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**SCHEDULING THE PRODUCTION OF
SEVERAL ITEMS WITH RANDOM
DEMANDS ON A SINGLE FACILITY¹**

by

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Scheduling The Production of Several Items with Random Demands on a Single Facility

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ABSTRACT

Consider the problem of scheduling the production of several items on a single facility that can produce only one item at a time. The goal is to reduce the long run average holding, backorder and setup costs. This problem occurs since it is often economic to produce several items on a single facility.

We assume that cumulative demands are Brownian Motion with constant, item dependent, drifts. We allow backorders and charge holding and backlogging costs at linear time weighted rates. Items are produced at continuous constant rates. Setup times and setup costs are item dependent constants. These parameters, however, are independent of the order of setups.

First, with demands replaced by their expectations, we compute an optimal or near-optimal target cyclic schedule by a procedure that generalizes techniques developed for the Economic Lot Scheduling Problem.

Next, we study the problem of scheduling the facility after a single disruption perturbs the inventories. The goal is to recover the target cyclic schedule at minimal excess over the average cost of the cyclic schedule. We formulate this as a discrete time-invariant infinite-horizon linear-quadratic control problem and obtain a linear recovery policy that is optimal for a large configuration of disruptions.

Finally, we add safety stocks to hedge against randomness in demands. Our goal is to select safety stocks to minimize the long run average cost of following the target schedule with the recovery policy. We show that optimal safety stocks are unique and have the property that in the long run the proportion of time that an item is in stock is the ratio of backorder to holding plus backorder cost.

We present an example that integrates the cyclic schedule, the control policy and the safety stocks.

1. Introduction.

Consider the problem of scheduling several items on a single facility that can produce only one item at a time. The problem occurs when economies of scale justify facilities that can produce different items. The facility must setup, at an expense of time and money, for a production run of an item while all inventories are being depleted by demand. The objective is to schedule the facility to reduce the long run average cost of performing setups, carrying inventory and incurring stockouts.

In practice these facilities are scheduled by myopic dispatching rules that often revert to expediting in a constant effort to keep up with demand.

In this paper we study the problem where demands are random and stockouts are backordered. We assume, however, that expected demand rates are product dependent constants and that backorders are charged at a linear time-weighted rate.

We develop a cost effective scheduling tool in three steps: a target cyclic schedule based on expected demands, a control policy that keeps inventories close to the target schedule, and safety stocks to hedge against randomness. The target schedule, the control policy and the safety stocks can be precomputed, so the tool can be used in real time. A complete example is given.

The literature abounds in papers on the closely related Economic Lot Scheduling Problem (ELSP). In the ELSP, a perfectly reliable facility is assumed to produce the items at fixed, item dependent, production rates. Setup costs and times are often assumed to be item dependent, but sequence independent, constants. Demand rates are assumed to be known item dependent constants, and stockouts are not allowed. The objective is to minimize the long run average holding and set up cost. See Maxwell (1964) for pioneering work and Elmaghraby (1978) for a review up to 1978. More recent contributors are Axsater (1983), Boctor (1982), Dobson (1987), Dobson (1988), Delporte and Thomas (1978), Fujita (1978), Graves and Haessler (1978), Graves (1979), Haessler (1979), Hsu (1983), Jones and Inman (1987), Lee and Denardo (1985), Maxwell and Singh (1983), Park and Yun (1984), Roundy (1985), Vemuganti (1978) and Zipkin (1987).

The research emphasis on the ELSP has been on finding cost effective cyclic schedules. In a cyclic schedule the facility is setup for the items in a sequence that repeats forever. The time

elapsed in setups, production runs and idle times determines the cycle length and the initial inventories are reached at the end of each cycle. Key to this property is the assumption of deterministic demands. In practice, however, demands are random. One reason is that random yield ratios, at the facility or anywhere downstream, result in random requirements. Moreover, the universal assumption that stockouts are not allowed is too stringent in the presence of randomness. Consequently the presence of randomness precludes the existence of a cyclic schedule. However, cyclic schedules based on expected demands, with finite backorder costs, form the backbone of our scheduling tool.

The assumptions in this paper are identical to those of the ELSP except that demands are random and backorders are allowed. We choose the backorder assumption because it is quite natural for items with external demands. The model also provides a reasonable approximation to the less tractable lost sales case. Note that our model reduces to the ELSP when the random variables are degenerate and the backorder costs are infinite. Intermediate combinations are of interest, for example constant demands and finite, however large, backorder costs. In this case the control policy is capable of scheduling the facility after a disruption, for instance after a breakdown, so that the target schedule is recovered effectively. In practice, it is important to have an effective recovery policy at hand since things seldom go as planned.

Recently, Gallego and Roundy (1988) extended the ELSP to allow backorders. Their procedure combines and extends state of the art heuristics for the ELSP that result in an optimal or near-optimal cyclic schedule. As a first step we apply their procedure to our problem where demand rates are replaced by their expectations. We call the resulting cyclic schedule the target cyclic schedule.

The rest of this paper is organized as follows. In Section 2 we introduce notation, characterize cyclic schedules under constant demands and show that the ratios of backorder to holding plus backorder costs are equal to the proportion of times that the items are in stock. Section 3 deals with the problem of recovering the target schedule after a disruption. In Section 4 the recovery policy obtained in Section 3 is used to keep inventories close to those of the target schedule plus safety stocks. We seek the configuration of safety stocks that minimize the long run average cost and show that at the optimal the proportion of time the items are in stock is again the ratios of

backorder to holding plus backorder costs. A complete example is presented in Section 5.

2. Notation.

In this section demands are replaced by their expectations.

Let i be the item index. For $i = 1, \dots, m$ let

$d'_i :=$ expected demand rate for item i ,

$p'_i :=$ production rate for item i ,

$h'_i :=$ redefined holding cost for item i ,

$b'_i :=$ redefined backorder cost for item i ,

$s_i :=$ set up time for item i ,

$a_i :=$ set up cost for item i .

