SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853-3801

TECHNICAL REPORT NO. 1013

July 1992

**LIMIT THEOREMS FOR
MARKOV RANDOM WALKS**

by

Loon Ching Tang¹

¹Department of Industrial and Systems Engineering, National University of Singapore, Singapore 0511, Republic of Singapore. Research done while the author was on leave at Cornell University.

Limit Theorems for Markov Random Walks

Loon Ching Tang *

Department of Industrial and Systems Engineering

National University of Singapore, Singapore 0511, Republic of Singapore

Abstract

For Markov Random walks we obtain the central limit theorem and law of iterated logarithm for the additive component of Markov random walks . The parameters involved are computed explicitly in terms of the transition measure of the Markov random walk. An application in finance is briefly discussed.

Keywords: Markov random walks, central limit theorem, law of iterated logarithm.

*Work done while the author was on leave at Cornell University

1 Introduction

Suppose that we are given a probability space $(\Omega, \mathcal{F}, P)$. We denote $\mathbf{R} = (-\infty, \infty)$ and $\mathcal{E} =$ countable set. A time-homogeneous Markov random walk $(S, J) = \{(S_n, J_n), n \geq 0\}$ is a time-homogeneous Markov process on the state space $\mathbf{R} \times \mathcal{E}$ whose transition distribution measure is given by

$$P\{(S_{m+n}, J_{m+n}) \in A \times \{k\} | (S_m, J_m) = (x, j)\} = P\{(S_n, J_n) \in (A - x) \times \{k\} | J_0 = j\} \quad (1)$$

$\forall j, k \in \mathcal{E}$ and a Borel subset A of $\mathbf{R}$. We denote this probability as $Q_{jk}^{(n)}\{A - x\}$ and assume that it satisfies the regularity conditions. Also we shall denote the conditional probabilities and expectations given $J_0 = j$ as P_j and E_j respectively. Henceforth, we assume that the stationary distribution, $\{\pi_k, k \in \mathcal{E}\}$ of J exists.

Keilson and Wishart [3] obtained a central limit theorem for the MRW using matrix analysis where the result holds only when the modulating Markov chain J is on finite state space and is ergodic. They expressed the parameters involved as functions of eigenvalues derived from the transition measure of the MRW, which are difficult to compute algebraically.

Recently, Prabhu, Tang and Zhu [4] presented some new results for the MRW. Among them is the strong law of large numbers for the additive component S_n of the MRW.

In this paper we establish the central limit theorem and law of iterated logarithm for the additive component of MRW. Our proofs are probabilistic using

i.i.d. sequences of random variable obtained from the MRW. This idea is analogous to that of additive functionals of Markov chains discussed in Chung [1]. We express the parameters involved in terms of the transition measure of the MRW. Finally, we indicate an application of these theorems to finance models.

2 The Sequence of Random Variables $\{Y_r^j\}$

To obtain the central limit theorem for the MRW we first construct the following i.i.d. sequence of random variables. For each $j \in \mathcal{E}$, let

$$Y_r^j = S_{r+1}^j - S_r^j - \mu(\tau_{r+1}^j - \tau_r^j), \quad r \geq 1 \quad (2)$$

where

$$\tau_r^j = \min\{n > \tau_{r-1}^j : (S_n, J_n) \in \{\mathbf{R}\} \times \{j\}\} \quad (r \geq 1), \quad \tau_0^j = 0$$

are the successive hitting times of state $j \in \mathcal{E}$, $S_r^j = S_{\tau_r^j}$ ($r \geq 0$) and $\mu = \pi_j E_j(S_1^j)$.

We also denote the number of such hits (visits) as

$$N_n^j = \max\{r : \tau_r^j \leq n\}. \quad (3)$$

In Prabhu, Tang and Zhu [4] it is shown that if $\sup_{j \in \mathcal{E}} E_j|X_1| < \infty$ and the stationary probability measure $\{\pi_k, k \in \mathcal{E}\}$ of the marginal chain J exists, then

$$\mu = \pi_j E_j(S_1^j) = \sum_{k \in \mathcal{E}} \pi_k E_k(X_1). \quad (4)$$

For the sequence $\{Y_r^j, r \geq 1\}$ defined above we have the following.

