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SUBEXPONENTIALITY OF THE PRODUCT
OF INDEPENDENT RANDOM VARIABLES

by

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Abstract.

Suppose X and Y are independent nonnegative random variables. We study the behavior of $P(XY > t)$, as $t \to \infty$, when X has a subexponential distribution. Particular attention is given to obtaining sufficient conditions on $P(Y > t)$ for XY to have a subexponential distribution.

The relationship between $P(X > t)$ and $P(XY > t)$ is further studied for the special cases where the former satisfies one of the extensions of regular variation.

1. Introduction.

In this paper we study products of independent nonnegative random variables in connection with the family of *subexponential distributions* and its various subfamilies. Formally, we have the following.

DEFINITION 1.1. *A distribution F on $[0, \infty)$ is called subexponential if $\overline{F}(t) > 0$ for every t and*

$$\lim_{t \to \infty} \frac{\overline{F*F}(t)}{\overline{F}(t)} = 2, \tag{1.1}$$

where $\overline{F}(t) = 1 - F(t)$ is the tail of the distribution function F and $*$ denotes convolution.

Examples of subexponential distributions include Pareto distributions,

$$F(t) = 1 - (1 + t/b)^{-\alpha}, \qquad \alpha > 0, \quad b > 0;$$

the lognormal distribution,

$$F(t) = \Phi\left(\frac{\log t - \mu}{\sigma}\right), \qquad \mu \in \mathbb{R}, \quad \sigma > 0,$$

where Φ is the standard normal distribution; and certain Weibull distributions,

$$F(t) = 1 - e^{-t^\beta}, \qquad 0 < \beta < 1.$$

Subexponential distributions have been found to be useful in the theory of branching processes (Chistyakov (1969), Athreya and Ney (1972), Chover, Ney and Wainger (1973a,b)), queueing theory (Pakes (1975)), renewal theory (Teugels (1975), Embrechts and Goldie (1982)), infinite variance time series (Davis and Resnick (1985b)) and large deviations theory (Pinelis (1985), Cline and Hsing (1990)).

The class of subexponential distribution is typically denoted by $\mathcal{S}$ (or $\mathcal{S}_0$); it has been studied rather extensively in Pitman (1980), Embrechts and Goldie (1980, 1982), Cline (1986, 1987), Goldie and Resnick (1988), Klüppelberg (1988) and others. The following is a selection of the results from the above papers and used in the present paper. Note that the first statement accounts for the name "subexponential" and defines a larger class, the long-tailed distributions $\mathcal{L}$.

THEOREM 1.1.

(i) (Athreya and Ney (1972)). If $F \in \mathcal{S}$, then $F \in \mathcal{L}$, where $\mathcal{L}$ is the class of distributions on $[0, \infty)$ satisfying

$$\lim_{t \to \infty} \frac{\overline{F}(t + u)}{\overline{F}(t)} = 1 \qquad \text{for any } u > 0, \tag{1.2}$$

and, consequently,

$$\lim_{t \to \infty} e^{\alpha t} \overline{F}(t) = \infty \qquad \text{for any } \alpha > 0.$$

(ii) (Embrechts and Goldie (1980)). Let $F \in \mathcal{S}$, $G \in \mathcal{L}$ and $\sup_{t>0} \overline{F}(t)/\overline{G}(t) < \infty$. Then $F * G \in \mathcal{S}$ iff $G \in \mathcal{S}$.

(iii) (Cline (1987)). Let $F \in \mathcal{S}$ and $G \in \mathcal{L}$. If $\sup_{t>0} \overline{G}(t)/\overline{F}(t) < \infty$ then $F * G \in \mathcal{S}$.

(iv) (Embrechts and Goldie (1980)). Let $F, G \in \mathcal{S}$. Then $F * G \in \mathcal{S}$ iff $pF + (1-p)G \in \mathcal{S}$ for some (equivalently, all) $p \in (0, 1)$.

(v) (Cline (1987)). Let $F, G \in \mathcal{S}$. If

$$\sup_{t>0} \sup_{1/2 \le r \le 2} \frac{\overline{F}(rt)\overline{G}(t)}{\overline{F}(t)\overline{G}(rt)} < \infty,$$

then $F * G \in \mathcal{S}$.

Remark. It is important to remember in this connection that $F \in \mathcal{S}$ and $G \in \mathcal{S}$ does not, in general, imply that $F * G \in \mathcal{S}$. See Leslie (1989).

The above remark notwithstanding, Theorem 1.1 gives us a taste of the closure properties of the family of subexponential distributions with convolutions. That is, if X and Y are independent random variables and the distribution of X is in $\mathcal{S}$, then, under appropriate conditions on the distribution of Y, the distribution of the sum $X + Y$ is also in $\mathcal{S}$.

The present research is concerned with a related problem. Let, as above, X and Y be independent nonegative random variables, and the distribution of X is in $\mathcal{S}$. Under what conditions on the distribution of Y, will the distribution of the *product* XY (the product convolution) be in $\mathcal{S}$?

Our interest in this problem has originated from two particular applications where the above question is of much importance.

The first example concerns infinite variance regression (Cline (1986), (1989)) and infinite variance time series (Davis and Resnick (1985 a,b, 1986)) and others). Consider the settings, say, of simple linear regression,

$$Y_i = X_i + \epsilon_i,$$

(X_i, ϵ_i) iid with X_i and ϵ_i independent and of moving average time series,

$$Y_j = \sum_{i=0}^{\infty} b_i \epsilon_{j-i},$$

ϵ_i iid. The statistical behavior of least squares estimators in these settings requires knowledge of the tail behavior of $X_1 \epsilon_1$ in the former case and of $\epsilon_1 \epsilon_2$ in the latter. Previous

work has been limited to distributions with regularly varying tails. However, consistency results in particular may extend to a broader class of subexponential random variables, for example those with dominated varying tails.

The second application is related to the theory of sample paths of infinitely divisible stochastic processes. Rosinski and Samorodnitsky (1991) have considered the following problem. Given an infinitely divisible stochastic process

$$X(t) = \int_E f_t(x)M(dx), \quad t \in T,$$

where $\{f_t, t \in T\}$ is a family of measurable functions and M is an infinitely divisible random measure, it is frequently of interest to characterize the distribution (or at least the tail behavior) of certain functionals of the sample paths of the process $\{X(t), t \in T\}$, e.g.

$$\sup_{t \in T} X(t), \qquad \sup_{t \in T} |X(t)|, \qquad \int_T |X(t)|^p \, \nu(dt).$$

Under certain conditions, Rosinski and Samorodnitsky (1991) were able to characterize the tail behavior of such distributions. It turns out that, in many cases, the only condition one needs to check is whether or not the distribution of the product of a certain two independent random variables belongs to the subexponential class $\mathcal{S}$. (One random variable describes the effect of the Levy measure of the random measure M while the second describes the combined effect of the kernel $\{f_t, t \in T\}$ and of the control measure of M.)

Although our original interest in the problem stems from the two applications described above, insight into it will improve our understanding of the subexponentiality property in general. In particular, how "robust" is subexponentiality? In this context we mention the following well known result due to Embrechts and Goldie (1980). First, we recall that a proper subclass of the subexponential family is the class of distributions with *regularly varying tails*, that is, of distributions F such that for any $\lambda > 0$,

$$\lim_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = \lambda^{-\alpha} \qquad \text{for some } \alpha \geq 0.$$

The Embrechts and Goldie result says that if F has regularly varying tails and $\overline{G}(t) = o(\overline{F}(t))$ as $t \to \infty$ then the distribution H of the product XY also has regularly varying tails with the same index α.

This result expresses a certain "robustness" of the family $RV_{-\alpha}$, under the product convolution. The underlying objective of this work is to study how much of this "robustness" is shared by the whole subexponential family $\mathcal{S}$ (Section 2) and by its various other subclasses (Section 3).

We conclude this introductory section by mentioning that, as the many positive results of the following sections show, the subexponential family is "robust" enough to have various closure properties under the product convolution; still the closure properties appear to be fewer (and harder to derive) than the closure properties under the sum convolution. This, of course, is natural if one recalls that the very definition of the subexponential family of distribution is in terms of sums (and not products) of independent random variables.

Henceforth X and Y will be independent nonnegative random variables with distributions F and G, respectively (not degenerate at 0). The product XY has distribution H, whose tail behavior we study.

2. Sufficient conditions for H to be in $\mathcal{L}$ or $\mathcal{S}$.

This section has two main purposes. The first is to show that $\mathcal{L}$ is closed under the product convolution and the second is to give a partial analysis for the subexponential case in the spirit of the above-mentioned Embrechts-Goldie result. For the latter, we are dealing with the question posed as follows. Suppose $F \in \mathcal{S}$. It is reasonable to believe that as long as the tail of the distribution G is "light enough" compared to the tail of F, the "smoothness" of the former won't matter, or the "perturbation" of F caused by multiplying X by Y, will not be serious enough to remove the product distribution from the subexponential class. We know that this is true when F has regularly varying tails, and Embrecht's and Goldie's result is an example of "light enough" in this case. The following exhibits one such situation in the general subexponential case and is the main result of this section.