Without loss of generality we redefine the item units so the all expected demands rates are one. This is accomplished by setting $d_i \equiv d'_i/d'_i = 1$, $p_i \equiv p'_i/d'_i$, $h_i \equiv h'_i d'_i$ and $b_i \equiv b'_i d'_i$. The transformation maps many equivalent problems to one that is easier to manipulate.

Note that $r_i \equiv 1/p_i$ is the long run proportion of time (exclusive of setups) that the facility is engaged in the production of item i . Consequently $\kappa \equiv 1 - \sum_i r_i$ is the long run proportion of time available for setups. If $\kappa = 0$ there is enough capacity to keep up with demand but not enough to perform setups. Thus, we assume that $\kappa > 0$.

Denote the corresponding m -dimensional vectors by $\mathbf{d} = \mathbf{e}, \mathbf{p}, \mathbf{h}, \mathbf{b}, \mathbf{s}, \mathbf{a}$ and $\mathbf{r}$, where $\mathbf{e}$ is the vector of ones. A cyclic schedule can be completely described by four vectors. A production sequence $\mathbf{f}$, initial inventories $\mathbf{w}$, and idle and production times, $\mathbf{u}$ and $\mathbf{t}$. Each item has at least one position in the sequence; so $\mathbf{f}$, $\mathbf{t}$, and $\mathbf{u}$ are n -dimensional vectors with $n \geq m$.

At the start of the cycle the inventories are given by $\mathbf{w}$. The cycle starts by idling the facility for u_1 units of time, then the facility is setup for the item in the first position of the sequence and it is produced for t_1 units of time. The facility goes through the idle periods, setups, and

production for each position of the sequence. The ending inventories are again w . Different components of t or u that correspond to the same item are not required to be equal. If $n = m$, the cyclic schedule is called a rotation schedule. The cycle length, say T , is obtained by adding idle, production run and set up times. An example of a cyclic schedule is given in figure 1.

Figure 1.

Given any production sequence f there exist vectors w , t , and u that together with f form a cyclic schedule. See Gallego and Roundy (1988), and Dobson (1987) for the ELSP. Moreover, given f , there exist a parametric quadratic algorithm (in T) to find optimal values of w , t , and u . The resulting cyclic schedule is called an optimal f -cyclic schedule. The above remains true even if setup times are sequence dependent constants. See Gallego and Roundy (1988), and Zipkin (1987) for the ELSP. The schedule in figure 1 is an instance of an optimal f -cyclic schedule.

We define the service rate of a item as the proportion of time that it is in stock. The following Lemma serves to establish the item service rates when they follow an optimal f-cyclic schedule. We use the notation $x^+ = \max\{0, x\}$, and $x^- = \max\{0, -x\}$.

Lemma 2.1 Let $\mu(t)$ be a real valued Lebesgue integrable function in the interval $[0, T]$. Let b and h be positive constants and define

$$\Gamma(y) = \int_0^T [h(\mu(t) + y)^+ + b(\mu(t) + y)^-] dt$$

Then $\Gamma(y)$ exists, is strictly convex and Lipchitz continuous. Moreover, if $\mu(t)$ is nowhere flat, except in a set of measure zero, then $\Gamma(y)$ is continuously differentiable, and, $y = 0$ minimizes $\Gamma(y)$ if and only if $\mu(t)$ is positive in a set of measure $bT/(b + h)$.

Proof: $\Gamma(y)$ exists because $\mu(t) + y$ is integrable. $\Gamma(y)$ is strictly convex since $h(\mu(t) + y)^+ + b(\mu(t) + y)^-$ is strictly convex in y . It is easy to verify that $|\Gamma(y') - \Gamma(y)| < (h + b)T|y' - y|$, so $\Gamma(y)$ is Lipchitz continuous and hence has a finite derivative almost everywhere. Differentiating and rearranging terms we obtain

$$\Gamma'(y) = (h + b) \int_0^T \mathbf{1}\{t : \mu(t) + y > 0\} dt - b \int_0^T \mathbf{1}\{t : \mu(t) + y \neq 0\} dt \quad (1)$$

almost everywhere.

Clearly, $\Gamma'(y)$ is increasing. Moreover, if $\mu(t)$ is nowhere flat, $\Gamma'(y)$ is continuous with range $[-bT, hT]$. By (1) $\Gamma'(y) = 0$ if and only if

$$\int_0^T \mathbf{1}\{t : \mu(t) + y > 0\} dt = bT/(b + h).$$

Consequently $y^* = 0$ is optimal if and only if the set where $\mu(t)$ is positive has measure $bT/(b + h)$. (Q.E.D.)

Corollary 2.2 The service rates of an optimal f-cyclic schedule are $b_i/(b_i + h_i), i = 1, \dots, m$.

Proof: The inventory level of item i through a cycle is a continuous, hence integrable function. Moreover, its slope is either $p_i - 1 > 0$ or -1 so it is nowhere flat. Because of the optimality

of the schedule the holding and backorder costs can not be reduced by shifting the inventories vertically. By Lemma 2.1, item i must be on stock $b_i T / (b_i + h_i)$ units of time during the cycle. (Q.E.D.)

We now present an alternative characterization of a cyclic schedule that will be useful in the sequel. Denote by γ_j (resp., δ_j), the inventory of the item produced at position j just before (resp., just after) the production run. Let γ and δ be the corresponding n -dimensional vectors. Note that the knowledge of f, t, u, w is equivalent to the knowledge of f, γ, δ, w .

