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ASYMPTOTIC EXPANSIONS FOR WAITING TIME  
PROBABILITIES IN AN M/G/1 QUEUE WITH  
LONGTAILED SERVICE TIME

by

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# Asymptotic expansions for waiting time probabilities in an M/G/1 queue with longtailed service time

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## Abstract

We consider an  $M/G/1$  queue with FCFS queue discipline. We present asymptotic expansions for tail probabilities of the stationary waiting time when the service time distribution is longtailed and we discuss an extension of our methods to the  $M^{[X]}/G/1$  queue with batch arrivals.

Keywords and Phrases:  $M/G/1$  queue, waiting time, regular variation, subexponentiality

## 1 introduction

Consider an  $M/G/1/FCFS$  queue in which customers arrive according to a Poisson process with intensity  $\lambda > 0$ . The interarrival times of the customers are gathered in the sequence  $(A_i)_{i=1}^{\infty}$  and the service times are denoted by  $(B_i)_{i=1}^{\infty}$ . The probability distribution function (p.d.f) of  $B_i$  is denoted by  $F$  and the mean service time is  $\mathbf{E}(B_i) = \int_0^{\infty} 1 - F(x) dx = 1/\mu$ . If we denote the successive waiting times (in the queue) of the customers by  $(W_i)_{i=1}^{\infty}$  then it is evident that  $W_i$  satisfies the following recursive relation (cf. Feller(1971))

$$W_{n+1} = \max(W_n + B_n - A_n, 0), \quad n \geq 0$$

where it is understood that  $W_0 = A_0 = B_0 = 0$ . Putting  $T_n = B_n - A_n$  and  $S_n = \sum_{i=0}^n T_i$ ,  $0 \leq n < \infty$ , it follows that the stationary waiting time  $W_{\infty}$  is given by

$$W_{\infty} \stackrel{d}{=} \max_{n \geq 0} S_n \tag{1}$$

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where  $\stackrel{d}{=}$  has to be interpreted as equality in distribution. This shows that  $W_\infty$  is the maximum of a random walk which is generated by the distribution of  $B_1 - A_1$ . It is well known that  $W_\infty$  is a proper random variable if the random walk drifts to  $-\infty$ , i.e. if  $\mathbf{E}(B_1 - A_1) < 0$  or  $\rho := \lambda/\mu < 1$  (see e.g. Feller(1971) or Rogozin(1966)). Under this condition the distribution of  $\max_{n \geq 0} S_n$  may be found from Wiener-Hopf theory and it is known that

$$P(W_\infty \leq x) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n H^{*n}(x), \quad x \geq 0. \quad (2)$$

Here  $H$  is the p.d.f of the first ascending ladder height in the random walk (or the right Wiener-Hopf factor of  $B_1 - A_1$ ) and  $H^{*n}$  denotes the  $n$ -fold convolution of  $H$  with itself, i.e.  $H^{*(n+1)} = H * H^{*n}$  and  $H^{*0}$  is the p.d.f of a random variable degenerate in 0. Since in an M/G/1 queue  $A_1$  is exponentially distributed,  $H$  is known explicitly and equals (cf. Feller(1971))

$$H(x) = \mu \int_0^x 1 - F(y) dy,$$

which is the stationary residual lifetime distribution of  $F$ . Except for some special cases, the infinite series in (2) is hard to interpret and almost impossible to write in a compact form. Efficient numerical procedures for calculating the waiting time distribution when the service time is of phase-type were developed by Tijms and Van Hoorn(1981). These authors also treat the situation of finite capacity and state dependent entrance rates. In that case we can still express  $W_\infty$  as in (1) but  $(S_n)_{n=0}^\infty$  is no longer a random walk as the summands are dependent and not identically distributed. Other approximations based on the matching of moments were proposed by Kühn(1972,1976).

In this paper we primarily deal with approximating waiting time probabilities when the underlying service time distribution is longtailed, i.e. is not of phase-type. We concentrate on distributions of Pareto-type, lognormal and Weibull-type with shape parameter smaller than 1. All these distributions share the property that their tail function decreases slower than any exponential function and for that reason they are called 'sub'exponential. We show in the next sections that the class of subexponential distributions is particularly useful in obtaining asymptotic approximations for tail probabilities of the waiting time. The second part of section 3 is concerned with an extension of our methods to the  $M^{[X]}/G/1$  queue with batch arrivals.

