B)$.

The next theorem relates geometrically stable laws on B to stable laws on B in the same way as Theorem 2.2 relates geometrically infinitely divisible laws to infinitely divisible laws.

Theorem 2.3 (i) *The class $\mathcal{GS}(B)$ is closed under weak convergence.*

- (ii) Let $\mu \in \mathcal{GS}(B)$. Then $\hat{\nu}(t) = \exp(1 - 1(\hat{\mu}(t)))$ is the ch.f. of a strictly stable law on B . Conversely, if ν is a strictly stable law on B , then $\hat{\mu}(t) = \frac{1}{1 - \log \hat{\nu}(t)}$ is the ch.f. of a $\mathcal{GS}$ law on B .
- (iii) A functional $\hat{\mu} : B' \rightarrow C$ is the ch.f. of a $\mu \in \mathcal{GS}(B)$ if and only if either

$$(2.12) \quad \hat{\mu}(t) = \frac{1}{1 + \frac{1}{2} \langle Rt, t \rangle}, t \in B'$$

where R is the covariance operator of a Gaussian symmetric measure on B , or

$$(2.13) \quad \hat{\mu}(t) = \frac{1}{1 + \int_{\partial U} |\langle x, t \rangle|^\alpha \sigma(dx) + Q_\alpha(\sigma, t)} \text{ for all } t \in B',$$

where $0 < \alpha < 2$ and

$$(2.14) \quad Q_\alpha(\sigma, t) = \begin{cases} -\tan(\pi\alpha/2) \int_{\partial U} |\langle x, t \rangle|^\alpha \text{sign}(\langle x, t \rangle) \sigma(dx), & \alpha \neq 1, \\ \frac{2}{\pi} \int_{\partial U} \langle x, t \rangle \ln |\langle x, t \rangle| \sigma(dx), & \alpha = 1, \end{cases}$$

where U is the unit ball of B and σ is the spectral measure of the strictly α -stable measure ν defined in (ii). Further $a(p) = p^{1/\alpha}$ in (2.10) and (2.11).

- (iv) $\mu \in \mathcal{GS}(B)$ if and only if there is a sequence X_i , $i = 1, 2, \dots$ of i.i.d. B -valued random variables, independent of geometric random variables $T(p)$, $0 < p < 1$, and positive numbers $a(p)$, $0 < p < 1$, such that, as $p \rightarrow 0$

$$(2.15) \quad a(p) \sum_{i=1}^{T(p)} X_i \xrightarrow{w} Y,$$

where $Y \sim \mu$. Moreover, if (2.15) holds, then, necessarily, $a(p) = p^{1/\alpha} L(p)$ for some $0 < \alpha \leq 2$, and L a slowly varying function.

Proof. (i) Let $\mu_n \in \mathcal{GS}(B)$, $n = 1, 2, \dots$, $X_n \sim \mu_n$, $n = 1, 2, \dots$, and $X_n \xrightarrow{w} X \sim \mu$ as $n \rightarrow \infty$. Fix $0 < p < 1$ and let $a_n(p)$, $p = 1, 2, \dots$ be the sequence of normalizing constants in (2.10) and (2.11). Since $\overline{\lim}_{n \rightarrow \infty} a_n(p) < \infty$, there is an $a(p) \geq 0$ such that $a_{n_k}(p) \xrightarrow{k \rightarrow \infty} a(p)$ along a subsequence n_k , $k = 1, 2, \dots$. Then, obviously, $a(p)$ satisfies (2.10).

(ii) Let $\mu \in \mathcal{GS}(B)$. By Theorem 2.2 (iii) $\hat{\nu}(t) = \exp(1 - \frac{1}{\hat{\mu}(t)})$ is the ch.f. of a probability law on B . Applying (2.10) with $p = 1/n$ we immediately conclude that for any $n = 1, 2, \dots$, $\hat{\nu}(t)^n = \hat{\nu}(ta(1/n)^{-1})$, and so ν is strictly stable. The proof of the other direction is similar.

(iii) (2.12)-(2.14) follow from part (ii) and the spectral representation of the ch.f.'s of strictly stable measures on B . Further, substituting (2.12)-(2.14) into (2.10) gives us $a(p) = p^{1/\alpha}$, $p > 0$.

