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COMBINING PHASE I AND PHASE II
IN A POTENTIAL REDUCTION ALGORITHM
FOR LINEAR PROGRAMMING

by

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Abstract

This paper describes an affine potential reduction algorithm for linear programming that simultaneously seeks feasibility and optimality. The algorithm is closely related to a similar method of Anstreicher. The new features are that we use a two-dimensional linear programming problem to derive better lower bounds than Anstreicher, that our direction-finding subproblem treats phase I and phase II more symmetrically, and that we do not need an initial lower bound. Our method also allows for the generation of a feasible solution (so that phase I is terminated) during the course of the iterations, and we describe two ways to encourage this behavior.

1. Introduction

Most interior-point algorithms for linear programming assume that an initial interior point is known. While this assumption imposes no limitation from the viewpoint of theoretical computational complexity (Karmarkar [13]), it is a severe drawback for practical computation. Most implementations use artificial variables in the primal, the dual, or both, with appropriate large costs, to circumvent this difficulty (see, e.g., [16]). However, a number of algorithms have considered directly this feasibility problem, based on path-following methods (Freund [8], Gill et al. [11], Lustig [14], Lustig et al. [15], Polyak [17]), projective methods (Anstreicher [2], de Ghellinck and Vial [10], Todd [18] -- see also Todd and Wang [19]), or affine potential reduction methods (Anstreicher [3], Freund [9]).

This paper is concerned with another algorithm for attaining feasibility and optimality together, based on the affine potential reduction methods of Gonzaga [12], Ye [20] and Freund [7]. These methods are among the simplest polynomial-time bounded algorithms for linear programming; the primal-dual method of Ye [20] (see also Freund [7, section 4]) attains the currently best bound of $O(\sqrt{n} \log t)$ iterations to attain a precision of t bits in the optimal value of a problem with n inequalities, given a suitable initial point; and the algorithms allow line searches based on primal or primal-dual potential functions. Thus we believe that they are excellent candidates for efficient and theoretically justified algorithms for linear programming. Enhancing their ability to handle general problems for which interior point solutions are not known is therefore a worthwhile goal.

Freund's algorithm [9] shifts the inequality constraints by an amount proportional to the duality gap in order to make an initial primal solution feasible. It therefore needs an initial lower bound on the objective value of the primal problem. In addition, a bound on the feasible region of the dual problem is needed to initialize the method. Given these assumptions, Freund decreases at each iteration a single potential function that drives the iterates toward feasibility and optimality.

Anstreicher [3] uses an artificial variable to attain feasibility, as in the projective methods of himself [2] and Todd [18] (as well as that of de Ghellinck and Vial [10], although this is implicit -- see Todd and Wang [19]). However, there is no large cost associated with this variable. Instead, as in [2] and [18],

two potential functions, one driving the artificial variable to zero and the other the objective function toward a lower bound, are simultaneously decreased. (In some sense, the unpleasantness of having to choose the parameters h_j and the initial lower bound B^0 in Freund’s method [9] to attain a single potential function is analogous to that of choosing a high cost for an artificial variable to have a single objective function.)

The algorithm we propose in this paper is a modification of Anstreicher’s method [3]. There are three main differences. First, we use a two-variable linear programming problem at each iteration to improve the current lower bound on the objective function value. Anstreicher uses the solution of a single quadratic equation. Our bound is always superior, given the same iterate, and one might hope, in analogy with similar methods for updating lower bounds in projective algorithms, that it is far more effective in practice. Further, we do not require that an initial lower bound be provided; under suitable conditions our algorithm is guaranteed to generate one, and progress is made in the phase-II potential function even before such a lower bound is produced. Finally, our algorithm uses a subproblem to generate the search direction that treats the phase-I and phase-II objectives more symmetrically. (The subproblem also ensures the monotonicity of the phase-II objective when possible, as in Anstreicher [1] and [4].) This allows the phase-I objective to decrease faster than the phase-II, so that feasibility can be attained exactly before optimality under certain conditions. However, this is not possible if the phase-I objective is a variable included in the barrier part of the phase-II potential function. We discuss two ways to circumvent this difficulty.

Section 2 describes the problem, introduces the two potential functions, and discusses the role of scaling. In section 3 we show how lower bounds on the (phase-II) objective are updated. Section 4 describes how our algorithm generates a search direction and proves a critical result for the convergence analysis, which constitutes section 5. Finally, section 6 shows how to terminate phase I.

Our algorithm is based on the “primal-only” potential reduction algorithms of Gonzaga [12] (which assumes the optimal value is zero) and Freund [7, section 3]. In the integer model, these require

$O(nL)$ iterations where n is the number of variables and L the length of the input. Our convergence results correspond to this complexity; roughly, $O(n)$ iterations are required for each additional bit of precision. Ye's primal-dual algorithm [20] (see also Freund [7, section 4]) needs only $O(\sqrt{n} L)$ iterations for the integer model as long as the parameter q in the potential function is $n + O(\sqrt{n})$. Thus far, no combined phase I-phase II algorithm using only an initial primal solution and an initial lower bound has been devised with convergence results corresponding to this improved complexity (see also the discussion at the end of Anstreicher [3]). However, larger values of q than $n + O(\sqrt{n})$ seem more effective in practice, and it seems preferable to use the known optimal value 0 of the phase-I objective than a sequence of negative lower bounds, which would be required in a primal-dual setting. We leave this question to future research.

2. Potential functions and scaling

We will deal with the linear programming problem in the form

$$\begin{aligned}
 \min_x \quad & c^T x \\
 \text{(P)} \quad & Ax = b, \\
 & \xi^T x = 0, \\
 & x \geq 0,
 \end{aligned}$$

where A is $m \times n$ and the vectors are of conforming dimensions. We let

$$F_+(P) := \{x \in \mathbb{R}^n: Ax = b, \xi^T x = 0, x > 0\}$$

be the set of strictly positive feasible solutions, and set

$$F'_+(P) := \{x \in \mathbb{R}^n: Ax = b, x > 0\}.$$

We assume that we know a point in $F'_+(P)$. By scaling the variables, we can suppose this to be $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$. We can assume without loss of generality that $\xi^T e \geq 0$, for otherwise we can change the sign of ξ . If $\xi^T e = 0$, then $e \in F_+(P)$ and we can proceed with the standard primal potential reduction algorithm of Gonzaga [12] and Freund [7]. Hence we will assume that $\xi^T e > 0$.