THEOREM 2.1. *Assume that* $F \in \mathcal{S}$ *If there is a function* $a : (0, \infty) \to (0, \infty)$ *satisfying the following, then* $H \in \mathcal{S}$.

    (a) $a(t) \uparrow \infty$ as $t \to \infty$,

    (b) $t/a(t) \uparrow \infty$ as $t \to \infty$,

    (c) $\lim_{t \to \infty} \overline{F}(t - a(t))/\overline{F}(t) = 1$,

    (d) $\lim_{t \to \infty} \overline{G}(a(t))/\overline{H}(t) = 0$.

Remark. The assumption $\overline{G}(a(tb)) = o(\overline{F}(t))$ for *some* $b > 0$ is sufficient for condition (d) in Theorem 2.1.

The proof of Theorem 2.1 is based on a sequence of lemmas. The first one provides conditions for H to be "long-tailed", including the closure result for $\mathcal{L}$.

Remark. Before we begin, however, we note that in proving either (1.1) or (1.2) the limits infinum hold automatically so it necessary only to obtain the limits supremum.

THEOREM 2.2.

  (i) Let H_ϵ represent the distribution of $X(Y \vee \epsilon)$. Fix $\delta > 0$. If $H_\epsilon \in \mathcal{L}$ for every $\epsilon \in (0, \delta)$ then $H \in \mathcal{L}$.

  (ii) $F, G \in \mathcal{L} \implies H \in \mathcal{L}$.

  (iii) If $F \in \mathcal{L}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for every $b > 0$ then $H \in \mathcal{L}$.

PROOF:

  (i) First we observe that it always suffices to assume $Y > 0$ a.s. Indeed, suppose $P(Y = 0)$ is positive (but less than 1). Let Y_+ have the conditional distribution of Y, given $Y > 0$, and let H_+ be the distribution of XY_+. Since $\overline{H}(t) = P(XY_+ > t)P(Y > 0)$, it is easy to see that $H \in \mathcal{L} \iff H_+ \in \mathcal{L}$.

  For any fixed $\epsilon > 0$,

$$
\begin{aligned}
\overline{H_\epsilon}(t) &= P\left(X(Y \vee \epsilon) > t\right) \\
&\geq \overline{H}(t) \\
&\geq P(XY > t, Y > \epsilon) \\
&= P(X(Y \vee \epsilon) > t) - P(Y \leq \epsilon)P(\epsilon X > t) \\
&\geq P(Y > \epsilon)\overline{H_\epsilon}(t).
\end{aligned}
\tag{2.1}
$$

Therefore, for every $u > 0$,

$$
\limsup_{t \to \infty} \frac{\overline{H}(t - u)}{\overline{H}(t)} \leq \frac{1}{P(Y > \epsilon)} \limsup_{t \to \infty} \frac{\overline{H_\epsilon}(t - u)}{\overline{H_\epsilon}(t)} = \frac{1}{P(Y > \epsilon)}
$$

Letting $\epsilon \to 0$, we conclude that $H \in \mathcal{L}$.

  (ii) Assume first that $X \geq \epsilon_1$ and $Y \geq \epsilon_2$ a.s. Fix $u > 0$, $\delta > 0$. For large enough t_0,

$$
\overline{F}\left(s - u/\epsilon_2\right) \leq (1 + \delta)\overline{F}(s) \quad \text{and} \quad \overline{G}\left(s - u/\epsilon_1\right) \leq (1 + \delta)\overline{G}(s)
$$

whenever $s > t_0$. Thus for $t \geq t_0^2$ and $\epsilon_2 \leq y \leq (t - u)^{1/2}$,

$$
\overline{F}((t - u)/y) \leq \overline{F}(t/y - u/\epsilon_2) \leq (1 + \delta)\overline{F}(t/y).
$$

With a similar inequality for $\overline{G}$,

$$\overline{H}(t-u) = \int_{\epsilon_2}^{(t-u)^{1/2}} \overline{F}((t-u)/y)G(dy) + \int_{\epsilon_1}^{(t-u)^{1/2}} \overline{G}((t-u)/y)F(dy)$$

$$+ \overline{F}\left((t-u)^{1/2}\right)\overline{G}\left((t-u)^{1/2}\right)$$

$$\leq (1+\delta)\left[\int_{\epsilon_2}^{(t-u)^{1/2}} \overline{F}(t/y)G(dy) + \int_{\epsilon_1}^{(t-u)^{1/2}} \overline{G}(t/y)F(dy)\right.$$

$$\left. + \overline{F}\left(t/(t-u)^{1/2}\right)\overline{G}\left(t/(t-u)^{1/2}\right)\right]$$

$$\leq (1+\delta)\left[\int_{\epsilon_2}^{t^{1/2}} \overline{F}(t/y)G(dy) + \int_{\epsilon_1}^{t^{1/2}} \overline{G}(t/y)F(dy) + \overline{F}\left(t^{1/2}\right)\overline{G}\left(t^{1/2}\right)\right]$$

$$= (1+\delta)\overline{H}(t).$$

This shows

$$\limsup_{t\to\infty} \frac{\overline{H}(t-u)}{\overline{H}(t)} \leq 1,$$

which is sufficient for $H \in \mathcal{L}$.

More generally, $X \geq 0$ and $Y \geq 0$ a.s. It is clear that $X \vee \epsilon_1$ and $Y \vee \epsilon_2$ have the same probability tails as do X and Y. The above shows that $(X \vee \epsilon_1)(Y \vee \epsilon_2)$ has a long-tailed distribution for any $\epsilon_1, \epsilon_2 > 0$. By part (i), applied twice, it follows that XY has a long-tailed distribution.

(iii) By (i) it suffices to show that the result holds whenever $Y \geq \epsilon$ a.s., regardless of the value of ϵ. Take $u > 0$ and $\delta > 0$. Choose $t_0 > 0$ so big that $\overline{F}(t - u/\epsilon) \leq (1+\delta)\overline{F}(t)$ for all $t > t_0$. Then

$$\overline{H}(t-u) \leq \overline{G}(t/t_0) + \int_\epsilon^{t/t_0} \overline{F}(t/y - u/\epsilon)G(dy)$$

$$\leq \overline{G}(t/t_0) + (1+\delta)\overline{H}(t).$$

Therefore,

$$\limsup_{t\to\infty} \frac{\overline{H}(t-u)}{\overline{H}(t)} \leq 1 + \delta$$

and, as this is true for every $\delta > 0$, $H \in \mathcal{L}$. $\qquad\square$

LEMMA 2.3. *Let H_ϵ be as in Theorem 2.2(i).*

(i) *Suppose $H, H_\epsilon \in \mathcal{L}$ where $P(Y > \epsilon) > 0$. Then $H \in \mathcal{S} \iff H_\epsilon \in \mathcal{S}$.*

(ii) *Let $\delta > 0$. $H_\epsilon \in \mathcal{S}$ for all $\epsilon \in (0, \delta)$ implies $H \in \mathcal{S}$.*

PROOF:

(i) Equation (2.1) is justified by using Theorem 1.1(ii,iii). From these we see that

$$H_\epsilon \in \mathcal{S} \implies H * H_\epsilon \in \mathcal{S} \implies H \in \mathcal{S}$$

and vice versa.

(ii) This follows from Theorem 2.2(i) and part (i) of this lemma. □

We now turn to the final lemma to be used in the proof of Theorem 2.1. A piece of notation: for $X \sim F$ and $r > 0$ we will denote the distribution of rX by F_r.

LEMMA 2.4. *Let $F \in \mathcal{S}$, and let $a: (0, \infty) \to (0, \infty)$ satisfy $(a) - (c)$ of Theorem 2.1. Let*

$$r(t) = \inf\{u: ua(t/u) \geq 1\}.$$

Then $r(t) \downarrow 0$ as $t \to \infty$ and

$$\lim_{t \to \infty} \sup_{r(t) \leq r \leq 1} \frac{\overline{F * F_r}(t)}{\overline{F}(t) + \overline{F}_r(t)} = 1. \tag{2.2}$$

PROOF: The fact that $r(t) \downarrow 0$ as $t \to \infty$ is elementary. Fix an $\epsilon > 0$ and choose an $s > 0$ so large that $\overline{F}(s) \leq \epsilon$. It is straightforward to check that our assumption implies

$$\lim_{t \to \infty} \frac{\overline{F}(t - qa(t))}{\overline{F}(t)} = 1$$

for every $q > 0$. It follows that there is a $t_0 \geq 2s$ large enough so that for every $t \geq t_0$, we have $r(t) \leq 1$, $a(t) \geq 1$ and $\overline{F}(t - sa(t)) \leq (1 + \epsilon)\overline{F}(t)$. Using the easily checked fact that for any $r \geq r(t)$ we have $ra(t/r) \geq 1$, we obtain for any $t \geq t_0$,

$$\sup_{0 \leq u \leq s} \sup_{r(t) \leq r \leq 1} \frac{\overline{F}_r(t - u)}{\overline{F}_r(t)} = \sup_{0 \leq u \leq s} \sup_{r(t) \leq r \leq 1} \frac{\overline{F}\left(\frac{t-u}{r}\right)}{\overline{F}\left(\frac{t}{r}\right)}$$

$$\leq \sup_{r(t) \leq r \leq 1} \frac{\overline{F}\left(\frac{t}{r} - sa\left(\frac{t}{r}\right)\right)}{\overline{F}\left(\frac{t}{r}\right)} \leq 1 + \epsilon.$$