The Extended Zero Switch Rule (EZSR) defined in Gallego and Roundy is satisfied when $\gamma \leq 0$ and $\delta \geq 0$. The EZSR reduces to the more familiar Zero Switch Rule (ZSR) ($\gamma = 0$) when backorders costs are infinity. There are optimal f -cyclic schedules for which the EZSR fails to hold. These instances are very rare and correspond to contrived choices of f . See Maxwell (1964) for a similar experience with the ZSR. In what follows we assume that we have obtained, by the procedure suggested in Gallego and Roundy (1988), an optimal f -cyclic schedule that satisfies the EZSR.

3. Recovery from an initial Disruption.

A disruption to a cyclic schedule may be caused by a multitude of unpredictable events. For instance variations in demands, machine failures, lack of raw materials, etc.. In this section we discuss how to recover efficiently from a single initial disruption. That is, how to bring the inventory levels back to w whereon the target schedule can be followed.

To simplify the presentation we assume that $f = (1, 2, \dots, m)$ i.e., the target optimal f -cyclic schedule is a rotation schedule. However, all the results of this section hold for more general production sequences provided that the corresponding optimal f -cyclic schedule satisfies the EZSR. It is worth pointing out that the results also hold when the setups are sequence dependent. See Dobson (1988). See the Appendix for the more general discussion.

We will show that in the absence of further disruptions it is always possible to recover in finite time. In that case, the long run average cost is independent of the recovery strategy so the problem is trivial with respect to that objective. A more appropriate criterion, however, is the

excess over average cost which is defined as the limit infimum of $[C(t) - at]$. Here $C(t)$ is the total cost incurred up to time t and a is the long run average cost of the target optimal f -cyclic schedule.

We restrict attention to the class of f -recovery policies that set up and produce in the order of the sequence f . There are practical reasons for this, for instance the procurement of raw materials and tools is often ancillary to the production sequence. There are, however, situations where it may be better to alter the sequence. For example, after a large disruption, we can save time and money by avoiding setups. Unfortunately allowing general sequences during the recovery phase gives rise to a strongly NP-complete problem, see Arkin Gallego and Roundy (1988). Later in this Section, we suggest a heuristic that allows jumps within the sequence f . This heuristic can be used to avoid setups after a large disruption.

The control variables are the production run times; they can vary in every repetition of f . Idle times are fixed to those prescribed in the target schedule. This is a mild restriction since facilities producing at or near capacity have little or no idle time in their target schedules. The advantage of idling is apparent when a disruption results in inventories that are high relative to those of the target schedule. At the end of these Section, we show how to optimally insert idle time immediately after a disruption, but before applying an f -recovery policy.

We now model the recovery problem as a discrete-time infinite-horizon control-problem. Let x be the vector of inventory levels after the single initial disruption. This is the vector from which we want to make an efficient transition into the target schedule. Define the initial state of the system, say z , as the difference between the initial inventories and those resulting after the disruption. That is $z = w - x$, note that if $z \geq 0$ the disruption results in inventories lower than w .

Decision epoch zero occurs at time $\tau_0 = 0$. From there on we follow the production sequence f . Decision epoch k occurs at the end of the k^{th} repetition of f , at time τ_k . At decision epoch k we observe the inventory levels, say x^k , and determine the state of the system $z^k = w - x^k$. We then select the production runs $t^k = t + v^k$, for the next repetition of f by specifying a control vector v^k . Note that v^k must satisfy $t + v^k \geq 0$. The next decision epoch occurs at time

$\tau_{k+1} = \tau_k + T + \mathbf{e}^T \mathbf{v}^k$. By the k^{th} control cycle we refer to the events occurring in the time interval $[\tau_k, \tau_{k+1}]$.

Formally a policy is a mapping from the state space into the control space. Given $\mathbf{z} = \mathbf{z}^0$ an $\mathbf{f}$ -recovery policy generates infinite sequences $\{\mathbf{z}^k\}$ and $\{\mathbf{v}^k\}$ with the property that $\mathbf{z}^k \rightarrow 0$ as $k \rightarrow \infty$. If there is a finite K such that $\mathbf{z}^k = 0$ for $k \geq K$ we say that the policy recovers in finite time.

We now obtain the state at the end of the k^{th} control cycle. The inventories at time τ_{k+1} , say $\mathbf{x}^{k+1}$, are obtained from the inventories at time τ_k , say $\mathbf{x}^k$, by adding production and subtracting demand through the k^{th} control cycle. Define $\text{diag}(\mathbf{p})$ as the diagonal matrix with $\mathbf{p}$ in the diagonal, then $\mathbf{x}^{k+1} = \mathbf{x}^k + \text{diag}(\mathbf{p})(\mathbf{t} + \mathbf{v}^k) - \mathbf{e}(T + \mathbf{e}^T \mathbf{v}^k) = \mathbf{x}^k + [\text{diag}(\mathbf{p}) - \mathbf{e}\mathbf{e}^T]\mathbf{v}^k$.

The last equation is justified by the nature of the cyclic schedule, i.e., $\text{diag}(\mathbf{p})\mathbf{t} - \mathbf{e}T = 0$. Let $Q \equiv \text{diag}(\mathbf{p}) - \mathbf{e}\mathbf{e}^T$, then $\mathbf{x}^{k+1} = \mathbf{x}^k + Q\mathbf{v}^k$, so the state at time epoch $k + 1$ is given by

$$\mathbf{z}^{k+1} = \mathbf{z}^k - Q\mathbf{v}^k. \quad (2)$$

Consequently the state update is linear in the control vector.

Lemma 3.1 The matrix Q has a nonnegative inverse.

Proof: Q is a rank one modification of $\text{diag}(\mathbf{p})$. By the Sherman-Morrison matrix identity we have

$$Q^{-1} = \text{diag}(\mathbf{r}) + \mathbf{r}\mathbf{r}^T / \kappa \geq 0. \quad (3)$$

Proposition 3.2 There exists an $\mathbf{f}$ -recovery policy that recovers in finite time. Moreover the infinite sum $\sum_k \mathbf{v}^k$ is independent of the $\mathbf{f}$ -recovery policy.