It is easy to transform a standard linear programming problem into the form of (P), with the assumptions met. Indeed, given the problem

$$\begin{aligned}
 (\tilde{P}) \quad & \min \tilde{c}^T \tilde{x} \\
 & \tilde{A} \tilde{x} = b, \\
 & \tilde{x} \geq 0,
 \end{aligned}$$

define $x = (\tilde{x}^T, \tilde{x}_+^T)^T$, $c^T = (\tilde{c}^T, 0)$, $\xi^T = (0^T, 1)$, and $A = (\tilde{A}, b - \tilde{A}\tilde{e})$, where $\tilde{e}$ is a vector of ones of the same dimension as $\tilde{x}$. Then (P) is equivalent to $(\tilde{P})$, since $\xi^T x = 0$ forces the last variable to be zero, $e \in F'_+(P)$ and $\xi^T e = 1 > 0$.

The algorithm we will describe generates a sequence $\{x^k\}$ of iterates in $F'_+(P)$, starting with $x^0 = e$. If at any stage we obtain a feasible point $x^k \in F_+(P)$, then we revert to the standard primal potential reduction algorithm. Otherwise, we wish to simultaneously approach feasibility (decrease $\xi^T x$) and optimality (decrease $c^T x$). However, since our iterates are not necessarily feasible, it is possible that $c^T x^k$ is below the optimal value of (P), denoted $v(P)$ (equal to $+\infty$ by convention if (P) is infeasible). In this case (we call x^k “superoptimal”) we may not be able to decrease $c^T x^k$. Instead, if z is a known lower bound on $v(P)$ with $c^T x^k \leq z$ (we say x^k is known to be superoptimal), then we will require that $c^T x^{k+1} \leq z$ also. Hence once a point known to be superoptimal is generated, all subsequent iterates will also be superoptimal, as in [2,3,18] (and hence, unless we are lucky enough to hit an exactly optimal solution, infeasible too).

Progress towards feasibility and optimality is measured by potential functions, introduced originally by Karmarkar [13]. Let q be a parameter, assumed throughout to be at least $n + \sqrt{n}$ but $O(n)$. Then the phase-I potential function is

$$\phi_1(x) := q \ln \xi^T x - \sum_j \ln x_j, \quad (1)$$

defined for all $x \in F'_+(P)$ with $\xi^T x > 0$. If z is a lower bound on $v(P)$ and $x \in F'_+(P)$ with $c^T x > z$, then the phase-II potential function is defined as

$$\phi_2(x; z) := q \ln(c^T x - z) - \sum_j \ln x_j. \quad (2)$$

A key property of these potential functions is their invariance under positive scalings of the variables. (More precisely, “potential differences” are invariant.) Let $\hat{x} \in F'_+(P)$ have $\xi^T \hat{x} > 0$, and let $\hat{X} := \text{diag}(\hat{x})$ be the diagonal matrix with the components of $\hat{x}$ down its diagonal. Then, under the affine transformation

$$x \rightarrow \bar{x} := \hat{X}^{-1}x, \quad (3)$$

(P) becomes the scaled problem

$$\begin{aligned} \min \quad & \bar{c}^T \bar{x} \\ & \bar{A} \bar{x} = b, \\ & \bar{\xi}^T \bar{x} = 0, \\ & \bar{x} \geq 0, \end{aligned} \quad (\bar{P})$$

where $\bar{A} = A\hat{X}$, $\bar{c} = \hat{X}c$, and $\bar{\xi} = \hat{X}\xi$. Moreover, $\hat{x}$ is transformed into e by (3). Suppose $x \in F'_+(P)$, and let $\bar{x} = \hat{X}^{-1}x$ be its image. Then, if $\xi^T x > 0$,

$$\begin{aligned} \bar{\phi}_1(\bar{x}) &:= q \ln \bar{\xi}^T \bar{x} - \sum_j \ln \bar{x}_j \\ &= \phi_1(x) + \sum_j \ln \hat{x}_j \end{aligned} \quad (4)$$

and if $c^T x > z$,

$$\begin{aligned}
\bar{\phi}_2(\bar{x}; z) &:= q \ln(\bar{c}^T \bar{x} - z) - \sum_j \ln \bar{x}_j \\
&= \phi_2(x; z) + \sum_j \ln \hat{x}_j.
\end{aligned} \tag{5}$$

Hence, if we wish to reduce ϕ_1 and ϕ_2 by constants, it is sufficient to rescale the problem and reduce the potential functions $\bar{\phi}_1$ and $\bar{\phi}_2$ by constants, and then transform back the new point. Of course, if x is known to be superoptimal, then so is $\bar{x}$ (using the objective function $\bar{c}^T \bar{x}$ and the same bound z), and clearly $\bar{c}^T \bar{x} = c^T x$.

From this discussion it is clear that we can assume at each iteration that the current solution is $e \in F'_+(P)$, since otherwise scaling will make it so. We make this assumption for most of this paper, and use A, c and ξ instead of the scaled $\bar{A}, \bar{c}$ and $\bar{\xi}$ for simplicity.

3. Lower bounds

In this section we show how lower bounds can be generated at each iteration. We assume that the current iterate is $e \in F'_+(P)$, and that the current lower bound is z (possibly $-\infty$). Let

$$\Delta_1 := \xi^T e \tag{6}$$

be the current infeasibility, assumed positive.