(iv) The proof of the fact that $a(p)$, $0 < p < 1$ is regularly varying at 0 repeats the corresponding proof in the stable case. Write for $0 < \theta < 1$, $0 < p < 1$

$$(2.16) \quad a(p\theta) \sum_{i=1}^{T(p\theta)} X_i \stackrel{d}{=} a(p\theta)X_1 + \frac{a(p\theta)}{a(p)} \sum_{j=1}^{T(\theta)} a(p) \sum_{i=1}^{T_j(p)-1} X_{i,j}$$

where $T_j(p), j = 1, 2, \dots$ are i.i.d. $Ge(p)$ random variables, independent of a $Ge(\theta)$ random variable $T(\theta)$, and all of them are independent of i.i.d. B -valued random variables $X_1, X_{i,j}, i = 1, 2, \dots, j = 1, 2, \dots$. Let $p \rightarrow 0$, and use (2.16) in the obvious way to show that $\overline{\lim}_{p \rightarrow 0} \frac{a(p\theta)}{a(p)} < \infty$, that $\underline{\lim}_{p \rightarrow 0} \frac{a(p\theta)}{a(p)} > 0$, unless the limit in (2.15) is degenerate, and, finally, that $\overline{\lim}_{p \rightarrow 0} \frac{a(p\theta)}{a(p)} = \underline{\lim}_{p \rightarrow 0} \frac{a(p\theta)}{a(p)}$. That is, $a(p) = p^d L(p)$, $d > 0$ and L slowly varying.

Further, if (2.15) holds then by Theorem 2.2 (v), $\mu \in \mathcal{GID}(B)$. By Theorem 2.2 (iii) $\hat{\nu}(t) = \exp(1 - 1/\hat{\mu}(t))$ is the ch.f. of a probability law. If we show that ν is a strictly stable law, then part (ii) of the present theorem will imply that μ is a $\mathcal{GS}$ law. To this end note that by (2.15), $\hat{\nu}(t) = \lim_{p \rightarrow 0} \hat{\mu}_0(ta(p))^{1/p}$, where μ_0 is the law of X_1 . Taking $p = 1/n$, we conclude that

$$\hat{\nu}(t) = \lim_{n \rightarrow \infty} (ta(\frac{1}{n}))^n,$$

implying both that ν is strictly stable and that $a(p) = p^{1/\alpha} L(p)$, $0 < \alpha \leq 2$. The second part of our statement is trivial. $\square$

Corollary 2.1 *A probability law μ on B is GS if and only if for every $t \in B'$ the probability law μ_t on R with characteristic function $\hat{\mu}_t(\cdot) = \hat{\mu}(\cdot t)$ is GS.*

Proof. Indeed, (2.15) immediately shows that $\mu \in \mathcal{GS}(B)$ implies that $\mu_t \in \mathcal{GS}(R)$ for every $t \in B'$. On the other hand, if $\mu_t \in \mathcal{GS}(R)$ for every $t \in B'$ then (2.15) will also show that $\mu \in \mathcal{GS}(B)$ if we show that for any $t, s \in B'$, $a_t(p) = a_s(p)$ in (2.10) for every $0 < p < 1$. By Theorem 2.3 (iii), $a_t(p) = p^{1/\alpha_t}$, $a_s(p) = p^{1/\alpha_s}$, some $0 < \alpha_t \leq 2, 0 < \alpha_s \leq 2$, and so we need to show that $\alpha_s = \alpha_t$. To this end, we may assume that μ_s and μ_t are non-degenerate, i.e. not concentrated at 0.