Proof: We exhibit an $\mathbf{f}$ -recovery policy that converges in finite time. If $\mathbf{z}^0 \geq 0$ then $\mathbf{v}^0 = Q^{-1}\mathbf{z}^0 \geq 0$, followed by $\mathbf{v}^k = 0, k \geq 1$ does the job. If $\mathbf{z}^0$ has negative components then K cycles with $\mathbf{t}^k = 0$, i.e $\mathbf{v}^k = -\mathbf{t}$ results in $\mathbf{z}^{K+1} = \mathbf{z}^0 + K\mathbf{e}^T(\mathbf{s} + \mathbf{u})\mathbf{e}$ which is clearly nonnegative for sufficiently large K . By (2) $\sum_{k \leq K} \mathbf{v}^k = Q^{-1}(\mathbf{z} - \mathbf{z}^{K+1}) \rightarrow Q^{-1}\mathbf{z}$ as $K \rightarrow \infty$. (Q.E.D.)

A policy π is said to satisfy EZSR at $\mathbf{z}$ if all the control cycles generated by π satisfy the

EZSR when applied to $\mathbf{z}$. That is, all production runs start (resp., end) with nonpositive (resp., nonnegative) inventories.

In the next paragraph we describe a scheme to obtain a linear $\mathbf{f}$ -recovery policy that is optimal for all $\mathbf{z}$ that satisfies the EZSR.

First, we obtain an upper bound, say $V^\pi(\mathbf{z})$, on the excess cost of an $\mathbf{f}$ -recovery policy π applied to the state $\mathbf{z}$ with the property that $V^\pi(\mathbf{z})$ is tight whenever π has the EZSR property at $\mathbf{z}$. We then show that the problem of minimizing $V^\pi(\mathbf{z})$ over the class of $\mathbf{f}$ -recovery policies is equivalent to a time-invariant infinite-horizon linear-quadratic control problem. The latter problem is then solved by standard techniques in control theory, resulting in a linear $\mathbf{f}$ -recovery policy

$$\mathbf{v}^k = G\mathbf{z}^k. \quad (4)$$

To obtain $V^\pi(\mathbf{z})$ we first study the cost over a control cycle. Let $\mu_i(t)$ denote the inventory of item i at time t . The cost over $[\tau_k, \tau_{k+1}]$ is given by

$$\int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^m [h_i(\mu_i(t))^+ + b_i(\mu_i(t))^-] dt + \mathbf{e}^T \mathbf{a}. \quad (5)$$

where the integrals represent holding and backorder costs and $\mathbf{e}^T \mathbf{a}$ is the setup cost.

To facilitate the computation of the integral in (5), let α_i (resp., β_i) be the minimum (resp., maximum) of $\mu_i(t)$ over the k^{th} control cycle. We obtain closed form expressions for $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$. As with the γ_i 's (resp., δ_i 's), the α_i 's (resp., β_i 's) are attained immediately before (resp., after) the production runs. Let $\mathbf{R}$ (resp., $\mathbf{S}$) be the lower (resp., strictly lower) triangular part of the matrix $\mathbf{Q}$. Then α and β are related to γ and δ by

$$\alpha = \gamma + \mathbf{S}\mathbf{v} - \mathbf{z}, \quad (6)$$

and

$$\beta = \delta + \mathbf{R}\mathbf{v} - \mathbf{z}. \quad (7)$$

Equation (6) holds because $-e_i^T S v$ additional units of item i are demanded before i is produced; (7) holds because $R - S = \text{diag}(p) - I$, and $(p_i - 1)v_i$ additional units of item i are produced.

Figure 2

Let B (resp., H) be the $m \times m$ diagonal matrix with $b_i p_i / (p_i - 1)$ (resp., $h_i p_i / (p_i - 1)$) in the i^{th} position of the diagonal. We now compute the integral in (5) by adding a quadratic and a potential function. The quadratic parts computes the weighted areas of the backorder and inventory triangles induced by α and β . The potential function adjusts for the areas of the triangles that are outside the interval of the control cycle (see figure 2). The potential function is $g(x) = \frac{1}{2} \sum_i \{h_i (x_i^+)^2 - b_i (x_i^-)^2\}$. The integral in (5) is then given by

$\frac{1}{2} \{\alpha^{-T} B \alpha^- + \beta^{+T} H \beta^+ - \alpha^{+T} H \alpha^+ - \beta^{-T} B \beta^-\} + g(w - z) - g(w - z + Qv)$, with the obvious upper bound

$$\frac{1}{2} \{\alpha^T B \alpha + \beta^T H \beta\} + g(w - z) - g(w - z + Qv). \quad (8)$$

Note that (8) is equal to the integral in (5) whenever $\alpha \leq 0$, and $\beta \geq 0$, i.e., whenever the EZSR is satisfied during the control cycle.

An upper bound, $C(\mathbf{z}, \mathbf{v})$, on the excess cost of a control cycle with state $\mathbf{z}$ and control $\mathbf{v}$ is obtained by substituting (6) and (7) into (8), by adding the setup cost $\mathbf{e}^T \mathbf{a}$ and by subtracting the average cost $a(T + \mathbf{e}^T \mathbf{v})$. Of course $aT = \frac{1}{2} \sum \{\gamma^T B \gamma + \delta^T H \delta\} + \mathbf{e}^T \mathbf{a}$ is just the cost over an unperturbed cycle of the target schedule. The algebra is simplified by

Lemma 3.3 The vector $\gamma^T B + \delta^T H$ is equal to 0.

Proof: The i^{th} component is proportional to $b_i \gamma_i + h_i \delta_i$. Because $\delta_i \geq 0$ and $\gamma_i \leq 0$, the proportion of time item i is in stock is $\delta_i / (\delta_i - \gamma_i)$. By Lemma 2.1 this proportion is $b_i / (b_i + h_i)$, so $b_i \gamma_i + h_i \delta_i = 0$. (Q.E.D.)