Lemma 1 *The sequence of $\{Y_r^j, r \geq 1\}$ are i.i.d. with mean zero and variance given by*

$$\begin{aligned} \text{Var}(Y_1^j) &= \sum_{k \in \mathcal{E}} {}^jP_{jk}^* E_k(X_1^2) + 2 \sum_{i, k \neq j} {}^jP_{ji}^* E_i(X_1; J_1 = k) \sum_{l \in \mathcal{E}} {}^jP_{kl}^* E_l(X_1) \\ &\quad + \mu^2 \left[\sum_{k \in \mathcal{E}} {}^jP_{jk}^* + 2 \sum_{k \neq j} {}^jP_{jk}^* \sum_{l \in \mathcal{E}} {}^jP_{kl}^* \right] \\ &\quad - 2\mu \sum_{k \in \mathcal{E}} E_k(X_1) [{}^jP_{jk}^* + {}^j\tilde{P}'_{jk}(1)] \end{aligned} \quad (5)$$

where

$${}^j\tilde{P}'_{jk}(1) = \lim_{s \rightarrow 1} \frac{d}{ds} [{}^j\tilde{P}_{jk}(s)], \quad {}^j\tilde{P}_{jk}(s) = \sum_{n=0}^{\infty} {}^jP_{jk}^{(n)} s^n$$

and

$${}^jP_{jk}^* = \sum_{n=0}^{\infty} {}^jP_{jk}^{(n)} = \sum_{n=0}^{\infty} P_l(J_n = k; \tau_1^j > n)$$

provided that all the sums converge.

Proof: (i) We note that the sequence $\{\tau_r^j, r \geq 1\}$ is a renewal process for each j and $\{S_r^j\}$ is a random walk on the state space $\mathbf{R}$, so that $\{Y_r^j\}$ are i.i.d sequence of random variables. Taking expectation on the right side of (2) we have

$$E(S_{r+1}^j - S_r^j) - \pi_j E_j(S_1^j) E(\tau_{r+1}^j - \tau_r^j) = E_j(S_1^j) - \pi_j E_j(S_1^j) \cdot \frac{1}{\pi_j} = 0.$$

(ii) We have

$$\text{Var}(Y_1^j) = E_j[(Y_1^j)^2] = E_j[(S_1^j)^2 + \mu^2(\tau_1^j)^2 - 2\mu S_1^j \tau_1^j]. \quad (6)$$

$$\begin{aligned} E_j[(S_1^j)^2] &= E_j \left[\sum_{m=1}^{\infty} (X_m) \mathbf{1}_{\{\tau_1^j \geq m\}} \right]^2 \\ &= E_j \left[\sum_{m=1}^{\infty} X_m^2 \mathbf{1}_{\{\tau_1^j \geq m\}} + 2 \sum_{m=1}^{\infty} X_m \sum_{r=1}^{\infty} X_{m+r} \mathbf{1}_{\{\tau_1^j \geq m+r\}} \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} E_j(X_m^2 \mathbf{1}_{\{\tau_1^j \geq m\}}) + 2E_j\left(\sum_{m=1}^{\infty} \sum_{i,k \in \mathcal{E}} X_m \mathbf{1}_{\{J_{m-1}=i, J_m=k\}} \sum_{r=1}^{\infty} X_{m+r} \mathbf{1}_{\{\tau_1^j \geq m+r\}}\right) \\
&= \sum_{m=1}^{\infty} E_j(X_m^2 \mathbf{1}_{\{\tau_1^j \geq m\}}) + 2 \sum_{m=1}^{\infty} \sum_{i,k \neq j} {}^jP_{ji}^{(m-1)} E_i(X_1; J_1 = k) \sum_{r=1}^{\infty} E_k(X_r \mathbf{1}_{\{\tau_1^j \geq r\}}) \quad (*) \\
&= \sum_{k \in \mathcal{E}} {}^jP_{jk}^* E_k(X_1^2) + 2 \sum_{i,k \neq j} \sum_{m=1}^{\infty} {}^jP_{ji}^{(m-1)} E_i(X_1; J_1 = k) \sum_{l \in \mathcal{E}} \sum_{r=1}^{\infty} {}^jP_{kl}^{(r-1)} E_l(X_1)
\end{aligned}$$