## 2 Asymptotic approximations of compound tail probabilities

Let us first take a closer look at (2) when the right tail of  $H$  is exponentially bounded. It follows from (2) that

$$\psi(x) := P(W_\infty > x | W_\infty > 0) = \frac{1-\rho}{\rho} \sum_{n=1}^{\infty} \rho^n (1 - H^{*n}(x))$$

and it is not hard to see that  $\psi$  satisfies the following renewal equation

$$\psi(x) = (1 - H(x)) + \rho \psi * H(x). \quad (3)$$

Applying the key-renewal theorem then yields that

$$\psi(x) \sim \frac{\hat{h}(-\gamma) \cdot (1 - \hat{h}(-\gamma))}{\gamma \hat{h}'(-\gamma)} \cdot \exp -\gamma x \quad , x \rightarrow \infty \quad (4)$$

with  $\hat{h}$  denoting the Laplace-Stieltjes transform of  $H$  and with  $\gamma$  a positive real number satisfying  $\hat{h}(-\gamma) = \rho$ . If the service time of the queue is of phase-type, i.e. its tail is decreasing exponentially fast, also  $H$  is of phase-type (Neuts(1981, p 52)) and then for any  $0 < \rho < 1$  there exists  $\gamma > 0$  with  $\hat{h}(-\gamma) = \rho$  such that (4) holds for any  $0 < \rho < 1$ . However, if  $\hat{h}$  is finite at its left abscis of convergence (denoted by  $\xi$ ), (4) is only valid for  $\rho$  restricted to the range  $[a, 1)$  with  $a = \hat{h}^{-1}(\xi)$ . If the service time is longtailed, i.e.  $\xi = 0$ , then (4) is not valid for any value of  $\rho$  and there is need for another method to approximate  $\psi(x)$  which we present below.

In order to generalize the subject a little we take a discrete probability measure  $(p_n)_{n=0}^{\infty}$  and we consider the compound p.d.f

$$G(x) = \sum_{n=0}^{\infty} p_n H^{*n}(x) \quad , x \geq 0. \quad (5)$$

To avoid trivialities we always assume that  $p_0 + p_1 < 1$ . We primarily deal with longtailed p.d.f  $H$ , which are defined as follows

**Definition 1** *Let  $H$  be a p.d.f on  $[0, \infty)$  such that  $H(x) < 1$  for every  $x \in \mathbb{R}$ . Then*

$$\begin{aligned} &H \text{ is longtailed } (H \in \mathcal{L}) \\ &\text{iff} \\ &\lim_{x \rightarrow \infty} \frac{1 - H(x-y)}{1 - H(x)} = 1 \quad , \forall y \in \mathbb{R}. \end{aligned}$$

Some archetypes of p.d.f in  $\mathcal{L}$  are

(a) The Pareto family, i.e.

$$H(x) = 1 - (x - x_0 + 1)^{-\alpha}, \quad x > x_0 > 0, \quad \alpha > 0. \quad (6)$$

(b) The lognormal distributions, i.e.

$$H'(x) = \frac{1}{\sigma x \sqrt{2\pi}} \exp -\frac{1}{2\sigma^2}(\log x - \log \alpha)^2, \quad x > 0, \quad \sigma, \alpha > 0. \quad (7)$$

(c) The Weibull family with shape parameter smaller than 1, i.e.

$$H(x) = 1 - \exp -(\alpha x)^\beta, \quad \alpha > 0, \quad 0 < \beta < 1. \quad (8)$$

The main aim of this paper is to approximate  $G(x)$  in (5) when the underlying p.d.f  $H$  is longtailed. In order to motivate the type of approximation one may expect, we turn to the following probabilistic interpretation of (5): let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with common p.d.f  $H$  and let  $N$  be a discrete random variable with  $P(N = n) = p_n$ ,  $n \geq 0$ . Then  $G(x)$  is the p.d.f of the random sum  $S_N = X_1 + X_2 + \dots + X_N$ . Now if  $H$  is longtailed, large observations  $X_i$  may occur with high probability and it is not unreasonable to think that the random sum  $S_N$  may be governed by just one outstanding summand. For that reason it might be possible to relate the tail behaviour of  $G$  to that of  $H$ . A lot of attempts have been made in the literature to do so and it was first shown by Cohen(1973) that

$$1 - G(x) \sim \left( \sum_{n=1}^{\infty} n p_n \right) \cdot (1 - H(x)) \quad , \quad x \rightarrow \infty \quad (9)$$

if  $(p_n)_{n=0}^{\infty}$  is a Poisson sequence and if

$$1 - H(x) \sim x^{-\alpha} L(x) \quad (10)$$

for some  $\alpha > 0$  and  $L$  slowly varying, i.e.  $L(xt)/L(x) \rightarrow 1$  as  $x \rightarrow \infty$ ,  $\forall t > 0$  (cf. Bingham et al.(1987)). If  $1 - H$  satisfies (10), it is called regularly varying with index  $-\alpha$  (notation  $1 - H \in \mathcal{R}_{-\alpha}$ ). Notice that every p.d.f of Pareto type belongs to  $\cup_{\alpha \geq 0} \mathcal{R}_{-\alpha}$  and that  $\cup_{\alpha \geq 0} \mathcal{R}_{-\alpha} \subset \mathcal{L}$ . One may ask whether the class of p.d.f with regularly varying tail can be extended so as to characterize the asymptotic behaviour in (9). It turns out that the proper class for this purpose is the class  $\mathcal{S}$  of *subexponential* distributions, defined below.