Suppose, to the contrary, that $\alpha_s < \alpha_t$ (say). Let $t_\rho = t + \rho s \in B'$, $X \sim \mu$. Then for any $\rho \neq 0$ and $\delta > 0$

$$E|<t_\rho, X>|^\delta = E|<t, X> + \rho<s, x>|^\delta = \infty$$

for any $\delta \geq \alpha_s$. (We have used Theorem 5 of Ramachandran (1969) to see that $\alpha_t < 2$ implies $E|<t, X>|^\delta < \infty$ iff $\delta < \alpha_t$, while $\alpha_t = 2$ implies $E|<t, X>|^\delta < \infty$ for every $\delta > 0$.) Therefore, $\alpha_{t_\rho} \leq \alpha_s, \rho \neq 0$. Further, since by assumption, μ_s and μ_t are non-degenerate, there is a sequence $\rho_k \xrightarrow{k \rightarrow \infty} 0$, $\rho_k \neq 0$, such that $\mu_{t\rho_k}$ is nondegenerate for any $k = 1, 2, \dots$. We have then

$$(2.17) \quad \alpha_{t_{\rho_k}} \leq \alpha_s < \alpha_t \text{ for any } k = 1, 2, \dots$$

Obviously, $\langle t_{\rho_k}, X \rangle \xrightarrow{w} \langle t, X \rangle$ as $k \rightarrow \infty$. By Theorem 2.3 (iii) with $B = R$ we conclude that $\alpha_{t_{\rho_k}} \xrightarrow{k \rightarrow \infty} \alpha_t$, contradicting (2.17). Therefore, the parameter α_t is the same for all $t \in B'$ such that μ_t is non-generate, and the proof is complete. $\square$

3 Rate of convergence in the domain of attraction of GS laws

In this section we discuss the rate of convergence of geometric sums (2.15) in the domain of attraction of a $\mu \in \mathcal{GS}(B)$ to their limiting distribution. We restrict ourselves to the “normal” domain of attraction. That is, the situation under consideration is

$$(3.1) \quad Z_p = p^{1/\alpha} \sum_{i=1}^{T(p)} X_i \xrightarrow{w} Y, \quad 0 < \alpha \leq 2,$$

$X_1, X_2, \dots, Y$ are B -valued random variables, Y is GS.

The rate-of-convergence problem clearly depends on the metric we use to compare the underlying distributions. While the uniform metric ρ

$$(3.2) \quad \rho(Z_p, Y) = \sup\{|P(Z_p < x) - P(Y < x)| : x \in \mathbf{R}^d, \}$$

is the natural metric to use for distributions on $\mathbf{R}^d$, $1 \leq d < \infty$, it does not make sense in the Banach space setting. In the latter case one usually explores the Lévy-Prokhorov metric

$$(3.3) \quad \pi(X, Y) = \inf\{\epsilon > 0 : P(X \in C) \leq P(Y \in C^\epsilon) + \epsilon \text{ for any closed } C \subset B\},$$

and the Kantorovich metric

$$(3.4) \quad \kappa(X, Y) = \sup_{f \in Lip_b(B)} |Ef(X) - Ef(Y)|,$$

where $Lip_b(B)$ is the set of all bounded real valued functions satisfying the Lipschitz condition

$$|f(x) - f(y)| \leq \|x - y\| \text{ for all } x, y \in B.$$

Recall that π metrizes the topology of weak convergence, $x \geq \pi^2$, and if $E\|X_n\| + E\|X\| < \infty$, then $\kappa(X_n, X) \rightarrow 0$ if and only if $X_n \xrightarrow{w} X$ and $E\|X_n\| \rightarrow E\|X\|$, see Dudley (1989), Rachev (1991).

In the one-dimensional case estimates for κ -rate of convergence of the geometric sum $p \sum_{i=1}^{T(p)} X_i$ with $X_i \geq 0$ and $EX_i < \infty$ to the exponentially distributed Y with $EY = EX_1$ were given in Kalashnikov and Vsekhsvjatskii (1985). For example,

$$(3.5) \quad \kappa(p \sum_{i=1}^{T(p)} X_i, Y) \leq 2p(\kappa(X_1, Y) + \kappa_2(X_1, Y)), 0 < p < 1,$$

where κ_s stands for the s^{th} difference pseudomoment, which for B -valued r.v.'s is defined by

$$\kappa_s(X, Y) = \sup\{|Ef(X) - Ef(Y)| : f : B \rightarrow \mathbf{R} \text{ (bounded)}\},$$

$$|f(x) - f(y)| \leq \|x\|^{s-1} - \|y\|^{s-1}, \text{ for all } x, y \in B\}.$$

In the finite dimensional case $B = \mathbf{R}^d$ the following estimates for κ -rate of convergence in (3.1) are derived in Rachev and Sengupta (1992):