Therefore, for every $t \geq t_0$ and $r(t) \leq r \leq 1$,

$$\left| \int_0^s \frac{\overline{F}_r(t - y)}{\overline{F}_r(t)} F(du) - 1 \right| \leq \int_0^s \left| \frac{\overline{F}_r(t - y)}{\overline{F}_r(t)} - 1 \right| F(du) + \overline{F}(s)$$

$$\leq 2\epsilon.$$

Similarly we obtain

$$\left| \int_0^s \frac{\overline{F}(t - u)}{\overline{F}(t)} F_r(du) - 1 \right| \leq 2\epsilon$$

and

$$\left| \int_0^s \frac{\overline{F}(t-u)}{\overline{F}(t)} F(du) - 1 \right| \leq 2\epsilon.$$

We then obtain

$$\overline{F*F_r}(t) = \overline{F}(t-s)\overline{F_r}(s) + \int_0^s \overline{F_r}(t-u)F(du)$$

$$+ \int_0^s \overline{F}(t-u)F_r(du) + \int_s^{t-s} \overline{F_r}(t-u)\overline{F}(du)$$

$$\leq \overline{F}(t-s)\overline{F}(s) + (1+2\epsilon)(\overline{F_r}(t) + \overline{F}(t)) + \int_s^{t-s} \overline{F}(t-u)F(du)$$

$$\leq \overline{F*F}(t) - 2(1-2\epsilon)\overline{F}(t) + (1+2\epsilon)(\overline{F_r}(t) + \overline{F}(t)).$$

Since $F \in \mathcal{S}$, we can choose $t_1 \geq t_0$ such that for every $t \geq t_1$, $\overline{F*F}(t) \leq 2(1+\epsilon)\overline{F}(t)$. Then, for every $t \geq t_1$ and $r(t) \leq r \leq 1$

$$\overline{F*F_r}(t) \leq (1+2\epsilon)(\overline{F_r}(t) + \overline{F}(t)) + 6\epsilon\overline{F}(t)$$

$$\leq (1+2\epsilon)(\overline{F_r}(t) + \overline{F}(t)) + 6\epsilon\overline{F*F_r}(t).$$

Thus,

$$\limsup_{t\to\infty} \sup_{r(t)\leq r\leq 1} \frac{\overline{F*F_r}(t)}{\overline{F}(t) + \overline{F_r}(t)} \leq \frac{1+2\epsilon}{1-6\epsilon}.$$

Letting $\epsilon \to 0$ proves the only non-trivial part of (2.2). $\square$

PROOF OF THEOREM 2.1: By Theorem 2.2(iii), $H \in \mathcal{L}$. Likewise, the distribution of $X(Y \vee 1)$ is long-tailed. If $P(Y > 1) > 0$, it suffices by Lemma 2.3(i) to show the latter is subexponential. Otherwise, we replace Y with $cY \vee 1$ where $c > 1$ and $P(cY > 1) > 0$. Note that condition (d) holds for the distribution of cY. Note also that both $\mathcal{L}$ and $\mathcal{S}$ are closed under scalar multiplication. We are free therefore to prove the result only for the case $Y \geq 1$ a.s.

Let X_i and Y_i, $i = 1, 2$, be independent copies of X and Y. We have

$$\overline{H*H}(t) = P(X_1Y_1 + X_2Y_2 > t)$$

$$\leq P(X_1Y_1 + X_2Y_2 > t, Y_2 \leq Y_1 \leq a(t)) + P(X_1Y_1 + X_2Y_2 > t, Y_1 < Y_2 \leq a(t))$$

$$+ 2P(Y_1 > a(t)). \tag{2.3}$$

Note that for every $1 \leq y_2 \leq y_1 \leq a(t)$ we have $y_2/y_1 \geq r(t/y_1)$. We now apply Lemma 2.4 to conclude that for every $\epsilon > 0$ there is a $t_0 > 0$ so large that for every $t > t_0$ and every $1 \leq y_2 \leq y_1 \leq a(t)$

$$\frac{\overline{F*F_{y_2/y_1}}(t/y_1)}{\overline{F}(t/y_1) + \overline{F}_{y_2/y_1}(t/y_1)} \leq \sup_{r(t/y_1)\leq r\leq 1} \frac{\overline{F*F_r}(t/y_1)}{\overline{F}(t/y_1) + \overline{F}_r(t/y_1)} \leq 1 + \epsilon. \tag{2.4}$$

It follows now from (2.3) and (2.4), conditioning on Y_1 and Y_2, that for any $\epsilon > 0$,

$$\overline{H*H}(t) \leq (1 + \epsilon)\left[P\left(X_1 Y_1 > t, Y_2 \leq Y_1 \leq a(t)\right) + P\left(X_2 Y_2 > t, Y_2 \leq Y_1 \leq a(t)\right)\right]$$
$$+ (1 + \epsilon)\left[P\left(X_1 Y_1 > t, Y_1 < Y_2 \leq a(t)\right) + P\left(X_2 Y_2 > t, Y_1 < Y_2 \leq a(t)\right)\right]$$
$$+ 2P\left(Y_1 > a(t)\right)$$
$$\leq 2(1 + \epsilon)\overline{H}(t) + 2\overline{G}(a(t)).$$

Therefore,

$$\limsup_{t \to \infty} \frac{\overline{H*H}(t)}{\overline{H}(t)} \leq 2(1 + \epsilon).$$

And since $\epsilon > 0$ can be taken arbitrarily small, we conclude that $H \in \mathcal{S}$. $\qquad \square$

We demonstrate applicability of Theorem 2.1 by several examples, the first of which is formulated as a corollary.

COROLLARY 2.5. *If $F \in \mathcal{S}$ and Y is a bounded random variable then H is in $\mathcal{S}$.*

PROOF: Condition (d) in Theorem 2.1 holds trivially but we need to verify that we may choose $a(t)$ to satisfy (a)-(c). One choice is

$$a(t) = \begin{cases} \dfrac{2}{r_1}t, & 0 < t \leq r_1 \\[2mm] n + \dfrac{t - r_{n-1}}{r_n - r_{n-1}}, & r_{n-1} < t \leq r_n, n = 2, 3, \ldots, \end{cases}$$

where

$$r_0 = 0, \quad r_1 = \inf\left\{t > r_0 : \frac{\overline{F}(u - 3)}{\overline{F}(u)} \leq 1 + \frac{1}{2} \quad \text{for all } u \geq t\right\}$$

and, inductively, for $n \geq 2$,

$$r_n = \inf\left\{t \geq \frac{n+1}{n}r_{n-1} : \frac{\overline{F}(u - (n + 2))}{\overline{F}(u)} \leq 1 + \frac{1}{n+1} \quad \text{for all } u \geq t\right\}.$$

Since $F \in \mathcal{L}$, the sequence $\{r_n\}_{n=1}^{\infty}$ is well-defined, and thus the proof is complete. $\qquad \square$

Example 2.1. Let X be a lognormal random variable with parameters μ and σ^2, i.e. the tail of F is

$$\overline{F}(t) = 1 - \Phi\left((\log t - \mu)/\sigma\right), \qquad t > 0,$$

where Φ is the standard normal distribution. Let Y be a nonnegative random variable, independent of X. We claim that if for some $\theta > 1$

$$P\left(Y > t\right) = o\left((\log t)^{-1}\exp\left\{-\frac{1}{2\sigma^2}\left(\log t + \theta\log\log t\right)^2\right\}\right),$$

as $t \to \infty$, then the distribution of the product XY belongs to the subexponential class $\mathcal{S}$. Indeed, Theorem 2.1 applies with

$$a(t) = t/(\log t)^{\theta_1} \quad \text{for} \quad 1 < \theta_1 < \theta.$$

Both Corollary 2.5 and Example 2.1 deal with situations in which the tail of the distribtion of a random variable Y is suitably lighter than the tail of the distribution of a random variable X. This is, indeed, the spirit of Theorem 2.1. It is also applicable to many situations discussed in the next section. However, Theorem 2.1 can be applied in certain situations when the tails of the two distributions are comparable, as our next example demonstrates.

Example 2.2. Let X and Y be i.i.d. random variables with common distribution F such that

$$\overline{F}(t) = \exp\left\{-t^p L(t)\right\}, \qquad 0 < p < 1/2,$$

where L is slowly varying at infinity and eventually decreasing. Let H be the distribution of the product XY. It follows from Cline (1986) and from Goldie and Resnick (1988) that $F \in \mathcal{S}$. Since $\overline{H}(t) \geq (\overline{F}(t^{1/2}))^2$, it follows that Theorem 2.1 applies with $a(t) = Mt^{1/2}$, where $M > 2^{1/p}$. Therefore $H \in \mathcal{S}$.