**Definition 2** *Let  $H$  be a p.d.f on  $[0, \infty)$  and let  $H(x) < 1$  for every  $x \in \mathbb{R}$ . Then*

$H$  is subexponential ( $H \in \mathcal{S}$ )

iff

$$\lim_{x \rightarrow \infty} \frac{1 - H^{*2}(x)}{1 - H(x)} = 2. \quad (11)$$

The class  $\mathcal{S}$  was originated independently by Chistyakov(1964) and Chover et al.(1973) and has been thoroughly investigated among others by Teugels(1975), Pitman(1980), Embrechts and Goldie(1980,1982) and Willekens(1988). It is known that  $\mathcal{S}$  is a proper subclass of  $\mathcal{L}$  though examples in  $\mathcal{L} \cap \mathcal{S}^c$  are mainly pathological. Furthermore the three archetypes of  $\mathcal{L}$  mentioned above also belong to  $\mathcal{S}$ . A very important probabilistic property of the class  $\mathcal{S}$  is the characterization of the asymptotic behaviour in (9). The most general result in this direction is due to Cline(1986):

**Theorem 1** *Let  $(p_n)_{n=0}^{\infty}$  be a discrete probability measure such that its probability generating function  $P(z) = \sum_{n=0}^{\infty} p_n z^n$  is analytic at 1 and let  $G$  be given by (5). Then the following assertions are equivalent:*

- (i)  $H \in \mathcal{S}$
- (ii)  $G \in \mathcal{S}$
- (iii)  $1 - G(x) \sim (\sum_{n=1}^{\infty} np_n)(1 - H(x)) \quad , x \rightarrow \infty.$

The fact that  $P(z)$  has to be analytic at 1 is no real restriction in practical problems and only means that the sequence  $(p_n)_n$  has to decrease exponentially fast, which is for example the case for Poisson, geometric or negative binomial sequences. On the contrary, when  $(p_n)_n$  is not exponentially decreasing, e.g.  $p_n \sim n^{-\alpha}$  for some  $\alpha > 0$ , then theorem 1 is no longer valid and it was shown by Stam(1973) that the asymptotic constant  $(\sum_{n=1}^{\infty} np_n)$  may be altered. Whereas theorem 1 provides a first order approximation to the tail of  $G$ , we are interested in its accuracy and we therefore investigate the behaviour of the remainder term

$$R(x) := 1 - G(x) - \left( \sum_{n=1}^{\infty} np_n \right) (1 - H(x)).$$

This was done by Omey and Willekens(1986,1987) and their main result reads as follows:

**Theorem 2** Suppose  $P(z)$  is analytic at 1. Let  $H \in \mathcal{S}$  be a p.d.f with density  $H'=h$  and let  $1/\mu = \int_0^\infty 1 - H(x) dx < \infty$ . Then the following assertions are equivalent:

$$(i) \quad \lim_{x \rightarrow \infty} \frac{1 - H^{*2}(x) - 2(1 - H(x))}{h(x)} = \frac{2}{\mu} \quad (12)$$

$$(ii) \quad \lim_{x \rightarrow \infty} \frac{1 - G(x) - (\sum_{n=1}^\infty np_n)(1 - H(x))}{h(x)} = \frac{2}{\mu} \sum_{n=2}^\infty \binom{n}{2} p_n.$$

Notice that theorem 2 is very powerful as it does not only yield the right rate of convergence of  $R(x)$  but also provides the correct asymptotic constant. The minor drawback for practical purposes is that condition (i) is hard to check for a given p.d.f  $H$ .

One may ask whether the above approximation procedure can be iterated, i.e. is it possible to determine the behaviour of

$$1 - G(x) - (\sum_{n=1}^\infty np_n)(1 - H(x)) - \frac{2}{\mu} \sum_{n=2}^\infty \binom{n}{2} p_n \cdot h(x) \quad \text{as } x \rightarrow \infty.$$

In view of theorems 1 and 2 one would expect that the rate of convergence in this case will be given by the derivative of  $h(h')$  if it exists. The following result was achieved by Omey and Willekens(1987):

**Theorem 3** Suppose  $P(z)$  is analytic at 1. Let  $H \in \mathcal{S}$  be a p.d.f with differentiable density  $h$  and let  $1/\mu = \int_0^\infty 1 - H(x) dx < \infty$ .

If

$$\lim_{x \rightarrow \infty} \frac{H^{*3}(x) - 3H^{*2}(x) + 3H(x) - 1}{h'(x)} = \frac{3}{\mu^2} \quad (13)$$

then

$$\lim_{x \rightarrow \infty} \frac{1 - G(x) - (\sum_{n=1}^\infty np_n)(1 - H(x)) + \sum_{n=2}^\infty \binom{n}{2} p_n \cdot (1 - H)^{*2}(x)}{-h'(x)}$$

$$= \frac{3}{\mu^2} \sum_{n=3}^\infty \binom{n}{3} p_n.$$

As indicated by Omey and Willekens(1987) it is possible to continue this procedure and expand the tail of  $G(x)$  in a series containing successive derivatives of  $H$ . However, the higher the order of approximation, the more impracticable

and cumbersome the conditions. We restrict ourselves to third order approximations and we show below that the condition in (13) is satisfied for the three main types of longtailed p.d.f which we introduced before.