By writing $x = e + x'$, (P) can be written as

$$\begin{aligned}
&\min c^T x' \quad (+c^T e) \\
&Ax' = 0 \\
&\xi^T x' = -\Delta_1 \\
&x' \geq -e.
\end{aligned}$$

Now x' lies in the null space of A . For any vector u , let u_p denote its projection into this null

space. Then $x'_p = x'$, and by the properties of projections,

$$u^T x' = u^T x'_p = u_p^T x'$$

for any u . Hence we can replace c and ξ by c_p and ξ_p above, and add the constraint

$(e - e_p)^T x' = 0$, to get the equivalent problem

$$\min c_p^T x' \quad (+c^T e)$$

$$Ax' = 0$$

$$\xi_p^T x' = -\Delta_1$$

$$(e - e_p)^T x' = 0$$

$$x' \geq -e.$$

Now we relax the constraints $Ax' = 0$. We call the result Fraley's relaxation, since its motivation (and especially the use of constraints like $(e - e_p)^T x' = 0$) came from Fraley [6]:

$$\min c_p^T x' \quad (+c^T e)$$

$$(FR) \quad \xi_p^T x' = -\Delta_1$$

$$(e - e_p)^T x' = 0$$

$$x' \geq -e.$$

The dual of this problem is

$$\max -\Delta_1 \lambda - e^T s \quad (+c^T e)$$

$$(FD)' \quad \xi_p \lambda + (e - e_p)^T \tau + s = c_p,$$

$$s \geq 0.$$

While it seems unrelated, in fact $(FD)'$ is very closely related to the dual

$$\begin{aligned}
& \max \quad b^T y \\
\text{(D)} \quad & A^T y + \xi \lambda + s = c, \\
& s \geq 0,
\end{aligned}$$

of (P). Indeed, since $b = Ae$, the objective function of (D) can be written as $e^T A^T y = c^T e - \Delta_1 \lambda - e^T s$, as in (FD)'. Also, for any u , $u - u_p$ is in the range of A^T . Let us write $u - u_p = A^T y_u$. Then (FD)' can be written as

$$\begin{aligned}
& \max \quad c^T e - \Delta_1 \lambda - e^T s \\
& A^T (y_c - \lambda y_\xi + \tau y_e) + \xi \lambda + s = c \\
& s \geq 0,
\end{aligned}$$

which is clearly a restriction of (D) since y is required to be a specified function of λ and τ . Thus $v(\text{FD})' \leq v(\text{D})$.

Let us simplify (FD)' some more. First, we eliminate the constant $c^T e$ and negate the objective function, to get $\min \Delta_1 \lambda + e^T s$; this represents the gap below the current value $c^T e$. Note that the gap may be negative, since e is not feasible. Next we substitute for s to eliminate it from the objective function, to get

$$\begin{aligned}
\text{(FD)} \quad & \min \quad (-\xi_p^T e + \Delta_1) \lambda + (e_p^T e - n) \tau + c_p^T e \\
& \xi_p \lambda + (e - e_p) \tau \leq c_p.
\end{aligned}$$

This is a two-dimensional linear programming problem which can be readily solved.

Suppose (FD) has an optimal solution, with value Δ . Then (FD)' has optimal value $c^T e - \Delta$. Hence there is a feasible solution to (D) with objective value $c^T e - \Delta$, certifying it as a valid lower bound. In this case, we set

$$z_+ := \max\{z, c^T e - \Delta\} \quad (7)$$

as our new lower bound.

Now suppose (FD) is unbounded. Then so is (FD'), and hence a fortiori (D). In this case, we have proved that (P) is infeasible and we stop.

Finally, if (FD) is infeasible, then so is (FD'). We can say nothing about (D) in this case, and are content to set

$$z_+ := z. \quad (8)$$

4. Search direction

We continue our assumption that the current iterate is $e \in F'_+(P)$, and we assume that z has been updated to z_+ as in the previous section. Recall that

$$\begin{aligned} \Delta_1 &:= \xi^T e, \text{ and let} \\ \Delta_2 &:= c^T e - z_+, \end{aligned} \quad (9)$$

with $\Delta_2 = +\infty$ if $z_+ = -\infty$. Δ_1 is assumed positive, but Δ_2 can have either sign.

We seek to reduce the infeasibility Δ_1 by decreasing the phase-I potential function ϕ_1 , defined in (1). Its gradient at the current point is

$$g_1 := \nabla \phi_1(e) = \frac{q}{\Delta_1} \xi - e, \quad (10)$$

whose projection into the null space of A is

$$d_1 := (g_1)_P = \frac{q}{\Delta_1} \xi_P - e_P.$$

Let

$$h_1 := \Delta_1 d_1 = q\xi_p - \Delta_1 e_p. \quad (11)$$

If $0 < \Delta_2 < \infty$, we seek to approach optimality by decreasing the phase-II potential function ϕ_2 , defined in (2). We set

$$g_2 := \nabla_x \phi_2(e; z_+) = \frac{q}{\Delta_2} c - e, \quad (12)$$

and let its projection into the null space of A be

$$d_2 := (g_2)_p = \frac{q}{\Delta_2} c_p - e_p.$$

When $\Delta_2 < \infty$ (see below for the case that $\Delta_2 = \infty$), we define

$$h_2 := qc_p - \Delta_2 e_p; \quad (13)$$

this is $\Delta_2 d_2$ if $\Delta_2 > 0$.

We seek a direction d that simultaneously yields a good decrease in ϕ_1 and (if $\Delta_2 > 0$) in $\phi_2(\cdot; z_+)$. Thus we require it to make inner product at most -1 with the gradients d_1 and (if $\Delta_2 > 0$) d_2 . Equivalently, we require $h_1^T d \leq -\Delta_1$ and $h_2^T d \leq -\Delta_2$. The latter constraint also turns out to be useful if $\Delta_2 \leq 0$; we discuss the interpretation of this constraint after proving Theorem 4.1. It is also helpful to maintain monotonicity if $\Delta_2 > 0$, so that $c^T d \leq 0$; this constraint was first introduced by Anstreicher [1] in projective algorithms, and also [4] in potential reduction methods. Since d will lie in the null space of A , this is equivalent to $c_p^T d \leq 0$. Hence we find d from

$$\begin{aligned}
& \min \|d\| \\
(\text{SP}) \quad & h_1^T d \leq -\Delta_1, \\
& h_2^T d \leq -\Delta_2, \\
& \text{and, if } \Delta_2 > 0, \quad c_p^T d \leq 0.
\end{aligned}$$

If $\Delta_2 = \infty$, we interpret the constraint $h_2^T d \leq -\Delta_2$ as its limiting consequence as $\Delta_2 \rightarrow \infty$, $-e_p^T d \leq -1$. Note that, since h_1, h_2 (or $-e_p$) and c_p are all in the null space of A , the solution to (SP), if it exists, is also in this null space.

We now aim to show that (SP) has an optimal solution with value at most one:

Theorem 4.1. If (FD) is not unbounded, and $q \geq n + \sqrt{n}$, then (SP) has an optimal solution d with $0 < \|d\| \leq 1$.

