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**TIME VARIANT LOT SIZING MODELS FOR
THE WAREHOUSE SCHEDULING PROBLEM**

by

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ABSTRACT

We consider the problem of scheduling the delivery of n products into a warehouse with limited space under the assumptions of continuous demands at constant rates, infinite horizon, and no backorders. The delivery schedule is described by a cyclic schedule with time varying lot sizes. The order frequencies and the order sequence are assumed to be given. We formulate a linear program that determines delivery times relative to the cycle length to minimize the relative maximum space used and show that the optimal solution is characterized by filling the warehouse at each order. We bound the optimal solution using worst case analysis and give conditions under which the linear program has the same optimal solution as a quadratic program that minimizes the holding cost. Under general conditions, we derive a bound on the cost penalty that results when using the optimal solution of the linear program as a solution to the quadratic program. Finally, we complete a solution to the nonlinear lot-sizing model by determining the best cycle length corresponding to the solution to the linear program and present a bound on the quality of this solution.

INTRODUCTION

High volume consumer products assembly plants are taking aggressive steps to reduce inventories and to reduce cycle times. Investments in warehouses to support these plants are seen as exercises in waste because large inventories are not needed if the production plans are stable and the delivery system is efficient. Managers of plants with existing warehouses have been known to paint lines on the floor and forbid the use of space beyond the painted limits to store inventory. The current philosophy in such plant is that production rates should be stable and the delivery of components and sub-assembly to the assembly line should be frequent. Our paper focuses on the development of cyclic delivery schedules to such an assembly line to satisfy the constraint of limited space and with the goal of optimally balancing the fixed delivery costs with the inventory holding costs. It is therefore directly relevant to a topic of current concern in the manufacturing sector.

We refer to the problem of scheduling deliveries into a warehouse to minimize inventory holding and order costs subject to a space constraint as the Warehouse Scheduling Problem (WSP).

Many authors studied WSP under different assumptions. Holt [12], Hadley and Whitin [7], Parsons [19], Johnson and Montgomery [15], Love [16], and Tersine [21] ignored the scheduling aspect of the problem and considered order sizing exclusively. Homer [13], Page and Paul [18], Zoller [22], and Hall [8] considered the scheduling aspect but restricted the reorder intervals to be equal for all products.

Hadley and Whitin [7], applied the Lagrangian Multipliers Technique, LMT, as a solution procedure to solve the problem when the scheduling aspect is ignored. This procedure minimizes the long run average inventory costs subject to the warehouse space limitation constraint. The constraint formulation allows for the possibility of receiving simultaneously all the parts at one point in time. Consequently, the maximum space of the

LMT solution is an upper bound on the optimal value obtained by any other technique.

In 1966, Homer renewed interest in the subject by considering the scheduling aspect of the problem. He used the Common Cycle, CC, approach (Elmaghraby [3]) where the reorder intervals are restricted to be equal for all parts. In addition, he determined the best way of spreading out the deliveries over the common cycle length to avoid simultaneous deliveries so as to minimize the maximum space required. Page and Paul [18] showed that the CC yields a better utilization of the warehouse space than LMT. However, the CC does not always produce a better cost solution than LMT. They refined the CC by developing a grouping heuristic where the parts within each group share the same reorder interval. After computing the maximum space used for each group, they used LMT to size the cycle of each group under the assumption that the maximum space used across all groups occurs simultaneously. Goyal [6] showed that the Page and Paul refined approach can be further improved by allowing parts to be ordered more than once. The same result of the Common Cycle approach with staggering was also obtained, independently, by Zoller [22], and by Hall [8]. Hall included in his cost function the cost related to the storage space used. Hartley and Thomas [11] solved the WSP to optimality for the case of two products using the staggering technique without the restriction of equal reorder intervals. Recently, Anily [1] derived a worst-case lower bound for the CC approach using the same cost function as Hall. She developed a dynamic grouping heuristic that generates a solution within 41% of the optimal solution in the worst case. Gallego, Queyranne, and Simchi-Levi [5] established the same bound for the Hall / Anily problem and showed that it applies to all feasible policies. They also designed a time invariant lot sizing algorithm that is sequence independent and compared their algorithm to the algorithm of our companion paper, Hariga and Jackson [10] and to a lower bound that they establish. This worst case lower bound is a factor of 2 and considers all feasible policies. Gallego, Shaw, and Simchi-Levi [4] showed that WSP is NP hard in the strong sense.

In this paper, we consider cyclic delivery schedules and focus on the problem of determining the timing or time-phasing of the deliveries within a cycle to minimize the maximum space used in the warehouse. We assume that the order frequencies within the cycle (the number of orders per cycle for each product) and the order sequence are given. In a companion paper (Hariga and Jackson [10]), we present algorithms to determine order frequencies and sequence. One of the algorithms presented there relies on the results of this paper.