**Lemma 1** *Let  $H$  be a p.d.f with a twice differentiable density and suppose that  $h'' \in \mathcal{R}_{-\alpha}$  with  $\alpha > 4$ . Let  $1/\mu = \int_0^\infty 1 - H(x) dx$  and  $1/\mu_2 = 2 \int_0^\infty x(1 - H(x)) dx < \infty$ . Then*

$$(i) \lim_{x \rightarrow \infty} \frac{1 - H^{*2}(x) - 2(1 - H(x)) - \frac{2}{\mu}h(x)}{-h'(x)} = \frac{1}{\mu_2}$$

and

$$(ii) \lim_{x \rightarrow \infty} \frac{H^{*3}(x) - 3H^{*2}(x) + 3H(x) - 1}{h'(x)} = \frac{3}{\mu^2}$$

**Proof**

Part (ii) was proved by Willekens(1986). As to (i), we prove the stronger statement

$$h^{\times 2}(x) - 2h(x) + \frac{2}{\mu}h'(x) \sim \frac{1}{\mu_2}h''(x) \quad (x \rightarrow \infty)$$

which implies (i) by de l'Hôpital's rule ( $\times$  denotes the density convolution product, i.e.  $h^{\times 2}(x) = \int_0^x h(x-y)h(y) dy$ ). Notice that

$$\begin{aligned} & h^{\times 2}(x) - 2h(x) + \frac{2}{\mu}h'(x) \\ &= 2 \int_0^{x/2} h(x-y)h(y) dy - 2h(x) + \frac{2}{\mu}h'(x) \\ &= 2 \int_0^{x/2} (h(x-y) - h(x) + yh'(x))h(y) dy - 2h(x)(1 - H(x/2)) \\ &\quad + 2h'(x) \int_{x/2}^\infty yh(y) dy \\ &= (I) + (II) + (III). \end{aligned}$$

By the regular variation of  $h''$  and Karamata's theorem(cf. Bingham et al. (1987)),

(II) + (III) =  $o(h''(x))$  as  $x \rightarrow \infty$ . Since

$$h(x-y) - h(x) + yh'(x) = \int_0^y \int_0^z h''(x-u) du dz,$$



it follows from the uniform convergence theorem for regularly varying functions (Bingham et al. (1987)) and Lebesgue's theorem on dominated convergence that

$$(I) \sim \left( \int_0^{x/2} y^2 h(y) dy \right) h''(x) \sim \frac{1}{\mu_2} h''(x) \quad (x \rightarrow \infty).$$

This proves the lemma.

**Lemma 2** *Let  $H \in \mathcal{S}$  with  $h''$  asymptotically decreasing. Suppose there exists a positive function  $\chi \in \mathcal{R}_\gamma$  ( $0 < \gamma \leq 1$ ) such that*

$$\lim_{x \rightarrow \infty} \frac{h''(x + y\chi(x))}{h''(x)} = \exp -y \quad \forall y \in \mathbb{R}. \quad (14)$$

*Finally, let  $h(x - y)h(y)$  be monotone decreasing for large  $x$  and  $c \leq y \leq x/2$  where  $c > 0$  is some constant. Then (i) and (ii) of lemma 1 hold.*

**Remark.**

It is readily verified that  $H$  in (7) or (8) satisfies the conditions of lemma 2 with  $\chi(x) = \sigma^2 x / \log x$  if  $H$  is given by (7) and  $\chi(x) = (\alpha^{-\beta} / \beta) x^{1-\beta}$  if  $H$  is given by (8).

**Proof of lemma 2**

Since  $h''$  satisfies (14) it follows from de Haan (1970) that  $\chi(x)$  may be chosen as  $-h'(x)/h''(x)$  and that

$$-h'(x)/h''(x) \sim h(x)/-h'(x) \sim (1 - H(x))/h(x) = o(x) \quad (x \rightarrow \infty). \quad (15)$$

Let  $A(x)$  be a positive function such that  $A(x) \rightarrow \infty$  but  $A(x) = o(\chi(x))$  as  $x \rightarrow \infty$  (Take e.g.  $A(x) = \frac{x}{(\log x)^2}$  if  $H$  is given by (7) and  $A(x) = \frac{x^{1-\beta}}{\log x}$  if  $H$  is as in (8)). Then by the asymptotic monotonicity of  $h''$  we have that

$$h''(x - y)/h''(x) \rightarrow 1 \quad (x \rightarrow \infty) \quad (16)$$

uniformly in  $y \in [0, A(x)]$ .

Proof of (i)

As in lemma 1 we show that

$$h^{x^2}(x) - 2h(x) + \frac{2}{\mu} h'(x) \sim \frac{1}{\mu_2} h''(x) \quad (x \rightarrow \infty).$$

Now

$$\begin{aligned}
& h^{\times 2}(x) - 2h(x) + \frac{2}{\mu}h'(x) \\
&= 2 \int_0^{A(x)} h(x-y)h(y) dy - 2h(x) + \frac{2}{\mu}h'(x) + \int_{A(x)}^{x-A(x)} h(x-y)h(y) dy \\
&= 2 \int_0^{A(x)} (h(x-y) - h(x) + yh'(x))h(y) dy - 2h(x)(1 - H(A(x))) \\
&\quad + 2h'(x) \int_{A(x)}^{\infty} yh(y) dy + \int_{A(x)}^{x-A(x)} h(x-y)h(y) dy \\
&= (I) + (II) + (III) + (IV).
\end{aligned}$$