After some algebra we obtain

$$C(\mathbf{z}, \mathbf{v}) = \frac{1}{2} \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix}^T \begin{pmatrix} C & D \\ D^T & E \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mathbf{v} \end{pmatrix} + \mathbf{c}^T \mathbf{v} + g(\mathbf{w} - \mathbf{z}) - g(\mathbf{w} - \mathbf{z} + Q\mathbf{v}) \quad (9)$$

where $C = H + B$, $D = BS^T + HR^T$, $E = S^T BS + R^T HR$ and the vector $\mathbf{c}^T = \gamma^T BS + \delta^T HR - \mathbf{a}e^T$.

Lemma 3.4 The matrix in (9) is positive definite.

Proof: The matrix in (9) is congruent to the diagonal matrix

$$\begin{pmatrix} C^{-1} & 0 \\ 0 & E - D^T C^{-1} D \end{pmatrix},$$

under the nonsingular transformation

$$\begin{pmatrix} C & D \\ 0 & I \end{pmatrix}.$$

C^{-1} (resp., $E - D^T C^{-1} D$) is in fact diagonal with positive diagonal elements $(p_i - 1)/p_i(b_i + h_i)$ (resp., $h_i b_i p_i / (b_i + h_i)(p_i - 1)$). (Q.E.D.)

An upper bound of the excess cost of π at $\mathbf{z}$ is given by

$$V^\pi(\mathbf{z}) = \sum_k C(\mathbf{z}^k, \mathbf{v}^k). \quad (10)$$

Theorem 3.5 Minimizing $V^\pi(\mathbf{z})$ over $\mathbf{f}$ -recovery policies is equivalent to the infinite-horizon linear-quadratic control problem

$$W^*(\mathbf{z}) = \inf_{\{\mathbf{v}^k\}} \frac{1}{2} \sum_{k=0}^{\infty} \begin{pmatrix} \mathbf{z}^k \\ \mathbf{v}^k \end{pmatrix}^T \begin{pmatrix} C & D \\ D^T & E \end{pmatrix} \begin{pmatrix} \mathbf{z}^k \\ \mathbf{v}^k \end{pmatrix} \quad (11a)$$

subject to

$$\mathbf{z}^{k+1} = \mathbf{z}^k - Q\mathbf{v}^k. \quad (11b)$$

Proof: Since π is a recovery policy $\mathbf{z}^k \rightarrow 0$; consequently the telescoping sum $\sum_k [g(\mathbf{w} - \mathbf{z}^k) - g(\mathbf{w} - \mathbf{z}^{k+1})] = g(\mathbf{w} - \mathbf{z}) - g(\mathbf{w})$. Also by Lemma 3.1 $\sum_k \mathbf{v}^k \rightarrow Q^{-1}\mathbf{z}$ as $k \rightarrow \infty$, then by (9) and (10) we have

$$V^\pi(\mathbf{z}) = \frac{1}{2} \sum_{k=0}^{\infty} \begin{pmatrix} \mathbf{z}^k \\ \mathbf{v}^k \end{pmatrix}^T \begin{pmatrix} C & D \\ D^T & E \end{pmatrix} \begin{pmatrix} \mathbf{z}^k \\ \mathbf{v}^k \end{pmatrix} + \mathbf{c}^T Q^{-1}\mathbf{z} + g(\mathbf{w} - \mathbf{z}) - g(\mathbf{w}).$$

The $\inf V^\pi(\mathbf{z})$ over $\mathbf{f}$ -recovery policies is then equal to $W^*(\mathbf{z}) + \mathbf{c}^T Q^{-1}\mathbf{z} + g(\mathbf{w} - \mathbf{z}) - g(\mathbf{w})$. (Q.E.D.)

Well known sufficient conditions for the existence and uniqueness of a solution to (11a-b) is that Q is invertible and that the matrix in (11a) be positive definite. These conditions were verified in Lemmas 3.1 and 3.4.

Problem (11a-b) can be solved by standard techniques in control theory resulting in the linear policy (4). The value function $W^*(\mathbf{z})$ is given by

$$W^*(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T M \mathbf{z}, \quad (12)$$

where the matrix M is the unique positive definite solution to the matrix algebraic Riccati equation (ARE)

$$M = M + C - (MQ - D)(E + Q^T M Q)^{-1} (MQ - D)^T. \quad (13)$$

G in (4) is called the gain matrix and is given by

$$G = (E + Q^T M Q)^{-1} (M Q - D)^T. \quad (14)$$

Equation (13) can be solve by successive approximation. See for example Dorato and Lewis (1971).

Corollary 3.6 The policy (4) with G given by (14) is optimal with respect to excess cost among all f-recovery policies that satisfy the EZSR at z .

Proof: By hypothesis (4) satisfies the EZSR at z so that $t + Gz^k \geq 0$ for all k , hence (4) is feasible. Let π be any f-recovery policy that satisfies the EZSR at z ; its excess cost is given by $V^\pi(z)$ which by Theorem 3.5 is at least as large as the excess cost of (4). (Q.E.D.)

The state updates under the optimal recovery policy (4) are obtained by substituting (4) in (2); i.e., $z^{k+1} = Xz^k = X^k z$ where $X = I - QG$. Because (4) is an f-recovery policy, X is a convergent matrix. Consequently, there is a norm, say $\|\cdot\|$, for which X is a contraction. We will use this fact to establish

Proposition 3.7 The set of initial disruptions z for which (4) satisfies the EZSR contains an open ball $B(\epsilon) = \{z : \|Xz\| < \epsilon\}$ with positive radius.

