SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7501

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RANDOM TRANSFORMATIONS FOR
POISSON PROCESSES AND
SUP-INTEGRAL PROCESSES

by

L. de Haan* and Sidney Resnick**

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Abstract

Consider a two dimensional Poisson process with points $\{(T_k, X_k)\}$ and mean measure $dt \cdot x^{-2} dx$ and a random function $\{F(t), t > 0\}$ non-anticipating in the sense that $F(t)$ depends on points for $T_k < t$. Then the randomly transformed point process with points

$$\{(\int_0^{T_k} F(u) du, F(T_k)X_k)\}$$

is still Poisson with the same mean measure. Define the sup-integral process by

$$I(t) = \bigvee_{T_k \leq t} F(T_k)X_k, \quad t > 0.$$

$I(\cdot)$ may be expressed as a randomly time changed extremal process. We investigate tail probabilities of $I(t)$ and give some circumstances where $I(\cdot)$ is a continuous time approximation of solutions of discrete time stochastic difference equations with random coefficients.

1. Introduction

Consider a Poisson process

$$N = \sum_k \epsilon_{(T_k, X_k)}$$

on $[0, \infty) \times (0, \infty]$ with mean measure $dt \cdot x^{-2} dx$. (Notation: For $(t, x) \in [0, \infty) \times (0, \infty]$ and $B \subset [0, \infty) \times (0, \infty]$ we have

$$\epsilon_{(t, x)}(B) = 1_B(t, x) = \begin{cases} 1 & \text{if } (t, x) \in B \\ 0 & \text{otherwise.} \end{cases}$$

This means that for disjoint Borel sets $A_1, \dots, A_j$ we have $N(A_1), \dots, N(A_j)$ independent and $N(A)$ is Poisson distributed with

$$EN(A) = \int \int_A dt x^{-2} dx.$$

Suppose that $\{F(t), t > 0\}$ is a non-negative stochastic process which is non-anticipating in the sense that for each t we have $F(t)$ determined by the realization of the Poisson process up to time t .

(For precise formulations see section 2.) From the ingredients N and F we form the sup-integral process

$$I(t) = \bigvee_{T_k \leq t} F(T_k) X_k.$$

Such processes may be considered as generalizations of max-stable processes (de Haan, 1984; de Haan and Pickands, 1986) and they may also arise as continuous time approximations of discrete time processes which satisfy recursions with random coefficients. This is illustrated in section 5 where we discuss approximating processes $\{\xi_n\}$ which for example may satisfy recursions of the form

$$\xi_{n+1} = A_{n+1} \xi_n \vee B_{n+1},$$

with $\{A_n, B_n\}$ random. Such random coefficient equations arise in modelling utilities and in stochastic hydrology (cf. Gade, 1973 and Helland and Nilsen, 1976).

Associated with the function F is its integral

$$\tau(t) = \int_0^t F(u)du, \quad t \geq 0.$$

If $F(t)$ is non-random for each t , then standard transformation theory for Poisson processes yields that

$$\sum_k \epsilon(\tau(T_k), F(T_k)X_k)$$

is still Poisson with the same mean measure. The main result of section 2 shows this is still true when the function F is random and non-anticipating. This allows the sup-integral process $\{I(t), t > 0\}$ to be expressed as a randomly time changed extremal process and also illuminates the time change formula of Rosinski and Woyczynski (1986) for stochastic integrals with respect to stable motion.

In Section 4, asymptotic formulas for

$$P[I(t) > x]$$

as $x \rightarrow \infty$ are given. These formulas depend on

$$P[\tau(t) > x].$$

In particular when $E\tau(t) < \infty$ we have

$$P[I(t) > x] \sim x^{-1}E\tau(t)$$

as $x \rightarrow \infty$. (In certain cases where $E\tau(t) = \infty$, it is still possible to specify the tail behavior of $P[I(t) > x]$. These results are illustrated for the case

$$F(t) = Y(t-).$$

Another interpretation of $\tau(t)$ in the case that $E\tau(t) < \infty$ is

$$E \sum_k 1_{[T_k \leq t, F(T_k)X_k > \epsilon]} = \epsilon^{-1} E\tau(t).$$

This is proven and used in section 2.

2. Random transformation of a Poisson process

Consider a Poisson process on $[0, \infty) \times (0, \infty]$

$$N = \sum_k \epsilon_{(T_k, X_k)}$$

with mean measure $dt \cdot x^{-2} dx$. Here N is the counting function and the points are $\{(T_k, X_k)\}$. Our notation is

$$\epsilon_{(t,x)}(A) = \begin{cases} 1 & \text{if } (t,x) \in A \\ 0 & \text{if } (t,x) \notin A. \end{cases}$$

Let $\{\mathcal{F}_t\}$ be an increasing family of σ -fields such that

$$N(A) \in \mathcal{F}_t$$

for all $A \in \mathcal{B}([0, t) \times (0, \infty])$ and such that for $B \in \mathcal{B}([t, \infty) \times (0, \infty])$ we have

$$\mathcal{F}_t \text{ and } N(B)$$

independent. Suppose $\{F(t), t > 0\}$ is a measurable non-negative stochastic process such that

$F(t) \in \mathcal{F}_t$ for all $t > 0$; i.e. $\{F(t)\}$ is adapted to $\{\mathcal{F}_t\}$. Define for $t \geq 0$

$$\tau(t) = \int_0^t F(u) du.$$

Then we show the random transformation

$$(t, x) \rightarrow (\tau(t), F(t)x)$$

of the state space of N preserves the distribution of N . (Note that if $\{F(t), t > 0\}$ is a non-random function, then this result is covered by standard transformation theory for Poisson processes. See for example Resnick, 1987, page 134.)

Theorem 2.1. Suppose for each $t > 0$ that $P[\tau(t) < \infty] = 1$ and that

$$\tau(t) \rightarrow \infty \text{ a.s.}$$

as $t \rightarrow \infty$. Then setting

$$N^* := \sum_k \epsilon_{(\tau(T_k), F(T_k)X_k)}$$

we have $N^* \stackrel{d}{=} N$, so that the transformed process N^* is Poisson with mean measure $dt \cdot x^{-2} dx$.

