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**MARKOV-MODULATED SINGLE
SERVER QUEUEING SYSTEMS**

by

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*Work done while the author was on leave at Cornell University.

Markov-Modulated Single Server Queueing Systems

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Abstract

We consider single server queueing systems that are modulated by a discrete time Markov chain on a countable state space. The underlying stochastic process is a Markov random walk (MRW) whose increments can be expressed as differences between service times and interarrival times. We derive the joint distributions of the waiting and idle times in the presence of the modulating Markov chain. Our approach is based on properties of the ladder sets associated with this MRW and its time-reversed counterpart. The special case of a Markov-modulated M/M/1 queueing system is then analysed and results analogous to the classical case are obtained.

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1 Introduction

In many practical situations, the customer arrivals and the service mechanism of a queueing system are both influenced by some factors external or internal to the system. This has motivated considerable interest in the study of Markov-modulated queueing models where the influence of these factors is conveyed through a secondary Markov chain.

In most of the models proposed so far the secondary Markov chain is formulated in continuous time. Recently Prabhu and Zhu [6] have given a comprehensive survey of the literature in this area and also investigated a Markov-modulated M/G/1 system. Asmussen [2] has also considered this M/G/1 system using a different approach. In another class of models the secondary Markov chain is formulated in discrete time. Specifically, let us denote by $\{u_k, k \geq 1\}$ and $\{v_k, k \geq 1\}$ the interarrival and service times respectively, $U_n = \sum_{k=1}^n u_k$ and $V_n = \sum_{k=1}^n v_k$ ($n \geq 1$). The basic assumption is that $\{U_n, V_n, J_n\}$ is a Markov renewal process (MRP), the secondary Markov chain being $J = \{J_n\}$. This implies that the distributions of both the interarrival and the service times of a customer are determined by the J -chain at the epoch of the customer's arrival, in contrast to the case of a continuous time J where the presence of the secondary Markov chain is felt even between arrival and departure epochs. The queue discipline is first come, first served. The first paper to treat a model of

this latter type is due to Arjas [1], who obtained some results using a matrix version of the Miller-Kemperman identity. In this paper, we further study this model by considering the underlying Markov random walk (MRW) $\{S_n, J_n\}$ where $S_0 = 0$ and $S_n = V_n - U_n$; ($n \geq 1$). Here the interarrival and the service times need not be conditionally independent of each other, given the state of J , as we are only considering their difference. The MRW considered here is assumed to be nondegenerate (see Prabhu *et al.* [7] for definition and discussion on degeneracy). Essentially, we are extending the techniques used in the classical case (where the state space of J consists of a single element) to the Markov-modulated case. For the purpose of this extension we need several new ideas, including the following.

(a) The class of nondegenerate MRW's we consider here is one whose increments X_k can be represented as the differences of two non-negative random variables. We denote this class as $\mathcal{C}_Q$.

(b) Associated with the given system Σ is what we have designated as the adjoint system $\hat{\Sigma}$ obtained by a time-reversal of the MRW. This latter MRW also belongs to the class $\mathcal{C}_Q$. This system $\hat{\Sigma}$ is not the dual of the system considered by Prabhu [5] for the classical case. In the dual considered by Arjas [1], in addition to time reversal, the sign of the increments of the given MRW are also reversed; thus this dual is different from our adjoint. The result of Lemma 1 indicates that it is our notion of $\hat{\Sigma}$ that gives the result analogous to that of the classical case; see Prabhu ([5] Lemma 1.3).

(c) Our starting point is Presman's [8] Wiener-Hopf factorization which we have expressed (see Prabhu *et al.* [7]) in terms of the ladder sets associated with the system Σ and $\hat{\Sigma}$. In

order to apply this factorization we obtain some new results, specifically concerning the distributions of the maximum and minimum functionals of the MRW. We derive these in section 2.

(d) For application to the special case of the Markov-modulated M/M/1 system we have extended the partial lack of memory property used by Prabhu [5] in the classical case. This property is expressed in terms of the increments of the given MRW and its time-reversed counterpart.

In section 3 we show that the distributions of the waiting time and the accumulated idle time jointly with the J -chain can be expressed in terms of the ladder measures of the MRWs corresponding to the given system Σ and its adjoint $\hat{\Sigma}$. We then obtain effortlessly the joint transform of the steady state waiting time, accumulated idle time and the J -chain.

The special case of a Markov-modulated M/M/1 queue is investigated in section 4. Using the partial lack of memory property mentioned in (d) above, we arrive at results analogous to those in the classical M/M/1 queue concerning the idle period. We then derive a matrix identity for this queueing system.

We would like to thank Yizin Zhu for several helpful comments and making available to us his unpublished manuscript that also contains Lemmas 1,2 and Theorem 1.

2 Preliminary Results

Consider a single server queueing system whose customers belong to certain types indexed by $j \in \mathcal{E}$, the switching mechanism between different types of customers being governed

by a Markov chain J with state space $\mathcal{E}$. Each customer type has his own interarrival and service time distributions determined by the state of the J chain at the epoch of arrival. The Markov chain J is assumed to be irreducible and persistent. The queue discipline is first come, first served.

We denote by $\{u_k, k \geq 1\}$ the sequence of interarrival times and by $\{v_k, k \geq 1\}$ the sequence of service times. Let $X_k = v_k - u_k; (k \geq 1)$, $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$ ($n \geq 1$). Then $(S, J) = \{S_n, J_n; n \geq 0\}$ is a time homogeneous MRW. It belongs to the class $\mathcal{C}_Q$ defined as follows.