Proof. Clearly, (SP) is either infeasible or it has an optimal solution d . Since $\Delta_1 > 0$, in the latter case $d \neq 0$.

If (SP) is infeasible, then a lemma of the alternative implies that there exist nonnegative multipliers μ, ν and π , with $\pi = 0$ if $\Delta_2 \leq 0$, such that

$$\begin{aligned}
\mu h_1 + \nu h_2 + \pi c_p &= 0 \quad \text{and} \\
\mu \Delta_1 + \nu \Delta_2 &> 0.
\end{aligned}$$

(Throughout the proof we assume $\Delta_2 < \infty$. If $\Delta_2 = \infty$, the proof goes through with ν replacing $\nu \Delta_2$ and 0 replacing νc_p everywhere.)

If (SP) has an optimal solution with value greater than one, then the Karush-Kuhn-Tucker conditions imply that it is also the optimal solution to

$$\min \|d\|$$

$$(\mu h_1 + \nu h_2 + \pi c_p)^T d \leq -\mu \Delta_1 - \nu \Delta_2$$

for some nonnegative multipliers μ , ν , and π , with $\pi = 0$ if $\Delta_2 \leq 0$. This solution is $\kappa(\mu h_1 + \nu h_2 + \pi c_p)$, where $\kappa = -(\mu \Delta_1 + \nu \Delta_2) / \|\mu h_1 + \nu h_2 + \pi c_p\|^2$.

Hence, in either case, we have that

$$\|\mu h_1 + \nu h_2 + \pi c_p\| < \mu \Delta_1 + \nu \Delta_2$$

$$\text{with } \mu, \nu, \pi \geq 0, \pi = 0 \text{ if } \Delta_2 \leq 0.$$
(14)

Thus, with $h = \mu h_1 + \nu h_2 + \pi c_p$, we have

$$h > -(\mu \Delta_1 + \nu \Delta_2)e, \quad e^T h < (\mu \Delta_1 + \nu \Delta_2)\sqrt{n}$$

(the latter by Cauchy-Schwartz), so that, substituting for h , we get

$$\xi_p(-\mu q) + (e - e_p)(-\mu \Delta_1 - \nu \Delta_2) < (\nu q + \pi)c_p$$
(15)

and

$$(\nu q + \pi)c_p^T e + \mu q \xi_p^T e - (\mu \Delta_1 + \nu \Delta_2)e_p^T e < (\mu \Delta_1 + \nu \Delta_2)\sqrt{n}.$$
(16)

We can assume without loss of generality that $\nu > 0$, since the strict inequalities in (15) and (16) remain true if we increase ν slightly. Thus $\nu q + \pi > 0$; by scaling μ , ν , and π we can assume that $\nu q + \pi = 1$. Then (15) implies that $\lambda = -\mu q$, $\tau = -\mu \Delta_1 - \nu \Delta_2$ is feasible in (FD), and its objective function value is

$$\begin{aligned}
& c_p^T e + (-\xi_p^T e + \Delta_1)(-\mu q) + (e_p^T e - n)(-\mu\Delta_1 - \nu\Delta_2) \\
& < (\mu\Delta_1 + \nu\Delta_2)(\sqrt{n} + n) - \mu\Delta_1 q \quad (\text{using (16), since } \nu q + \pi = 1) \\
& = (\mu\Delta_1 + \nu\Delta_2)(\sqrt{n} + n - q) + \nu q \Delta_2 \\
& \leq \nu q \Delta_2 \\
& \leq (\nu q + \pi)\Delta_2 = \Delta_2 \quad (\text{since } \pi = 0 \text{ if } \Delta_2 \leq 0).
\end{aligned}$$

This contradicts (7) and (9). The proof is complete.

To conclude this section, we discuss the effect of the constraint $h_2^T d \leq -\Delta_2$ in the case $\Delta_2 \leq 0$ and compare our direction-finding subproblem with that of Anstreicher [3].

If $\Delta_2 \leq 0$, the constraint $c_p^T d \leq 0$ is not present. Moreover, since $d = 0$ satisfies $h_2^T d \leq -\Delta_2$, the constraint $h_1^T d \leq -\Delta_1$ must be tight. Hence $h_2^T d \leq -\Delta_2$ has the same effect as

$$\begin{aligned}
& (h_2 - \frac{\Delta_2}{\Delta_1} h_1)^T d \leq 0, \text{ or} \\
& (c_p - \frac{\Delta_2}{\Delta_1} \xi_p)^T d \leq 0, \text{ or} \\
& (c - \frac{\Delta_2}{\Delta_1} \xi)^T d \leq 0,
\end{aligned} \tag{17}$$

since d is in the null space of A . If we move from e to $e + \alpha d$ with $\xi^T(e + \alpha d) \geq 0$, then $\alpha \xi^T d \geq -\xi^T e = -\Delta_1$. Thus

$$\begin{aligned}
c^T(e + \alpha d) &= c^T e + \alpha c^T d \\
&\leq c^T e + \alpha \frac{\Delta_2}{\Delta_1} \xi^T d \\
&\leq c^T e - \Delta_2 \\
&= z_+.
\end{aligned}$$

Hence the constraint ensures that the new point will have objective function value possibly bigger than the current value but certainly no bigger than the current lower bound; the new point will still be superoptimal.

Anstreicher [3] uses exactly (17). His subproblem (rewritten, since he moves in the direction $-d$) is

$$\begin{aligned} \min \quad & g_1^T d \\ \text{Ad} = & 0 \\ (c - \frac{\Delta_2}{\Delta_1} \xi)^T d \leq & 0 \\ \|d\| \leq & 1, \end{aligned}$$

which can be expressed as

$$\begin{aligned} \min \quad & \|d\| \\ \text{(ASP)} \quad & h_1^T d \leq -\Delta_1 \\ (h_2 - \frac{\Delta_2}{\Delta_1} h_1)^T d \leq & 0. \end{aligned}$$

Here we have first used $Ad = 0$ to replace g_1 by $d_1 = (g_1)_p$ and $c - \frac{\Delta_2}{\Delta_1} \xi$ by $c_p - \frac{\Delta_2}{\Delta_1} \xi_p$; then eliminated $Ad = 0$, since it is automatically satisfied; then replaced the constraint $\|d\| \leq 1$ and the objective $d_1^T d$ with the objective $\|d\|$ and the constraint $d_1^T d \leq -1$ (the resulting optimal solution is a positive scalar multiple of the previous one); and finally scaled the constraints by $q > 0$.