The central assumption of this paper is that we are free to choose any delivery times that satisfy the given sequence. A more typical assumption in the literature of this problem has been to require that the intervals between deliveries of the same product be identical. Stated another way, since we assume constant demand and the so-called *Zero Switch Rule*, ZSR (Maxwell [17]), it is more typical to insist that the lot size for a product order be the same each time the product is ordered. However, Dobson [2] has noted in studying the related Economic Lot Scheduling Problem that by relaxing this restriction the problem becomes much simpler. We also have observed that the problem of determining delivery times to minimize the maximum space used will be simplified if lot sizes are allowed to be time variant. We prove that a single matrix inversion is all that is required. This follows from proving that in an optimal schedule with time variant lot sizes, the warehouse is full immediately after each delivery. We relate this problem to the time invariant lot size case (equal reorder interval) by establishing conditions under which the optimal schedule naturally exhibits time invariant lot sizes. There are managerial advantages to restricting attention to time-invariant lot sizes. For this reason, we devote another companion paper (Jackson and Hariga [14]) to studying the problem with the time invariant lot size restriction. A manager could thus compare the results of these two approaches and decide if the benefits of time variant lot sizes outweigh the difficulty of implementing them.

In this paper, we also consider the problem of determining the cycle length to minimize

average annual inventory ordering and holding costs subject to the constraint on warehouse space available. We propose restricting attention to the space minimizing delivery schedule in solving this problem and establish bounds on the quality of the resulting solution similar to those found by Dobson for the Economic Lot Scheduling Problem.

Notation

The following are the symbols used for the terms mentioned against each.

- n = number of products to be ordered;
- m = total number of orders to be placed;
- m_i = number of orders for product i , $i=1, \dots, n$;
- λ_i = demand rate for product i in unit space/unit time, $i=1, \dots, n$;
- h_i = holding cost rate for product i in \$/unit space/unit time, $i=1, \dots, n$;
- K_i = setup cost(ordering cost) in \$/order, $i=1, \dots, n$.
- V = warehouse space available;
- Z_{ij}^a = inventory of product i immediately after delivery of the j th order;
- Z_j = total inventory immediately after delivery of the j th order;
- S_{ij} = time interval until product i is ordered again for the first time after the j th order;
- W = maximum space used(since the demand rate is expressed in unit space per unit time, maximum space used and maximum inventory will be used interchangeably throughout this paper);
- U_j = time interval between j th and $(j+1)$ st order, interpreted cyclically;
- T_j = next reorder interval of the product ordered on the j th order;
- τ = cycle length;
- w = W/τ , the relative maximum inventory;
- u_j = U_j/τ , the relative time interval between j th and $(j+1)$ st order;
- t_j = T_j/τ , the relative reorder interval of the product ordered on the j th order;

$$\delta_{ij} = \begin{cases} 1 & \text{if product } i \text{ is ordered on the } j\text{th order,} \\ 0 & \text{otherwise; and} \end{cases}$$

$$\delta_{ijk} = \begin{cases} 1 & \text{if the next order of product } i \text{ after the } j\text{th order is on the } k\text{th order,} \\ 0 & \text{otherwise.} \end{cases}$$

The demand rates and holding cost rates have been expressed using units of space. Conversion from units of product to units of space is a trivial matter given the space required per unit of product.

Linear Programming Model

The objective function to be minimized for the LP model is the maximum inventory, i.e.

$$W = \text{Max} \left\{ \sum_{i=1}^n Z_{ij}^a : j=1, \dots, m \right\}. \quad (1)$$

The initial inventory for each product must be set so that backorders will not be incurred. Since the ZSR is assumed, the stock of any product on the j th order should be enough to last until its next order. Thus, Z_{ij}^a can be written as:

$$Z_{ij}^a = \lambda_i S_{ij}, \quad (2)$$

and the total inventory on the j th order is:

$$Z_j = \sum_{i=1}^n \lambda_i S_{ij}. \quad (3)$$

Now let

$$\Delta_{ijk} = \sum_{l=j+1}^k \delta_{ijl}.$$

Throughout this paper, such summations are to be interpreted in a cyclic fashion. That is, if $k \leq j$, then

$$\Delta_{ijk} = \sum_{l=j+1}^m \delta_{ijl} + \sum_{l=1}^k \delta_{ijl}.$$

Note that $\Delta_{ijk}=1$ if and only if the next order of product i after the j th order is one of $\{j+1, j+2, \dots, k\}$, interpreted cyclically if $k \leq j$. Otherwise $\Delta_{ijk}=0$. Using this fact, S_{ij} can be expressed as:

$$S_{ij} = U_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk}) U_k. \quad (4)$$