Since

$$h(x-y) - h(x) + yh'(x) = \int_0^y \int_0^z h''(x-u) du dz,$$

it follows from (16) that  $(I) \sim \frac{1}{\mu^2}h''(x)$  ( $x \rightarrow \infty$ ). Also, by (15) and the choice of  $A(x)$  we have that  $(II) + (III) = o(h''(x))$  ( $x \rightarrow \infty$ ). As to  $(IV)$  it follows from the conditions of the lemma that  $h(x-y)h(y)$  is monotone decreasing on the interval  $[A(x), x/2]$  which implies that  $|(IV)| \leq xh(x-A(x))h(A(x)) = o(h''(x))$  ( $x \rightarrow \infty$ ). This proves (i).

Proof of (ii)

Put  $R_2(x) := 1 - H^{\ast 2}(x) - 2(1 - H(x))$ . Then from the proof of (i),

$$\lim_{x \rightarrow \infty} \frac{R_2'(x)}{h'(x)} = \frac{2}{\mu}. \quad (17)$$

Now

$$\begin{aligned}
& 1 - 3H(x) + 3H^{\ast 2}(x) - H^{\ast 3}(x) \\
&= \int_0^x (R_2(x-y) - R_2(x))h(y) dy - R_2(x)(1 - H(x)) \\
&= \left\{ \int_0^{A(x)} + \int_{A(x)}^{x-A(x)} + \int_{x-A(x)}^x \right\} (R_2(x-y) - R_2(x))h(y) dy \\
&\quad - R_2(x)(1 - H(x)) \\
&= \{(I) + (II) + (III)\} + (IV).
\end{aligned}$$

It follows from (15) and (17) that  $(IV) = o(h'(x))$  ( $x \rightarrow \infty$ ). Again by (17),

$$(I) = - \int_0^{A(x)} \int_0^y R_2'(x-z) dz h(y) dy \sim -\frac{2}{\mu^2}h'(x) \quad (x \rightarrow \infty). \quad (18)$$

Also

$$\begin{aligned}
(III) &= \int_0^{A(x)} R_2(y)(h(x-y) - h(x)) dy \\
&\quad + h(x) \int_0^{A(x)} R_2(y) dy - R_2(x)(H(x) - H(x - A(x))) \\
&= (III)_a + (III)_b + (III)_c.
\end{aligned}$$

Clearly by (17),  $(III)_c = o(h'(x))$  ( $x \rightarrow \infty$ ). Since  $\int_0^{A(x)} R_2(y) dy = -\int_{A(x)}^{\infty} R_2(y) dy$ , it follows from (15) and (17) that  $(III)_b = o(h'(x))$  ( $x \rightarrow \infty$ ). Finally by (15) and (16), we have that  $(III)_a \sim -h'(x) \int_0^{A(x)} y R_2(y) dy \sim -\frac{1}{\mu^2} h'(x)$  ( $x \rightarrow \infty$ ). Combining the estimates shows that

$$(III) \sim -\frac{1}{\mu^2} h'(x) \quad (x \rightarrow \infty). \quad (19)$$

It follows from (18) and (19) that the proof is finished if we can show that

$$(II) = o(h'(x)) \quad (x \rightarrow \infty).$$

Since  $y \in [A(x), x - A(x)]$ ,  $x - y \geq A(x)$  such that by (17),

$$R_2(x - y) - R_2(x) = -\int_0^y R_2'(x - z) dz \sim \frac{2}{\mu}(h(x - y) - h(x)) \quad (x \rightarrow \infty).$$

Using this together with (15) and (17) then yields

$$\begin{aligned}
\frac{\mu}{2}(II) &\sim \int_{A(x)}^{x-A(x)} h(y)(h(x-y) - h(x)) dy \\
&= h^{\times 2}(x) - 2 \int_0^{A(x)} h(y)h(x-y) dy - h(x) \int_{A(x)}^{x-A(x)} h(y) dy \\
&= (h^{\times 2}(x) - 2h(x)) - 2 \int_0^{A(x)} h(y)(h(x-y) - h(x)) dy \\
&\quad + 2h(x)(1 - H(A(x))) - h(x)(H(x - A(x)) - H(A(x))) \\
&\sim -\frac{2}{\mu}h'(x) + \frac{2}{\mu}h'(x) + o(h'(x)) \quad (x \rightarrow \infty). \\
&= o(h'(x)) \quad (x \rightarrow \infty).
\end{aligned}$$

This completes the proof.

Combining lemma's 1 and 2 together with theorem 3 yields the following expansion for  $G(x)$ .