There is clearly a close relationship between Anstreicher's subproblem (ASP) and ours; the argument above shows they are equivalent if $\Delta_2 \leq 0$. The second constraint of (ASP) implies $h_2^T d \leq \frac{\Delta_2}{\Delta_1} h_1^T d \leq -\Delta_2$ if $\Delta_2 \geq 0$, while if $\Delta_2 < 0$ the second constraint of (SP) similarly implies that of (ASP). We also include the monotonicity constraint $c_p^T d \leq 0$ if $\Delta_2 > 0$, which has proved to be very important in projective algorithms (see, e.g., Anstreicher and Watteyne [5]). Another

important difference is that we treat feasibility and optimality in a more symmetric fashion. (It is not hard to check that a monotonicity constraint $\xi_p^T d \leq 0$ can also be added to (SP) without falsifying Theorem 4.1. See also section 6. We omit this constraint to keep (SP) simple.) Finally, if $\Delta_2 > 0$ it is possible that only the second constraint of (SP) is binding, so that $h_1^T d < -\Delta_1$, $h_2^T d = -\Delta_2$. In this case, the algorithm approaches feasibility faster than the objective approaches its lower bound, and it is possible that, for some α , $\xi^T(e + \alpha d) = 0$ with $c^T(e + \alpha d) > z_+$ and $e + \alpha d > 0$. Then phase I can be terminated and the algorithm can proceed, generating only feasible iterates henceforth. In this regard see also Todd [18] and Todd and Wang [19]. We will return to this question in section 6.

Anstreicher uses his subproblem (AP(Δ)) (equivalently, (ASP(Δ)), where $\Delta = \Delta_2$ is regarded as a parameter) also to improve his lower bound. He tries first $\Delta = c^T e - z$; if the solution to (ASP(Δ)) has norm less than one, $z_+ = z$. Otherwise, $z_+ = c^T e - \Delta$, where Δ is chosen so that the solution to (ASP(Δ)) has norm one. Finding such a Δ requires the solution of a single quadratic equation, as opposed to our two-variable linear programming problem (FD). However, in the case of projective algorithms, updating lower bounds by solving one or more quadratic equations seems far less powerful than by solving one- or two-variable linear programming problems; see for instance Todd and Wang [19]. One can show as in the proof of Theorem 4.1 that, if the lower bound is updated using (FD), which is not unbounded, and if $q \geq n + \sqrt{n}$, then (ASP) has an optimal solution d with $0 < \|d\| \leq 1$. Indeed, if not, then proceeding exactly as in that proof we find that

$$\|\mu' h_1 + \nu(h_2 - \frac{\Delta_2}{\Delta_1} h_1)\| < \mu' \Delta_1$$

for some μ' , $\nu \geq 0$. Setting $\mu := \mu' - \nu \Delta_2 / \Delta_1$ we obtain (14) with $\pi = 0$, and the rest of the proof is the same. This shows that (FD) must give a better lower bound than Anstreicher's scheme, given the same iterate.

It seems at first sight aesthetically more pleasing to base both lower-bound updates and direction-finding on the same problem. Our proof of theorem 4.1 shows that there is however a close relationship between the (few variable) linear programming problem (FD) to update the lower bound and the (few constraint) quadratic programming problem (SP) to compute the search direction.

5. Convergence

Here we summarize the algorithm and prove its convergence. The key result we use is Theorem 4.1.

We are assuming that $x^0 := e$ lies in $F'_+(P)$, and that z_0 (possibly $-\infty$) is a lower bound on $v(P)$. Given $x^k \in F'_+(P)$ and a lower bound z_k on $v(P)$, the k th iteration is as follows:

- 1) Let $X_k = \text{diag}(x^k)$ and define the scaled data $\bar{A} = AX_k$, $\bar{c} = X_k c$, and $\bar{\xi} = X_k \xi$.
- 2) Compute the projections $\bar{c}_p$, $\bar{\xi}_p$ and e_p of $\bar{c}$, $\bar{\xi}$ and e into the null space of $\bar{A}$.
- 3) Solve the two-variable problem (FD) with $\bar{\xi}_p$ and $\bar{c}_p$ replacing ξ_p and c_p . If it is unbounded, stop; (P) is infeasible. If it is infeasible, set $z_{k+1} = z_k$. If it has optimal value Δ , let $z_{k+1} = \max\{z_k, c^T e - \Delta\}$.
- 4) Let $\Delta_1 = \bar{\xi}^T e = \xi^T x^k$ and $\Delta_2 = \bar{c}^T e - z_{k+1} = c^T x^k - z_{k+1}$. Define $h_1 = q\bar{\xi}_p - \Delta_1 e_p$ and (if $\Delta_2 < \infty$) $h_2 = q\bar{c}_p - \Delta_2 e_p$. Compute the optimal solution $\bar{d}$ of (SP).
- 5) Choose $\alpha > 0$ so that $\bar{x}^+ = e + \alpha \bar{d}$ satisfies the following:

$$\bar{x}^+ > 0; \tag{18}$$

$$\bar{\phi}_1(\bar{x}^+) \leq \bar{\phi}_1(e) - \frac{1}{4}; \tag{19}$$

if $0 < \Delta_2 < \infty$,

$$\bar{\phi}_2(\bar{x}^+; z^{k+1}) \leq \bar{\phi}_2(e; z^{k+1}) - \frac{1}{4}; \tag{20}$$

if $\Delta_2 = \infty$,

$$\bar{\phi}_2(\bar{x}^+; z) \leq \bar{\phi}_2(e; z) - \frac{1}{4} \tag{21}$$

for any $z < \bar{c}^T \bar{x}^+$.

- 6) Set $x^{k+1} = X_k \bar{x}^+$.