Proof: The minimum and maximum inventory vectors under (4) in the k^{th} cycle are obtained from (6) and (7) by substituting $z^k = X^k z$

$$\alpha^k = \gamma - (I - SG)X^k z, \quad (15)$$

$$\beta^k = \delta - (I - RG)X^k z \quad (16).$$

Let ϵ^* be the largest value of ϵ for which $B(\epsilon)$ satisfies $\alpha^0 \leq 0$ and $\beta^0 \geq 0$. Because γ and δ are bounded away from zero, $\epsilon^* \geq 0$. Because X is a contraction it follows that $B(\epsilon^*)$ is contained in the set that satisfies $\alpha^k \leq 0$ and $\beta^k \geq 0$ for all $k = 0, 1, 2, \dots$. Consequently, there is a nonempty set of initial states for which (4) is the optimal f-recovery strategy.(Q.E.D.)

When (4) does not satisfy the EZSR at z the runs may be set to $(t + Gz)^+$, eventually resulting in a state for which the EZSR property holds.

We now consider idling for $\theta \geq 0$ before we apply the f-recovery policy (4).

Proposition 3.8. The optimal idle time after a disruption, before applying the linear control policy (4), is given by

$$\theta^* = [a - c^T Q^{-1} e - e^T M z]^+ / e^T M e. \quad (17)$$

Proof: The recovery starts at time $\theta \geq 0$ from the state $z + \theta e$. We obtain θ^* by minimizing the excess cost over the interval $[0, \theta]$ plus the upper bound of the excess cost thereon; i.e., by minimizing $W^*(z + \theta e) + c^T Q^{-1}(z + \theta e) - a\theta$ over $\theta \geq 0$. This yields (17).

Of course (17) can be used heuristically at any time when the inventories are high.

The policy (4) was derived under the assumption that the first setup corresponds to the item in the first position of the production sequence. The sequence can be rotated, so a recovery policy exists starting from every position of the sequence. For instance if $f = (1, 2, 3)$ we can obtain G' and G'' for $f' = (2, 3, 1)$ and $f'' = (3, 1, 2)$. In fact both G' and G'' can be obtained from G by a simple transformation. What is ultimately important is to know how to react to a disruption from any position of the sequence. We can compute the excess cost of recovering from each position and start from that with smallest excess cost. This procedure can be used heuristically when convenient to generate jumps in f .

The linear policy obtained from (7) by forcing $\beta = \delta$, i.e., by setting $v = R^{-1}z$, is called the produce-up-to policy. Under a produce-up-to policy production of item i continues until its inventory reaches δ_i ; regardless of the inventories of the other items. Produce-up-to policies are very easy to implement. Surprisingly, produce-up-to policies are f-recovery policies; i.e., $z^k \rightarrow 0$ as $k \rightarrow \infty$. It can be shown that produce-up-to policies are optimal when backorders are proportional to processing times; i.e., b_i proportional to r_i . Moreover, because $R^{-1} \geq 0$, the set of states for which the produce-up-to policy satisfies the EZSR strictly contains the nonnegative orthant. See Gallego (1988b).

4. Random Demands and Safety Stocks.

In this section we assume that cumulative demands for the items are independent stochastic

processes governed by Brownian Motion. In particular, the cumulative demand of item i up to time t , $D_i(t)$ say, is normally distributed with mean t and variance $\sigma_i^2 t$.

We assume that after each repetition of $\mathbf{f}$, the state $\mathbf{Z}^k = \mathbf{z}^k$ is observed and the production runs are fixed to $\mathbf{t} + G\mathbf{z}^k$. The state $\mathbf{Z}^{k+1}$ is therefore normally distributed with mean $X\mathbf{z}^k$ and variance-covariance matrix $(T + \mathbf{e}^T G\mathbf{z}^k) \text{diag}(\sigma)^2$.

Proposition 4.1 The sequence of states $\{\mathbf{Z}^k\}$ converges to a normal random vector $\mathbf{Z}$ with mean zero and variance-covariance matrix Σ satisfying $\Sigma = T \text{diag}(\sigma)^2 + X\Sigma X^T$.

Proof: Because X is convergent $E\mathbf{Z}^{k+1} = XE\mathbf{Z}^k = X^{k+1}E\mathbf{Z}^0 \rightarrow 0$ as $k \rightarrow \infty$. Clearly, $E\mathbf{Z}^{k+1}(\mathbf{Z}^{k+1})^T = (T + \mathbf{e}^T G E\mathbf{Z}^k) \text{diag}(\sigma)^2 + X E\mathbf{Z}^k (\mathbf{Z}^k)^T X^T$. The result follows by taking the limit as $k \rightarrow \infty$. (Q.E.D.)

Given $\mathbf{Z} = \mathbf{z}$, set time $t = 0$ and the production runs to $\mathbf{t} + G\mathbf{z}$.

Let $X_i(t | \mathbf{z})$ be the inventory of item i at time t and let $p_i(t | \mathbf{z})$ be its cumulative production up to time t . Clearly, $X_i(t | \mathbf{z}) = w_i - z_i + p_i(t | \mathbf{z}) - D_i(t)$, and $E X_i(t | \mathbf{z}) = w_i - z_i + p_i(t | \mathbf{z}) - t \equiv \mu_i(t | \mathbf{z})$.

Given $\mathbf{Z} = \mathbf{z}$, the expected cost per cycle is given by

$$\sum_{i=1}^m \int_0^{T(\mathbf{z})} [h_i E X_i(t | \mathbf{z})^+ + b_i E X_i(t | \mathbf{z})^-] dt + \mathbf{e}^T \mathbf{a} \quad (18)$$

where $T(\mathbf{z}) = T + \mathbf{e}^T G\mathbf{z}$. The interchange of integration and expectation is justified since $X_i(t | \mathbf{z})^+$ and $X_i(t | \mathbf{z})^-$ are nonnegative. Note that since $hX^+ + bX^-$ is convex in X , Jensens' inequality implies that $hEX^+ + bEX^- \geq h(EX)^+ + b(EX)^-$. Consequently, (18) is at least as large as the cycle cost when demands are constant. In fact, if backorder (resp., holding) costs are expensive relative to holding (resp., backorders) costs then (18) can be large relative to its deterministic counterpart. Thus, it may pay to carry positive (resp., negative) safety stocks.