We leave it for the reader to observe the numerous ways in which the above assumption on F can be relaxed.

We conclude this section with an observation that even the spirit of the results discussed above seems nowhere to be found in the *multivariate* case. Apart from shedding light on the multivariate subexponentiality, this shows that the property of subexponentiality is fragile indeed where taking products of independent random variables is concerned.

The extension of the notion of a subexponential distribution to the multivariate care is due to Cline and Resnick (1991); it is stated in terms of vague convergence of measures, which is, in the context of $\mathbb{R}^d$, a language preferable to that of distribution tail functions. To this end let "$\overset{v}{\longrightarrow}$" stand for the vague convergence of measures on $E = [-\infty, -\infty]^d - \{(-\infty, -\infty, \dots, -\infty)\}$ and let $\underline{b}(t) = (b_1(t), \dots, b_d(t)) : \mathbb{R}_+ \to \mathbb{R}_+^d$, with $b_i(t) \to \infty$ as $t \to \infty$ for every $i = 1, \dots, d$.

DEFINITION 2.1. *A distribution F on $\mathbb{R}_+^d$ is called subexponential if*

$$tF(\underline{b}(t) + \cdot) \overset{v}{\longrightarrow} \nu \tag{2.5}$$

and

$$tF*F(\underline{b}(t) + \cdot) \overset{v}{\longrightarrow} 2\nu,$$

where ν is a finite measure concentrated on the $2^d - 1$ points in $\Xi = \{-\infty, \infty\}^d - \{(-\infty, -\infty, \ldots, -\infty)\}$ and satisfying

$$\sum_{(x_1, \ldots, x_d) \in \Xi, x_i = +\infty} \nu\{(x_1, \ldots, x_d)\} > 0$$

for every $i = 1, \ldots, d$.

In the one-dimensional case this definition reduces to the usual definition of subexponentiality in terms of distribution functions (Cline and Resnick (1991)).

The following example shows that multivariate subexponentiality is not necessarily preserved when we multiply componentwise independent random vectors in $\mathbb{R}^2$, one with a subexponential distributions and the other one bounded. Compare this fact with Corollary 2.5 above.

Example 2.3. Let $\underline{X} = (X_1, X_2)$ have distribution function F satisfying

$$P(X_1 > x_1, X_2 > x_2) = \frac{1 + \gamma \sin\left(\log(1 + x_1 + x_2)\right) \sin\left(\pi \frac{x_1 - x_2}{1 + x_1 + x_2}\right)}{1 + x_1 + x_2}$$

for $x_1 > 0$, $x_2 > 0$, and $0 < |\gamma| \le 1/12$. Cline and Resnick (1991) exhibit this distribution and show that it is subexponential.

Let $\underline{Y} = (Y_2, Y_2)$ be a random vector independent of $\underline{X}$ such that

$$P(\underline{Y} = (1,1)) = P(\underline{Y} = (2,1)) = \frac{1}{2}$$

and let $\underline{Z} = (X_1 Y_1, X_2 Y_2)$. We contend that the distribution of $\underline{Z}$ is not subexponential. Indeed, denoting $Z_i = X_i Y_i, i = 1, 2$, it is obvious that

$$P(Z_1 > z) \sim \frac{3}{2} z^{-1} \qquad \text{as } z \to \infty,$$

$$P(Z_2 > z) \sim z^{-1} \qquad \text{as } z \to \infty.$$

It follows then from Proposition 4.2 of Cline and Resnick (1991) that if the distribution of $\underline{Z}$ were subexponential, it must satisfy (2.5) with $b_i(t) = c_i t$, $i = 1, 2$, for some $c_1 > 0$, $c_2 > 0$. That is, $tP(Z_1 > z_1 + c_1 t, Z_z > z_2 + c_2 t)$ must converge to a limit as $t \to \infty$. However,

$$tP(Z_1 > z_1 + c_1 t, Z_2 > z_2 + c_2 t) - A(t) \to 0,$$

where

$$\begin{aligned}
A(t) = &\frac{1}{2}\left(\frac{1}{c_1 + c_2} + \frac{1}{c_1/2 + c_2}\right) \\
&+ \frac{\gamma}{2}\sin(\log t)\left[\frac{\sin(\theta_1)\cos(\log(c_1 + c_2))}{c_1 + c_2} + \frac{\sin(\theta_2)\cos(\log(c_1/2 + c_2))}{c_1/2 + c_2}\right] \\
&+ \frac{\gamma}{2}\cos(\log t)\left[\frac{\sin(\theta_1)\sin(\log(c_1 + c_2))}{c_1 + c_2} + \frac{\sin(\theta_2)\sin(\log(c_1/2 + c_2))}{c_1/2 + c_2}\right]
\end{aligned}$$

and

$$\theta_1 = \pi \frac{c_1 - c_2}{c_1 + c_2} \neq \theta_2 = \pi \frac{c_1/2 - c_2}{c_1/2 + c_2}.$$

We therefore must have the coefficients of both $\sin(\log t)$ and $\cos(\log t)$ to be equal to 0. Since it is straightforward to check that no choice of $c_1 > 0$ and $c_2 > 0$ will ensure that, our argument is complete.

3. Closure properties of subclasses of $\mathcal{S}$.

Because subexponentiality is a difficult property to characterize we consider in this section subclasses of $\mathcal{S}$ which, being more easily characterized, lead to more refined results. We will look at classes whose tails have the regular variation property or one of its extensions.

As before, we are principally interested in two questions: a) if F is in some class $\mathcal{F}$, what conditions on G ensure that H is in $\mathcal{F}$? and b) in particular, is $\mathcal{F}$ closed under the product convolution? A related question is the so-called factorization problem: how can $\overline{H}(t)$ be approximated with a "relatively simple" expression (such as a linear combination) of $\overline{F}(t)$ and $\overline{G}(t)$? This is a much harder problem, difficult even in the situation of regularly varying tails (Cline, 1986) and we only consider certain special cases here.

These questions have been thoroughly studied in the case of regularly varying tails (Breiman (1965), Embrechts and Goldie (1980), Cline (1986)). An investigation of situations involving the extensions of regular variation leads to the general conclusion that the behavior of $\overline{H}$ is determined principally by two features. The first of these is the behavior of the heavier of the two tails, $\overline{F}$ and $\overline{G}$. Thus, if G has light enough tails then H and F are in the same class. The second feature is the behavior of the least "regular" of the two tails. Thus, if F has dominated varying tails and G has regularly varying tails we generally can say only that H has dominated varying tails. There are, however, several special classes such that F's membership implies H's membership regardless of G. (The class with dominated varying tails is one of these.) We begin by defining the classes of interest.

DEFINITION 3.1.

(i) $F \in \mathcal{R}$ if $\overline{F}$ is regular varying, i.e.

$$\lim_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = \lambda^{-\alpha} \qquad \text{for some } \alpha \geq 0, \text{ all } \lambda \geq 1.$$

(ii) $F \in \mathcal{E}$ if $\overline{F}$ is extended regular varying, i.e.

$$\liminf_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} \geq \lambda^{-c} \qquad \text{for some } c \geq 0, \text{ all } \lambda \geq 1.$$

(iii) $F \in \mathcal{I}$ if $\overline{F}$ is intermediate regular varying, i.e.

$$\lim_{\lambda \downarrow 1} \liminf_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = 1.$$

(iv) $F \in \mathcal{D}$ if $\overline{F}$ is dominated varying, i.e.

$$\liminf_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} > 0, \qquad \text{for some } \lambda > 1.$$

For detailed discussion of regular variation, extended regular variation and dominated variation, see Bingham, Goldie and Teugels (1989, Ch. 2-3) (hereafter referred to as BGT). For discussion of intermediate regular variation, see Cline (1991). Due to monotonicity, $F \in \mathcal{I}$ is equivalent to

$$\liminf_{\lambda \downarrow 1, t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = 1. \tag{3.1}$$

For continuous F, this is the defining property of regular oscillation (Berman, 1982, 1988).

From Definition 3.1, it is evident that $R \subset \mathcal{E} \subset \mathcal{I} \subset \mathcal{D}$. These inclusions are proper and furthermore $\mathcal{I} \subset (\mathcal{D} \cap \mathcal{L}) \subset \mathcal{S}$ (Embrechts and Omey, 1984; Cline, 1991). For example, let $\rho(t) = -\log \overline{F}(e^t - 1)$. Then $\overline{F}(t) = \exp(-\rho(\log(1 + t)))$ and

$$
\begin{array}{ll}
F \in \mathcal{R} & \text{if } \rho(t) = t; \\
F \in \mathcal{E} \text{ but } F \notin \mathcal{R} & \text{if } \rho(t) = [t] + (t - [t])^2; \\
F \in \mathcal{I} \text{ but } F \notin \mathcal{E} & \text{if } \rho(t) = [t] + (t - [t])^{1/2}; \\
F \in \mathcal{D} \cap \mathcal{L} \text{ but } F \notin \mathcal{I} & \text{if } \rho(t) = [t] + (t(t - [t]) \wedge 1); \\
F \in \mathcal{D} \text{ but } F \notin \mathcal{L} & \text{if } \rho(t) = [t] + \left(e^t(t - [t]) \wedge 1\right); \\
F \in \mathcal{S} \text{ but } F \notin \mathcal{D} \cap \mathcal{L} & \text{if } \rho(t) = t^2.
\end{array}
$$

Two further subclasses we refer to are the following.