To see this, suppose $\delta_{ijj'}=1$, i.e. the first time product i is delivered after the j th order is on the j' th order. In this case, $\Delta_{ijk}=0$ for $k=j+1, j+2, \dots, j'-1$ and $\Delta_{ijk}=1$ for $k=j', j'+1, \dots, j$. Hence, $S_{ij} = U_j + U_{j+1} + \dots + U_{j'-1}$, as desired. To simplify the above expression of S_{ij} , we let γ_{ijk} be defined as follows:

$$\gamma_{ijk} = \begin{cases} 1 & \text{if } j=k, \\ 1 - \Delta_{ijk} & \text{if } j \neq k. \end{cases}$$

It follows that:

$$S_{ij} = \sum_{k=1}^m \gamma_{ijk} U_k. \quad (4')$$

By definition, T_j is the time interval between the j th order and the next order of the product delivered on the j th order; i.e.

$$T_j = \sum_{i=1}^n \delta_{ij} S_{ij}, \quad j = 1, \dots, m. \quad (5)$$

Moreover, the reorder intervals for each product should sum to τ .

$$\sum_{j=1}^m \delta_{ij} T_j = \tau, \quad i=1, \dots, n. \quad (6)$$

Next, define

$$\begin{aligned}
A_{jk} &= \sum_{i=1}^n \lambda_i \gamma_{ijk} \quad \text{for } j,k = 1,\dots,m, \\
L_{jk} &= \sum_{i=1}^n \delta_{ij} \gamma_{ijk} \quad \text{for } j,k = 1,\dots,m, \text{ and} \\
R_{ij} &= \delta_{ij} \quad \text{for } i = 1,\dots,n, \text{ and } j = 1,\dots,m.
\end{aligned}$$

Then, the above expressions can be rewritten as:

$$Z_j = \sum_{k=1}^m A_{jk} U_k, \quad j=1,\dots,m, \quad (7)$$

$$T_j = \sum_{k=1}^m L_{jk} U_k, \quad j=1,\dots,m, \quad (8)$$

$$\sum_{j=1}^m R_{ij} T_j = \tau, \quad i=1,\dots,n. \quad (9)$$

Let A , L , R , Z , U , and T be the matrices consisting of elements A_{jk} , L_{jk} , R_{ij} , Z_j , U_j , and T_j , respectively. Z , U , and T are column vectors. In this section, we assume the cycle length, τ , is given and we work with the relative decision vectors $z = Z / \tau$, $u = U / \tau$, and $t = T / \tau$. Let e_m and e_n denote m and n column vectors of 1's, respectively. Then the problem that minimizes the maximum inventory, when written in terms of the relative maximum inventory, w , and the vector of relative time intervals, u , can be expressed as:

LP:

$$\begin{aligned}
&\text{Min } w & (10) \\
&\text{s.t}
\end{aligned}$$

$$A u - w e_m \leq 0, \quad (11)$$

$$-L u + t = 0, \quad (12)$$

$$R t - e_n = 0, \quad (13)$$

$$u \geq 0.$$

Lemma 1: The above linear program can be simplified to:

SLP:

$$\text{Min } w \quad (10)$$

s.t

$$A u - w e_m \leq 0, \quad (11)$$

$$e_m' u = 1, \quad (14)$$

$$u \geq 0.$$

Proof: Substitution of (12) into (13) yields $R L u = e_n$. Moreover, the elements of the matrix $(R L)$ are given by:

$$(RL)_{ij} = \sum_{l=1}^m R_{il} L_{lj} = \sum_{\{l: \delta_{il}=1\}} L_{lj} = 1,$$

since $R_{il} = 1$ if product i is ordered on the l th order, and the rows of L that correspond to the orders on which product i is delivered partition the columns of L . Therefore, the system of equations $R L u = e_n$ is composed of n identical equations $e_m' u = 1$. $\square$

Finally, it is clear that the minimum value of w is positive, since the matrix A is a nonnegative matrix with positive diagonal elements all equal to the sum of the demand rates over all products. Then, if we let $x = u/w$, (14) becomes

$$e_m' x = w^{-1}.$$

Moreover, note that minimizing w is equivalent to maximizing w^{-1} . Thus, the above linear program can be transformed to:

TLP:

$$\text{Min } e_m' x \quad (15)$$

s.t

$$A x \leq e_m, \quad (16)$$

$$x \geq 0.$$

The next theorem gives a characterization of the optimal solution that minimizes the maximum inventory.

Theorem 1: The schedule that minimizes the maximum inventory fills the warehouse at each order; ie,

$$Z_1 = Z_2 = \dots = Z_m . \quad (17)$$

Proof : The most intuitive proof relies on marginal analysis for which many cases have to be examined. We illustrate the proof for only one case to convey the essential idea.

Suppose that the total inventories that correspond to the optimal schedule do not satisfy (17). Then, let:

k be the order number such that $Z_k = \text{Min} \{ Z_j : j=1, \dots, m \}$;

i be the number of the product ordered on the kth order;

$y^{(k)}$ be the ordering time of the kth order derived from the optimal schedule.