Proof. Suppose $f \in C_K([0, \infty) \times (0, \infty])$ is continuous with compact support. We must prove

$$\begin{aligned} (2.1) \quad E \exp\{-\int f dN^*\} &= E \exp\{-\sum_k f(\tau(T_k), F(T_k)X_k)\} \\ &= \exp\left\{-\int_{[0, \infty) \times (0, \infty]} (1 - e^{-f(t, x)}) dt \cdot x^{-2} dx\right\} \end{aligned}$$

(cf. Resnick, 1987; Kallenberg, 1983; Neveu, 1976). We begin by supposing F is simple; i.e. F is of the form

$$(2.2) \quad F(t) = \sum_{i=0}^{\infty} \phi_i 1_{(t_i, t_{i+1}]}(t)$$

(cf. Rosinski and Woyczynski, 1986) where $0 \leq \phi_i < \infty$, $\phi_i \in \mathfrak{F}_{t_i}$, $0 = t_0 < t_1 < \dots$

and $\lim_{n \rightarrow \infty} t_n = \infty$. For simple F we have

$$(2.3) \quad \tau(t) = \sum_{i=0}^{j-1} \phi_i(t_{i+1} - t_i) + \phi_j(t - t_j) \text{ for } t_j \leq t \leq t_{j+1}.$$

We show for simple F that for $n \geq 1$

$$(2.4) \quad \begin{aligned} & E \exp \left\{ - \sum_k f(\tau(T_k), F(T_k)X_k) 1_{[t_{n-1}, t_n)}(T_k) \mid \mathfrak{F}_{t_{n-1}} \right\} \\ &= \exp \left\{ - \int_{s=\tau(t_{n-1})}^{\tau(t_n)} \int_{x=0}^{\infty} (1 - e^{-f(s,x)}) ds x^{-2} dx \right\}. \end{aligned}$$

To see this, use (2.3) and the left side of (2.4) is

$$E \left(\exp \left\{ - \sum_k f(\tau(t_{n-1}) + \phi_{n-1}(T_k - t_{n-1}), \phi_{n-1}X_k) 1_{[t_{n-1}, t_n)}(T_k) \right\} \mid \mathfrak{F}_{t_{n-1}} \right)$$

and because $\{(T_k, X_k), t_{n-1} < T_k \leq t_n\}$ are Poisson points on $[t_{n-1}, t_n) \times (0, \infty]$ we have the conditional expectation equal to

$$\begin{aligned} & \exp \left\{ - \int_{s=t_{n-1}}^{t_n} \int_{x=0}^{\infty} (1 - e^{-f(\tau(t_{n-1}) + \phi_{n-1}(s-t_{n-1}), \phi_{n-1}x)}) ds x^{-2} dx \right\} \\ &= \exp \left\{ - \int_{s=t_{n-1}}^{t_n} \int_{y=0}^{\infty} (1 - e^{-f(\tau(s), y)}) \phi_{n-1} ds y^{-2} dy \right\} \end{aligned}$$

(where we set $y = \phi_{n-1}x$)

$$= \exp \left\{ - \int_{s=t_{n-1}}^{t_n} \int_{y=0}^{\infty} (1 - e^{-f(\tau(s), y)}) \tau(ds) y^{-2} dy \right\}$$

$$= \exp \left\{ - \int_{u=\tau(t_{n-1})}^{\tau(t_n)} \int_{y=0}^{\infty} \left(1 - e^{-f(u,y)} \right) du y^{-2} dy \right\}$$

which gives (2.4).

From (2.4) it is routine to see that $(n \geq 0)$

$$M_{t_n} := \exp \left\{ - \left[\sum_k f(\tau(T_k), F(T_k)X_k) 1_{[0, t_n)}(T_k) - \int_{s=0}^{\tau(t_n)} \int_{x=0}^{\infty} (1 - e^{-f(s,x)}) ds x^{-2} dx \right] \right\}$$

is an $\{\mathcal{F}_{t_k}\}$ -martingale and a bit of extra work gives in fact that

$$(2.5) \quad M_t = \exp \left\{ - \left[\sum_k f(\tau(T_k), F(T_k)X_k) 1_{[0, t)}(T_k) - \int_{s=0}^{\tau(t)} \int_{x=0}^{\infty} (1 - e^{-f(s,x)}) ds x^{-2} dx \right] \right\}$$

is an $\{\mathcal{F}_t\}$ -martingale. The martingale is uniformly bounded by

$$\begin{aligned} M_t &\leq 1 \cdot \exp \left\{ \int_0^{\infty} \int_0^{\infty} (1 - e^{-f(s,x)}) ds x^{-2} dx \right\} \\ &< \infty \end{aligned}$$

so $M_t \rightarrow M_{\infty}$ a.s. and in L_1 , as $t \rightarrow \infty$. In particular

$$(2.6) \quad 1 = EM_0 = EM_{\infty} = E e^{-\int f dN^*} e^{\int_0^{\infty} \int_0^{\infty} (1 - e^{-f(s,x)}) ds x^{-2} dx}$$

on account of the assumption $\tau(t) \rightarrow \infty$ a.s. This result (2.6) is equivalent to the desired (2.1).

Before proceeding further with the proof of Theorem 2.1 we stop to give an interpretation of the function $\tau(t)$.

Proposition 2.2. Suppose as before that

$$N = \sum_k \epsilon_{(T_k, X_k)}$$

is Poisson on $[0, \infty) \times (0, \infty]$ with mean measure $dt \cdot x^{-2} dx$ and that for every $T > 0$, $F(t, \omega) \in L_1([0, T] \times \Omega)$; w.r.t. $dt \times P$, i.e.

$$(2.7) \quad E \int_0^T F(u) du = E\tau(T) < \infty, \text{ for all } T > 0.$$

Then for any $\epsilon > 0$

$$\begin{aligned} (2.8) \quad & E \sum_k \epsilon_{(T_k, F(T_k)X_k)}([0, T] \times (\epsilon, \infty]) \\ &= E\{\#k: F(T_k)X_k > \epsilon, T_k \leq T\} \\ &= \epsilon^{-1} E\tau(T). \end{aligned}$$

Proof. We may suppose without loss of generality that N , when restricted to $[0, T] \times (0, \infty]$, is of the form

$$\sum_k \epsilon_{(U_k, T\Gamma_k^{-1})}$$

where $(U_k, k \geq 1)$ are iid $U(0, T)$ random variables independent of $\{\Gamma_k^{-1}\}$ and

$$\Gamma_k = E_1 + \dots + E_k$$

where $\{E_n, n \geq 1\}$ are iid unit exponentials (cf. Resnick, 1987, Lemma 4.10).