**Corollary 1** *Suppose  $P(z)$  is analytic at 1 and let  $H$  satisfy the conditions of lemma 1 or 2. Then*

$$\begin{aligned} 1 - G(x) &= \left( \sum_{n=1}^{\infty} n p_n \right) (1 - H(x)) + \frac{2}{\mu} \sum_{n=2}^{\infty} \binom{n}{2} p_n \cdot h(x) \\ &\quad - \left( \frac{3}{\mu^2} \sum_{n=3}^{\infty} \binom{n}{3} p_n + \frac{1}{\mu_2} \sum_{n=2}^{\infty} \binom{n}{2} p_n \right) \cdot h'(x) \\ &\quad + o(h'(x)) \quad (x \rightarrow \infty) \end{aligned}$$

The results above are valid for a large class of compound p.d.f and may be applied in various domains of stochastic processes where random sums do appear as process characteristics (cf. risk-theory, dam-theory, queueing-theory, branching-processes etc ...).

In the next section we apply our results to the M/G/1 queue when the service time distribution is of the form (6)-(8). We concentrate on these types of p.d.f since among all p.d.f. in the class  $\mathcal{S}$  they arise most frequently in practise.

### 3 expansions for the waiting time distribution in an M/G/1 queue

Recall that the waiting time distribution of a delayed customer is given by (3). In this case  $p_n = (1 - \rho)\rho^{n-1}$ ,  $n \geq 1$ , and it is not hard to check that

$$\sum_{n=k}^{\infty} \binom{n}{k} p_n = \frac{\rho^{k-1}}{(1-\rho)^k} \quad (k \geq 1).$$

Furthermore if the service time distribution  $F$  is of the form (6)- (8) straightforward calculation shows that  $H(x) := \mu \int_0^x 1 - F(y) dy$  satisfies the conditions of lemma 1 or 2. Applying corollary 1, we thus get the following expansion of  $\psi(x)$ .

**Corollary 2** *Let  $\psi(x)$  be given by (3) and suppose  $F$  is of the form (6)- (8). Let  $\frac{1}{\mu_i} := \int_0^{\infty} x^i f(x) dx$  and  $\frac{1}{\mu_1} = \frac{1}{\mu}$ . Then*

$$\begin{aligned} \psi(x) &= \frac{\mu}{1-\rho} \left( \int_x^{\infty} (1 - F(y)) dy \right) + \frac{\mu^2 \rho}{\mu_2 (1-\rho)^2} (1 - F(x)) \\ &\quad + \left( \frac{3\mu^3 \rho^2}{4\mu_2^2 (1-\rho)^3} + \frac{\mu^2 \rho}{3\mu_3 (1-\rho)^2} \right) f(x) + o(f(x)) \quad (x \rightarrow \infty) \end{aligned}$$

It follows from the theory of the previous section that  $o(f(x))$  may be interpreted as  $O(f'(x))$ . Notice that the proportional descend rate of successive order terms

in the expansion is governed by the failure rate  $f(x)/(1 - F(x))$ ; the faster the rate at which  $f(x)/(1 - F(x))$  tends to zero, the better the approximation of  $\psi(x)$  for smaller  $x$ . However, the coefficients of the order terms grow geometrically fast with a factor  $\rho/(1 - \rho)$ , so for  $\rho$  close to 1, it may be expected that corollary 2 will only provide a decent approximation for  $x$  large. It turns out in practise that the approximation starts to perform quite well beyond the 0.999 percentile of the waiting time distribution. If one is interested in approximating central and intermediate waiting time probabilities, an alternative is to act as if the underlying service time distribution is exponentially bounded (e.g. by changing the original distribution into a phase-type distribution which results from a two moment fit) and then apply the exponential estimate in (4), see e.g. Seelen et al.(1985). Although such estimate is asymptotically not correct, it is superior to the approximation in corollary 2 for a large range of  $x$  values. This is partly explainable from the stability of the waiting time in an  $M/G/1$  queueing system with respect to changes in the service time and from the fact that a two moment fit of a distribution of the form (6)-(8) by a mixture of Erlang distributions performs very well on 99 % of the range of the distribution. Only (far) out in the tail the deviation becomes important, implying that in that range the expansion in corollary 2 should be used, rather than the exponential estimate.

### 3.1 extension to batch arrivals

It is interesting to note that the results of section 2 can also be applied to approximate the waiting time distribution of an individual customer in the  $M^{[X]}/G/1$  queue.

Suppose that the batch size distribution is given by  $(g_k)_{k=1}^{\infty}$  and denote the mean batch size as  $\gamma = \sum_{k=1}^{\infty} kg_k$  (As to other quantities we stick to the notation of the previous sections). Cohen(1976) showed that if  $\rho\gamma < 1$ , the stationary waiting time distribution is given by

$$P(W_{\infty} \leq x) = \int_0^x W_{SC}(x - y) dW_F(y) \quad (20)$$

with

$$W_F(x) = \sum_{k=1}^{\infty} F^{*k-1}(x) \left( \frac{1}{\gamma} \sum_{j=k}^{\infty} g_j \right)$$

and with  $W_{SC}$  the waiting time of a so called *super customer* (a complete batch). As the batches follow an ordinary  $M/G/1$  queue with service time distribution

$$F_{SC}(x) = \sum_{n=1}^{\infty} g_n F^{*n}(x),$$

we have that

$$W_{SC}(x) = (1 - \gamma\rho) \sum_{n=1}^{\infty} (\gamma\rho)^n H_{SC}^{*n}(x)$$

where

$$H_{SC}(x) = \frac{\mu}{\gamma} \int_0^x 1 - F_{SC}(y) dy.$$

In case the service time distribution is exponentially decreasing, it was recently shown by Van Ommeren(1988) that  $P(W_\infty > x | W_\infty > 0)$  satisfies an exponential estimate comparable to (4). The results we obtain below are complementary to the result of Van Ommeren as we concentrate on longtailed service time distributions. The following theorem provides a second order expansion for the waiting time in an  $M^{[X]}/G/1$  queue.