(i) If $0 < \alpha < 1$, then

$$(3.6) \quad \kappa(Z_p, Y) \leq p^{-1+\alpha} \kappa(X_1, Y).$$

(ii) If $1 \leq \alpha \leq 2$ and $r \geq 2$ is an integer such that

$$(3.7) \quad E(X_1^{(1)})^{i_1} \dots (X_1^{(d)})^{i_d} = E(Y^{(1)})^{i_1} \dots (Y^{(d)})^{i_d}$$

for all powers $i_j \geq 0, j = 1, \dots, d, i_1 + \dots + i_d \leq r - 1$, and if Y has a density p_Y with r^{th} Fréchet derivative $p_Y^{(r)}$ existing almost everywhere, then

$$(3.8) \quad \kappa(Z_p, Y) \leq p^{1/\alpha} \kappa(X_1, Y) + qp^{\frac{r}{\alpha}-1} \frac{K_r}{(r-1)!} \kappa_r(X_1, Y)$$

where

$$K_r := \int_{\mathbf{R}^d} |p_Y^{(r)}(x)| dx.$$

For $d = 1, \alpha = 1, r = 2$, (3.8) is slightly better than (3.5).

The bound in (3.8) utilizes the smoothing effect of convolving a random vector with the GS random vector Y having a “nice” density and, therefore, cannot be extended to an infinite-dimensional B .

Let now B be a separable Banach space, and $\mathcal{P}(B)$ the set of all Borel probability measures on B .

We begin with the study on the rate of convergence in (3.1) in terms of ideal metrics in the space $\mathcal{P}(B)$. Recall that a metric

$$\mu_r(X, Y) = \mu_r(P^X, P^Y)$$

in $\mathcal{P}(B)$ is said to be ideal of order $r \in \mathbf{R}$, if

$$\mu_r(cX + Z, cY + Z) \leq |c|^r \mu_r(X, Y)$$

for any Z independent of X and Y and $c \in \mathbf{R}$. We refer to Zolotarev (1983), Kalashnikov and Rachev (1990), (Chapter 3), and Rachev (1991) (Chapter 19) for more facts on ideal metrics.

Examples of ideal metrics of order $r \geq 0$:

(a) *Zolotarev ζ_r -metric*

$$(3.9) \quad \zeta_r(X, Y) = \sup_{f \in \mathcal{F}_r} |Ef(X) - Ef(Y)|,$$

where $\mathcal{F}_r$ is the set of all bounded continuous functions $f : B \rightarrow \mathbf{R}$ having m -th Fréchet derivative and

$$\|f^{(m)}(x) - f^{(m)}(y)\| \leq \|x - y\|^\beta,$$

$r = m + \beta, m = 0, 1, \dots, \beta \in (0, 1]$;

(b) *ideal metric in the space of ch.f.'s $f_X(t) = E \exp\{i \langle X, t \rangle\}, t \in B'$:*

$$(3.10) \quad \chi_r(X, Y) := \chi_r(f_X, f_Y) := \sup_{t \in B'} \|t\|^{-r} |f_X(t) - f_Y(t)|;$$

(c) *weighted ideal metrics:* Let μ_r be an ideal metric of order $r \geq 0$ and

$$(3.11) \quad \mu_r^{(h)}(X, Y) = |h|^k \mu_r(X + h\theta, Y + h\theta), h \in \mathbf{R},$$

where θ is independent of X and Y . In (3.11), taking the convolution

$$\mu_r(X + h\theta, Y + h\theta) = \mu_r(P^X * P^{h\theta}, P^Y * P^{h\theta})$$

corresponds to an appropriate “smoothing” of the laws X and Y , allowing to increase the ideality of μ_r . In fact, if $\|\cdot\|_{r'}$, ($1 \leq r' \leq \infty$) is the usual $L_{r'}$ -norm of real-valued functions on $\mathbf{R}$, taking into account (3.11), the metric

$$(3.12) \quad \mu_{r,k,r'}(X, Y) = \|\mu^{(\cdot)}(X, Y)\|_{r'}, 1 \leq r' \leq \infty,$$

is an ideal metric of order $r + k + 1/r'$. One typically chooses θ to satisfy certain smoothness requirements. In the sequel we will work with a particular choice of the metric (3.12), defined by choosing $\mu_0(X, Y) = \sup_{A \text{ Borel}} |P^X(A) - P^Y(A)|$ (the total variation distance) and θ is a non-degenerate α -stable random variable. The resulting ideal metric of order $k \geq 0, \mu_{0,k,\infty}$, has been studied by Rachev and Yukich (1989).