The proof rests on two formulas valid for $F \in L_1([0, T] \times \Omega)$. The first is

$$(2.9) \quad E(\chi(F(U_k), U_k) \mid U_k = u) = E\chi(F(u), u)$$

for each k , where $\chi: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is bounded and continuous and non-random. If F is simple with form (2.2) we have for Borel $A \subset [0, \infty)$

$$\begin{aligned} & \int_{u \in A} E(\chi(F(U_k), U_k) \mid U_k = u) P[U_k \in du] \\ &= E\chi(F(U_k), U_k) 1_{[U_k \in A]} \\ &= \sum_j E\chi(\phi_j, U_k) 1_{[U_k \in A \cap (t_j, t_{j+1}]]} \end{aligned}$$

and since $\phi_j \in \mathfrak{F}_{t_j}$ is independent of Poisson points in $(t_j, t_{j+1}] \times (0, \infty]$ this becomes

$$\begin{aligned} & \sum_j E \int_{[u \in A \cap (t_j, t_{j+1}]]} \chi(\phi_j, u) P[U_k \in du] \\ &= \sum_j E \int_{[u \in A]} \chi(F(u), u) 1_{(t_j, t_{j+1}]}(u) P[U_k \in du] \\ (2.10) \quad &= \int_{u \in A} E\chi(F(u), u) P[U_k \in du]. \end{aligned}$$

If $F \in L_1([0, T] \times \Omega)$ is not simple, there exist $\{F_n\}$ simple with $F_n \xrightarrow{L_1} F$ in $L_1([0, T] \times \Omega)$ and thus there is some subsequence $F_{n'} \rightarrow F$ ae $dt \times dP$. Thus

$$\begin{aligned} & \int_{[u \in A]} E(\chi(F(U_k), U_k) \mid U_k = u) P[U_k \in du] \\ &= E\chi(F(U_k), U_k) 1_{[U_k \in A]} \\ &= \lim_{n' \rightarrow \infty} E\chi(F_{n'}(U_k), U_k) 1_{[U_k \in A]} \\ &= \lim_{n' \rightarrow \infty} \int_{[u \in A]} E\chi(F_{n'}(u), u) P[U_k \in du] \end{aligned}$$

(from (2.10) and the fact that χ is bounded and continuous)

$$= \int_{[u \in A]} E\chi(F(u), u) P[U_k \in du].$$

Summarizing we have for arbitrary Borel $A \subset [0, \infty)$

$$\begin{aligned} & \int_{[u \in A]} E(\chi(F(U_k), U_k) \mid U_k = u) P[U_k \in du] \\ &= \int_{[u \in A]} E\chi(F(u), u) P[U_k \in du] \end{aligned}$$

whence

$$E(\chi(F(U_k), U_k) \mid U_k = u) = E\chi(F(u), u) \text{ a.e.}$$

This proves (2.9).

The second fact needed is that for each k

$$(2.11) \quad (F(U_k), U_k) \text{ and } \Gamma_k$$

are independent. Let $\chi: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ and $g: [0, \infty) \rightarrow [0, \infty)$ be bounded. If F is simple and given by (2.2) we have

$$\begin{aligned} E\chi(F(U_k), U_k) g(\Gamma_k) &= \sum_j E\chi(F(U_k), U_k) 1_{(t_j, t_{j+1}]}(U_k) g(\Gamma_k) \\ &= \sum_j E\chi(\phi_j, U_k) 1_{(t_j, t_{j+1}]}(U_k) g(\Gamma_k). \end{aligned}$$

Because $\phi_j \in \mathcal{F}_{t_j}$ independent of Poisson points in $(t_j, t_{j+1}] \times (0, \infty]$ and because $\{\Gamma_k\}$ is independent of $\{U_k\}$ we have the above equal to

$$\begin{aligned} & \sum_j E\chi(\phi_j, U_k) 1_{(t_j, t_{j+1}]}(U_k) E g(\Gamma_k) \\ &= \sum_j E\chi(F(U_k), U_k) 1_{(t_j, t_{j+1}]}(U_k) E g(\Gamma_k) \\ &= E\chi(F(U_k), U_k) E g(\Gamma_k), \end{aligned}$$

yielding the desired independence when F is simple. When $F \in L_1([0, T] \times \Omega)$ the result follows by approximation. There exist simple $F_n \rightarrow F$ a.e. $dt \times dP$ and $(F_n(T_k), T_k)$ and X_k are independent. Independence is preserved by taking limits.

Now for the proof of Proposition 2.2 we have

$$\begin{aligned} E \sum_k \mathbb{1}_{(T_k, F(T_k)X_k)}([0, T] \times (\epsilon, \infty]) \\ = \sum_k P[F(T_k)X_k > \epsilon, T_k \leq T] \\ = \sum_{k=1}^{\infty} P[F(U_k) T \Gamma_k^{-1} > \epsilon] \end{aligned}$$

(recall $\{U_k\}$ are iid $U(0, T)$)

$$\begin{aligned} &= \sum_{k=1}^{\infty} P[\Gamma_k \leq TF(U_k)/\epsilon] \\ &= \sum_{k=1}^{\infty} E(P[\Gamma_k \leq TF(U_k)/\epsilon \mid F(U_k), U_k]) \end{aligned}$$

and utilizing (2.11) this is

$$\begin{aligned} &= \sum_{k=1}^{\infty} E \int_0^{TF(U_k)/\epsilon} \frac{e^{-x} x^{k-1}}{(k-1)!} dx \\ &= \sum_{k=1}^{\infty} \int_0^T E \left(\int_0^{TF(U_k)/\epsilon} \frac{e^{-x} x^{k-1}}{(k-1)!} dx \mid U_k = u \right) P[U_k \in du] \end{aligned}$$

whence applying (2.9) we get

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_0^T E \int_0^{TF(u)/\epsilon} \frac{e^{-x} x^{k-1}}{(k-1)!} dx P[U_k \in du] \\ &= \int_0^T E \frac{TF(u)}{\epsilon} P[U_k \in du] \\ &= \frac{1}{\epsilon} E \int_0^T F(u) du \end{aligned}$$

as desired. $\square$

We now return to the proof of Theorem 2.1. Assume $F \in L_1([0, T] \times \Omega)$ for all T . Suppose that the support of $f \in C_k([0, \infty) \times (0, \infty))$ is contained in $[0, t_0] \times (\epsilon, \infty]$. Fix T and choose simple F_n such that $F_n \rightarrow F$ in $L_1([0, T] \times \Omega)$. Since

$$E \sup_{s \leq T} |\tau_n(s) - \tau(s)| \leq E \int_0^T |F_n(u) - F(u)| du \rightarrow 0$$

and there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$(2.12) \quad \sup_{s \leq T} |\tau_{n'}(s) - \tau(s)| \rightarrow 0 \text{ a.s.}$$

$$(2.13) \quad F_{n'} \rightarrow F \text{ ae } dt \Big|_{[0, T]} \times dP.$$

From Proposition 2.2

$$\#\{k: T_k \leq T, F(T_k)X_k > \epsilon\} < \infty$$

a.s. and thus the number of terms in the sum

$$\sum_k f(\tau(T_k), F(T_k)X_k) 1_{[0, T]}(T_k)$$

is finite a.s. Hence as $n' \rightarrow \infty$

$$\exp \left\{ - \left[\sum_k f(\tau_{n'}(T_k), F_{n'}(T_k)X_k) 1_{[0, T]}(T_k) - \int_0^{\tau_{n'}(T)} \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right\}$$

converges almost surely and boundedly to the corresponding expression with n' removed. This shows that the process $\{M_t, \mathcal{F}_t\}$ given in (2.5) is again a martingale when $F \in L_1([0, T] \times \Omega)$ for every $T \geq 0$ and (2.1) follows as in the simple case.