(For simplicity, we are assuming here that all iterates satisfy $\xi^T \bar{x}^k > 0$. This follows from (18), for example, if $\xi \geq 0$, $\xi \neq 0$. In the general case, we may not be able to find α satisfying (18)-(21) because $\bar{\xi}^T(e + \alpha \bar{d})$ hits zero too soon. Then we can choose α so that (18) holds, $\bar{\xi}^T \bar{x}^+ = 0$, and (20) or (21) holds with the $\frac{1}{4}$ omitted. We proceed using the standard primal potential reduction algorithm.)

Sections 4 and 5 of [18] describe efficient ways to solve problems of the form of (FD) and (SP). Section 3 above shows that step 3 leads to correct termination or a valid lower bound z_{k+1} . Then Theorem 4.1 proves that (SP) has an optimal solution $\bar{d}$ with $0 < \|\bar{d}\| < 1$. We now show that step 5 can be carried out:

Lemma 5.1. There is a step size $\alpha > 0$ so that (18)-(21) hold.

Proof. We let $\alpha = 1/2\|\bar{d}\|$, so that $\bar{x}^+ = e + \beta \bar{d}/\|\bar{d}\|$ with $\beta = 1/2$. Now $\nabla \bar{\phi}_1(e) = \frac{qc}{\Delta_1} - e$, so that by the constraints of (SP),

$$\nabla \bar{\phi}_1(e)^T \bar{d}/\|\bar{d}\| = \frac{1}{\Delta_1} h_1^T \bar{d}/\|\bar{d}\| \leq -1/\|\bar{d}\|.$$

But Theorem 4.1 implies that $\|\bar{d}\| \leq 1$, so that $\nabla \bar{\phi}_1(e)^T \bar{d}/\|\bar{d}\| \leq -1$. Now we use a standard argument (see for instance Ye [20] or Freund [7]) to get

$$\begin{aligned} \bar{\phi}_1(\bar{x}^+) &\leq \bar{\phi}_1(e) + \beta \nabla \bar{\phi}_1(e)^T \bar{d}/\|\bar{d}\| + \frac{\beta^2}{2(1-\beta)} \\ &\leq \bar{\phi}_1(e) - \frac{1}{2} + \frac{(\frac{1}{2})^2}{2(1 - \frac{1}{2})} \\ &= \bar{\phi}_1(e) - \frac{1}{4}. \end{aligned}$$

If $0 < \Delta_2 < \infty$, a similar argument shows that

$$\bar{\phi}_2(\bar{x}^+; z^{k+1}) \leq \bar{\phi}_2(e; z^{k+1}) - \frac{1}{4}.$$

Now assume that $\Delta_2 = \infty$. Then the second constraint of (SP) is $-e_p^T \bar{d} \leq -1$. Let

$$\bar{\phi}_0(\bar{x}) := - \sum_j \ell_n \bar{x}_j, \quad (22)$$

the logarithmic barrier function. Then $\nabla \bar{\phi}_0(e) = -e$, so that $\nabla \bar{\phi}_0(e)^T \bar{d} / \|\bar{d}\| \leq -1$ also, and hence

$$\bar{\phi}_0(\bar{x}^+) \leq \bar{\phi}_0(e) - \frac{1}{4}. \quad (23)$$

Also, the last constraint of (SP) implies that $\bar{c}^T \bar{d} \leq 0$, so that $\bar{c}^T \bar{x}^+ - z \leq \bar{c}^T e - z$ for any z . With (23) this yields (21) as desired.

We can now prove our convergence result, which is very similar to that of Anstreicher [3] -- see also [18].

Theorem 5.1. Suppose that $x^0 = e \in F'_+(P)$ and that $z_0 \leq v(P)$. Let the algorithm above generate the sequence $\{x^k\}$. Then, unless or until infeasibility of (P) is demonstrated by an unbounded problem (FD), we have

- a) $x^k \in F'_+(P)$ and $z_k \leq v(P)$ for all k ;
- b) $\xi^T x^k \leq \exp(-k/4q) \cdot (e^T x^k / n)^{n/q} \cdot \xi^T e$;
- c) $c^T x^k - z \leq \exp(-k/4q) \cdot (e^T x^k / n)^{n/q} \cdot (c^T e - z)_+$,

where $(c^T e - z)_+ = \max\{0, c^T e - z\}$ and z is z_k if $z_k > -\infty$ and any scalar less than $c^T x^k$ otherwise; and

- d) $c^T x^k \leq \max\{c^T x^{k-1}, z_k\}$.

Moreover, if $\{x \in \mathbb{R}^n: Ax = 0, \xi^T x = 0, c^T x \leq 0, x \geq 0\} = \{0\}$, then, unless infeasibility is demonstrated,

- e) if $v(P) < \infty$, $e^T x^k / n$ is uniformly bounded, and
- f) a finite lower bound z_k is generated at some iteration.

Proof. Part (a) follows from (18) and since $\bar{d}$ is in the null space of $\bar{A}$, and because of the argument of section 3. Part (d) follows from the argument of section 4.

Next, (19) shows that

$$\phi_1(x^k) \leq \phi_1(e) - k/4.$$

From the definition of ϕ_1 and since

$$-\sum_j \ell_n x_j \geq -n \ell_n(e^T x/n)$$

from the arithmetic-geometric mean inequality for any $x \in F'_+(P)$, we easily deduce (b). Part (c) is trivial if x^k is known to be superoptimal, so assume $c^T x^k > z$, where z is as in part (c). Then by (20) or (21),

$$\phi_2(x^k; z) \leq \phi_2(x^{k-1}; z) - \frac{1}{4}. \quad (24)$$

We now prove by induction that

$$\phi_2(x^k; z) \leq \phi_2(e; z) - k/4. \quad (25)$$

If $z_k = -\infty$ so that $z_{k-1} = -\infty$, or if $z_k > -\infty$ but $z_{k-1} = -\infty$, then by the inductive hypothesis

$$\phi_2(x^{k-1}; z) \leq \phi_2(e; z) - (k-1)/4,$$

so that (24) yields (25). Finally, if $z_{k-1} > -\infty$ then

$$\phi_2(x^{k-1}; z_{k-1}) \leq \phi_2(e; z_{k-1}) - (k-1)/4,$$

by the inductive hypothesis, which implies, since $c^T x^{k-1} \leq c^T e$ and $z_k \geq z_{k-1}$, that

$$\phi_2(x^{k-1}; z_k) \leq \phi_2(e; z_k) - (k-1)/4.$$

With (24) we again have (25). From (25) we easily deduce (c) as before.