we then have

$$E_j[(S_1^j)^2] = \sum_{k \in \mathcal{E}} {}^jP_{jk}^* E_k(X_1^2) + 2 \sum_{i,k \neq j} {}^jP_{ji}^* E_i(X_1; J_1 = k) \sum_{l \in \mathcal{E}} {}^jP_{kl}^* E_l(X_1). \quad (7)$$

Substitute 1 for $E_k(X_1)$ and $E_k(X_1^2)$ and proceed from (*) we have

$$E_j(\tau_1^j)^2 = \sum_{k \in \mathcal{E}} {}^jP_{jk}^* + 2 \sum_{k \neq j} {}^jP_{jk}^* \sum_{l \in \mathcal{E}} {}^jP_{kl}^*. \quad (8)$$

Finally, we have

$$\begin{aligned}
E_j(S_1^j \tau_1^j) &= E_j\left[\sum_{n=1}^{\infty} \left(n \sum_{m=1}^n X_m; \tau_j^1 = n\right)\right] = \sum_{n=1}^{\infty} n \sum_{m=1}^n E_j(X_m; \tau_j^1 = n) \\
&= \sum_{n=1}^{\infty} \sum_{r=1}^n \sum_{m=1}^n E_j(X_m; \tau_j^1 = n) = \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} \sum_{m=1}^n E_j(X_m; \tau_j^1 = n) \\
&= \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} \sum_{n=m}^{\infty} E_j(X_m; \tau_j^1 = n) = \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} E_j(X_m; \tau_j^1 \geq m) \\
&= \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} \sum_{k \in \mathcal{E}} E_j(X_m; \tau_j^1 > m-1; J_{m-1} = k) \\
&= \sum_{k \in \mathcal{E}} E_k(X_1) \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} P_j\{\tau_j^1 > m-1; J_{m-1} = k\} \\
&= \sum_{k \in \mathcal{E}} E_k(X_1) \sum_{m=1}^{\infty} \sum_{r=1}^m P_j\{\tau_j^1 > m-1; J_{m-1} = k\} \\
&= \sum_{k \in \mathcal{E}} E_k(X_1) \sum_{m=0}^{\infty} (m+1) P_j\{\tau_j^1 > m; J_m = k\} \\
&= \sum_{k \in \mathcal{E}} E_k(X_1) [{}^jP_{jk}^* + \sum_{m=0}^{\infty} m P_j\{\tau_j^1 > m; J_m = k\}].
\end{aligned}$$

Therefore

$$E_j(S_1^j \tau_1^j) = \sum_{k \in \mathcal{E}} E_k(X_1) [{}^jP_{jk}^* + {}^j\tilde{P}'_{jk}(1)]. \quad (9)$$

The result follows by substituting (7), (8) and (9) into (6). $\square$

Remark 1 *Using the fact that*

$${}^jP_{kl}^* = \pi_l(E_k(\tau_1^j) + E_j(\tau_1^l) - E_k(\tau_1^l)) \text{ and } \sum_{l \in \mathcal{E}} {}^jP_{kl}^* = E_k(\tau_1^j)$$

we may express (7) and (8) in terms of the stationary distribution of J as follows.

$$E_j[(S_1^j)^2] = \frac{1}{\pi_j} \left\{ \sum_{k \in \mathcal{E}} \pi_k E_k(X_1^2) + 2 \sum_{i, k \neq j} \pi_i E_i(X_1; J_1 = k) \cdot \left[\sum_{l \in \mathcal{E}} E_l(X_1) \pi_l (E_k(\tau_1^j) + E_j(\tau_1^l) - E_k(\tau_1^l)) \right] \right\}. \quad (10)$$

$$E_j(\tau_1^j)^2 = \frac{1}{\pi_j} [1 + 2 \sum_{k \neq j} \pi_k E_k(\tau_1^j)]. \quad (11)$$

3 Limit Theorems

Theorem 1 *(Central Limit Theorem for MRW). If the stationary distribution $\{\pi_j\}$ of J exists and $0 < \sigma < \infty$, then*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}$$

where $\sigma^2 = \pi_j \text{Var}(Y_1^j)$ and $\mathcal{N}$ denotes standard normal distribution.