Definition 1 *The class $\mathcal{C}_Q$ of MRWs is the one in which the increments X_k can be represented by the difference of two non-negative random variables.*

The transition distribution measure of this MRW is given by

$$\begin{aligned} Q_{jk}^{(n)}\{A - x\} &= P\{(S_{m+n}, J_{m+n}) \in A \times \{k\} | (S_m, J_m) = (x, j)\} \\ &= P\{(S_{m+n} - S_m, J_{m+n}) \in (A - x) \times \{k\} | J_m = j\} \quad \forall j, k \in \mathcal{E} \end{aligned} \quad (1)$$

where A is any Borel subset of $\mathbf{R}$. Clearly, the state space of the MRW is $\mathbf{R} \times \mathcal{E}$. We assume that the MRW is nondegenerate. As usual, we write Q_{jk} for $Q_{jk}^{(1)}$ and denote the conditional probabilities and expectations given $J_0 = j$ as P_j and E_j respectively.

It is clear that the MRW resulting from a given Markov-modulated queue is unique. Conversely from the uniqueness (up to translation) of the difference between two non-negative random variables, given a MRW in class $\mathcal{C}_Q$, there exists an Markov-modulated queueing system, say Σ , which is also unique (upto translation).

In addition, consider the time-reversed version $\{(\hat{S}_n, \hat{J}_n), n \geq 0\}$ of the given MRW whose transition probability measure is given by

$$\hat{Q}_{jk}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}\{A\} \quad \forall k, j \in \mathcal{E} \quad (2)$$

where $Q_{kj}\{\cdot\}$ is the transition probability measure of (S, J) , $\{\pi_i\}$ is a sequence of positive numbers and A is any Borel subset of $\mathbf{R}$.

Consider the queueing system $\hat{\Sigma}$ whose associated MRW has the transition measure given by (2). This system is unique up to translation, in view of the fact that $X_k \stackrel{d}{=} (v_k + c) - (u_k + c)$ for any $c \geq 0$. We call $\hat{\Sigma}$ the *adjoint system* of the given system Σ . We shall use $\widehat{(\cdot)}$ to denote the counterpart of $(\cdot)$ for the adjoint system. By setting $A = \mathbf{R}$ in the above equation, we see that

$$\pi_j \hat{P}_{jk} = \pi_k P_{kj},$$

which shows that $\{\pi_j\}$ is necessarily the stationary measure of both J and $\hat{J}$.

We denote W_n as the waiting time of the n^{th} customer, I_n as the idle time (if any) that just terminates before serving the n^{th} customer and $\mathcal{I}_n = I_1 + I_2 + \cdots + I_n$, the accumulated idle time. It is clear that W_n and $\mathcal{I}_n$ are related to S_n and m_n in the same way as in the classical case; thus for $W_0 = 0$ we have

$$W_n = S_n - m_n \quad \text{and} \quad \mathcal{I}_n = -m_n \quad (3)$$

where $m_n = \min_{0 \leq k \leq n} S_k$ ($n \geq 0$) is the minimum functional of the underlying MRW. Similarly we denote the maximum functional by $M_n = \max_{0 \leq k \leq n} S_k$.

In order to obtain the distribution of W_n and $\mathcal{I}_n$, as represented by (3), we shall first

establish the following result relating the extremum functionals of the given MRW and the MRW of $\hat{\Sigma}$.

Lemma 1 *We have*

$$P_j\{S_n - m_n \leq x, -m_n \leq y, J_n = i\} = \frac{\pi_i}{\pi_j} P_i\{\widehat{M}_n \leq x, \widehat{M}_n - \widehat{S}_n \leq y, \widehat{J}_n = j\}, \quad x > 0, y \geq 0. \quad (4)$$

Proof: The probability on the right side of (4) is

$$P_i\{\widehat{M}_n \leq x, \widehat{M}_n - \widehat{S}_n \leq y, \widehat{J}_n = j\} = P_i\{\max_{r \leq n} \widehat{S}_r \leq x, \max_{r \leq n} (\widehat{S}_r - \widehat{S}_n) \leq y, \widehat{J}_n = j\}.$$

We now cite a result from Prabhu *et al.* [7] stating that

$$P_j\{\mathcal{A}_n; J_n = i\} = \frac{\pi_i}{\pi_j} P_i\{\widehat{\mathcal{A}}_n; \widehat{J}_n = j\} \quad (5)$$

$\forall \mathcal{A}_n \in \sigma(X_1, \dots, X_n)$, where $\widehat{\mathcal{A}}_n = \{\omega : (X_n(\omega), \dots, X_1(\omega)) \in \mathcal{A}_n\}$; ie. $\widehat{\mathcal{A}}_n$ is the corresponding set for the time-reversed MRW. It follows that

$$\begin{aligned} & P_i\{\max_{r \leq n} \widehat{S}_r \leq x, \max_{r \leq n} (\widehat{S}_r - \widehat{S}_n) \leq y, \widehat{J}_n = j\} \\ &= \frac{\pi_j}{\pi_i} P_j\{\max_{r \leq n} (S_n - S_{n-r}) \leq x, \max_{r \leq n} (-S_{n-r}) \leq y, J_n = i\} \\ &= \frac{\pi_j}{\pi_i} P_j\{S_n - m_n \leq x, -m_n \leq y, J_n = i\}. \end{aligned}$$