Now suppose $\{x \in \mathbb{R}^n: Ax = 0, \xi^T x = 0, c^T x \leq 0, x \geq 0\} = \{0\}$. Let

$$S_\alpha := \{x \in \mathbb{R}^n: \|Ax\| \leq \alpha\|b\|, 0 \leq \xi^T x \leq \alpha, \\ c^T x \leq \alpha \max\{0, c^T e, v(P)\}, e^T x = n, x \geq 0\}.$$

By assumption, S_0 is empty. Hence there is some $0 < \alpha^* < 1$ such that S_α is empty for $0 \leq \alpha \leq \alpha^*$; otherwise, if we take $x^k \in S_{\alpha_k}$ for a sequence $\alpha_k \rightarrow 0$ then all x^k 's lie in the compact set $\{x \in \mathbb{R}^n: e^T x = n, x \geq 0\}$, and any limit point lies in S_0 , a contradiction.

We can now show (e). Indeed, suppose $k \geq 4q(|\ln \alpha^*| + \ln(\xi^T e))$, and assume $e^T x^k/n =: 1/\alpha \geq 1/\alpha^*$. Let $x = x^k/(e^T x^k/n) = \alpha x^k$. By (a), $\|Ax\| = \|\alpha b\| \leq \alpha^* \|b\|$ and $x \geq 0$. By definition $e^T x = n$. Now (b) implies $0 \leq \xi^T x \leq \exp(-k/4q) \cdot (e^T x^k/n)^{n/q-1} \cdot \xi^T e \leq \exp(-k/4q) \cdot \xi^T e \leq \alpha^*$ by our choice of k . Finally, (d) implies $c^T x^k \leq \max\{0, c^T e, v(P)\}$, so $c^T x \leq \alpha^* \max\{0, c^T e, v(P)\}$. This shows that x lies in S_{α^*} , a contradiction. Hence $e^T x^k/n < 1/\alpha^*$ for all sufficiently large k .

Finally, for (f), assume that no lower bound is generated in the first K iterations. Then $c^T x^k \leq c^T e$ for $k \leq K$, and we can repeat the analysis of the previous two paragraphs with $\max\{0, c^T e\}$ replacing $\max\{0, c^T e, v(P)\}$, to obtain that $e^T x^k/n \leq 1/\hat{\alpha}$ for all $K \geq k \geq 4q(|\ln \hat{\alpha}| + \ln(\xi^T e))$. Here $\hat{\alpha}$ is the new value of α^* with the new “max” term. But if $K \geq \max\{4q(|\ln \hat{\alpha}| + \ln(\xi^T e)), 4n|\ln \hat{\alpha}|\}$, then from (c), letting $z \rightarrow -\infty$, we obtain

$$1 \leq \exp(-K/4q) \cdot (e^T x^k/n)^{n/q},$$

so that $e^T x^K/n \geq \exp(K/4n) \geq 1/\hat{\alpha}$. This contradiction shows that a finite lower bound must be generated in at most K iterations, completing the proof.

The proofs of parts (e) and (f) are considerably simplified if we add the monotonicity constraint $\xi_p^T d \leq 0$ for the phase-I objective to (SP). However, as we stated earlier, we opted to keep the subproblem as simple as possible, and this complicates the analysis. Also, our proof shows how it may be possible in some cases to get explicit bounds for the term $e^T x^k/n$ and for the number of iterations until a finite lower bound is generated.

6. Finite termination of Phase I

We remarked in section 4 that one may be able to take a step so that $\xi^T(e + \alpha d) = 0$ while the Phase II-potential function is reduced. In this case one can obtain a feasible solution in $F_+(P)$ and continue with the standard primal potential reduction algorithm. Alternatively, we can proceed with the algorithm described here with the following modifications. We incorporate the constraint $\xi^T x = 0$ into $Ax = b$, so that the subscript p denotes projection into the null space of the (suitably scaled) matrix $\begin{bmatrix} A \\ \xi^T \end{bmatrix}$. In (FD), the variable λ is deleted. In (SP), the constraint $h_1^T d \leq -\Delta_1$ is omitted. In the selection of α in step (5) of the algorithm, (19) is ignored. The convergence results of section 5 remain valid for these variants.

This modification may be useful for certain classes of problem (P); as discussed in [18], the violated constraint $\xi^T x = 0$ may arise from the addition of cutting planes. However, when (P) results from $(\tilde{P})$ as in section 2, $\xi^T x$ is the final component of x , so if $\xi^T x$ is zero, $\phi_2(e + \alpha d)$ must be infinite unless by great fortune we have reached the exact optimal solution. Hence it appears that phase I can never terminate in this case. Here we describe two methods to circumvent this difficulty. The first method is certainly applicable to other phase I - phase II algorithms, while the second is more algorithm dependent.

Recall that $(\tilde{P})$ is

$$\begin{aligned} \min \tilde{c}^T \tilde{x} \\ \tilde{A} \tilde{x} &= b, \\ \tilde{x} &\geq 0. \end{aligned}$$

In section 2, we reformulated $(\tilde{P})$ in the form of (P) using an artificial variable $\tilde{x}_+$ that is required to be zero. Our second method again uses the same reformulation and changes the algorithm, while the first introduces an artificial variable $\tilde{x}_+$ that is required to take a positive value. (This approach was suggested by Rob Freund (private communication) as a simpler alternative to a scheme of the author using two artificial variables.) Consider the problem

$$\begin{aligned} (\hat{P}) \quad \min(\tilde{c}^T, 0,) \quad & \begin{bmatrix} \tilde{x} \\ \tilde{x}_+ \end{bmatrix} \\ A\tilde{x} + (2b - 2Ae)\tilde{x}_+ &= 2b - Ae \\ \tilde{x}_+ &= \frac{1}{2} \\ \tilde{x} \geq 0, \tilde{x}_+ &\geq 0. \end{aligned}$$

This is almost of the form of (P) , and e is feasible in all but the last equation. It suffices to use any one of the first equations with a nonzero right-hand side (if there are none, $\tilde{x} = \frac{1}{2}e$ is feasible in $(\tilde{P})$) to eliminate the right-hand side of the final equation and hence reduce it to the form $\xi^T x = 0$. Moreover, it is easily seen that $(\hat{P})$ is equivalent to $(\tilde{P})$. Thus in this reformulation it is certainly possible to have $\xi^T x = 0$ with $x > 0$ at some iteration; we have allowed for finite termination of phase I.