DEFINITION 3.2.

(i) $F \in \mathcal{E}'$ if F is absolutely continuous and $tF'(t)/\overline{F}(t)$ is bounded,

(ii) $F \in \mathcal{I}'$ if F is continuous and $F \in \mathcal{I}$.

That $\mathcal{E}' \subset \mathcal{E}$ follows from the representation theorem for extended regular variation (BGT, Thm 2.2.6). The class $\mathcal{I}'$ is the class of distributions with regularly oscillating tails, i.e. with tail satisfying (3.1) and continuous. Furthermore this is equivalent to "$\log \overline{F}(e^t)$ is uniformly continuous on $[0, \infty)$ and continuous elsewhere" (Berman, 1982).

Associated with $F \in \mathcal{D}$ are the Matuszewska indices of F, $-\alpha_F$ and $-\beta_F$ where $\alpha_F \geq \beta_F \geq 0$, which are the most narrowly defined constants satisfying the following. For every $\epsilon > 0$ there exist C and t_0 so that

$$\frac{\lambda^{-\alpha_F - \epsilon}}{C} \leq \frac{\overline{F}(\lambda t)}{\overline{F}(t)} \leq C \lambda^{\epsilon - \beta_F} \tag{3.2}$$

for all $\lambda \geq 1$, $t \geq t_0$ (cf. definition in BGT, p. 68). These constants may be defined for any F, but $F \in \mathcal{D}$ if and only if $\alpha_F < \infty$.

More precisely, we define the generalized index functions, for $\lambda > 0$,

$$\overline{F}_*(\lambda) = \liminf_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} \qquad \text{and} \qquad \overline{F}^*(\lambda) = \limsup_{t \to \infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)}.$$

If $\overline{F}(t) = 0$ for some finite t, these limits are taken to be 0, 1 and ∞, for $\lambda > 1$, $\lambda = 1$ and $\lambda < 1$, respectively. (Note: $\overline{F}^*(1/\lambda) = (\overline{F}_*(\lambda))^{-1}$). The Matusweska indices may be determined by (BGT, Thm 2.1.5)

$$\alpha_F = \lim_{\lambda \to \infty} \frac{-\log \overline{F}_*(\lambda)}{\log \lambda}; \qquad \beta_F = \lim_{\lambda \to \infty} \frac{-\log \overline{F}^*(\lambda)}{\log \lambda}. \tag{3.3}$$

Also, $F \in \mathcal{I}$ if and only if $\overline{F}_*(\lambda) \uparrow 1$ as $\lambda \downarrow 1$; and $F \in \mathcal{E}$ if and only if $\overline{F}_*(\lambda) \geq \lambda^{-c}$, for some $c < \infty$, all $\lambda \geq 1$. Moreover, there exist the Karamata indices of $\overline{F}$, $-c_F \leq -d_F \leq 0$, for which

$$(1 - \epsilon) \lambda^{-c_F} \leq \frac{\overline{F}(\lambda t)}{\overline{F}(t)} \leq \lambda^{-d_F}(1 + \epsilon)$$

uniformly for $\lambda \in [1, \Lambda]$ and all large t (def. in BGT, pp. 66-67). They may be determined by

$$c_F = \lim_{\lambda \downarrow 1} \frac{-\log \overline{F}_*(\lambda)}{\log \lambda}; \qquad d_F = \lim_{\lambda \downarrow 1} \frac{-\log \overline{F}^*(\lambda)}{\log \lambda} \tag{3.4}$$

and $F \in \mathcal{E}$ if and only $c_F < \infty$. Finally, we have the relationship (BGT, Thm. 2.1.8)

$$\lambda^{-c_F} \leq \overline{F}_*(\lambda) \leq \lambda^{-\alpha_F} \leq \lambda^{-\beta_F} \leq \overline{F}^*(\lambda) \leq \lambda^{-d_F} \leq 1, \quad \lambda > 1. \tag{3.5}$$

As much as possible we derive results in terms of the index functions. To this end, also let $\overline{G}^*$, $\overline{G}_*$, $\overline{H}^*$ and $\overline{H}_*$ be the corresponding index functions of G and H. These functions are nondecreasing but not necessarily continuous.

First, we try to get a handle on $\overline{H}_*$ and $\overline{H}_*$. To this end define,

$$m_{F,G} = \lim_{s \to \infty} \liminf_{t \to \infty} \int_s^\infty \frac{\overline{F}(t/y)}{\overline{H}(t)} G(dy)$$

and for $\lambda > 1$,

$$R_{F,G}(\lambda) = \limsup_{s \to \infty} \limsup_{t \to \infty} \int_s^{\lambda s} \frac{\overline{F}(t/y)}{\overline{H}(t)} G(dy),$$

with similar definitions for $m_{G,F}$ and $R_{G,F}(\lambda)$. Note that each of these values is in $[0,1]$. Further points are given in the next lemma.

LEMMA 3.1.

    (i) $1/2 \leq m_{F,G} \vee m_{G,F} \leq 1 \leq m_{F,G} + m_{G,F}.$

    (ii) For any $\lambda > 1$, $\bigl(R_{F,G}(\lambda) \wedge R_{G,F}(\lambda)\bigr) \leq 1/2.$

    (iii) If $\overline{G}(t) = o(\overline{H}(bt))$ for all $b > 0$, or if $H \in \mathcal{D}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for some $b > 0$, then $m_{G,F} = 1$ and $R_{G,F}(\lambda) = 0.$

PROOF:

    (i) Since, for $t > s^2$,

$$\int_s^\infty \overline{F}(t/y)G(dy) + \int_s^\infty \overline{G}(t/y)F(dy) = \overline{H}(t) + \int_s^{t/s} \overline{F}(t/y)G(dy) + \overline{F}(s)\overline{G}(t/s)$$

we immediately have $m_{F,G} + m_{G,F} \geq 1$. This in turn implies $m_{F,G} \vee m_{G,F} \geq 1/2$ and we have already noted that each value is bounded above by 1.

    (ii) Using (i),

$$\bigl(R_{F,G}(\lambda) \wedge R_{G,F}(\lambda)\bigr) \leq (1 - m_{F,G}) \wedge (1 - m_{G,F}) \leq 1/2.$$

    (iii) Let $\lambda > 1$. First suppose $\overline{G}(t) = o(\overline{H}(bt))$ for all $b > 0$.

$$
\begin{aligned}
1 \geq m_{G,F} &= 1 - \lim_{s \to \infty} \limsup_{t \to \infty} \int_0^s \frac{\overline{G}(t/x)}{\overline{H}(t)} F(dx) \\
&\geq 1 - \lim_{s \to \infty} \lim_{t \to \infty} \frac{\overline{G}(t/s)}{\overline{H}(t)} \\
&= 1.
\end{aligned}
$$

    Secondly, suppose $H \in \mathcal{D}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for some $b > 0$. Then $\overline{H}(bt)/\overline{H}(b_1 t)$ is bounded for any fixed $b_1 > 0$. Hence, the first condition holds and $m_{G,F} = 1$.

    Finally, $R_{G,F}(\lambda) \leq 1 - m_{G,F} = 0.$ □

LEMMA 3.2. For any F and G and for each $\lambda > 1$,

    (i)

$$
\begin{aligned}
\overline{H}_*(\lambda) &\geq \begin{cases} m_{G,F}\overline{F}_*(\lambda) + (1 - m_{G,F})\overline{G}_*(\lambda), & \overline{F}_*(\lambda) \geq \overline{G}_*(\lambda), \\ (1 - m_{F,G})\overline{F}_*(\lambda) + m_{F,G}\overline{G}_*(\lambda), & \overline{F}_*(\lambda) < \overline{G}_*(\lambda), \end{cases} \\
&\geq \overline{F}_*(\lambda) \wedge \overline{G}_*(\lambda)
\end{aligned}
$$

and

    (ii)

$$
\begin{aligned}
\overline{H}^*(\lambda) &\leq \bigl(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda)\bigr) + \bigl(R_{F,G}(\lambda) \wedge R_{G,F}(\lambda)\bigr)\bigl(\overline{F}^*(\lambda) \wedge \overline{G}^*(\lambda)\bigr) \\
&\leq \bigl(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda)\bigr) + \frac{1}{2}\bigl(\overline{F}^*(\lambda) \wedge \overline{G}^*(\lambda)\bigr).
\end{aligned}
$$

PROOF: We prove (i) when $\overline{F}^*(\lambda) \leq \overline{G}^*(\lambda)$. Express $\overline{H}(t)$ in the convenient form, with arbitrary $s > 0$,

$$\overline{H}(t) = \int_0^s \overline{F}(t/y)G(dy) + \int_0^{t/s} \overline{G}(t/x)F(dx) + \overline{F}(t/s)\overline{G}(s).$$

Let $\lambda > 1$ and $\epsilon > 0$. For large enough s and t/s,

$$\overline{H}(\lambda t) = \int_0^{\lambda s} \overline{F}(\lambda t/y)G(dy) + \int_0^{t/s} \overline{G}(\lambda t/x)F(dx) + \overline{F}(t/s)\overline{G}(\lambda s) \qquad (3.6)$$

$$\geq (1 - \epsilon)\left[\overline{F}_*(\lambda)\int_0^{\lambda s}\overline{F}(t/y)G(dy) + \overline{G}_*(\lambda)\int_s^\infty \overline{F}(t/y)G(dy)\right]$$

$$\geq (1 - \epsilon)\left[\overline{F}_*(\lambda)\overline{H}(t) + (\overline{G}_*(\lambda) - \overline{F}_*(\lambda))\int_s^\infty \overline{F}(t/y)G(dy)\right].$$

Since s, and then ϵ, are arbitrary, it follows that

$$\overline{H}_*(\lambda) = \liminf_{t\to\infty}\frac{\overline{H}(\lambda t)}{\overline{H}(t)} \geq \overline{F}_*(\lambda) + (\overline{G}_*(\lambda) - \overline{F}_*(\lambda))\,m_{F,G},$$

as was to be shown.