Now, consider another schedule where the kth order is placed at time $y^{(k)} - \epsilon$, where ϵ is a very small real number.

Suppose that product i is ordered only once, then

$$Z_j^{\text{new}} = Z_j - \epsilon \lambda_i \text{ for all } j \text{ such that } j \neq k,$$

$$\text{and } Z_k^{\text{new}} = Z_k + \epsilon \sum_{r=1, r \neq i}^n \lambda_r .$$

$$\text{So, for a small } \epsilon, \text{Max}_j(Z_j^{\text{new}}) < \text{Max}_j(Z_j),$$

which contradicts that $\{Z_j : j=1, \dots, m\}$ is optimal.

Similar arguments hold when product i is ordered more than once per cycle. An alternative proof of the theorem using the theory of M-matrices can be found in Hariga [9]. $\square$

By the above theorem, the optimal schedule that minimizes the maximum inventory can be obtained without using the simplex method to solve LP. The following Corollary gives the explicit optimal solution of LP.

Corollary 1: The optimal vector of relative time intervals, u , is uniquely given by:

$$u^* = \frac{A^{-1} e_m}{e_m' A^{-1} e_m}, \quad (18)$$

and the optimal relative maximum inventory is:

$$w^* = \frac{1}{e_m' A^{-1} e_m}, \quad (19)$$

Proof: By Theorem 1, the optimal policy is to fill the warehouse at each order. Hariga[9] established the nonsingularity of A as well as the nonnegativity of A and its inverse. Thus, the optimal solution to TLP corresponds to binding constraints; i.e.

$A x^* = e_m$, or equivalently, $x^* = A^{-1} e_m$.

Moreover, since $(w^*)^{-1} = e_m' x^*$, then $w^* = (e_m' A^{-1} e_m)$, and

$u^* = w^* x^* = (A^{-1} e_m) / (e_m' A^{-1} e_m)$. $\square$

The following Lemmas and Propositions are also a consequence of Theorem 1.

Lemma 2: No more than one order can be placed at any point in time during the cycle when minimizing the maximum inventory; i.e. $u^* > 0$.

Proof: The proof is trivial since A^{-1} is nonnegative and nonsingular. Therefore, $u^* > 0$. Alternatively, observe that if the warehouse is filled after each order, then the second of two orders occurring simultaneously must be for a zero quantity. Under the ZSR and positive demand rates, this is not allowed.

Proposition 1: The average warehouse utilization rate, ρ , is given by:

$$\rho = 1 - \frac{1}{2} \frac{\sum_{j=1}^m \sum_{i=1}^n \delta_{ij} (\lambda_i t_j^*)^2}{w^* \lambda_s}, \quad (20)$$

where t_j^* is the optimal reorder interval of the product ordered on the j th order ($t^* = L u^*$) and λ_s is the total demand rate of all products.

Proof: Using the result of Theorem 1, the reduction in space after the j th order should be equal to the space needed for the product ordered on the $(j+1)$ st order (see Figure 1). Thus

$$u_j^* \lambda_s = t_{j+1}^* \sum_{i=1}^n \lambda_i \delta_{i,j+1}, \quad j=1, \dots, m, \quad (21)$$

where (21) is to be interpreted cyclically. Therefore, the shaded area of unused capacity between the j th order and the $(j+1)$ st order is:

$$\frac{1}{2} u_j^* \left(t_{j+1}^* \sum_{i=1}^n \lambda_i \delta_{i,j+1} \right) = \frac{1}{2} \frac{\left(t_{j+1}^* \sum_{i=1}^n \lambda_i \delta_{i,j+1} \right)^2}{\lambda_s}. \quad (22)$$

Hence, the total space used during the cycle of length 1 is:

$$w^* - \frac{1}{2 \lambda_s} \sum_{j=1}^m \sum_{i=1}^n \delta_{ij} (\lambda_i t_j^*)^2.$$

Finally, (20) is obtained by dividing the above expression by w^* . $\square$

Figure 1. Total inventory for an optimal schedule of LP

In the next proposition, an explicit formula for the optimal relative maximum inventory will be derived using the above proposition.

Proposition 2: The optimal relative maximum inventory is given by:

$$w^* = \frac{1}{2} \sum_{j=1}^m \left(\frac{\sum_{i=1}^n \lambda_i^2 \delta_{ij}}{\lambda_s} + \sum_{i=1}^n \lambda_i \delta_{ij} \right) (t_j^*)^2. \quad (23)$$

Proof: Using the definition of the warehouse space utilization as the ratio of the average

inventory to the maximum inventory, the warehouse space utilization can be rewritten as:

$$\rho = \frac{\frac{1}{2} \sum_{j=1}^m \left(\sum_{i=1}^n \lambda_i \delta_{ij} \right) (t_j^*)^2}{w^*}, \quad (24)$$

where the numerator is the average inventory. Then, equating (24) to (20) yields (23). $\square$

In the following Lemma, the results of Homer[13], Page and Paul[18], Zoller[22], and Hall[8] will be derived using the above formula.