The last step is to assume that F merely satisfies

$$P[\tau(t) < \infty] = 1, \text{ for all } t > 0.$$

We finish the proof by localization. Define

$$F_n(t) = F(t) 1_{[\tau(t) \leq n]}$$

for $n \geq 1$. Then with $\tau_n(t) = \int_0^t F_n(u) du$ we have $E\tau_n(T) < \infty$ for all $T > 0$. Let $M_{n,t}$ be given by (2.5) with F, τ replaced by F_n, τ_n . The previous part of this proof demonstrated that

$$(M_{n,t}, \mathcal{F}_t, t \geq 0)$$

is a martingale. Furthermore for any $t > 0$

$$\begin{aligned} E |M_{n,t} - M_t| &= E |M_{n,t} - M_t| 1_{[\tau(t) \leq n]} + E |M_{n,t} - M_t| 1_{[\tau(t) > n]} \\ &\leq E 0.1 1_{[\tau(t) \leq n]} + 2 \exp \left\{ \int_0^\infty \int_0^\infty (1 - e^{-f(s,x)}) ds x^{-2} dx \right\} P[\tau(t) > n] \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. It follows that $(M_t, \mathcal{F}_t)$ is a martingale also and (2.1) is obtained as in the case F is simple. This finishes the proof. $\square$

In case $P[\tau(t) < \infty] = 1$ for all t but the condition

$$P[\tau(\infty) = \infty] = 1$$

fails we may fix things in a standard way. (Cf. McKean, 1969; Karatzas and Shreve, 1988.) Suppose we have our standard Poisson process $N = \sum_k \epsilon_{(T_k, X_k)}$ with mean measure $dt \cdot x^{-2} dx$ and our random transformation F satisfying

$$P[\tau(t) < \infty] = 1, \text{ for all } t.$$

Take an independent Poisson process $\sum \epsilon_{(S_k, Y_k)}$ with mean measure $dt \cdot x^{-2} dx$ and define the augmented process

$$(2.14) \quad N^+ = \sum_k \epsilon_{(\tau(T_k), F(T_k)X_k)} + \sum_k \epsilon_{(\tau(\infty)+S_k, Y_k)}.$$

Proposition 2.3. The augmented process N^+ on $[0, \infty) \times (0, \infty]$ is Poisson with mean measure $dt \cdot x^{-2} dx$: $N^+ \underline{d} N$.

Proof. We have for $f \in C_K([0, \infty) \times (0, \infty])$ and any $t > 0$

$$E \exp \left\{ - \left[\sum_k f(\tau(T_k), F(T_k)X_k) 1_{[0, t]}(T_k) - \int_0^{\tau(t)} \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right\} = 1$$

and letting $t \rightarrow \infty$ yields

$$(2.15) \quad 1 = E \exp \left\{ - \left[\sum_k f(\tau(T_k), F(T_k)X_k) - \int_0^{\tau(\infty)} \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right\}.$$

Thus (recall $N^* := \sum_k \epsilon_{(\tau(T_k), F(T_k)X_k)}$)

$$(2.16) \quad \begin{aligned} & E \exp \left\{ - \left[\int f dN^+ - \int_0^\infty \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right\} \\ &= E \left(\exp \left\{ - \left[\int f dN^* - \int_0^{\tau(\infty)} \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right. \right. \\ & \quad \left. \left. E \left(\exp \left\{ - \left[\int f d \left(\sum_k \epsilon_{(\tau(\infty)+S_k, Y_k)} \right) - \int_{\tau(\infty)}^\infty \int_0^\infty (1 - e^{-f(s, x)}) ds x^{-2} dx \right] \right\} \mid \mathcal{F}_{\tau(\infty)} \right) \right) \right). \end{aligned}$$

Conditional on $\mathcal{F}_{\tau(\infty)}$ the process $\sum \epsilon_{(\tau(\infty)+S_k, Y_k)}$ is Poisson on $[\tau(\infty), \infty) \times (0, \infty]$ with

mean measure $dt \Big|_{[\tau(\infty), \infty)} x^{-2} dx$. So the left side of (2.16) equals

$$E \exp \left\{ - \int \left[fdN^* - \int_0^{\tau(\infty)} \int_0^\infty (1 - e^{-f(s,x)}) ds x^{-2} dx \right] \right\} \cdot 1$$

which by applying (2.15) equals 1. Thus N^+ has the correct Laplace functional and the result is proved.

3. Stochastic sup-integrals

As usual let $N = \sum_k \epsilon_{(T_k, X_k)}$ be Poisson on $[0, \infty) \times (0, \infty]$ with mean measure $dt \times x^{-2} dx$. Let F be our random transformation satisfying $F(t) \in \mathfrak{F}_t$ and $P[\tau(t) < \infty] = 1$ for all $t > 0$. The sup-integral process $(I(t), t > 0)$ is defined by

$$(3.1) \quad I(t) = \bigvee_{T_k \leq t} F(T_k) X_k.$$

Let N^+ be the augmented Poisson process defined at the end of section 2. If $P[\tau(\infty) = \infty] = 1$, then $N^+ = N^*$. If we represent N^+ as

$$N^+ = \sum \epsilon_{(t_k, j_k)}$$

and define for $t > 0$

$$(3.2) \quad E(t) = \bigvee_{t_k \leq t} j_k,$$

then we get an extremal- Φ_1 process. This is a process satisfying ($t > 0, x \geq 0$)

$$P[E(t) \leq x] = \Phi_1^t(x) = \exp\{-tx^{-1}\}$$

which is stochastically continuous and has its paths in $D[0, \infty)$, the space of right continuous functions on $[0, \infty)$ with finite left hand limits. If we define

$$\tau^{\leftarrow}(s) = \begin{cases} \inf\{t > 0: \tau(t) > s\}, & s < \tau(\infty) \\ \infty & s \geq \tau(\infty) \end{cases}$$

then $\tau^{\leftarrow}$ is non-decreasing, right-continuous on $(0, \tau(\infty))$ with values in $[0, \infty)$ and $\tau(t) \leq s$ iff $t \leq \tau^{-1}(s)$. Thus for $t < \tau(\infty)$

$$\begin{aligned} E(t) &= \bigvee_{\tau(T_k) \leq t} F(T_k)X_k \\ (3.3) \quad &= \bigvee_{T_k \leq \tau^{\leftarrow}(t)} F(T_k)X_k = I(\tau^{\leftarrow}(t)) \end{aligned}$$

so $P[I(t) < \infty, \text{ for all } t < \tau(\infty)] = 1$. Furthermore for any $t > 0$

$$E(\tau(t)) = I(\tau^{\leftarrow}(\tau(t))).$$

If $\tau^{\leftarrow}(\tau(t)) > t$, then $\tau(\tau^{\leftarrow}(\tau(t))) = \tau(t)$ and $F(s, \omega) = 0$ a.e. on $(t, \tau^{\leftarrow}(\tau(t))]$. We now show that for any t :

$$(3.4) \quad t < T_k \leq \tau^{\leftarrow}(\tau(t)) \implies F(T_k)X_k = 0 \text{ a.s.}$$

This last statement is checked by means of the following lemma.