(ii) Let $\lambda > 1$ and $\epsilon > 0$. From (3.6) we have, for large enough s and t/s,

$$\overline{H}(\lambda t) \leq (1 + \epsilon)\left[\left(\overline{F}^*(\lambda) \vee \epsilon\right)\int_0^{\lambda s}\overline{F}(t/y)G(dy) + \left(\overline{G}^*(\lambda) \vee \epsilon\right)\int_s^\infty \overline{F}(t/y)G(dy)\right]$$

$$\leq (1 + \epsilon)\left(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda) \vee \epsilon\right)\overline{H}(t)$$

$$+ (1 + \epsilon)\left(\left(\overline{F}^*(\lambda) \wedge \overline{G}^*(\lambda)\right) \vee \epsilon\right)\int_s^{\lambda s}\overline{F}(t/y)G(dy).$$

Hence,

$$\overline{H}^*(\lambda) = \limsup_{t\to\infty}\frac{\overline{H}(\lambda t)}{\overline{H}(t)} \leq \left(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda)\right) + R_{F,G}(\lambda)\left(\overline{F}^*(\lambda) \wedge \overline{G}^*(\lambda)\right).$$

Likewise,

$$\overline{H}^*(\lambda) \leq \left(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda)\right) + R_{G,F}(\lambda)\left(\overline{F}^*(\lambda) \wedge \overline{G}^*(\lambda)\right).$$

The second inequality holds by Lemma 3.1(ii). □

We now state and prove our theorems, one each for the classes $\mathcal{D}$, $\mathcal{I}$, and $\mathcal{E}$. Recall we assume both F and G are not degenerate at 0.

THEOREM 3.3.

    (i) *For any F and G,*

$$(\beta_F \wedge \beta_G) \leq \beta_H \leq \alpha_H \leq (\alpha_F \wedge \alpha_G).$$

    (ii) $F \in \mathcal{D} \implies H \in \mathcal{D}.$

    (iii) *If $F \in \mathcal{D}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for some $b > 0$ then for each $\lambda > 1$,*

$$\overline{F}_*(\lambda) \leq \overline{H}_*(\lambda) \leq \overline{H}^*(\lambda) \leq \overline{F}^*(\lambda). \tag{3.7}$$

    (iv) *If $F \in \mathcal{D}$ and $EY^{\alpha_F + \epsilon} < \infty$ for some $\epsilon > 0$, then (3.7) holds and*

$$0 < E\left[\overline{F}_*(1/Y)\right] \leq \liminf_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \leq \limsup_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \leq E\left[\overline{F}^*(1/Y)\right] < \infty.$$

Remark. Theorems 2.2 and 3.3 together imply that the subexponential subclass $\mathcal{D} \cap \mathcal{L}$ is closed under products.

PROOF:

    (i) Using (3.3) and Lemma 3.2(ii),

$$\begin{aligned}
\beta_H &= \lim_{\lambda \to \infty} \frac{-\log \overline{H}^*(\lambda)}{\log \lambda} \\
&\geq \lim_{\lambda \to \infty} \frac{-\log\left(\frac{3}{2}\left(\overline{F}^*(\lambda) \vee \overline{G}^*(\lambda)\right)\right)}{\log \lambda} \\
&= \beta_F \wedge \beta_G.
\end{aligned}$$

Likewise, we can show $\alpha_H \leq \alpha_F \vee \alpha_G$ but in fact we want to show more. In order to accomplish this, we resort to the representation for distributions in $\mathcal{D}$. It suffices to show $\alpha_H \leq \alpha_F$ in the case α_F is finite. Let $\alpha > \alpha_F$. By the representation theorem for O-regularly varying functions (BGT, Thm. 2.2.7),

$$-\log \overline{F}(t) = \eta_F(t) + \int_0^t \frac{\zeta_F(u)}{u} du,$$

where η_F is bounded and $\zeta_F(t) \leq \alpha$. Furthermore, since $\overline{F}$ is bounded and monotone, we may in fact choose η_F and ζ_F so that the latter is nonnegative. (This is evident from the proof of the representation theorem.) Now let

$$\rho_F(t) = \int_0^t \frac{\zeta_F(u)}{u} du$$

and

$$\overline{H}_0(t) = \int_0^\infty e^{-\rho_F(t/y)} G(dy).$$

Then we note that we may represent $-\log \overline{H}(t) = \eta_H(t) + \int_0^t \frac{\zeta_H(u)}{u} du$, where

$$\eta_H(t) = -\log\left(\overline{H}(t)/\overline{H}_0(t)\right)$$

is bounded and

$$\zeta_H(t) = -\frac{t\overline{H}_0'(t)}{\overline{H}_0(t)}$$
$$= \frac{1}{\overline{H}_0(t)} \int_0^\infty \zeta_F(t/y) e^{-\rho_F(t/y)} G(dy) \in [0, \alpha].$$

This shows that satisfies the representation for $\mathcal{D}$ with some $\alpha_H \le \alpha$. Since α may be chosen at will in (α_F, ∞), we conclude $\alpha_H \le \alpha_F$.

(ii) This follows immediately from (i), since $\alpha_H \le \alpha_F < \infty$.

(iii) By (ii), $H \in \mathcal{D}$. From the proof of Lemma 3.1(iii), we have that $\overline{G}(t) = o(\overline{H}(bt))$ for all $b > 0$ and not just for some $b > 0$.

Fix any $\lambda > 0$ and choose t_0 so large that $\overline{F}(\lambda t) \le (1 + \epsilon)\overline{F}^*(\lambda)\overline{F}(t)$ when $t \ge t_0$. Choose $t_1 > t_0$ so that $\overline{G}(t/t_0) \le \epsilon\overline{H}(t)$ when $t \ge t_1$. Then for such t,

$$\overline{H}(\lambda t) \le \int_0^{t/t_0} \overline{F}(\lambda t/y) G(dy) + \overline{G}(t/t_o)$$
$$\le (1 + \epsilon)\overline{F}^*(\lambda) \int_0^{t/t_0} \overline{F}(t/y) G(dy) + \epsilon\overline{H}(t)$$
$$\le \left((1 + \epsilon)\overline{F}^*(\lambda) + \epsilon\right) \overline{H}(t).$$

Thus, $\overline{H}^*(\lambda) \le \overline{F}^*(\lambda)$. As this is true for any $\lambda > 0$, we also have $\overline{H}_*(\lambda) \ge \overline{F}_*(\lambda)$.

(iv) Let $\alpha \in (\alpha_F, \alpha_F + \epsilon)$. Then $t^\alpha \overline{G}(t) \to 0$ and $t^\alpha \overline{F}(t) \to \infty$ (follows from BGT, Prop. 2.2.1). Hence $\overline{G}(t) = o(\overline{F}(bt))$, all $b > 0$. This is sufficient for the condition in part (iii), so (3.7) holds.