Lemma 3: If each product is ordered only once during the cycle, then

$$u_j^* = \frac{\sum_{i=1}^n \lambda_i \delta_{ij+1}}{\lambda_s}, \quad j=1, \dots, m, \quad (25)$$

and

$$w^* = \frac{1}{2} \left(\lambda_s + \frac{\sum_{i=1}^n \lambda_i^2}{\lambda_s} \right). \quad (26)$$

Proof: If $n=m$, then

$$t_j = 1, \quad j=1, \dots, m, \quad (27)$$

and (25) and (26) are obtained by substituting (27) into (21) and (23) respectively. $\square$

We now proceed to derive a lower bound for the relative maximum inventory. (See also Anily [1]. Gallego, Queyranne, and Simchi-Levi [5] generalized this theorem to apply to all policies.)

Theorem 2: The relative maximum inventory is bounded from below by:

$$w^{(lb)} = \frac{1}{2} \sum_{i=1}^n \left[\frac{\lambda_i^2}{\lambda_s} + \lambda_i \right] \frac{1}{m_i}.$$

Proof : The relative maximum inventory given by (23) can be rewritten as:

$$w^* = \frac{1}{2} \sum_{i=1}^n \left[\left(\frac{\lambda_i^2}{\lambda_s} + \lambda_i \right) \sum_{j=1}^m (t_j^*)^2 \delta_{ij} \right].$$

Now consider the following nonlinear problem.

$$\begin{aligned}
& \text{Min } \sum_{j=1}^m t_j^2 \delta_{ij} \\
& \text{s.t.} \\
& \sum_{j=1}^m t_j \delta_{ij} = 1.
\end{aligned}$$

The optimal solution of this nonlinear problem is given by ($t_j = 1/m_i$ such that $\delta_{ij}=1$).

Therefore, $\sum_{j=1}^m (t_j^*)^2 \delta_{ij} \geq \sum_{j=1}^m \left(\frac{1}{m_i}\right)^2 \delta_{ij} = \frac{1}{m_i}$, and $w^* \geq w^{(lb)}$. $\square$

Note that $w^{(lb)}$ is a lower bound over all possible sequences with the same number of replenishments for each product since it is sequence independent. In Theorem 4, we give some conditions under which this lower bound is achieved. Next, we provide a worst case analysis for the relative maximum inventory by comparing it to the above lower bound.

For any order j at which product i is ordered, the optimal relative reorder interval obtained from the solution of the linear program can be written as:

$$t_j^* = \frac{1}{m_i} + \theta_{ij} \text{ for all } j \text{ such that } \delta_{ij}=1 \text{ and } i=1,\dots,n,$$

$$\text{where } \sum_{j=1}^m \theta_{ij} \delta_{ij} = 0 \text{ for } i=1,\dots,n.$$

Theorem 3: If $|\theta_{ij}| < (\epsilon/m_i)$ for all i and j such that $\delta_{ij}=1$, then the worst-case gap of the relative maximum inventory is bounded by ϵ^2 .

$$\begin{aligned}
\text{Proof : } w^* &= \frac{1}{2} \sum_{i=1}^n \left[\left(\frac{\lambda_i^2}{\lambda_s} + \lambda_i \right) \sum_{j=1}^m \left(\frac{1}{m_i} + \theta_{ij} \right)^2 \delta_{ij} \right] \\
&< \frac{1}{2} \left[\left(\frac{\lambda_i^2}{\lambda_s} + \lambda_i \right) \left(\frac{1}{m_i} + \sum_{j=1}^m \frac{\epsilon^2}{m_i^2} \delta_{ij} \right) \right] = w^{(lb)} + w^{(lb)} \epsilon^2.
\end{aligned}$$

Therefore, $\frac{w^*}{w^{(lb)}} < 1 + \epsilon^2$. $\square$

Quadratic Programming Model

The problem considered in this section differs from LP in that the space capacity is no longer a decision variable and the objective to be minimized is the average holding cost. Since it is assumed that the sequence is given, and therefore the frequencies are fixed, the long run setup cost per unit time is constant for a given cycle length. In this case, the objective function to be minimized is:

$$\frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} T_j^2. \quad (28)$$

Using the notation introduced above, the Quadratic Program, QP, can be formulated as:

QP :

$$\text{Min } T' H T$$

s. t.

$$A U \leq V e_m, \quad (29)$$

$$R T = \tau e_n, \quad (30)$$

$$L U - T = 0, \quad (31)$$

$$U \geq 0, \quad (32)$$

where H is an m by m diagonal matrix with $H_{jj} = \sum_{i=1}^n \delta_{ij} \lambda_i h_i$ and τ is assumed to be

given. Constraints (29) state that the total inventory immediately after the delivery of each order should not exceed the space capacity of the warehouse. The remaining constraints were introduced during the formulation of the linear programming model.