Proof: We write

$$S_n - n\mu = \sum_{k=1}^{\tau_1^j} X_k + \sum_{r=1}^{N_n^j} Y_r^j + \sum_{k=\tau_{N_n^j}^j+1}^n X_k + \mu(\tau_{N_n^j+1}^j - \tau_1^j - n). \quad (12)$$

Dividing it by $\sigma\sqrt{n}$ and let $n \rightarrow \infty$ we have the 1st term tends to zero as it does not depends on n . For the 3rd term we note that

$$E\left[\sum_{k=\tau_{N_n^j}^j+1}^n X_k\right] \leq E\left[\sum_{k=\tau_{N_n^j}^j+1}^{\tau_{N_n^j}^j+2} |X_k|\right] < \infty.$$

since J is persistent and the first moments of $\{X_k\}$ are finite. It then follows that $\sum_{k=\tau_{N_n^j}^j+1}^n X_k$ is bounded in probability and thus $\sum_{k=\tau_{N_n^j}^j+1}^n X_k/n$ converges to 0. For the last term, we have

$$\frac{\tau_{N_n^j+1}^j - \tau_1^j}{n} = \frac{N_n^j}{n} \cdot \frac{\sum_{r=1}^{N_n^j} \tau_r^j}{N_n^j} \rightarrow \pi_j \cdot \frac{1}{\pi_j} = 1;$$

so it also converges to 0. Finally, we have the 2nd term converges in distribution to the standard normal distribution by the central limit theorem with random index; i.e.

$$\sqrt{\frac{N_n^j}{n\pi_j}} \frac{\sum_{r=1}^{N_n^j} Y_r^j}{\sqrt{(N_n^j) \text{Var}(Y_1^j)}} \xrightarrow{\mathcal{L}} \mathcal{N} \quad \text{as} \quad \frac{N_n^j}{n\pi_j} \rightarrow 1.$$

□

Theorem 2 (*Law of Iterated Logarithm for MRW*). *Under the conditions as Theorem 1, we have*

$$\limsup \frac{S_n - n\mu}{\sqrt{2\sigma^2 n \log \log n}} = 1 \quad \text{a.s.} \quad (13)$$

$$\text{and} \quad \liminf \frac{S_n - n\mu}{\sqrt{2\sigma^2 n \log \log n}} = -1 \quad \text{a.s.} \quad (14)$$

Proof: Applying the law of iterated logarithm for random walks on the sequence $\{Y_r^j\}$, we have

$$\limsup \frac{S_n - n\mu}{\sqrt{2\sigma^2 N_n^j \log \log N_n^j}} = 1 \quad \text{a.s.}$$

and

$$\liminf \frac{S_n - n\mu}{\sqrt{2\sigma^2 N_n^j \log \log N_n^j}} = -1 \quad \text{a.s.}$$

since $N_n^j \rightarrow \infty$. Now we have

$$\frac{\log N_n^j}{\log n} \rightarrow \frac{\log n}{\log n} + \frac{\log \pi_j}{\log n} \rightarrow 1$$

and

$$\log \log N_n^j - \log \log n \rightarrow \log\left(1 + \frac{\log \pi_j}{\log n}\right) \rightarrow 0.$$

Using (12) and by the same argument as in the proof of Theorem 1 the desired result follows. $\square$

To see that $\sigma^2 = \pi_j \text{Var}(Y_1^j)$ is actually independent of π_j we need to consider the time-reversed J defined as follows.