Since $F \in \mathcal{D}$ and G is not degenerate at 0, it must be that $E\left[\overline{F}_*(1/Y)\right] > 0$. The lower bound follows by Fatou's lemma,

$$\liminf_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \ge \int_0^\infty \overline{F}_*(1/y) G(dy).$$

To obtain the upper bound we first use (3.2). For any $\epsilon' \in (0, \epsilon)$, there is $C < \infty$ and t_0 such that

$$\frac{\overline{F}(t/y)}{\overline{F}(t)} \le \begin{cases} Cy^{\beta_F - \epsilon'}, & \text{if } y \le 1, t \ge t_0 \\ Cy^{\alpha_F + \epsilon'}, & \text{if } 1 \le y \le t/t_0, t \ge t_0. \end{cases}$$

Hence $\overline{F}^*(1/y) \leq C\left(y^{\alpha_F + \epsilon'} \vee y^{\beta_F - \epsilon'}\right)$ so that $E\left[\overline{F}^*(1/Y)\right] < \infty$. Furthermore, when $t \geq t_0$,

$$\overline{H}(t) \leq \int_0^1 \overline{F}(t/y)G(dy) + \int_1^{t/t_0} \overline{F}(t/y)G(dy) + \overline{G}(t/t_0)$$

$$\leq C\left[\int_0^1 y^{\beta_F - \epsilon'}G(dy) + \int_1^\infty y^{\alpha_F + \epsilon'}G(dy) + \overline{G}(t/t_0)/\overline{F}(t)\right]\overline{F}(t).$$

Since $\overline{G}(t/t_0) = o(\overline{F}(t))$, as $t \to \infty$, we conclude by dominated convergence that

$$\limsup_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \leq \limsup_{t \to \infty} \int_0^{t/t_0} \frac{\overline{F}(t/y)}{\overline{F}(t)}G(dy)$$

$$\leq \int_0^\infty \overline{F}^*(1/y)G(dy).$$

□

THEOREM 3.4.

(i) $F, G \in \mathcal{I} \implies H \in \mathcal{I}$.

(ii) $F \in \mathcal{I}$ and $G(t) = o(H(bt))$ for some $b > 0$ implies $H \in \mathcal{I}$.

(iii) $F \in \mathcal{I}' \implies H \in \mathcal{I}'$.

PROOF:

(i) The assumption is that $\overline{F}_*(\lambda) \uparrow 1$ and $\overline{G}_*(\lambda) \uparrow 1$ as $\lambda \downarrow 1$. By Lemma 3.2(i), this implies $\overline{H}_*(\lambda) \uparrow 1$ as $\lambda \downarrow 1$. Hence $H \in \mathcal{I}$.

(ii) By Theorem 3.3(iii), $\overline{F}_*(\lambda) \leq \overline{H}_*(\lambda)$. Thus, $\overline{F}_*(\lambda) \uparrow 1$ implies $\overline{H}_*(\lambda) \uparrow 1$ and $H \in \mathcal{I}$.

(iii) As noted after Definition 3.2, $F \in \mathcal{I}'$ is equivalent to $\log \overline{F}(e^t)$ is uniformly continuous on $[0, \infty)$ and continuous elsewhere. We must therefore show that $\log \overline{H}(e^t)$ shares this property. The assumption on F is the same as: for each $\epsilon > 0$ there is $\delta > 0$ so that $|\lambda - 1| \leq \delta$ implies

$$\left|\frac{\overline{F}(\lambda t)}{\overline{F}(t)} - 1\right| < \epsilon \qquad \text{for all } t \geq 0.$$

By this,

$$1 - \epsilon \leq \frac{\int_0^\infty \overline{F}(\lambda t/y)G(dy)}{\int_0^\infty \overline{F}(t/y)G(dy)} \leq 1 + \epsilon$$

whenever $|\lambda - 1| \leq \delta$. And this demonstrates that H has the desired property. □

THEOREM 3.5.

(i) For any F and G, $c_H \leq c_F \vee c_G$.

(ii) $F, G \in \mathcal{E} \implies H \in \mathcal{E}$.

(iii) $F \in \mathcal{E}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for some $b > 0$ then $H \in \mathcal{E}$ and

$$d_F \leq d_H \leq c_H \leq c_F. \tag{3.8}$$

(iv) $F \in \mathcal{E}' \implies H \in \mathcal{E}'$.

(v) If $F \in \mathcal{E}$ and $EY^{c_F + \epsilon} < \infty$ for some $\epsilon > 0$, then (3.8) holds and

$$E\left[Y^{c_F} \wedge Y^{d_F}\right] \leq \liminf_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \leq \limsup_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} \leq E\left[Y^{c_F} \vee Y^{d_F}\right]. \tag{3.9}$$

Furthermore, there exist $\eta_F(t)$ and $\zeta_F(t)$ such that

$$-\log \overline{F}(t) = \eta_F(t) + \int_0^t \frac{\zeta_F(u)}{u} du \tag{3.10}$$

where $\eta_F(t) \to m \in \mathbb{R}$, as $t \to \infty$, and ζ_F is bounded. If in addition ζ_F is slowly varying, then

$$\lim_{t \to \infty} \frac{\overline{F}(t) E\left[Y^{\zeta_F(t)}\right]}{\overline{H}(t)} = 1. \tag{3.11}$$

Remark. We owe the idea for (3.11) to Berman (1991, Thm. 3.1) who assumes F and G are continuously differentiable and whose result is expressed in terms of the density of $\log X + \log Y$. We have also weakened his conditions on F and G in other ways.

PROOF:

    (i) Using (3.4) and Lemma 3.2(i),

$$\begin{aligned}
c_H &= \lim_{\lambda \downarrow 1} \frac{-\log \overline{H}_*(\lambda)}{\log \lambda} \\
&\leq \lim_{\lambda \downarrow 1} \frac{-\log \left(\overline{F}_*(\lambda) \wedge \overline{G}_*(\lambda)\right)}{\log \lambda} \\
&= c_F \vee c_G.
\end{aligned}$$

    (ii) This follows immediately from (i).

    (iii) By Theorem 3.3(iii),

$$\overline{F}_*(\lambda) \leq \overline{H}_*(\lambda) \leq \overline{H}^*(\lambda) \leq \overline{F}^*(\lambda).$$

Thus (3.8), follows from this by applying (3.4) as in part (i).

    (iv) By assumption, $\zeta_F(t) = tF'(t)/\overline{F}(t)$ exists and is bounded, say by c. Then H' exists and

$$\begin{aligned}
\frac{tH'(t)}{\overline{H}(t)} &= \frac{1}{\overline{H}(t)} \int_0^\infty \zeta_F(t/y) \overline{F}(t/y) G(dy) \\
&\leq c.
\end{aligned}$$

Hence $H \in \mathcal{E}'$.

(v) Let $c \in (c_F, c_F + \epsilon)$. Then $t^c \overline{G}(t) \to 0$ and $t^c \overline{F}(t) \to \infty$ (follows from **BGT**, Prop. 2.2.3). Hence $\overline{G}(t) = o(\overline{F}(bt))$, for all $b > 0$ which is sufficient for $\overline{G}(t) = o(\overline{H}(bt))$, all $b > 0$. By part (iii), this ensures (3.8). Since $\overline{F}^*(1/y) = \left(\overline{F}_*(y)\right)^{-1} \le y^{c_F}$ for $y > 1$ by (3.5) and since $\mathcal{E} \subset \mathcal{D}$, then Theorem 3.3(iv) applies and (3.9) is immediate.

The representation (3.10) is provided by BGT (Thm. 2.2.6).

Now assume ζ_F is slowly varying. This implies

$$\lim_{t \to \infty} \left(\int_{t/y}^{t} \frac{\zeta_F(u)}{u} du - (\log y)\zeta_F(t) \right) = 0, \tag{3.12}$$

uniformly for compact y-sets in $(0, \infty)$. With the assumption of convergence for η_F,

$$\lim_{t \to \infty} \left| \frac{\overline{F}(t/y)}{\overline{F}(t)} y^{-\zeta_F(t)} - 1 \right| = \lim_{t \to \infty} \left| \exp\left(\eta_F(t) - \eta_F(t/y) + \int_{t/y}^{t} \frac{\zeta_F(u)}{u} du - (\log y)\zeta_F(t) \right) - 1 \right|$$
$$= 0. \tag{3.13}$$

From (3.12) and BGT (Thm. 2.2.6),

$$\limsup_{t \to \infty} \zeta_F(t) = \lim_{\lambda \downarrow 1} \limsup_{t \to \infty} \frac{1}{\lambda - 1} \int_{t}^{\lambda t} \frac{\zeta_F(u)}{u} du = c_F.$$

Since ζ_F is also eventually positive (to be slowly varying) choose $t_0 > 1$ so that $0 \le \zeta_F(t) \le (c_F + \epsilon)$ and $|\eta_F(t) - m| \le \epsilon$ for all $t \ge t_0$. Then it follows that, for some $C \in (0, \infty)$ and all $t \ge t_0$,

$$\frac{1}{C} < E\left[Y^{\zeta_F(t)} \right] < C.$$

It also follows, if both $t \ge t_0^2$ and $y \le t/t_0$,

$$\frac{\overline{F}(t/y)}{\overline{F}(t)} \bigvee y^{\zeta_F(t)} = \exp\left(\eta_F(t) - \eta_F(t/y) + \int_{t/y}^{t} \frac{\zeta_F(u)}{u} du \right) \bigvee y^{\zeta_F(t)}$$
$$\le e^{2\epsilon} y^{c_F + \epsilon}.$$

With this bound and with (3.13), dominated convergence is allowed for

$$\lim_{t \to \infty} \int_{0}^{t/t_0} \left| \frac{\overline{F}(t/y)}{\overline{F}(t)} - y^{\zeta_F(t)} \right| G(dy) = 0.$$

Finally,

$$\left| \frac{\overline{H}(t)}{\overline{F}(t)} - E\left[Y^{\zeta_F(t)} \right] \right| \le \int_{0}^{t/t_0} \left| \frac{\overline{F}(t/y)}{\overline{F}(t)} - y^{\zeta_F(t)} \right| G(dy) + \frac{\overline{G}(t/t_0)}{\overline{F}(t)} + E\left[Y^{\zeta_F(t)} 1_{Y > t/t_0} \right].$$

The first term, we have just shown, vanishes as $t \to \infty$. The second term also vanishes by a remark made above and the third term clearly follows suit. Therefore,

$$\lim_{t \to \infty} \left| \frac{\overline{H}(t)}{\overline{F}(t)} - E\left[Y^{\zeta_F(t)}\right] \right| = 0.$$

Since $E\left[Y^{\zeta_F(t)}\right]$ is bounded away from 0, (3.11) is proven. $\qquad\qquad\qquad \square$

As a corollary, we provide the results for $\mathcal{R}$. All three are well-known, though the second is a slight extension of the result by Embrechts and Goldie (1980) discussed in the introduction. Note that $F \in \mathcal{R}$ means $\alpha_F = \beta_F = c_F = d_F < \infty$ and is implied by $c_F = d_F < \infty$.