Note that ideal metrics may take infinite values. The finiteness condition on μ_r may require moment assumptions, see Zolotarev (1983), Rachev and Yukich (1989), Rachev and Rüschendorf (1991), Theorem 5.3. In particular, for

any of the metrics $\zeta_r, \chi_r, \mu_{r_1, k, r'}$, $r = r_1 + k + 1 \setminus r'$, the finiteness condition $\zeta_r(X, Y) < \infty$ (resp. $\chi_r(X, Y) < \infty$, or $\mu_{r_1, k, r'}(X, Y) < \infty$) for d -dimensional vectors X and Y implies equality of all mixed moments up to order $r - 1$. That is, (3.7) must hold.

Moreover, if Banach valued random variables X and Y have equal moments $Ef^{(k)}(0)(X) = Ef^{(k)}(0)(Y)$, $k = 0, 1, \dots, m$ for all $f \in \mathcal{F}_r$ then

$$(3.13) \quad \zeta_r(X, Y) \leq \frac{\Gamma(1 + \beta)}{\Gamma(1 + r)} \nu_r(X, Y),$$

where

$$(3.14) \quad \nu_r(X, Y) = \int \|x\|^r |P^X - P^Y|(dx)$$

is the r^{th} absolute pseudomoment (Zolotarev (1983)).

From now on C denotes an absolute constant which may be different in different places.

Let $Y, Y_1, Y_2, \dots$ be GS i.i.d. B -valued random variables, $X_1, X_2, \dots$ i.i.d. B -valued random variables and Z_p as in (3.1).

Theorem 3.1 *Suppose μ_r is one of the three ideal metrics of order $r > 0$ described by (3.9), (3.10) or $\mu_{0, k, \infty}$ described by (3.12). Then for any $p \in (0, 1)$*

$$(3.15) \quad \mu_r(Z_p, Y) \leq p^{-1+r/\alpha} \mu_r(X, Y).$$

Proof. Let $a = p^{1/\alpha}$. For any ideal metric of order $r > 0$,

$$(3.16) \quad \mu_r(Z_p, Y) = \mu_r(a \sum_{i=1}^{T(p)} X_i, a \sum_{i=1}^{T(p)} Y_i) \leq a^r r_\tau \left(\sum_{i=1}^{T(p)} X_i, \sum_{i=1}^{T(p)} Y_i \right).$$

Observe that if μ_r is one of the three metrics in the theorem then

$$(3.17) \quad \mu_r \left(\sum_{i=1}^{T(p)} X_i, \sum_{i=1}^{T(p)} Y_i \right) \leq \sum_{k=1}^{\infty} P(T(p) = k) \mu_r \left(\sum_{i=1}^k X_i, \sum_{i=1}^k Y_i \right).$$

Finally, $\mu_r(\sum_{i=1}^k X_i, \sum_{i=1}^k Y_i) \leq k \mu_r(X_1, Y_1)$, by invoking again the “ideality” of μ_r . Combining the above estimates we get our result. $\square$

In certain Banach spaces the result of Theorem 3.1 can be used to obtain the rate of convergence in the Lévy-Prokhorov metric π . Write as before $r = m + \beta$, $m = 0, 1, 2, \dots$ and $\beta \in (0, 1]$, and suppose that the Banach space B has the following property: the norm $\|\cdot\|$, considered as a function $\eta(x) = \|x\|$ is m times Fréchet differentiable at every $x \neq 0$, and for every $k = 1, 2, \dots, m$ there is an $A_k < \infty$ such that

$$(3.18) \quad \|\eta^{(k)}(x)\| \leq A_k \|x\|^{k-1}.$$

It is well known that a Hilbert space has this property for every $r > 0$. Since we are mostly interested in the case $0 < r \leq 2$, it is worth noting that every Banach space has this property for $0 < r \leq 1$, whereas in the case $1 < r \leq 2$ all Orlicz spaces $L_\Phi(\Omega, \mathcal{F}, \mu)$ satisfy (3.18) if

- (a) Φ is continuously differentiable on $(0, \infty)$ and $\Phi'(x) > 0$ for $x > 0$.
- (b) The complementary function

$$\Psi(x) = \sup_{y \in R} (|x|y - \Phi(y))$$

satisfies the Δ_2 condition.