Clearly, the optimal T is unique since H is a positive diagonal matrix. However, U need not be unique since L is of rank $(m-n+1)$.

Proposition 3: A lower bound for QP is given by $\hat{T}' H \hat{T}$, where

$$\hat{T}_j = \sum_{i=1}^n \delta_{ij} \frac{\tau}{m_i} \text{ for } j=1, \dots, m.$$

Proof: The quadratic program $\{\text{Min THT, s.t. } RT = \tau e_n\}$, whose optimal solution is $\hat{T}_j$, constitutes a lower bound for QP. This optimal solution states that the quantity to be replenished for each product i whenever it is ordered is $\tau \lambda_i / m_i$. $\square$

The next two Lemmas and Theorem give conditions under which the same schedule minimizes both the maximum inventory and the average holding cost.

Lemma 4: Suppose that the optimal maximum inventory of LP is equal to the warehouse space available, V , then both LP and QP have the same optimal solutions.

Proof: Suppose that one of the constraints, say the j th constraint, of (29) has a nonzero slack in the optimal solution to QP. Then, the total inventory on the j th order is smaller than $V = \tau w^*$, where w^* is the optimal relative maximum inventory of LP, which contradicts that w^* is optimal. Moreover, by the uniqueness of the optimal solution of LP, (u^*, w^*) , then the two problems have the same optimal solutions. $\square$

Lemma 5: Suppose that the solution to LP yields a schedule with equal lot sizes and that the optimal maximum inventory does not exceed the warehouse space available; that is,

$$U^* = \tau \frac{A^{-1} e_m}{e_m' A^{-1} e_m}, T_j^* = \sum_{i=1}^n \delta_{ij} \frac{\tau}{m_i}, \text{ and } w^* \tau \leq V, \quad (33)$$

then (U^*, T^*) is also optimal for QP.

Proof: T^* is a feasible solution to QP since it is a solution to LP and constraint (29) is satisfied by assumption. Moreover, T^* is an optimal solution to QP by proposition 3 since

$$T^* = \hat{T}_j. \quad \square$$

In the above Lemma, it has been supposed that the solution to LP yielded a schedule with equal lot sizes. This is not always the case. In the remainder of this section, we establish a set of conditions under which the optimal solution to LP exhibits equal reorder intervals.

For notation simplicity, let square brackets around an order number represent the index of the product ordered on that order. For example,

$$\lambda_{[j]} = \sum_{i=1}^n \lambda_i \delta_{ij}.$$

Now, consider the solution given by

$$u_j^s = \frac{\lambda_{[j+1]}}{\lambda_s} \frac{1}{m_{[j+1]}}, \quad j=1, \dots, m. \quad (34)$$

It is easily seen that $\sum_{j=1}^m u_j^s = 1$ and $u_j^s \geq 0$, $j=1, \dots, m$, so that the solution is feasible for

SLP. If this solution exhibits equal reorder intervals then it is the optimal solution to SLP.

To be precise, we have

Proposition 4 : If $t_j^s = 1 / m_{[j]}$ for all j , where $t^s = L u^s$, then u^s is the optimal solution to SLP.

Proof : If $t_j^s = 1 / m_{[j]}$ then (34) implies

$$u_j^s \lambda_s = t_{j+1}^s \lambda_{[j+1]},$$

for all j . That is, the space released between the j th order and the $(j+1)$ st order is exactly equal to the space required by the $(j+1)$ st order, for all j . Hence, each order brings the relative space used in the warehouse back to exactly the same level. By Theorem 1 and the uniqueness of the optimal solution (Corollary 1), this implies that $u^s = u^*$. $\square$

Corollary 2 : Under the conditions of the proposition, $U^s = u^s \tau$ is an optimal solution to QP.

Proof : Immediate from Lemma 5. $\square$

We are led to search for conditions under which the simple solution u^s , given by (34), exhibits equal reorder intervals. The conditions we have identified involve very special restrictions on the order frequencies and order sequence.

Let $m^+ = \text{Max}\{m_i : i=1, \dots, n\}$. If m^+ / m_i is an integer for every product i , then we say that the ordering frequencies are integer-ratio frequencies. A particular class of integer-ratio frequencies are the so called power-of-two frequencies: $m_i = 2^{\alpha_i}$, where α_i is an integer,

$i=1,\dots,n$. (Other integer bases besides 2 could also be used.) The power-of-two frequencies lend themselves to a simple sequencing heuristic, as the Power-of-Two Bin Packing Heuristic, presented in Dobson[2] for the Economic Lot Scheduling Problem and in Hariga and Jackson[10] for the Warehouse Scheduling Problem. Without presenting the heuristic here, we will summarize several key properties of the sequences that it generates. The sequence of orders can be partitioned into m^+ ordered sub-sequences, called bins, such that:

- (P1) Product i appears in m_i distinct bins, for all i ;
- (P2) The bins in which product i is ordered are equally spaced; that is, if b_{ik} is the index of the bin in which the k th order of product i appears, then $b_{i,k+1} = b_{ik} + m^+/m_i$, interpreted cyclically.
- (P3) The set of products ordered before product i is the same in every bin in which product i is ordered, for all i .