COROLLARY 3.6.

    (i) (Embrechts and Goldie (1980)). $F, G \in \mathcal{R} \implies H \in \mathcal{R}$ with $\alpha_H = \alpha_F \wedge \alpha_G$.

    (ii) (Embrechts and Goldie (1980)). If $F \in \mathcal{R}$ and $\overline{G}(t) = o(\overline{H}(bt))$ for some $b > 0$, then $H \in \mathcal{R}$ and $\alpha_H = \alpha_F$.

    (iii) (Breiman (1965)). If $F \in \mathcal{R}$ and $E\left[Y^{\alpha_F + \epsilon}\right] < \infty$ for some $\epsilon > 0$, then

$$\lim_{t \to \infty} \frac{\overline{H}(t)}{\overline{F}(t)} = E\left[Y^{\alpha_F}\right].$$

PROOF:

    (i) If $\alpha_F < \alpha_G$ then it is well known that $\overline{G}(t) = o(\overline{F}(bt))$, for all $b > 0$, and the condition for part (ii) holds. Thus $\alpha_H = \alpha_F$. Likewise, $\alpha_H = \alpha_G$ if $\alpha_G < \alpha_F$.

    Otherwise, assume $\alpha_F = \alpha_G$. Let F_* be the measure satisfying $F_*(x, \infty) = \overline{F}_*(x) = x^{-\alpha_F}$ and give a similar definition to G_*. By a double application of Fatou's lemma,

$$\liminf_{s \to \infty} \liminf_{t \to \infty} \frac{\overline{H}(t)}{\overline{G}(s)\overline{F}(t/s)} \geq \liminf_{s \to \infty} \frac{1}{\overline{G}(s)} \int_0^\infty \overline{F}_*(s/y) G(dy)$$

$$= \liminf_{s \to \infty} \int_0^\infty \frac{\overline{G}(s/x)}{\overline{G}(s)} F_*(dx)$$

$$\geq \int_0^\infty \overline{G}_*(1/x) F_*(dx)$$

$$= \infty.$$

Thus

$$R_{F,G}(\lambda) = \limsup_{s \to \infty} \limsup_{t \to \infty} \int_s^{\lambda s} \frac{\overline{F}(t/y)}{\overline{H}(t)} G(dy)$$

$$\leq \limsup_{s \to \infty} \limsup_{t \to \infty} \frac{\overline{F}(t/\lambda s)}{\overline{F}(t/s)} \frac{\overline{F}(t/s)\overline{G}(s)}{\overline{H}(t)}$$

$$= 0.$$

Since $\alpha_F = \beta_F = \alpha_G = \beta_G$ then Lemma 3.2 says,

$$\lambda^{-\alpha_F} = \left(\overline{F}_*(\lambda) \wedge \overline{G}_*(\lambda)\right) \leq \overline{H}_*(\lambda) \leq \overline{H}^*(\lambda) \leq \left(\overline{F}_*(\lambda) \vee \overline{G}_*(\lambda)\right) = \lambda^{-\alpha_F}.$$

Therefore $H \in \mathcal{R}$ with $\alpha_H = \alpha_F$.

(ii) By Theorem 3.5(iii),

$$\alpha_F = d_F \leq d_H \leq c_H \leq c_F = \alpha_F.$$

Since $d_H = c_H$ then $H \in \mathcal{R}$.

(iii) This follows from Theorem 3.5(v) since $\alpha_F = c_F = d_F$. $\qquad\square$

References.

Athreya, K. B. and Ney, P. E. (1972). *Branching Processes*, Springer-Verlag, New York.

Balkema, A. A. and de Haan, L. (1972). On R. von Mises condition for the domain of attraction of $\exp\{e^{-x}\}$, *Ann. Math. Stat.* 43, 1352–1354.

Berman, S. M. (1982). Sojourns and extremes of a diffusion process on a fixed interval, *Adv. Appl. Prob.* 14, 811–832.

Berman, S. M. (1988). Extreme sojourns of diffusion processes, *Ann. Prob. 16*, 361–374.

Berman, S. M. (1991). The tail of the convolution of densities and its application to a model of HIV-latency time, *Adv. Appl. Prob.*, to appear.

Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1989). Regular Variation. *Encyclopedia of Mathematics and its Applications*, paperback with additions. Cambridge University Press, Cambridge, UK.

Breiman, L. (1965). On some limit theorems similar to the arc-sine law, *Theor. Prob. Appl.* 10, 323–331.

Chistyakov, V. P. (1964). A theorem on sums of independent positive random variables and its applications to branching random processes, *Theor. Prob. Appl.* 9, 640–648.

Cline, D. B. H. (1986). Convolution tails, product tails and domains of attraction, *Prob. Th. Rel. Fields* 72, 529–557.

Cline, D. B. H. (1987). Convolutions of distributions with exponential and subexponential tails, *J. Austral. Math. Soc.* (Series A) 43, 347–365.

Cline, D. B. H. (1989). Consistency for least squares regression estimators with infinite variance, *J. Stat. Plan. Infer.* 23, 163–179.

Cline, D. B. H. (1991). Intermediate regular- and Π-variation, Tech. Rpt., Dept. of Stat., Texas A&M University.

Cline, D. B. H. and Hsing, T. (1990). Large deviation probabilities for sums of random variables with heavy or subexponential tails, Tech. Rpt., Texas A&M Univeresity.

Cline, D. B. H. and Resnick, S. I. (1991). Multivariate subexponential distributions, *Stoch. Proc. Appl.*, to appear.

Chover, J., Ney, P. and Wainger, S. (1973a). Functions of probability measures, *J. Analyse Math.* 26, 255–302.

Chover, J., Ney, P. and Wainger, S. (1973b). Degeneracy properties of subcritical branching processes, *Ann. Prob.* 1, 663–673.

Davis, R. A. and Resnick, S. I. (1985a). Limit theory for moving averages of random variables with regularly varying tail probabilities, *Ann. Prob.* 13, 179–195.

Davis, R. A. and Resnick, S. I. (1985b). More limit theory for the sample correlation function of moving averages, *Stoch. Proc. Appl.* 20, 257–279.

Davis, R. A. and Resnick, S. I. (1986). Limit theory for the sample covariance and correlation functions of moving averages, *Ann. Stat.* 14, 533–558.

Embrechts, P. and Goldie, C. M. (1980). On closure and factorization properties of subexponential distributions, *J. Austral. Math. Soc.* (Series A) 29, 243–256.

Embrechts, P. and Goldie, C. M. (1982). On convolution tails, *Stoch. Proc. Appl.* 13, 263–278.

Embrechts, P. and Omey, E. (1984). A property of long tailed distributions, *J. Appl. Prob.* 21, 80–87.

Goldie, C. M. and Resnick, S. I. (1988). Distributions that are both exponential and in the domain of attraction of an extreme-value distribution, *Adv. Appl. Prob.* 20, 706–718.

Leslie, J. R. (1989). On the non-closure under convolution of the subexponential family, *J. Appl. Prob.* 26, 58–66.

Klüppelberg, C. (1988). Subexponential distributions and integrated tails, *J. Appl. Prob.* 25, 132–141.

Pakes, A. G. (1975). On the tails of waiting-time distributions, *J. Appl. Prob.* 12, 555–564.

Pinelis, I. F. (1985). Asymptotic equivalence of the probabilities of large deviations for sums and maxima of independent random variables (in Russian), *Limit Theorems of Probability Theory,* ed. A. A. Borovkov, Trudy Inst. Mat. 5, "Nauka" Sibirsk. Otdel. Novosibirsk, 144–173.

Pitman, E. I. G. (1980). Subexponential distribution functions, *J. Austral. Math. Soc.* (Series A) 29, 337–343.

Rosinski, J. and Samorodnitsky, G. (1991). Distributions of subadditive functionals of sample paths of infinitely divisible processes, Tech. Rpt., Math. Sciences Inst., Cornell University.

Teugels, J. L. (1975). The class of subexponential distributions, *Ann. Prob.* 1, 1001–1011.