See Rao and Ren (1991), Theorem 7.2.2. In particular, all $L_p(\Omega, \mathcal{F}, \mu)$ spaces with $p > 1$ satisfy (3.18) for $1 < r \leq 2$.

It has been proved by Zolotarev (1976) that for any Banach space B satisfying (3.18) for some $r > 0$, there is a $C < \infty$ such that

$$\pi(\|X\|, \|Y\|)^{1+r} \leq C \zeta_r(X, Y).$$

The following is then an immediate consequence of Theorem 3.1.

Corollary 3.1 *For any Banach space B satisfying (3.18) there is a constant $C < \infty$ such that for any $0 < p < 1$,*

$$(3.19) \quad \pi(\|Z_p\|, \|Y\|) \leq Cp^{\frac{-1+r/\alpha}{1+r}} (\zeta_r(X_1, Y))^{\frac{1}{1+r}}.$$

Remark 1. One cannot expect that the rate of convergence in (3.19) can be improved in general. Indeed, if $\alpha = 2$, that is, Y is a “Laplace distributed” B -valued random variable,

$$Y \stackrel{d}{=} \sqrt{p} \sum_{i=1}^{T(p)} Y_i,$$

and $r = 3$, the bound (3.19) implies $\pi(\|Z_p\|, \|Y\|) \leq O(p^{1/8})$. This result is in accordance with Senatov’s (1981) result showing that under $\zeta_r(X_1, N) < \infty$ where N is a B -valued Gaussian random variable, (i.e.

$$N \stackrel{d}{=} \frac{1}{\sqrt{n}}(N_1 + \dots + N_n), N_i \stackrel{d}{=} N \text{ i.i.d. })$$

the “correct” rate is $\pi(\frac{1}{\sqrt{n}} \|\sum_{i=1}^n X_i\|, \|N\|) = O(n^{-1/8})$ for a sequence $X_1, X_2, \dots$ of i.i.d. B -valued r.v.’s.

To get the rate-of-convergence for $\pi(Z_p, Y) \rightarrow 0$ rather than the rate of $\pi(\|Z_p\|, \|Y\|) \rightarrow 0$, we shall explore metrics which are ideal of order $r = 1$ but act as ideal of order $r > 1$.

For any $r > 0$ define the minimal metric $\hat{\mathcal{L}}_r$ with respect to $\mathcal{L}_r(X, Y) = (E\|X - Y\|^r)^{r'}$, $r' = 1 \wedge 1/r$, by

$$(3.20) \quad \hat{\mathcal{L}}_r(X, Y) = \hat{\mathcal{L}}_r(P^X, P^Y) = \inf\{\mathcal{L}_r(\tilde{X}, \tilde{Y}) : \tilde{X} \stackrel{d}{=} X, \tilde{Y} \stackrel{d}{=} Y\}.$$

For the properties of $\hat{\mathcal{L}}_r$ we refer to de Haan and Rachev (1989), Rachev (1991); here we only note that for $B = \mathbf{R}$, $r \geq 1$,

$$\hat{\mathcal{L}}_r(X, Y) = \|F_X^{inv} - F_Y^{inv}\|_r,$$

where F_X^{inv} is the generalized inverse of the d.f. F_X , cf. Cambanis, Simons and Stout (1976).