Lemma 3.1. a) If $\tau(t) = 0$ a.s. then $I(t) = 0$ a.s.

b) If $\tau(t_1) = \tau(t_2)$ for some $t_1 < t_2$,

then

$$\bigvee_{T_k \in (t_1, t_2]} F(T_k)X_k = 0, \text{ a.s.}$$

Proof. a) We have

$$\begin{aligned} I(t) &= \bigvee_{T_k \leq t} F(T_k)X_k \leq \bigvee_{\tau(T_k) \leq \tau(t)} F(T_k)X_k \\ &= \bigvee_{k: \tau(T_k)=0} F(T_k)X_k. \end{aligned}$$

Since $\sum_k \epsilon(\tau(T_k), F(T_k)X_k)$ is Poisson, the cardinality of $\{k: \tau(T_k) = 0\}$ is 0 almost surely. The proof of b) is almost the same. $\square$

To verify (3.4) define

$$G(s) = F(s) 1_{(t, \tau \leftarrow (\tau(t))]}(s).$$

Then for every s we have $G(s) \in \mathfrak{F}_s$. For $s \leq t$ this is obvious. For $s > t$

$$\begin{aligned} [s \leq \tau \leftarrow (\tau(t))] &= [\int_t^s F(u)du = 0] \\ &\subset [t < \tau \leftarrow (\tau(t))] \end{aligned}$$

so

$$\begin{aligned} [1_{(t, \tau \leftarrow (\tau(t))]}(s) = 1] &= [1_{[t < \tau \leftarrow (\tau(t))] \cap [s \leq \tau \leftarrow (\tau(t))]} = 1] \\ &= [1_{[\int_t^s F(u)du = 0]} = 1] \in \mathfrak{F}_s. \end{aligned}$$

Furthermore for any T we have

$$\int_0^T G(u)du = 0$$

so by the Lemma

$$\bigvee_{T_k \leq T} G(T_k)X_k = 0 \text{ a.s.}$$

Let $T \rightarrow \infty$ to conclude

$$\bigvee_{T_k < \infty} G(T_k)X_k = \bigvee_{t < T_k \leq \tau \leftarrow (\tau(t))} F(T_k)X_k = 0 \text{ a.s.}$$

Therefore

$$\begin{aligned} I(\tau^{\leftarrow}(\tau(t))) &= \bigvee_{T_k \leq t} F(T_k)X_k \vee \bigvee_{t < T_k \leq \tau^{\leftarrow}(\tau(t))} F(T_k)X_k \\ &= \bigvee_{T_k \leq t} F(T_k)X_k = I(t) \text{ a.s.} \end{aligned}$$

since

$$\bigvee_{t < T_k \leq \tau^{\leftarrow}(\tau(t))} F(T_k)X_k = 0 \text{ a.s.}$$

So for any $t > 0$

$$(3.5) \quad I(t) = E(\tau(t)) \text{ a.s.}$$

The paths of I are in $D[0, \infty)$. To check this let

$$F_n(t) = F(t)1_{[\tau(t) \leq n]}.$$

From section 2, for any ϵ

$$\begin{aligned} &\#\{k: T_k \leq s, F_n(T_k)X_k \geq \epsilon\} \\ &= \#\{k: T_k \leq s, F(T_k)X_k \geq \epsilon\} < \infty \end{aligned}$$

for any $s \leq t$ on the set $[\tau(t) \leq n]$. This implies (Resnick, 1987, p. 181)

$$I_n(s) = \bigvee_{T_k \leq s} F_n(T_k)X_k = \bigvee_{T_k \leq s} F(T_k)X_k = I(s)$$

is right continuous for $s < t$ on the set $[\tau(t) \leq n]$. Let $n \rightarrow \infty$ and conclude $I(s)$ is right continuous for $s \in (0, t)$ on $[\tau(t) < \infty]$. Since we assume $P[\tau(t) < \infty] = 1$ for all t , we get $I \in D[0, \infty)$.

A standard application of (3.5) is as follows. The process

$$\{\log E(t), t > 0\}$$

is an extremal process governed by $\Lambda(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$:

$$P[\log E(t) \leq x] = \Lambda^t(x),$$

and as is well known for such processes

$$\frac{\log E(s)}{\log s} \rightarrow 1$$

a.s. as $s \rightarrow \infty$ (Resnick, 1974). We therefore conclude from (3.5) that

$$\log I(t) \sim \log \tau(t)$$

a.s. as $t \rightarrow \infty$.

4. Tail probabilities of sup-integrals

Consider the sup-integral $I(t)$ defined by (3.1). We consider the asymptotic form, as $x \rightarrow \infty$, of

$$P[I(t) > x].$$

Because of (3.4), this is equivalent to investigating

$$P[E(\tau(t)) > x]$$

as $x \rightarrow \infty$. The method of attack relies on the discussion in Resnick, 1986, section 6.

Theorem 4.1. Let $\{\tau_n, n \geq 1\}$ be iid copies of $\tau(t)$. Suppose

$$\int_0^x P[\tau_1 > u] du = L(x)$$

is slowly varying at infinity so that (Feller, 1970) the weak law of large numbers holds

$$\sum_{j=1}^n \tau_j / a(n) \xrightarrow{P} 1$$

with

$$U(t) = t/L(t), \quad a(t) = U^{\leftarrow}(t).$$

Then as $x \rightarrow \infty$

$$P[I(t) > x] \sim (1/U(x)).$$

In particular, if $E\tau(t) < \infty$ we have as $x \rightarrow \infty$

$$P[I(t) > x] \sim E\tau(t)x^{-1}.$$

Proof. Recall $N^+ = \sum \epsilon_{(t_k, j_k)}$ is Poisson with mean measure $dt \times x^{-2}dx$ and define for $0 < s < t$, $E(s, t] := \bigvee_{t_k \in (s, t]} j_k$ so that $E(t) = E(0, t]$. Create iid copies of $\tau(t)$ as follows. Let $\tau_1 = \tau(t)$ and let τ_2 have the same definition as τ_1 but relative to the process $\sum_k 1_{[t_k > \tau_1]}$ and so on. Then

$$\left\{ E \left(\sum_{i=0}^{j-1} \tau_i, \sum_{i=0}^j \tau_i \right), j \geq 1 \right\}$$

are iid copies of $E(\tau_1) = E(\tau(t)) = I(t)$. Since for any n

$$\frac{E(a(n)\cdot)}{a(n)} \stackrel{d}{=} E(\cdot)$$

in $D[0, \infty)$ and because of the weak law for $\left\{ \sum_1^n \tau_i \right\}$ we get

$$\sum_{i=0}^{[n\cdot]} \tau_i / a(n) \xrightarrow{P} (\cdot),$$

the identity function in $D[0, \infty]$. Billingsley, 1968, Theorem 4.4 gives joint convergence

$$\left(\frac{E(a(n)\cdot)}{a(n)}, \frac{\sum_1^{[n\cdot]} \tau_i}{a(n)} \right) \Rightarrow (E(\cdot), (\cdot)).$$