We will obtain a result similar to Lemma 2.1 relating safety stocks to service levels. Let $\mathbf{y} = (y_i)$ be a vector of safety stocks. Redefine the state so that $\mathbf{w} + \mathbf{y} - \mathbf{z}$ represents the actual inventories. Let $\Gamma(\mathbf{y} | \mathbf{z})$ be the expected holding and inventory costs when carrying $\mathbf{y}$ as safety stock and $\mathbf{Z}$

= z. Then the expected cost is

$$\Gamma(\mathbf{y}) = E \int_0^{T(\mathbf{Z})} \sum_i [h_i E(X_i(t | \mathbf{Z}) + y_i)^+ + b_i E(X_i(t | \mathbf{Z}) + y_i)^-] dt, \quad (19)$$

with expected cycle length $T + \mathbf{e}^T G E \mathbf{Z} = T$. By the reward renewal theorem the long run average cost converges, with probability one, to $(\Gamma(\mathbf{y}) + \mathbf{e}^T \mathbf{a})/T$. Thus, minimizing the long run average cost is equivalent to minimizing (19).

Proposition 4.2 $\Gamma(\mathbf{y})$ is separable and strictly convex in $\mathbf{y}$.

Proof: Separability is obvious. Moreover

$$hE(X + y)^+ + bE(X + y)^- = (h + b)E(X + y)^+ - by.$$

Taking derivative of the last expression, we get

$$(h + b)\Phi((y + \mu)/\sigma\sqrt{t}) - b,$$

where Φ is the cumulative distribution of the standard normal. The second derivative is $(h + b)\phi((y + \mu)/\sigma\sqrt{t})/\sigma\sqrt{t} > 0$ where ϕ is the density of the standard normal.

We now seek the vector $\mathbf{y}$ of safety stocks that minimizes $\Gamma(\mathbf{y})$.

Theorem 4.3 There is a unique vector of safety stocks $\mathbf{y}^* = (y_i^*)$ that minimizes $\Gamma(\mathbf{y})$. Moreover, y_i^* is optimal if and only if the expected proportion of time item i is in stock is $b_i/(b_i + h_i)$ for all i .

Proof: By proposition 4.2 $\Gamma(\mathbf{y})$ is strictly convex, so it has a unique minimum satisfying the first order conditions which are equivalent to

$$E \int_0^{T(\mathbf{Z})} E \mathbf{1}\{t : X_i(t | \mathbf{Z}) + y_i \geq 0\} dt = b_i T / (b_i + h_i).$$

The result is satisfied by y_i^* after dividing by T , the expected cycle length. Note that these service rates are identical to those of the target schedule! Hence, safety stocks are carried to preserve that property.

How do you compute it?

Remark: Demands need not be Brownian Motion. All that is needed is a continuous distribution with finite second moment, constant expected demand rates and independent stationary increments.

5. EXAMPLE.

We illustrate the scheduling tool with an example. The data was obtained as follows: holding costs $U(5,100)$, backorder cost $U(5000,50000)$, production rates $U(4,8)$, set up times $U(1,5)$ and set up cost $U(50,500)$. Additional products were included as long as $\kappa > 0.15$.

Data: $m = 5$, $h = (57, 77, 55, 73, 72)$, $b = (16495, 10613, 46907, 39997, 38774)$, $s = (1, 4, 3, 2, 2)$, $p = (6, 5, 5, 6, 6)$, $a = (188, 308, 179, 404, 69)$.

We used the procedure developed in Gallego and Roundy (1988) to compute the target cyclic schedule. This resulted in $f = (4, 5, 1, 2, 4, 5, 1, 3)$, $t = (14.295, 14.252, 14.223, 34.000, 14.039, 14.081, 14.110, 34.000)$, $w = (33.305, 50.790, 135.841, 1.872, 18.164)$. The target schedule had no idle time and a cycle length of 170. The proportion of time available for setups was 10%. The average cost per unit time was \$16,082.66. A lower bound on the average cost developed in Gallego and Roundy (1988) was \$15,672.17 for a ratio of 1.0262.

The optimal linear control policy was computed by solving the Matrix Ricatti Equation (13) resulting in

$$M = \begin{pmatrix} 48283.696 & 35874.056 & 41738.979 & 31551.926 & 31862.901 \\ 35874.056 & 53112.605 & 51022.447 & 35228.877 & 36135.278 \\ 41738.979 & 51022.447 & 82117.526 & 43172.108 & 40981.590 \\ 31551.926 & 35228.877 & 43172.108 & 77339.632 & 36763.250 \\ 31862.901 & 36135.278 & 40981.590 & 36763.250 & 74836.479 \end{pmatrix}.$$

The gain matrix (14) is given by

$$G = \begin{pmatrix} 0.008005 & 0.023333 & -0.056388 & 0.191382 & -0.018039 \\ 0.010040 & 0.026552 & -0.056557 & 0.032556 & 0.182747 \\ 0.181386 & 0.010007 & -0.051487 & 0.029222 & 0.006205 \\ -0.054597 & 0.147695 & -0.203368 & -0.035185 & -0.065656 \\ 0.033625 & 0.030715 & 0.003266 & 0.025340 & 0.059093 \\ 0.039006 & 0.034027 & 0.007473 & 0.022310 & 0.031044 \\ 0.018261 & 0.030434 & -0.041354 & 0.006845 & 0.026886 \\ 0.105283 & 0.128406 & 0.217835 & 0.111230 & 0.104626 \end{pmatrix}.$$