Definition 1 $\{\hat{J}_n, n \geq 0\}$ is defined to be a time-reversed Markov chain induced by J if its transition probability measure is given by

$$\hat{P}_{jk} = \frac{\pi_k}{\pi_j} P_{kj} \quad \forall k, j \in \mathcal{E} \quad (15)$$

where P_{kj} is the transition matrix of J , $\{\pi_i\}$ is necessarily the stationary measure of both J and $\hat{J}$.

Lemma 2 We may express (9) in terms of the stationary distribution of J as follows.

$$E_j(S_1^j \tau_1^j) = \frac{1}{\pi_j} \sum_{k \in \mathcal{E}} \pi_k E_k(X_1) [1 + E_k(\widehat{\tau_1^j})]. \quad (16)$$

Proof: In terms of the time-reversed J , we have

$$\begin{aligned} \sum_{m=0}^{\infty} m P_j \{ \tau_j^1 > m; J_m = k \} &= \sum_{m=1}^{\infty} m \frac{\pi_k}{\pi_j} P_k \{ \widehat{\tau_j^1} > m-1; \widehat{J}_m = j \} \\ &= \sum_{m=1}^{\infty} m \frac{\pi_k}{\pi_j} \widehat{f}_{kj}^{(m)} = \frac{\pi_k}{\pi_j} E_k(\widehat{\tau_1^j}) \end{aligned}$$

The result follows by substituting this into (9).

Remark 2 *Substitute (10), (11) and (16) into (6), we have*

$$\begin{aligned} \sigma^2 &= \sum_{k \in \mathcal{E}} \pi_k E_k(X_1^2) + 2 \sum_{i, k \neq j} \pi_i E_i(X_1; J_1 = k) \left[\sum_{l \in \mathcal{E}} E_l(X_1) \pi_l (E_k(\tau_1^j) + E_j(\tau_1^l) - E_k(\tau_1^l)) \right] \\ &\quad + \mu^2 \left[1 + 2 \sum_{k \neq j} \pi_k E_k(\tau_1^j) \right] - 2\mu \sum_{k \in \mathcal{E}} \pi_k E_k(X_1) [1 + E_k(\widehat{\tau_1^j})]. \end{aligned} \quad (17)$$

It can be seen that σ^2 is actually independent of π_j and thus independent of j .

4 Application

In Prabhu and Tang [5] the Central Limit Theorem is applied to obtain the heavy traffic limit distribution of the waiting time of some Markov modulated single server queueing systems. Here, we discuss another possible application in finance.

Suppose that the profits or losses from a single trading can be modelled by the increments of the MRW. This is plausible under discrete trading and when the prices of stocks, for example, depend on a number of different market conditions which in turn can be modelled by a Markov chain J . After identifying different types of market conditions with the elements of $\mathcal{E}$ and with sufficient data of the profits and losses trading under these conditions, one may use the method given in Karr [2] to estimate the transition measure $\|Q\|$ of the MRW.

Using the estimated $\|Q\|$, we can use Theorem 1 to compute the probability of the long term gain or losses from the investment. From Theorem 2, we can deduce the long term bounds for the gain and losses. The upper bound which is equal to $\sqrt{2\sigma^2 n \log \log n}$ assuming that $\mu = 0$, may be used to devise some stopping rule. Moreover, having a sufficient large number of trading in mind, one may determine the capital required to cover all possible losses using the same bound.

References

- [1] Chung, K.L. (1967) *Markov Chains with Stationary Transition Probabilities*. 2nd. ed., Springer-Verlag, Berlin Heidelberg.
- [2] Karr, A.F. (1991) *Point Processes and Their Statistical Inference* 2nd ed., Marcel Dekker, New York.
- [3] Keilson, J. and Wishart, D.M.G. (1964) Central limit theorem for processes defined on a finite Markov chain. *Proc. Camb. Philo. Soc.*, **60**, 547-567.
- [4] Prabhu, N.U., Tang, L.C. and Zhu, Y. (1992) Some new results for the Markov random walk. to appear in *J. Math. Phys. Sc.*
- [5] Prabhu, N.U. and Tang, L.C. (1992) Markov modulated single server queueing systems. submitted to *Adv. Appl. Prob.*