Composition of the components and evaluation at $t = 1$ yields

$$\frac{E(\sum_1^n \tau_i)}{a(n)} \Rightarrow E(1).$$

Since $E(\sum_1^n \tau_i) = \bigvee_{j=1}^n E(\sum_{i=0}^{j-1} \tau_i, \sum_{i=0}^j \tau_i)$, a maximum of a iid random variables, we conclude

$$\begin{aligned} P[E(\sum_1^n \tau_i) \leq a(n)x] &= P^n[E(\tau_1) \leq a(n)x] \\ &= P^n[I(t) \leq a(n)x] \\ &\rightarrow P[E(1) \leq x] = \Phi_1(x) = e^{-x^{-1}} \end{aligned}$$

whence

$$1/P[I(t) > x] \sim a^{\leftarrow}(x) \sim U(x)$$

as $x \rightarrow \infty$ (cf. de Haan, 1970; Resnick, 1987).

If $E\tau(t) < \infty$ then

$$L(x) \sim E\tau(t), \quad x \rightarrow \infty$$

so

$$U(x) \sim x/E\tau(t)$$

completing the proof. $\square$

Example. Let

$$F(t) = E(t-) = \bigvee_{t_k < t} j_k.$$

Then

$$\tau(t) = \int_0^t E(s-)ds = \int_0^t E(s)ds$$

Observe that for fixed $0 < s \leq 1$

$$P[E(s) > x] = 1 - e^{-sx^{-1}} \sim sx^{-1}$$

as $x \rightarrow \infty$. Thus

$$\int_0^x P[E(s) > u]du = L(x) \sim s \log x \quad \text{as } x \rightarrow \infty.$$

Set $U(x) = \frac{x}{L(x)} \sim \frac{x}{s \log x}$, whence

$$a(x) := U^{\leftarrow}(x) \sim s x \log x.$$

Let $\{E_j(t), j \geq 1, t > 0\}$ be iid copies of $\{E(s), s > 0\}$. The weak law of large numbers gives for each $s > 0$

$$\frac{\sum_{j=1}^n E_j(s)}{a(n)} \xrightarrow{P} 1,$$

i.e.

$$\sum_{j=1}^n \frac{E_j(s)}{n \log n} \xrightarrow{P} s.$$

Since this is true for every s , one readily checks that

$$\frac{\sum_{j=1}^n E_j(\cdot)}{n \log n} \xrightarrow{P} (\cdot),$$

in $D[0, \infty)$ and applying the functional

$$x \in D[0, \infty) \rightarrow \int_0^t x(u) du$$

we get for any $t > 0$

$$\begin{aligned} \frac{\int_0^t \sum_{j=1}^n E_j(u) du}{n \log n} &= \frac{\sum_{j=1}^n \int_0^t E_j(u) du}{n \log n} \\ &\rightarrow \int_0^t u du = \frac{t^2}{2}. \end{aligned}$$

Since $\left\{ \int_0^t E_j(u) du, j \geq 1 \right\}$ are iid replicates of $\tau(t) = \int_0^t E(s) ds$, Theorem 4.1 implies as $x \rightarrow \infty$

$$(\alpha(x) = \frac{t^2}{2} x \log x)$$

$$P[I(t) > x] \sim 1/\alpha^{\leftarrow}(x)$$

$$\sim \frac{t^2 \log x}{2x}.$$

There is an alternative, more explicit way to do this calculation. Observe that

$$\begin{aligned} I(t) &= \bigvee_{T_k \leq t} E(T_k^-) X_k \\ &= \bigvee_{T_k \leq t} \bigvee_{T_j < T_k} X_j X_k = \bigvee_{0 < T_j < T_k \leq t} X_j X_k. \end{aligned}$$

Suppose as in section 2 that N restricted to $[0, t] \times (0, \infty]$ is of the form

$$N = \sum_{k=1}^{\infty} \epsilon_{(tU_k, t\Gamma_k^{-1})}.$$

Then $I(t) = t^2 \Gamma_1^{-1} \Gamma_2^{-1}$ and

$$P[I(t) > x] = P[\Gamma_1^{-1} \Gamma_2^{-1} > t^{-2}x]$$

$$= P[\Gamma_1 \Gamma_2 \leq t^2 x^{-1}]$$

can be computed explicitly by using the fact that

$$P[\Gamma_2 \leq x] = \int_0^x u e^{-u} du$$

and

$$\mathcal{L}(\Gamma_1 | \Gamma_2 = t) = U(0, t).$$

5. Applications and examples

Consider a random sequence $\{Z_n\}_{n=0}^{\infty}$ defined by

$$(5.1) \quad Z_n := (A_n Z_{n-1}) \vee B_n \quad \text{for } n \geq 1,$$

where $\{(A_n, B_n)\}_{n=1}^{\infty}$ is a random sequence with:

$$(5.2) \quad B_1, B_2, \dots \text{ are i.i.d. with distribution function } \exp\{-x^{-1}\} \quad (x > 0),$$

$$A_n > 0 \text{ is independent of } B_n, B_{n+1}, \dots \quad (n = 1, 2, \dots).$$

and Z_0 is some initial random variable.

One can think of Z_n as the utility of, say, the car one owns at time n . When going from time n to time $n+1$ the following changes occur:

1. The utility of the car decreases by a factor A_{n+1} .
2. One gets an offer of an alternative car with utility B_{n+1} .
3. One buys the other car if its utility is higher than the one owned, otherwise one keeps the old one.

Equation (5.1) is similar to a stochastic difference equation considered by Vervaat (1979).

We shall show that if one applies a suitable scaling and lets $n \rightarrow \infty$, the sequence $\{Z_n\}$ converges to a sup-integral process under suitable conditions. Write

$$Y_0 := Z_0$$

$$Y_n := Z_n / (A_1 A_2 \dots A_n) \quad (n = 1, 2, \dots),$$

so that with $C_n := 1/(A_1 A_2 \dots A_n)$

$$Y_n = Y_{n-1} \vee (C_n B_n)$$

$$= Y_0 \vee \bigvee_{i=1}^n (C_i B_i).$$

In order to be able to formulate the right scaling for the limiting procedure, we now change notation to shrink the time scale and apply the scaling at the same time. Imagine we have a sequence of $D[0, \infty)$ valued stochastic processes $\{C_n(t), t \geq 0\}$. We define the point processes

$$N_n = \sum_k \epsilon_{\left(\frac{k}{n}, \frac{B_k}{n}\right)}$$

in $M_P = M_P([0, \infty) \times (0, \infty])$, the space of Radon point measures on $[0, \infty) \times (0, \infty]$, knowing that in M_P

$$(5.3) \quad N_n \Rightarrow N,$$

the Poisson process with mean measure $dt \cdot x^{-2} dx$. We also define the processes

$$(5.4) \quad Y_n(t) = (Y_0 \vee \bigvee_{i=1}^{[nt]} C_n(\frac{i}{n}) B_i) / n$$

and we hope Y_n converges to a sup-integral process. In special cases, this is achieved using the following theorem.