□

We now define the semi-recurrent sets

$$\zeta^+ = \{(n, j) : S_n = M_n, J_n = j, \rho_n = n\} \quad \text{and} \quad \zeta^- = \{(n, j) : S_n = m_n, J_n = j\} \quad (6)$$

where ρ_n is the first epoch $\leq n$ to attain M_n . Associated with these sets are the following measures. Let

$$u_{ij}^{(n)}(x) = P_i\{(n, j) \in \zeta_+, S_n \leq x\} \quad (n \geq 1, x > 0); \quad u_{ij}^{(0)}(x) = \delta_{ij}1_{\{x \geq 0\}} \quad (7)$$

$$v_{ij}^{(n)}(x) = P_i\{(n, j) \in \zeta_-, S_n \geq x\} \quad (n \geq 1, x \leq 0); \quad v_{ij}^{(0)}(x) = \delta_{ij}1_{\{x \leq 0\}} \quad (8)$$

$$u_{ij}(x) = \sum_{n=0}^{\infty} u_{ij}^{(n)}(x) \quad \text{and} \quad v_{ij}(x) = \sum_{n=0}^{\infty} v_{ij}^{(n)}(x). \quad (9)$$

These measures are all finite. To see this we define the two sequences of random variables

$$T_k = \min\{n > T_{k-1} : S_n > S_{T_{k-1}}\}, \quad \bar{T}_k = \min\{n > \bar{T}_{k-1} : S_n \leq S_{\bar{T}_{k-1}}\} \quad (k \geq 1) \quad (10)$$

with $T_0 = \bar{T}_0 = 0$. In the queueing system Σ , $\bar{T}_k - \bar{T}_{k-1}$ represents the number of customers served during the k^{th} busy period. It is clear that the collection of these ladder points with the J state are exactly the two sets ζ_+ and ζ_- ; i.e.

$$\zeta_+ = \{(T_0, J_0), (T_1, J_{T_1}), \dots\}, \quad \zeta_- = \{(\bar{T}_0, J_0), (\bar{T}_1, J_{\bar{T}_1}), \dots\}. \quad (11)$$

It turns out that (9) are Markov renewal measures of the MRP's $\{(S_{T_k}, J_{T_k})\}$ and $\{(-S_{T_k}, J_{T_k})\}$ respectively. It also follows from Markov renewal theory that $\sum_{j \in \mathcal{E}} u_{ij}(I) < \infty$ for all finite $I \in \mathbf{R}_+$.

For the adjoint system these measures are related to the number of customers served in a busy period in the following manner.

Lemma 2 *For $y \leq 0$ and $x > 0$, we have*

$$(i) \hat{u}_{jk}^{(n)}(x) = \frac{\pi_k}{\pi_j} P_k\{\bar{T}_1 > n, S_n \leq x, J_n = j\}; \quad (12)$$

$$(ii) \hat{v}_{jk}^{(n)}(y) = \frac{\pi_k}{\pi_j} P_k\{T_1 > n, S_n \geq y, J_n = j\}. \quad (13)$$

Proof: (i) By definition we have

$$\begin{aligned}\hat{u}_{jk}^{(n)}(x) &= P_j\{(n, k) \in \hat{\zeta}_+, \hat{S}_n \leq x\} \\ &= P_j\{\hat{S}_n > \hat{S}_1, \hat{S}_n > \hat{S}_2, \dots, \hat{S}_n > \hat{S}_{n-1}, \hat{S}_n \leq x, \hat{J}_n = k\}\end{aligned}$$

Using the result (5), this last probability can be rewritten as

$$\begin{aligned}& \frac{\pi_k}{\pi_j} P_k\{S_n > S_n - S_{n-1}, S_n > S_n - S_{n-2}, \dots, S_n > S_n - S_1, S_n \leq x, J_n = j\} \\ &= \frac{\pi_k}{\pi_j} P_k\{S_{n-1} > 0, S_{n-2} > 0, S_1 > 0, S_n \leq x, J_n = j\} \\ &= \frac{\pi_k}{\pi_j} P_k\{\bar{T}_1 > n, S_n \leq x, J_n = j\}.\end{aligned}$$

This proves (12) and the proof of (13) is similar. $\square$

We can also express the joint distributions of the extremum functionals as functions of these measures.

Theorem 1 *We have*

$$(i) P_i\{M_n \leq x, M_n - S_n \leq y, J_n = j\} = \pi_j \sum_{k \in \mathcal{E}} \sum_{m=0}^n u_{ik}^{(m)}(x) \frac{1}{\pi_k} \hat{v}_{jk}^{(n-m)}(-y), \quad x > 0, y \geq 0 \quad (14)$$

$$(ii) P_j\{S_n - m_n \leq x, -m_n \leq y, J_n = i\} = \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-y), \quad x > 0, y \geq 0 \quad (15)$$

In particular, we have

$$(i) P_i\{M_n \leq x, J_n = j\} = \pi_j \sum_{k \in \mathcal{E}} \sum_{m=0}^n u_{ik}^{(m)}(x) \frac{1}{\pi_k} \hat{v}_{jk}^{(n-m)}(-\infty), \quad x > 0. \quad (16)$$

$$(ii) P_j\{m_n \geq -y, J_n = i\} = \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(\infty) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-y), \quad y \geq 0. \quad (17)$$