**Theorem 4** *Suppose  $\sum_{n=1}^{\infty} g_n z^n$  is analytic at 1 and let  $F$  be given by (6)-(8). Then*

$$\begin{aligned} & (1 - 1/\gamma + \rho) \psi(x) \\ &= \frac{\mu\gamma\rho}{1 - \gamma\rho} \int_x^\infty 1 - F(y) dy \\ &+ \left\{ \frac{1}{\gamma} \left( \sum_{j=2}^{\infty} \binom{j}{2} g_j \right) \left\{ \frac{3 - 2\gamma\rho}{1 - \gamma\rho} + 2 \left( \frac{\gamma\rho}{1 - \gamma\rho} \right)^2 \right\} + \frac{1}{\mu_2} \left( \frac{\mu\gamma\rho}{1 - \gamma\rho} \right)^2 \right\} (1 - F(x)) \\ &+ o(1 - F(x)) \quad (x \rightarrow \infty). \end{aligned}$$

Notice that the higher order term also involves higher moments of the batch size distribution. If the batch sizes are degenerate at 1, we end up with the first two terms of the expansion in corollary 2. The proof of theorem 4 is split up in several lemma's.

**Lemma 3** *Let  $F$  satisfy the conditions of lemma 1 or 2, and suppose that  $F$  has an ultimately decreasing failure rate. Then*

$$\lim_{x \rightarrow \infty} \frac{1 - H_{SC}^{*2}(x) - 2(1 - H_{SC}(x))}{h_{SC}(x)} = \frac{\mu}{\mu_2} + \frac{2}{\gamma\mu} \sum_{j=2}^{\infty} \binom{j}{2} g_j.$$

**Proof**

We will show that under the conditions of the lemma,

$$\lim_{x \rightarrow \infty} \frac{h_{SC}^{*2}(x) - 2h_{SC}(x)}{-h'_{SC}(x)} = \frac{\mu}{\mu_2} + \frac{2}{\gamma\mu} \sum_{j=2}^{\infty} \binom{j}{2} g_j.$$

Let  $A(x)$  be a positive function such that  $A(x) \rightarrow \infty$  ( $x \rightarrow \infty$ ) but  $A(x) \leq x/2$ . Then

$$\begin{aligned}
& h_{SC}^{\times 2}(x) - 2h_{SC}(x) \\
&= 2 \left(\frac{\mu}{\gamma}\right)^2 \int_0^{A(x)} (F_{SC}(x) - F_{SC}(x-y))(1 - F_{SC}(y)) dy \\
&\quad - 2 \left(\frac{\mu}{\gamma}\right)^2 (1 - F_{SC}(x)) \int_{A(x)}^{\infty} (1 - F_{SC}(y)) dy \\
&\quad + 2 \left(\frac{\mu}{\gamma}\right)^2 \int_{A(x)}^{x/2} (1 - F_{SC}(x-y))(1 - F_{SC}(y)) dy \\
&= (I) - (II) + (III).
\end{aligned}$$

From the conditions on  $F$ , we have that  $f_{SC}(x) \sim \gamma f(x)$  (see e.g. Chover et al.(1973)) such that  $F_{SC}$  inherits the asymptotic properties of  $F$ . By appropriate choice of  $A(x)$  and proceeding in exactly the same way as in the proof of lemma 2, it follows that

$$(I) \sim 2f_{SC}(x) \left(\frac{\mu}{\gamma}\right)^2 \int_0^{\infty} y(1 - F_{SC}(y)) dy \quad (x \rightarrow \infty)$$

and that  $(II) + (III) = o(f_{SC}(x))$  ( $x \rightarrow \infty$ ). Since

$$2f_{SC}(x) \left(\frac{\mu}{\gamma}\right)^2 \int_0^{\infty} y(1 - F_{SC}(y)) dy = -h'_{SC}(x) \left(\frac{\mu}{\mu_2} + \frac{2}{\gamma\mu} \sum_{j=2}^{\infty} \binom{j}{2} g_j\right),$$

the proof is finished.