As an example of a sequence of product orders that satisfies properties (P1)-(P3), consider the following:

(1, 2, 4, 1, 3, 5, 7, 1, 2, 4, 1, 3, 6, 1, 2, 4, 1, 3, 5, 8, 1, 2, 4, 1, 3, 6).

Observe that $m^+ = m_1 = 8$ and that the 8 sub-sequences that begin with product 1 form bins that satisfy the stated properties. In particular, we verify the properties for product 5. P1: it is ordered exactly twice, once in the second bin (1, 3, 5, 7) and once in the sixth bin (1, 3, 5, 8). P2: it is ordered in a repeated pattern, such that, every fourth bin will contain an order of product 5, beginning with the second bin. P3: in both bins, the order for product 5 is preceded by orders for product 1 and 3.

A consequence of properties (P1)-(P3) is that there exists one product that is ordered, and ordered first, in every bin. For convenience, we assume this product is labelled with index 1, as in the example.

For an order sequence satisfying properties (P1)-(P3), let B_b denote the b th sub-

sequence, or bin, of orders, for $b = 1, \dots, m^+$, represented as an ordered set of product indices. In the example above, $B_3 = (1, 2, 4)$. Let O_b denote the set of order numbers of the b th sub-sequence. In the example $O_3 = (8, 9, 10)$.

Theorem 4 : If the order sequence satisfies properties (P1)-(P3) and if

$$\sum_{i \in B_b} \frac{\lambda_i}{m_i} = \frac{\lambda_s}{m^+} \quad (35)$$

for every order sub-sequence, B_b , $b=1, \dots, m^+$, then u^s exhibits equal reorder intervals and is the optimal solution to SLP.

Proof : For the b th bin, we have

$$\begin{aligned} \sum_{j \in O_b} u_j^s &= \sum_{i \in B_b} \frac{\lambda_i}{\lambda_s} \frac{1}{m_i} \\ &= \frac{1}{m^+}, \end{aligned}$$

by (35). This implies that product 1, the first product in each bin, has equal reorder intervals. For any other product, $i \neq 1$, suppose the k th order of that product is order number l in the overall sequence and the $(k+1)$ st order of that product is order number l' in the overall sequence. Then, it is easily seen that

$$t_l = - \sum_{j \in O_{b_{ik}} \text{ and } j < l} u_j + \frac{m^+}{m_i} \frac{1}{m^+} + \sum_{j \in O_{b_{ik+1}} \text{ and } j < l'} u_j,$$

by (P2) and the result that all reorder intervals for product 1 equal $1/m^+$. By (P3) and (34), the first and third terms in the above expression cancel, leaving $t_l = 1/m_i$, as desired. Since i and k were chosen arbitrarily, the result is proved. $\square$

Theorem 4 motivates the Power-of-two Bin Packing heuristic proposed by Hariga and Jackson[10]. That heuristic generates sequences that satisfy (P1)-(P3) and it attempts to

equalize $\sum_{i \in B_b} \lambda_i/m_i$ across all bins. This is equivalent to attempting to satisfy (35). Thus,

the sequencing heuristic is directed at solving both LP and QP.

Nonlinear Programming Model

In this section, we consider the problem of minimizing the long run inventory costs per unit time subject to the space availability constraints. This problem can be formulated as a nonlinear program, NLP, where the objective function is the sum of the long run ordering (or setup) cost per unit time and the long run inventory holding cost per unit time. The constraints are the same as for the quadratic program formulated in the previous section. However, in this section, the cycle length, τ , is considered as a decision variable. As before, the order frequencies (number of orders per cycle) and the order sequence are given. Then, NLP can be formulated as:

NLP:

$$\begin{aligned} \text{Min } & \frac{1}{\tau} \left(K R e_m + \frac{1}{2} T' H T \right) \\ \text{s.t. } & \begin{aligned} A U & \leq V e_m, \\ L U - T & = 0, \\ e_m' U & = \tau, \\ R T & = \tau e_n, \\ U & \geq 0. \end{aligned} \end{aligned}$$

It is clear that a lower bound for NLP is given by the following problem:

RNLP:

$$\begin{aligned} \text{Min } & \frac{1}{\tau} \left(K R e_m + \frac{1}{2} T' H T \right) \\ \text{s.t. } & R T = \tau e_n. \end{aligned}$$

Lemma 6: The optimal solution to the lower bound problem is given by:

$$T_j^* = \frac{\tau^*}{m_{[j]}}, \quad (36)$$

$$\tau^* = \sqrt{\frac{2 \sum_{i=1}^n m_i K_i}{\sum_{i=1}^n \frac{\lambda_i h_i}{m_i}}}. \quad (37)$$

Proof: The proof is straightforward by showing that (36) and (37) satisfy the Karush-Kuhn-Tucker conditions of NLP. $\square$

As a consequence of the above Lemma, a near optimal solution to NLP can be obtained by the following simple algorithm:

1- Solve LP to get $w^{(LP)}$, $u^{(LP)}$, and $t^{(LP)} = L u^{(LP)}$.