Proof: (i) Let $T(n) = \max\{k : T_k \leq n\}$. Then

$$P_i\{M_n \leq x, M_n - S_n \leq y, J_n = j\} = \sum_{r=0}^n P_i\{T(n) = r, M_n \leq x, M_n - S_n \leq y, J_n = j\}$$

$$\begin{aligned}
&= \sum_{r=0}^n \sum_{k \in \mathcal{E}} \sum_{m=r}^n P_i\{T_r = m, T_{r+1} > n, S_{T_r} \leq x, S_{T_r} - S_n \leq y, J_n = j, J_{T_r} = k\} \\
&= \sum_{k \in \mathcal{E}} \sum_{m=0}^n \sum_{r=0}^m \int_0^x P_i\{T_{r+1} > n, S_{T_r} - S_n \leq y, J_n = j | T_r = m, S_{T_r} = z, J_{T_r} = k\} \cdot \\
&\quad P_i\{T_r = m, S_{T_r} \in dz, J_{T_r} = k\} \\
&= \sum_{k \in \mathcal{E}} \sum_{m=0}^n \sum_{r=0}^m P_i\{T_r = m, S_m \leq x, J_m = k\} P_k\{T_1 > n - m, S_{n-m} \geq -y, J_{n-m} = j\} \\
&= \sum_{k \in \mathcal{E}} \sum_{m=0}^n P_i\{(m, k) \in \zeta_+, S_m \leq x\} \frac{\pi_j}{\pi_k} P_j\{(n - m, k) \in \hat{\zeta}_-, \hat{S}_{n-m} \geq -y\} \text{ from (13)} \\
&= \pi_j \sum_{k \in \mathcal{E}} \sum_{m=0}^n u_{ik}^{(m)}(x) \frac{1}{\pi_k} \hat{v}_{jk}^{(n-m)}(-y).
\end{aligned}$$

(ii) From Lemma 1 and (i) above, we have

$$\begin{aligned}
P_j\{S_n - m_n \leq x, -m_n \leq y, J_n = i\} &= \frac{\pi_i}{\pi_j} P_i\{\widehat{M}_n \leq x, \widehat{M}_n - \widehat{S}_n \leq y, \widehat{J}_n = j\} \\
&= \frac{\pi_i}{\pi_j} \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{\pi_j}{\pi_k} v_{jk}^{(n-m)}(-y) = \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-y).
\end{aligned}$$

Letting $y \rightarrow \infty$ in (14), we obtain (16). Similarly letting $x \rightarrow \infty$ in (15), we get (17). $\square$

3 Waiting and Idle Times

In this section we obtain the joint distribution of the waiting time and accumulated idle time and its transform. We also give explicit expression for their limit distributions.

Theorem 2 *We have*

$$P_j\{W_n \leq x, \mathcal{I}_n \leq y, J_n = i\} = \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-y), \quad x > 0, y \geq 0. \quad (18)$$

Moreover

$$(i) \quad \lim_{n \rightarrow \infty} P_j\{W_n \leq x, J_n = i\} = \pi_i \sum_{k \in \mathcal{E}} \hat{u}_{ik}(x) P_k\{\widehat{T}_1 = \infty\}, \quad x > 0. \quad (19)$$

$$(ii) \quad \lim_{n \rightarrow \infty} P_j \{ \mathcal{I}_n \leq y, J_n = i \} = \pi_i \sum_{k \in \mathcal{E}} v_{jk}(-y) P_k \{ \bar{T}_1 = \infty \}, \quad y \geq 0. \quad (20)$$

Proof: From (3) and Theorem 1, we find that

$$\begin{aligned} P_j \{ W_n \leq x, \mathcal{I}_n \leq y, J_n = i \} &= P_j \{ S_n - m_n \leq x, -m_n \leq y, J_n = i \} \\ &= \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-y) \end{aligned}$$

as desired. Letting $y \rightarrow \infty$ we obtain

$$\lim_{n \rightarrow \infty} P_j \{ W_n \leq x, J_n = i \} = \lim_{n \rightarrow \infty} \pi_i \sum_{k \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{ik}^{(m)}(x) \frac{1}{\pi_k} v_{jk}^{(n-m)}(-\infty).$$

Also,

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{v}_{jk}^{(n)}(-\infty) &= \lim_{n \rightarrow \infty} P_j \{ (n, k) \in \hat{\zeta}_-, \hat{S}_n \geq -\infty \} \\ &= \lim_{n \rightarrow \infty} \frac{\pi_k}{\pi_j} P_k \{ T_1 > n, S_n \geq -\infty, J_n = j \} \quad (\text{from Lemma 2}) \\ &= \frac{\pi_k}{\pi_j} \cdot \pi_j P_k \{ T_1 = \infty \} \end{aligned}$$

where the last equality follows from the fact that the event $\{T_1 = \infty\} \in \bigcup_k \sigma \{X_1, \dots, X_k\} \quad \forall k$ and the event $\{J_\infty = j\}$ belongs to the tail σ -field which is independent of the former for each fixed k . By interchanging the summation and limit (as the sum is finite) we find that

$$\lim_{n \rightarrow \infty} P_j \{ W_n \leq x, J_n = i \} = \pi_i \sum_{k \in \mathcal{E}} \hat{u}_{ik}(x) P_k \{ \hat{T}_1 = \infty \}$$

which proves (19). The proof of (20) is similar. $\square$

It can be seen that the limit distributions of W_n and $\mathcal{I}_n$ can be interpreted as the weighted average of the measures $\hat{u}_{ik}(x)$ with weights $P_k \{ \hat{T}_1 = \infty \}$ and $v_{jk}(-y)$ with weights $P_k \{ \bar{T}_1 = \infty \}$ respectively. Both distributions are independent of J_0 . The following result shows that the contributions of these weights are “mutually exclusive”.

Proposition 1 *We have*

$$P_j\{\bar{T} = \infty\}P_j\{\hat{T} = \infty\} = 0 \quad \forall j. \quad (21)$$

i.e. $\bar{T}$ and $\hat{T}$ cannot be both defective.