Theorem 3.2 (i) *If $0 < \alpha < 1$ then for any $r > \alpha$*

$$(3.21) \quad \hat{\mathcal{L}}_r(Z_p, Y) \leq p^{\bar{r}}(1 - q^{r'})^{-2} \hat{\mathcal{L}}_r(X_1, Y) \leq C p^{\bar{r}-2} \hat{\mathcal{L}}_r(X_1, Y),$$

where $q = 1 - p$, $\bar{r} = (1 \wedge r^{-1})(1 + r/\alpha)$ and where the first inequality holds for every $0 < p < 1$, while the second inequality holds for any $0 < p \leq 1/2$ (say). Moreover,

$$(3.22) \quad \begin{aligned} \pi(Z_p, Y) &\leq [p^{\bar{r}}(1 - q^{r'})^{-2} \hat{\mathcal{L}}_r(X_1, Y)]^{\frac{1}{1+\bar{r}}} \\ &\leq C [p^{\bar{r}-2} \hat{\mathcal{L}}_r(X_1, Y)]^{\frac{1}{1+\bar{r}}} \end{aligned}$$

with the second inequality holding when $0 < p \leq 1/2$.

(ii) *Suppose B is of Rademarker type $r \in [1, 2]$. If $1 \leq \alpha < r$ then for any $0 < p < 1$*

$$(3.23) \quad \hat{\mathcal{L}}_r(Z_p, Y) \leq c p^{1/\alpha - 1/r} \hat{\mathcal{L}}_r(X_1, Y)$$

and

$$(3.24) \quad \pi(Z_p, Y) \leq C [p^{1/\alpha - 1/r} \hat{\mathcal{L}}_r(X_1, Y)]^{\frac{1}{1+\bar{r}}}.$$

Proof. (i) $\hat{\mathcal{L}}_r$ is an ideal metric of order $r_1 = \min(1, r)$ and from the definition (3.20) we obtain readily:

$$\begin{aligned} \hat{\mathcal{L}}_r(Z_p, Y) &\leq p^{r_1/\alpha} \inf \left\{ \mathcal{L}_r(\tilde{X}, \tilde{Y}) : \tilde{X} \stackrel{d}{=} \sum_{i=1}^{T(p)} X_i, \tilde{Y} \stackrel{d}{=} \sum_{i=1}^{T(p)} Y_i \right\} \\ &\quad (\tilde{Y}_i \text{'s are i.i.d. copies of } Y) \end{aligned}$$

$$\begin{aligned}
&\leq p^{r_1/\alpha} \inf \left\{ \mathcal{L}_r \left(\sum_{i=1}^{T(p)} \tilde{X}_i, \sum_{i=1}^{T(p)} \tilde{Y}_i \right) : \right. \\
&\quad \tilde{X}_i \text{'s are i.i.d. } \tilde{X}_i \stackrel{d}{=} X_i, \\
&\quad \tilde{Y}_i \text{'s are i.i.d. } \tilde{Y}_i \stackrel{d}{=} Y \left. \right\} \\
&= p^{r_1/\alpha} \inf \left\{ \left(\sum_{k=1}^{\infty} p q^{k-1} E \left\| \sum_{i=1}^k (\tilde{X}_i - \tilde{Y}_i) \right\|^r \right)^{r'} : \right. \\
&\quad \tilde{X}_i \text{'s are i.i.d. } \tilde{X}_i \stackrel{d}{=} X_i, \\
&\quad \tilde{Y}_i \text{'s are i.i.d. } \tilde{Y}_i \stackrel{d}{=} Y \left. \right\} \\
&\leq p^{r_1/\alpha} \inf \left\{ \left(\sum_{k=1}^{\infty} (p q^{k-1}) \right)^{r'} \sum_{i=1}^k \mathcal{L}_r(\tilde{X}_i, \tilde{Y}_i) : \right. \\
&\quad (\tilde{X}_i, \tilde{Y}_i) \text{ are i.i.d. pairs, } \tilde{X}_i \stackrel{d}{=} X_1, \tilde{Y}_i \stackrel{d}{=} Y \left. \right\} \\
&= p^{\bar{r}} \inf \left\{ \sum_{k=1}^{\infty} q^{(k-1)r'} k \mathcal{L}_r(\tilde{X}_1, \tilde{Y}); \tilde{X}_1 \stackrel{d}{=} X_1, \tilde{Y}_1 = Y \right\} \\
&= p^{\bar{r}} \left(\sum_{k=1}^{\infty} q^{(k-1)r'} k \right) \hat{\mathcal{L}}_r(X_1, Y) \\
&= (1 - q^{r'})^{-2} p^{\bar{r}} \hat{\mathcal{L}}_r(X_1, Y),
\end{aligned}$$

which proves (3.21). The estimate (3.22) follows from (3.21) and the bound

$$\pi \leq \hat{\mathcal{L}}_r^{(1 \vee r)/(1+r)},$$

see for example Rachev (1991), Lemma 8.2.1.