Set $D^+[0, \infty) \subset D[0, \infty)$ to be the non-negative functions of $D[0, \infty)$.

Theorem 5.1. Suppose

$$T: M_P \rightarrow D^+[0, \infty)$$

is a.s. continuous with respect to the distribution of N and suppose

$$C_n = TN_n.$$

Suppose $\int_0^t TN(u) du < \infty$ a.s. for all $t > 0$ and that $TN(t) \in \mathcal{F}_t$ for all t , where $\{\mathcal{F}_t\}$ are the increasing fields of section 2. Then in $D[0, \infty)$

$$(5.5) \quad Y_n \Rightarrow Y$$

where Y is the sup-integral process

$$Y(t) = \bigvee_{T_k \leq t} TN(T_k) X_k.$$

Proof. By the continuous mapping theorem (Billingsley, 1968), (5.3) implies

$$(5.6) \quad (TN_n, N_n) = (C_n, N_n) \Rightarrow (TN, N)$$

in $(D[0, \infty)) \times M_P$. Now define the map

$$\oplus : D^+[0, \infty) \times M_P \rightarrow D^+[0, \infty)$$

by

$$\oplus(x(\cdot), \sum_k \epsilon_{(t_k, j_k)}) = \bigvee_{t_k \leq (\cdot)} x(t_k) j_k.$$

This map is almost surely continuous and when applied to (5.6) yields via the continuous mapping theorem the desired result (5.5). (To check the a.s. continuity of $\oplus$ note if $(x(\cdot), \mu) \in D^+[0, \infty) \times M_P$ is close to $(y(\cdot), \nu) \in D^+[0, \infty) \times M_P$, then after applying a homeomorphism λ of $[0, 1]$ onto $[0, 1]$ we get $x, y \circ \lambda$ uniformly close on $[0, 1]$. Now show

$$\bigvee_{t_i \leq t} x(t_i) j_i, \quad \bigvee_{t'_i \leq \lambda(t)} y(t'_i) j'_i$$

are uniformly close on $[0, 1]$, where

$$\mu = \sum_i \epsilon_{(t_i, j_i)}, \quad \nu = \sum_i \epsilon_{(t'_i, j'_i)}.$$

Next let us consider the analogue of a stochastic differential equation. Let as before

$$I(t) := \bigvee_{T_k \leq t} F(T_k) X_k.$$

Let $\{F_n\}$ be a sequence of simple functions with $F_n(t)$ measurable with respect to $\mathcal{F}_t$, converging to F in the sense that

$$\int_0^t |F_n(u) - F(u)| du \rightarrow 0 \quad (n \rightarrow \infty)$$

in probability. Then, according to Theorem 5.2 below,

$$I_n(t) := \bigvee_{T_k \leq t} F_n(T_k) X_k \rightarrow \bigvee_{T_k \leq t} F(T_k) X_k \quad (n \rightarrow \infty)$$

in probability. Let F_n be constant on $(t_r, t_{r+1}]$ ($r = 1, 2, \dots$). Then

$$I_n(t_{r+1}) = I_n(t_r) \vee F_n(t_r) \bigvee_{t_r \leq T_k \leq t_{r+1}} X_k.$$

This suggests the following analogue of the stochastic differential equation. Let $\{F(t)\}$ be a non-negative random process with $F(t)$ measurable with respect to $\mathcal{F}_t$ ($t > 0$). We write

$$X(t) = X(0) \vee \bigvee_{T_k \leq t} F(T_k) X_k$$

in differential form as follows

$$X(t+\Delta t) = X(t) \vee F(t) \bigvee_{t < T_k \leq t+\Delta t} X_k.$$

Example of a stochastic differential equation with respect to the operation $\vee$. Let the non-negative stochastic process $X(t)$ satisfy the equation

$$X(t) = X(0) \vee \bigvee_{T_k \leq t} X(T_k^-) X_k$$

for $t \geq 0$. We claim that the solution is

$$X(t) := \bigwedge_{T_k \leq t} (1 \vee X_k).$$

For the proof it is sufficient to verify

$$X(t) = X(0) \vee \bigvee_{T_k \leq t} X(T_k^-) X_k$$

i.e.

$$T_k^{\prod_{i \leq t} (1 \vee X_k)} = 1 \vee \bigvee_{\substack{T_k \leq t \\ X_k \geq 1}} T_i^{\prod_{i < T_k} (1 \vee X_i) X_k}.$$

This can be proved in two ways:

1. The right hand side equals

$$(5.7) \quad 1 \vee \bigvee_{\substack{T_k \leq t \\ X_k \geq 1}} T_i^{\prod_{i < T_k} (1 \vee X_i) X_k},$$

since, if $X_k < 1$, there are two cases: either $[0, T_k) \times [1, \infty)$ is empty, in which case

$$T_i^{\prod_{i < T_k} (1 \vee X_i) X_k} < 1,$$

or there is a last $T_r < T_k$ with $X_r > 1$, in which case

$$T_i^{\prod_{i < T_k} (1 \vee X_i) X_k} < T_i^{\prod_{i < T_r} (1 \vee X_i) X_r},$$

so in both cases $T_i^{\prod_{i < T_k} (1 \vee X_i) X_k}$ does not contribute to the right hand side.

To see that

$$T_k^{\prod_{i \leq t} (1 \vee X_k)} = 1 \vee \bigvee_{\substack{T_k \leq t \\ X_k \geq 1}} T_i^{\prod_{i < T_k} (1 \vee X_i) X_k}$$

it is sufficient to realize that the number of points (T_k, X_k) in $[0, t] \times [1, \infty]$ is a.s. finite and to verify

that for $x_1, x_2, \dots, x_n > 0$ (with the empty product defined to be one)

$$\begin{aligned} 1 \vee \bigvee_{k=1}^n \prod_{i=1}^{k-1} (1 \vee x_i) x_k &= (1 \vee x_1) \vee \bigvee_{k=2}^n \prod_{i=1}^{k-1} (1 \vee x_i) x_k \\ &= (1 \vee x_1) \left\{ 1 \vee \bigvee_{k=2}^n \prod_{i=2}^{k-1} (1 \vee x_i) x_k \right\} \end{aligned}$$

and to proceed by induction.