Proof: First we cite the following from Presman [8]; i.e.

$$\mathbf{I} - s\Phi(\omega) = \Pi^{-1}[\mathbf{I} - \hat{\chi}^+(s, \omega)]' \Pi[\mathbf{I} - \chi^-(s, \omega)] \quad (22)$$

where

$$\Phi(\omega) = \{[\phi(\omega)]_{\mathcal{E} \times \mathcal{E}} : \phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} Q_{jk}\{dx\} \} \quad (23)$$

$$\chi^+(s, \omega) = \{[\chi_{jk}^+(s, \omega)]_{\mathcal{E} \times \mathcal{E}} : \chi_{jk}^+(s, \omega) = E_j(s^T e^{i\omega S_T}, J_T = k)\} \quad (24)$$

$$\chi^-(s, \omega) = \{[\chi_{jk}^-(s, \omega)]_{\mathcal{E} \times \mathcal{E}} : \chi_{jk}^-(s, \omega) = E_j(s^T e^{i\omega S_T}, J_T = k)\} \quad (25)$$

$$\Pi = \text{diag}(\pi) \quad (26)$$

and as usual, $T = T_1$, $\bar{T} = \bar{T}_1$, while $\hat{\chi}^+(s, \omega)$ and $\hat{\chi}^-(s, \omega)$ denote the counterparts for the corresponding time-reversed MRW. The prime denotes the transpose.

Setting $\omega = 0$ in (22) we obtain

$$I - sP = \Pi^{-1}[I - \|E_j(s^{\hat{T}}, J_{\hat{T}} = k)\|]' \Pi[I - \|E_j(s^T, J_T = k)\|].$$

Post-multiplying both sides of this last identity by $e = (1, 1, \dots, 1)$ and letting $s \rightarrow 1$, we arrive at

$$\Pi(\dots, 0, \dots)' = [I - \|P_j\{\hat{T} < \infty; J_{\hat{T}} = k\}\|]' \Pi(\dots, 1 - P_j\{\bar{T} < \infty\}, \dots)'.$$

Pre-multiplying both sides of this equation by $e = (1, 1, \dots, 1)$ we obtain

$$0 = \pi_j P_j \{\hat{T} = \infty\} P_j \{\bar{T} = \infty\} \quad \forall j.$$

□

The following classification of the underlying MRW shows that the existence of the limit distributions in (19)-(20) is mutually exclusive.

Proposition 2 *Suppose that the underlying MRW of Σ is nondegenerate with persistent J .*

Then we have

$$(i) P_j \{\hat{T} = \infty\} = P_j \{\bar{T} = \infty\} = 0 \quad \forall j \Leftrightarrow \limsup_n S_n = \infty, \liminf_n S_n = -\infty \Leftrightarrow \mu = 0.$$

$$(ii) P_j \{\hat{T} = \infty\} = 0 \quad \forall j, P_j \{\bar{T} = \infty\} > 0 \text{ for some } j \Leftrightarrow \lim_n S_n = \infty \Leftrightarrow \mu > 0.$$

$$(iii) P_j \{\hat{T} = \infty\} > 0 \text{ for some } j, P_j \{\bar{T} = \infty\} = 0 \quad \forall j \Leftrightarrow \lim_n S_n = -\infty \Leftrightarrow \mu < 0;$$

where $\mu = \sum_{j \in \mathcal{E}} \pi_j E_j(X_1)$, provided that $E_j(X_1)$ exists for all $j \in \mathcal{E}$, is the aggregated mean.

Proof: First we observe that $S_n = \hat{S}_n$ and so $\limsup_n S_n = \limsup_n \hat{S}_n$. We can therefore replace $\hat{T}$ by T in (i), (ii) and (iii). The results then follows from a classification of non-degenerate MRW with persistent J given by Prabhu *et al.* [7]. Here $P_j \{T = \infty\} = 0 (> 0)$ corresponds to the case where ζ^+ is nonterminating (terminating) and $P_j \{\bar{T} = \infty\} = 0 (> 0)$ corresponds to the case where ζ^- is nonterminating (terminating). □

From (18), it is easy to obtain the transform of $\{W_n, \mathcal{I}_n, J_n\}$ as the transforms of the measures $\hat{u}_{jk}^{(n)}(x), v_{jk}^{(n)}(-y)$ are easy to evaluate.

Theorem 3 *Let $\|\Psi(z, \omega_1, \omega_2)\|_{jk} = \|\sum_{n=0}^{\infty} z^n E_j(e^{i\omega_1 W_n + i\omega_2 \mathcal{I}_n}; J_n = k)\|$. Then we have*

$$\Psi(z, \omega_1, \omega_2) = [\mathbf{I} - \chi^-(z, -\omega_2)]^{-1} [\mathbf{I} - \chi^-(z, \omega_1)] [\mathbf{I} - z\Phi(\omega_1)]^{-1}. \quad (27)$$

Proof: From (18) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} z^n E_j(e^{i\omega_1 W_n + i\omega_2 \mathcal{I}_n}; J_n = k) &= \sum_{n=0}^{\infty} z^n \int_0^{\infty} \int_0^{\infty} \pi_k \sum_{l \in \mathcal{E}} \sum_{m=0}^n \hat{u}_{kl}^{(m)}(dx) \frac{1}{\pi_l} v_{jl}^{(n-m)}(-dy) e^{i\omega_1 x + i\omega_2 y} \\ &= [v^*(z, -\omega_2) \Pi^{-1} \hat{u}^*(z, \omega_1)' \Pi]_{jk} \end{aligned}$$

where

$$\hat{u}_{jk}^*(z, \omega) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{i\omega x} \hat{u}_{jk}^{(n)}(dx)$$

and

$$v_{jk}^*(z, \omega) = \sum_{n=0}^{\infty} z^n \int_{-\infty}^0 e^{i\omega x} v_{jk}^{(n)}(dx).$$