The approach above is certainly easy, but it is rather ad hoc, although less so than the heuristic proposed at the end of [18]. We now show a perhaps more natural technique, which omits the

artificial variable from the barrier terms in the potential functions; after all, we are trying to drive it to zero. We reformulate $(\tilde{P})$ into the form of (P) using an artificial variable $\tilde{x}_+$ required to be zero as in section 2. We now make the following changes to the algorithm.

First, we omit the term $-\ell \ln x_n$ from the potential functions ϕ_1 and ϕ_2 (see (1) and (2)). (Similarly, we omit $-\ell \ln \bar{x}_n$ from $\bar{\phi}_0$ in (22).) (4) and (5) remain valid if the sums are over the first $n - 1$ indices j only. Then $g_1 = \nabla \phi_1(e)$ becomes $(q/\Delta_1)\xi - \hat{e}$ and, if $\Delta_2 > 0$, $g_2 = \nabla \phi_2(e)$ becomes $(q/\Delta_2)c - \hat{e}$, where

$$\hat{e} := (1, 1, \dots, 1, 0)^T = e - \xi.$$

The vectors d_1, h_1, d_2 and h_2 are similarly changed -- e_p becomes $\hat{e}_p$; see (10)-(13).

In (FR) and $(FD)'$ we replace $(e - e_p)$ with $(\hat{e} - \hat{e}_p)$. Thus in (FD) , $(e - e_p)$ becomes $(\hat{e} - \hat{e}_p)$ and $(e_p^T e - n)$ becomes $(\hat{e}_p - \hat{e})^T e = (\hat{e}_p^T e - n + 1)$.

In the direction-finding subproblem (SP) , h_1 and h_2 take their new values, and we add the constraint $\xi_p^T d \leq 0$ to ensure that the artificial variable is non-increasing. If $\Delta_2 = \infty$, $h_2^T d \leq -\Delta_2$ is interpreted as $\hat{e}_p^T d \leq -1$.

Proposition 6.1. With the changes above, Theorem 4.1 remains true. Hence Theorem 5.1 remains true also.

Proof. We merely outline the changes to the proof of Theorem 4.1.

If (SP) is infeasible, then there exist nonnegative multipliers μ, ν, π , and ρ with $\pi = 0$ if $\Delta_2 < 0$ such that

$$\begin{aligned} h &:= \mu h_1 + \nu h_2 + \pi c_p + \rho \xi_p = 0 \\ \mu \Delta_1 + \nu \Delta_2 &> 0. \end{aligned}$$

If (SP) is feasible but its optimal value is greater than one, then its optimal solution $\bar{\mathbf{d}}$ is also optimal to

$$\begin{aligned} \min \quad & \|\mathbf{d}\| \\ (\mu\mathbf{h}_1 + \nu\mathbf{h}_2 + \pi\mathbf{c}_p + \rho\xi_p)^\top \mathbf{d} \leq & -\mu\Delta_1 - \nu\Delta_2 \end{aligned}$$

for some nonnegative μ, ν, π , and ρ , with $\pi = 0$ if $\Delta_2 < 0$. This solution is $\kappa(\mu\mathbf{h}_1 + \nu\mathbf{h}_2 + \pi\mathbf{c}_p + \rho\xi_p)$, where $\kappa = -(\mu\Delta_1 + \nu\Delta_2)/\|\mu\mathbf{h}_1 + \nu\mathbf{h}_2 + \pi\mathbf{c}_p + \rho\xi_p\|^2$. Thus

$$\mathbf{h} := \mu\mathbf{h}_1 + \nu\mathbf{h}_2 + \pi\mathbf{c}_p + \rho\xi_p \tag{26}$$

has

$$\|\mathbf{h}\| < \mu\Delta_1 + \nu\Delta_2. \tag{27}$$

Also, since $\bar{\mathbf{d}}$ is a negative multiple of $\mathbf{h}$, and $\xi^\top \bar{\mathbf{d}} = \xi_p^\top \bar{\mathbf{d}} \leq 0$,

$$\xi^\top \mathbf{h} \geq 0. \tag{28}$$

Hence, in either case, there exist nonnegative μ, ν, π , and ρ , with $\pi = 0$ if $\Delta_2 < 0$, such that (26)-(28) hold. Thus

$$\mathbf{h} \begin{pmatrix} > \\ \geq \end{pmatrix} - (\mu\Delta_1 + \nu\Delta_2)\hat{\mathbf{e}},$$

meaning that strict inequality holds in all but the last component, and

$$\mathbf{e}^\top \mathbf{h} < (\mu\Delta_1 + \nu\Delta_2) \sqrt{n}.$$

Therefore,

$$\xi_p(-\mu q - \rho) + (\hat{e} - \hat{e}_p)(-\mu\Delta_1 - \nu\Delta_2) \left(\begin{smallmatrix} < \\ \geq \end{smallmatrix} \right) (\nu q + \pi)c_p \quad (29)$$

and

$$(\nu q + \pi)c_p^T e + (\mu q + \rho)\xi_p^T e - (\mu\Delta_1 + \nu\Delta_2)\hat{e}_p^T e < (\mu\Delta_1 + \nu\Delta_2)\sqrt{n}, \quad (30)$$

cf. (15)-(16),

If ν and π are zero, we first increase ρ slightly to obtain strict inequalities in all inequalities in (29) and (30) (note that $\xi^T \xi_p > 0$, so that the last inequality in (29) becomes strict). Then we can increase ν slightly maintaining the strict inequalities. Now $\nu q + \pi$ is positive, and by scaling μ, ν, π and e we can assume $\nu q + \pi = 1$ and then proceed as in the proof of Theorem 4.1 to a contradiction. (The extra term $-\rho\Delta_1$ in the objective function only helps.) This concludes the proof.

Rob Freund (private communication) has pointed out that $\tilde{x}_+$ should be regarded as a free variable, and that therefore a different projection should be employed; eliminating $\tilde{x}_+$ from the barrier terms only deals with half the problem. We agree, but the corresponding analysis is unfortunately rather complicated and hence we omit it.

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