2- Calculate τ^* using (38).

$$\tau^* = \text{Min} \left\{ \sqrt{\frac{2 \sum_{i=1}^n m_i K_i}{\sum_{i=1}^n \frac{\lambda_i h_i}{m_i}}}, \frac{V}{w^{(LP)}} \right\} \quad (38)$$

Note that if $T^{(LP)} \neq T^*$, where $T^{(LP)} = \tau^* t^{(LP)}$, then $T^{(LP)}$ can be written as:

$$T_j^{(LP)} = T_j^* + \theta_{ij} \quad \text{for all } j \text{ such that } [j] = i, \quad i=1, \dots, n, \quad (39)$$

$$\text{where } \sum_{j=1}^m \delta_{ij} \theta_{ij} = 0, \quad i=1, \dots, n. \quad (40)$$

Lemma 7: If $|\theta_{ij}| \leq \epsilon T_j^*$, that is; if the reorder intervals obtained by using the above algorithm differ by at most $100\epsilon\%$ of the optimal reorder intervals of the relaxed problem, RNLP, then the error incurred for NLP is at most $100\epsilon^2\%$.

Proof: Let $C^{(lb)}$ be the lower bound on the long run inventory costs of NLP.

$$C^{(lb)} = \frac{\sum_{i=1}^n m_i K_i}{\tau^*} + \frac{\tau^*}{2} \sum_{i=1}^n \frac{\lambda_i h_i}{m_i}.$$

The long run inventory costs per unit time of NLP using the algorithm is:

$$\frac{1}{\tau^*} \left\{ \sum_{i=1}^n m_i K_i + \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} (T_j^{(LP)})^2 \right\}$$

$$\begin{aligned}
&= \frac{1}{\tau^*} \left\{ \sum_{i=1}^n m_i K_i + \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} (T_j^* + \theta_{ij})^2 \right\} \\
&= C^{(lb)} + \frac{1}{2 \tau^*} \sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} \theta_{ij}^2.
\end{aligned}$$

Dividing the right hand side by $C^{(lb)}$ yields:

$$\begin{aligned}
1 + \frac{\frac{1}{2 \tau^*} \sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} \theta_{ij}^2}{C^{(lb)}} &< 1 + \frac{\sum_{j=1}^m \sum_{i=1}^n \lambda_i h_i \delta_{ij} \theta_{ij}^2}{(\tau^*)^2 \sum_{i=1}^n \frac{\lambda_i h_i}{m_i}} \\
&< 1 + \frac{\sum_{i=1}^n \lambda_i h_i \sum_{j=1}^m \delta_{ij} \epsilon^2 (\tau^*/m_i)^2}{(\tau^*)^2 \sum_{i=1}^n \frac{\lambda_i h_i}{m_i}} \\
&= 1 + \epsilon^2. \quad \square
\end{aligned}$$

Thus, the error cost incurred by using the optimal solution of the time variant lot sizes may not be important. A 20% maximum error of adjustment of the reorder intervals would cause an error of at most 4% for the long run average inventory costs per unit time.

Conclusion

We highlight the following results of this paper. Given the product order frequencies and the order sequence, the problem of minimizing the maximum space used can be formulated as a linear program, LP. The optimal solution to LP is characterized by the property that every order fills the warehouse to the same level of space used. Furthermore, no two orders occur at the same time. A single matrix inversion step is all that is required

to solve LP. Explicit formulas are possible for all variables of interest, including the average warehouse utilization rate. These results generalize the results obtained by Homer, Page and Paul, Zoller, and Hall under the Common Cycle assumption.

The solution to LP can be used to generate a solution to the nonlinear program, NLP, that minimizes the long run inventory cost per unit time subject to the space constraint. Under certain circumstances, the solution to LP has the property of equal lot sizes even though the equal lot sizes restriction is not imposed in LP. In these circumstances, it follows that the solution to LP yields an optimal solution to NLP. These circumstances correspond to achievement of a lower bound to the power-of-two bin packing problem used to generate the ordering sequence. Hence, this result motivates the use of this particular sequencing heuristic.

In other cases, the solution to LP may not have the property of equal lot sizes or equal reorder intervals, but the cost penalty of using the LP solution in NLP is less than $100\epsilon^2\%$ of the optimal NLP solution, where $100\epsilon\%$ is the maximum percentage deviation of the LP solution from equal reorder intervals.

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