Since

$$\hat{u}_{jk}^*(z, \omega) = \sum_{m=0}^{\infty} E_j[z^{\hat{T}_m} e^{i\omega S_{\hat{T}_m}}; J_{\hat{T}_m}] = \sum_{m=0}^{\infty} [\hat{\chi}^+(z, \omega)]_{jk}^m$$

we find that

$$\hat{u}^*(z, \omega) = [\mathbf{I} - \hat{\chi}^+(z, \omega)]^{-1}$$

and similarly

$$v^*(z, \omega) = [\mathbf{I} - \chi^-(z, \omega)]^{-1}.$$

From (22) we have

$$\Pi^{-1}[\mathbf{I} - \hat{\chi}^+(z, \omega)]^{-1} \Pi = [\mathbf{I} - \chi^-(z, \omega)][\mathbf{I} - z\Phi(\omega)]^{-1}.$$

Combining the above results we arrive at (27). □

It can be seen that if we set $\omega_1 = -\omega_2$, we get $\Psi(z, \omega_1, -\omega_1) = [\mathbf{I} - z\Phi(\omega_1)]^{-1}$ which is expected since $W_n - \mathcal{I}_n = S_n$.

Similarly from (19), we obtain the following.

Theorem 4 Suppose that $\mu < 0$ so that $(W, J) = \lim_n (W_n, J_n)$ exists. Then $E_j(e^{i\omega W}; J = k)$ is the k^{th} entry of the vector

$$\Pi[\mathbf{I} - \hat{\chi}^+(1, \omega)]^{-1}[\mathbf{I} - \hat{\chi}^+(1, 0)]e \quad (28)$$

and this is independent of J_0 .

Proof: From (19) we find that

$$E_j(e^{i\omega W}; J = k) = \pi_k \sum_{l \in \mathcal{E}} \int_0^\infty e^{i\omega x} \hat{u}_{kl}(dx) P_l\{\hat{T} = \infty\} = \pi_k \sum_{l \in \mathcal{E}} P_l\{\hat{T} = \infty\} \hat{u}_{kl}^*(1, \omega).$$

The desired result follows from the facts that $\hat{u}^*(z, \omega) = [\mathbf{I} - \hat{\chi}^+(z, \omega)]^{-1}$ and $P_l\{\hat{T} = \infty\}$ is the l^{th} entry of $[\mathbf{I} - \hat{\chi}^+(1, 0)]e$. $\square$

Next we give the limit distributions of the waiting and accumulated idle times when $\mu > 0$ and $\mu < 0$ respectively in which case we do not have steady state distributions for these random variables. However, as in the classical system, one may use the central limit result for the underlying Markov random walk so that the waiting and accumulated times, when suitably normed, have limit distributions.

Theorem 5 (i) If $\mu > 0, \sigma < \infty$, we have

$$\lim_{n \rightarrow \infty} P\left\{\frac{W_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \mathcal{N}(x) \quad (29)$$

(ii) If $\mu < 0, \sigma < \infty$, we have

$$\lim_{n \rightarrow \infty} P\left\{\frac{\mathcal{I}_n + n\mu}{\sigma\sqrt{n}} \leq x\right\} = \mathcal{N}(x) \quad (30)$$

Proof: From Lemma 3 we have

$$\frac{W_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - m_n - n\mu}{\sigma\sqrt{n}}.$$

Since m_n converges to a finite random variable for $\mu > 0$, the result (i) follows from the central limit theorem for MRW (see Tang [9]). The result (ii) follows in a similar manner. $\square$

Remark 1 *In the case where $\sigma = \infty$, one may expect the limit distribution of S_n to be in the domain of attraction of a stable law. The results given by Heyde [3] can then be invoked to obtain the limit distribution of the waiting time.*

4 Markov-modulated M/M/1 Queue

In this section we consider a Markov-modulated M/M/1 queueing system where the inter-arrival time u_n and the service time v_n have the following joint exponential density given J_n and J_{n-1} . i.e.

$$P\{u_n \leq x, v_n \leq y | J_{n-1} = j, J_n = k\} = (1 - e^{-\lambda_{jk}x})(1 - e^{-\mu_{jk}y}). \quad (31)$$

The resulting transition measure of the associated MRW is given by

$$\begin{aligned} Q_{jk}\{dx\} &= \frac{p_{jk}\lambda_{jk}\mu_{jk}}{\lambda_{jk} + \mu_{jk}} e^{\lambda_{jk}x} & x \leq 0; \\ &= \frac{p_{jk}\lambda_{jk}\mu_{jk}}{\lambda_{jk} + \mu_{jk}} e^{-\mu_{jk}x} & x \geq 0. \end{aligned} \quad (32)$$

We consider the following four conditions.

C_1 : arrival rate of the next customer does not depend on the present customer-type (the J -state); ie. $\lambda_{jk} = \lambda_k \ \forall j$

C_2 : service rate of the present customer does not depend on the next customer-type, ie.

$$\mu_{jk} = \mu_j \quad \forall k.$$

C_3 : arrival rate of the next customer only depends on the present J -state; ie. $\lambda_{jk} = \lambda_j \quad \forall k$

C_4 : service rate of the present customer only depends on the next customer-type, ie.

$$\mu_{jk} = \mu_k \quad \forall j.$$

When all these conditions hold it reduces to the classical M/M/1 system. Conditions C_1 and C_2 are more often met in most manufacturing systems as they imply that the arrival rate of a job to a machine is independent of the preceding one and that the service rate of a job under process is independent of its successor. Condition C_3 implies that the arrival rate of the job depends only on the preceding one and condition C_4 implies that the rate of service depends on the next job to be done.