**Lemma 4** *Let  $F$  satisfy (12) and suppose  $\sum_{n=1}^{\infty} g_n z^n$  is analytic at  $z=1$ . Then*

$$\begin{aligned}
& 1 - H_{SC}(x) \\
&= \mu \int_x^{\infty} 1 - F(y) dy + \frac{2}{\gamma} \left( \sum_{j=2}^{\infty} \binom{j}{2} g_j \right) (1 - F(x)) \\
&\quad + o(1 - F(x)) \quad (x \rightarrow \infty).
\end{aligned}$$

**Proof**

By theorem 2 and de l'Hôpital's rule,

$$1 - H_{SC}(x) - \mu \int_x^{\infty} 1 - F(y) dy$$

$$\begin{aligned}
&= \frac{\mu}{\gamma} \int_x^\infty [(1 - F_{SC}(y)) - \gamma(1 - F(y))] dy \\
&\sim \frac{2}{\gamma} \left( \sum_{j=2}^\infty \binom{j}{2} g_j \right) (1 - F(x)) + o(1 - F(x)) \quad (x \rightarrow \infty).
\end{aligned}$$

**Lemma 5** *Suppose the conditions of lemma 4 hold. Then*

$$\begin{aligned}
\text{(i)} \quad 1 - W_F(x) &= \frac{1}{\gamma} \left( \sum_{j=2}^\infty \binom{j}{2} g_j \right) (1 - F(x)) + o(1 - F(x)) \quad (x \rightarrow \infty) \\
\text{(ii)} \quad W'_{SC}(x) &= \frac{\mu\gamma\rho}{1 - \gamma\rho} (1 - F(x)) + o(1 - F(x)) \quad (x \rightarrow \infty).
\end{aligned}$$

**Proof**

- (i) Straightforward from theorem 1.  
(ii) Clearly

$$W'_{SC}(x) = (1 - \gamma\rho) \sum_{n=0}^\infty (\gamma\rho)^n h_{SC}^{\times n}(x)$$

with  $h_{SC}(x) = \frac{\mu}{\gamma}(1 - F_{SC}(x))$ . Since  $F$  satisfies (12), it follows from de l'Hôpital's rule that also

$$\lim_{x \rightarrow \infty} \int_0^x \frac{(1 - F(x-y))(1 - F(y))}{1 - F(x)} dy = \frac{2}{\mu}.$$

Furthermore, by theorem 1,  $1 - F_{SC}(x) \sim \gamma(1 - F(x)) \quad (x \rightarrow \infty)$  such that

$$\lim_{x \rightarrow \infty} \int_0^x \frac{(1 - F_{SC}(x-y))(1 - F_{SC}(y))}{1 - F_{SC}(x)} dy = \frac{2\gamma}{\mu}$$

or

$$\lim_{x \rightarrow \infty} \frac{h_{SC}^{\times 2}(x)}{h_{SC}(x)} = 2.$$

Hence  $h_{SC}(x)$  is a subexponential density and it was shown by Chover et al.(1973) that in this case,

$$\begin{aligned}
W'_{SC}(x) &= \frac{\gamma\rho}{1 - \gamma\rho} h_{SC}(x) + o(h_{SC}(x)) \quad (x \rightarrow \infty) \\
&= \frac{\mu\gamma\rho}{1 - \gamma\rho} (1 - F(x)) + o(1 - F(x)) \quad (x \rightarrow \infty).
\end{aligned}$$



**Proof of theorem 4**

From (20),

$$P(W_\infty > x) = \int_0^x (1 - W_F(x-y)W'_{SC}(y)) dy + 1 - W_{SC}(x).$$

By lemma 3,  $H_{SC}$  satisfies (12) such that by theorem 2,

$$\begin{aligned} & 1 - W_{SC}(x) \\ &= \frac{\gamma\rho}{1-\gamma\rho}(1 - H_{SC}(x)) \\ &+ \left(\frac{\gamma\rho}{1-\gamma\rho}\right)^2 \left(\frac{\mu}{\mu_2} + \frac{2}{\gamma\mu} \sum_{j=2}^{\infty} \binom{j}{2} g_j\right) h_{SC}(x) \\ &+ o(1 - F(x)) \quad (x \rightarrow \infty). \end{aligned}$$

By lemma 5 and the fact that  $F$  satisfies (12), we may apply lemma 3.1.1 of Omev and Willekens(1987) to find that

$$\begin{aligned} & \int_0^x \frac{1 - W_F(x-y)}{1 - F(x)} W'_{SC}(y) dy \\ &= \lim_{x \rightarrow \infty} \frac{W'_{SC}(x)}{1 - F(x)} \int_0^\infty (1 - W_F(y)) dy \\ &+ \lim_{x \rightarrow \infty} \frac{1 - W_F(x)}{1 - F(x)} \int_0^\infty W'_{SC}(y) dy + o(1) \quad (x \rightarrow \infty) \\ &= \left\{ \frac{\rho}{1-\gamma\rho} \sum_{j=2}^{\infty} \binom{j}{2} g_j + \frac{1}{\gamma} \sum_{j=2}^{\infty} \binom{j}{2} g_j \right\} \\ &+ o(1) \quad (x \rightarrow \infty) \\ &= \frac{1}{1-\gamma\rho} \left\{ \frac{1}{\gamma} \sum_{j=2}^{\infty} \binom{j}{2} g_j \right\} + o(1) \quad (x \rightarrow \infty). \end{aligned}$$

The desired expression now follows from a combination of the results above and the fact that  $P(W_\infty = 0) = \frac{1}{\gamma}(1 - \gamma\rho)$ .

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