(ii) As in the proof of (i) we obtain

$$\hat{\mathcal{L}}_r(Z_p, Y) \leq p^{1/\alpha} \inf \left\{ \left(\sum_{k=1}^{\infty} p q^{k-1} \mathcal{L}_r^r \left(\sum_{i=1}^k \tilde{X}_i, \sum_{i=1}^k \tilde{Y}_i \right) \right)^{1/r} : \right.$$

$$(3.25) \quad \left. (\tilde{X}_i, \tilde{Y}_i) \text{ are i.i.d. pairs , } \tilde{X}_i \stackrel{d}{=} X_1, \tilde{Y}_i \stackrel{d}{=} Y \right\}.$$

To bound $\mathcal{L}_r(\sum_{i=1}^k \tilde{X}_i, \sum_{i=1}^k \tilde{Y}_i)$ we use the following result of Woysczynski (1980): if B is of type r then for some constant B_r and any independent B -valued r.v.'s $Z_1, Z_2, \dots$ with mean 0,

$$(3.26) \quad E \left\| \sum_{i=1}^n Z_i \right\|^{r^*} \leq B_r^{r^*} E \left(\sum_{i=1}^n \|Z_i\|^r \right)^{r^*/r}, \quad r^* \geq 1.$$

Thus, by (3.26) with $r^* = r$,

$$(3.27) \quad \mathcal{L}_r \left(\sum_{i=1}^k \tilde{X}_i, \sum_{i=1}^k \tilde{Y}_i \right) \leq B_r k^{1/r} \mathcal{L}_r(\tilde{X}_1, \tilde{Y}_1).$$

Combining (3.25) with (3.27) we obtain (3.23), which implies the corresponding bound on π in the same way as in part (i). $\square$

The comparison between (3.19) and (3.24) shows that the rate of convergence in (3.19), $\pi(\|Z_p\|, \|Y\|) \leq O(p^{\frac{-1+r/\alpha}{1+r}})$, is the same as in (3.24), $\pi(Z_p, Y) \leq O(p^{\frac{-1+r/\alpha}{1+r}})$. However, while (3.24) implies $Z_p \xrightarrow{w} Y$ under the assumption $\hat{\mathcal{L}}_r(X_1, Y) < \infty$, the bound (3.19) implies only $\|Z_p\| \xrightarrow{w} \|Y\|$ provided $\zeta_r(X_1, Y) < \infty$. Moreover, $\hat{\mathcal{L}}_r$ is bounded by the r th difference pseudomoment. Namely,

$$\hat{\mathcal{L}}_r(X, Y)^r \leq k_r(X, Y) \leq 2^r \nu_r(X, Y),$$

(Zolotarev (1978)). On the other hand, it is not known whether $\zeta_r(X, Y)$ can be estimated from above by $k_r(X, Y)$, without the moment assumption $E\|X\|^r + E\|Y\|^r < \infty$. The answer of the above question is more than likely negative. Also while ζ_r -convergence implies the weak convergence in $\mathcal{P}(B)$ for B being a Hilbert space (Giné and Leon (1990)), there is no inequality of the type $\zeta_r \geq C\pi^a, a > 0$ (Senatov (1981), Bentkus and Rachkauskas (1985)). The problem whether ζ -convergence implies weak convergence in $\mathcal{P}(B)$ for a general separable Banach space B is still open and again, the answer is most likely negative. All that shows that (3.24) is a more informative bound than (3.19).

Of course, (3.24) assumes more about the geometry of B than (3.19) does.

Consider next the case $0 < \alpha < 1$. The comparison between (3.19) and (3.22) shows that the former bound gives a better rate of convergence than the latter. Nevertheless, (3.22) is valid in any Banach space B , and it provides the rate of convergence of Z_p to Y , while (3.19) deals only with $\|Z_p\| \xrightarrow{w} \|Y\|$.

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