Let

$$\mathbf{A} = \|a_{jk}\| = \left\| \frac{p_{jk}\lambda_{jk}}{\mu_j + \lambda_{jk}} \left(\sum_i \frac{p_{ji}\lambda_{ji}}{\mu_j + \lambda_{ji}} \right)^{-1} \right\| \quad (33)$$

$$\mathbf{B} = \|b_{kj}\| = \left\| \frac{\pi_k p_{kj} \mu_{kj}}{\mu_{kj} + \lambda_j} \left(\sum_i \frac{\pi_i p_{ij} \mu_{ij}}{\mu_{ij} + \lambda_j} \right)^{-1} \right\| \quad (34)$$

$$\mathbf{C} = \|c_{jk}\| = \left\| \frac{p_{jk}\mu_{jk}}{\lambda_j + \mu_{jk}} \left(\sum_i \frac{p_{ji}\mu_{ji}}{\lambda_j + \mu_{ji}} \right)^{-1} \right\| \quad (35)$$

$$\mathbf{D} = \|d_{kj}\| = \left\| \frac{\pi_k p_{kj} \lambda_{kj}}{\lambda_{kj} + \mu_j} \left(\sum_i \frac{\pi_i p_{ij} \lambda_{ij}}{\lambda_{ij} + \mu_j} \right)^{-1} \right\|. \quad (36)$$

Then we have

Lemma 3 (*Partial lack of memory property*)

(a) Under condition C_2 ,

$$P_j\{X_1 \leq x + y, J_1 = k | X_1 > y\} = a_{jk}[1 - \exp(-\mu_j x)]. \quad \forall x > 0, y \geq 0. \quad (37)$$

(b) Under condition C_1 ,

$$P_j\{\widehat{X}_1 \geq x - y, \widehat{J}_1 = k | \widehat{X}_1 \leq -y\} = b_{jk}[1 - \exp(\lambda_j x)] \quad \forall \quad x \leq 0, \quad y > 0. \quad (38)$$

(c) Under condition C_3 ,

$$P_j\{X_1 \geq x - y, J_1 = k | X_1 \leq -y\} = c_{jk}[1 - \exp(\lambda_j x)] \quad \forall \quad x \leq 0, \quad y > 0. \quad (39)$$

(d) Under condition C_4 ,

$$P_j\{\widehat{X}_1 \leq x + y, \widehat{J}_1 = k | \widehat{X}_1 > y\} = d_{jk}[1 - \exp(-\mu_j x)] \quad \forall \quad x > 0, \quad y \geq 0. \quad (40)$$

Proof: $\forall \quad x, y \geq 0$, we have

$$\begin{aligned} P_j\{X_1 \leq x + y, J_1 = k | X_1 > y\} &= \frac{P_j\{y < X_1 \leq x + y, J_1 = k\}}{P_j\{X_1 > y\}} \\ &= \frac{p_{jk}\lambda_{jk}}{\mu_{jk} + \lambda_{jk}}[e^{-\mu_{jk}y} - e^{-\mu_{jk}(x+y)}][\sum_i \frac{p_{ji}\lambda_{ji}}{\lambda_{ji} + \mu_{ji}}e^{-\mu_{ji}y}]^{-1} \end{aligned}$$

which reduce to (37) by (33) and condition C_2 .

The proofs for (b), (c) and (d) are similar. □

Theorem 6 *For the Markov-modulated M/M/1 queue satisfying we have*

$$\chi^+(z, \omega) = \eta(z) \cdot \Theta_a(\omega) \quad \text{if condition } C_2 \text{ is satisfied;} \quad (41)$$

$$\widehat{\chi}^-(z, \omega) = \widehat{\xi}(z) \cdot \Theta_b(\omega) \quad \text{if condition } C_1 \text{ is satisfied;} \quad (42)$$

$$\chi^-(s, \omega) = \xi(z) \cdot \Theta_c(\omega) \quad \text{if condition } C_3 \text{ is satisfied;} \quad (43)$$

$$\widehat{\chi}^+(z, \omega) = \widehat{\eta}(z) \cdot \Theta_d(\omega) \quad \text{if condition } C_4 \text{ is satisfied;} \quad (44)$$

where

$$\eta(z) = \|E_j[z^T; J_{T-1} = k]\|; \quad \xi(z) = \|E_j[z^T; J_{T-1} = k]\|; \quad (45)$$

$$\Theta_a(\omega) = \|a_{jk} \frac{\mu_j}{\mu_j - i\omega}\|; \quad \Theta_b(\omega) = \|b_{jk} \frac{\lambda_j}{\lambda_j + i\omega}\|; \quad (46)$$

$$\Theta_c(\omega) = \|c_{jk} \frac{\lambda_j}{\lambda_j + i\omega}\|; \quad \Theta_d(\omega) = \|d_{jk} \frac{\mu_j}{\mu_j - i\omega}\|. \quad (47)$$

Proof: Let

$$f_{jk}^{(n)}(x) = P_j\{T = n, 0 < S_T \leq x, J_T = k\}.$$

We have

$$f_{jk}^{(n)}(x) = \int_{-\infty}^0 \sum_{l \in \mathcal{E}} P_j\{S_1 \leq 0, \dots, S_{n-1} \in dy, S_n > 0, J_{n-1} = l\} P_l\{X_1 \leq x + y, J_1 = k | X_1 > y\}$$

From (37) we know that

$$P_l\{X_1 \leq x + y, J_1 = k | X_1 > y\} = a_{lk}[1 - \exp(-\mu_l x)]$$

which is independent of y . So we have

$$f_{jk}^{(n)}(x) = \sum_{l \in \mathcal{E}} P_j\{T = n, J_{n-1} = l\} a_{lk}[1 - \exp(-\mu_l x)].$$

Now

$$\begin{aligned} \chi_{jk}^+(z, \omega) &= E_j[z^T e^{i\omega S_T}; J_T = k] = \int_0^\infty \sum_{n=1}^\infty z^n e^{i\omega x} f_{jk}^{(n)}(dx) \\ &= \sum_{l \in \mathcal{E}} E_j[z^T; J_{T-1} = l] \int_0^\infty a_{lk} \mu_l \exp[(-\mu_l + i\omega)x] dx \\ &= \sum_{l \in \mathcal{E}} \eta_{jl}(z) \Theta_a(\omega)|_{lk}. \end{aligned}$$

This proves (41) and the proofs for (42), (43) and (44) are similar. $\square$

Equation (43) shows that the number of customers served in a busy period is conditionally independent of the duration of the idle period that follows it, given the last customer-type served. Moreover, we have the following result.