2. Apply Theorem 5.1 with

$$T \sum_k \epsilon_{(t_k, j_k)} = \bigvee_{t_k \leq (\cdot)} (1 \vee j_k).$$

The function T applied to the point process N is a non-decreasing and locally bounded process

$$T_{k \leq t}^{\Pi} (1 \vee X_k).$$

This is really only a finite product since there are only finitely many (T_k, X_k) in $[0, 1] \times [1, \infty]$.

We remark that the r.v. $T_{k \leq t}^{\Pi} (1 \vee X_k)$ does not have a finite expectation since it is bounded below by

$$T_{k \leq t}^{\vee} (1 \vee X_k) \text{ whose distribution function is } \exp\{-\frac{1}{x}\} \text{ for } x > 1.$$

We finish this section with a convergence result of a different type.

Theorem 5.2. Suppose

$$I_n(t) := \bigvee_{T_k \leq t} F_n(T_k) X_k$$

for $n = 1, 2, \dots$ and

$$\int_0^T |F_n(u) - F(u)| du \rightarrow 0 \quad (n \rightarrow \infty)$$

in probability for each $T > 0$. Then

$$\sup_{t \leq T} |I_n(t) - I(t)| \rightarrow 0 \quad (n \rightarrow \infty)$$

in probability, with $I(t) := \bigvee_{T_k \leq t} F(T_k) X_k$.

Proof. Write $(n = 1, 2, \dots)$

$$A_{n,\eta} := \left\{ \int_0^T |F_n(u) - F(u)| du \leq \eta \right\}.$$

Then

$$\begin{aligned} P\left\{\sup_{t \leq T} |I_n(t) - I(t)| > \epsilon\right\} \\ \leq P\left\{\sup_{t \leq T} |I_n(t) - I(t)| > \epsilon, A_{n,\eta}\right\} + P\{A_{n,\eta}^c\}. \end{aligned}$$

Clearly $\lim_{n \rightarrow \infty} P\{A_{n,\eta}^c\} = 0$. The first term at the right hand side is at most

$$(5.8) \quad P\left\{\bigvee_{T_k \leq T} |F_n(T_k) - F(T_k)| X_k > \epsilon, A_{n,\eta}\right\}.$$

Write

$$G_n(s) := |F_n(s) - F(s)| 1_{\left\{\int_0^s |F_n(u) - F(u)| du \leq \eta\right\}}.$$

Then (5.8) is at most

$$\begin{aligned} & P\left\{\bigvee_{T_k \leq T} G_n(T_k) X_k > \epsilon, A_{n,\eta}\right\} \\ & \leq \sum_k P\{T_k \leq T, G_n(T_k) X_k > \epsilon\} = \epsilon^{-1} E \int_0^T G_n(s) ds \\ & = \epsilon^{-1} E \int_0^T |F_n(s) - F(s)| ds 1_{\left\{\int_0^T |F_n(u) - F(u)| du \leq \eta\right\}} \\ & \leq \epsilon^{-1} \eta P\left\{\int_0^T |F_n(u) - F(u)| du \leq \eta\right\} \rightarrow \epsilon^{-1} \eta \quad (n \rightarrow \infty). \end{aligned}$$

Since η is arbitrary, the proof is complete.

6. Concluding remarks

We have defined the sup-integral process $\{I(t)\}$ with respect to a Poisson process on $\mathbb{R}_+ \times (0, \infty]$ with mean measure $dx \cdot dx^{-2}dx$. Suppose $\{(S_k, Y_k)\}$ are the points of a Poisson process on $\mathbb{R}_+ \times (0, \infty]$ with mean measure $dt \cdot \alpha x^{-\alpha-1}dx$. Then $\{(S_k, Y_k^\alpha)\}$ are the points of a Poisson process with mean measure $dt \cdot x^{-2}dx$. Since obviously

$$S_{k \leq t}^{\vee} F(S_k)Y_k = \left(S_{k \leq t}^{\vee} F^\alpha(S_k)Y_k^\alpha \right)^{1/\alpha},$$

the properties of this process follow closely those of $\{I(t)\}$ if $\int_0^t F^\alpha(s)ds < \infty$ a.s. for all $t > 0$.

Specifically, if

$$H(x) := x \left[\int_0^x P \left\{ \int_0^t F^\alpha(s)ds > u \right\} du \right]^{-1}$$

is a regularly varying function at infinity with index 1, then

$$P \left\{ S_{k \leq t}^{\vee} F(S_k)Y_k > x \right\} \sim [H(x^\alpha)]^{-1}$$

as $x \rightarrow \infty$ (Theorem 4.1). In particular, if $E \int_0^t F^\alpha(s)ds < \infty$, then

$$P \left\{ S_{k \leq t}^{\vee} F(S_k)Y_k > x \right\} \sim x^{-\alpha} \cdot E \int_0^t F^\alpha(s)ds.$$

We may also write $S_{k \leq t}^{\vee} F(S_k)Y_k$ as a time changed extremal process.

We now state the corresponding results for stochastic integrals with respect to completely asymmetric stable motion with stability parameter $\alpha < 1$. The proofs are completely analogous to the proofs for the sup-integral. If $\int_0^t F^\alpha(s)ds < \infty$ a.s. for $t > 0$, then

$$\left\{ \sum_{S_k \leq t} F(S_k) Y_k, \quad t \geq 0 \right\}$$

is a well-defined stochastic process. Moreover, if the function $H(x)$ defined in (5.1) is regularly varying at infinity with index 1, then

$$P\left\{ \sum_{S_k \leq t} F(S_k) Y_k > x \right\} \sim \frac{1}{H(x^\alpha)}$$

($x \rightarrow \infty$). In particular, if $E \int_0^t F^\alpha(s) ds < \infty$, then

$$P\left\{ \sum_{S_k \leq t} F(S_k) Y_k > x \right\} \sim \frac{E \int_0^t F^\alpha(s) ds}{x^\alpha}.$$

Compare this with the estimate given in Theorem 2.1, Rosinski and Woyczynski (1986). A time change result for the process $\left\{ \sum_{S_k \leq t} F(S_k) Y_k, \quad t \geq 0 \right\}$ follows from the transformation Theorem 2.1 as in section 3. (Cf. Rosinski and Woyczynski, 1986, Theorem 3.1).

It is possible to formulate two sided versions of many of the results of this paper. Such two sided formulations would be more readily applicable to stochastic integration with respect to stable motion. Thus we could have started with a Poisson process

$$\sum_k \epsilon_{(T_k, X_k)}$$

on $[0, \infty) \times ([-\infty, \infty] \setminus \{0\})$ with mean measure $dt \times (x^{-2} dx \, 1_{(0, \infty]}(x) + |x|^{-2} dx \, 1_{[-\infty, 0)}(x))$. Given $(F(t), t > 0)$ where $F(t)$ has range $(-\infty, \infty)$ we would need to define

$$\tau(t) = \int_0^t |F(u)| du$$

and with these changes Theorem 2.1 still holds.

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