We now assume that demands are Brownian Motion and that $\sigma = (0.4, 0.3, 0.5, 0.4, 0.3)$; i.e. demands are fairly variable. The long run average cost of following the target schedule with the recovery policy was \$33,276.32. To obtain the optimal safety stocks we used Monte Carlo simulation to find unbiased estimates of the gradient of $\Gamma(\mathbf{y})$. We used the Robbins-Monro procedure to find the root of the gradient. Each estimation of the gradient took 1000 cycles where the first 50 were ignored to avoid initialization bias. The step sizes were chosen as $\frac{1}{k}$ starting with $k = 10$ in the negative direction of the gradient. These resulted in safety stocks $\mathbf{y} = (9.87, 17.41, 12.36, 7.83, 5.83)$. The proportion of times the products were in stock with these safety stocks was $(0.99655, 0.99280, 0.99882, 0.99818, 0.99815)$. Compare with the proportions of the target schedule $(0.99656, 0.99280, 0.99883, 0.99818, 0.99815)$. The long run average cost with the optimal safety stocks was \$20,781.01, a savings of 38%. Note that the long run average cost of scheduling the facility with random demands is only 23% more than the cost of the target schedule and only 6% more than the cost of the target schedule with safety stocks under deterministic demands.

The effort of computing and using the optimal safety stocks pays handsome dividends when backorders are expensive relative to holding costs, as they are in the ELSP. Since relatively large backorder costs are related to large service rates through the ratio $b/(b+h)$ the scheduling tool can be used by specifying the service rates rather than the backorder costs. Those who object to the backorder assumption made in this paper could set $b = 9999h$ thus achieving a service rate of 99.99%, in which case backorders are indeed a rare event.

Appendix

We extend the results of section 3 to the case where $\mathbf{f}$ is not restricted to permutations of $\{1, 2, \dots, m\}$.

Let $\mathbf{f} = (f_1, f_2, \dots, f_n)$, recall that $n \geq m$. Let F be the $n \times m$ matrix where the ji^{th} component, F_{ji} , is equal to 1 if $f_j = i$ and 0 otherwise. Since all the unit vectors in m -space appear as rows of F , F has full row rank, and, hence, has a left inverse.

The state $\mathbf{z}$ remains in m -space and the control $\mathbf{v}$ now lives in n -space. The inventories at the end of the control cycle are $(\mathbf{w} - \mathbf{z}) + [\text{diag}(\mathbf{p}) - \mathbf{e}\mathbf{e}^T]F^T\mathbf{v} = \mathbf{w} - (\mathbf{z} - QF^T\mathbf{v})$ so the state updates

are given by

$$\mathbf{z}^{k+1} = \mathbf{z}^k - QF^T \mathbf{v}^k. \quad (\text{A1})$$

The above is justified by $\text{diag}(\mathbf{p})F^T \mathbf{t} = T\mathbf{e}$; i.e., total production is equal to total demand through a cycle.

Proposition A.1 The set of f-recovery policies is non-empty. Moreover, $F \sum_k \mathbf{v}^k$ is independent of the f-recovery policy.

Proof: Identical to Proposition 3.2 using the fact that F has full row rank. For any f-recovery policy we have, by (A1), $F^T \sum_k \mathbf{v}^k = Q^{-1}\mathbf{z}$. (Q.E.D.)

Consider the $n \times n$ matrix FQF^T . The jk^{th} component is equal to $p_i - 1$ if item i is produced at both positions j and k , else it is -1 .

Let $R(\text{resp.}, S)$ be the lower (resp., strictly lower) triangular part of FQF^T . Then the vectors of minimum and maximum inventories in a control cycle, α and β , are related to the vectors of minimum and maximum inventories in the target schedule, γ and δ , by $\alpha = \gamma + S\mathbf{v} - F\mathbf{z}$, and $\beta = \delta + R\mathbf{v} - F\mathbf{z}$.

Let B and H be $m \times m$ diagonal matrices as in Section 3. Let $B'(\text{resp.}, H')$ be the $n \times n$ diagonal matrix obtained by setting to zero all the nondiagonal elements of $FBF^T(\text{resp.}, FHF^T)$. Then an upper bound on the cost of the control cycle is given by $\mathbf{e}^T F^T F \mathbf{a} + \frac{1}{2} \{ \alpha^T B' \alpha + \beta^T H' \beta \} + g(\mathbf{w} - \mathbf{z}) - g(\mathbf{w} - \mathbf{z}')$. The bound is tight whenever $\alpha \leq 0$ and $\beta \geq 0$. An upper bound on the excess cost over the control cycle is given by (9) where now $C = F^T(B' + H')F$, $D = -F^T(B'S + H'R)$, $E = S^T B'S + R^T H'R$ and $\mathbf{c}^T = \gamma^T B'S + \delta^T H'R - \mathbf{e}^T F^T = \mathbf{c}'^T F^T$. Here we use the fact that $(\gamma^T B' + \delta^T H')F = 0$ and the fact that $\mathbf{c}^T$ is in the column space of F . Both facts follow from the definition of an optimal f-cyclic schedule that satisfies the EZSR and from Corollary 2.2.

Combining the above with Proposition A.1 and following the arguments of Section 3, we obtain an infinite-horizon linear quadratic linear-control problem similar to (11a-b) where (11b) is replaced by (A1) and in (11a) we use the new definition for C, D and E . The value function $W^*(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T M \mathbf{z}$ is obtained by solving the algebraic matrix Ricatti equation

$$M = M + C - (MQF^T - D)(E + FQ^T MQF)^{-1}(MQF^T - D)^T.$$

With corresponding optimal gain matrix

$$G = (E + FQ^T MQF)^{-1}(MQF^T - D)^T.$$

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