Lemma 4 *Under condition C_3 the transition density of the idle period given the last customer type served in an busy period is given by*

$$H_{jk}\{dx\} = c_{jk}\lambda_j \exp(-\lambda_j x) \quad (48)$$

i.e. given that the last customer type served in an busy period is j , the length of the idle period is exponentially distributed with rate λ_j .

Proof: First we note that $I_n = -S_T$. From (47) we see that $\Theta_c(\omega)|_{jk}$ is the characteristic function for the random variable having the density function

$$c_{jk}\lambda_j \exp(\lambda_j x).$$

The result follows by changing the sign and the assertion follows from the fact that the row sum of c_{jk} is 1. $\square$

Next, for the purpose of evaluating η and $\hat{\xi}$, we state the following matrix identities which hold under conditions C_1 and C_2 . Similar identities involving $\hat{\eta}$ and ξ can be obtained under conditions C_3 and C_4

Proposition 3 *Under conditions C_1 and C_2 , η and $\hat{\xi}$ satisfy the following matrix equations.*

$$z\mathbf{P} = \mathbf{B}'^{-1}\Pi^{-1}\hat{\xi}'\Pi + \eta\mathbf{A}^{-1} - \mathbf{B}'^{-1}\Pi^{-1}\hat{\xi}'\Pi \eta\mathbf{A}^{-1} \quad (49)$$

$$z\mathfrak{S} = \mathbf{B}'^{-1}\Pi^{-1}\mathbf{L}\hat{\xi}'\Pi - \eta\mathbf{M}\mathbf{A}^{-1} - \mathbf{B}'^{-1}\Pi^{-1}\mathbf{L}\hat{\xi}'\Pi\eta\mathbf{A}^{-1} + \mathbf{B}'^{-1}\Pi^{-1}\hat{\xi}'\Pi\eta\mathbf{M}\mathbf{A}^{-1} \quad (50)$$

where

$$\mathfrak{S}_{jk} = \frac{p_{jk}(\lambda_j - \mu_k)}{\lambda_j \mu_k} \quad \mathbf{M} = \text{diag}(\mu_j^{-1}) \quad \mathbf{L} = \text{diag}(\lambda_j^{-1}) \quad (51)$$

Proof: From (22) we have

$$\mathbf{I} - z\Phi(\omega) = \Pi^{-1}[\mathbf{I} - \hat{\xi}(z)\Theta_b(\omega)]' \Pi[\mathbf{I} - \eta(z)\Theta_a(\omega)].$$

On the left side of this identity we have

$$\delta_{jk} - z \frac{p_{jk} \lambda_j \mu_k}{(\lambda_j + i\omega)(\mu_k - i\omega)} = \delta_{jk} - z \left[\frac{p_{jk} \lambda_j \mu_k (\lambda_j \mu_k + \omega^2)}{(\lambda_j^2 + \omega^2)(\mu_k^2 + \omega^2)} + \frac{i\omega p_{jk} \lambda_j \mu_k (\lambda_j - \mu_k)}{(\lambda_j^2 + \omega^2)(\mu_k^2 + \omega^2)} \right],$$

while on the right side we have

$$\begin{aligned} \delta_{jk} - \left(\sum_m \frac{b_{jm} \lambda_m (\lambda_m - i\omega)}{\pi_j (\lambda_m^2 + \omega^2)} \hat{\xi}'_{mk} \pi_k + \sum_m \eta_{jm} \frac{a_{mk} \mu_m (\mu_m + i\omega)}{\mu_m^2 + \omega^2} \right. \\ \left. - \sum_m \left[\sum_q \frac{b_{jq} \lambda_q (\lambda_q - i\omega)}{\pi_j (\lambda_q^2 + \omega^2)} \hat{\xi}'_{qm} \pi_k \right] \left[\sum_r \eta_{mr} \frac{a_{rk} \mu_r (\mu_r + i\omega)}{\mu_r^2 + \omega^2} \right] \right). \end{aligned}$$

For the real part we let $\omega \rightarrow 0$ on both sides. For imaginary part we divide both sides by ω then we let $\omega \rightarrow 0$. The result then follows. $\square$

Remark 2 In the case where $\mathcal{E} = \{j\}$, we have from (49) and (50)

$$z = \eta + \xi - \eta\xi \quad \text{and} \quad \frac{z(\lambda - \mu)}{\lambda\mu} = \frac{\eta}{\mu} - \frac{\xi}{\lambda} - \frac{\eta\xi(\lambda - \mu)}{\lambda\mu}$$

which give the results

$$\xi = \frac{(\lambda + \mu) - \sqrt{(\lambda + \mu)^2 - 4z\mu\lambda}}{2\lambda}$$

and

$$\eta = \frac{(\lambda + \mu) - \sqrt{(\lambda + \mu)^2 - 4z\mu\lambda}}{2\mu}.$$

These agree with the known results for the classical $M/M/1$ system.

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